POSITIVE BRAIDS AND LORENZ LINKS

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EL-SAYED AHMED EL-RIFAI

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ABSTRACT

POSITIVE BRAIDS AND LORENZ LINKS by EL-SAYED EL-RIFAI

In this work a new foundation for the study of positive braids in Artin's braid groups is given. The basic braids considered are the set SB_n of positive permutation braids, defined as those positive braids where each pair of arcs cross at most once. These are shown to be in 1-1 correspondence with the permutations in S_n. A canonical form for positive braids as products of braids in SB_n is given, together with an explicit algorithm for writing every positive braid in canonical form and a practical test for use in the algorithm. This is a useful approach to braid theory because permutations can be particularly easily handled.

Applications of this canonical form are:

(1) An easily handled approach to Garside's solution of the word problem in B_n.

(2) An algorithm to decide whether $(\Delta_n)^k$ is a factor of a positive braid; this happens if and only if at most k canonical factors have equal to Δ_n (where Δ_n is the positive braid with each pair of arcs cross exactly once).

(3) A proof that a positive braid is a factor of $(\Delta_n)^k$ if and only if its canonical form has at most k factors.

(4) An improvement of Garside's solution of the conjugacy problem, this is by reducing the summit set to a much smaller invariant class under conjugation (super summit set). This includes a necessary and sufficient condition for positive braid to contain Δ_n up to conjugation.

The linear generators of the Hecke algebras used by Morton and Short are in 1-1 correspondence with the elements of SB_n. The canonical forms above give a quick proof that the number of strands in a twist positive braid (one of the form $(\Delta_n)^{2m}P$ for positive braid P and for positive integer m) is the braid index of the closure of that braid, which was first proved by Franks and Williams. It is also shown that if the 2-variable link invariant $P_L(v,z)$ for an oriented link L has width k in the variable v, then it is the same as the polynomial of a closed k-braid, for k = 1, 2. A complete list of 3-braids of width 2, which close to knots, is given. It is also shown that twist positive 3-braids do not admit exchange moves (in the sense of Birman). Consequently the conjugacy class of a twist positive 3-braid representative is a complete link invariant, provided that Birman's conjecture about Markov's moves and exchange moves holds.

Lorenz knots and links are studied as an example of positive links. It is proved that a positive permutation braid π is a Lorenz braid if and only if all braid words which equal π have the same single starting letter. A semicanonical form for a minimal braid representative of a

Lorenz link is given. It is proved that every algebraic link of c components is a Lorenz link, for c = 1, 2. (The case for knots was first proved by Birman and Williams). Consequently a necessary and sufficient condition for a knot to be algebraic is given, together with a canonical form for a minimal braid representative for every algebraic knot. To some extent the relation between Lorenz knots and their companions is studied.

It is shown that Lorenz knots and links of braid index 3 are determined by conjugacy classes in B_3. A complete list of 3-braids which close to Lorenz knots and links is given and a complete list of pure 4-braids which close to Lorenz links is also given. It is shown that Lorenz knots and links of braid index 3 are determined by their Alexander polynomials. As a further analogy with the properties of algebraic links it is shown that the linking pattern of a Lorenz link L with pure braid representative and braid index $t \leq 4$, determines a unique braid representative for L and so determines L.

This Thesis is dedicated to

my daughter Ranya on her third birthday

and in loving memory of

my uncle Abu Al-Futuh El-Rifai

and my mother-in-law,

may *ALLAH* forgive me, them and my parents.

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Finally, I wish to express my gratitude to my wife for her support and constant encouragement throughout the period of my research.

CONTENTS

CHAPTER 0

PRELIMINARIES

§0.1 FOUNDATIONS :

(0.1) Definition: (Knots and links), [Ro]

A link $L \subset X$ in a space X is the union of μ-simple closed polygonal curves, embedded in X, where the case $\mu=1$ is called a knot. A polygonal knot is one which is the union of finite number of closed straight-line segments. A knot is tame if it is equivalent to a polygonal knot, otherwise it is wild. All knots and links in this work are assumed to be classical $S^1 \subset \mathbb{R}^3$, or $S^1 \subset S^3$, and tame.

(0.2) Definition: (Equivalent knots), [Ro]

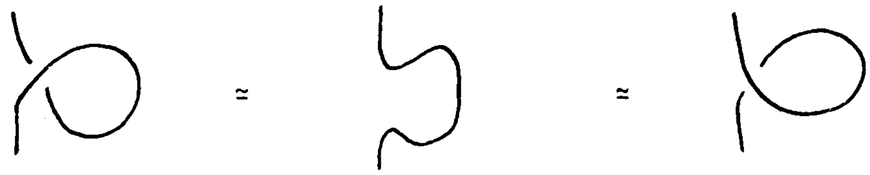
Two links L, $L' \subset S^3$ are equivalent if there is a homeomorphism $h: S^3 \to S^3$, such that $h(L) = h(L')$. i.e. $(S^3, L) \simeq (S^3, L')$.

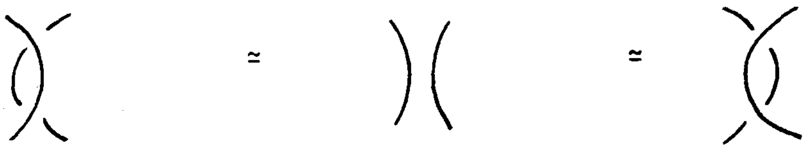
(0.3) Definition: (Link diagram, regular projection), [R]

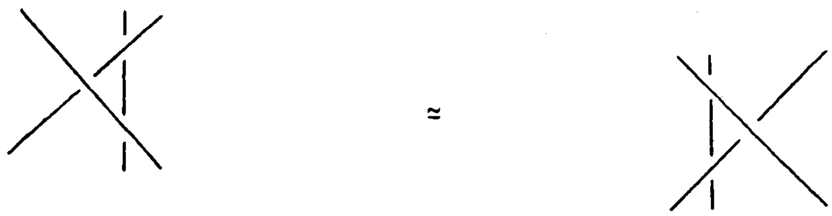
A link diagram $D(L)$ for a link L is a projection to $\mathbb{R}^3$ with only a finite number of crossings, such that at the neighbourhood of each crossing only two arcs cross transversely.

(0.4) Theorem: (Reidemeister), [R]

Two links L_1, and L_2 are ambient isotopic if and only if a diagram of L_1 can be altered to a diagram of L_2 by a sequence of three moves:

(i): 

(ii): 

(iii): 

<u>(0.5) Definition</u>: (Braid group), [B2]

Define the braid group B_n as the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations:

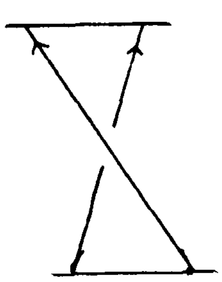
(i) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\quad 1 \leqslant i \leqslant n-2$

(ii) $\sigma_i\sigma_j = \sigma_j\sigma_i$ $\quad |i-j| \geqslant 2,\ 1 \leqslant i,j \leqslant n-1$

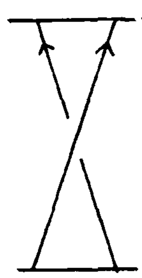
The pair (α, n) will be referred to as a word $\alpha \in B_n$ for some $n \in \mathbb{Z}^+$. The geometric braids representing σ_i and $(\sigma_i)^{-1}$ are illustrated in figure (0-1a).

<u>(0.6) Definition</u>: (Closed braids), [B2]

The closed braid β^c from a braid word β is formed by tying the top end to each string to the same position on the bottom of the braid β as shown in figure (0-1b).

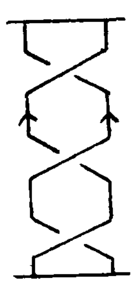


geometric braid $(\sigma_1)^{-1}$

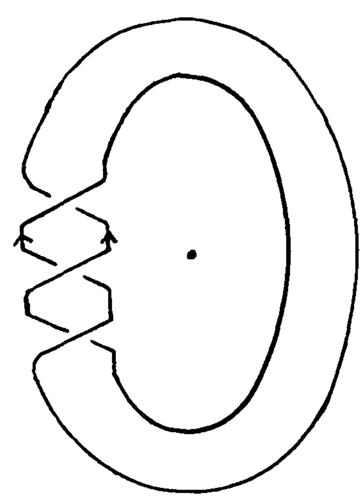


geometric braid σ_1

Figure (0-1a)



$(\sigma_1)^3$



$((\sigma_1)^3)^c \simeq$ trefoil

Figure (0-1b)

<u>(0.7) Theorem</u>: (Alexander), [B2]

Every oriented link L can be represented as the closure β^c of some (β,n).

<u>(0.8) Theorem</u>: (Markov), [B2]

Any two braids whose closures are the same oriented link, up to isotopy, are related by a sequence of moves of type:

$$(i): \ (\beta,n) \sim (\alpha^{-1}\beta\alpha,n), \text{ for some } (\alpha,n)$$
$$(ii): \ (\beta,n) \sim (\beta(\sigma_n)^{\pm 1},n+1)$$

<u>(0.9) Definition</u>: (Braid index), [B2]

A link β^c has braid index n if it can be represented by a braid (β,n), but can not be represented by a braid $(\beta',n-1)$.

<u>(0.10) Definition</u>:

For a braid (α,n), let $\rho[\alpha]$ denote to the rotation through angle π about the centre axis (perpendicular to the plane of the diagram of α) followed by arrow reversed. Then $\rho[\alpha]$ is the reverse of α. Also let $\tau[\alpha]$ be the reflection in the plane of the diagram of α, followed by changing the sign of crossings, i.e. rotation about vertical axis. Hence as a braid word, $\rho[\alpha]$ is the word α read backwards, and $\tau[\alpha]$ is the result of turning over α.

<u>(0.11) Definition</u>: (Symmetric group)

Define the symmetric group S_n as the group generated by the transpositions $\tau_1, \tau_2, \ldots, \tau_{n-1}$ where $\tau_i = (i,i+1)$ subject to the following relations:

(i) $\tau_i^2 = e$ $\quad 1 \leqslant i \leqslant n-1$

(ii) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ $\quad 1 \leqslant i \leqslant n-2$

(iii) $\tau_i \tau_j = \tau_j \tau_i$ $\quad |i-j| \geqslant 2, \quad 1 \leqslant i,j \leqslant n-1$

<u>(0.12) Definition</u>: (Companion, satellite, and cable knots), [Ro]

Let K be a knot in a 3-space S^3 and V an unknotted solid torus in S^3 with $K \subset V \subset S^3$. Assume that K is geometrically essential (not contained in a 3-ball of V). A homeomorphism $h: V \to U \subset S^3$ onto a tubular neighbourhood U of a non-trivial knot $C \subset S^3$ which maps a meridian of S^3-V onto a longitude of U and maps K onto a knot $K_1 = h(K)$. The knot K_1 is called a satellite of C and C is its companion. If K is the (p,q) torus knot on the boundary of V, and h is faithful, then K_1 is called the (p,q) cable on its companion C, or simply a cable knot.

<u>(0.13) Definition</u>: (Algebraic knots and links), [E-N]

Let f(x,y) be a complex plane polynomial vanishing at the origin, and let

$$V = \{(x,y) \in \mathbb{C}^2 \mid f(x,y)=0\} = f^{-1}(0)$$

be the corresponding plane curve. For all sufficiently small $\varepsilon > 0$, the 3-space

$$S_\varepsilon^3 = \{(x,y) \in \mathbb{C}^2 \mid |(x,y)| = \varepsilon\}$$

meets V transversely in a link, which has a natural orientation coming from that of V, i.e. $L = V \cap S_\varepsilon^3$ a union of closed curves. An oriented link (S_ε^3, L) determined in this way is said to be an algebraic links. If L is connected, it is called the algebraic knot. Then solving f(x,y) = 0 for y in terms of x, obtaining a set of solutions which are fractional power series in x. Each fractional power series solution gives rise to a branch of the curve, and thus to one component of the link. It is known that all but finitely many terms of the power series can be removed without changing the topology of the link. Also the resulting minimal series are written in the form

$$y = x^{(q_1/p_1)}[a_1 + x^{(q_2/p_1p_2)}[a_2 + \dots + [a_{s-1} + x^{(q_s/p_1p_2 \dots p_s)}[a_s + \dots] \dots]] \qquad (0.1)$$

with p_i, $q_i > 0$, and (p_i, q_i) respectively prime for all i.

(0.14) Proposition: Murasugi.K, [Mu2]

Any element of B_3 is conjugate to one and only one element of some Λ_i, where

$\Lambda_0 = \{(\Delta_3)^{2n} \mid n = 0, \pm 1, \dots \}$

$\Lambda_1 = \{(\Delta_3)^{2n}\sigma_1\sigma_2 \mid n = 0, \pm 1, \dots \}$

$\Lambda_2 = \{(\Delta_3)^{2n}(\sigma_1\sigma_2)^2 \mid n = 0, \pm 1, \dots \}$

$\Lambda_3 = \{(\Delta_3)^{2n+1} \mid n = 0, \pm 1, \dots \}$

$\Lambda_4 = \{(\Delta_3)^{2n}(\sigma_1)^{-p} \mid n = 0, \pm 1, \dots ; p = 1, 2, \dots \}$

$\Lambda_5 = \{(\Delta_3)^{2n}(\sigma_2)^{q} \mid n = 0, \pm 1, \dots ; q = 1, 2, \dots \}$

$\Lambda_6 = \{(\Delta_3)^{2n}(\sigma_1)^{-(p_1)}(\sigma_2)^{(q_1)} \dots (\sigma_1)^{-(p_r)}(\sigma_2)^{(q_r)} \mid n = 0, \pm 1, \dots ;$
p_i, q_i are positive integers$\}$

§0.2. LIST OF SYMBOLS

B_n	The Artin's braid group of n-strands
$B(\beta)$	The base of braid word β
$B(k,r)$	The class of Lorenz braids of type $\beta(k,r)$
$BL(\beta)$	The base length of a braid word β
$c(\alpha)$	The exponent sum of the braid α
$CL(P)$	The canonical length of a positive braid P
$\mathbb{C}$	The set of complex numbers
e	The identity element in B_n and in S_n
$F(P)$	The finishing set of a positive braid P
I_P	The set of all initial positive permutation braid factors of a positive braid P
$L(\alpha)$	The length of α, for $\alpha \in B_n$, or $\alpha \in S_n$
$\mathbb{R}$	The set of real numbers
$S(P)$	The starting set of a positive braid P
S_n	The symmetric group
SB_n	The set of all positive permutation braids in B_n
$SS(\beta)$	The summit set of a braid word β
$SSS(\beta)$	The super summit set of a braid word β
$\cup_b$	The join bottom operator in SB_n
$W(\beta)$	The maximal number of Δ_n in a braid word β
X_i	The Lorenz braid $\beta(1,i)$
Y_i	The Lorenz braid $\beta(i,1)$
$\mathbb{Z}$	The set of integers
$\mathbb{Z}^+$	The set of positive integers, $0 \in \mathbb{Z}^+$
$=^c$	Equal up to conjugation, in B_n

(α,n)	A braid $\alpha\in B_n$, for some $n\in\mathbb{Z}^+$
α^c	A closed braid $\alpha\in B_n$, for some $n\in\mathbb{Z}^+$
$\beta(k,r)$	The Lorenz braid of permutation $\pi(k,r)$
β_π	The associated positive permutation braid to the permutation $\pi\in S_n$
$(\beta_\pi)_-$	The associated negative permutation braid to the permutation $\pi\in S_n$
δ	The permutation $\delta\in S_n$, with $\delta(i) = n-i+1$
Δ_n	The half twist braid in B_n, $\beta_\delta = \Delta_n$
$\Delta_{i,\leftarrow}$	The half twist braid in the last i-strands in B_n, $i \leqslant n$
π^*	The upper complement of π in δ, i.e. $(\pi^*)\pi = \delta$
π_*	The lower complement of π in δ, i.e. $\pi(\pi_*) = \delta$
$\tau[\alpha]$	The conjugation of α by Δ_n
$\rho[\alpha]$	The result of reading α backwards

INTRODUCTION

The central theme of this thesis is the study of positive braids in Artin's braid groups. Positive braids are particularly attractive to link theory since positive links, the closure of positive braids, are fibred and include the torus links, the algebraic links which occur as isolated singularities of algebraic equations and the Lorenz links of periodic orbits of dynamical systems. The class of positive links was first studied by Burau, [Bu], and later studied by many researchers, e.g. [Mu1], [S].

In chapter 1 we introduced a construction for factoring every positive braid, as a product of positive permutation braids, those braids where each pair of arcs cross at most once.

Section 1.1 deals with a characterisation of positive permutation braids. A complete list of factors and factor pairs for the fundamental braid Δ_n is given, where it is shown that SB_n (the set of all positive permutation braids in B_n) is the set of all possible factors of Δ_n. The characteristic properties for the braid Δ_n are also explored.

Section 1.2 is concerned firstly, with the proof of the main result of chapter 1, the canonical form for every positive braid. Secondly, this section provides a method for shortening the work required to decide whether $(\Delta_n)^k$ is a factor of a positive braid; this happens if and only if at most k canonical factors have equal to Δ_n. This includes a proof that a positive braid is a factor of $(\Delta_n)^k$ if and only if its canonical form has at most k factors.

Section 1.3 describes a practical algorithm for finding the canonical form for every positive braid. It also gives a practical test for use in the algorithm.

Section 1.4 is devoted to applications of the canonical form for every positive braid. An efficient normal form for Garside's solution of the word problem, in B_n, is given. While an algorithm to decide whether a positive braid contains Δ_n (up to conjugation) is given, together with a necessary and sufficient condition for a positive braid to contain Δ_n (up to conjugation) is given. Finally this section contains an improvement of Garside's solution of the conjugacy problem in B_n, this is by reducing the summit set to an invariant, under conjugation, subclass (super summit set). It is also shown that any two super summit forms, for a given braid, are conjugate through such these forms, by a sequence of positive permutation braid conjugations. Consequently it is proved that two braids are conjugate if and only if their super summit sets are identical.

Chapter 2 is concerned to the study of twist positive braids (those of the form $(\Delta_n)^{2m}P$ for a positive braid P and for a positive integer m) which are interested subclass of positive braids.

In section 2.1, it is noticed that the linear generators of the Heche algebras used by Morton and Short, [M-S1], are in 1-1 correspondence with the elements of SB_n. The canonical forms above give a quick proof that the number of strands in a twist positive braid is the braid index of the closure of that braid, which was first proved by Franks and Williams, [F-W].

Section 2.2 is concerned to the width of the 2-variable link invariant $P(v,z)$ in the variable v, where the width is the minimal number of strands allowed by the index bound. It is proved that if the polynomial $P(v,z)$ of width i, then it is the same as the polynomial of a closed i-braid, for i = 1, 2. A complete list of 3-braids of width 2, which close to knots, is given.

Section 2.3 is devoted to the study of Birman's "exchange moves" in B_3. It is proved that twist positive 3-braids do not admit non trivial exchange moves. Consequently the conjugacy class of a twist positive 3-braid representative is a complete link invariant, provided that Birman's conjecture about Markov's moves and exchange moves holds.

Chapters 3, and 4 are devoted to the study of Lorenz knots and links, those which represent the periodic orbits in the solutions of Lorenz differential equations.

In section 3.1 the class of Lorenz braids is widened to include all positive permutation braids which can not written as positive words in B_n with more than one starting letter. It is proved that every algebraic link of c components is a Lorenz link, for c = 1, 2. (The case for knots was first proved by Birman and Williams, [B-W1]). Consequently a necessary and sufficient condition for a knot to be algebraic is given. Finally a semicanonical form for a minimal braid representative for every Lorenz link is given, with a canonical form for a minimal braid representative for an algebraic knot.

Section 3.2 is devoted to the study of the possible satellites of a Lorenz knot. It is shown in section 3.1 that every Lorenz link is a closed braid, which must follow some pattern (called Lorenz presentation pattern). Hence the Lorenz knots which are satellites of other Lorenz knots should also follow that presentation pattern. It is also shown in section 3.1 that the only way in which a Lorenz knot appears as a represented cable in Lorenz presentation pattern is when it is an algebraic knot. So it is a very plausible conjecture that these are the only ways in which a Lorenz knot can be presented as a satellite, although attempts to prove it using an extension of Williams methods, [W2], have so far been unsuccessful.

Chapter 4 is concerned to the study of Lorenz links of pure braid representatives. As a further analogy with the properties of the algebraic links, it is shown that every Lorenz link of braid index k with k components is determined by the associated linking pattern of its components, for $k \leqslant 4$.

Section 4.1 is concerned to Lorenz knots and links in B_3. It is proved that Lorenz knots and links with braid index 3 are determined by Alexander polynomial. In fact it is shown that Alexander polynomial for a Lorenz knot or link L with braid index 3 determines a unique braid representative for L and so determines L. A complete list of 3-braids which close to Lorenz knots and links is given.

Section 4.2 is devoted to the study of Lorenz links of braid index 4 with 4 components. A complete list of pure 4-braids which close to Lorenz links is given. It is also proved that the linking pattern of a Lorenz link L of braid index 4 with 4 components determines a unique 4-braid representative for L and so determines L.

At the introduction of each chapter, in more specific and technical detail the results achieved are described with the problems led to this work and with their historical settings.

CHAPTER 1

ON BRAID GROUPS

§1.0 INTRODUCTION :

In the braid group B_n of Artin, [Ar1] and [Ar2], the word problem was solved by Artin himself, but the conjugacy problem waited many years for the solution of Garside, [G1] and [G2], where he also gave a further solution of the word problem. Garside's solutions are purely algebraic and mainly depend on a diagram (Cayley diagram) which represents the generators and the relators of a group, [C1] and [C2]. The solutions also depend on a special braid word Δ_n, called fundamental, as defined below.

In Cayley diagram, the multiplication table of a given group G with given presentation can be represented in a diagram having one vertex for each element of the group, where edges represent generators and its inverses. Any vertex may be taken as origin and the others may be traced out from there. The drawn diagram can show the initial letter on the bottom and the others extending in order to top. The factors of a word P in the group are the possible subdiagrams of the diagram of P.

In Garside's solution of the word problem in B_n, it is shown that each braid word α admit a unique normal form, called standard, $\alpha = (\Delta_n)^m P$, $m \in \mathbb{Z}$ and P is a positive word. The integer m is called the power of α and the word P is called the base of α. For conjugacy problem in B_n, Garside also shown that one may choose a unique representative $(\Delta_n)^r Q$ for the conjugacy class of the braid (α,n). The integer r is called the summit power of α and the positive word Q is called the summit base of α, [G1] and [G2].

Let $(\Delta_n)^r R$ be the standard form for a braid (α,n). But the braid Δ_n itself has many factorizations into two factors. If PQ is one of these factorizations, then $P(\Delta_n)^r RP^{-1}$ may equal $(\Delta_n)^{r+1}T$ in the standard form of $P\alpha P^{-1}$. So that take all conjugations of $(\Delta_n)^r R$, where conjugators are all possible factors of Δ_n. From yielding words consider those which are of power $\geqslant r$ and which are distinct from $(\Delta_n)^r R$ and from each other. Now repeat the process for each of the words yield by previous step, where the condition being always that each new word must be of power $\geqslant r$. Continue to repeat the process for every new word arising, hence Garside shown that a stage must be reached when further applications of the process will yield no new words. So the device of generating conjugate braids by working through the factors of Δ_n can be used to raise the power of Δ_n only so far. The braids containing Δ_n raised to that highest power, called the summit power, form the summit set of all the braids from which it can be so reached. Garside proved that two braids are conjugate if and only if their summit sets are identical, [G1] and [G2].

Then within each braid group B_n, both solutions of Garside require the use of extensive lists of factor pairs for Δ_n. Throughout upon the definitions and the notations of Garside, Thomas.R.S.D gave an algorithm for writing down the factor lists of Δ_n, [T]. He proved inductively that each Cayley diagram of Δ_n is made up of n copies of Cayley diagram of Δ_{n-1} linked by σ_{n-1} and so on down to the Cayley diagram of Δ_2. He also shown that the Cayley diagram of Δ_n has n! vertices, which is the all possible factors of Δ_n. But Thomas's algorithm has the same nature of Garside's technique which is completely algebraic, quite long and quite difficult to apply.

In the following paragraph some definitions are stated, where the abstract definition of braid group B_n is given in definition (0.5).

(1.0.1) Definition: (The fundamental word or the half twist), [B2]

In B_n, the braid which is accomplished by holding the top of the braid fixed and attaching the string bottoms to a rod which is then turned over once (in a positive sense), is known as a half twist positive braid Δ_n and

$$\Delta_n = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1})$$

(1.0.2) Definition: (Positive braids and twist positive braids)

A braid (p,n) consisting of an ordered sequence of the generators only, in which no inverse of any generator occurs will be called a positive braid. A positive braid P is a twist positive braid if $P = (\Delta_n)^{2m}Q$, for $m \in \mathbb{Z}^+$, $m \neq 0$ and Q is a positive braid.

(1.0.3) Definition: (Factor pairs for a positive braid)

For a positive braid (α, n), the positive braid β is called a factor of α if and only if there exists a positive braid γ such that either $\alpha = \gamma\beta$ or $\alpha = \beta\gamma$, the pair $\{\beta, \gamma\}$ is called a factor pair for α.

Section 1 is devoted to the study of the factors of Δ_n. It is proved in theorem (1.1.14) that a positive braid (α,n) is a factor of Δ_n if and only if each pair of arcs in the diagram of α cross at most once. In theorem (1.1.4) it is proved that the permutations in S_n are in 1-to-1 correspondence with the set of factors of Δ_n, denoted SB_n. But the geometric relations between S_n and SB_n are shown in definition (1.1.1) and in lemma (1.1.3), where the elements of SB_n are called the <u>positive permutation braids</u>. In fact the conception of positive permutation braids was first introduced by Morton and Traczyk, [M-T]. In corollary (1.1.15) the list of all possible factor pairs for Δ_n is given, i.e. the list of all possible positive braids P's and Q's such that $\Delta_n = PQ$. In fact Q is the lower complement of P in Δ_n, where Δ_n is the largest positive permutation braid, in B_n, as shown in lemma (1.1.10).

A necessary and sufficient condition for a generator $\sigma_i \in B_n$ to be a starter and a finisher of an element in SB_n is given in lemma (1.1.8), where σ_i is a starter for a positive braid (P,n) if there exists a positive braid Q, such that $P = \sigma_i Q$ and σ_i is a finisher for P if there is a positive braid R such that $P = R\sigma_i$, as in definition (1.1.7). If σ_i is a starter or finisher for a positive braid, then simply let i denote to σ_i.

The recognition results of the fundamental braid Δ_n are given, in lemma (1.1.10), where it is shown that every i, $1 \leqslant i \leqslant n-1$, is a starter and finisher for Δ_n. It is shown also that each two arcs in Δ_n cross exactly once. The conjugation of a braid by Δ_n is shown in lemma (1.1.11). Following that it is proved in corollary (1.1.12) that

$(\Delta_n)^2$ lies in the centre of B_n, which is geometrically obvious where passing a braid α through $(\Delta_n)^2$, the full twist, it means that the diagram of α has turned over twice. At the end of section 1 the recognition results for the factor pairs for Δ_n are given in lemma (1.1.16).

In fact section 1 contains some technical results on factors of Δ_n and on the starters (finishers) of a braid $\pi \epsilon SB_n$ which are used frequently in this work.

Section 2 is devoted to find a canonical form for positive braids as products of positive permutation braids. In theorem (1.2.1) it is proved that every positive braid P can be written uniquely as a product of positive permutation braids, $P = \pi_1\pi_2 \ldots \pi_k$, where π_i is the largest possible positive permutation braid as a starter of $(\pi_i\pi_{i+1} \ldots \pi_k)$, for $1 \leqslant i \leqslant k$.

The proof of theorem (1.2.1) is begun with some definitions and lemmas. In definition (1.2.4) the set I_P of all initial positive permutation braid factors of a positive braid P is given with some ordering operator. It is proved, in proposition (1.2.7), that I_P has a maximal element for each two of its elements, while corollary (1.2.8) provided a unique maximal element in I_P, i.e. a unique maximal starter for P. The relation between the starting set of a positive braid and the starting set of its maximal starter is given in corollary (1.2.9) Now let $P = \pi_1 P_1$ where $\pi_1 \epsilon SB_n$ and P_1 is a positive braid, then π_1 is the maximal starter for P if and only if $S(P_1) \subseteq F(\pi_1)$, as shown in proposition (1.2.10). At this stage the proof of theorem (1.2.1) is also given.

As a further analogy with the left-hand canonical form of a positive braid, it is shown that every positive braid has also a right-hand canonical form with similar properties, as shown in remark (1.2.14). Now let $AB = \Delta_n R$ for positive braids A, B and R, then it is shown that $A = A_1\pi$ and $B = \pi_* B_1$ for some positive braids A_1, $B_1 \in B_n$ and $\pi \in SB_n$ (as in lemma (1.2.15)) and it is proved that $F(A) \cup S(B) = \{1,2, \ldots ,n-1\}$ (as in corollary (1.2.16)).

The positive braid (P,n) contains Δ_n if and only if $P = A\Delta_n B$ for some positive braids A and B, as in definition (1.2.11). Necessary and sufficient conditions for the positive braid P to contain Δ_n are given; this happens if and only if Δ_n is the maximal element in I_P (as shown in lemma (1.2.12)) and it is also happens if and only if $S(P) = \{1,2, \ldots ,n-1\}$, as shown in corollary (1.2.13).

For a positive braid P, if at most k canonical factors of P have equal to Δ_n, then $P = (\Delta_n)^k Q$ for some positive and prime (to Δ_n) braid Q, hence $(\Delta_n)^k$ is a factor of P. An algorithm to decide whether a positive braid $P = AB$ contains $(\Delta_n)^k$ is given; this is by writing A and B in their canonical factorizations, then look to the factors, as in theorem (1.2.17). This algorithm provides a method for shortening the work required to decide if a given positive braid has $(\Delta_n)^k$ as a factor or not.

Following that it is shown some properties of the factors of $(\Delta_n)^k$. A necessary and sufficient condition for a positive braid P to be a factor of $(\Delta_n)^k$ is given; this happens if and only if the canonical form of P has at most k factors, as in theorem (1.2.18).

It is also shown that every factor of $(\Delta_n)^k$ has property that every pair of arcs cross at most k times, as in proposition (1.2.19). But not every such positive braid with each two arcs cross at most k times is a factor for $(\Delta_n)^k$, an example to show that is given in example (1.2.20). Finally a geometric view of the factors of $(\Delta_n)^2$ is presented in proposition (1.2.21).

In section 3, a practical algorithm for writing a positive braid in its canonical form is given, as in (1.3.1). Starting with a positive braid (P,n), write P as a successive product of generators. Then bracket the successive letters of the word P as a product of positive permutation braids $(\pi_1\pi_2 \ldots \pi_k)$. Hence investigate the crossings of the arcs of the first factor π_1, to decide which arcs do not cross in the braid π_1. If a pair of such these arcs cross in π_2 and if it is possible to pull that crossing at the end of π_1 then do it. Do that with the other pair of arcs. Hence finish with new positive permutation braids $(\pi_1)'$ and $(\pi_2)'$. Repeat that again on $(\pi_2)'$ and π_3 to finish with $(\pi_2)''$ and $(\pi_3)'$. Repeat that again on $(\pi_3)'$ and (π_4), and so on. Then the braid P has the new factorization, $[(\pi_1)'(\pi_2)''(\pi_3)'' \ldots (\pi_{k-1})''(\pi_k)']$. Note that the number of factors does not increase under the algorithm, because it is possible that some of the factors vanish. But L(P) is finite and SB_n is also a finite set. Then ultimately a stage must be reached when further applications of the process will yield no new factorizations.

A practical test for use in the algorithm above is given in theorem (1.3.2), where it is proved that $(\pi_1\pi_2 \ldots \pi_k)$ is the canonical (left-hand) factorization for a positive braid P if and only $S(\pi_{i+1}) \subseteq$

$F(\pi_i)$, for $1 \leqslant i \leqslant k-1$. Following that an example for applying the algorithm is given in example (1.3.3). Consequently it is proved that the number of factors in the left-hand canonical form of a positive braid equals the number of factors in its right-hand form, as in corollary (1.3.4). Then the number of factors in the canonical form of a positive braid P is called the <u>canonical length</u> of P and denoted CL(P). For a positive braid P with CL(P) = k, it is also proved that $P^{-1} = (\Delta_n)^{-k}Q$ where Q is positive and prime to Δ_n, i.e. the power of P^{-1} equals -CL(P), as in corollary (1.3.4).

Section 4 is devoted to discus some contributions of the canonical form for every positive braid . An efficient normal form for Garside's solution of the word problem is given. It is shown in theorem (1.4.2) that any word can be uniquely determined by a sequence of permutations called base and an integer called power.

An algorithm to decide whether a positive braid P is conjugate (or not) to $\Delta_n Q$ for some positive braid Q is given in (1.4.3). The idea of that is to write P in its canonical form $(\pi_1\pi_2 \ldots \pi_k)$, then cycle the first factor π_1 to the end of $(\pi_2\pi_3 \ldots \pi_k)$, i.e. conjugate by π_1 and put the resulting word, $P_1 = (\pi_1)^{-1}P(\pi_1)$, in its canonical form $(\eta_1\eta_2 \ldots \eta_{k_1})$, say. If $\eta_1 = \Delta_n$, then P contains Δ_n up to conjugation, hence stop the algorithm. But if $\eta_1 \neq \Delta_n$ repeat the previous process by cycling η_1 at the end of $(\eta_2\eta_3 \ldots \eta_{k_1})$, i.e. conjugate P_1 by η_1 and write $P_2 = (\eta_1)^{-1}P_1(\eta_1)$ in its canonical factorization $(\alpha_1\alpha_2 \ldots \alpha_{k_2})$, say, and so on. But $k \geqslant k_1 \geqslant \ldots \geqslant k_i$ and SB_n is finite, then ultimately a stage must be reached when further applications will either factor out Δ_n or yield no new words.

The algorithm above is proved, in theorem (1.4.4), where a positive braid (P,n) is conjugate to $\Delta_n R$, for some positive braid R, if and only if the algorithm above produces Δ_n. This result reduces the required calculations to decide whether P is in the summit set of some braid α, or not. In lemma (1.4.5) it is proved a result (due to Garside), that if braids P and Q are conjugate by a positive braid A and power of P = power of Q = k, then power of $\alpha^{-1}P\alpha \geq k$, where α is the maximal starter for A, which is the key for constructing the summit set.

Finally the Garside's solution of the conjugacy problem is improved by reducing Garside's invariant class (summit set), under conjugation, to an invariant subclass (super summit set). The summit set of a braid P is defined as $SS(P) = \{(R,n) \mid R$ conjugate to P and $R = (\Delta_n)^m Q$, for m maximal and Q positive braid$\}$. But the super summit set of a braid P is defined as $SSS(P) = \{(R,n) \mid R = (\Delta_n)^m Q$ is summit form with minimal $CL(Q)\}$

In theorem (1.4.8) it is proved that if P and Q are super summit forms (for a given braid α), then there are a sequence of elements $R_0 = P, R_1, \ldots, R_s = Q$ in super summit set of α such that R_{i+1} conjugate to R_i by a positive permutation braid. Using theorem (1.4.8) and lemma (1.4.5), it is proved, in theorem (1.4.9), that two braids are conjugate if and only if their super summit sets are identical.

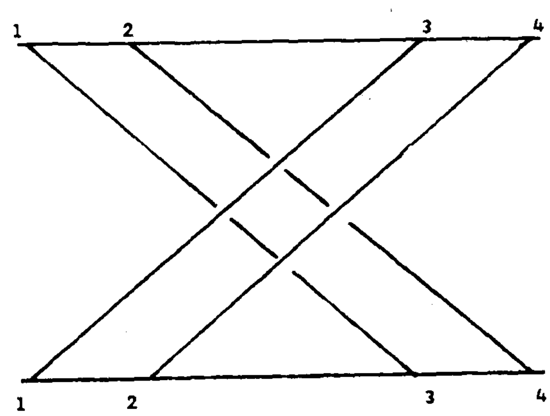
§1.1. THE FACTORS AND THE FACTOR PAIRS OF Δ_n

(1.1.1) Definition: (Positive permutation braids)

Given a permutation π in the symmetric group S_n, make a diagram $D(\pi)$ of π by joining the points 1, 2, ... , n by lines to the points $\pi(1)$, $\pi(2)$, ... , $\pi(n)$ respectively, such that only two pair of lines are crossed at each crossing. Then each pair of lines cross at most once. Make each pair cross in the positive sense, then read the resulting braid β_π from $D(\pi)$. The positive braid when each pair of strings cross at most once, will called a positive permutation braid and SB_n denote the set of all positive permutation braids in B_n.

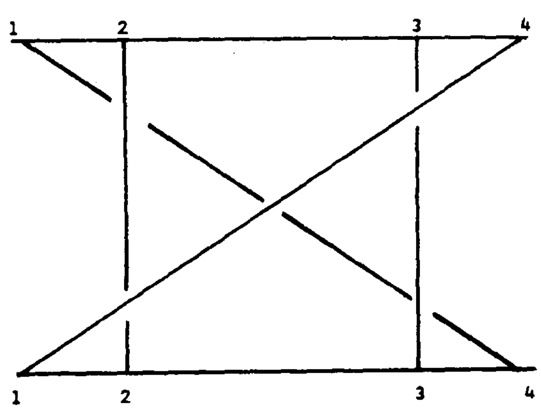
(1.1.2) Example:

In S_4, the cases when $\pi_1 = (13)(24)$ and $\pi_2 = (14)$ are illustrated in figures (1-1a) and (1-1b) respectively. Then $\beta_{(\pi_1)} = \sigma_2\sigma_1\sigma_3\sigma_2$ and $\beta_{(\pi_2)} = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3$.



$D(\pi_1 = (13)(24))$

Figure (1-1a)



$D(\pi_2 = (14))$

Figure (1-1b)

(1.1.3) Lemma:

For a permutation $\pi \in S_n$, the associated positive permutation braid β_π depends only on π but not on the choice of the diagram $D(\pi)$.

Proof:

Number the strands $1, 2, \ldots, n$ on the bottom of β_π, from left to right. But each pair of strings cross at most once, then the string labelled 1, at the bottom of β_π, lies over (not under) each other string. So we can isotop it to lie at level t_1, with respect to the braid axis $x_{(\beta_\pi)}$ of P. Do that again with the arc labelled 2 and so on. Then the string i lies at level t_i, where $t_1 > t_2 > \ldots > t_n$, with respect to $x_{(\beta_\pi)}$ as in figure (1-2). So that string i always crosses over (not under) string j for $i < j$, hence β_π depends only on π □

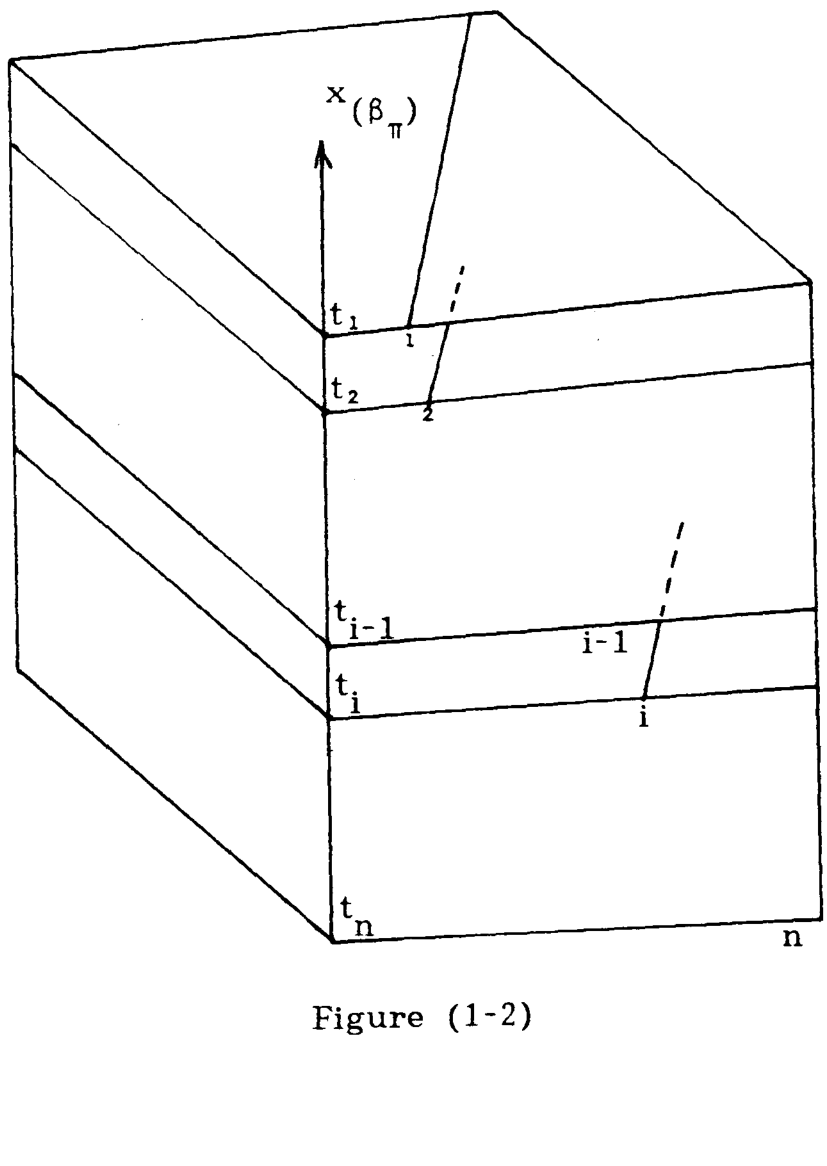


Figure (1-2)

(1.1.4) Theorem:

Let $\pi, \gamma \in S_n$, then $\beta_\pi = \beta_\gamma$ if and only if $\pi = \gamma$.

Proof:

The necessity of the condition is clear. To establish sufficiency, draw β_π and β_γ with strings in levels as in lemma (1.1.3). Then the two braids are isotopic, because we can move string i of β_π to string i of β_γ within level t_i □

(1.1.5) Corollary:

For each $\pi \in S_n$ there is a unique braid $\beta_\pi \in SB_n$ with permutation of β_π equals π.

Proof:

The proof is a direct consequence of theorem (1.1.4) □

(1.1.6) Remarks:

(a): As a result of theorem (1.1.4) we can think of every positive permutation braid β_π, simply as a permutation π without any care how the arcs in β_π cross. <u>In fact this is the key of the main result of this chapter, "The canonical form for every positive braid", as in section 2</u>.

(b): For every permutation $\pi \in S_n$ we can use a negative crossing to read the resulting braid from the diagram $D(\pi)$ of π. As a further analogy with the previous process we can call such this braid <u>"the negative permutation braid"</u>. Hence for every permutation $\pi \in S_n$, let β_π and $(\beta_\pi)_-$ be its associated positive and negative permutation braids respectively. So that

$$(\beta_\pi)_- = \rho[(\beta_\pi)^{-1}]$$

where $\rho[\alpha]$ is the braid α in reverse, as in definition (0.10).

(1.1.7) Definition:

For a positive braid (P,n), define the starting and the finishing sets, respectively as:

$$S(P) = \{i \mid P = \sigma_i Q, \text{ for some positive braid } Q\}$$

$$F(P) = \{i \mid P = Q\sigma_i, \text{ for some positive braid } Q\}$$

The following lemma presents a necessary and sufficient condition for starters and finishers of positive permutation braids.

(1.1.8) Lemma:

Every permutation $\pi \in S_n$ satisfies the following:

(i): $i \in S(\beta_\pi)$ if and only if $\pi(i) > \pi(i+1)$

(ii): $i \in F(\beta_\pi)$ if and only if $\pi^{-1}(i) > \pi^{-1}(i+1)$.

Proof:

(i) For necessity: let $i \in S(\beta_\pi)$, then the strings labelled i and i+1 on top of β_π never cross in β_γ , where

$$\beta_\pi = \sigma_i \beta_\gamma$$

otherwise $\beta_\pi \notin SB_n$, hence

$$\pi(i) > \pi(i+1)$$

For sufficiency: let $\pi \in S_n$, such that

$$\pi(i) > \pi(i+1)$$

then the two arcs labelled i and i+1, on the top of β_π, cross in β_π. Draw the braid β_π in levels as in lemma (1.1.3), hence the string labelled i at bottom of β_π always cross over (not under) string labelled j at bottom of β_π for $i < j$. Therefore we can draw a pattern for β_π, as in figure (1-3), where $\alpha_i \in SB_n$ for all $1 \leqslant i \leqslant 4$. Hence from the diagram and under isotopy, there is some $\alpha \in SB_n$ such that

$$\beta_\pi = \sigma_i \alpha$$

then $i \in S(\pi)$. The proof of case (ii) is similar to that in case (i) □

Now we are going to find out the characteristic properties for the fundamental braid (Δ_n, n), which is defined in definition (1.0.1). A picture of the braid $(\Delta_5, 6)$ is given in figure (1-4a).

β_π:

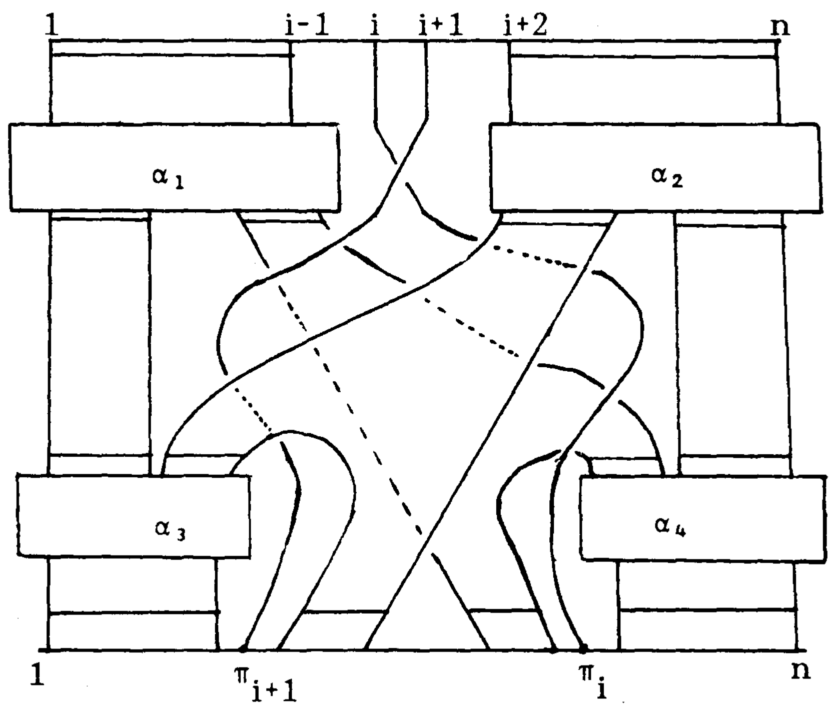


Figure (1-3)

<u>(1.1.9) Remark</u>:

In B_n, let $(\Delta_{i,\leftarrow})$ denote the half twist (fundamental braid) in the last i strands, for $i \leqslant n$, then $(\Delta_{i,\leftarrow})$ is the result of turning over of the half twist Δ_i in the first i strands in B_n, i.e. $\tau[\Delta_i] = (\Delta_{i,\leftarrow})$. Hence

$$F(\Delta_{i,\leftarrow}) \subseteq \{n-1, n-2, \ldots, n-i+1\}$$

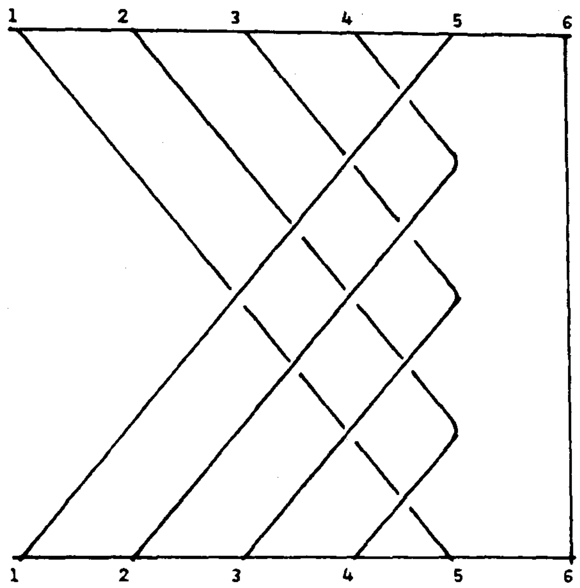
where

$$S(\Delta_i) \subseteq \{1, 2, \ldots, i-1\}$$

Then in B_n and for $i = n$ we have

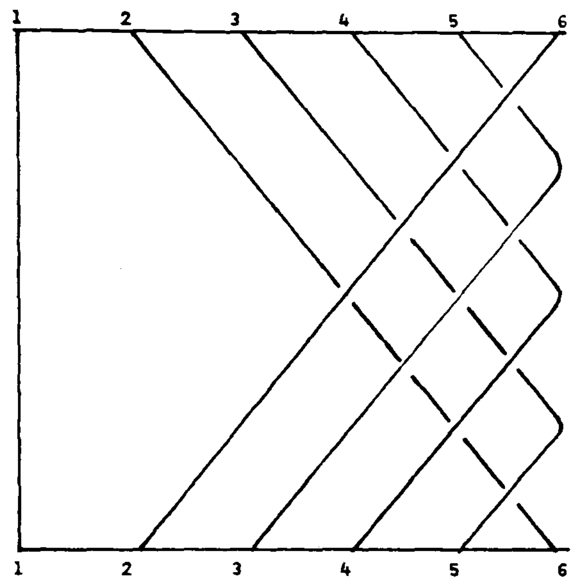
$$\Delta_i = (\Delta_{i,\leftarrow})$$

An example for $n = 6$ and $i = 5$ is given in figure (1-4b).



Δ_5

Figure (1-4a)



$(\Delta_{5,\leftarrow})$

Figure (1-4b)

(1.1.10) Lemma: (Recognition results for Δ_n)

For a permutation $\pi \in S_n$, the following statements are equivalent:

(i): $\beta_\pi = \Delta_n$

(ii): Each pair of strings in β_π cross exactly once

(iii): $S(\beta_\pi) = F(\beta_\pi) = \{1, 2, \ldots, n-1\}$.

Proof:

Consider the permutation π such that

$$\pi(i) = n-i+1, \text{ for } 1 \leqslant i \leqslant n$$

then

$$\pi^{-1}(i) = n-i+1, \text{ for } 1 \leqslant i \leqslant n$$

hence

$$\pi(i) > \pi(i+1) \text{ and } \pi^{-1}(i) > \pi^{-1}(i+1), \text{ for } 1 \leqslant i \leqslant n$$

So lemma (1.1.8) tells us that

$$i \in S(\beta_\pi) \text{ and } i \in F(\beta_\pi), \text{ for } 1 \leqslant i \leqslant n-1$$

i.e.

$$S(\beta_\pi) = F(\beta_\pi) = \{1, 2, \ \ldots \ , n-1\}$$

But, from the definition of π, we can write

$$\pi = \eta(\tau_1\tau_2 \ \ldots \ \tau_{n-1})$$

for some $\eta \in S_n$, such that

$$\eta(i) = \pi(i-1) = n-i+2, \text{ for } 2 \leqslant i \leqslant n$$

so that

$$\beta_\pi = (\sigma_{n-1}\sigma_{n-2} \ \cdots \ \sigma_1)\beta_\eta$$

Replace π by η to get $\eta'' \in S_n$, such that

$$\eta''(i) = \eta(i-1) = n-i+3, \text{ for } 3 \leqslant i \leqslant n$$

Continuing this process we can finish with,

$$\begin{aligned}\beta_\pi &= (\sigma_{n-1}\sigma_{n-2} \ \cdots \ \sigma_1)(\sigma_{n-1}\sigma_{n-2} \ \cdots \ \sigma_2) \ \ldots \ (\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1}) \\ &= \Delta_n\end{aligned}$$

Now let δ denote the permutation where $\beta_\delta = \Delta_n$, i.e. $\delta(i) = n-i+1$, for $1 \leqslant i \leqslant n$.

(i) $\to$ (ii): Since $\beta_\pi = \Delta_n$, then the geometric definition of Δ_n, shown in definition (1.0.1), tells us that each pair of arcs in β_π cross exactly once, but theorem (1.1.4) tells us that π is unique, then $\pi = \delta$.

(ii) $\to$ (iii): If each pair of arcs, in β_π, cross exactly once, then definition of Δ_n tells us that $\beta_\pi = \beta_\delta = \Delta_n$, hence as shown above

$$S(\beta_\pi) = F(\beta_\pi) = \{1,2, \ldots ,n-1\}$$

(iii) $\to$ (ii): Let $S(\beta_\pi) = \{1,2, \ldots ,n-1\}$, then

$$\pi(i) > \pi(i+1), \text{ for all } 1 \leqslant i \leqslant n-1$$

hence

$$\pi(i) > \pi(j), \text{ for } i < j$$

But $\beta_\pi \in SB_n$, then each two arcs cross exactly once, otherwise $\pi(i) < \pi(j)$, for some $i < j$.

(ii) $\to$ (i): Given a permutation π such that each two arcs in β_π cross exactly once, then $\beta_\pi = \Delta_n$, so $\pi = \delta$ □

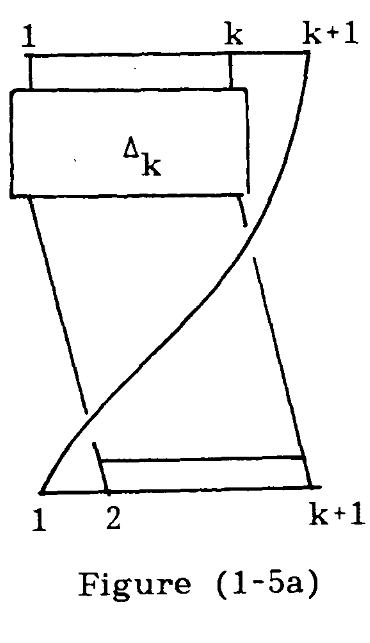


Figure (1-5a)

$\simeq$

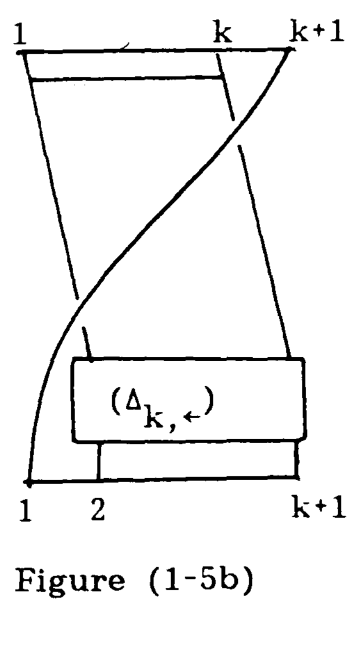


Figure (1-5b)

<u>(1.1.11) Lemma</u>:

In B_n, let $\beta = \prod_{j=1}^{k} [(\sigma_{(i_j)})^{(\varepsilon_j)}]$, then $\tau[\beta] = \prod_{j=1}^{k} [(\sigma_{(n-i_j)})^{(\varepsilon_j)}]$, where $\tau[\beta]$ is the conjugate of β by Δ_n (as in definition (0.10)), $\varepsilon_j \in \mathbb{Z}$ and $1 \leqslant i_j \leqslant n-1$, for $1 \leqslant j \leqslant k$.

<u>Proof</u> :

It is enough to show that $\tau[\sigma_i] = \sigma_{n-i}$, for τ is a homomorphism. Using (iii) in lemma (1.1.10), then for every $1 \leqslant i \leqslant n-1$, we have

$$\beta_\delta = \Delta_n = \sigma_i \beta_{(\delta_i)}$$

where in β_{δ_i} each pair of strings cross exactly once, except those which labelled i and i+1 (on top of β_δ) they never cross. Draw $\Delta_n = \sigma_i \beta_{(\delta_i)}$ with strings in levels, so if the string from i to $\delta(i)$ lies in level t_i (say), then the string from i+1 to $\delta(i+1)$ lies in the successive level t_{i+1}, just over the level t_i, hence we can isotop these two arcs to cross at the end of Δ_n as in figure (1-6), then

$$\Delta_n = \sigma_i \beta_{(\delta_i)} = \beta_{(\delta_i)} \sigma_{n-i}$$

Now if $\beta = \sigma_i$, then

$$\begin{aligned}
\tau(\sigma_i) = (\Delta_n)^{-1} \sigma_i \Delta_n &= (\Delta_n)^{-1} \sigma_i \beta_\delta \\
&= (\Delta_n)^{-1} \sigma_i (\sigma_i \beta_{\delta_i}) \\
&= (\Delta_n)^{-1} (\sigma_i \beta_{\delta_i}) \sigma_{n-i} \\
&= (\Delta_n)^{-1} \Delta_n \sigma_{n-i} \\
&= \sigma_{n-i}
\end{aligned}$$

But τ is a homomorphism, then the proof follows by repeated applications of the previous process on the successive letters of β □

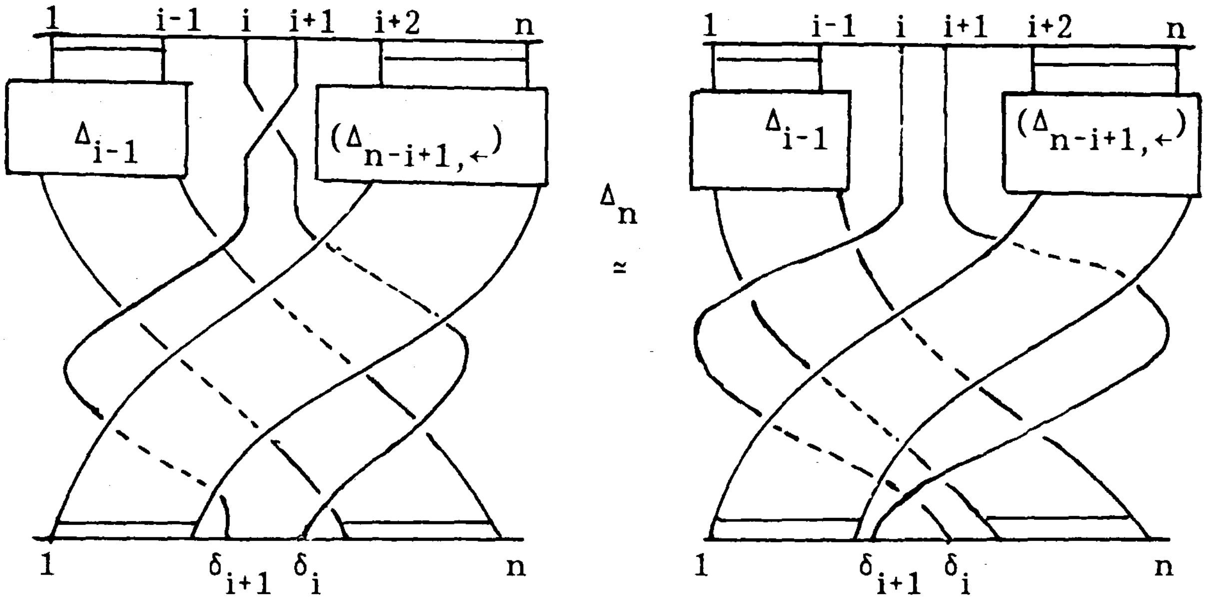


Figure (1-6)

(1.1.12) Corollary:

In B_n, $(\Delta_n)^2$ lies in centre of B_n, i.e. $(\Delta_n)^2$ commutes with every thing.

Proof :

Since τ is a homomorphism, then as in lemma (1.1.11) it is enough to show that,

$$\tau^2[\sigma_i] = \sigma_i, \text{ for all } 1 \leqslant i \leqslant n-1$$

where τ^2 is the conjugation by $(\Delta_n)^2$. Using lemma (1.1.11), since

$$\tau[\sigma_i] = \sigma_{n-i}, \text{ for all } 1 \leqslant i \leqslant n-1$$

then applying τ again, we have

$$\tau^2[\sigma_i] = \tau[\sigma_{n-i}] = \sigma_i$$

This is geometrically obvious, because passing a braid P through $(\Delta_n)^2$ means that P has turned over twice □

(1.1.13) Remark:

Given a positive braid (P,n) and let π be the associated permutation of P, then the number of crossings of any two strings labelled i and j (at top of P) equals the number of crossings of the same strings in π (mod 2). So that if P and Q are two positive braids with the same permutation π, then strings i and j (at top of both P and Q) cross the same times (mod 2). The number of crossings (in a positive braid) of two strings i and j is also odd if $\pi(i) < \pi(j)$ for $i > j$ and it is even if $\pi(i) > \pi(j)$ for $i > j$.

(1.1.14) Theorem: (The factors of Δ_n)

A positive braid (β,n) is a factor of Δ_n if and only if $\beta\in SB_n$.

Proof:

For necessity: Let P be a factor of Δ_n, then there exists a positive braid Q (as in definition (1.0.3)) such that

$$\Delta_n = PQ$$

Then number of crossings of string labelled i with string labelled j, in PQ, is $\geqslant$ the number of crossings in P, then each two arcs in P cross at most once, so $P\in SB_n$.

For sufficiency: Let $\alpha\in SB_n$ with permutation $\pi\in S_n$ and choose $\gamma\in S_n$, such that $\pi\gamma = \delta$ and $\beta_\delta = \Delta_n$, i.e. $\beta_{(\pi\gamma)}$, $\beta_\pi = \alpha$ and β_γ are all in SB_n, then each two arcs in each one of them cross at most once, so each two arcs in $\beta_\pi\beta_\gamma$ cross 0, 1 or 2 times. Compare $\beta_\pi\beta_\gamma$ and $\beta_{\pi\gamma} = \Delta_n$. So using remark (1.1.13) and the fact that each two arcs in Δ_n cross exactly once, then each two arcs in $\beta_\pi\beta_\gamma$ cross an odd number of times, so

$$\Delta_n = \beta_{\pi\gamma} = \beta_\pi\beta_\gamma = \alpha\beta_\gamma$$

i.e. α is a factor of Δ_n, which completes the sufficiency and so completes the proof $\square$

<u>(1.1.15) Corollary</u>: (The factor pairs for Δ_n)

For each permutation $\pi \in S_n$, there exist two permutations π_* and π^*, such that $\beta_{(\pi^*)}\beta_\pi = \beta_\pi \beta_{(\pi_*)} = \Delta_n$.

<u>Proof</u>:

Theorem (1.1.14) tells us that β_π is a factor of Δ_n for every $\pi \in S_n$. i.e. there are two positive permutation braids $\beta_{(\pi^*)}$ and $\beta_{(\pi_*)}$, such that

$$\beta_{(\pi^*)}\beta_\pi = \beta_\pi \beta_{(\pi_*)} = \Delta_n$$

In fact $\beta_{(\pi^*)}$ and $\beta_{(\pi_*)}$ are the upper and lower complements of π in δ, where $\beta_\delta = \Delta_n$ $\square$

<u>(1.1.16) Lemma</u>: (Recognition results for the factor pairs of Δ_n)

Every $\pi \in S_n$ satisfies the following:

(i): $\tau[\beta_{(\pi_*)}] = \beta_{(\pi^*)}$, $\tau[\beta_{(\pi^*)}] = \beta_{(\pi_*)}$

(ii): $F(\beta_\pi) \cap S(\beta_{(\pi_*)}) = \Phi$, $F(\beta_\pi) \cup S(\beta_{(\pi_*)}) = \{1,2, \ldots ,n-1\}$

(iii): $F(\beta_{(\pi^*)}) \cap S(\beta_\pi) = \Phi$, $F(\beta_{(\pi^*)}) \cup S(\beta_\pi) = \{1,2, \ldots ,n-1\}$

<u>Proof</u>:

For any $\beta_\pi \in SB_n$, corollary (1.1.15) tells us that,

$$\Delta_n = (\beta_{\pi^*})\beta_\pi = \beta_\pi(\beta_{\pi_*})$$

For (i):

$$\begin{aligned}\tau[\beta_{\pi_*}] &= (\Delta_n)\beta_{\pi_*}(\Delta_n)^{-1}\\ &= (\beta_{\pi^*}\beta_\pi)\beta_{\pi_*}(\Delta_n)^{-1}\\ &= \beta_{\pi^*}(\beta_\pi\beta_{\pi_*})(\Delta_n)^{-1}\\ &= \beta_{\pi^*}(\Delta_n)(\Delta_n)^{-1}\end{aligned}$$

$$= \beta_{\pi^*}$$

hence by using corollary (1.1.12),

$$\tau[\beta_{\pi^*}] = \beta_{\pi_*}$$

For (ii):

Clearly $F(\beta_\pi) \cap S(\beta_{\pi_*}) = \Phi$, otherwise there exist α, $\beta \in SB_n$ such that for some integer j,

$$\beta_\pi = \alpha\sigma_j \text{ and } \beta_{\pi_*} = \sigma_j\beta, \text{ for } 1 \leqslant j \leqslant n-1$$

hence,

$$\Delta_n = \beta_\pi\beta_{\pi_*} = \alpha(\sigma_j)^2\beta$$

so that in Δ_n there are two arcs cross twice, which is impossible as in lemma (1.1.10). Now let $j \notin [F(\beta_\pi) \cup S(\beta_{\pi_*})]$, then by using lemma (1.1.8),

$$\pi^{-1}(j) < \pi^{-1}(j+1) \text{ and } \pi_*(j) < \pi_*(j+1)$$

i.e. there are two arcs, which labelled j and j+1 in bottom of β_π, never cross each other in $\beta_\pi\beta_{\pi_*} = \Delta_n$, which is impossible, hence

$$F(\beta_\pi) \cup S(\beta_{\pi_*}) = \{1, 2, \ldots, n-1\}$$

For (iii):

Follows from (ii) with π^* in place of π □

§1.2 A CANONICAL FORM FOR EVERY POSITIVE BRAID

(1.2.1) Theorem: (A canonical form for every positive braid)

Every positive braid (P,n) has a unique left-hand, [right-hand], canonical form as a product of positive permutation braids. More precisely:

Every positive braid (P,n) can be written uniquely as a product $P = (\pi_1\pi_2 \dots \pi_k)$, $[P = (\alpha_r\alpha_{r-1} \dots \alpha_1)]$, where π_i, $[\alpha_i]$, is the largest possible positive permutation braid as a starter, [finisher], of $(\pi_i\pi_{i+1} \dots \pi_k)$, for $1 \leqslant i \leqslant k$, $[(\alpha_r\alpha_{r-1} \dots \alpha_i)$, for $1 \leqslant i \leqslant r]$.

To proof the theorem, we begin with several definitions and lemmas.

(1.2.2) Definition:

In B_n, let $\pi_1 = \alpha\sigma_i$ and $\pi_2 = \alpha\sigma_j$, then define the join bottom of π_1 and π_2, as

$$(\pi_1) \cup_b (\pi_2) = \begin{cases} \alpha\sigma_i & \text{if } i=j \\ \alpha\sigma_i\sigma_j & \text{if } |i-j|\geqslant 2 \\ \alpha\sigma_i\sigma_{i+1}\sigma_i & \text{if } |i-j|=1 \end{cases}$$

(1.2.3) Lemma:

The set SB_n is closed under the join bottom operator $\cup_b$, i.e. $[(\alpha\sigma_i) \cup_b (\alpha\sigma_j)] \in SB_n$, for all $\alpha\sigma_i, \alpha\sigma_j \in SB_n$.

Proof:

Order the strings at bottom of α, from left to right. Number them $1,2, \dots ,n$. Then the pair $\{i,i+1\}$ of strings does not cross in α as do the pair $\{j,j+1\}$, otherwise $(\alpha\sigma_i)$, $(\alpha\sigma_j)\notin SB_n$. Then consider the following three cases:

Case (1): If $i = j$, then directly from the definition of the join bottom, we have

$$(\alpha\sigma_i) \cup_b (\alpha\sigma_j) = (\alpha\sigma_i)\in SB_n$$

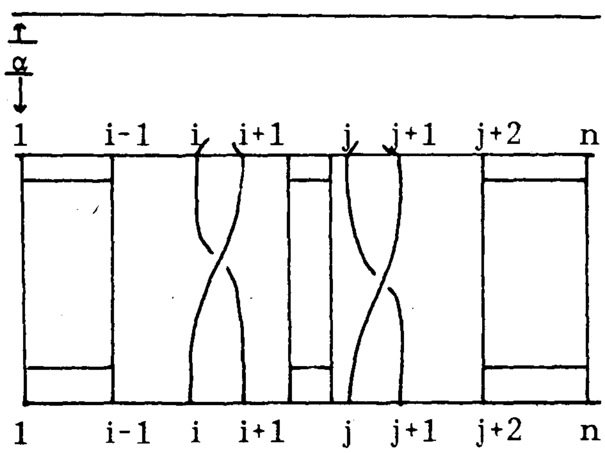
Case (2): If $|i-j|\geqslant 2$, let $i < j$, then the pair $\{i,j\}$ of strings do not cross in $\alpha\sigma_i$, as in figure (1-7a), hence

$$(\alpha\sigma_i) \cup_b (\alpha\sigma_j) = (\alpha\sigma_i\sigma_j)\in SB_n$$

Case (3): If $|i-j|=1$, let $j = i+1$, then the pair $\{i,i+2\}$ of strings never cross in α, as in figure (1-7b), hence

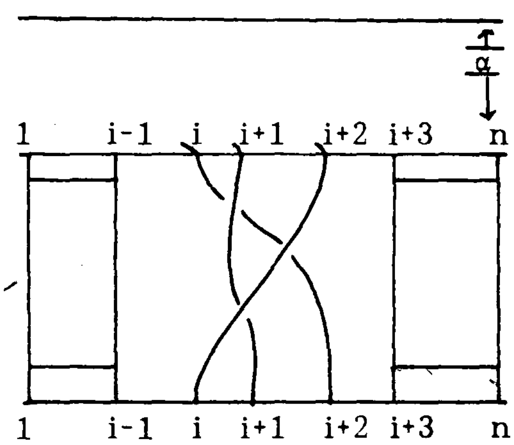
$$(\alpha\sigma_i) \cup_b (\alpha\sigma_j) = (\alpha\sigma_i\sigma_{i+1}\sigma_i)\in SB_n$$

which completes the proof □



$\alpha\sigma_i\sigma_j$

Figure (1-7a)



$\alpha\sigma_i\sigma_{i+1}\sigma_i$

Figure (1-7b)

<u>(1.2.4) Definition</u>:

Given a positive braid (P,n), let I_P be the set of all possible initial positive permutation braid factors of P, i.e. if $\alpha\in I_P$, then $P = (\alpha P_1)$ for some positive braid P_1, hence $L(P_1) \leqslant L(P)$. The set I_P is called the starter set for P and every $\alpha\in I_P$ is called a starter of P. For two elements π, $\gamma\in SB_n$, if π is a starter of γ, then there exists $\alpha\in SB_n$ such that $\gamma = \pi\alpha$ and denoted $\gamma \geqslant_s \pi$. The positive permutation braid γ is also a maximal element in I_P if $\beta = \gamma$ for all $\beta \geqslant_s \gamma$.

(1.2.5) Example:

In cases when $P = \Delta_3$ and $P = \Delta_4$, the starter sets $I_{(\Delta_3)}$ and $I_{(\Delta_4)}$ are illustrated diagrammatically in figures (1-8a) and (1-8b), respectively. Note that $I_{(\Delta_n)} = SB_n$, because Δ_n is the largest positive permutation braid in B_n.

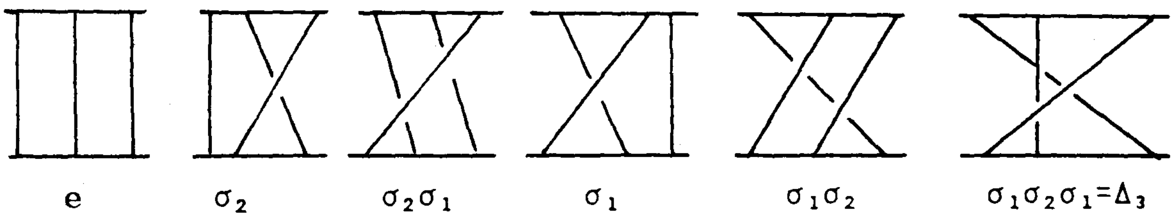


Figure (1-8a)

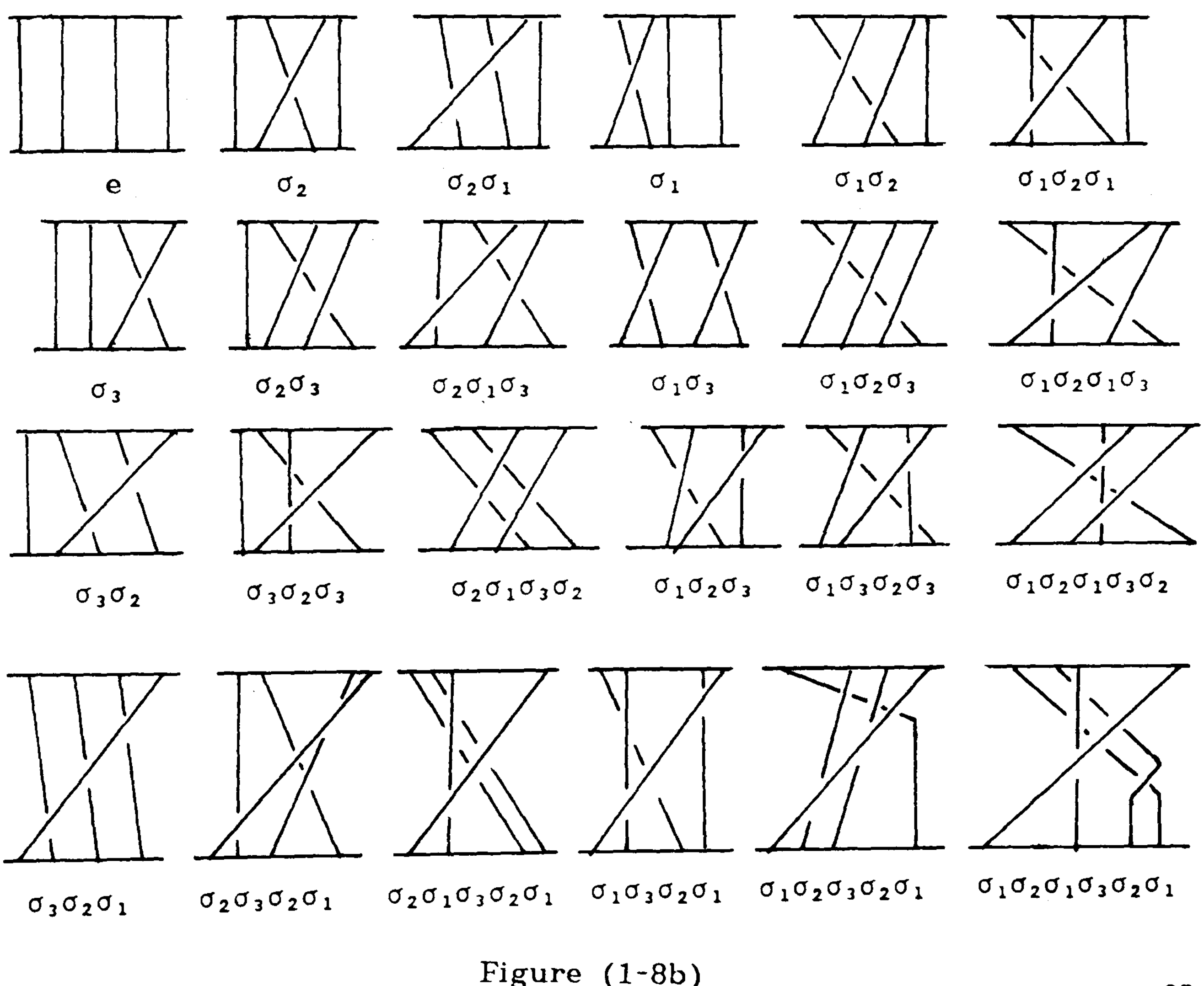


Figure (1-8b)

This work generally is concerned to braids rather than words and here is the place where we used to work particularly with words. The following lemma, due to Garside, is precisely concerned to words in B_n.

<u>(1.2.6) Lemma</u>: (Garside) [G2]

For positive words P, $Q \in B_n$, suppose that $\sigma_i P = \sigma_j Q$, then

$$\begin{cases} P = Q & \text{if } i=j \\ P = \sigma_j Z,\ Q = \sigma_i Z & \text{if } |i-j| \geqslant 2 \\ P = \sigma_j \sigma_i Z,\ Q = \sigma_i \sigma_j Z & \text{if } |i-j|=1 \end{cases}$$

for some positive word $Z \in B_n$, where $1 \leqslant i,j \leqslant n-1$.

<u>Outline of the proof of lemma (1.2.6)</u>:

The braid relators (i) and (ii) of definition (0.5) have the property that no inverse of a generator appears in either relation. Hence there is a semigroup

$$A_n = \left\{ a_i,\ 1 \leqslant i \leqslant n-1 \;\middle|\; \begin{array}{l} a_i a_j = a_i a_j,\ |i-j| > 1 \\ \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1},\ \text{for } 1 \leqslant i \leqslant n-2 \end{array} \right\}$$

where the mapping $a_i \to \sigma_i$, for $1 \leqslant i \leqslant n-1$ induces a natural embedding of A_n in B_n, [B2]. Garside's idea is to transfer from A_n to B_n information easily obtained in A_n. Now for positive braid words V_i, for $0 \leqslant i \leqslant r$, if $V_0 = V_1 = \ldots = V_r$ and if each V_i can be obtained from V_{i-1} by a single application of one of the braid relators (i) or (ii) of definition (0.5) (without involving inverses), then one says that V_r can be obtained from V_0 by a transformation of chain-length r. i.e. simply the words V_i, $0 \leqslant i \leqslant r$ are equal in A_n. For transformations of chain-length one, the proof is straightforward. For transformations of greater chain-length, one factors into transformations of smaller length, first applies the inductive hypothesis about chain-length then the inductive hypothesis about letter length, and checks the all possibilities. The complete calculations of this proof are given by Garside in [G2].

(1.2.7) Proposition:

For a positive braid (P,n) and for every π, $\eta \in I_P$, there exists $\xi \in I_P$ such that $\xi \geqslant_s \pi$ and $\xi \geqslant_s \eta$.

Proof:

Given two positive permutation braids π and η in I_P (i.e. $P = \pi P_1 = \eta P_2$, for positive braids P_1 and P_2), write $\pi = \sigma_i \pi_1$ and $\eta = \sigma_j \eta_1$, for some π_1, $\eta_1 \in SB_n$. Using lemma (1.2.6) one can find a common starter α for both π and η. Now Define $m(\alpha) = L(\Delta_n) - L(\alpha)$, for all $\alpha \in SB_n$, so then $m(\alpha) \geqslant 0$. Refer to proposition (1.2.7), when π and η have a common starter α with $m(\alpha) = k$, as $(\text{Prop.})_k$,

$(\text{Prop.})_0$: Then $m(\alpha) = 0$, so $\alpha = \Delta_n$. But both π, $\eta \in SB_n$, then

$$\pi = \eta = \Delta_n$$

hence,

$$\xi = \Delta_n$$

So the proof of the general proposition follows by induction on k. For our induction hypothesis we assume that $(\text{Prop.})_r$ holds. Suppose that π and η have common starter α with $m(\alpha) = r+1$. Let

$$\pi = \alpha\sigma_i\pi' \ , \quad \eta = \alpha\sigma_j\eta', \text{ for } \pi', \ \eta' \in SB_n$$

and write

$$P = \alpha Q$$

where

$$Q = \sigma_i\pi' P_1 = \sigma_j \eta' P_2, \text{ i.e. } i,j \in S(Q)$$

for some positive words P_1 and P_2. Now lemma (1.2.6) tells us that

$$\begin{cases} \pi' P_1 = \eta' P_2 & \text{if } i=j \\ \pi' P_1 = \sigma_j Z, \ \eta' P_2 = \sigma_i Z & \text{if } |i-j| \geq 2 \\ \pi' P_1 = \sigma_j \sigma_i Z, \ \eta' P_2 = \sigma_i \sigma_j Z & \text{if } |i-j| = 1 \end{cases} \tag{1.2.1}$$

for some positive braid $Z \in B_n$, where $1 \leq i,j \leq n-1$. So

$$P = \alpha Q = [(\alpha\sigma_i) \cup_b (\alpha\sigma_j)](R_{i,j})$$

where $R_{i,j}$ is a positive braid depends on i and j (as in equation (1.2.1)) with

$$L(R_{i,j}) \leq L(\pi' P_1) = L(\eta' P_2)$$

Now the pair $\{\pi, (\alpha\sigma_i) \cup_b (\alpha\sigma_j)\}$ and the pair $\{\eta, (\alpha\sigma_i) \cup_b (\alpha\sigma_j)\}$, of braids, have the common starters $(\alpha\sigma_i)$ and $(\alpha\sigma_j)$, respectively. But lemma (1.2.3) tells us that $[(\alpha\sigma_i) \cup_b (\alpha\sigma_j)] \in SB_n$, therefore

$$\begin{aligned} m(\alpha\sigma_i) &= L(\Delta_n) - L(\alpha\sigma_i) \\ &= L(\Delta_n) - L(\alpha) - 1 \\ &= m(\alpha) - 1 \\ &= r \\ &= m(\alpha\sigma_j) \end{aligned}$$

Then by induction hypothesis, there exists $\chi_1, \chi_2 \in I_P$ such that

$$\chi_1 = \pi\theta_1 = [(\alpha\sigma_i) \cup_b (\alpha\sigma_j)]\theta_2$$

and

$$\chi_2 = \eta\varepsilon_1 = [(\alpha\sigma_i) \cup_b (\alpha\sigma_j)]\varepsilon_2$$

for some θ_i, $\varepsilon_i \in SB_n$, $i = 1, 2$. We can also apply the induction process again, because χ_1 and χ_2 have $[(\alpha\sigma_i) \cup_b (\alpha\sigma_j)]$ as a common starter with

$$\begin{aligned} m[(\alpha\sigma_i) \cup_b (\alpha\sigma_j)] &= L(\Delta_n) - L((\alpha\sigma_i) \cup_b (\alpha\sigma_j)) \\ &< L(\Delta_n) - L(\alpha) \\ &= m(\alpha) \\ &= r+1 \end{aligned}$$

Then there exists $\xi \in I_P$ such that

$$\xi = \chi_1\gamma_1 = \pi\theta_1\gamma_1 = \chi_2\eta_1 = \eta\varepsilon_1\eta_1$$

which completes the proof of $(\text{Prop.})_{r+1}$, hence completes the proof of the general proposition. The relations between these braids are represented diagrammatically in figure (1-9) □

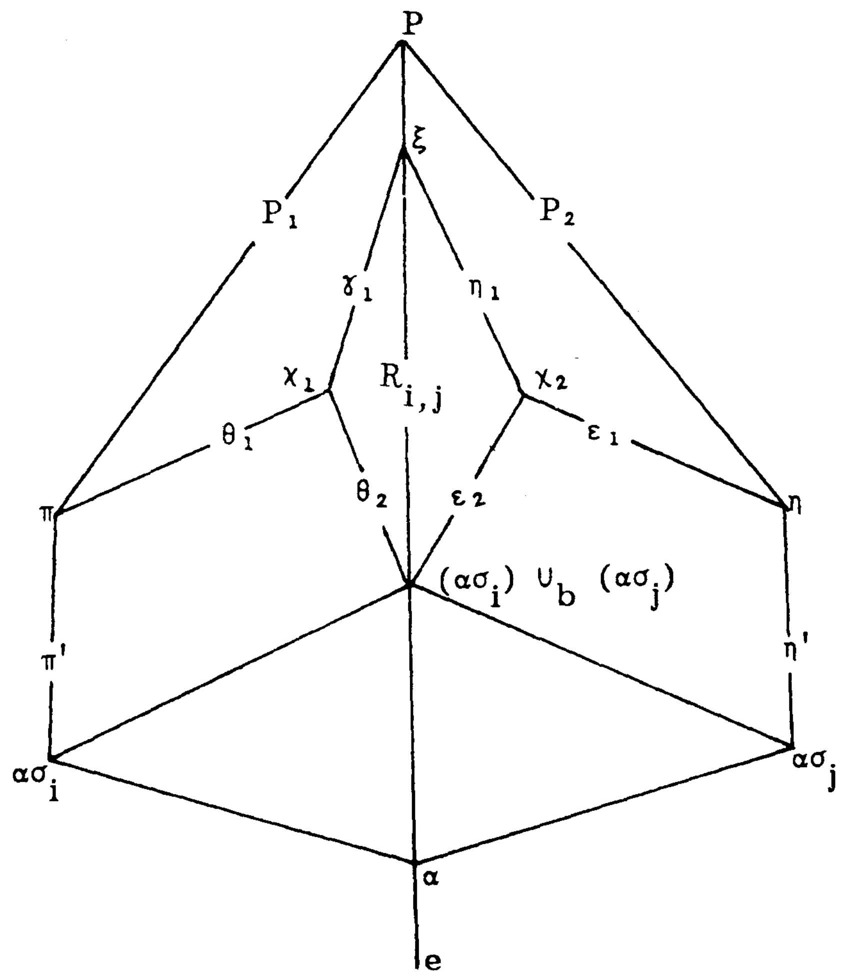


Figure (1-9)

(1.2.8) Corollary:

For every positive braid (P,n), I_P contains a unique maximal element, i.e. $P = \pi_1 P_1$, for some positive braid (P_1,n), such that $(\pi_1\sigma_i) \notin SB_n$, for all $i \in S(P_1)$.

Proof:

Let γ, $\eta \in I_P$, then proposition (1.2.7) tells us that there exists an element $\alpha \in I_P$ such that

$$\alpha \geqslant_s \gamma \text{ and } \alpha \geqslant_s \eta$$

Now assuming that both γ and η are two maximal elements in I_P, then

$$\alpha = \gamma \text{ and } \alpha = \eta$$

hence

$$\alpha = \gamma = \eta$$

Assuming that π_1 is the unique maximal element in I_P, then

$$P = \pi_1 P_1$$

for some positive braid $P_1 \in B_n$ □

(1.2.9) Corollary:

For a positive braid (P,n), if π_1 is the unique maximal element in I_P, then $S(P) = S(\pi_1)$.

Proof:

Corollary (1.2.8) tells us that

$$P = \pi_1 P_1$$

where π_1 is the unique maximal element in I_P and P_1 is a positive braid, then

$$S(\pi_1) \subseteq S(P)$$

For the converse, let $i \in S(P)$, then there is a positive braid P' such that

$$P = \sigma_i P'$$

So

$$\sigma_i \in I_P$$

But π_1 is the unique maximal element in I_P, hence

$$\pi_1 \geqslant_s \sigma_i$$

then

$$i \in S(\pi_1)$$

so

$$S(P) \subseteq S(\pi_1)$$

which completes the proof □

For a positive braid (P,n), the following proposition presents a practical test to decide whether an element $\alpha \in I_P$ is the unique maximal starter of the braid P, or not.

<u>(1.2.10) Proposition</u>:

In B_n, let $P = \pi_1 P_1$ for a positive braid $P_1 \in B_n$ and for $\pi_1 \in SB_n$, then π_1 is the unique maximal element in I_P if and only if $S(P_1) \subseteq F(\pi_1)$.

<u>Proof</u>:

For necessity: Order the strings on top of P_1, from left to right. Let $j \in S(P_1)$, then $P_1 = \sigma_j Q$ for some positive braid $Q \in B_n$. But π_1 is the unique maximal element in I_P, then $\alpha = \pi_1 \sigma_j \notin SB_n$, i.e. the strings labelled j and j+1, at bottom of α, cross twice in α. Let λ and μ be the permutations of α and π_1 respectively, then

$$\lambda = \tau_j \mu \in S_n$$

and

$$\lambda^{-1}(j) < \lambda^{-1}(j+1)$$

i.e.

$$(\tau_j \mu)^{-1}(j) < (\tau_j \mu)^{-1}(j+1)$$

hence

$$\mu^{-1}(j+1) < \mu^{-1}(j)$$

then lemma (1.1.8) implies that $j \in F(\pi_i)$. Now to establish the sufficiency, let

$$S(P_1) \subseteq F(\pi_1)$$

If $j \in S(P_1)$, then there exist a positive braid (Q,n) and some $\theta_j \in SB_n$, such that

$$P_1 = \sigma_j Q$$

and

$$\pi_1 = \theta_j \sigma_j$$

so

$$\pi_1 \sigma_j = \theta_j \sigma_j^2, \text{ for } j \in S(P_1)$$

i.e. $\pi_1 \sigma_j \notin SB_n$, for all $j \in S(P_1)$, hence π_1 is the largest positive permutation braid in I_P, which completes the sufficiency condition and hence completes the proof of the proposition □

Proof of theorem (1.2.1):

Given a positive braid (P,n), then find I_P and using corollary (1.2.8), we can write

$$P = \pi_1 P_1$$

where π_1 is the unique maximal element in I_P and (P_1,n) is a positive braid. Hence find π_2, the unique maximal element in $I_{(P_1)}$ and write

$$P_1 = \pi_2 P_2$$

for some positive braid P_2. But

$$L(P_2) < L(P_1) < L(P)$$

then continuing this process, we have

$$P = \pi_1\pi_2 \ldots \pi_k$$

for some $k \in \mathbb{Z}^+$ and $k \geq 1$, with unique maximal factor π_i as a starter of $(\pi_i\pi_{i+1} \ldots \pi_k)$, $1 \leq i \leq k$. But as in remark (1.1.6a), we can think of π_i simply as a permutation in S_n without any care how the arcs in π_i cross. Therefore P is uniquely determined by an ordered sequence of permutations □

(1.2.11) Definition:

A positive braid (P,n) is said to contain Δ_n if and only if $P = A\Delta_n B$, for some positive braids A and B. If P does not contain Δ_n, then P is prime to Δ_n and said P has power zero, [B2].

(1.2.12) Lemma:

A positive braid (P,n) is said to contain Δ_n if and only if Δ_n is the maximal element in I_P, i.e. P contains Δ_n if and only if $P = \Delta_n R$ for some positive braid R.

Proof:

For necessity: Let P contain Δ_n, then

$$P = A\Delta_n B$$

for some positive braids A and B. So that

$$P = \Delta_n[(\Delta_n)^{-1}A\Delta_n]B$$

$$= \Delta_n\tau[A]B$$

But $\tau[A]$ is a positive braid, then take $R = \tau[A]B$.

For sufficiency: Let $P = \Delta_n R$ for positive braid R, then write $P = A\Delta_n B$ for $A = e$ and $R = B$, hence P contains Δ_n, which completes the sufficiency, hence completes the proof □

(1.2.13) Corollary:

A positive braid (P,n) contains Δ_n if and only if $S(P) = \{1,2, \ldots ,n-1\}$.

Proof:

The necessity is a direct consequence from lemma (1.1.10). To establish the sufficiency: corollary (1.2.8) tells us that

$$P = \pi_1 P_1$$

where π_1 is the unique maximal element in I_P and P_1 is a positive braid in B_n and corollary (1.2.9) tells us that

$$S(\pi_1) = S(P)$$

So if

$$S(P) = \{1,2, \ldots ,n-1\}$$

then

$$S(\pi_1) = \{1,2, \ldots ,n-1\}$$

hence lemma (1.1.10) tells us that $\pi_1 = \Delta_n$, which completes the sufficiency, hence completes the proof □

(1.2.14) Remark:

As a further analogy with theorem (1.2.1), every positive braid (P,n) has a unique right-hand factorization as product of positive permutation braids

$$P = \alpha_r \alpha_{r-1} \ldots \alpha_1$$

for some $r \in \mathbb{Z}^+$ and $r \geqslant 1$, with unique maximal factor α_i at the end of $(\alpha_r \alpha_{r-1} \ldots \alpha_i)$, $1 \leqslant i \leqslant r$. Similarly, as in corollary (1.2.9),

$$F(P) = F(\alpha_1)$$

and if

$$F(P) = \{1,2, \ldots ,n-1\}$$

then

$$P = P_1 \Delta_n$$

for some positive braid P_1. Let $P = P_1 \alpha_1$ where P_1 is a positive braid and $\alpha_1 \in SB_n$, then α_1 is the unique maximal positive permutation braid at the end of P if and only if $F(P_1) \subseteq S(\alpha_1)$. Finally for a positive braid, the number of Δ_n factors in its canonical form is called the power of P.

(1.2.15) Lemma: (Garside, Appendix of [G2])

In B_n, let A and B be two positive braids, such that $P = AB = \Delta_n R$, for some positive braid R, then

(i): $A\pi_1$ contains Δ_n, where π_1 is the maximal starter of B

(ii): $\alpha_1 B$ contains Δ_n, where α_1 is the maximal finisher of A.

Proof:

The proof will be done by induction on length of A.

For $L(A) = 0$, then

$$P = B = \Delta_n R$$

so Δ_n is the maximal starter of B. Now for our induction hypothesis we assume that $A\pi_1$ contains Δ_n for $L(A) = r$ and π_1 is the maximal element in I_B. For $L(A) = r+1$, let $i \in F(A)$ for some $1 \leqslant i \leqslant n-1$, then we can write $A = A_1\sigma_i$, So

$$P = AB = A_1(\sigma_i B)$$

Let η be the maximal starter for $B_1 = \sigma_i B$, but $i \in S(B_1)$, then

$$\sigma_i \leqslant \eta$$

i.e. η contains σ_i, so η can be written as

$$\eta = \sigma_i \gamma$$

Then γ is a starter for B and

$$\gamma \leqslant \pi_1$$

so the maximal starter of B_1 is contained in $\sigma_i\pi_1$. Now since $L(A_1) = r$, then our induction hypothesis implies that $A_1\eta$ contains Δ_n and so $A\pi_1$ contains Δ_n, which completes the proof of case (i).

The proof of case (ii) follows by considering reverse elements in case (i) □

(1.2.16) Corollary:

In B_n, let A and B be two positive braids such that $AB = \Delta_n R$, for some positive braid R, then $F(A) \cup S(B) = \{1,2, \ldots ,n-1\}$.

Proof:

Applying (i) of lemma (1.2.15), then $A\pi_1$ contains Δ_n, where π_1 is the maximal starter for B. Again apply (ii) of lemma (1.2.15), on

$A\pi_1$, then $\alpha_1\pi_1$ contains Δ_n, where α_1 is the maximal finisher for A. Now write $\Delta_n = \alpha_1(\alpha_1)_*$, then

$$(\alpha_1)_* \leqslant \pi_1$$

hence

$$\begin{aligned} F(\alpha_1) \cup S((\alpha_1)_*) &\subseteq F(\alpha_1) \cup S(\pi_1) \\ &= F(A) \cup S(B) \end{aligned}$$

But lemma (1.1.16) tells us that

$$F(\alpha_1) \cup S((\alpha_1)_*) = \{1, 2, \dots , n-1\}$$

so

$$F(A) \cup S(B) = \{1, 2, \dots , n-1\}$$

which completes the proof □

For a positive braid P, if at most k canonical factors of P have equal to Δ_n, then $P = (\Delta_n)^k Q$ for some positive braid Q and prime to Δ_n, i.e. $(\Delta_n)^k$ is a factor of P. Now the following theorem provides an algorithm to decide whether a given braid P = AB contains $(\Delta_n)^k$, or not. This is by writing A and B in their canonical factorizations, hence look to factors.

<u>(1.2.17) Theorem</u>:

In B_n, let A and B be two positive braids, such that $P = AB = (\Delta_n)^k R$, for some positive braid R, then

(i): $A = A_1(\pi_1\pi_2 \dots \pi_k)$

(ii): $B = (\eta_1\eta_2 \dots \eta_k)B_1$

such that $(\pi_1\pi_2 \dots \pi_k)(\eta_1\eta_2 \dots \eta_k) = (\Delta_n)^k$, where $\pi_i, \eta_i \in SB_n$ for every $1 \leqslant i \leqslant k$.

Proof:

Let $(\mathrm{Th.})_k$ refer to (i) of the theorem.

$(\mathrm{Th.})_1$ follows directly from lemma (1.2.15), the proof of the general theorem follows by induction on k. For our induction hypothesis we assume that $(\mathrm{Th.})_r$ holds.

For $(\mathrm{Th.})_{r+1}$: Let $(\Delta_n)^{r+1}R = AB$, then AB contains Δ_n, hence lemma (1.2.15) tells us that there exists some $\pi, \eta \epsilon SB_n$ such that $A = A'\pi$ and $B = \eta B'$, with $\pi\eta = \Delta_n$. Then

$$\begin{aligned}(\Delta_n)^{r+1}R &= AB \\ &= A'\pi\eta B' \\ &= A'\Delta_n B' \\ &= A'\tau[B']\Delta_n\end{aligned}$$

So that

$$(\Delta_n)^r\tau[R]\Delta_n = A'\tau[B']\Delta_n$$

i.e.

$$(\Delta_n)^r\tau[R] = A'\tau[B']$$

Then the induction hypothesis tells us that

$$A' = A_1\alpha_1\alpha_2 \ldots \alpha_r, \text{ say}$$

for some $\alpha_i \epsilon SB_n$, $1 \leqslant i \leqslant r$. So that

$$\begin{aligned}(\Delta_n)^{r+1}R &= (\Delta_n)^r\tau[R]\Delta_n \\ &= A_1(\alpha_1\alpha_2 \ldots \alpha_r)\tau[B']\Delta_n \\ &= A_1(\alpha_1\alpha_2 \ldots \alpha_r)\Delta_n(B')\end{aligned}$$

$$= A_1(\alpha_1\alpha_2 \ \dots \ \alpha_r)(\pi\eta)(B')$$

$$= A_1(\alpha_1\alpha_2 \ \dots \ \alpha_r\pi)B$$

which completes the induction hypothesis, hence completes the proof of (i) of the general theorem. Case (ii) also follows by considering reverse elements in (i) □

<u>(1.2.18) Theorem</u>:

A positive braid (P,n) is a factor of $(\Delta_n)^k$ if and only if its canonical form has at most k factors.

<u>Proof</u>:

For necessity: Let P be a factor of $(\Delta_n)^k$, then there exist a positive braid Q such that $PQ = (\Delta_n)^k$. But for k = 1 the proof follows directly from corollary (1.1.15). Then the proof of the necessity follows by induction on k. Assume that the theorem holds for k = r. Now let

$$PQ = (\Delta_n)^{r+1}$$

then lemma (1.2.15) tells us that $P = P_1\pi_1$ and $Q = \xi_1Q_1$ for positive braids P_1, Q_1 and for π_1, $\xi_1 \in SB_n$, such that $\pi_1\xi_1 = \Delta_n$. So

$$\begin{aligned}(\Delta_n)^{r+1} &= PQ \\ &= P_1\Delta_nQ_1 \\ &= P_1\tau[Q_1]\Delta_n\end{aligned}$$

hence

$$(\Delta_n)^r = P_1\tau[Q_1]$$

Then from our induction hypothesis, the canonical form of P_1 has at most r factors. Therefore $P_1\pi_1$ has at most r+1 factors, which completes the induction process, and so completes the proof of the necessity.

For sufficiency: Let

$$P = (\pi_1\pi_2 \ldots \pi_k)$$

for $\pi_i \in SB_n$, $1 \leqslant i \leqslant k$, then by switching the factors π_i by either $(\pi_i)_*$ or $(\pi_i)^*$ and using lemma (1.1.16), for $1 \leqslant i \leqslant k.$, we have

$$P(\pi_k)_* = (\pi_1\pi_2 \ldots \pi_k)(\pi_k)_*$$

$$= (\pi_1\pi_2 \ldots \pi_{k-1})(\Delta_n)$$

hence

$$P(\pi_k)_*(\pi_{k-1})^* = (\pi_1\pi_2 \ldots \pi_{k-2})(\pi_{k-1})(\Delta_n)(\pi_{k-1})^*$$

$$= (\pi_1\pi_2 \ldots \pi_{k-2})(\pi_{k-1})(\pi_{k-1})_*\Delta_n$$

$$= (\pi_1\pi_2 \ldots \pi_{k-2})(\Delta_n)^2$$

Then continuing this process we finish with positive braid Q such that $PQ = (\Delta_n)^k$. Then P is a factor of $(\Delta_n)^k$, which completes the proof of sufficiency, hence completes the proof of the theorem □

(1.2.19) Proposition:

Every factor of $(\Delta_n)^k$ has property that each pair of arcs cross at most k times.

Proof:

Let P be a factor of $(\Delta_n)^k$, then theorem (1.2.18) tells us that the canonical form of P has at most k factors. But every pair of arcs in a positive permutation braid cross at most once, hence the proof follows directly □

But not every such positive braid with each two arcs cross at most k times is a factor of $(\Delta_n)^k$, an example to show that is given below.

<u>(1.2.20) Example</u>:

In B_n, the braids $\beta_{i,n} = (\sigma_{i+1})^2\sigma_i\sigma_{i+2}(\sigma_{i+1})^2$ and $\alpha_{i,n} = (\sigma_i)^2(\sigma_{i+1})^2$ are not factors for $(\Delta_n)^2$.

<u>Proof</u>:

It is enough to look at $(\alpha_{1,3})\epsilon B_3$ and $(\beta_{1,4})\epsilon B_4$, because we can have $(\Delta_{n-1})^2$ from $(\Delta_n)^2$ by deleting any string in $(\Delta_n)^2$ as in figure (1-10a). So that order the arcs in top of $\alpha_{1,3} = (\sigma_1)^2(\sigma_2)^2$ from left to right, then as in figure (1-10b) the pair {1,2} of arcs cross each other twice, as do the pair {2,3} of arcs. So $\alpha_{1,3}$ is a factor of $(\Delta_3)^2$ only if the pair {1,3} of arcs cross each other twice, which is impossible without crossing the middle arc, hence $\alpha_{1,3}$ is not a factor for $(\Delta_3)^2$.

Similarly order the arcs in the top of $\beta_{1,4} = (\sigma_2)^2\sigma_1\sigma_3(\sigma_2)^2$ from the left to the right. Then as in figure (1-10c), the pair {2,3} of arcs cross each other twice, as do the pair {1,4} of arcs and the pair {1,3} of arcs never cross each other. But the pair {1,3} never cross each other without crossing either the second arc or the fourth one, which means that two arcs crossed more than twice, hence $\beta_{1,4}$ is not a factor for $(\Delta_4)^2$ □

(1.2.21) Proposition: (A geometric view of the factorization of $(\Delta_n)^2$)

In B_n, if $(\Delta_n)^2 = PQ$ for two positive braids P and Q, then $P = \alpha_1\alpha_2$ and $Q = \beta_1\beta_2$, where $\alpha_i, \beta_i \in SB_n$, for $i = 1,2$.

Proof:

Write $P = \alpha_1\alpha_2$ such that α_2 is the largest positive permutation braid as a finisher of P. Assume that $\alpha_1 \notin SB_n$, but P is a factor for $(\Delta_n)^2$, then there are two arcs crossed twice in α_1 and they never cross in α_2. Hence the proof follows by induction on such these arcs which cross twice in α_1. So let α_1 has only two arcs i and j (labelled on top of α_1) such that they cross each other twice. Then

$$\alpha_1 = A\sigma_k B\sigma_m C$$

for some generators σ_k and σ_m which represent the crossings of the arcs i and j. Now take $\alpha = A\sigma_k B$ and $\gamma = \sigma_m C\alpha_2$, then $P = \alpha\gamma$, where $\alpha \in SB_n$, Hence to prove that $\gamma \in SB_n$, it is enough to show that the arcs which crossed in $\sigma_m C$ never cross in α_2.

Now as in figure (1-11), we can arrange the arcs labelled i and j on the top of P to cross at the end of $A\sigma_k B$. We can also arrange the braid word C to contain a Lorenz braid $\beta(a,b)$, (which is a positive permutation braid in B_{a+b}, with single starter ,see definition (3.1.1)). Then we only need to prove that the arcs which crossed in $\beta(a,b)$ they never cross in α_2. But in α_2 the out strands from the tangle $\beta(a,b)$ never cross the arcs labelled i,j in the top of the braid $(\Delta_n)^2$. Therefore assume the contrary. Hence we have a contradiction with example (1.2.19), where $\alpha_{k,n}$ and $\beta_{r,n}$ are factors of $(\Delta_n)^2$ respectively, see figures (1-12a) and (1-12b) □

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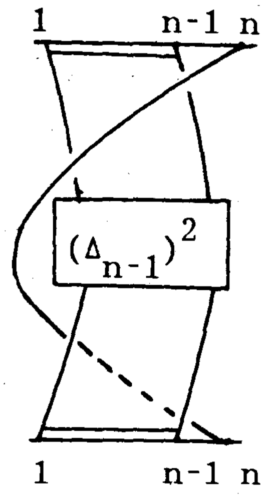
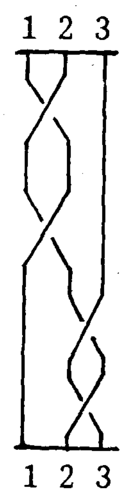


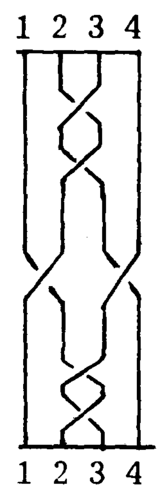
$(\Delta_n)^2$

Figure (1-10a)



$\alpha_{1,3}$

Figure (1-10b)



$\beta_{1,4}$

Figure (1-10c)

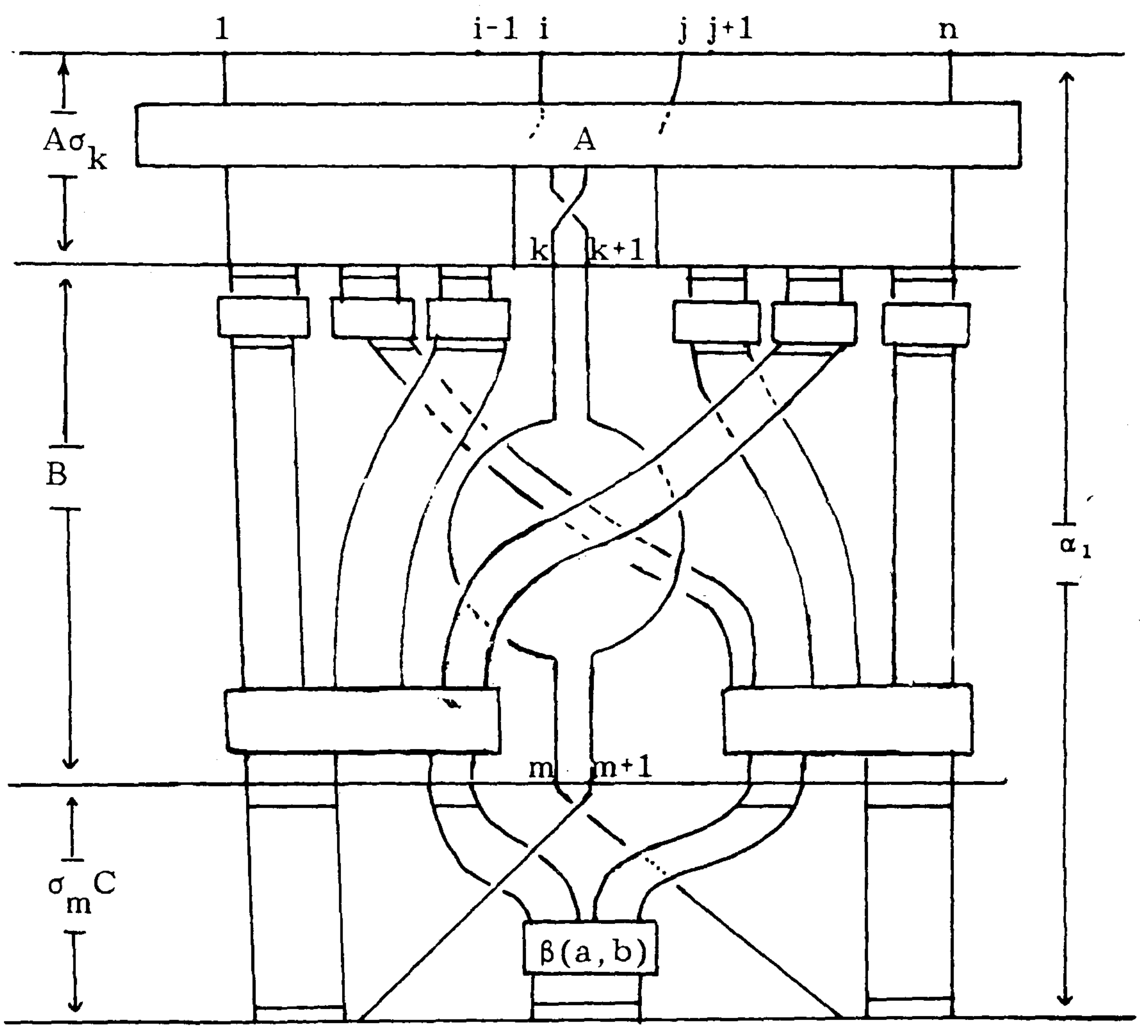


Figure (1-11)

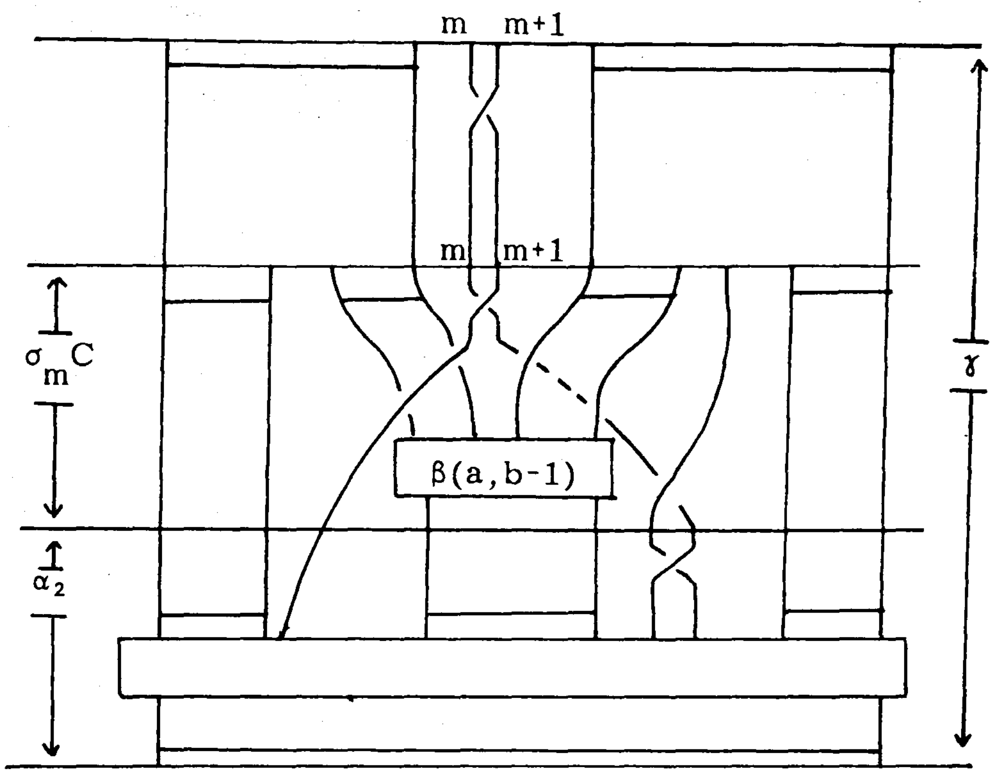


Figure (1-12a)

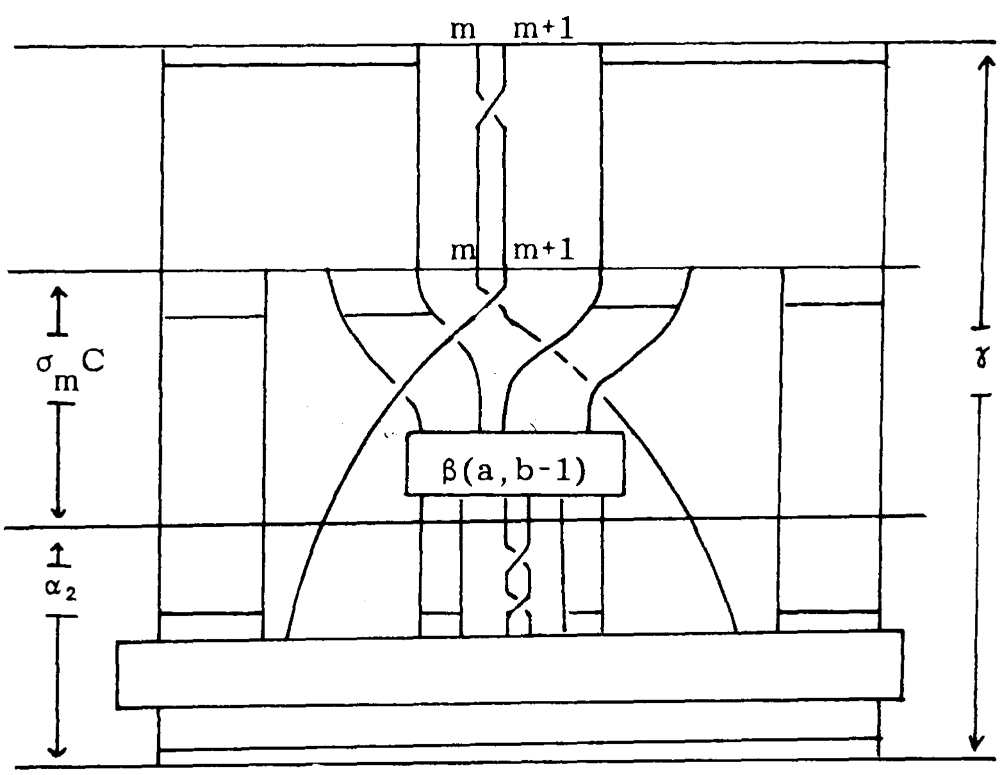


Figure (1-12b)

§1.3. AN ALGORITHM FOR FINDING THE CANONICAL FORM FOR A POSITIVE BRAID

(1.3.1) Algorithm:

Starting with a positive braid (P,n), then without any application of the braid relators (i) and (ii) of definition (0.5), write P as a successive product of generators, i.e

$$P = \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_m}$$

where

$$1 \leqslant i_j \leqslant n-1,\ 1 \leqslant j \leqslant m$$

Again without any application of relators (i) and (ii) of definition (0.5), rewrite P as a product of positive permutation braids, i.e.

$$P = \pi_1\pi_2 \dots \pi_r, \text{ for } \pi_i \epsilon SB_n,\ 1 \leqslant i \leqslant r$$

where

$$\pi_{j+1} = (\sigma_{i_{s_j+1}})(\sigma_{i_{s_j+2}}) \dots (\sigma_{i_{s_{j+1}}})$$

for $0 \leqslant j \leqslant r-1$, $s_0 = 0$ and $s_r = m$. Find the starter set I_{π_i} for every π_i, $2 \leqslant i \leqslant r$, then find the largest element $\alpha \epsilon I_{\pi_2}$ such that $(\pi_1\alpha)\epsilon SB_n$, so for some $\gamma \epsilon SB_n$, we have

$$\pi_1\pi_2 = (\pi_1\alpha)\gamma$$

where $\pi_2 = \alpha\gamma$. Again find the largest starter β for π_3 such that $(\gamma\beta)\epsilon SB_n$ and write

$$\gamma\pi_3 = (\gamma\beta)\eta$$

where $\pi_3 = \beta\eta$. Continuing this process we can have a new factorization

$$P = \eta_1\eta_2 \ldots \eta_s, \text{ say}$$

for some $s \in \mathbb{Z}^+$. Again find the starter set for each η_i and then repeat the previous steps.

In other words bracket the successive letters of the word P as a product of positive permutation braids $(\pi_1\pi_2 \ldots \pi_k)$. Investigate the crossings of the arcs of the first factor π_1, to decide which arcs do not cross in the braid π_1. If a pair of such these arcs cross in π_2 and if it is possible to pull that crossing at the end of π_1 then do it. Do that with the other pair of arcs, hence finish with new positive permutation braids $(\pi_1)'$ and $(\pi_2)'$. Repeat that again on $(\pi_2)'$ and π_3 to finish with $(\pi_2)''$, and $(\pi_3)'$. Repeat that again on $(\pi_3)'$ and (π_4), and so on. Then the braid P has the new factorization, $[(\pi_1)'(\pi_2)''(\pi_3)'' \ldots (\pi_{k-1})''(\pi_k)']$. Note that the number of factors does not increase under the algorithm, because it is possible that some of the factors vanish. But $L(P)$ is finite and SB_n is also a finite set. Then ultimately a stage must be reached when further applications of the process will yield no new factorizations.

The condition is that a starter of a factor should be a finisher of the previous factor. i.e. if $i \in S(\pi_{i+1})$ then $i \in F(\pi_i)$, otherwise we can increase the length of π_i. Note also that the number of factors of P never increase under the algorithm. An example for applying this algorithm is given in example (1.3.3).

(1.3.2) Theorem : (A practical test for use in the algorithm)

Given a positive braid (P,n) with the factorization $P = (\pi_1\pi_2 \ldots \pi_k)$, where $\pi_i \in SB_n$, $1 \leqslant i \leqslant k$, then the given factorization is the left-hand canonical form of P if and only if $S(\pi_{i+1}\pi_{i+2} \ldots \pi_k) \subseteq F(\pi_i)$, for $1 \leqslant i \leqslant k-1$.

Proof:

Put $P_i = (\pi_i\pi_{i+1} \ldots \pi_k)$, for $1 \leqslant i \leqslant k$, with $P_1 = P$. Using proposition (1.2.10) for P_1, then P_2 and so on. Then π_i is the maximal factor for P_i if and only if $S(\pi_{i+1}\pi_{i+2} \ldots \pi_k) \subseteq F(\pi_i)$, for $1 \leqslant i \leqslant k-1$, which completes the proof □

(1.3.3) Example:

$$\text{Let} \quad P = [\sigma_1\sigma_3(\sigma_2\ )^2\sigma_3\sigma_1\sigma_3\sigma_2\sigma_3\sigma_2] \in B_4$$

Then write P as a product of positive permutation braids,

$$P = (\sigma_1\sigma_3\sigma_2)(\ \sigma_2\sigma_3\sigma_1)(\sigma_3\sigma_2\sigma_3)(\sigma_2)$$

Find the starting and finishing sets for each factor, then

$$F(\pi_1=\sigma_1\sigma_3\sigma_2) = \{2\}$$

$$F(\pi_2=\sigma_2\sigma_3\sigma_1) = \{1,3\} \text{ and } S(\pi_2) = \{2\}$$

$$F(\pi_3=\sigma_3\sigma_2\sigma_3) = S(\pi_3) = \{2,3\} \text{ and } S(\pi_4=\sigma_2) = \{2\}$$

Then applying the algorithm we can write P in its canonical factorization as $P = (\sigma_1\sigma_3\sigma_2\sigma_1)(\ \sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_2)(\sigma_2)$, where the applications are illustrated diagrammatically in figure (1-13) □

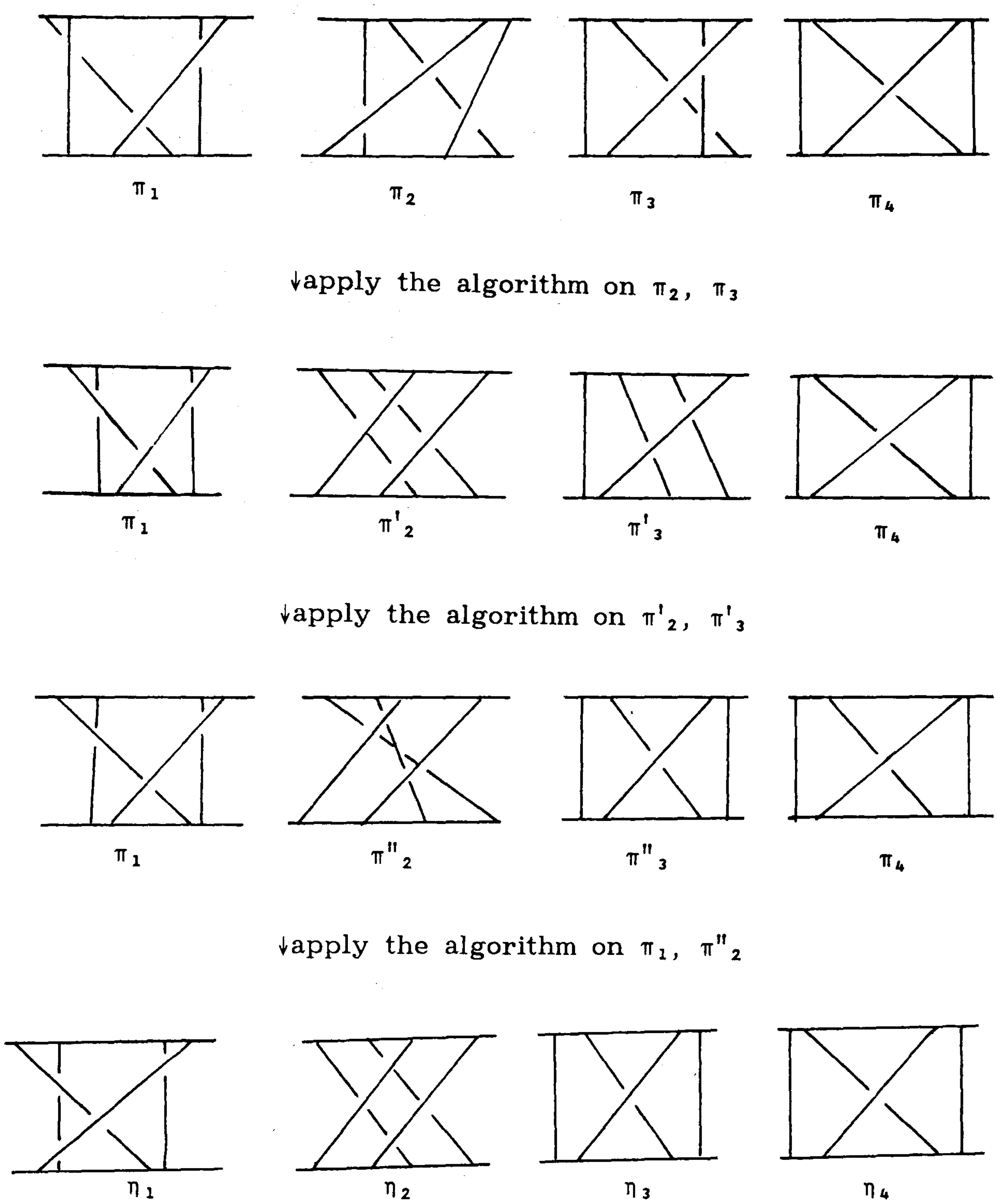


Figure (1-13)

<u>(1.3.4) Corollary</u>: (Canonical length for a positive word)

For a positive braid (P,n), the number of factors in the right-hand canonical form of P equals the numbers of factors in its left-hand canonical form.

Proof:

Let P have left-hand canonical form with k terms and right-hand canonical form with r terms, then start with the left-hand canonical form of P (which has k terms) and apply the algorithm above to write the right-hand canonical form of P. But the algorithm never increase the number of factors, hence $r \leqslant k$. Similarly if we start with the right-hand canonical form of P, then we have $k \leqslant r$, so $k = r$. The number of factors in a canonical form of a positive braid P is called the canonical length of P and denoted CL(P) □

(1.3.5) Corollary:

If (P,n) is a positive braid with CL(P) = k, then $P^{-1} = (\Delta_n)^{-k}Q$, where Q is positive and prime to Δ_n, i.e. the power of P^{-1} equals -CL(P).

Proof:

Theorem (1.2.18) tells us that P is a factor of $(\Delta_n)^k$. Then there exists a positive braid Q such that

$$PQ = (\Delta_n)^k$$

and CL(Q) $\leqslant$ k, because Q is also a factor of $(\Delta_n)^k$. Then

$$P^{-1} = (\Delta_n)^{-k}Q$$

But Q does not contain Δ_n, otherwise P is a factor of $(\Delta_n)^{(k-1)}$ which contradicts theorem (1.2.18), hence Q has power 0 □

§1.4. APPLICATIONS

(I): A NORMAL FORM FOR GARSIDE'S SOLUTION OF THE WORD PROBLEM IN B_n :

Let β be any word in B_n, then from corollary (1.1.15), we can replace every negative permutation braid π^{-1} (which occurs in the braid word β) by

$$(\Delta_n)^{-1}\pi^*$$

Now using the property, of lemma (1.1.11),

$$\tau[(\sigma_i)^{\pm 1})] = (\sigma_{n-i})^{\pm 1}$$

then collect all $[(\Delta_n)^{-1}]$'s (introduced in the further step) at the left. So that β is represented by a word of the form

$$\beta = (\Delta_n)^m P, \ m \leqslant 0$$

for a positive word P. Now find the left-hand canonical form of P, as in theorem (1.2.1),

$$P = \pi_1\pi_2 \ \ldots \ \pi_k, \text{ say}$$

Let P has power r, i.e. each one of the first r factors in the canonical form equals Δ_n, hence

$$P = (\Delta_n)^r(\pi_{r+1}\pi_{r+2} \ \cdots \ \pi_k)$$

So that

$$\beta = (\Delta_n)^{m+r}(\pi_{r+1}\pi_{r+2} \ \cdots \ \pi_k) \tag{1.4.1}$$

Then the form in equation (1.4.1) is called the standard form for β and $(m+r)$ is called the power of β, which denoted $W(\beta)$. Since every positive permutation braid is only determined, as in lemma (1.1.3), by its associated permutation, then β is determined by its power and the corresponding tuple

$$B(\beta) = (\beta_{r+1}, \beta_{r+2}, \ldots, \beta_k)$$

where β_i is the associated permutation of π_i, in equation (1.4.1). Such $B(\beta)$ is called the base of β and the number of components in $B(\beta)$ is called the base length of β, denoted $BL(\beta)$.

(1.4.1) Proposition :

In B_n every word β is uniquely determined by its power and base.

Proof

Let α be a braid word with two powers a, b and with two corresponding bases

$$B_1(\alpha) = (\alpha_1, \alpha_2, \ldots, \alpha_k) \text{ and } B_2(\alpha) = (\beta_1, \beta_2, \ldots, \beta_r)$$

then

$$\alpha = (\Delta_n)^a(\pi_1\pi_2 \ldots \pi_k) = (\Delta_n)^b(\eta_1\eta_2 \ldots \eta_r)$$

where π_i and η_j are the corresponding positive permutation braids for the permutations α_i and β_j respectively, for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant r$. Now assuming that $a \neq b$ and $(a-b) > 0$, then

$$(\Delta_n)^{a-b}(\pi_1\pi_2 \ldots \pi_k) = (\eta_1\eta_2 \ldots \eta_r)$$

which contradicts that $(\eta_1\eta_2 \ldots \eta_r)$ is prime to Δ_n, otherwise $(\beta_1, \beta_2, \ldots, \beta_r)$ does not a base for β, so $a = b$ and

$$(\pi_1\pi_2 \ldots \pi_k) = (\eta_1\eta_2 \ldots \eta_r) = P \text{ , say}$$

But theorem (1.2.1) tells us that P has a unique (left-hand) canonical form, so

$$\eta_i = \pi_i \in SB_n, \text{ for } 1 \leqslant i \leqslant k = r$$

hence

$$\alpha_i = \beta_i \in S_n, \text{ for } 1 \leqslant i \leqslant k = r$$

Therefore the two standard forms are identical □

<u>(1.4.2) Theorem</u>: (The solution of the word problem)

In B_n, two words are equal if and only if their standard forms are identical.

<u>Proof</u>

The sufficiency is clear and the necessity has been shown in proposition (1.4.1) □

<u>(II): ON CONJUGACY PROBLEM IN B_n</u> :

An algorithm is now given to decide whether a positive braid P is conjugate (or not) to $\Delta_n Q$ for some positive braid Q.

<u>(1.4.3) Algorithm</u>:

In B_n, let P be positive and prime to Δ_n. Then we can decide whether P is conjugate to $\Delta_n Q$ (or not) for a positive braid word Q as follows:

Put P in its (left-hand) canonical form

$$P = \pi_1\pi_2 \ldots \pi_k$$

then $\pi_1 \neq \Delta_n$, because P is prime to Δ_n. Now conjugate P by π_1. Then

$$(\pi_1)^{-1}(P)\pi_1 = \pi_2\pi_3 \ldots \pi_k\pi_1 = P_1, \text{ say}$$

Hence write P_1 in its canonical form as

$$P_1 = \alpha_1\alpha_2 \ldots \alpha_{k_1}$$

If $\alpha_1 = \Delta_n$, then the algorithm will stop. Otherwise repeat by conjugating P_1 by α_1, then

$$(\alpha_1)^{-1}P_1\alpha_1 = \alpha_2\alpha_3 \ldots \alpha_{k_1}\alpha_1 = P_2, \text{ say}$$

Hence write P_2 in its canonical form as

$$P_2 = \beta_1\beta_2 \ldots \beta_{k_2}$$

If $\beta_1 \neq \Delta_n$, then continue the process. Hence on repetition the algorithm either stops or cycles, i.e. at some stage either P_i contains Δ_n or P_i is prime to Δ_n with $P_i = P_j$, for some $j < i$. This is because $k_i \leqslant k_{i-1} \leqslant \ldots \leqslant k_1 \leqslant k$ and SB_n is a set of finite order, i.e. a stage must be reached when further applications of the process will either factor out Δ_n or yield no new words □

The following theorem provides a proof of the algorithm above:

(1.4.4) Theorem:

In B_n, the positive braid P is conjugate to $\Delta_n R$, for a positive braid R if and only if the algorithm above produces Δ_n.

Proof:

Let $P = \pi_1\pi_2 \ldots \pi_k$ be the (left-hand) canonical form of P and write

$$P_{i+1} = (\alpha_i)^{-1}P_i\alpha_i, \quad 1 \leqslant i \leqslant k-1$$

and

$$P_1 = (\pi_1)^{-1} P \pi_1$$

where α_i is the first factor in the (left-hand) canonical form of P_i. For sufficiency: Each P_i is conjugate to P, so if Δ_n appears in P_i then take $\alpha_i = \Delta_n$ and $R = Q_i$, where $P_i = \alpha_i Q_i$.

For necessity: Let P be conjugate to $\Delta_n R$ for a positive braid R, then put $P = \pi_1 Q_1$, for positive braid Q_1 and π_1 is the maximal starter for P. So let $\pi_1 \neq \Delta_n$, otherwise P contains Δ_n, hence the proof is trivial. Now let P and $\Delta_n R$ are conjugate by a braid word γ. But $(\Delta_n)^{2m}\gamma = A$, say, is positive for large enough m and $(\Delta_n)^2$ commutes with every thing, then

$$AP = \Delta_n RA$$

So if $L(A) = 0$, then P contains Δ_n up to conjugation, hence the proof follows by induction on the length of the conjugator A. Then for our induction hypothesis we assume that the theorem holds for conjugators of length $\leqslant$ r. Then assume that $L(A) = r+1$. But

$$A\pi_1 Q_1 = \Delta_n RA$$

Then using lemma (1.2.15), we can write

$$A = A_1(\pi_1)^*$$

for a positive word A_1, so

$$\begin{aligned} AP &= A_1(\pi_1)^*(\pi_1)Q_1 \\ &= A_1(\Delta_n)Q_1 \\ &= (\Delta_n)RA \end{aligned}$$

$$= (\Delta_n)RA_1(\pi_1)^*$$

Hence

$$A_1(\Delta_n)Q_1\pi_1 = (\Delta_n)RA_1(\pi_1)^*\pi_1$$

$$= (\Delta_n)RA_1(\Delta_n)$$

$$= (\Delta_n)^2\tau[RA_1]$$

So

$$\tau[A_1]Q_1\pi_1 = (\Delta_n\tau[R])\tau[A_1]$$

Then $Q_1\pi_1$ is conjugate (by $\tau[A_1]$) to $\Delta_n\tau[R]$, with $L(A_1) < L(A)$. So by induction hypothesis, cycling the factors $Q_1\pi_1$ will produce Δ_n, i.e. applying the algorithm (on P) produces Δ_n, which completes the induction process and so completes the proof □

(1.4.5) Lemma:

In B_n, if braids P and Q are conjugate by a positive braid A and if power of P = power of Q = k, then the power of $\alpha^{-1}P\alpha \geqslant k$, where α is the maximal starter for A.

Proof

Let α be the maximal starter for A. i.e. $A = \alpha A_1$, for a positive word A_1. Then proposition (1.2.10) tells us that

$$F(\alpha) \supseteq S(A_1)$$

But

$$\alpha^* PA = \alpha^* P\alpha A_1 = \alpha^* \alpha A_1 Q = \Delta_n A_1 Q$$

i.e.

$$(\alpha^* P\alpha)A_1 = \Delta_n A_1 Q$$

Now put P and Q in their standard forms $(\Delta_n)^k P'$ and $(\Delta_n)^k Q'$, respectively, for positive braids P' and Q'. Then for k-even

$$(\alpha^* P'\alpha)A_1 = \Delta_n A_1 Q'$$

So

$$F(\alpha^* P'\alpha) \supseteq F(\alpha) \supseteq S(A_1)$$

But, using corollary (1.2.16),

$$S(A_1) \cup F(\alpha^* P'\alpha) = \{1, 2, \ldots, n-1\}$$

Then corollary (1.2.13) tells us that $(\alpha^* P'\alpha)$ contains Δ_n, i.e.

$$\alpha^* P'\alpha = \Delta_n R$$

for some positive word R. So

$$\alpha^* P'\alpha = \alpha^* \alpha R$$

then

$$P'\alpha = \alpha R$$

so $\alpha^{-1}P'\alpha = R$ is a positive word, then $\alpha^{-1}P\alpha$ of power $\geq k$. Now for k-odd and using lemma (1.1.16), we have

$$\alpha_* P'\alpha A_1 = \Delta_n \tau[A_1]Q'$$

Similarly $(\alpha_* P'\alpha)$ contains Δ_n, i.e.

$$(\alpha_* P'\alpha) = \Delta_n R'$$

for some positive braid R'. Then

$$\Delta_n(\alpha_* P'\alpha) = \alpha^*(\Delta_n P')\alpha = (\Delta_n)^2 R' = \alpha^*(\alpha)(\Delta_n R')$$

so

$$\Delta_n P'\alpha = \alpha(\Delta_n R')$$

i.e. $\alpha^{-1}(\Delta_n P')\alpha$ is positive and contains Δ_n, hence $\alpha^{-1}P\alpha$ of power $\geqslant$ k, which completes the proof □

(1.4.6) Algorithm: (summit forms and summit set), [G1] and [G2]

In B_n, every word α has a standard form $(\Delta_n)^r P$, for a positive word P which is uniquely determined by its canonical form. Let $(\Delta_n)^r P = W_1$, say. Define

$$W^{(1)} = \{\pi W_1 \pi^{-1} \mid \pi \in SB_n\}$$

Let those words in $W^{(1)}$ which are of power $\geqslant$ r, which are distinct from W_1 and from each other, be $W_2, W_3, \ldots, W_t$. Now repeat the process for each of the words $W_2, W_3, \ldots, W_t$ in turn, denoting successively by $W_{t+1}, W_{t+2}, \ldots$ any new words occurring. The condition being always that each new word must be of power $\geqslant$ r. Continue to repeat the process for every new distinct word arising, as the sequence $W_1, W_2, \ldots, W_{t+2}, \ldots$ expands. Now each word of the sequence is of the same index length as α. So let $(\Delta_n)^k Q$ be the standard form for any W_i, then $k \geqslant r$. But

$$L(\alpha) = L((\Delta_n)^k Q) = k(L(\Delta_n)) + L(Q) \geqslant L(\Delta_n)$$

So

$$[L(\alpha)/L(\Delta_n)] \geqslant k \geqslant r$$

Then the number of values of k is finite and the possible values for Q are also finite for fixed k, hence the sequence $W_1, W_2, \ldots$ is finite. So ultimately a stage must be reached when further applications of the process will yield no new words. Suppose that the highest power reached is s and that the words of power s form the subset $V_1, V_2, \ldots$, then any V_r will called a summit form of α. The set V_1, V_2,

... will called the summit set of α, denoted $SS(\alpha)$. The power s of any summit form will called the summit power of α.

(1.4.7) Definition: (Super summit forms and super summit set)

For a braid word α in B_n, apply Garside's algorithm above with the condition that; choose those words where their associated basis (in their canonical forms) have the smallest canonical length among those words at each stage. Then define the super summit forms of α as those summit forms with basis of the smallest canonical length among the summit set of α. The set of super summit forms of α will be called the super summit set of α, denoted $SSS(\alpha)$. Hence for every braid word α there is an associated number (the canonical length of the base of any super summit form of α), called the summit length of α and denoted $SL(\alpha)$.

(1.4.8) Theorem:

For a braid word α, let P and Q be two super summit forms, then there are a sequence of elements $R_0 = P$, R_1, ... , $R_s = Q$ in the super summit set of α such that R_{i+1} conjugate to R_i by a positive permutation braid.

Proof:

Let P and Q have summit length r, i.e. $SL(P) = SL(Q) = r$, then P and Q have standard forms

$$P = (\Delta_n)^k P' \text{ and } Q = (\Delta_n)^k Q'$$

where P' and Q' are positive braids with $CL(P') = CL(Q') = r$ and k is the summit power of α. Now let P and Q be conjugate by braid W, then they are conjugate by a positive braid $X = (\Delta_n)^{2m} W$ for large enough positive integer m, where $(\Delta_n)^2$ commutes with every thing. So put X in its left-hand canonical form, as

$$X = \pi_1\pi_2 \ldots \pi_s, \text{ say}$$

and let

$$W_i = (\pi_i)^{-1}(W_{i-1})\pi_i, \text{ for } 2 \leqslant i \leqslant s$$

with

$$W_1 = (\pi_1)^{-1}P\pi_1 \text{ and } W_s = Q$$

Then lemma (1.4.5) tells us that each W_i is of power $\geqslant k$, hence of power k. i.e. each W_i is a summit form for α. Now find the inverse of each W_i and use corollary (1.3.5), then

$$P^{-1} = (\Delta_n)^{-(k+r)}P_1 \text{ and } Q^{-1} = (\Delta_n)^{-(k+r)}Q_1$$

where P_1 and Q_1 are positive and prime words to Δ_n. Now let $W_1 = (\Delta_n)^kR$, with $SL(W_1) = t \geqslant r$, then $(W_1)^{-1}$ has power $-(k+t)$. But the braids P^{-1} and Q^{-1} have the same power $-(k+r)$ and they are conjugate by the positive braid X, then lemma (1.4.5) tells us that $(\pi_1)^{-1}P^{-1}\pi_1 = (W_1)^{-1}$ has power $\geqslant -(k+r)$, so that $-(k+t) \geqslant -(k+r)$, i.e. $r \geqslant t$, hence $r = t$. Repeat this process with W_1 and Q, and so on. Then each W_i has summit length r, i.e. each W_i is a super summit form for α, which completes the proof □

The following theorem provides an improvement of Garside's solution to the conjugacy problem in B_n, where the invariant class (summit set) under conjugacy is reduced to a much smaller invariant subclass (super summit set).

(1.4.9) Theorem:

In B_n, two words are conjugate if and only if their super summit sets are identical.

Proof:

The sufficiency is clear. To establish the necessity, suppose that the words α and β are conjugate in B_n. Let $(\Delta_n)^r A$ and $(\Delta_n)^t B$ be any super summit forms for α and β, respectively, hence $(\Delta_n)^r A$ and $(\Delta_n)^t B$ are conjugate through such forms (by a braid word R), as in theorem (1.4.8). But $(\Delta_n)^{2m} R = X$, say, is positive for large enough m and $(\Delta_n)^2$ commutes with every thing, then

$$X^{-1}(\Delta_n)^r AX = (\Delta_n)^t B$$

Now assume that $t \geqslant r$, then put X in its left-hand canonical form, as

$$X = \pi_1\pi_2 \ldots \pi_k, \text{ say}$$

Put

$$W_i = (\pi_i)^{-1}(W_{i-1})\pi_i, \text{ for } 2 \leqslant i \leqslant k$$

and

$$W_1 = (\pi_1)^{-1}(\Delta_n)^r A\pi_1$$

Then lemma (1.4.5) tells us that each W_i is of power at least r, for $1 \leqslant i \leqslant k$. But theorem (1.4.8) also tells us that each W_i still in the super summit set. Therefore $W_k = (\Delta_n)^t B$ is of power at least r. So $(\Delta_n)^t B$ is a super summit form for α, hence we can not have $t >$ r. Similarly we can not have $r > t$, so $r = t$ and $(\Delta_n)^t B$ is a super summit form for α. Similarly any super summit form of α is a super summit form of β. So that the super summit sets of α and β are identical, which completes the proof □

CHAPTER 2

TWIST POSITIVE BRAIDS WITH THE 2-VARIABLE LINK INVARIANT AND THE ALGEBRAIC LINK PROBLEM

§2.0. INTRODUCTION

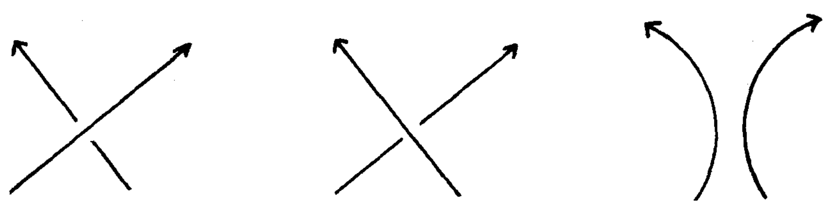
I: On the 2-variable link invariant

A link invariant is a function from the isotopy classes of links to some algebraic structure. Alexander.J, [A], has been introduced the first link invariant $\Delta_L(t)$ of an oriented link L, which is a Laurent polynomial in the variable t. Alexander had explained how to calculate $\Delta_L(t)$ by taking the determinant of a matrix associated with a projection of the link suitably chosen in a special position in a plane. In fact $\Delta_L(t)$ is a link invariant up to sign and multiplication by powers of the variable t and can be normalised so that $\Delta_L(t) = \Delta_L(t^{-1})$.

The Conway polynomial $\nabla_L(z)$ is a direct link invariant, in fact it generalise the normalised Alexander polynomial, where $\Delta_L(t) = \nabla_L(\sqrt{t}-\sqrt{t}^{-1})$, $\Delta_L(t)$ is normalised. The polynomial $\nabla_L(z)$, first introduced by Conway.J, [Co], has remarkable properties that allow its computation from a link diagram without recourse to matrices or determinants. Conway has proved that, if L_+, L_- and L_0 are planar projections of three oriented links that are exactly the same except near one point where they are as in figure (2-1), then the Conway polynomial satisfies the formula:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z) \tag{2.0.1}$$

But the unknot O has $\Delta_O(t) = 1$ and that the Alexander polynomial for the unlink of unknots is zero, then Conway's algorithm for calculating the polynomial $\nabla_L(z)$ is given by changing cross-overs, in sequence, any link can be changed to an unlink of unknots, where the polynomial is known.



L_+ L_- L_0

Figure (2-1)

Using representations of the braid groups, Jones.V.F.R introduced a Laurent polynomial invariant $V_L(t)$ for an oriented link L in S^3, [J]. Jones began with a link L expressed as a closed braid α^c, for some (α,n). He then defined a representation, Φ, of B_n to the group of units of a certain Hecke-algebra over the field of fractions of $\mathbb{Z}[\sqrt{t}]$ on which is defined a trace function, then he defined,

$$V_L(t) = -(\sqrt{t}+\sqrt{t}^{-1})^{(n-1)}\mathrm{trace}[\Phi(\alpha)] \tag{2.0.2}$$

By using the structure of the braid group, and Markov moves, Jones showed that $V_L(t)$ is indeed a link invariant. He also proved that $V_L(t)$ satisfies,

$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) + (\sqrt{t}-\sqrt{t}^{-1})V_{L_0}(t) = 0 \tag{2.0.3}$$

where L_+, L_- and L_0 are closed braids that are exactly the same except near one point where they are related as in figure (2-1). It is also true that $V_O(t) = 1$, O is the unknot. The formula in equation (2.0.3) could be employed to calculate $V_L(t)$ for any link, just as in the case of Alexander and Conway polynomials. The similarity between $\Delta_L(t)$ and $V_L(t)$ raised the question: Are there a more general polynomial invariant for isotopy classes of oriented links, which specialise $\Delta_L(t)$ and $V_L(t)$?. In fact the question has been answered by many authors, where Freyd.P, Yetter.D, Hoste.J, Lickorish.W.B.R, Millett.K and Ocneanu.A, [F-Y-H-L-M-O], independently realised that $V_L(t)$ could be generalised to produce a link invariant $P_L(v,z)$ which is a Laurent polynomial of 2-variables and which specialises to give $\Delta_L(t)$, $\nabla_L(t)$ and $V_L(t)$. Every discoverer of the 2-variable polynomial $P_L(v,z)$, gave his own approach which is either completely combinatorial, [L-M], or combinatorial and algebraic, [O]. Hence there are different constructions of $P_L(v,z)$, where they are related by simple change of parameters. Here it is followed the construction given by Morten.H, [Mo3], and Morton.H & Short.H, [Mo-S1], to compute the polynomial $P_K(v,z)$ by representing K as a closed braid. They developed the theory based on the approach of Ocneanu, where a braid (β,n) closing to the given oriented link K is represented as $\rho_v(\beta)$ in an algebra H(z), Hecke algebra as in theorem (2.0.1) below. So that after normalisation by a suitable constant μ, (because of the similarity between condition (iii) of theorem (2.0.2), below and Markov moves of type (ii) of theorem (0.8)), the number

$$P(\beta) = (1/\mu^{(n-1)})\mathrm{Tr}(\rho_v(\beta)) \qquad (2.0.4)$$

depends only on K and not on the representing braid β. This number $P(\beta)$ is a polynomial with integer coefficients, $P_K(v,z)$, in two parameters $v^{\pm 1}$, $z^{\pm 1}$ which are involved in the construction of $H(z)$ and the representation ρ_v. The polynomial $P(\beta)$ provides a link invariant of K which is to be calculated from a given choice of β.

It follows from relation (i) of theorem (2.0.1) below, that c_i is invertible with $(c_i)^{-1} = c_i - z$, then B_{n+1} can be represented in H_n, for any choice of v, by a homomorphism ρ_v, where $\rho_v(\sigma_i) = vc_i$. Starting with $Tr(1) = 1$ and since σ_i, $(\sigma_i)^{-1}$ close to the same closure, hence using relation (iii) of theorem (2.0.2) we have,

$$Tr(\rho_v((\sigma_i)^{-1})) = Tr(\rho_v(\sigma_i)) = Tr(vc_i) = vTr(c_i)) = vT$$

But

$$Tr(\rho_v((\sigma_i)^{-1})) = Tr(v^{-1}(c_i)^{-1}) = v^{-1}Tr(c_i-z) = v^{-1}(T-z) = \mu, \text{ say}$$

hence

$$T = (z/1-v^2), \ \mu = vT = (z/v^{-1}-v) \tag{2.0.5}$$

It is shown that $P_K(v,z)$ is a Laurent polynomial in $Z[v^{\pm 1}, z^{\pm 1}]$, satisfying the recurrence relation,

$$v^{-1}P_{L_+}(v,z) - vP_{L_-}(v,z) = zP_{L_0}(v,z) \tag{2.0.6}$$

where L_+, L_- and L_0 are links that are exactly the same except near one point where they are related as in figure (2-1), see for example [Mo3]. This formula in fact gives a good method for recursively computing $P_L(v,z)$ together with the normalisation that $P_O(v,z) = 1$ and the unlink O^n of n components has $P_{(O^n)}(v,z) = [(v^{-1}-v)/z]^{n-1}$. Now given an oriented link L, write

$$P_L(v,z) = v^{(e_{min})}[Q_0(z) + v^2Q_1(z) + \dots]$$

then Morton.H, [Mo3], proved that

$$c(\beta)-(n-1) \leqslant e_{min} \leqslant e_{max} \leqslant c(\beta)+(n-1)$$

for any braid (β,n) with $\beta^c \simeq L$ and $[(e_{max}-e_{min})/2 + 1]$ is the lower bound for the braid index n of any braid with closure L. Also if L can be represented by a positive braid (β,n), then $e_{min} = c(\beta) - (n-1)$. So Morton.H asked if $e_{max} = c(\beta) + (n-1)$ for a twist positive braid. This inequality is shown in [Mo4] to apply also where n is the Seifert circles arising from any diagram of L. A similar bounds for $c(\beta) \pm (n-1)$ and for the braid index is given by Franks.J and Williams.K, [F-W]. A different upper bound for $c(\beta) - (n-1)$ was also given before by Bennquin, [Be], since he proved that $c(\beta) - (n-1) \leqslant 1 - \chi$, where χ is the Euler characteristic for a minimal genus spanning surface for K.

<u>(2.0.1) Theorem</u>: (Ocneanu.A, [O])

We can construct; for each $z \in \mathbb{C}$, an algebra H(z) with generators c_i, $1 \leqslant i$ and relations

(i): $(c_i)^2 = zc_i + 1$ for all i

(ii): $c_ic_j = c_jc_i$ $|i-j| > 1$

(iii): $c_{i+1}c_ic_{i+1} = c_ic_{i+1}c_i$ $1 \leqslant i$;

a Hecke algebra, which is the group algebra $\mathbb{C}[S_\infty]$ when $z = 0$.

<u>(2.0.2) Theorem</u>: (Ocneanu.A, [O])

We can construct for any given $T \in \mathbb{C}$ a linear function Tr: $H(z) \to \mathbb{C}$ with the following properties:

(i): $Tr(1) = 1$

(ii): $Tr(ab) = Tr(ba)$

(iii): $Tr(Wc_n) = T(Tr(W))$, for all $W \in H_{n-1}$

(2.0.3) Theorem:

(i): $P_L(\sqrt{t}^{-1}, \sqrt{t}-\sqrt{t}^{-1}) = 1$, $P_L(\sqrt{t}, \sqrt{t}-\sqrt{t}^{-1}) = (-1)^{1-c}$

(ii): $P_L(1,z) = \nabla_L(z)$

(iii): $P_L(1, \sqrt{t}-\sqrt{t}^{-1}) = \Delta_L(t)$

$$= (\sqrt{t}-\sqrt{t}^{-1})^{1-c} T((\sqrt{t}-\sqrt{t}^{-1})^2)$$

$$= \nabla_L(\sqrt{t}-\sqrt{t}^{-1})$$

(iv): $P_L(t, \sqrt{t}-\sqrt{t}^{-1}) = V_L(t)$

(v): If the braid (β,n) closes to an amphicheiral knot then,

$$c(\beta)-(n-1) \leqslant e_{min} \leqslant 0 \leqslant e_{max} \leqslant c(\beta)+(n-1), \text{ so } |c(\beta)| < n$$

(vi): $e_{min} = (c-1)\mathrm{mod}(2)$

where c is the number of components of the oriented link L and e_{max}, e_{min} are the largest and the smallest degrees of v in $P(v,z)$.

II: On the algebraic link problem:

The central theme in the link theory is to find an algorithm to decide "whether any given links are equivalent or not". This geometric problem is translated to an algebraic form after the approach of braid theory to the link theory, where Alexander.J proved that every oriented link can be represented as a closed braid, [B2]. Markov also proved that two closed braids are the same oriented link if they are related by a sequence of moves of types (i) and (ii) of theorem (0.8), [B2]. Hence the geometric problem, cited above, can be formulated in an algebraic form as "given two closed braids α^c and β^c does there exist an algorithm to decide whether (α,n) can be obtained from (β,m) by a sequence of Markov moves", this form is known as the algebraic link problem. In fact there are several examples of non conjugate braids which define the same link type, e.g. for any (α,n) the two braids $\alpha\sigma_n$ and $\alpha(\sigma_n)^{-1}$ are not conjugate, but they represent the same link type, [B2]. There are also much more complicated examples of non conjugate braids which define the same link type, see for example, [B1] and [Mu-Th]. Even for minimal braid index, the conjugacy classes are not link invariant, e.g. $\alpha = (\sigma_1)^3(\sigma_2)^5(\sigma_3)^7$ and $\beta = (\sigma_1)^3(\sigma_2)^7(\sigma_3)^5$, are not conjugate in B_4, but α^c and β^c have the same isotopic closure, [B2]. The existence of such examples show that the solution of the algebraic link problem is not simple. Recently Birman.J introduced a new move between isotopic links, called "exchange move", which takes one closed braid to the another, [B-Me]. In fact the exchange move is a generalisation of Conway's "flype move", which is defined as in figure (2-2a) below, by replacing each individual strand by parallel copies and replace the braids U, V, R by braids on more than two strands, as in figure (2-2b) below.

(2.0.4) Conjecture: (Birman.J and Menasco.W), [B-Me]

Exchange moves are possible alternative to Markov moves. More precisely:

Let (α,n) be a braid with α^c a link of braid index $m \leqslant n$, then there is a finite sequence of n-braids $\alpha=\alpha_1\to\alpha_2\to \dots \to\alpha_k$, such each α_{i+1} is obtained from α_i by either conjugation or exchange move such that α_k admits an exchange move which is strictly index reducing. Consequently when a link L, of braid index n, is a closure of two braids (α,n) and (β,n), then the two closed braids α^c and β^c are related by a sequence of n-braids $\alpha=\alpha_1\to\alpha_2\to \dots \to\alpha_k=\beta$, obtainable as above.

Because of the intrinsic properties of the twist positive braids, such as in theorem (2.1.10), (where the number of strands in a twist positive braid is a link invariant and the 2-variable polynomial determines $c(\beta)$, the crossing number of any positive braid (β,n)), we can ask: Can one decide whether a braid type could be written as a twist positive braid up to conjugation? e.g. consider the following conjecture:

<u>(2.0.5) Conjecture</u>: (Morton.H)

Braids in B_n admitting non-trivial exchange move, not simply conjugation, can not be written as twist positive braids. In other words the conjugacy class of twist positive braid representative is a link invariant, provided that Birman's conjecture in (2.0.4) holds.

Section 1 is devoted to the study of twist positive braids with the 2-variable link invariant $P_L(v,z)$, for some link L. Starting with the observation that the elements of SB_n have the property that each two arcs cross at most once, then one can combine a pair π_g, π_h of elements of SB_n where π_h is π_g with two adjacent pairs of arcs crossed in π_h, while they do not cross in π_g, i.e. for some r, $1 \leqslant r \leqslant n-1$, we can write $\pi_h = \sigma_r \pi_g$. Starting also with a result due to Morton.H and Short.H, [M-S1], where the subalgebra $H_n(z)$ of the algebra $H(z)$ in theorem (2.0.1) has dimension $(n+1)!$ as a vector space generated by $\rho_1(SB_{n+1})$, with $\rho_1(\sigma_i) = c_i$, $1 \leqslant i \leqslant n$. Consequently one can think of $\rho_1(\beta)$, for any braid (β,n), as a linear combination of the basis elements, i.e. we can write $\rho_1(\beta) = (W_1(z)b_1 + W_2(z)b_2 + \ldots + W_{n!}(z)b_{n!})$, where $b_g = \rho_1(\pi_g)$, and W_g is a polynomial of z with integer coefficients.

In lemma (2.1.3) it is proved that $\rho_1(Q)b_h$ is a positive combination of b_g in H_{n-1}, i.e. no cancellation of factors. Moreover it is proved in lemma (2.1.4) that $\rho_1(\pi\rho[\pi])$ is a linear combination of generators b_h's with leading coefficient $(1 + zf(z))$, for a polynomial $f(z)$ with non-negative coefficients, where $\rho[\pi]$ is π reversed. In fact this approach gives a quick proof that the number of strands in a twist positive braid is the braid index, which was first proved in [F-W1].

Consequently it is shown in lemma (2.1.5) that $\rho_1(\pi\Delta_n)$ contains $\rho_1(\Delta_n)$, for every $\pi \in SB_n$, a generalisation of that is given, in corollary (2.1.6), by replacing π by Q, for any positive braid Q. Following that it is proved in proposition (2.1.8) that twist positive braid is always full, where the braid (α,n) is called full braid if $[e_{max} - e_{min}] = 2(n-1)$, where e_{max} and e_{min} are the largest and the smallest degrees of v in $P_K(v,z)$, for $K \simeq \alpha^c$, as defined in definition (2.1.7). Consequently the full braid is always minimal. Hence it is concluded in theorem (2.1.10) that the number of strands in a twist positive braid is the braid index.

Section 2 is devoted to the study of the possible 2-variable polynomials $P_K(v,z)$ of width 2, where width $P_K(v,z)$ is the minimal number of strings allowed by the index bound, shown in definition (2.1.7). It is shown in lemma (2.2.2) that, if the polynomial $P_K(v,z)$ has width 1, then it is the same as the polynomial of the closed 1-braid (the unknot). But no examples for a knot with a polynomial of width 1 and braid index > 1 are known. In theorem (2.2.3) it is proved that if the polynomial has width 2, then it is the same as the polynomial of a closed 2-braid. There are examples where the width is strictly less than the braid index, for width 2 the braid [$\alpha = (\sigma_2\sigma_3\sigma_1\sigma_2)^3$] with width 2, [F-W2], but it has braid index 4, [Mo-S2]. Therefore not every link of polynomial of width 2 is a closed 2-braid. It is not known (in general) if a knot of width k must have some polynomial as some closed k-braid.

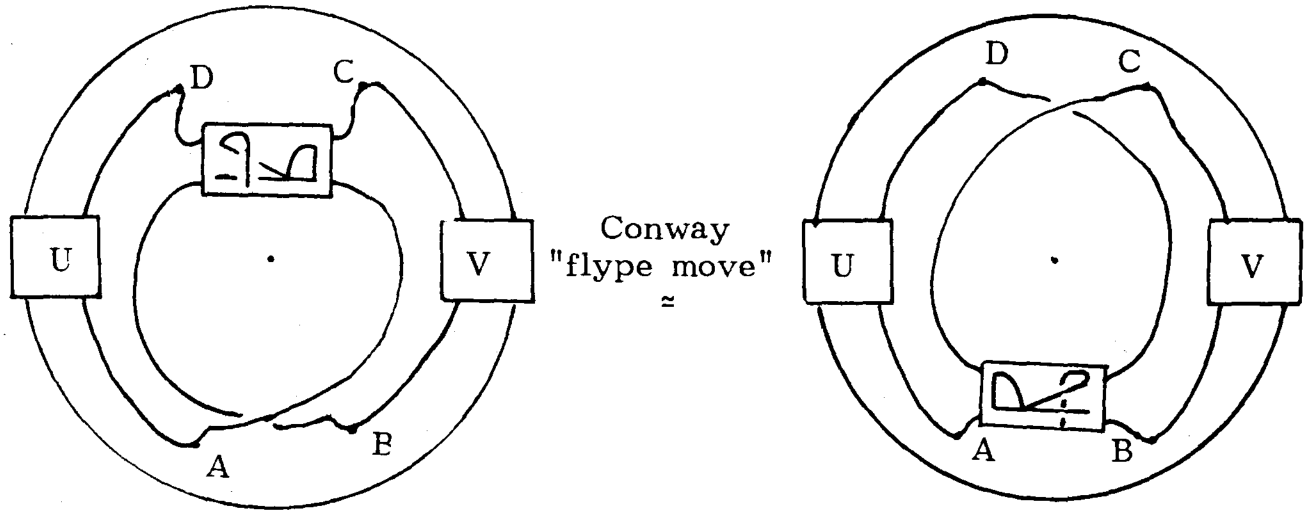
In theorem (2.2.4) a complete list of 3-braids of width 2, which close to knots, are given. Consequently it is shown that $P_K(v,z)$ determines $c(\beta)$ for full 3-braid β, where $\beta^c \simeq K$, as in corollary (2.2.10). The 2-variable polynomial for non-full 3-braid is calculated in proposition (2.2.11).

These results, recovering $P(v,z)$ from the Alexander polynomial and crossing number, are observed independently of Murakami.H, [Mur].

Section 3 is devoted to the study of Morton's conjecture cited above, in case $n = 3$. It is shown, in remark (2.3.2), that Birman's "exchange move" includes Markov's "stabiliser move" and exchange move preserve braid index, hence preserve the exponent sum, in a particular case. In figure (2-5) it is illustrated an isotopic sequence of closed braids to represent the general exchange move.

Using the canonical form approach for every positive braid (shown in theorem (1.2.1)) it is formulated, in lemma (2.3.4), the standard form for any positive braid word $(\alpha,3)$. Following that it is given a nice representative for the conjugacy class of a twist positive braid in B_3, as in lemma (2.3.6). Investigating the exchangeable 3-braids, as in remark (2.3.7), it is excluded the cases of trivial exchangeable (conjugation) braids and some cases which never conjugate to twist positive 3-braids.

A complete list of those non-trivial exchangeable 3-braids, which might contain $(\Delta_3)^k$ up to conjugation, $k \geqslant 1$, is given in lemma (2.3.8). Using Murasugi's classification of the conjugacy classes in B_3 (shown in proposition (0.14)) it is given an affirmative answer, in proposition (2.3.3), for Morton's conjecture cited above, for 3-braids.



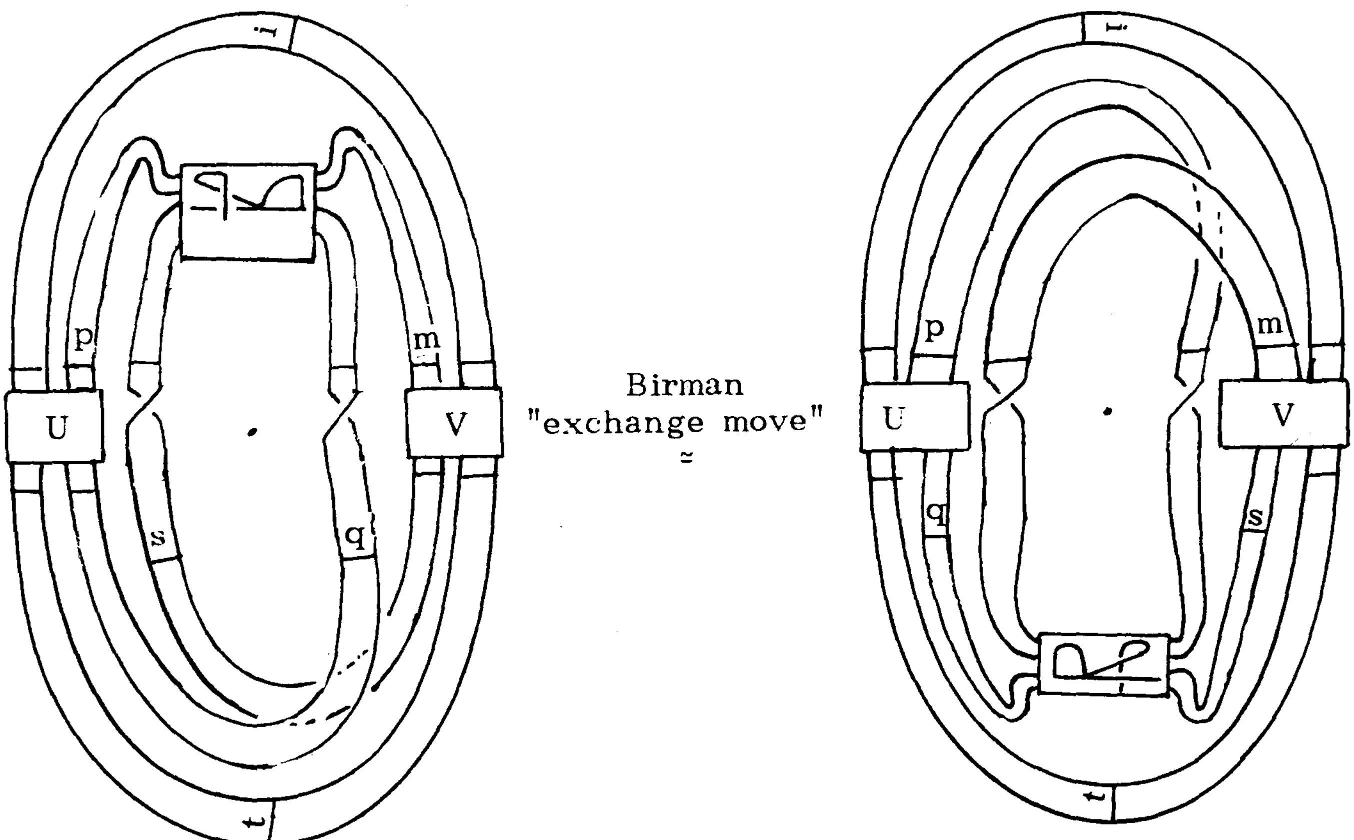
α

flype(α)

"The flype move is in fact a half twist, where

4 points A, B, C, D are left fixed"

Figure (2-2a)



β

Exch(α)

Figure (2-2b)

§2.1. TWIST POSITIVE BRAIDS ARE MINIMAL REPRESENTATIVES FOR KNOTS AND LINKS

(2.1.1) Remarks:

(a): For every $\pi \in SB_n$, each two arcs cross at most once, then each adjacent pair of arcs either cross once or not at all. So if the two arcs labelled r and r+1 (at top of π) cross in π, then $r \in S(\pi)$, see figure (1-6), i.e.

$$\pi = \sigma_r \pi' \tag{2.1.1}$$

for some $\pi' \in SB_n$. But if arcs labelled r, r+1 do not cross in π, then

$$\pi' = \sigma_r \pi \tag{2.1.2}$$

still in SB_n. Hence in SB_n and for a given r with $r < n$, the elements π and π' are paired by σ_r as in equations (2.1.1) and (2.1.2).

(b): It was proved by Morton and Short that the subalgebra $H_n(z)$ of the algebra H(z) (in theorem (2.0.1)) has dimension (n+1)! as a vector space, [M-S1]. In fact $H_n(z)$ is generated by $\rho_1(SB_{n+1})$, where ρ_1 is the linear representation $\rho_v\colon B_{n+1} \to H_n(z)$, with $\rho_v(\sigma_i) = vc_i$, for $v = 1$.

Now starting with a braid $\pi_g \in SB_n$, then we can construct (n+1) elements in SB_{n+1} by fixing a string (the heavy string as illustrated in figure (2-3)) at position (n+1) at top of the geometric braid π_g and fixe the other end of the added string at bottom of the geometric braid

π_g to give a braid in SB_{n+1}. If the added arc crosses r arcs of π_g, then it gives the braid $\pi_g(\sigma_n\sigma_{n-1} \dots \sigma_{n-r+1}) \in SB_{n+1}$.

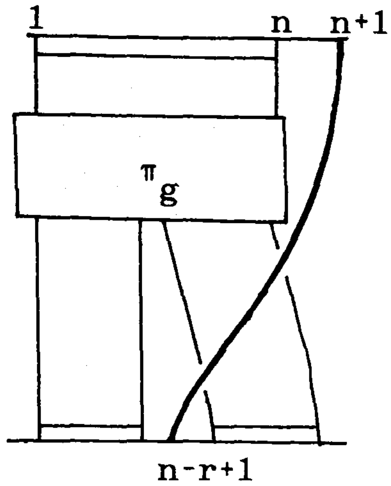


Figure (2-3)

Now let $\sigma(r,n) = \sigma_n\sigma_{n-1} \dots \sigma_{n-r+1}$ and take $\sigma(0,n) = e$ when the added arc does not cross arcs of π_g. Therefore for every $\pi_g \in SB_n$ there are associated (n+1) elements ($\sigma(r,n)$, $r = 0, 1, 2, \dots, n$) in SB_{n+1}.

The technique above provides an algorithm to order the elements of SB_n. Suppose that, starting with $\pi_1 = 1$, we have already constructed elements π_g, $g \leqslant n!$ for SB_n. Then define $\pi_h = \pi_g\sigma(r,n)$ for $h = g + r(n!)$ with $1 \leqslant g \leqslant n!$, hence $1 \leqslant g \leqslant (n+1)!$. Therefore we can write $\pi_h \in SB_{n+1}$ uniquely as $\pi_h = \sigma(r_1,1)\sigma(r_2,2) \dots \sigma(r_n,n)$, with $g = 1+r_1+r_2+ \dots +r_n$ and $0 \leqslant r_j \leqslant j$. Hence h is uniquely determined by the factorial expansion $(r_1, r_2, \dots, r_n)$.

It was proved by Morton and Short that; for two braids π_g and π_h in SB_{n+1} with factorial expansions $(g_1, g_2, \dots, g_n)$ and $(h_1, h_2, \dots, h_n)$, respectively, if $g_r \leqslant g_{r-1}$, then $\pi_g\sigma_r = \pi_h$, where $h_{r-1} = g_r$, $h_r = g_{r-1} + 1$, $h_j = g_j$ otherwise. Then $h_r > h_{r-1}$ and h

$> g$. This result means that for a given choice of r, the braids in SB_{n+1} can be paired as in (a) above.

Now let $b_g = \rho_1(\pi_g)$ for $\pi_g \in SB_{n+1}$, $1 \leqslant g \leqslant (n+1)!$, then using (a) above we can pair the generators, b_h, $1 \leqslant h \leqslant (n+1)!$, of the vector space $H_n(z)$, as

$$b_g = c_i b_h, \text{ if } g > h$$

and

$$b_h = c_i b_g, \text{ if } h > g$$

where b_h and b_g correspond $\rho_1(\pi)$ and $\rho_1(\pi')$ for π and π' as in equations (2.1.1) and (2.1.2) above.

(c): Consider the subset

$$H^+ = \{W = \sum_h W_h(z) b_h \mid W \neq 0\}$$

of the algebra $H_n(z)$, where $W_h(z)$ is a polynomial of z with non negative coefficients. Hence it is clear that H^+ is closed under linear combinations of elements of H^+ with polynomials of non negative coefficients. i.e. $[X_1(z)W_1 + X_2(z)W_2] \in H^+$, for all W_1 and W_2 in H^+ and for all $X_1(z)$ and $X_2(z)$ of non negative coefficients.

<u>(2.1.2) Lemma</u>:

The subset H^+ of the subalgebra $H_n(z)$ is closed under the multiplication, i.e. $W_1W_2 \in H^+$, for all W_1 and $W_2 \in H^+$.

<u>Proof</u>:

It is enough to show that $(c_r W) \in H^+$, for $W \in H^+$ and for any, c_r, generator of the subalgebra $H_n(z)$, then write

$$c_r W = \sum_h W_h(z)(c_r b_h)$$

Then using (b) of remark (2.1.1) we can write

$$c_r b_h = (c_r)^2 b_g$$

$$= zc_r b_g + b_g, \text{ if } h > g$$

and

$$c_r b_h = b_g, \text{ if } g > h$$

then the proof can be completed by induction on length of generators b_h's and by use of (c) in remark (2.1.1), where H^+ is closed under addition, so that $c_r W \epsilon H^+$ □

<u>(2.1.3) Lemma</u>:

In H_n, the element $(\rho_1(Q))b_g \epsilon H^+$, for every positive braid Q and for every generator b_h of the vector space H_n.

<u>Proof</u>

If $L(Q) = 1$, then $Q = \sigma_i$ for some i and $\rho(Q)b_g = c_i b_g$ which is in H^+ as in lemma (2.1.2). Then the proof follows by induction on the length of Q. Now assume that the lemma holds for $L(Q) = r$. Take positive braid Q with $L(Q) = r+1$, then write $Q = \sigma_i Q'$ for some positive braid Q' and for some integer i, so

$$\rho_1(Q)b_g = c_i \rho_1(Q')b_g$$

Then from our induction hypothesis $\rho_1(Q')b_g \epsilon H^+$. Then using lemma (2.1.2) we have $\rho_1(Q)b_g \epsilon H^+$, which completes the induction process, hence completes the proof □

The following two lemmas explore some properties of positive permutation braids in the algebra H_n, which are the keys to give a quick proof that the number of strands in a twist positive braid is the braid index for the closure of that braid.

(2.1.4) Lemma:

For every $\pi \in SB_n$, $\rho_1(\pi\rho[\pi]) \in H^+$, with leading coefficient $W_1(z) = 1+zf(z)$, where $\rho[\pi]$ is the reverse of π.

Proof

Let

$$\pi = \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_k}$$

where

$$1 \leqslant i_j \leqslant n-1, \text{ for } j = 1, 2, \dots k$$

then

$$\rho[\pi] = \sigma_{i_k}\sigma_{i_{k-1}} \dots \sigma_{i_1} = \alpha, \text{ say}$$

So

$$\begin{aligned}\rho_1(\pi\alpha) &= (c_{i_1} \dots c_{i_{k-1}}(c_{i_k})^2 c_{i_{k-1}} \dots c_{i_1}) \\ &= z(c_{i_1} \dots c_{i_{k-1}} c_{i_k} c_{i_{k-1}} \dots c_{i_1}) \\ &\qquad + (c_{i_1} \dots c_{i_{k-2}}(c_{i_{k-1}})^2 c_{i_{k-2}} \dots c_{i_1})\end{aligned}$$

Hence

$$\rho_1(\pi\alpha) = 1 + z\{\sum_{j=1}^{k}(c_{i_1} \dots c_{i_{j-1}} c_{i_j} c_{i_{j-1}} \dots c_{i_1})\}$$

But $\pi\alpha$ is a positive braid, then lemma (2.1.3) tells us that no cancellation of factors, so

$$\rho_1(\pi\alpha) = [1 + zf(z)] + W$$

where $W \in H^+$ with leading coefficient zero and $f(z)$ is polynomial of z with non-negative coefficients □

(2.1.5) Lemma:

For every $\pi \in SB_n$, $\rho_1(\pi\Delta_n) = z^{L(\pi)}\rho_1(\Delta_n) + W$, where $W \in H^+$ and $L(\pi)$ is the length of π.

Proof

This lemma means that $\rho_1(\pi\Delta_n)$ always contain $\rho_1(\Delta_n)$ with non-zero coefficient when written as a linear combination of generators of H_{n-1}, for every $\pi \in SB_n$. To proof that it is enough to prove it for $\pi = \sigma_i$. Hence given any $\pi \in SB_n$, write $\pi = \pi'\sigma_i$ for some $i \in F(\pi)$, so

$$\begin{aligned}\rho_1(\pi\Delta_n) &= \rho_1(\pi'\sigma_i\Delta_n)\\ &= \rho_1(\pi')\rho_1(\sigma_i\Delta_n)\\ &= \rho_1(\pi')[z\rho_1(\Delta_n) + W]\\ &= z\rho_1(\pi'\Delta_n) + W'\end{aligned}$$

where $W \in H^+$ and lemma (2.1.2) tells us that $W' \in H^+$. Again rewrite $\pi' = \pi''\sigma_j$, for some $j \in F(\pi')$, then by induction on length of π, we can complete the proof. Now let $\pi = \sigma_i$, but $i \in S(\Delta_n)$ for all $1 \leqslant i \leqslant n-1$, (as in (iii) of lemma (1.1.10)), i.e.

$$\Delta_n = (\sigma_i)(\sigma_i)_* \text{ , for all } 1 \leqslant i \leqslant n-1$$

where $\Delta_n = (\pi)(\pi_*)$, for every $\pi \in SB_n$ (as in corollary (1.1.15)), so

$$\begin{aligned}\rho_1(\sigma_i\Delta_n) &= \rho_1((\sigma_i)^2(\sigma_i)_*)\\ &= (c_i)^2\rho_1((\sigma_i)_*)\end{aligned}$$

$$= zc_i \rho_1((\sigma_i)_*) + \rho_1((\sigma_i)_*)$$

$$= z\rho_1(\sigma_i(\sigma_i)_*) + \rho_1((\sigma_i)_*)$$

$$= z\rho_1(\Delta_n) + \rho_1((\sigma_i)_*)$$

which completes the induction process, hence completes the proof □

(2.1.6) Corollary:

In H_n, $\rho_1(Q\Delta_n) = z^{L(Q)}\rho_1(\Delta_n) + W$, for every positive braid Q, where $W \in H^+$ and L(Q) is the length of Q.

Proof

The proof is similar to that in lemma (2.1.5), i.e. by replacing Q by π and use induction on L(Q) □

(2.1.7) Definition:

The braid (β, n) is called a full braid if $[e_{max} - e_{min}] = 2(n-1)$, where e_{max} and e_{min} are the largest and the smallest degrees of v in the 2-variable polynomial $P_L(v,z)$, respectively, with $L \approx \beta^c$. Hence define width $P_L(v,z))$ or simply width β as $W(\beta) = [(e_{max} - e_{min})/2] + 1$. i.e. $W(\beta)$ is the minimal number of strings allowed by the index bound, hence the braid (β, n) is full if and only if $W(\beta) = n$.

(2.1.8) Proposition:

Twist positive braids are always full.

Proof

We need to show that $\rho_1((\Delta_n)^2) \in H^+$ with all the cofficients are non zero polynomials Now $\Delta_n = \pi\pi_*$ for any given $\pi \in SB_n$. Let $\alpha = \rho[\pi]$ and $\alpha_* = \rho[\pi_*]$. But $\rho[\Delta_n] = \Delta_n$, i.e. $\pi\pi_* = \alpha_*\alpha = \Delta_n$, then

$$(\Delta_n)^2 = (\alpha_*\alpha)\Delta_n$$

hence

$$(\Delta_n)^2\alpha_* = \alpha_*\alpha\Delta_n\alpha_*$$

But $(\Delta_n)^2$ commutes with every thing, then

$$\alpha_*(\Delta_n)^2 = \alpha_*\alpha\Delta_n\alpha_*$$

so

$$(\Delta_n)^2 = \alpha\Delta_n\alpha_*$$

Then lemma (2.1.5) tells us that

$$\begin{aligned}\rho_1((\Delta_n)^2) &= \rho_1(\alpha\Delta_n)\rho_1(\alpha_*)\\ &= [z^{L(\pi)}\rho_1(\Delta_n) + W]\rho_1(\alpha_*)\\ &= z^{L(\pi)}\rho_1(\Delta_n\alpha_*) + W'\\ &= z^{L(\pi)}\rho_1(\pi\pi_*\alpha_*) + W'\\ &= z^{L(\pi)}\rho_1(\pi)\rho_1(\pi_*\rho[\pi_*]) + W'\end{aligned}$$

where $W \in H^+$ and lemma (2.1.2) tells us that $W' \in H^+$. Now using lemma (2.1.4), we have

$$\rho_1((\Delta_n)^2) = z^{L(\pi)}\rho_1(\pi)[1 + zf(z) + V] + W'$$

with $V \in H^+$, then

$$\rho_1((\Delta_n)^2) = z^{L(\pi)}(1 + zf(z))\rho_1(\pi) + X \qquad (2.1.3)$$

for every $\pi \in SB_n$, where $f(z)$ is a polynomial of z with non-negative coefficients and $X \in H^+$. Therefore $\rho_1((\Delta_n)^2)$ contains $\rho_1(\pi)$ for every $\pi \in SB_n$, with non zero coefficients. So given a twist positive braid $\beta = (\Delta_n)^2Q$, for a positive braid Q, as in definition (1.0.2), then

$$\rho_1(\beta) = \rho_1((\Delta_n)^2 Q) = \rho_1((\Delta_n)^2)\rho_1(Q)$$

But lemma (2.1.3) tells us that $\rho_1(Q) \in H^+$ and equation (2.1.3) above also tells us that all the generators $\rho_1(\pi)$, $\pi \in SB_n$, appear in $\rho_1((\Delta_n)^2)$ with positive coefficients, hence no cancellation of the factors. So $\rho_1(\beta)$ is a linear combination of all the generators of $H_n(z)$, where the coefficients are positive polynomials. Now

$$P_L(v,z) = v^{c(\beta)}(\mu^{1-n}) \, Tr(\rho_1(\beta))$$

and

$$\mu = z/(v^{-1} - v)$$

as in equations (2.0.4) and (2.0.5) where $L \simeq \beta^c$. Then $P_L(v,z)$ contains the factor

$$v^{c(\beta)}[(v^{-1}-v)/z]^{n-1} z^{L(Q)}(1 + zf(z))$$

so that $[v^{c(\beta)-(n-1)}]$ and $[v^{c(\beta)+(n-1)}]$ have non-zero coefficients. Therefore

$$e_{max} = c(\beta) + (n-1) \text{ and } e_{min} = c(\beta) - (n-1)$$

where e_{max} and e_{min} are the largest and the smallest degrees of v in the 2-variable polynomial $P_L(v,z)$, respectively and $L \simeq \beta^c$. This completes the proof that the twist positive braids are full braids □

<u>(2.1.9) lemma</u> :

A full braid is always minimal.

<u>Proof</u> :

Let $\beta = \gamma\sigma_{n-1}$, for some $(\gamma, n-1)$, then

$$c(\beta)-(n-1) \leqslant e_{min} \leqslant e_{max} \leqslant c(\beta)+(n-1)$$

But β^c and γ^c are isotopic. Hence have the same invariant polynomial, then

$$c(\gamma)-(n-2) \leqslant e_{min} \leqslant e_{max} \leqslant c(\gamma)+(n-2)$$

which implies that

$$\eta = e_{max} - e_{min} \leqslant 2(n-2)$$

Then $\eta \neq 2(n-1)$, hence β does not a full braid, which completes the proof □

(2.1.10) Theorem:

If a link L is represented as a closed twist positive braid (α,n), then L has braid index n, i.e the number of strands in a twist positive braid is a link invariant.

Proof

The proof is a direct consequence of proposition (2.1.8) and lemma (2.1.9) □

§2.2. THE 2-VARIABLE LINK INVARIANTS OF WIDTH 2 AND 3-BRAIDS

(2.2.1) Remark:

Given a polynomial $P(v,z)$ of width n (where the width of $P(v,z)$ is the minimal number of strings allowed by the index bound as defined in definition (2.1.7).. Then the polynomial looks like

$$P(v,z) = v^k[Q_0(z) + v^2Q_1(z) + \ldots + v^{2n-2}Q_{n-1}(z)]$$

which can be written as

$$P(v,z) = v^k \begin{pmatrix} 1 & v^2 & \ldots & v^{2n-2} \end{pmatrix} \times \begin{bmatrix} Q_0(z) \\ Q_1(z) \\ \\ \\ Q_{n-1}(z) \end{bmatrix}$$

So if we know $P(v,z)$ for n different values of v, e.g. $v_0, v_1, \ldots, v_{n-1}$ and if we know $k = e'_{min}$, then we know $P(v,z)$, because

$$A \times \begin{bmatrix} Q_0(z) \\ Q_1(z) \\ \cdot \\ \cdot \\ \cdot \\ Q_{n-1}(z) \end{bmatrix} = \begin{bmatrix} (v_0)^{-k}P(v_0,z) \\ (v_1)^{-k}P(v_1,z) \\ \cdot \\ \cdot \\ \cdot \\ (v_{n-1})^{-k}P(v_{n-1},z) \end{bmatrix} \qquad (2.2.1)$$

for an invertible n×n matrix A, where

$$A = \begin{bmatrix} 1 & (v_0)^2 & \cdots & (v_0)^{2n-2} \\ 1 & (v_1)^2 & \cdots & (v_1)^{2n-2} \\ \cdot & \cdot & \cdots & \cdot \\ \\ \cdot & \cdot & \cdots & \cdot \\ 1 & (v_{n-1})^2 & \cdots & (v_{n-1})^{2n-2} \end{bmatrix}$$

(2.2.2) Lemma:

If the 2-variable link invariant $P(v,z)$ has width 1 then it is the same as the polynomial of the closed 1-braid.

Proof:

The polynomial of width 1 has the form

$$P(v,z) = v^k Q(z), \text{ with } k = e_{min}$$

Then using (i) of theorem (2.0.3), we have

$$Q'(s)(s)^{-k} = 1$$

and

$$Q'(s)(s)^k = (-1)^{c-1}$$

where $Q'(s) = Q(s-s^{-1})$, $s = \sqrt{t}$ and c is the number of components. But $k = (c-1)\mathrm{mod}(2)$, as in (vi) of theorem (2.0.3), then $s^k = 1$, for every s, hence $k = 0$. Then using (ii) of theorem (2.0.3), we have $P(v,z) = Q(z) = \nabla(z) = 1$, which completes the proof. Moreover the link of width 1 has odd number of components □

(2.2.3) Theorem:

If the 2-variable polynomial $P_K(v,z)$ has width 2, then it is the same as the polynomial of a closed 2-braid $[(\sigma_1)^k]^c$, for $|k| \neq 1$.

Proof:

Given a polynomial $P_L(v,z)$ of width 2, then

$$P_L(v,z) = v^{(e_{min})}[Q_0(z) + v^2Q_1(z)]$$

But as in (vi) of theorem (2.0.3), $e_{min} = (c-1) \bmod(2)$, where c is the number of components of L, then using (i) of theorem (2.0.3), we have

$$P(\surd t^{-1},\surd t-\surd t^{-1}) = 1 \text{ and } P(\surd t,\surd t-\surd t^{-1}) = (-1)^{c-1}$$

Now put $v_0 = s = \surd t$, $v_1 = s^{-1}$, $z = s - s^{-1}$ and $k = e_{min}$ in equation (2.2.1), but $s^{-k}(-1)^{c-1} = (-s)^{-k}$, so

$$\begin{bmatrix} Q_0(s) \\ Q_1(s) \end{bmatrix} = 1/(s^2-s^{-2}) \begin{bmatrix} s^{-2} & -s^2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s^k \\ (-s)^{-k} \end{bmatrix} \qquad (2.2.2)$$

Now the 2-variable polynomial $P(v,z)$ for the 2-closed braid $[(\sigma_1)^k]^c$, $|k| \neq 1$, has the form

$$v^{k-1}[W_0(z) + v^2W_1(z)]$$

where $W_0(z)$ and $W_1(z)$ can be determined by employing the formula in equation (2.0.6), then applying the observation above, to see that the 2-variable polynomial of width 2 and $e_{min} = k$ is the same as the polynomial of the closed 2-braid $[(\sigma_1)^{k+1}]^c$ □

In the following theorem, we give a complete list of 3-braids which are not full, i.e. have width 2.

(2.2.4) Theorem

If a closed 3-braid $\beta^c \simeq L$, has Alexander polynomial equals the Alexander polynomial of a (2,p) torus knot times a power of t, then β is conjugate to one of the following braids:

$\alpha = (\Delta_3)^{4k}(\sigma_1)(\sigma_2)^{-(6k+1)}$, $\gamma = (\Delta_3)^2\alpha$, $\eta = (\Delta_3)^2(\sigma_1)^p(\sigma_2)^{-1}$, α^{-1}, γ^{-1} or η^{-1}, for $k \in \mathbb{Z}^+$.

The proof of theorem (2.2.4) will start with the following two lemmas.

(2.2.5) Lemma:

The closed 3-braid $[\beta = (\Delta_3)^{2n}(\sigma_1)^p(\sigma_2)^{-q}]^c \simeq K$, has the Alexander polynomial

$$(1-s+s^2)\Delta_K(s) = 1-(\pm 1)^n s^{3n-q}\{1 + s^{p+q} + s[\sum_{i=0}^{p-1} s^i][\sum_{i=0}^{q-1} s^i]\} + s^{6n-q+p}$$

Hence if $\Delta_K(s) = \pm s^k \Delta_{(2,p)}(s)$, then pq = m if n-even and pq + 4 = m if n-odd.

Proof:

The reduced Burau matrix B(t) of the braid β is the image of β under the reduced Burau representation Φ: $B_n \to GL(n-1, \mathbb{Z}[t,t^{-1}])$, [B2]. In this presentation

$$\Phi(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}$$

Now let t = -s, then

$$\Phi((\Delta_3)^2) = -s^3 I_{2\times 2}$$

and β has the Burau matrix,

$$B(s) = (\pm 1)^n s^{3n-q} \begin{bmatrix} s^p & \sum_{i=0}^{p-1} s^i \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} s^q & 0 \\ s\sum_{i=0}^{q-1} s^i & 1 \end{bmatrix}$$

Then

$$B(s) = (\pm 1)^n s^{3n-q} \begin{bmatrix} s^{p+q} + s[\sum_{i=0}^{p-1} s^i][\sum_{i=0}^{q-1} s^i] & \sum_{i=0}^{p-1} s^i \\ s\sum_{i=0}^{q-1} s^i & 1 \end{bmatrix}$$

So

$$\begin{aligned} tr[B(s)] &= (\pm 1)^n s^{3n-q}\{1+s^{p+q}+s[\sum_{i=0}^{p-1} s^i][\sum_{i=0}^{q-1} s^i]\} \\ &= (\pm 1)^n s^{3n-q}\{1+s^{p+q}+s[1+2s+3s^2+ \dots +3s^{p+q-4}+2s^{p+q-3}+s^{p+q-2}]\} \\ &= (\pm 1)^n s^{3n-q}\{1+s+2s^2+3s^3+ \dots +3s^{p+q-3}+2s^{p+q-2}+s^{p+q-1}+s^{p+q}]\} \end{aligned}$$

But the Alexander polynomial of the link $\beta^c \cup L_\beta$, the closure of β together with its axis, is given by

$$\Delta(x,t) = \det[xI-B(t)] = x^2 - trB(t).x + \det B(t)$$

where the variable x refers to the meridians of the axis L_β of the closed braid $\beta^c \simeq L$ and t refers to all meridians of the oriented closed braid L. Then for a link $L \simeq \beta^c$, the Alexander polynomial satisfies,

$$(1+t+t^2)\Delta_L(t) = \Delta(1,t)$$

Hence for $t = -s$, we have

$$(1-s+s^2)\Delta_K(s) = 1 - (\pm 1)^n s^{3n-q}\{1 + s^{p+q} + s[\sum_{i=0}^{p-1} s^i][\sum_{i=0}^{q-1} s^i]\} + s^{6n-q+p}$$

hence $\Delta_K(1) = 2 - (\pm 1)^n\{pq + 2\}$. Now assume that $\Delta_K(s) = \pm s^k\Delta_{(2,m)}(s)$, then $pq = m$ if n-even and $pq + 4 = m$ if n-odd, which completes the proof □

(2.2.6) Lemma:

If the closed 3-braid $\beta^c \simeq K$ for

$$\beta = (\Delta_3)^{2n}(\sigma_1)^{p_1}(\sigma_2)^{-q_1} \ldots (\sigma_1)^{(p_r)}(\sigma_2)^{-q_r}$$

has the Alexander polynomial of a (2,m) torus knot, then $r = 1$; where p_i, $q_i \in \mathbb{Z}^+$, for $1 \leqslant i \leqslant r$ and for every $n \in \mathbb{Z}$. Moreover $n = \pm(q+2)/3$ if n-odd and $n = \pm(q-1)/3$ if n-even.

Proof:

Let $\Delta(x,s)$ be the Alexander polynomial of the link $K \cup L_\beta$ (the closure of β together with its axis), [M1], then

$$\Delta_L(s) = \{\Delta(1,s)/(1-s+s^2)\} = \pm s^k\Delta_{(2,m)}(s)$$

and

$$\Delta(1,s) = 1 \pm s^{3n-Q}[1+(rs) \bmod(s^2)] + s^{6n-Q+P}$$

hence

$$1 \pm s^{3n-Q}[1+(rs) \bmod(s^2)] + s^{6n-Q+P}$$

$$= \pm s^k(1-s+s^2)[(1-s^m)/(1-s)]$$

$$= \pm s^k[1+s^2+s^3+ \ldots +s^{m-1}+s^{m+1}] \qquad (2.2.3)$$

where

$$P = p_1 + p_2 + \cdots + q_r$$

and

$$Q = q_1 + q_2 + \cdots + q_r$$

Let e_s and e_t be the smallest and the largest power of s in equation (2.2.3), respectively. Then in the right-hand side, $e_s = k$ with coefficient equals ±1 and $e_t = k+m+1$ with coefficient equals ±1. Hence consider the two cases:

<u>Case (1): $n \geqslant 0$</u> :

(1a): For the left-hand side, $e_s = 0$ with coefficient equals 1, then $k = 0$ and from equation (2.2.3) we have,

$$s^{3n-Q}[1+(rs) \bmod (s^2)] + s^{6n-Q+P} = s^2+s^3+ \dots +s^{m-1}+s^{m+1}$$

So $3n-Q = 2$ and $r = 1$, i.e. $Q = q$ and $n = (q+2)/3$ should be odd.

(1b): $e_s = k = 3n-Q < 0$, with coefficient equals ±1 :

Multiplying both sides of equation (2.2.3) by s^{Q-3n}, we have

$$s^{Q-3n} \pm [1+rs \bmod (s^2)] + s^{3n+P} = \pm(1+s^2+s^3+ \dots +s^{m-1}+s^{m+1})$$

But the right hand side does not contain s, then $Q - 3n = 1$, otherwise $r = 0$, which leads to a contradiction. Hence $r = 1$, $k = -1$ and $n = (q-1/3)$ should be even integer.

<u>Case (2): $n < 0$</u>

Multiplying both sides of equation (2.2.3) by s^{Q-6n} we have,

$$s^{Q-6n} \pm s^{-3n}[1+rs \bmod (s^2)] + s^P$$

$$= \pm s^{k+Q-6n}(1+s^2+s^3+ \dots +s^{m-1}+s^{m+1})$$

Then comparing the smallest power of s in both sides of the equation above, we have the following two cases:

(2a): $n < 0$ and $k+Q-6n = -3n$, then

$$s^{Q-6n} \pm s^{-3n}[1+rs \bmod(s^2)]+s^P = \pm s^{-3n}(1+s^2+s^3+\ldots+s^{m-1}+s^{m+1})$$

Now let $P = -3n+1$, otherwise $r = 0$, then comparing the coefficients in both sides, we have $r = 1$.

(2b): $n < 0$, and $k+Q-6n = P$, then

$$s^{Q-6n} \pm s^{-3n}[1+rs \bmod(s^2)] = \pm s^P(s^2+s^3+ \ldots +s^{m-1}+s^{m+1})$$

Thus comparing the smallest power of s in both sides, we have $p+2 = -3n$, then $r = 1$. This completes the proof □

Proof of theorem (2.2.4) :

There is a nice representative of each conjugacy class in B_3, where the classes are divided to different seven patterns, as in proposition (0.14), [Mu2]. So the proof will be done by investigating each pattern individually. In proposition (0.14), the types Λ_0, Λ_3, Λ_4 and Λ_5 represent links. The types Λ_1, Λ_2 also represent (3,3n+1) and (3,3n+2) tours knots, respectively. Hence proposition (0.14) tells us that, the non full 3-braids (which close to knots) lie in Λ_6, with representative

$$\beta = (\Delta_3)^{2n}(\sigma_1)^{p_1}(\sigma_2)^{-q_1} \ldots (\sigma_1)^{(p_r)}(\sigma_2)^{-q_r}$$

where p_i, $q_i \in \mathbb{Z}^+$, for $1 \leqslant i \leqslant r$ and for every $n \in \mathbb{Z}$. Now let

$$\Delta_L(t) = t^k \Delta_{(2,m)}(t)$$

for some k, $m \in \mathbb{Z}$ and $\beta^c \simeq L$, then lemma (2.2.6) tells us that $r = 1$ and gives relations between n, p and q. So consider the cases:

Case (1): $n \geqslant 0$:

(I); Let n-odd, then lemma (2.2.6) tells us that $n = \pm(q+2)/3$, so $n \neq 0$. Now let $n > 0$, i.e. $n = (q+2)/3$, then lemma (2.2.5) tells us that $4 + pq = m$ and

$$1+s+2s^2+ \dots +2s^{p+q-2}+s^{p+q-1}+s^{p+q}+s^{p+q+2}$$

$$= 1+s+s^2+s^3+ \dots +s^{m-3}+s^{m-1}$$

hence

$$p+q+3 = m$$

and so

$$(q-1) = p(q-1)$$

If $q = 1$, then $n = 1$ and

$$\beta = (\Delta_3)^2(\sigma_1)^p(\sigma_2)^{-1}$$

for $p \in \mathbb{Z}^+$ and $m = p + 4$. In fact β^c is isotopic to (2,p+4) torus knot. Now let $q \neq 1$, then $p = 1$ and

$$\beta = (\Delta_3)^{2[(q+2)/3]}\sigma_1(\sigma_2)^{-q}, \text{ for } q \in \mathbb{Z}^+, \text{ with}$$

$$\Delta_\beta(t) = \Delta_{(2,q+4)}(t)$$

Similarly if $n < 0$, then $n = -(p+2)/3$ and lemma (2.2.5) tells us that

$$s^{q+2p+4} + s^{p+2}(1+s+2s^2+ \dots +2s^{p+q-2}+s^{p+q-1}+s^{p+q})$$

$$= s^p(s^2+s^3+ \dots +s^{m-1}+s^{m+1})$$

then

$$p+m+1 = q+2p+4, \text{ i.e. } m = p + q + 3$$

hence

$$p(q-1) = (q-1)$$

So the resulting braid is conjugate to the inverse of the given, above, braid β for $n > 0$.

Case (II); Let n-even, then lemma (2.2.6) tells us that $n = \pm(q-1)/3$. Now if $n = 0$, then $q = 1$ and so $\beta = (\sigma_1)^p(\sigma_2)^{-1}$, hence β^c is isotopic to (2,p) torus knot. For $n > 0$, i.e. $n = (q-1)/3$ with $q \neq 1$, then lemma (2.2.5) tells us that $pq = m$ and

$$s - (1+s+2s^2+ \ \ldots \ +2s^{p+q-2}+s^{p+q-1}+s^{p+q}) + s^{p+q-1}$$

$$= -(1+s^2+s^3+ \ \ldots \ +s^{m-1}+s^{m+1})$$

then

$$p+q = m+1$$

hence

$$p(q-1) = (q-1), \ q > 1$$

so $p = 1$ and $q = m$, hence

$$\beta = (\Delta_3)^{2[(q-1)/3]}\sigma_1(\sigma_2)^{-q}, \text{ for } q \in \mathbb{Z}^+$$

with

$$\Delta_\beta(t) = t^{-1}\Delta_{(2,q)}(t)$$

Similarly if $n < 0$, then $n = -(p-1)/3$ with $p > 1$ and lemma (2.2.5) tells us that

$$s^{q-6n}-(s^{-3n}+s^{-3n+1}+2s^{-3n+2}+ \ \ldots \ +2s^{q-6n-1}+s^{q-6n}+s^{q-6n+1})+s^{-3n+1}$$

$$= -(s^{-3n}+s^{-3n+2}+ \ \ldots \ +s^{-3n+m-1}+s^{-3n+m+1})$$

then

$$q-6n+1 = -3n+m+1, \text{ i.e. } m = q-3n = p+q-1$$

hence

$$q(p-1) = (p-1)$$

so $q = 1$ and $m = p$. Therefore our β is conjugate to the inverse of the given braid β, above, with

$$\Delta_\beta(t) = t^{-p}\Delta_{(2,p)}(t)$$

which completes the proof of theorem (2.2.4) □

<u>(2.2.7) Corollary</u>:

The non full 3-braids close to non amphicheiral knots.

<u>Proof</u>:

The non full 3-braid α has $|c(\alpha)| > 3$, shown in theorem (2.2.4). Hence as a direct consequence of (v) of theorem (2.0.3), α^c is not amhpicheiral □

Through the proof of theorem (2.2.4), it is proved the following results:

<u>(2.2.8) Proposition</u> :

The closed 3-braid $[\beta = (\Delta_3)^{4k}\sigma_1(\sigma_2)^{-(6k+1)}]^c \simeq K$ has the Alexander polynomial, $\Delta_K(t) = t^{-1}\Delta_{(2,6k+1)}(t)$ and $\Delta_L(t) = t^{-6k}\Delta_K(t)$, where L is the inverse of K and $k \in \mathbb{Z}^+$.

<u>(2.2.9) Corollary</u>:

Since the maximum spread of $P_{(2,p)}(v,z)$ is 2, then the closed 3-braid K has a full representative if and only if $\Delta_K(t) \neq t^k\Delta_{(2,p)}(t)$, for any p, $k \in \mathbb{Z}$.

(2.2.10) Corollary:

If K is the closure of a full 3-braid β, then $P_K(v,z)$ determines $c(\beta)$, where $c(\beta)$ is the exponent sum of β and $\beta^c \simeq K$.

Proof:

Since K has a full 3-braid representative, then

$$P_K(v,z) = v^{c(\beta)-2}[Q_0(z) + v^2 Q_1(z) + v^4 Q_2(z)]$$

for non-zero polynomials $Q_i(z)$, i=0,1,2, which determines $c(\beta)$ □

(2.2.11) Proposition :

The closed 3-braid $[\beta = (\Delta_3)^{4k}(\sigma_1)(\sigma_2)^{-(6k+1)}]^c \simeq K$, has the 2-variable invariant $P_K(v,z) = v^{6k}[Q_1(z)+v^2Q_2(z)]$, where $k \in \mathbb{Z}^+$ and $P_1(\sqrt{t}-\sqrt{t}^{-1})=[t^{3k+2}-t^{-3k}]/(t^2-1)$, $P_2(\sqrt{t}-\sqrt{t}^{-1})=[t^{3k+1}-t^{-3k+1}]/(1-t^2)$

Proof :

Let

$$P_K(v,z) = v^{c(\beta)-(n-1)}[Q_0(z) + v^2Q_1(z)+v^4Q_2(z)]$$

but

$$\Delta_K(t) = t^{-1}[\Delta_{(2,6k+1)}(t)]$$

$$= t^{3k-1}[t^{3k}-t^{3k-1}+ \dots -t^{-3k+1}+t^{-3k}]$$

then using (i) and (iii) of theorem (2.0.3), we have

$$A(t) = S_0(t)+S_1(t)+S_2(t) = t^{-3k}[(t^{6k+1}+1)/(t+1)]$$

$$B(t) = S_0(t)+tS_1(t)+t^2S_2(t) = t^{-3k+1}$$

$$C(t) = S_0(t)+t^{-1}S_1(t)+t^{-2}S_2(t) = t^{3k-1}$$

Then solving A(t), B(t) and C(t) of $S_0(t)$, $S_1(t)$ and $S_2(t)$, we have

$$A(t)- B(t) = (1-t)S_1(t)+(1-t^2)S_2(t)$$

$$= t^{-3k}[(t^{6k+1}+1)/(t+1)]-t^{-3k+1}$$

and

$$A(t)-C(t) = (1-t^{-1})S_1(t)+(1-t^{-2})S_2(t)$$

$$= t^{-3k}[(t^{6k+1}+1)/(t+1)]-t^{3k-1}$$

Then

$$[A(t)-B(t)]+t[A(t)-C(t)] = (1-t^2+t-t^{-1})S_2(t)$$

$$= t^{3k+1}-t^{3k}-t^{-3k+1}+t^{-3k}$$

So that

$$S_2(t) = [t^{3k+1}-t^{-3k+1}]/[1-t^2]$$

hence

$$S_1(t) = [t^{3k+2}-t^{-3k}]/[t^2-1]$$

then

$$S_1(t)+S_2(t) = \Delta_K(t)$$

Therefore

$$S_0(t) = 0$$

which completes the proof □

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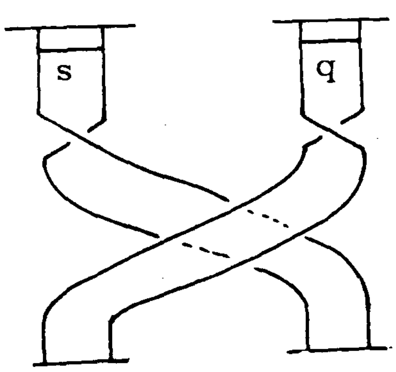
§2.3. TWIST POSITIVE 3-BRAIDS DO NOT ADMIT NON TRIVIAL EXCHANGE MOVES

(2.3.1) definition:

A braid α is exchangeable (admits exchange move) if it is conjugate to a braid of the form $[URV\Sigma_{q,s}]$, where $(U,i+p)$, $(R,p+s)$ and $(V,i+m)$ are braids as in figure (2-2b) and $\Sigma_{q,s}$ is the $(q+s)$-braid, shown in figure (2-4a), where a group of q-strands pass over a group of s-strands and a negative half twist occurs in each group of strands. The exchange of exchangeable braid $\alpha = [URV\Sigma_{q,s}]$ is $exch(\alpha) = [U\Sigma_{p,m}\tau[R]V]$.

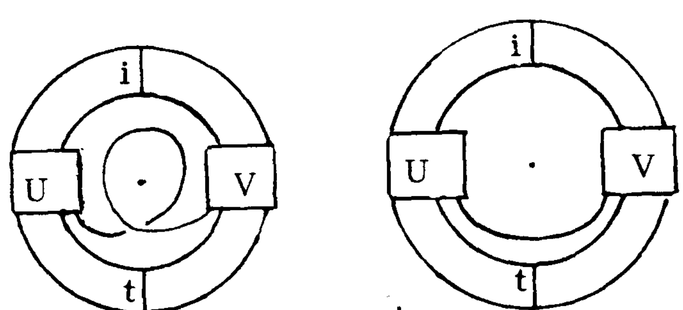
(2.3.2) Remark:

Suppose β is exchangeable as in the definition above, then the number of strands of the braids U, V and R are related as, $[t+s+q = i+q+m = p+s+i = n]$ and $[t+s+p = t+q+m = p+m+i = n']$. Hence $n = n'$ if $t = i$, i.e. the braid index preserved by exchange move when $t = i$ and so the exponent sum is preserved. Exchange moves also include Markov's move of type (ii) of theorem (0.8) as a special case. This occurs when $q = s = 1$ and $p = m = 0$, so $t+1=i$, as in figure (2-4b). An isotopic sequence of closed braids is illustrated in figure (2-5) to represent the general exchange move.



$\Sigma_{q,s}$

Figure (2-4a)



Markov's "stabiliser move"

Figure (2-4b)

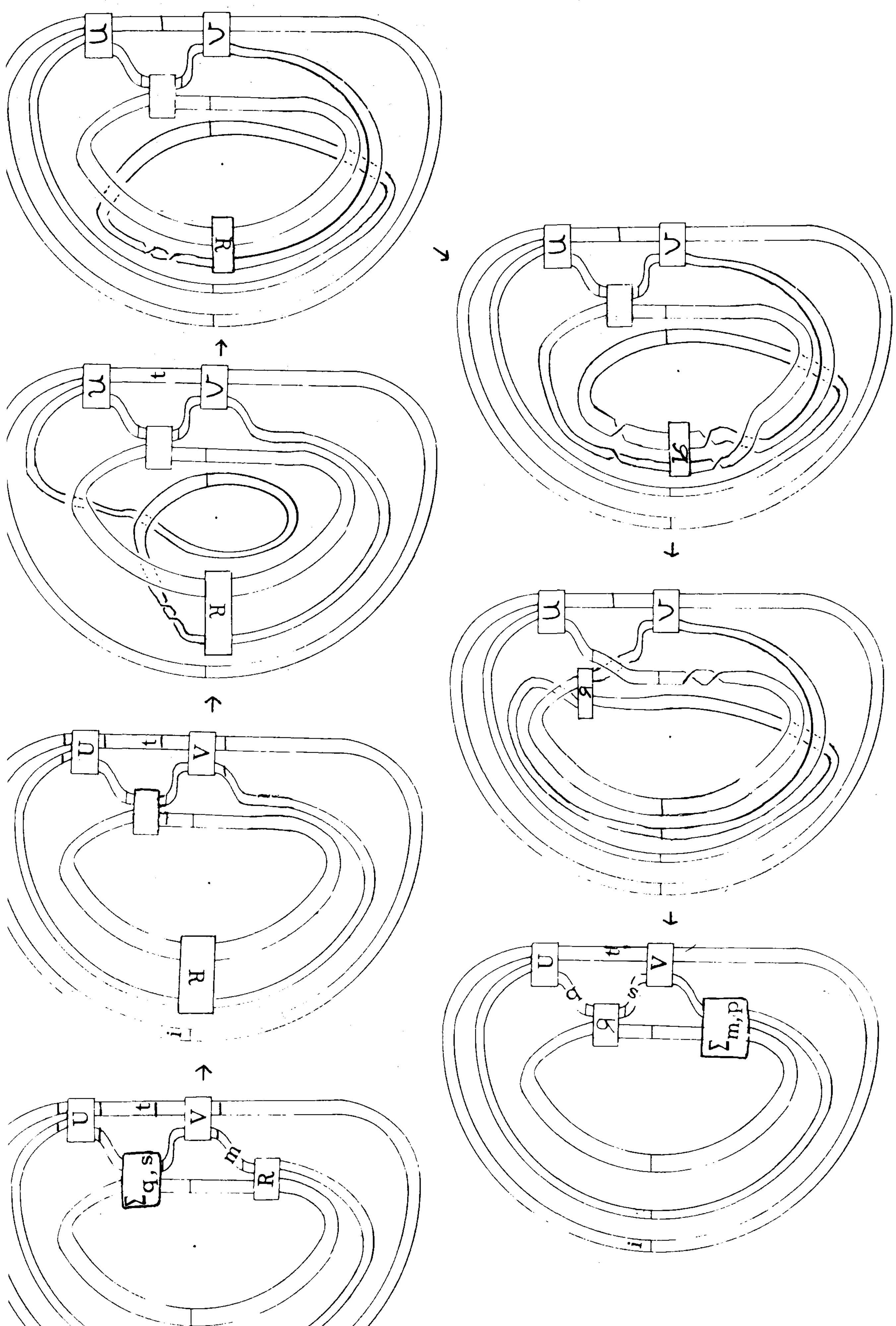


Figure (2-5)

(2.3.3) Theorem:

Braids in B_3 admitting non-trivial exchange move, not simply conjugation, can not be written as twist positive braids. Hence the conjugacy class of twist positive braid representative is a link invariant, provided that Birman's conjecture in (2.0.4) holds.

The proof of the theorem will start with some lemmas.

(2.3.4) Lemma:

Any 3-braid has the standard form

$$\alpha = (\Delta_3)^m[\prod_{i=1}^{r}(\sigma_1)^{(s_i)}(\sigma_1\sigma_2)(\sigma_2)^{(t_i)}(\sigma_2\sigma_1)]^{\theta}\beta$$

or $\tau[\alpha]$ (the conjugate of α by Δ_3), where $\beta\in\{e, (\sigma_1)^s, (\sigma_1)^s(\sigma_1\sigma_2), (\sigma_1)^s(\sigma_1\sigma_2)(\sigma_2)^t\}$, for s_i, t_i, s, $t\in\mathbb{Z}^+$, $1 \leqslant i \leqslant r$, $m\in\mathbb{Z}$ and $\theta\in\{0, 1\}$.

Proof:

The braid $(\pi_1\pi_2 \ldots \pi_k)$ is the canonical form for a given braid α if and only if $S(\pi_{i+1}) \subseteq F(\pi_i)$, for $1 \leqslant i \leqslant k-1$, as shown in theorem (1.3.1). But

$$SB_3 = \{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$$

So that $S(\pi)$, $F(\pi) \subseteq \{1, 2\}$, for all $\pi\in SB_3$. So if $\pi_1 = \sigma_1$, then either $\pi_2 = \sigma_1$ or $\pi_2 = \sigma_1\sigma_2$ and if $\pi_1 = \sigma_2$, then either $\pi_2 = \sigma_2$ or $\pi_2 = \sigma_2\sigma_1$. But if $\pi_1 = \sigma_1\sigma_2$, then either $\pi_2 = \sigma_2$ or $\pi_2 = \sigma_2\sigma_1$ and if $\pi_1 = \sigma_2\sigma_1$, then either $\pi_2 = \sigma_1$ or $\pi_2 = \sigma_1\sigma_2$. Hence the general pattern for a positive 3-braid, which is prime to Δ_3, is

$$\alpha = [\prod_{i=1}^{r}(\sigma_1)^{(s_i)}(\sigma_1\sigma_2)(\sigma_2)^{(t_i)}(\sigma_2\sigma_1)]^{\theta}\beta$$

or $\tau[\alpha]$, where $\beta\in\{e, (\sigma_1)^s (\sigma_1)^s(\sigma_1\sigma_2), (\sigma_1)^s(\sigma_1\sigma_2)(\sigma_2)^t\}$, for s_i, t_i, s, $t\in\mathbb{Z}^+$, $1 \leqslant i \leqslant r$ and $\theta\in\{0, 1\}$, which completes the proof □

(2.3.5) Remark:

The braid group B_3 has a presentation $\{\sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$, shown in definition (0.5). One can also introduce a new generators $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$, then B_3 has a new presentation $\{a, b \mid a^2 = b^3\}$, where $\sigma_1 = a^{-2}(b^2a)$ and $\sigma_2 = a^{-2}(ab^2)$. So $(\sigma_1)^{-1} = a^{-2}(ab)$ and $(\sigma_2)^{-1} = a^{-2}(ba)$. Therefore $(\sigma_1\sigma_2) = b$ and $(\sigma_2\sigma_1) = a^{-2}(aba)$, [Mu2].

(2.3.6) Lemma:

The twist positive braid $(\alpha,3)$ is conjugate to the braid

$$(\Delta_3)^{m+r\theta}[\prod_{i=1}^{r}(\sigma_2)^{(s_i)}(\sigma_1)^{-1}]^{\theta}\beta$$

where $\theta\in\{0, 1\}$, $\beta\in\{e, (\sigma_2)^s\}$, s_i, $s\in\mathbb{Z}^+$, for $1 \leqslant i \leqslant r$ and m is a positive integer such that $m \geqslant 2$.

Proof:

Using lemma (2.3.4) we can write the standard form for twist positive braid $(\alpha,3)$ as

$$\alpha = (\Delta_3)^m(\prod_{i=1}^{r}[(\sigma_2)^{(s_i)}(\sigma_1\sigma_2)(\sigma_2)^{(t_i)}(\sigma_2\sigma_1)])^{\theta}\beta$$

or $\tau[\alpha]$, where $\beta\in\{e, (\sigma_1)^s, (\sigma_1)^s(\sigma_1\sigma_2), (\sigma_1)^s(\sigma_1\sigma_2)(\sigma_2)^t\}$, for s_i, t_i, s, m and $t\in\mathbb{Z}^+$, $1 \leqslant i \leqslant r$, such that $m \geqslant 2$ and $\theta\in\{0, 1\}$. Using the presentation $\{a, b \mid a^2 = b^3\}$ (shown in remark (2.3.5)) and the fact that a^2 commutes with every thing, then α can be written as

$$\alpha = a^{m-2(S+T+r)}(\prod_{i=1}^{r}[(b^2a)^{(s_i)}(b)(ab^2)^{(t_i)}(ab)(a)])^{\theta}\beta'$$

where

$\beta' \in \{e,\ (a^{-2s})(b^2a)^s,\ (a^{-2s})(b^2a)^s(b),\ (a^{-2s})(b^2a)^s(b)(a^{-2t})(ab^2)^t\}$

with

$$\theta \in \{0,\ 1\},\ S = s_1+s_2+\dots+s_r \text{ and } T = t_1+t_2+\dots+t_r$$

Now rewrite $(b^2a)^n = [b^2(ab^2)^{n-1}a]$ and consider the following cases:

<u>Case (a): $\beta' = e$ and $\theta = 1$</u> , then

$$\alpha = a^{m-2(S+T+r)}[b^2(ab^2)^{(s_1-1)}(ab)(ab^2)^{(t_1)}(ab)(a)]$$
$$[b^2(ab^2)^{(s_2-1)}(ab)(ab^2)^{(t_2)}(ab)(a)]$$
$$[b^2(ab^2)^{(s_r-1)}(ab)(ab^2)^{(t_r)}(ab)(a)]$$
$$= a^{m-2(S+T+r)}(b^2)[(ab^2)^{(s_1-1)}(ab)(ab^2)^{(t_1)}(ab)]$$
$$[(ab^2)^{(s_2)}(ab)(ab^2)^{(t_2)}(ab)]$$
$$[(ab^2)^{(s_r)}(ab)(ab^2)^{(t_r)}(ab)](a)$$

This is conjugate by a to

$$a^{m-2(S+T+r)}(\prod_{i=1}^{r}[(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)])$$

Then using the relations between the two presentations of B_3, shown in remark (2.3.5), we can write α (up to conjugation) as

$$\alpha =^{c} (\Delta_3)^{m+2r}\prod_{i=1}^{r}[(\sigma_2)^{(s_i)}(\sigma_1)^{-1}(\sigma_2)^{(t_i)}(\sigma_1)^{-1}] \qquad (2.3.1)$$

where s_i, $t_i \in \mathbb{Z}^+$, for $1 \leqslant i \leqslant r$, m is a positive integer such that $m \geqslant 2$ and $=^c$ means equal up to conjugation.

<u>Case (b): $\beta' = a^{-2s}(b^2a)^s$ and $\theta = 1$</u> , then

$$\alpha = a^{m-2(S+T+r+s)}[b^2(ab^2)^{(s_1-1)}(ab)(ab^2)^{(t_1)}(ab)] \times$$

$$\prod_{i=2}^{r-1}[(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)] \times$$

$$[(ab^2)^{(s_r)}(ab)(ab^2)^{(t_r)}(ab)](a)(b^2a)^s$$

This is conjugate by a to

$$= a^{m-2(S+T+s+r)}\prod_{i=1}^{r}([(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)](ab^2)^s$$

Then using the relations between the two presentations of B_3, shown in remark (2.3.5), we can write α (up to conjugation) as

$$\alpha =^c a^{m+2r}\prod_{i=1}^{r}([(\sigma_2)^{(s_i)}(\sigma_1)^{-1}(\sigma_2)^{(t_i)}(\sigma_1)^{-1}](\sigma_2)^s \qquad (2.3.2)$$

<u>Case (c): $\beta' = [a^{-2}(b^2a)]^s(b)$ and $\theta = 1$</u> , then

$$\alpha = a^{m-2(S+T+s+r)}[b^2(ab^2)^{(s_1-1)}(ab)(ab^2)^{(t_1)}(ab)] \times$$

$$\prod_{i=2}^{r-1}[(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)] \times$$

$$[(ab^2)^{(s_r)}(ab)(ab^2)^{(t_r)}(ab)](a)(b^2a)^s(b)$$

This is conjugate by a to

$$\alpha = a^{m-2(S+T+s+r)-1}\prod_{i=1}^{r}([(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)](ab^2)^s(ab)$$

Then using the relations between the two presentations of B_3, shown in remark (2.3.5), we can write α (up to conjugation) as

$$\alpha =^c a^{m+2r+1}\prod_{i=1}^{r}([(\sigma_2)^{(s_i)}(\sigma_1)^{-1}(\sigma_2)^{(t_i)}(\sigma_1)^{-1}][(\sigma_2)^s(\sigma_1)^{-1}]$$

$$\text{-------}\rightarrow \ (2.3.3)$$

<u>Case (d): $\beta' = [a^{-2(t+s)}(b^2a)^s(b)(ab^2)^t$ and $\theta = 1$</u> , then

$$\alpha = a^{m-2(S+T+s+t+r)}[b^2(ab^2)^{(s_1-1)}(ab)(ab^2)^{(t_1)}(ab)] \times$$
$$\prod_{i=2}^{r-1}[(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)] \times$$
$$[(ab^2)^{(s_r)}(ab)(ab^2)^{(t_r)}(ab)](a)(b^2a)^s(b)(ab^2)^t$$

This is conjugate by a to

$$a^{m-2(S+T+s+t+r)-1}\prod_{i=1}^{r}([(ab^2)^{(s_i)}(ab)(ab^2)^{(t_i)}(ab)](ab^2)^s(ab)(ab^2)^t$$

Then using the relations between the two presentations of B_3, shown in remark (2.3.5), we can write α (up to conjugation) as

$$\alpha =^c a^{m+2r+1}\prod_{i=1}^{r}([(\sigma_2)^{(s_i)}(\sigma_1)^{-1}(\sigma_2)^{(t_i)}(\sigma_1)^{-1}][(\sigma_2)^s(\sigma_1)^{-1}](\sigma_2)^t \quad \text{-------}\rightarrow (2.3.4)$$

<u>Case (e): $\theta = 0$</u> , then

$$\alpha = a^m,\ \alpha = a^{m-2s}(b^2a)^s,\ \alpha = a^{m-2s-1}a[b^2(ab^2)^{s-1}(a)(b)]$$

or

$$\alpha = a^{m-2s-2t-1}a[b^2(ab^2)^{s-1}(a)(b)(ab^2)^t]$$

Hence using the relations between the two presentations of B_3, shown in remark (2.3.5), we can write α (up to conjugation) as

$$\alpha\in\{(\Delta_3)^m,\ (\Delta_3)^m(\sigma_2)^s,\ (\Delta_3)^{m+1}(\sigma_2)^s(\sigma_1)^{-1}\ (\Delta_3)^{m+1}(\sigma_2)^s(\sigma_1)^{-1}(\sigma_2)^t\} \quad \text{-------}\rightarrow (2.3.5)$$

Therefore the equations (2.3.1) - (2.3.4) give the general pattern for the representative of the twist positive braid $(\alpha,3)$, (up to conjugation) as

$$(\Delta_3)^{m+r\theta}[\prod_{i=1}^{r}[(\sigma_2)^{(s_i)}(\sigma_1)^{-1}]^\theta\beta$$

where $\theta \in \{0, 1\}$, $\beta \in \{e, (\sigma_2)^s\}$, s_i, $s \in \mathbb{Z}^+$, for $1 \leqslant i \leqslant r$ and m is a positive integer such that $m \geqslant 2$.

(2.3.7) Remark:

Using remark (2.3.2), we can write the general pattern for exchangeable braid $(\alpha,3)$ as

$$\alpha = (\sigma_2)^k(\sigma_1)^\varepsilon(\sigma_2)^s(\sigma_1)^r$$

where $k,s,r \in \mathbb{Z}$ and $\varepsilon = \pm 1$, then

$$\mathrm{Exch}(\alpha) = (\sigma_2)^s(\sigma_1)^\varepsilon(\sigma_2)^k(\sigma_1)^r$$

So $\alpha = \mathrm{Exch}(\alpha)$, for $k = s$ or $\varepsilon = r$. Now to see that the twist positive 3-braids do not admit non-trivial exchangeable moves, we consider the different cases according to the sign of the powers of k, s, r and ε. It is obvious that α does not conjugate to a twist positive 3-braid if k, s and r are all negative, because the length of the braid (the algebraic crossing number) is invariant under conjugacy. Then consider the following cases:

(a): If $r = \varepsilon$, then α and $\mathrm{exch}(\alpha)$ are conjugate. This can be seen by cycling the letters of α. So let $\varepsilon = 1$ and $r = -1$, then for positive integers k, s, we have

$$\begin{aligned}\alpha &= (\sigma_2)^k(\sigma_1)(\sigma_2)^s(\sigma_1)^{-1}\\ &= (\sigma_2)^{k-1}(\Delta_3)(\sigma_2)^{s-1}(\sigma_2\sigma_1)(\Delta_3)^{-1}\\ &=^c (\sigma_1)^{k-1}(\sigma_2)^{s-1}(\sigma_2\sigma_1)\\ &=^c (\sigma_1)^k(\sigma_2)^s\end{aligned}$$

But

$$\begin{aligned}\mathrm{exch}(\alpha) &= (\sigma_2)^k(\sigma_1)^{-1}(\sigma_2)^s(\sigma_1)\\ &= (\sigma_2)^k(\Delta_3)^{-1}(\sigma_1\sigma_2)(\sigma_2)^s(\sigma_1)\\ &= (\Delta_3)^{-1}(\sigma_1)(\sigma_1)^k(\sigma_2)^s(\sigma_2\sigma_1)\\ &=^c (\sigma_1)^k(\sigma_2)^s\end{aligned}$$

Then α and $\mathrm{exch}(\alpha)$ are conjugate. In this case we call the exchange move trivial.

(b): If k, s, r are positive integers and $\varepsilon = 1$, then we can easily check that α and $\mathrm{exch}(\alpha)$ are conjugate if $k \leqslant 2$, $s \leqslant 2$ or $r = 1$.

(c): If $k < 0$, $\varepsilon = 1$, $s > 2$ and $r > 1$, then

$$(\sigma_2)^k = (\Delta_3)^k \begin{cases} (\sigma_2\sigma_1)[(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{-(k+1)/2} & \text{if } k\text{-odd} \\ [(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{-k/2} & \text{if } k\text{-even} \end{cases}$$

so

$$\alpha =^c (\Delta_3)^k(\sigma_2\sigma_1)^{(\theta_k)}[(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{(t_k)}(\sigma_1)(\sigma_2)^s(\sigma_1)^r \tag{2.3.6}$$

where $\theta_k = 1$, $t_k = -(k+1)/2$ for k-odd and $\theta_k = 0$, and $t_k = -(k)/2$ for k-even.

(2.3.8) Lemma:

If α admits a non-trivial exchange move, then α is conjugate to:

(i): $(\Delta_3)^4(\sigma_2)^p(\sigma_1)^{-1}(\sigma_2)^q(\sigma_1)^{-1}(\sigma_2)^i(\sigma_1)^{-1}$

for different non-negative integers p, q, i.

(ii): $(\Delta_3)^2(\sigma_2)^p(\sigma_1)^{-1}(\sigma_2)^q(\sigma_1)^{-j}$

for non-negative integers p, q, i, such that $p \neq q$

(iii): $(\sigma_2)(\sigma_1)^{-p}(\sigma_2)^{i}(\sigma_1)^{-q}$

for non-negative integers p, q, i, such that $p \neq q$.

Proof:

Using remark (2.3.7), it is enough to consider the following cases:

Case (i): When k, s, r are positive integers such that $k, s \geqslant 3$, $r \geqslant 2$ and $\varepsilon = 1$. Using the presentation $\{a, b \mid a^2 = b^3\}$ of B_3, shown in remark (2.3.5), we can write α as

$$\alpha = (\sigma_2)^{k}(\sigma_1)(\sigma_2)^{s}(\sigma_1)^{r}$$

$$= a^{-2(k+s+r+1)}(ab^2)^{k}(b^2a)(ab^2)^{s}(b^2a)^{r}$$

$$= a^{-2(k+s+r+1)}(ab^2)[(ab^2)^{k-3}][(ab^2)^2(b^2a)(ab^2)^2] \times$$

$$[(ab^2)^{s-3}][(ab^2)(b^2)][(ab^2)^{r-2}][(ab^2)a]$$

But using the relations between the two presentations of B_3, shown in remark (2.3.5), then

$$[(ab^2)^2(b^2a)(ab^2)^2] = a^{10}(ab)$$

and

$$(ab^2)(b^2) = a^2(ab)$$

so

$$\alpha = a^{-2(k+s+r+1)}(ab^2)[(ab^2)^{k-3}][a^{10}(ab)] \times$$

$$[(ab^2)^{s-3}][a^2(ab)][(ab^2)^{r-2}][(ab^2)a]$$

$$= a^{-2(k+s+r-5)}(ab^2)[(ab^2)^{k-3}](ab) \times$$

$$[(ab^2)^{s-3}](ab)[(ab^2)^{r-2}][(ab^2)a]$$

$$=^c a^{-2(k+s+r-5)}[(ab^2)^{k-3}](ab) \times$$

$$[(ab^2)^{s-3}](ab)[(ab^2)^{r-2}][ab^2a^2b^2]$$

where $=^c$ means equal up to conjugacy. But

$$ab^2a^2b^2 = a^4(ab)$$

then

$$\alpha =^c a^{-2(k+s+r-7)}[(ab^2)^{k-3}](ab)[(ab^2)^{s-3}](ab)[(ab^2)^{r-2}](ab)$$

Hence using relations between the two presentations of B_3, shown in remark (2.3.5), we can write α as

$$\alpha =^c (\Delta_3)^4(\sigma_2)^{k-3}(\sigma_1)^{-1}(\sigma_2)^{s-3}(\sigma_1)^{-1}(\sigma_2)^{r-2}(\sigma_1)^{-1}$$

Case (ii): Let k negative and all other powers are positive integers, then using equation (2.3.6), we have

$$\alpha = (\sigma_2)^k(\sigma_1)(\sigma_2)^s(\sigma_1)^r$$

$$= (\Delta_3)^k(\sigma_2\sigma_1)^{(\theta_k)}[(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{(t_k)}(\sigma_1)(\sigma_2)^s(\sigma_1)^r$$

where $\theta_k = 1$, $t_k = -(k+1)/2$ for k-odd and $\theta_k = 0$, $t_k = -(k)/2$ for k-even. Now if k-even, then

$$\alpha = (\Delta_3)^k[(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{-(k/2)}(\sigma_1)(\sigma_2)^s(\sigma_1)^r$$

so using the presentation in remark (2.3.5), we have

$$\alpha = (a)^k[(b)a^{-2}(aba)]^{-(k/2)}(a^{-2}b^2a)(a^{-2}ab^2)^s(a^{-2}b^2a)^r$$

$$= (a)^{2(k-s-r-1)}(ba)^{-k}(b^2a)(ab^2)^s(b^2a)^r$$

$$= a^{2(k-s-r-1)}b(ab)^{-k-1}ab^2aab^2(ab^2)^{s-2}(ab^2)(b^2a)(b^2a)^{r-1}$$

$$= a^{2(k-s-r-1)}b(ab)^{-k}(a^4)(ab^2)^{s-2}(a^2\ ab)(ab^2)^{r-1}a$$

$$=^c a^{2(k-s-r-1)+6}(ab)^{-k+1}(ab^2)^{s-2}(ab)(ab^2)^{r-1}$$

Using again the presentations of B_3, shown in remark (2.3.5), we can write α as

$$\alpha =^c (\Delta_3)^2(\sigma_2)^{s-2}(\sigma_1)^{-1}(\sigma_2)^{r-1}(\sigma_1)^{k-1}$$

But if k-odd, then

$$\alpha = (\Delta_3)^k(\sigma_2\sigma_1)[(\sigma_1\sigma_2)(\sigma_2\sigma_1)]^{-(k+1/2)}(\sigma_1)(\sigma_2)^s(\sigma_1)^r$$

Then using the presentation in remark (2.3.5), we have

$$\alpha = a^k(a^{-2}\ aba)[(b)\ a^{-2}(aba)]^{-(k+1/2)}(a^{-2}b^2a)(a^{-2}\ ab^2)^s(a^{-2}b^2a)^r$$

$$= a^{2(k-s-r-1)}(aba)(ba)^{-(k+1)}(b^2a)(ab^2)^s(b^2a)^r$$

$$= a^{2(k-s-r-1)}(ab)^{-k}(a.b^2a.ab^2)(ab^2)^{s-2}(ab^2.b^2a)(b^2a)^{r-1}$$

$$= a^{2(k-s-r-1)}(ab)^{-k}(a^4ab)(ab^2)^{s-2}(a^2\ ab)(ab^2)^{r-1}a$$

$$=^c a^{2(k-s-r+2)}(ab)^{-k+1}(ab^2)^{s-2}(ab)(ab^2)^{r-1}$$

Using again the presentations of B_3, shown in remark (2.3.5), we can write α as

$$\alpha =^c (\Delta_3)^2(\sigma_2)^{s-2}(\sigma_1)^{-1}(\sigma_2)^{r-1}(\sigma_1)^{k-1}$$

Case (iii): when k, s are negative, and ε, r are positive, then similarly as in case (ii), and using equation (2.3.5), we have

$$\alpha = (\sigma_2)^k(\sigma_1)(\sigma_2)^s(\sigma_1)^r$$

$$= (\Delta_3)^k(\sigma_2\sigma_1)^{(\theta_k)}[(\sigma_1\sigma_2)(\ \sigma_2\sigma_1)]^{(t_k)}(\sigma_1) \times$$

$$(\sigma_2\sigma_1)^{(\theta_s)}[(\sigma_1\sigma_2)(\ \sigma_2\sigma_1)]^{(t_s)}(\sigma_1)^r$$

where $\theta_k = \theta_s = 1$, $t_k = -(k+1)/2$, $t_s = -(s+1)/2$ for odd integers k, s and $\theta_k = \theta_s = 0$, $t_k = -(k)/2$, $t_s = -(s)/2$ for even integers k, s. Then following the previous calculations as in case (ii), one can check that α is conjugate to a braid with pattern as

$$(\sigma_2)(\sigma_1)^{-p}(\sigma_2)^i(\sigma_1)^{-q}$$

for non-negative integers p, q, i, such that $p \neq q$, which completes the proof □

Proof of theorem (2.3.3):

Lemma (2.3.6) tells us that the selected conjugacy representative α for a twist positive braid contains $(\Delta_3)^{m+r}$, where $m \geq 2$ and r is the number of factors $[(\sigma_2)^s(\sigma_1)^{-1}]$ in α, for $s \in \mathbb{Z}^+$. Hence using Murasugi's result on classifying the conjugacy classes in B_3 (as in proposition (0.14)), then comparing the conjugacy representatives for twist positive braids and for the non-trivial exchangeable braids, we conclude that non-trivial exchangeable braids can not conjugate to twist positive braids, which completes the proof □

CHAPTER 3

ON LORENZ KNOTS AND LINKS

§3.0 INTRODUCTION

In all known examples of differential equations the solutions appeared to fall into two categories, those which ultimately settled down to some sort of steady state behaviour and those which are periodic in time.

Starting with the Navier-Stokes equation, [M], which governs the motion of a viscous, incompressible fluid, Lorenz introduced a truncation which enabled him to reduce the Navier-Stokes equation to a system of ordinary differential equations in 3-space variables x,y,z as a function of time, [L].

For a system of ordinary differential equations such as Lorenz differential equations as t changes the points of $\mathbb{R}^3$ move simultaneously along trajectories, defining a flow $\Phi_t:\mathbb{R}^3 \to \mathbb{R}_t$, for $t \in \mathbb{R}$. Williams.F.R has found structures, Lorenz attractors (Lorenz knot holder), in $\mathbb{R}^3$ relative to the flow Φ_t which allow the periodic orbits, in the solution of Lorenz equations, to be collapsed onto a 2-dimensional branched manifold in $\mathbb{R}^3$, for $t \geqslant 0$, [W1].

The concept of Lorenz knots, Lorenz links and Lorenz braids (which are the subject matter of this chapter and chapter 4), have been introduced by Birman.J and Williams.F.R, in a series of papers, [B-W1], [B-W2] and [W2]. They have investigated the periodic orbits in the solution of Lorenz equations and so they shown that knots and links do occur, which called "Lorenz knots and Lorenz links". They also proved that there are infinitely many inequivalent Lorenz knots, where the relation between the class of Lorenz links and other classes such as fibred links, algebraic links and closed positive braids have been studied.

Section 1 is devoted to the study of minimal braid representatives of a Lorenz link. It is an attempt to formulate a canonical form for a minimal braid representative of every Lorenz link. An example of Lorenz braids is given, in example (3.1.7), where a Lorenz braid, shown in definition (3.1.1), is two groups of strands cross with positive crossings only, such that no self crossing in each group and each two strands cross at most once.

In remark (3.1.2) it is noted that the class B(k,r) of Lorenz braids $\beta(k,r)$, by the conception cited above, is much wider than [B-W1]'s class. In [B-W1]'s conception, it is necessary that each arc in any group of strands should cross some arcs in the other group, whereas here is not. moreover let $\pi(k,r)$ denote to the associated permutation to the braid $\beta(k,r)$, then in [B-W1] it is excluded those braids of permutation π with $\pi = \mu_1\mu_2 \ldots \mu_s$ as a product of disjoint s cycles such that no two cyclic factors μ_i, μ_j of the same length r, with $\mu_i(p) = \mu_j(p) + t$, p=1,2, ... ,r, for some integer t, whereas here is included. Two examples to explain that widen, of the conception of Lorenz braids, are illustrated in figures (3-1a) and (3-1b).

Recalling the concept of positive permutation braids, it is proved in lemma (3.1.3) that every Lorenz braid $\beta(k,r)$ is in SB_{k+r}, hence it is shown in corollary (3.1.4) that a Lorenz braid depends only on its associated permutation. Using this approach to positive permutation braids, a necessary and sufficient condition for a positive permutation braid to be a Lorenz braid is established in proposition (3.1.5), (in fact this provides an alternative definition for Lorenz braids), where it is proved that a positive braid π is a Lorenz braid $\beta(k,r)$ if and only if $S(\pi) = \{k\}$, for some $k \in \mathbb{Z}^+$. A formula for a Lorenz braid in terms of its associated permutation is given in lemma (3.1.8).

A technical combinatorial method for representing a Lorenz link by a braid, not a Lorenz braid, in fewer strands is established in lemma (3.1.11). For a Lorenz link L with Lorenz braid β and $S(\beta) = \{k\}$, let π be the associated permutation of β, then consider the number $t = k+1-i$, where i is the least integer $\leqslant k$ such that $\pi(i) > k$, such this number is called the trip number of L. Following the technique presented in lemma (3.1.11), it is shown in corollary (3.1.14) that every Lorenz link of trip number t, has a twist positive braid representative in B_t. As a consequence of corollary (3.1.14) and Corollary (2.1.12), where every twist positive braid is a minimal representative for some link, it is given an affirmative answer for [B-W1]'s conjecture about trip number of Lorenz links, (conjecture 11.6. page 81, of [B-W1]), where it is shown in corollary (3.1.15) that the trip number is the braid index, hence it is a link invariant.

Following that the relation between the class of algebraic knots and links and the class of Lorenz knots and links is investigated. In proposition (3.1.17) it is proved that every algebraic link with $\leqslant 2$ components is a Lorenz, the same was proved in [B-W1] for algebraic knots only. In proposition (3.1.18) it is given a necessary and sufficient condition for a knot to be algebraic, where it is shown that a knot is algebraic if and only if it is the closure of a braid with some specific pattern, shown in equation (3.1.5). Hence it is shown in corollary (3.1.19) that the only algebraic knots with minimal braid representative in B_n, for n prime, is the (n,r) torus knots for all integer r, such that $n \neq r$. An example, in example (3.1.20), to show that not every algebraic link is a Lorenz link is given.

Finally a semi-canonical form for a minimal braid representative of a Lorenz link is established in theorem (3.1.22), where a canonical form for a minimal braid representative for every algebraic knot is established in corollary (3.1.23). An attempt to formulate a canonical form (from that form in theorem (3.1.22)) for minimal braid representatives of a subclass of Lorenz links is done. It shown in lemma (3.1.25) that every Lorenz link of trip number equals to the number of components has an interested semicanonical form, such these links were the field of work in chapter 4.

Section 2 is devoted to the study of the possible satellites of a Lorenz knot. In fact every Lorenz link is a closed braid, which must follow some pattern (as in figure (3-7a)), hence the Lorenz knots which are satellites of other Lorenz knots should also follow that presentation pattern. But the construction of algebraic knots, as in remark (3.1.16) and proposition (3.1.18), tell us that the only way in which a Lorenz knot appears as a represented cable in this presentation is when it is an algebraic knot, hence it is a very plausible conjecture that these are the only ways in which a Lorenz knot can be presented as a satellite.

Now given a Lorenz link C with Lorenz braid $\beta(a,b)$, then using the combinatorial method in lemma (3.1.11), we can represent C as closures of b-braid $[L^{(a)}(\beta(a,b))]$ and a-braid $[R^{(b)}(\beta(a,b))]$, where $L(\beta(a,b))$, $R(\beta(a,b))$, as defined in definition (3.1.9). Then it is shown, in proposition (3.2.2), that for every Lorenz knot C the satellite constructed with pattern as a closed braid α^c, for $\alpha = (\Delta_r)^{2k}[L^{(a)}(\beta(a,r))][R^{(b)}(\beta(r,b))]$ is again a Lorenz knot, with positive integers a, b and r. The idea is modifying the Lorenz knot constructed by running r parallel strands around C in the knot holder H (of C) and including $L^{(a)}(\beta(a,r))$ and $R^{(b)}(\beta(r,b))$, (for some Lorenz braids $\beta(a,r)$ and $\beta(r,b)$).

The pattern given in proposition (3.2.2) is a closed r-braid α^c, where $\alpha = (\Delta_r)^{2k}AB$, with $A = (X_1)^{(a_1)}(X_2)^{(a_2)} \ \dots \ (X_{r-1})^{(a_{r-1})}$, and $B = (Y_1)^{(b_1)}(Y_2)^{(b_2)} \ \dots \ (Y_{r-1})^{(b_{r-1})}$, for positive integers a_i, b_i, for all $1 \leqslant i \leqslant r-1$, as shown in corollary (3.1.14). Then the case with $A = (X_i)^a$ and $B = (Y_i)^b$ gives a cable about C. So algebraic knots are built up successively, starting from the case when C is a torus knot.

It is likely to say that the satellites of Lorenz knots can only constructed by the pattern in proposition (3.2.2), although attempts to prove it using an extension of Williams methods, [W2], have so far been unsuccessful.

§3.1 A SEMICANONICAL FORM FOR A LORENZ BRAID

(3.1.1) Definition:

A Lorenz link L is a closed braid $\beta \epsilon B_n$ for some integer n, where in β the strands have a natural ordering from left to right. Number them, 1,2, ... ,n, on the top and on the bottom. These strings fall into two groups of parallel strands, a left group of k strands and a right group of r strands, k + r = n, where the strands in the right group always pass over (not under) those in the left group, but strands in the same group never cross one another. This braid β is called a Lorenz braid of type (k,r) and denoted $\beta(k,r)$.

(3.1.2) Remark:

Let $\pi(k,r)$ denote to the associated permutation for the braid $\beta(k,r)$ and let B(k,r) denote to the class of all Lorenz braids of type $\beta(k,r)$. Note that the class B(k,r) is much wider than the class of Lorenz braids in [B-W1]. In our definition it is not necessary that each arc in any group of strands should cross some arcs in the other group. e.g. the example illustrated in figure (3-1a) is not a Lorenz braid from point of view of [B-W1], because the left-hand strand in the left group does not cross any arcs from the right group. In [B-W1] it is also excluded those braids of permutation π with $\pi = \mu_1\mu_2 \ldots \mu_s$ as a product of disjoint s cycles such that no two cyclic factors μ_i, μ_j of the same length r, with $\mu_i(p) = \mu_j(p) + t$, p = 1,2, ... ,r, for some integer t, e.g. the Lore braid $\beta(n,n)$ of permutation $\pi(a) = a+n$ for $1 \leqslant a \leqslant n$ and $\pi(b) = b-n$ for $n+1 \leqslant b \leqslant 2n$ (which closes to the (n,n) torus link, i.e. the closure of the positive braid $(\Delta_n)^2$) is not a Lorenz braid from point of view of [B-W1], because $\pi = (1\ n+1)(2\ n+2)\ \ldots\ (n\ 2n)$. An example of such these braids is illustrated in figure (3-1b) for n = 5.

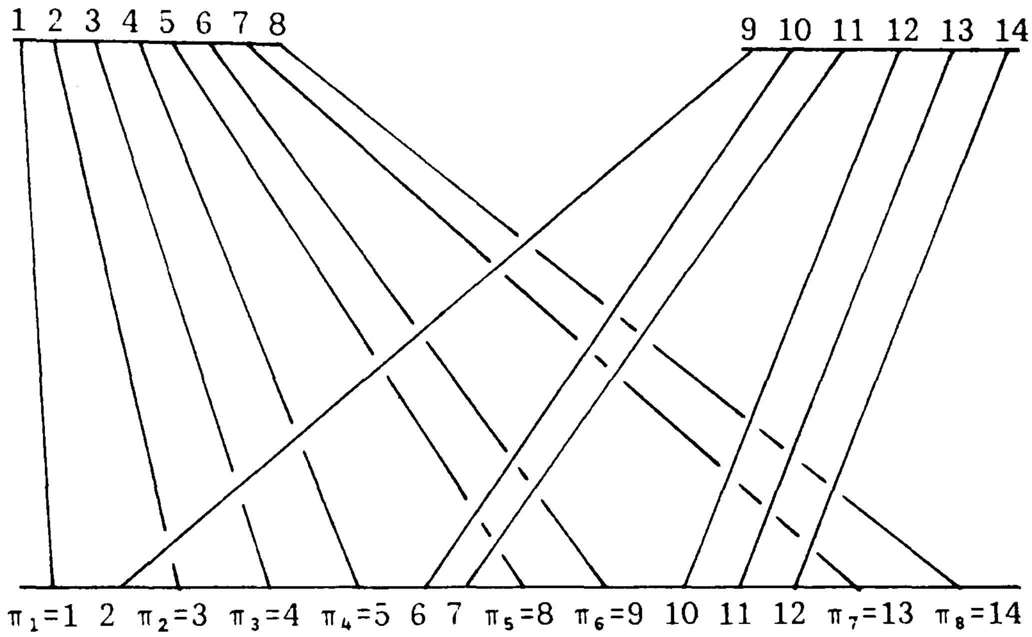


Figure (3-1a)

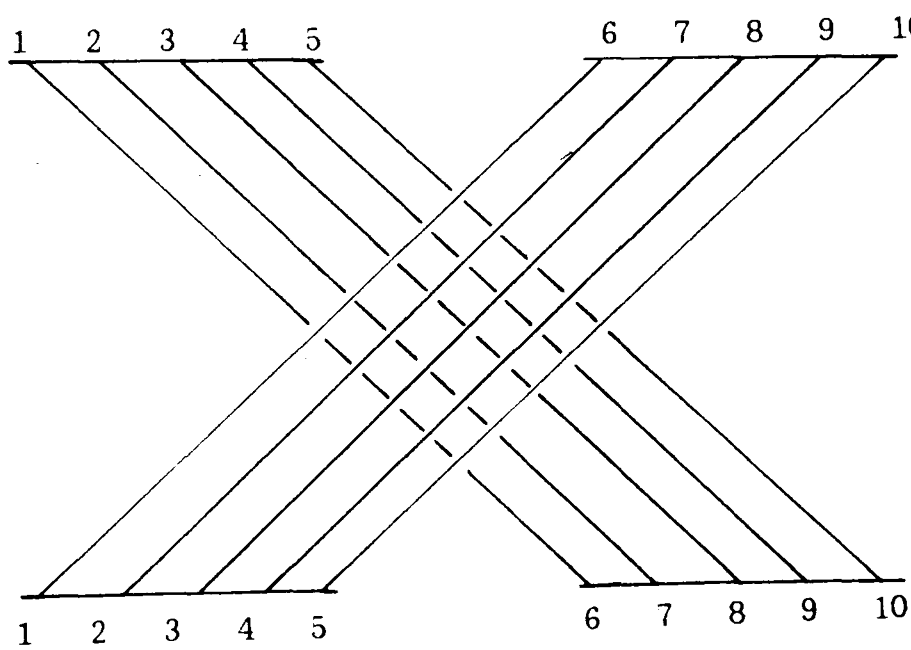


Figure (3-1b)

(3.1.3) Lemma:

Every Lorenz braid $\beta(k,r)$ is a positive permutation braid in B_{k+r}, i.e. $\beta(k,r)\in SB_{k+r}$.

Proof:

In Lorenz braids the strands in the right group always pass over (not under), in a positive sense, those in the left group, i.e. each strand in the right group cross at most once with each strand in the left group. But strands in the same group never cross one another. Hence in $\beta(k,r)$ each two strands cross at most once. The crossings also occur in a positive sense, so definition (1.1.1) tells us that $\beta(k,r)\in SB_{k+r}$ □

(3.1.4) Corollary:

The Lorenz braid $\beta(k,r)$ depends only on its associated permutation $\pi(k,r)$ and on the ordered pair (k,r) of integers.

Proof:

The ordered pair (k,r) determines a left group of k strands and a right group of r strands, where strands in the same group never cross one another. But $\beta(k,r)\in SB_{k+r}$, then lemma (1.1.3) tells us that $\pi(k,r)$ depends only on $\pi(k,r)$ □

The following proposition provides a necessary and sufficient condition for a positive permutation braid to be a Lorenz braid. In fact it can be considered as an alternative definition for Lorenz braids.

(3.1.5) Proposition:

In B_n, a positive permutation braid π is a Lorenz braid if and only if π is an actual word, i.e. π has a single starter ($S(\pi) = \{i\}$, for some i, $1 \leqslant i \leqslant n-1$, as in definition (1.1.7)).

Proof

For necessity: Let π be a Lorenz braid, then lemma (3.1.3) tells us that $\pi \in SB_n$. Assume that $i, j \in S(\pi)$, then lemma (1.2.6) tells us that

$$\pi = \begin{cases} \sigma_i\alpha = \sigma_j\alpha & \text{if } i = j \\ \sigma_i\sigma_j\alpha = \sigma_j\sigma_i\alpha & \text{if } |i - j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i\alpha = \sigma_j\sigma_i\sigma_j\alpha & \text{if } |i - j| = 1 \end{cases}$$

for some $\alpha \in SB_n$. So that we can not break up the strands, in π, to two groups such that no self crossings in each group, i.e. π does not a Lorenz braid, hence π has a single starter.

For sufficiency: Let $\pi \in SB_n$, with $S(\pi) = \{i\}$, for some $1 \leqslant i \leqslant n-1$, then clearly π is a Lorenz braid, because at some stage we can break up the strands to two groups where no self crossings occur in each group of strands □

(3.1.6) Remark:

For a Lorenz braid $\beta(k,r)$ with permutation $\pi(k,r)$, write $\pi(k,r) = (\pi_1, \pi_2, \ldots, \pi_{k+r})$, where $\pi_i = \pi(i)$, $1 \leqslant i \leqslant k+r$. But strands in the same group never cross one another, then

$$1 \leqslant \pi_1 < \pi_2 < \ldots < \pi_k \leqslant k+r, \text{ i.e. } \pi_i \geqslant i, \text{ for } 1 \leqslant i \leqslant k$$

and

$$1 \leqslant \pi_{k+1} < \pi_{k+2} < \ldots < \pi_{k+r} \leqslant k+r, \text{ i.e. } \pi_j \leqslant j, \text{ for } k+1 \leqslant j \leqslant k+r$$

So

$$\pi_i - i \leqslant r, \text{ for } 1 \leqslant i \leqslant k$$

and

$$j - \pi_j \leqslant k, \text{ for } k+1 \leqslant j \leqslant k+r$$

But no crossings occur in each group of strands, then it is clear that the permutation $\pi(k,r)$ is either determined by the tuple $(\pi_1, \pi_2, \ldots, \pi_k)$, denoted $L_1\pi(k,r)$ (simply $L_1\pi$) or the tuple $(\pi_{k+1}, \pi_{k+2}, \ldots, \pi_{k+r})$, denoted $L_2\pi(k,r)$ (simply $L_2\pi$). Note that there are braids in SB_{k+r}, which are not Lorenz braids, e.g. Δ_{k+r} does not a Lorenz braid in B_{k+r}.

(3.1.7) Example:

The example in figure (3-1a) of a Lorenz braid $\beta(8,6)$ has the Lorenz permutation

$$\pi(8,6) = (1,3,4,5,8,9,13,14,2,6,7,10,11,12), \text{ with } S(\pi) = \{8\}$$

hence

$$L_1\pi(8,6) = (1,3,4,5,8,9,13,14) \text{ and } L_2\pi(8,6) = (2,6,7,10,11,12)$$

Then by using $L_2\pi$, we can write $\beta(8,6)$ as a braid in B_{14}

$$\beta(8,6) = (\sigma_8\sigma_7 \cdots \sigma_2)(\sigma_9\sigma_8\sigma_7\sigma_6)(\sigma_{10}\sigma_9\sigma_8\sigma_7)\times$$

$$(\sigma_{11}\sigma_{10})(\sigma_{12}\sigma_{11})(\sigma_{13}\sigma_{12}) \ \square$$

Now let $\beta(k,r)$ refer to any Lorenz braid of type (k,r). A specific $\beta(k,r)$ is determined by an associated permutation $\pi(k,r)$ or simply by $L_1\pi$ or $L_2\pi$.

(3.1.8) Lemma:

The Lorenz braid $\beta(k,r)$ with permutation $\pi(k,r)$ has the positive braid representative,

$$\beta(k,r) = [\prod_{i=1}^{r} (\sigma_{k+i-1}\sigma_{k+i-2} \cdots \sigma_{\pi_{k+i}})] \in SB_{k+r}.$$

Proof:

The string from the position k+i at the top of the braid to the position π_{k+i} at its bottom pass over $(k-\pi_{k+i}+i)$ strands, hence by using the permutation $L_2\pi$, shown in figure (3-2), where boxes in the diagram represent some other Lorenz braids, we can write $\beta(k,r)$ as

$$\beta(k,r) = [\prod_{i=1}^{r} (\sigma_{k+i-1}\sigma_{k+i-2} \cdots \sigma_{\pi_{k+i}})] \in SB_{k+r} \quad \square$$

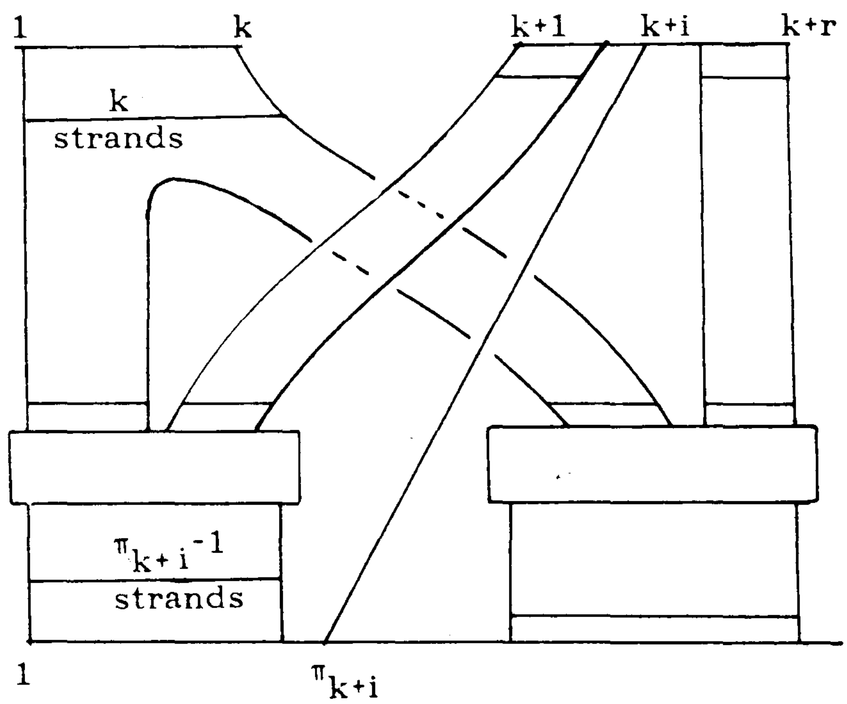


Figure (3-2)

(3.1.9) Definition:

Define the two operators L and R on B(k,r), such that L(β(k,r)), (R(β(k,r)))), is the tying the top of the first, (last), string of the left, (right), hand side to the same position on the bottom of β, then define

$$L^{(i)}(\beta) = L(L^{(i-1)}(\beta)) \text{ and } R^{(i)}(\beta) = R(R^{(i-1)}(\beta)).$$

(3.1.10) Remark:

Let X_i be the Lorenz braid β(1,i) with permutation π(1,i) and $L_1\pi = (\pi_1 = i+1)$ and let Y_i be the Lorenz braid β(i,1) with permutation η(i,1) and $L_2\eta = (\eta_{i+1} = 1)$, then clearly as in figure (3-3) and as braids of (i+1) strands, Y_i is the result of turning over X_i, i.e.

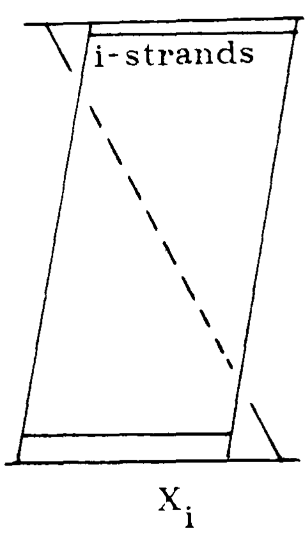
$$Y_i = \tau[X_i] = (\Delta_{i+1})X_i(\Delta_{i+1})^{-1}$$

with

$$X_i = \sigma_1\sigma_2 \ldots \sigma_i, \text{ and } Y_i = \sigma_i\sigma_{i-1} \ldots \sigma_1$$

where,

$$(\Delta_{i+1})^2 = (X_i)^{i+1} = (Y_i)^{i+1}$$



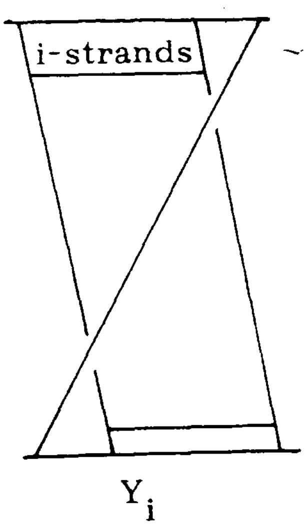


Figure (3-3)

The following lemma provides a technical combinatorial method for representing a Lorenz link by a braid (not a Lorenz braid) in fewer strands. In fact this method is the key to formulate a semicanonical form for a minimal braid representative for a Lorenz link.

(3.1.11) Lemma:

Given a Lorenz braid $\beta(k,r)$ with a permutation $\pi(k,r)$, then

$$\text{(i)}: \quad L(\beta(k,r)) = \begin{cases} O \cup \beta(k-1,r) & \text{when } \pi_1 = 1 \\ X_{\pi_1-2}\beta(k-1,r) & \text{otherwise} \end{cases}$$

and

$$\text{(ii)}: \quad R(\beta(k,r)) = \begin{cases} O \cup \beta(k,r-1) & \text{when } \pi_{k+r} = k+r \\ Y_{(k+r)-\pi_{k+r}-1}\beta(k,r-1) & \text{otherwise} \end{cases}$$

where O is the unknot, X_i is in the first (i+1) strands of the right group of β and Y_i is in the last (i+1) strands of the left group of β, while $\beta(k-1,r)$ and $\beta(k,r-1)$ have permutations in terms of the permutation $\pi(k,r)$.

Proof:

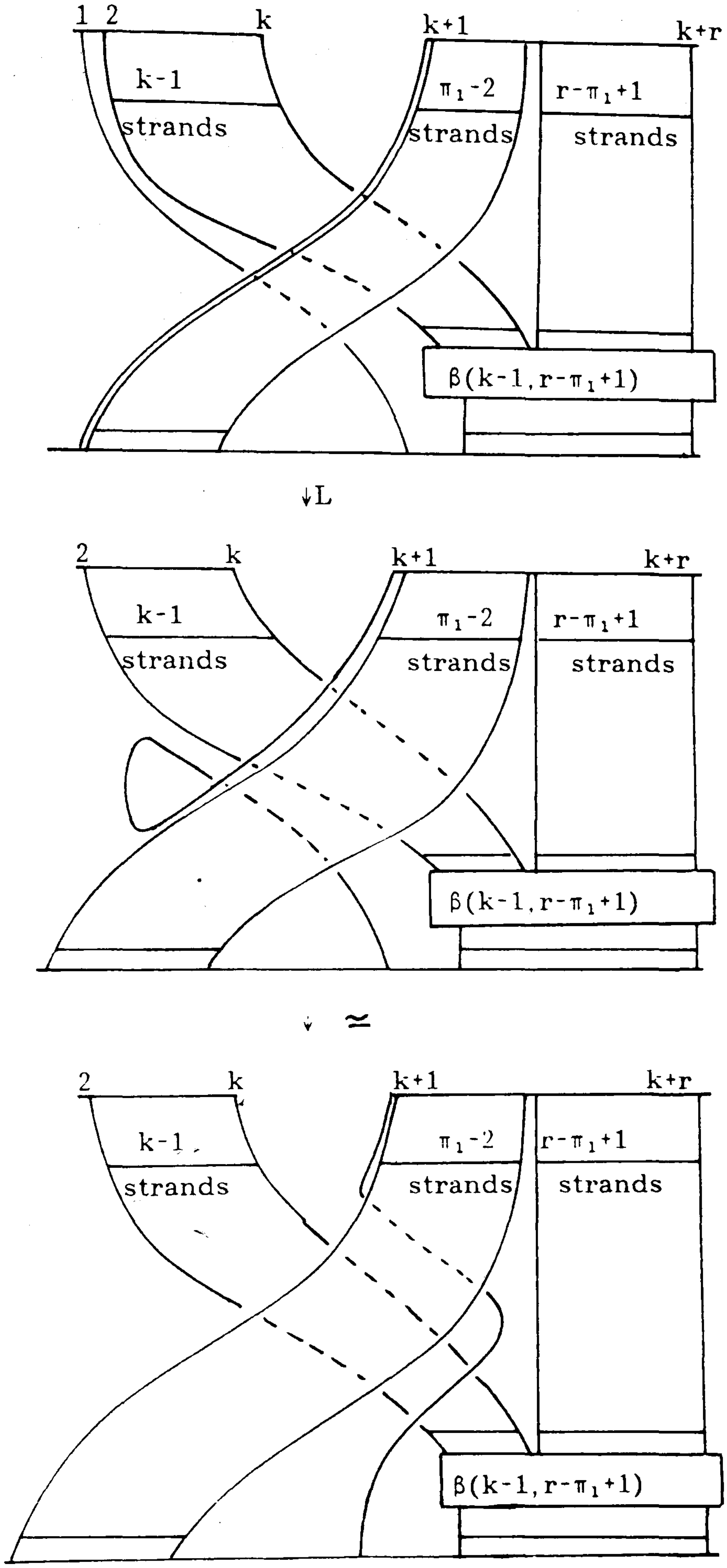
For (i): If $\pi_1 = 1$, then the left-hand string in β has no crossings with the others, hence

$$L(\beta(k,r)) = O \cup \beta(k-1,r)$$

where $\beta(k-1,r)$ has permutation $\pi'(k-1,r)$, such that

$$(\pi')_i = \pi_{i+1} - 1, \quad 1 \leqslant i \leqslant k-1$$

Now let $\pi_1 > 1$, then $\pi_{k+1} = 1$, so the first (π_1-1) strands from the right group of β pass over the left hand string of β. Then by



1 2 k k+1 k+r
k-1 strands
π1-2 strands
r-π1+1 strands
β(k-1, r-π1+1)
↓L
2 k k+1 k+r
k-1 strands
π1-2 strands
r-π1+1 strands
β(k-1, r-π1+1)
≃
2 k k+1 k+r
k-1 strands
π1-2 strands
r-π1+1 strands
β(k-1, r-π1+1)

Figure (3-4)

using Reidemeister moves, shown in theorem (0.4), we can isotop $L(\beta(k,r))$ to the required braid as in figure (3-4), so

$$L(\beta(k,r)) = X_{\pi_1-2}\beta(k-1,r)$$

where this $\beta(k-1,r)$ has the same permutation π', such that

$$(\pi')_i = \pi_{i+1} - 1, \quad 1 \leqslant i \leqslant k-1$$

But Y_i is the result of turning over X_i, then similarly we can conclude case (ii) □

<u>(3.1.12) Definition</u>:

From the diagram of a Lorenz braid $\beta(k,r)$ with permutation $\pi(k,r)$ we can read a number t, which is the maximum number of t strands in the left-hand side of the right group , which pass over t-strands in the right-hand side of the left group. This number t is called <u>the trip number</u> of $\beta(k,r)$, i.e. $S(\beta) = \{k\}$, so $t = k+1-i$, where i is the least integer $\leqslant k$, with $\pi(i) > k$. The example in figure (3-1a) has trip number equals 3.

<u>(3.1.13) Remark</u>:

Given a Lorenz braid $\beta(k,r)$ with trip number t and permutation $\pi(k,r)$, then $\beta(k,r)$ is the product of three Lorenz braids, shown in figure (3-5), i.e.

$$\beta(k,r) = [\beta(t,t)][\beta(k-t,t)][\beta(t,r-t)]$$

where $\beta(t,t)$ is a Lorenz braid in the last t-strands of the left group and the first t strands of the right group of β with permutation $\chi(t,t)$, such that

$$\chi_i = t+i, \text{ for } k-t+1 \leqslant i \leqslant k$$

$\beta(k-t,t)$ is in the first k strands with permutation $\xi(k-t,t)$, such that

$$\xi_i = \pi_i, \text{ for } 1 \leqslant i \leqslant k-t$$

and $\beta(t,r-t)$ is in the last r strands with permutation $\eta(t,r-t)$, such that,

$$\pi_i = (\eta\chi)_i = \eta(\chi_i) = \eta_{t+i}, \text{ for } k-t+1 \leqslant i \leqslant k$$

also, as in figure (3-5),

$$1 \leqslant \pi_i \leqslant k, \text{ for } 1 \leqslant i \leqslant k-t$$

and

$$k+1 \leqslant \pi_i \leqslant k+r, \text{ for } k+t+1 \leqslant i \leqslant k+r$$

But remark (3.1.6) tells us that,

$$\pi_i \geqslant i, \text{ for } 1 \leqslant i \leqslant k \text{ and } \pi_j \leqslant j, \text{ for } k+1 \leqslant j \leqslant k+r$$

hence

$$\pi^2(i) \geqslant \pi(i), \text{ for } 1 \leqslant i \leqslant k-t$$

and

$$\pi^2(j) \leqslant \pi(j), \text{ for } k+t+1 \leqslant j \leqslant k+r$$

The example in figure (3-1a), can be written as a product of three braid words,

$$\beta(8,6) = [(\sigma_8\sigma_7\sigma_6)(\sigma_9\sigma_8\sigma_7)(\sigma_{10}\sigma_9\sigma_8)] \times$$

$$[(\sigma_5\sigma_4\sigma_3\sigma_2)(\sigma_6)(\sigma_7)] \times$$

$$[(\sigma_{11}\sigma_{10})(\sigma_{12}\sigma_{11})(\sigma_{13}\sigma_{12})]$$

where the last two words commute □

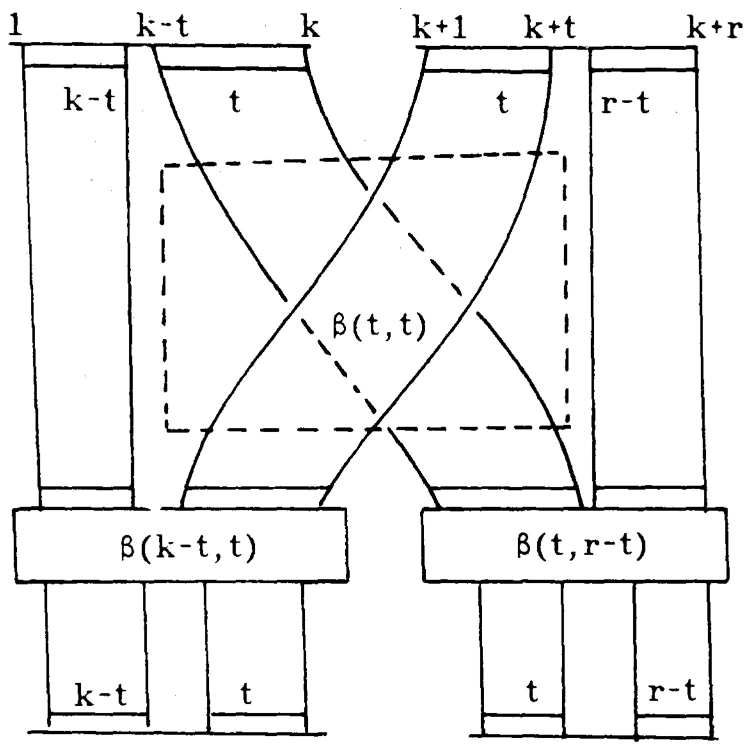


Figure (3-5)

(3.1.14) Corollary:

A Lorenz link $(\beta(k,r))^c$ with permutation $\pi(k,r)$ and trip number t, has a braid representative $[(\Delta_t)^2XY]\in B_t$, provided that $\pi_1 > 1$ and $\pi_{k+r} < k+r$, where

$$X = \prod_{i=1}^{t-1} [(X_i)^{(n_i)}], \qquad Y = \prod_{i=1}^{t-1} [(Y_i)^{(m_i)}],$$

with

$$n_i = \text{card}\{j \mid \pi_j - j - 1 = i,\ \pi^2(j) > \pi(j)\}$$

and

$$m_i = \text{card}\{j \mid j - \pi_j - 1 = i,\ \pi^2(j) < \pi(j)\}$$

Proof:

Since $\pi_1 > 1$ and $\pi_{k+r} < k+r$, then no trivial links in $L^{(i)}(\beta)$ and $R^{(i)}(\beta)$ for any i. Illustrate the braid $\beta(k,r)$ as in figure (3-5), where by successive application of lemma (3.1.11), as illustrated in figure (3-6a), yield

$$L(\beta(k-t,t)) = \prod_{i=1}^{k-t} (X_{[\pi_i-(i+1)]}) = X, \text{ say}$$

and

$$R(\beta(t,r-t)) = \prod_{i=1}^{r-t} (Y_{[(k+r-i)-\pi_{(k+r)-(i-1)}]}) = Y, \text{ say}$$

Now for a fixed integer i, as in figure (3-6b), with $1 \leqslant i \leqslant t$, there exist n_i, $\lambda_i \in \mathbb{Z}^+$ such that,

$$1 \leqslant \lambda_i + n_i < k-t$$

$$\pi_{\lambda_i+1} - (\lambda_i+1) = i$$

$$\pi_j + 1 = \pi_{j+1}, \text{ for } \lambda_i + 1 \leqslant j \leqslant \lambda_i + n_i - 1$$

$$\pi_{\lambda_i} + 1 - \pi_{\lambda_i} = a_i + 1 \geqslant 2$$

and

$$\pi_{\lambda_i} + n_i + 1 - \pi_{\lambda_i} + n_i = b_i + 1 \geqslant 2$$

Using remark (3.1.13), where $\pi^2(i) \geqslant \pi(i)$, for $1 \leqslant i \leqslant k-t$, then

$$n_i = \operatorname{card}\{j \mid \pi_j - j - 1 = i,\ \pi^2(j) > \pi(j)\}$$

and

$$\prod_{i=1}^{t-1} (n_i) = (k-t)$$

then

$$\prod_{j=1}^{n_i} [X_{(\pi_{\lambda_i}+j)-(\lambda_i+j)}] = [X_{\pi_{\lambda_i}-(\lambda_i+1)}]^{(n_i)}$$

$$= (X_i)^{(n_i)}$$

So

$$X = \prod_{i=1}^{t-1} [(X_i)^{(n_i)}], \text{ with } n_i \geq 0$$

But Y_i is the result of turning over X_i, then similarly

$$Y = \prod_{i=1}^{t-1} (Y_i)^{(m_i)}, \text{ with } m_i \geq 0$$

where

$$m_i = \mathrm{card}\{j \mid j-\pi_j-1 = i \ , \ \pi^2(j) < \pi(j)\}$$

such that

$$\prod_{i=1}^{t-1} (m_i) = (r-t)$$

The resulting braid is represented diagrammatically as in figure (3-7a) □

(3.1.15) Corollary:

For a Lorenz link $(\beta(k,r))^c$, the trip number is the braid index.

Proof:

Corollary (3.1.14) tells us that the link $(\beta(k,r))^c$ has a twist positive braid representative, then the proof is a direct consequence of corollary (2.1.10), where the closure of a twist positive braid $(\Delta_n)^{2m}Q$ has braid index n, for $m \geq 1$ and for a positive braid Q □

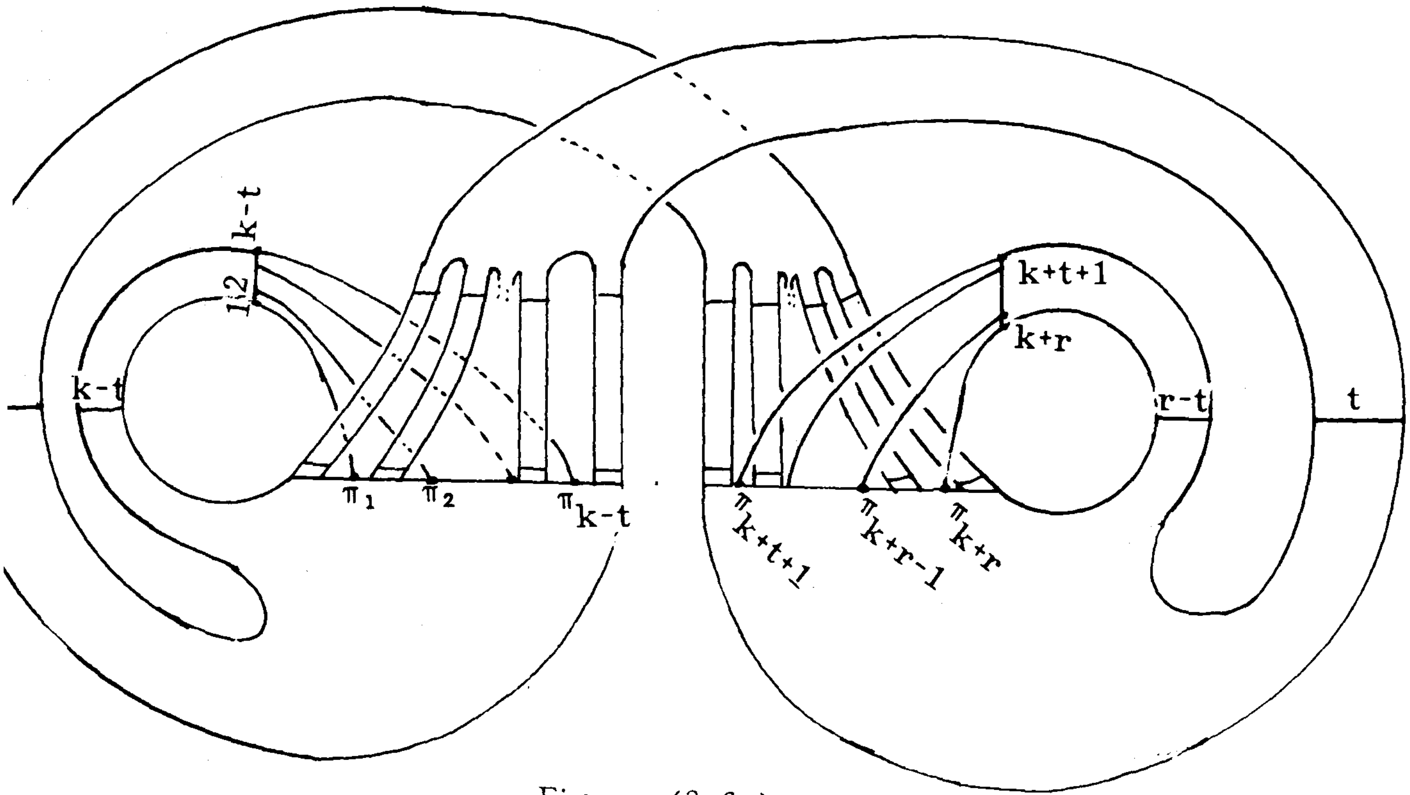


Figure (3-6a)

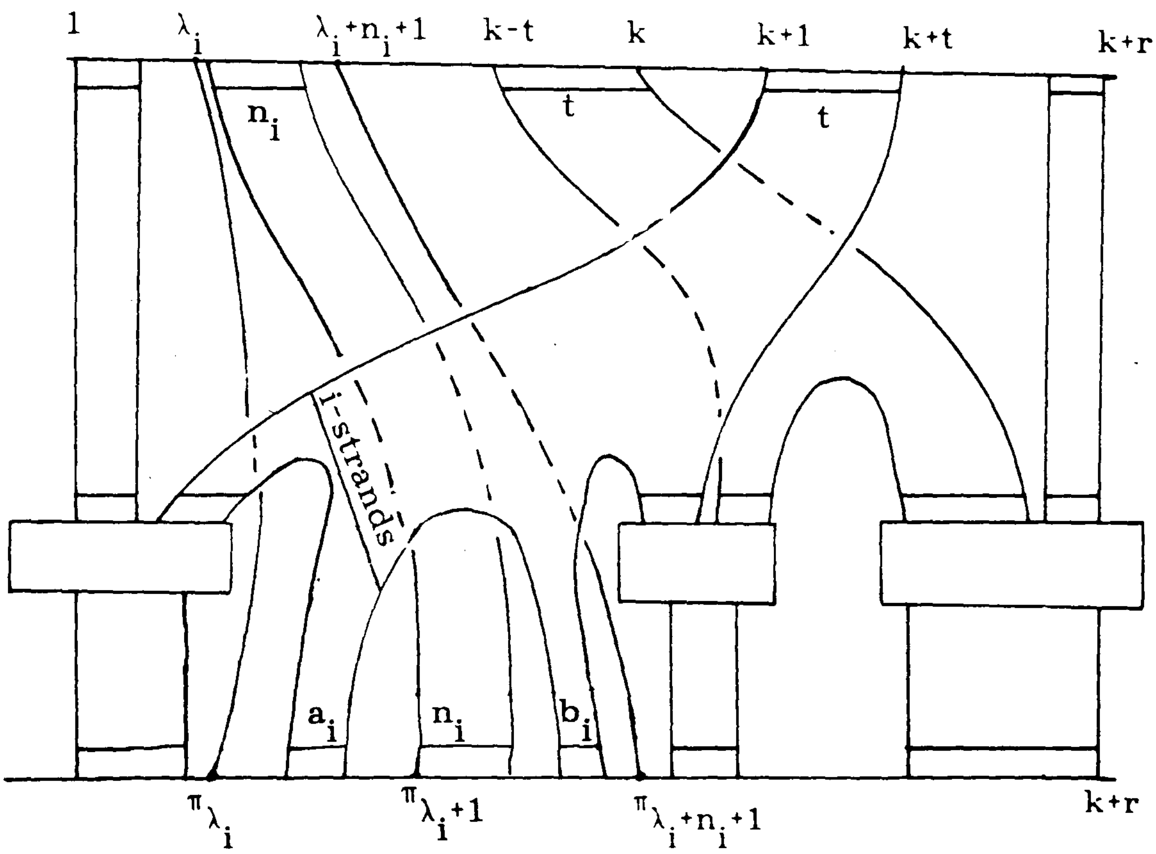


Figure (3-6b)

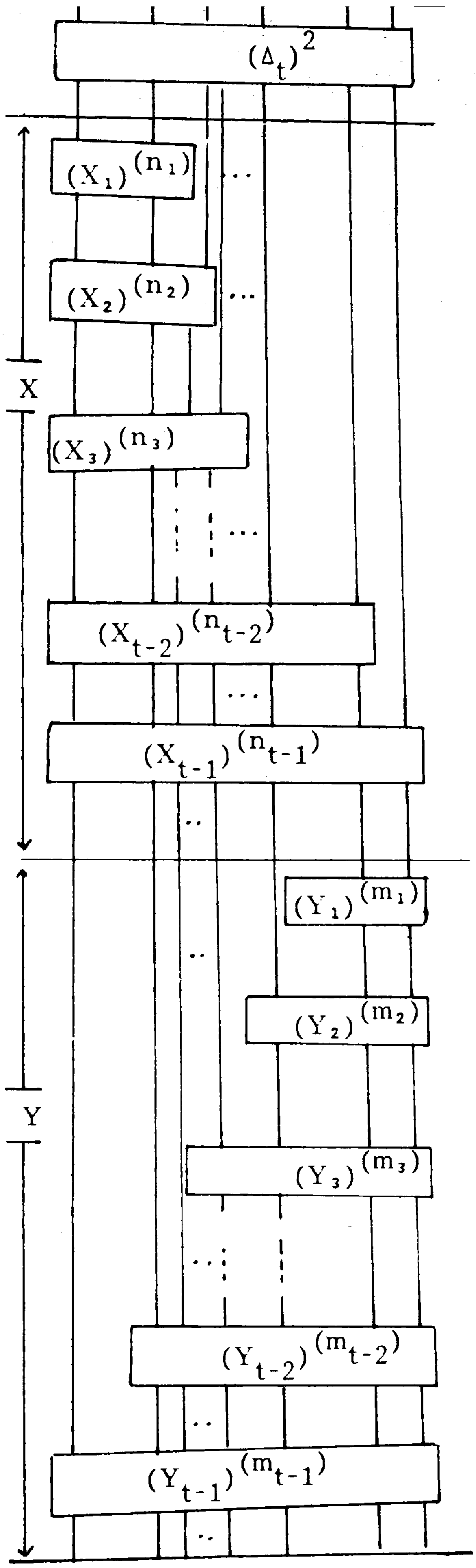


Figure (3-7a)

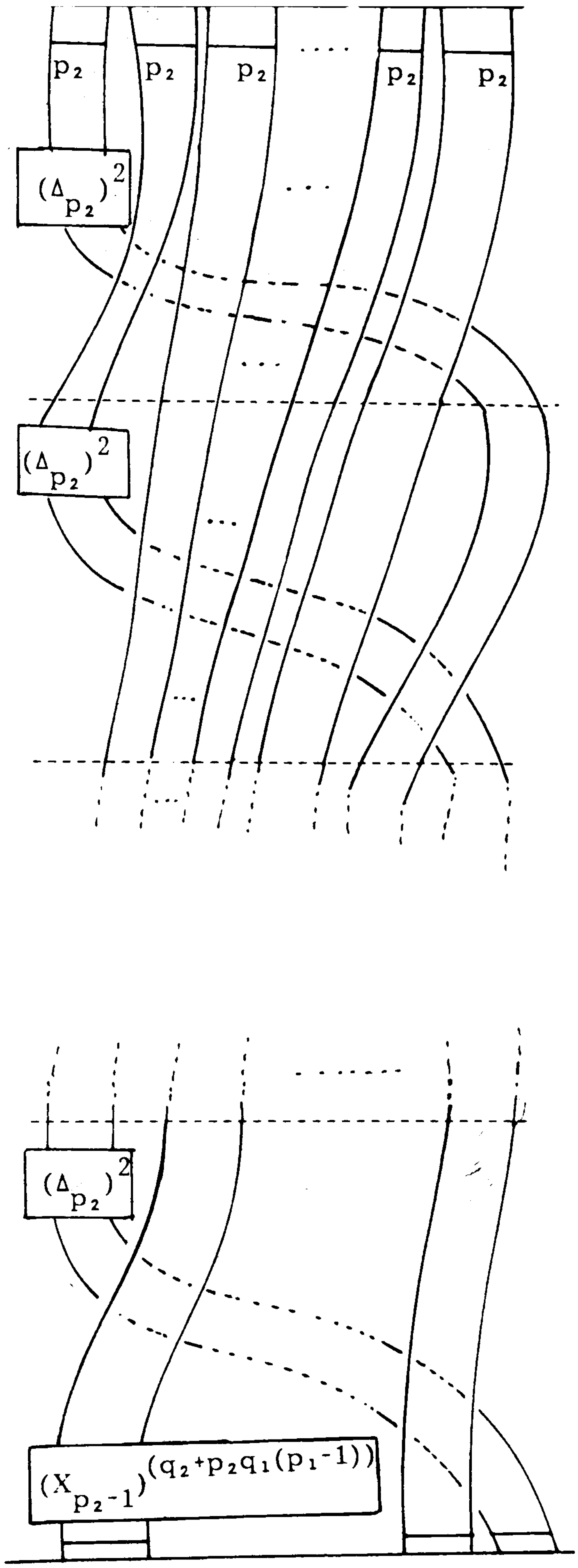


Figure (3-7b)

(3.1.16) Remark:

Given an algebraic knot K, as defined in definition (0.13), then $K \subseteq [(S_\varepsilon^{\,3}) \cap f^{-1}(0)]$, for some complex plane curve $f(x,y)$, with $f(0,0) = 0$, where $(S_\varepsilon^{\,3} = \{(x,y)\in\mathbb{C}^2 \mid |(x,y)| = \varepsilon\}$ for sufficiently small ε and K can be described by the fractional power series as in equation (0.1). Now consider the first approximation to equation (0.1), i.e. let

$$y = a_1 x^{(q_1/p_1)} \tag{3.1.1}$$

then take $x = \varepsilon t^{\theta}$, where t runs once around the complex unit circle $S^1 \subset f^{-1}(0)$, so y is a constant times $t^{(q_1)}$ and (x,y) runs p_1 times around in the longitudinal direction $S_\varepsilon^{\,3}$ (the x-axis) while running q_1 times around the meridianal direction of $S_\varepsilon^{\,3}$ (the y-axis). Hence the first approximation is the (p_1,q_1) torus knot K_1. Therefore K_1 is the closure of the p_1-braid $[\beta_1 = (X_{p_1-1})^{(q_1)}]$. Again consider the second approximation to the equation (0.1), i.e. let

$$y = x^{(q_1/p_1)}(a_1 + a_2 x^{(q_2/p_2)}) \tag{3.1.2}$$

then change the parameterisation to put $x = \varepsilon t^{(p_1 p_2)}$, so x will follow K_1 around p_2 times in a longitudinal direction in $S_\varepsilon^{\,3}$ (the x-axis). Hence the second approximation knot K_2 is the (p_2,a_2) cable on K_1, for some integer a_2. Continuing this process then the knot K represented by equation (0.1) is the (p_s,a_s) cable on the (p_{s-1},a_{s-1}) cable on the ... (p_1,a_1) cable on the unknot, for suitable integers $a_1, a_2, \ldots, a_s$. It is known that ([E-N], proposition 1A.1, page 51),

$$a_{i+1} = q_{i+1} + p_i p_{i+1} a_i, \text{ for } i \geq 1 \text{ and } a_1 = q_1 \tag{3.1.3}$$

But the pair (p_i, q_i) is relatively prime, then the pair (p_i, a_i) is also relatively prime and

$$a_{i+1} > p_i p_{i+1} a_i, \text{ for } i \geq 1, \tag{3.1.4}$$

Now back again to the case in equation (3.1.2) above, which can be represented by using equation (3.1.3) as a $(p_2, q_2+p_1p_2q_1)$ cable over (p_1, q_1), then K_2 is the closure of the braid β_2 illustrated in figure (3-7b), which can be written as

$$\beta_2 = [(X_{p_2-1})^{(q_2+p_2q_1(p_1-1))}][(X_{p_1p_2-1})^{(p_2q_1)}]$$

But we can start with $q_1 > p_1$, because the (p,q) torus knot is unchanged by interchanging p and q. Continuing the previous process we can see that the arbitrary algebraic knot, which represented by equation (0.1), has the n-braid representative

$$(X_{m_1-1})^{(k_1)}(X_{m_1m_2-1})^{(k_2)} \ldots (X_{m_1m_2\ldots m_r-1})^{(k_r)} \tag{3.1.5}$$

where

$$k_1 > k_2 > \ldots > k_r > n$$

and

$$k_i > m_i, \text{ for } 1 \leq i \leq r$$

such that

$$k_i = m_i d_i \text{ and } k_i = c_i k_{i-1}, \text{ for } 2 \leq i \leq r$$

for some integers c_i, d_i. Now let $L = K \cup K'$ be an algebraic link with two components, then L corresponds to two distinct equations such as in equation (0.1). Let y and y' be the first approximation of L, where y as in equation (3.1.1) above and y' equals y by replacing (p_1, q_1) by (p_1', q_1'), then consider the two different cases:

If $(p_1,q_1) = (p_1',q_1')$, take $x = \varepsilon t^{\theta}$, as above, then L represented by two parallel strands run p_1 times around in the longitudinal direction in S_ε^3 while q_1 times around the meridianal direction of S_ε^3. Therefore the two components K and K' have linking number q_1 (which gives the feature of the fact that every algebraic link is determined by the isotopic type of each component and the linking numbers of each pair of components, [E-N]), then L is the closure of the $(2p_1)$-braid $(X_{2p_1-1})^{(q_1)}$.

But if $(p_1,q_1) \neq (p_1',q_1')$, then L is a splitable link of two components (p_1,q_1) and (p_1',q_1') torus knots. Therefore L can be represented as a closure of the (p_1+p_2)-braid $[(X_{p_1-1})^{(q_1)}(Y_{p_1'-1})^{(q_1')}]$. Then by repeating the same construction adopted for the successive approximations to K used above, we can see that every algebraic link of two components has a braid representative of pattern such as in figure (3-7a) with some restrictions on the powers (n_i)'s and (m_i)'s. Therefore we can conclude the following result:

<u>(3.1.17) Proposition</u>:

Every algebraic link with two (or one) components is a Lorenz link.

<u>(3.1.18) Proposition</u>:

A knot is algebraic if and only if it is the closure of the braid in equation (3.1.5).

<u>Proof</u>:

The necessity is established in remark (3.1.16). To establish the sufficiency it is quite enough to check that the given knot K is (m_1,b_1) cable on (m_2,b_2) cable on ... (m_r,b_r) cable on the unknot, such that b_i's (replaced by a_i's) satisfy the equation (3.1.3). Now by sketching a diagram, simply as in figure (3-7b), we can see that

$$(X_{m_1 m_2 \ldots m_r} - 1)^{(k_r)} = (X_{m_1 m_2 \ldots m_r} - 1)^{(m_1 d_1)}$$

$$= [(X_{m_1 - 1})^{(m_1)} \; Z_{m_1, m_1(m_2 m_3 \ldots m_r - 1)}]^{(d_1)}$$

where $Z_{a,b}$ is the Lorenz braid $\beta(a,b)$, with permutation $\pi(a,b)$ such that $\pi(i) = i+b$, for $1 \leqslant i \leqslant a$. Therefore K is (m_1, b_1) cable on $(m_2 m_3 \ldots m_r, b_2)$ for some integers b_1 and b_2, such that $b_1 = (k_1 + k_2 + \ldots k_r)$ and $b_2 = d_r$. Continuing this process and by induction on r we can check that the given knot satisfies the condition in equation (3.1.3), hence it is an algebraic knot □

(3.1.19) Corollary

The only algebraic knots with minimal braid representatives in B_n, for n prime, is the (n,r) torus knots for any integer r, $n \neq r$.

Proof:

Using proposition (3.1.18) and corollary (3.1.15), we can see that the algebraic knot with minimal braid representative in B_n, for prime n, can be represented as the closure of the braid

$$\alpha = (X_{n-1})^m, \text{ for } m > n$$

because n is prime, so we can not factor it as a product of integers, hence α closes to the (n,m) torus knot, which completes the proof □

(3.1.20) Example

Since every Lorenz link has a braid representative, as illustrated in figure (3-7a), then recalling the construction in remark (3.1.16), one can establish many examples of algebraic links which do not follow the pattern in figure (3-7a), i.e. not Lorenz. e.g. consider the 4-braid

$$\alpha = (X_3)^4 (X_2)^3 (Y_2)^3 (\sigma_2)^3$$

which closes to an algebraic link of 3 components □

(3.1.21) Proposition:

In B_t, $(X_{i+1})^m = (X_i)^m(\sigma_{i+1}\sigma_i \dots \sigma_{i+2-m})$, for $1 \leqslant m \leqslant i+1$ and $1 \leqslant i \leqslant t-2$, hence $(Y_{i+1})^m = (Y_i)^m(\sigma_{t-i-1}\sigma_{t-i} \dots \sigma_{t-i-2+m})$.

Proof:

From remark (3.1.10)

$$X_{i+1} = \sigma_1\sigma_2 \dots \sigma_{i+1} = X_i\sigma_{i+1}$$

then the proposition is true for m = 1. Now refer to the proposition when m = k as $(\text{prop.})_k$. The proof of the general proposition follows by induction on k. For our induction hypothesis we assume that $(\text{prop.})_k$ holds, i.e.

$$(X_{i+1})^k = (X_i)^k(\sigma_{i+1}\sigma_i \dots \sigma_{i+2-k})$$

Then

$$\begin{aligned}(X_{i+1})^{(k+1)} &= (X_{i+1})^k X_{i+1} \\ &= (X_i)^k(\sigma_{i+1}\sigma_i \dots \sigma_{i+2-k})X_{i+1} \\ &= (X_i)^k(\sigma_{i+1}\sigma_i \dots \sigma_{i+2-k})(\sigma_1\sigma_2 \dots \sigma_{i+1})\end{aligned}$$

Now using the braid relator (ii) of definition (0.5), then

$$(X_{i+1})^{k+1} = (X_i)^k(\sigma_1\sigma_2 \dots \sigma_j)(\sigma_{i+1}\sigma_i \dots \sigma_{i+2-k})(\sigma_{j+1}\sigma_{j+2} \dots \sigma_{i+1}).$$

where

$$i - k = j$$

Now take

$$\eta = (\sigma_{i+1}\sigma_i \cdots \sigma_{j+2})(\sigma_{j+1}\sigma_{j+2} \cdots \sigma_{i+1})$$

$$= (\sigma_{i+1}\sigma_i \cdots \sigma_{j+3})(\sigma_{j+1}\sigma_{j+2}\sigma_{j+1})(\sigma_{j+3}\sigma_{j+4} \cdots \sigma_{i+1})$$

then using the braid relators (i) and (ii) of definition (0.5), we have

$$\eta = \sigma_{j+1}(\sigma_{i+1}\sigma_i \cdots \sigma_{j+4})(\sigma_{j+3}\sigma_{j+2}\sigma_{j+3})(\sigma_{j+4}\sigma_{j+5} \cdots \sigma_{i+1})\sigma_{j+1}$$

$$= \sigma_{j+1}(\sigma_{i+1}\sigma_i \cdots \sigma_{j+4})(\sigma_{j+2}\sigma_{j+3}\sigma_{j+2})(\sigma_{j+4}\sigma_{j+5} \cdots \sigma_{i+1})\sigma_{j+1}$$

then continuing this process we have,

$$\eta = (\sigma_{j+1}\sigma_{j+2} \cdots \sigma_i)(\sigma_{i+1})(\sigma_i\sigma_{i-1} \cdots \sigma_{j+1})$$

But

$$j = i - k$$

then

$$(X_{i+1})^{k+1} = (X_i)^k(\sigma_1\sigma_2 \cdots \sigma_j\sigma_{j+1} \cdots \sigma_i)(\sigma_{i+1}\sigma_i\sigma_{i-1} \cdots \sigma_{i+1-k})$$

$$= (X_i)^{k+1}(\sigma_{i+1}\sigma_i\sigma_{i-1} \cdots \sigma_{i+1-k})$$

which completes the proof of $(\text{prop.})_{k+1}$, hence completes the proof of the general proposition □

<u>(3.1.22) Theorem</u>:

Every Lorenz link of trip number t has a semicanonical form for its minimal braid representative in B_t. More precisely:

The Lorenz link $(\beta(k,r))^c$ of trip number t and a permutation $\pi(k,r)$ has the minimal representative,

$$\gamma = (\Delta_t)^{(2p_0 + 2q_0 + 2)} \cdot [\prod_{i=1}^{t-2} (\Delta_{t-i})^{(2p_i)}](\alpha)[\prod_{i=1}^{t-2} (\Delta_{t-i,\leftarrow})^{(2q_i)}](\beta)$$

where α, $\beta \in SB_t$ and p_i, $q_i \in \mathbb{Z}^+$ for all $0 \leqslant i \leqslant t-2$, such that $p_0 \neq 0$.

Proof:

Corollary (3.1.14) tells us that the link $(\beta(k,r))^c$ has the minimal braid representative

$$\gamma = (\Delta_t)^2 XY$$

where

$$X = \prod_{i=1}^{t-1} [(X_i)^{(n_i)}], \ n_i \geqslant 0, \text{ for } 1 \leqslant i \leqslant t-1$$

and

$$Y = \prod_{i=1}^{t-1} [(Y_i)^{(m_i)}], \ m_i \geqslant 0, \text{ for } 1 \leqslant i \leqslant t-1$$

Now let

$$n_{t-1} = \varepsilon_{t-1} \bmod(t)$$

i.e.

$$n_{t-1} = tp_0 + \varepsilon_{t-1}, \quad \varepsilon_{t-1} \leqslant t-1 \text{ and } p_0 \geqslant 0$$

then using remark (3.1.6) and proposition (3.1.21)

$$\begin{aligned} (X_{t-1})^{(n_{t-1})} &= (X_{t-1})^{(tp_0)} (X_{t-1})^{(\varepsilon_{t-1})} \\ &= (\Delta_t)^{(2p_0)} (X_{t-1})^{(\varepsilon_{t-1})} \\ &= (\Delta_t)^{(2p_0)} (X_{t-2})^{(\varepsilon_{t-1})} (\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{t-\varepsilon_{t-1}}) \end{aligned}$$

where, as in remark (3.1.10),

$$(\Delta_t)^2 = (X_{t-1})^t$$

But $(\Delta_n)^2$ commutes, shown in corollary (1.1.12), with every word in B_n, so using proposition (3.1.21), we have

$$(X_{t-2})^{(n_{t-2})}(X_{t-1})^{(n_{t-1})}$$

$$= (X_{t-2})^{(n_{t-2})}(\Delta_t)^{(2p_0)}(X_{t-2})^{(\varepsilon_{t-1})}(\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{t-\varepsilon_{t-1}})$$

$$= (\Delta_t)^{(2p_0)}[(X_{t-2})^{(n_{t-2}+\varepsilon_{t-1})}](\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{t-\varepsilon_{t-1}})$$

Repeat the same process again with

$$n_{t-2} + \varepsilon_{t-1} = \varepsilon_{t-2} \bmod(t-1), \text{ i.e.}$$

$$n_{t-2} + \varepsilon_{t-1} = (t-1)p_1 + \varepsilon_{t-2},\ p_1 \geqslant 0 \text{ and } \varepsilon_{t-2} \leqslant t-2$$

then

$$(X_{t-2})^{(n_{t-2})}(X_{t-1})^{(n_{t-1})}$$

$$= (\Delta_t)^{(2p_0)}[(X_{t-2})^{(t-1)p_1}(X_{t-2})^{(\varepsilon_{t-2})}](\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{t-\varepsilon_{t-1}})$$

$$= [(\Delta_t)^{(2p_0)}(\Delta_{t-1})^{(2p_1)}(X_{t-3})^{(\varepsilon_{t-2})}]\times$$

$$[(\sigma_{t-2}\sigma_{t-3} \cdots \sigma_{t-1-\varepsilon_{t-2}})(\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{t-\varepsilon_{t-1}})]$$

Then continuing this process we have

$$X = P\alpha$$

such that

$$P = \prod_{i=0}^{t-2}(\Delta_{t-i})^{(2p_i)}$$

and

$$\alpha = \alpha_1\alpha_2 \ \ldots\ \alpha_{t-1}$$

with

$$\alpha_i = (\sigma_i\sigma_{i-1} \cdots \sigma_{i+1-\varepsilon_i}) \in SB_{i+1} \tag{3.1.6}$$

where

$$n_i + \varepsilon_{i+1} = (i+1)p_{t-(i+1)} + \varepsilon_i \tag{3.1.7}$$

and

$$0 \leqslant \varepsilon_i \leqslant i, \ p_i \geqslant 0$$

for

$$1 \leqslant i \leqslant t-1, \ \varepsilon_t = 0$$

But Y_i is the result of turning over X_i, then

$$Y = Q\beta$$

such that

$$Q = \prod_{i=0}^{t-2} (\Delta_{t-i,\leftarrow})^{(2q_i)}$$

and

$$\beta = \beta_1\beta_2 \ \ldots \ \beta_{t-1}$$

with

$$\beta_i = (\sigma_{t-i}\sigma_{t-i+1} \ \cdots \ \sigma_{t-i-1+\delta_i}) \in SB_{i+1}$$

where

$$m_i + \delta_{i+1} = (i+1)q_{t-(i+1)} + \delta_i \tag{3.1.8}$$

and

$$q_i \geqslant 0, \ 0 \leqslant \delta_i \leqslant t-i$$

for

$$1 \leqslant i \leqslant t-1, \ \delta_t = 0$$

Which completes the proof of the theorem. This semicanonical form for representatives of a Lorenz link is represented diagrammatically in figure (3-8) □

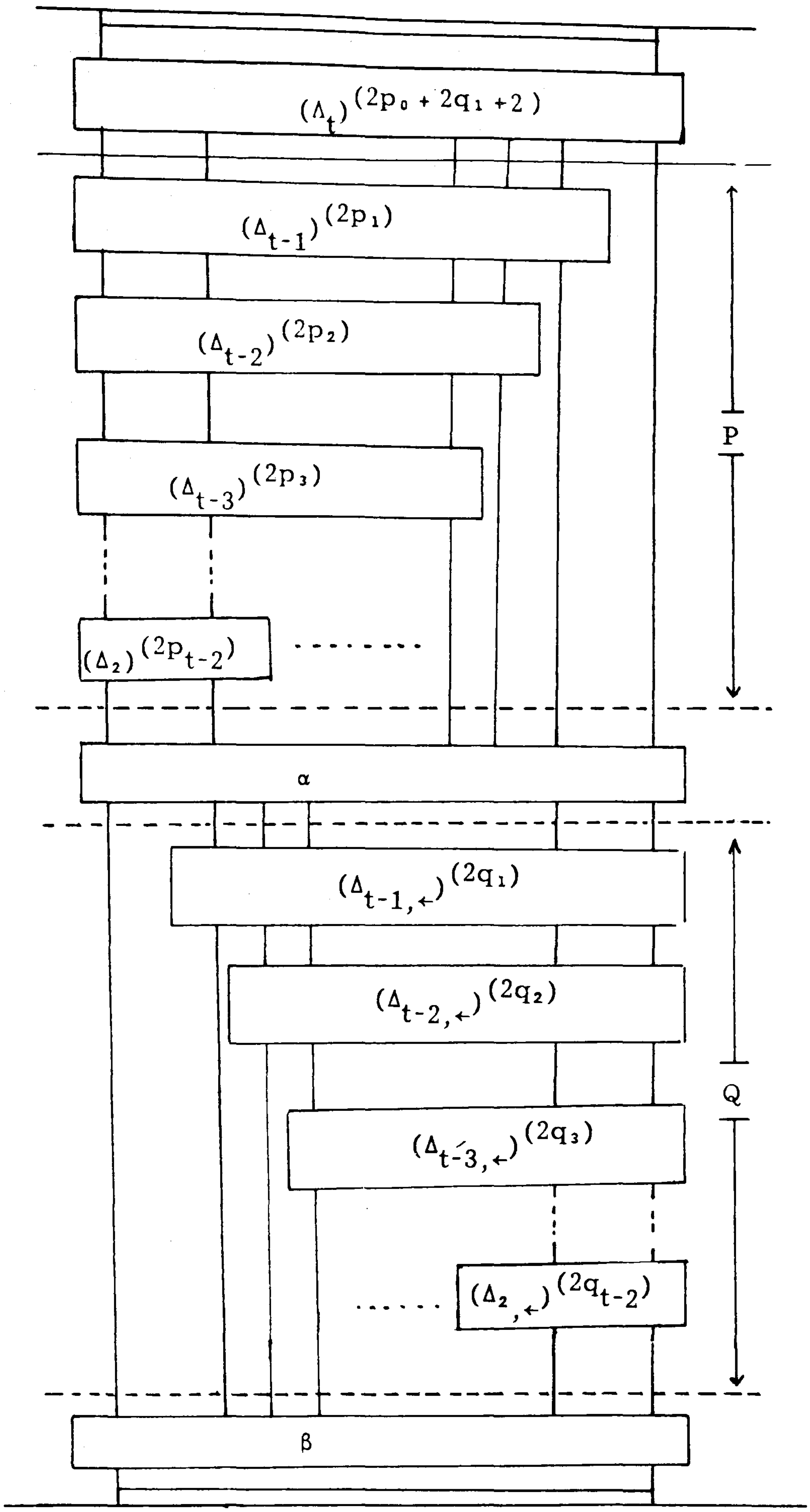

$(\Delta_t)^{(2p_0+2q_1+2)}$
$(\Delta_{t-1})^{(2p_1)}$
$(\Delta_{t-2})^{(2p_2)}$
$(\Delta_{t-3})^{(2p_3)}$
$(\Delta_2)^{(2p_{t-2})}$
P
α
$(\Delta_{t-1,\leftarrow})^{(2q_1)}$
$(\Delta_{t-2,\leftarrow})^{(2q_2)}$
Q
$(\Delta_{t-3,\leftarrow})^{(2q_3)}$
$(\Delta_{2,\leftarrow})^{(2q_{t-2})}$
β

Figure (3-8)

(3.1.23) Corollary

Every algebraic knot has a positive braid representative of the canonical form

$$(\alpha_1)\left(\prod_{i=2}^{t-1}[(\Delta_i)^{(2p_{t-i})-1}(\Delta_i\alpha_i)]\right)(\Delta_t)^{(2p_0)}$$

for positive integer p_i, $0 \leqslant i \leqslant t-2$, and α_i's as in equation (3.1.6).

Proof:

Using proposition (3.1.17) and theorem (3.1.22) we can see that every algebraic knot has a positive braid representative $P\alpha$ as illustrated diagrammatically in figure (3-8). But $(\Delta_i)^2$ commutes with every thing in i strands (or less) and recalling the construction of the canonical form for a positive braid, one can rewrite $P\alpha$ as in the required form, which is in fact a right hand canonical form for the positive braid $P\alpha$ □

(3.1.24) Remark:

Using the recurrence relations in equations (3.1.7) and (3.1.8), we have

$$\varepsilon_{t-j} = \sum_{i=1}^{j}(n_{t-i}) - \sum_{i=0}^{j-1}(t-i)p_i$$

and

$$\delta_{t-j} = \sum_{i=1}^{j}(m_{t-i}) - \sum_{i=0}^{j-1}(t-i)q_i \tag{3.1.9}$$

where n_i, m_i as in corollary (3.1.14), with

$$\sum_{i=1}^{t-1}(n_i) = k-t \text{ and } \sum_{i=1}^{t-1}(m_i) = r-t$$

Hence the two positive permutation braids, in the associated semicanonical form for a Lorenz braid $\beta(k,r)$, are determined by the associated permutation $\beta(k,r)$. In fact theorem (3.1.22) tells us that the number of of components in a Lorenz link is determined by the two positive permutation braids α and β in the associated semicanonical form for its Lorenz braid representative. Then the number of components of a Lorenz link equals the trip number if and only if $\alpha\beta$ has the identity permutation, i.e. $\beta = \rho[\alpha]$, where $\rho[\alpha]$, shown in definition (0.10), has the inverse permutation of the permutation α. Finally equation (3.1.6) tells us that

$$\alpha_i \in \{e,\ \sigma_i,\ \sigma_i\sigma_{i-1},\ \ldots\ ,\ \sigma_i\sigma_{i-1}\ \ldots\ \ \sigma_1\}$$

where α_i has the braid diagram as illustrated in figure (3-9a).

<u>(3.1.25) Lemma</u>:

Every Lorenz link with trip number t and t components, has a semicanonical form $(\Delta_t)^{2p}Q$ for its Lorenz braid representative, such that Q is prime to $(\Delta_t)^2$. More precisely:

Given a Lorenz link $(\beta(k,r))^c$ with permutation $\pi(k,r)$, trip number t and t components, then $\beta(k,r)$ has a semicanonical form $(\Delta_t)^{2p}R$, as in theorem (3.1.22), where either R has two strings with linking number zero or $R = (\Delta_t)^2R'$, R' has two strings with linking number zero.

<u>Proof</u>:

Take $R = P(\alpha)Q(\rho[\alpha])$, where P, Q, α and $\beta = \rho[\alpha]$ as in theorem (3.1.22), then exclude α beginning with σ_1 or ending with σ_{t-1}, otherwise we can extract $(\Delta_2)^2$ or $(\Delta_{2,\leftarrow})^2$ respectively. To exclude the turning over of R, if α and $\tau[\alpha]$ are different, select one of them and if they are equal, arrange P and Q such that,

$$(p_1,p_2, \ldots ,p_{t-2}) > (q_1,q_2, \ldots ,q_{t-2})$$

if and only if the number $p_1p_2 \ldots p_{t-2}$ (considered as a numerically expanded number) is greater than the number $q_1q_2 \ldots q_{t-2}$ (considered as a numerically expanded number). Then consider the following two cases:

Case (1):

$\alpha = \alpha_1 Y_{t-1}$, as in figure(3-9b), then

$$R = P(\alpha_1)Y_{t-1}Q(\rho[Y_{t-1}])(\rho[\alpha_1])$$

$$= P(\alpha_1)Q^*(\rho[\alpha_1])^*(Y_{t-1})(\rho[Y_{t-1}])$$

where

$$Q^*(\sigma_i) = Q(\sigma_{i+1})$$

and

$$(\rho[\alpha_1])^*(\sigma_i) = (\rho[\alpha_1])(\sigma_{i+1})$$

as a braid words (functions) of σ_i, $1 \leqslant i \leqslant t-2$. So if

$$p_1 + q_1 > 0$$

then we can extract $(\Delta_t)^2$ to finish with

$$R = (\Delta_t)^2 R', \quad R' = P'\alpha_1 Q'(\rho[\alpha_1])'$$

where either

$$P' = P, \ (q')_1 = q_1 - 1$$

and

$$q_i = (q')_i, \ 2 \leqslant i \leqslant t-2$$

or

$$Q = Q', \ (p')_1 = p_1 - 1$$

and

$$p_i = (p')_i \ , 1 \leqslant i \leqslant t-2$$

So that the last string in R' does not link any thing. But if

$$p_1 = q_1 = 0$$

then the largest full twist in P and Q is in (t-2) strands, hence

$$R = R'(Y_{t-1}\rho[Y_{t-1}])$$

R' is the end of a semicanonical form in (t-1) strands. So by induction either there exist two strings in R' with zero crossing, hence is too in R, so R is prime to $(\Delta_t)^2$ or

$$R' = (\Delta_{t-1})^2 R''$$

where R" has two strings with linking number zero, then

$$R = (\Delta_t)^2 R''$$

Case (2):

The corner strings in α are different, so

$$\alpha = \alpha_1 Y_i \alpha_2 Y_{[t-(i+k+2)],\leftarrow}$$

where α_1 and α_2 are positive permutation braids in the first (i-1) strands and last (t-j) strands respectively, as in figure(3-9c). So the corner strings does not cross in α, hence they do not cross in R, then R is prime to $(\Delta_t)^2$, which completes the proof □

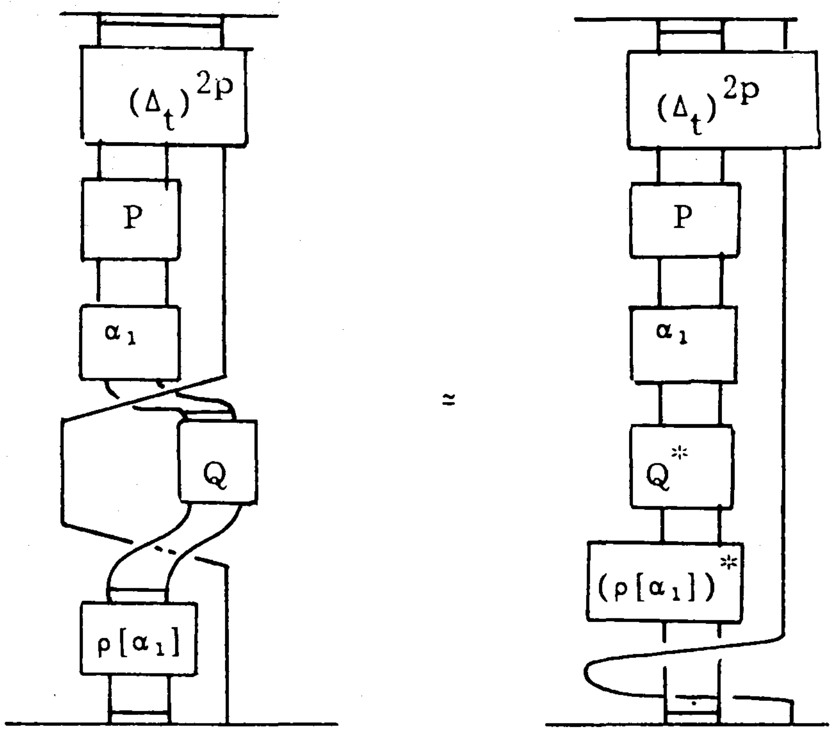


Figure (3-9b)

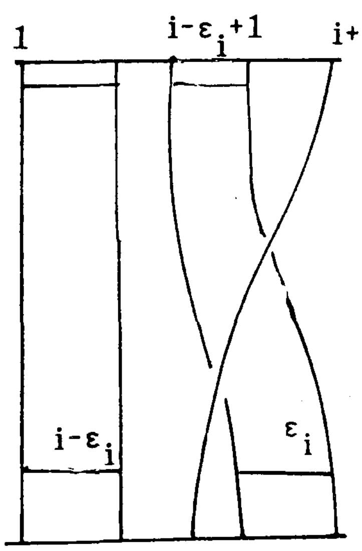


Figure (3-9a)

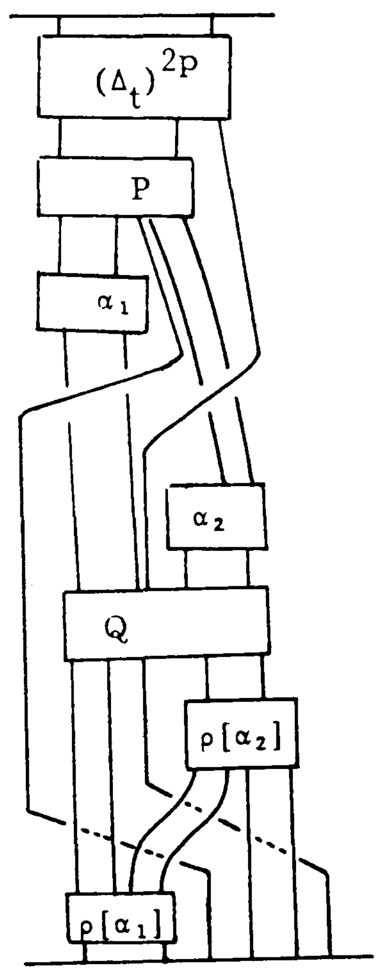
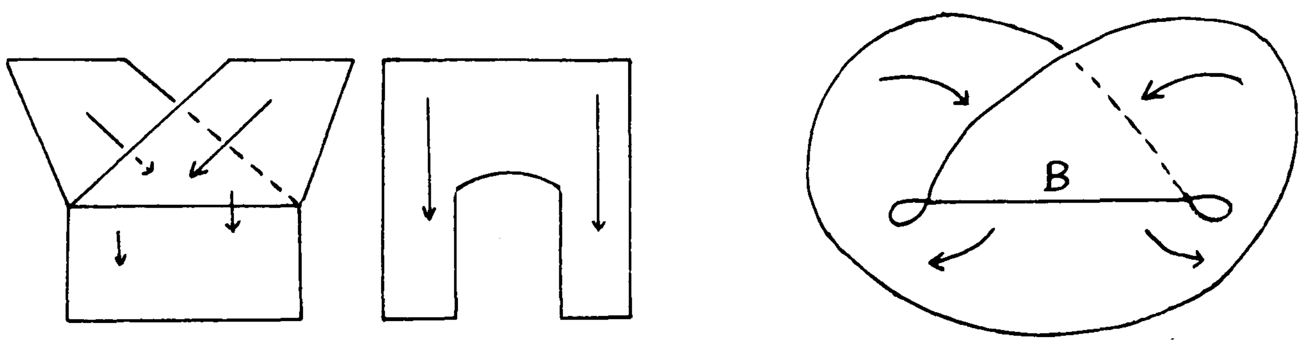


Figure (3-9c)

§3.2 ON LORENZ KNOTS AND LINKS WHICH ARE SATELLITES

(3.2.1) Definition: (Lorenz knot holder), [B-W2]

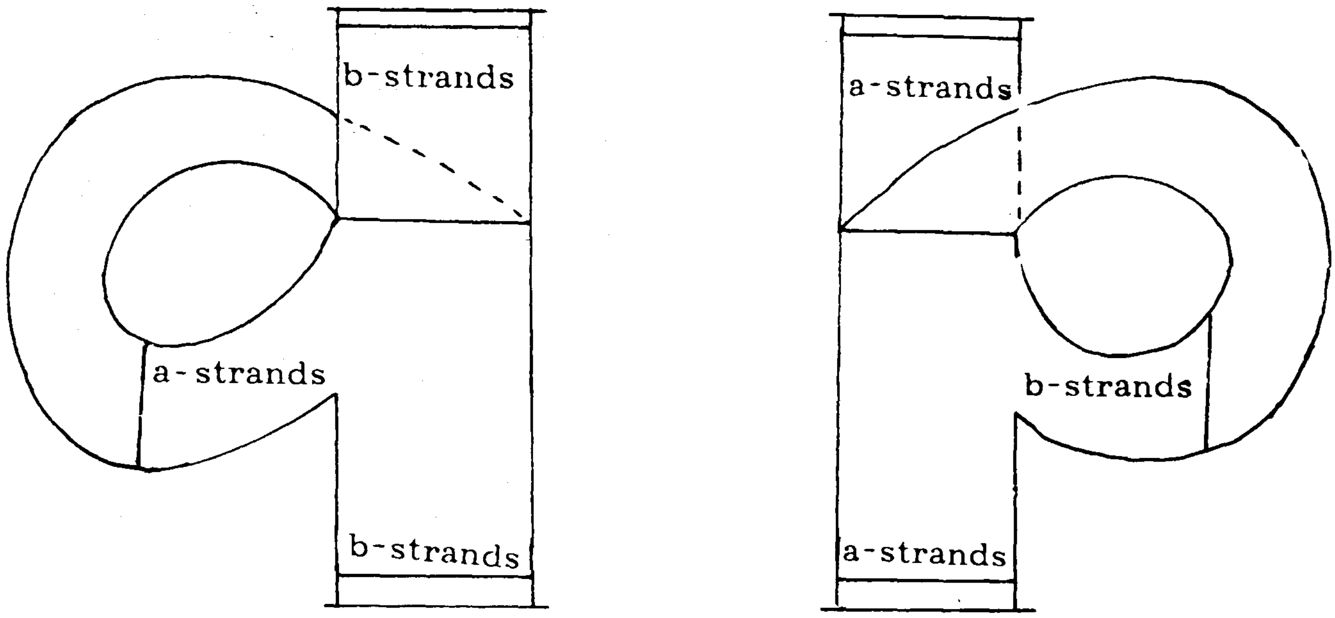
The Lorenz Knot holder is a branched 2-manifold H with boundary, in S^3, consisting of one "joining" and one "splitting" charts put together, as in figure (3-10), by sewing each bottom to exactly one top and vice versa. The joining chart has the defect that flow lines come together along the branch line B, likewise the flow leaves splitting chart at the bottom.



joining chart   splitting chart   Lorenz knot holder

Figure (3-10)

Now given a Lorenz link C with Lorenz braid $\beta(a,b)$, then using the combinatorial method in lemma (3.1.11), we can represent C as closures of b-braid $[L^{(a)}(\beta(a,b))] = B$ (say) and a-braid $[R^{(b)}(\beta(a,b))] = A$ (say), as in figure (3-11), where $L(\beta(a,b))$ is the tying the top of the first string of the left hand side to the same position on the bottom of $\beta(a,b)$, with $L^{(i)}(\beta(a,b)) = L(L^{(i-1)}(\beta(a,b)))$, as defined in definition (3.1.9).



$L^{(a)}(\beta(a,b))$ $R^{(b)}(\beta(a,b))$

Figure (3-11)

(3.2.2) Proposition:

For every Lorenz knot C the satellite constructed with pattern as a closed braid α^c, for

$$\alpha = (\Delta_r)^{2k}[L^{(a)}(\beta(a,r))][R^{(b)}(\beta(r,b))]$$

is again a Lorenz knot, where k = crossing number of C, for positive integers a, b and r.

Proof:

Modify the Lorenz knot constructed by running r-parallel strands around C in the knot holder H (of C) and including $L^{(a)}(\beta(a,r))$, $R^{(b)}(\beta(r,b))$, (for some Lorenz braids $\beta(a,r)$ and $\beta(r,b)$), at the ends of the branch line of H. Then the resulting knot K is again a Lorenz knot, as in figure (3-12a), so K can be represented by braid with pattern as in figure (3-12b), which completes the proof □

(3.2.3) Remark:

Note that the pattern given in proposition (3.2.2) is a closed r-braid α^c, where $\alpha = (\Delta_r)^{2k}AB$, with

$$A = (X_1)^{(a_1)}(X_2)^{(a_2)} \ldots (X_{r-1})^{(a_{r-1})}$$

and

$$B = (Y_1)^{(b_1)}(Y_2)^{(b_2)} \ldots (Y_{r-1})^{(b_{r-1})}$$

for positive integers a_i, b_i, for $1 \leqslant i \leqslant r-1$, as shown in corollary (3.1.14). Then the case with $A = (X_i)^a$ and $B = (Y_i)^b$ gives a cable about C, so algebraic knots are built up successively, starting from the case when C is a torus knot. It is likely to say that the satellites of Lorenz knots can only constructed by the pattern in proposition (3.2.2), although attempts to prove it using an extension of Williams methods, [W2], have so far been unsuccessful.

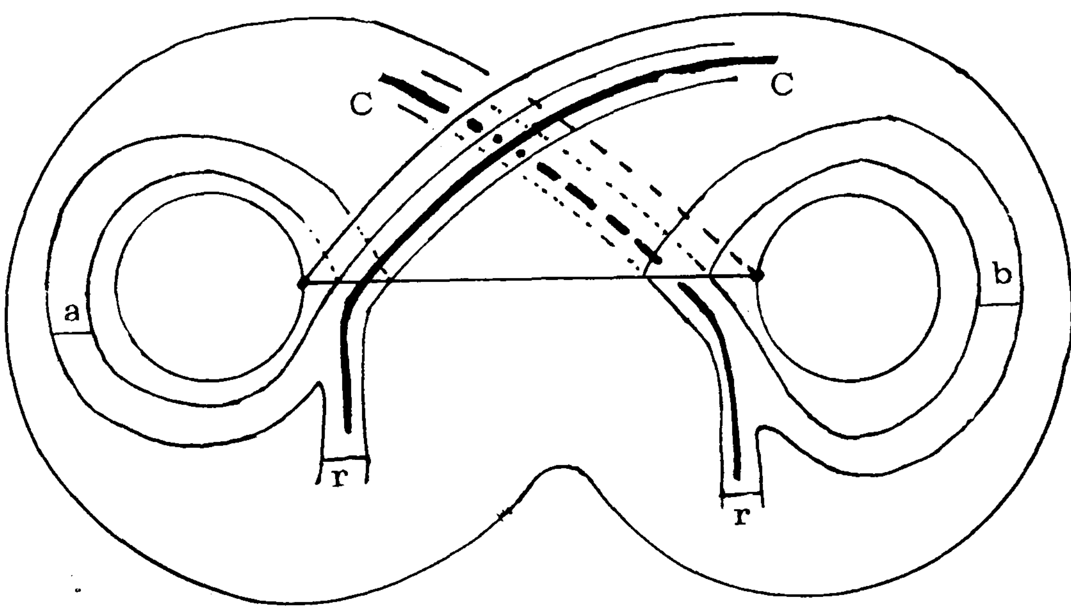


Figure (3-12a)

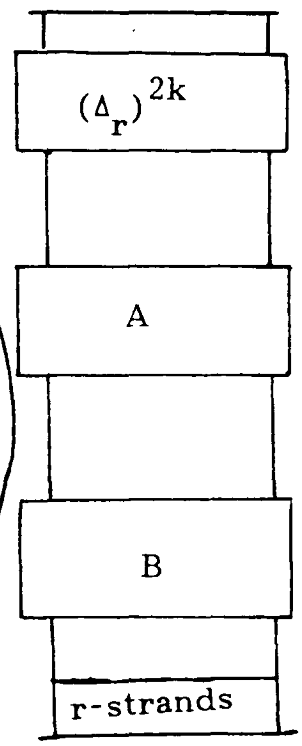


Figure (3-12b)

CHAPTER 4

ON LORENZ LINKS OF TRIP NUMBER $\leqslant$ 4 WITH THEIR ASSOCIATED LINKING PATTERNS

§4.0. INTRODUCTION :

Every algebraic knot is a Lorenz knot and every algebraic link of two components is also a Lorenz link, as shown early in proposition (3.1.17). Then in this chapter it is followed on with the properties of the algebraic knots and links, hence it is compared with those Lorenz knots and links. The algebraic knot is determined by its Alexander polynomial, [Y]. But this is not generally true even for class of closed twist positive braids, where algebraic knots belong to, [Mo5]. Algebraic link is also determined by its associated linking pattern and the isotopy type of each component, [Y] and [E-N]. But this does not hold (in general) for Lorenz links. An example of two non isotopic Lorenz links with some knotted components is given by H.Morton, as in (4.1.3). The central theme of this chapter is the study of the following conjecture:

(4.0.1) Conjecture: [Morton.H]

A Lorenz link L of unknotted components is determined by its associated pattern of the linking numbers.

To some extent an affirmative answer of the conjecture cited above is given. It is proved that the conjecture holds for those Lorenz links of trip number (braid index) $\leqslant$ 4.

Section 1 is devoted to the study of conjecture (4.0.1) for Lorenz links of trip number 3. It is proved, in theorem (4.1.4), that the conjecture holds in B_3. Furthermore it is proved, in theorem (4.1.2), that the 3-braid representatives for a Lorenz link of trip number 3 lie in one conjugacy class. A coplete list of 3-braids which close to Lorenz knots and links is given, as in lemma (4.1.1). Moreovor it is shown that the reduced Alexander polynomial $(\nabla_L(t))^{\sim}$ (for a Lorenz link L of trip number 3) determines a unique braid representative for L and so determines L itself. The reduced Alexander polynomial for any Lorenz link of trip number 3 is also calculated, as in theorem (4.1.2).

Section 2 is devoted to the study of conjecture (4.0.1) for Lorenz links of trip number 4. It is proved that conjecture (4.0.1) holds in B_4. A complete list of 4-braids which close to Lorenz links of 4 components is given. It is defined a six mutually disjoint subsets Ω_i of 4-braid representatives (each of which consists of non-conjugate braids) for all Lorenz links of 4 components, as in proposition (4.2.1). Following that it is proved that $\{(\Omega_i)^c \mid 1 \leqslant i \leqslant 6\}$ (the set of all closures of braids in $\{\Omega_i \mid 1 \leqslant i \leqslant 6\}$) represent different link types and they are determined by their associated linking patterns as in theorem (4.2.2). Furthermore the linking pattern of a Lorenz link, of trip number 4 with 4 components, determines a unique 4-braid representative for L (the braids $\{\Omega_i \mid 1 \leqslant i \leqslant 6\}$ and so determines L.

§4.1. LORENZ KNOTS AND LINKS OF TRIP NUMBER 3

(4.1.1) Lemma:

Every Lorenz link of trip number 3 has a minimal braid representative $\gamma\in\Omega = \{(\Delta_3)^{2k}(\sigma_1)^n(\sigma_2)^m | k,n,m\in\mathbb{Z}^+,\ k\geq 1 \text{ and } n\geq m\} \subset B_3$, where Ω has no two conjugate elements.

Proof:

Theorem (3.1.22) tells us that, every Lorenz link of trip number 3 has a minimal braid representative,

$$\gamma = (\Delta_3)^{2p}(\sigma_1)^{2p_1}\alpha(\sigma_2)^{2q_1}\beta$$

where α, $\beta\in SB_3$ p, p_1, $q_1\in\mathbb{Z}^+$, such that $p \geq 1$ and

$$\alpha,\ \beta\in SB_3 = \{e,\ \sigma_1,\ \sigma_2,\ \sigma_1\sigma_2,\ \sigma_2\sigma_1,\ \sigma_1\sigma_2\sigma_1\}$$

as illustrated in figure (1-8a). Hence, up to conjugacy,

$$\gamma = (\Delta_3)^{2k}(\sigma_1)^n(\sigma_2)^m \qquad (4.1.1)$$

where $k,n,m\in\mathbb{Z}^+$, $k \geq 1$ and $n \geq m$. The class Ω has no two conjugate elements, because the conjugation of braids in equation (4.1.1) is simply the cycling of the factors $(\sigma_1)^n$, and $(\sigma_2)^m$, since $(\Delta_3)^2$ commutes with every thing □

(4.1.2) Theorem:

Let L be a Lorenz link of trip number 3, then the 3-braid representatives of L lie in one conjugacy class and the word γ in equation (4.1.1) is the conjugacy representative of its class. Moreover the reduced Alexander polynomial $(\nabla_L(t))^\sim$ for a Lorenz link $L \simeq \gamma^c$ determines γ and so determines L itself, where

$$(1+t)^2(1+t+t^2)(\nabla_L(t))^\sim =$$

$$(1+t)^2[1+(-t)^{6k+n+m}]-t^{3k+1}[1+(-t)^{n+m}] -t^{3k}(1+t+t^2)[(-t)^n+(-t)^m].$$

Proof:

The reduced Burau matrix B(t) of the braid β is the image of β under the reduced Burau representation Φ: $B_n \to GL(n-1,\mathbb{Z}[t,t^{-1}])$, [B2]. In this presentation,

$$\Phi(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}$$

Then

$$\Phi((\sigma_1)^n) = \begin{bmatrix} (-t)^n & \sum_{i=0}^{n-1} (-t)^i \\ 0 & 1 \end{bmatrix}$$

and

$$\Phi((\sigma_2)^{(m)}) = \begin{bmatrix} 1 & 0 \\ t\sum_{i=0}^{m-1} (-t)^i & (-t)^m \end{bmatrix}$$

But

$$\Phi((\Delta_3)^{2k}) = t^{3k}\, I_{2\times 2}$$

then the braid word,

$$\gamma = [(\Delta_3)^{2k}(\sigma_1)^n(\sigma_2)^m]\in\Omega$$

has the Burau matrix

$$B(t) = t^{3k} \begin{bmatrix} (-t)^n + t[\sum_{i=0}^{n-1} (-t)^i][\sum_{i=0}^{m-1} (-t)^i] & (-t)^m[\sum_{i=0}^{n-1} (-t)^i] \\ t[\sum_{i=0}^{m-1} (-t)^i] & (-t)^m \end{bmatrix}$$

So

$$\det(B(t)) = t^{6k}(-t)^{n+m} = (-t)^{6k+n+m}$$

and

$$\operatorname{trace}(B(t)) = t^{3k}\{(-t)^n + (-t)^m + t[\sum_{i=0}^{n-1} (-t)^i][\sum_{i=0}^{m-1} (-t)^i]\}$$

But

$$(1+t+t^2)[\nabla_L(t)]^{\sim} = 1 - \operatorname{trace}(B(t)) + \det(B(t))$$

where $(\nabla_L(t))^{\sim}$ is the reduced Alexander polynomial for the link $\gamma^c \simeq L$, [B2]. Now consider the following cases, according to the number of components of γ^c:

Case(1): If $\gamma^c \simeq$. K is a knot :

Then both n and m are odd integers and $(\nabla_K(t))^{\sim} = \Delta_K(t)$. Hence

$$(1+t+t^2)\Delta_K(t) = 1+t^{3k}[t^n+t^m-t((t^n+1)(t^m+1)/(t+1)^2)] + t^{6k+n+m}$$

Then

$$(1+t)^2(1+t+t^2)\ \Delta_K(t)$$

$$= (1+t)^2+(t^{3k})^2[t^{n+m}(t+1)^2] + t^{3k}[(1+t+t^2)(t^n+t^m)-t(t^{n+m}+1)]$$

$$= (1+t)^2(1+t^{6k+n+m}) - t^{3k+1}(t^{n+m}+1) + t^{3k}(1+t+t^2)(t^n+t^m)$$

Therefore given a Lorenz knot K of trip number 3 and given its Alexander polynomial $\Delta_K(t)$, then find the polynomial

$$f_K(t) = (1+t)^2(1+t+t^2)\Delta_K(t)$$

$$= (1+t)^2(1+t^a) + t^b(1+t+t^2)(t^n+t^m) - t^{b+1}(t^{a-2b}+1)$$

where $(a+2)$ is the largest exponent of t in $f_K(t)$ and $n \geqslant m \geqslant 0$. Then $a = 2b+n+m$, $b = 3k$ and K has the braid representative,

$$[(\Delta_3)^{2(b/3)}(\sigma_1)^n(\sigma_2)^m] \in \Omega,$$

where n, m and $b = 3k \in \mathbb{Z}^+$, $n \geqslant m$ and $k \geqslant 1$.

Case(2): If $\gamma^c \simeq L$ is a link of two components :

So let n be even and m be odd, then $n > m \geqslant 0$. Hence

$$(1+t+t^2)(\nabla_L(t))^\sim$$

$$= 1 - t^{3k}\{t^n - t^m + t[1-t(\sum_{i=0}^{n-2}(-t)^i)][\sum_{i=0}^{m-1}(-t)^i]\} - t^{6k+n+m}$$

$$= 1 - t^{6k+n+m} - t^{3k}\{t^n - t^m + t[1-t((t^{n-1}+1)/(t+1))][(t^m+1)/(t+1)]\}$$

Then

$$(1+t)^2(1+t+t^2)(\nabla_L(t))^\sim$$

$$= (1+t)^2(1-t^{6k+n+m}) - t^{3k+1}(1-t^{n+m}) - t^{3k}(1+t+t^2)(t^n-t^m)$$

Therefore given a Lorenz link L of trip number 3 with two components. and reduced Alexander polynomial $(\nabla_L(t))^\sim$, then find the polynomial,

$$f_L(t) = (1+t)^2(1+t+t^2)(\nabla_L(t))^\sim$$

$$= (1+t)^2(1-t^a) - t^b(1+t+t^2)[(-t)^n+(-t)^m] - t^{b+1}(1-t^{a-2b})$$

where $(a+2)$ is the largest exponent of t in $f_L(t)$ and $n > m \geqslant 0$. Then $a = 2b+n+m$, $b = 3k$ and L has the braid representative,

$$[(\Delta_3)^{2(b/3)}(\sigma_1)^n(\sigma_2)^m]\in\Omega$$

where n, m and $(b/3)\in\mathbb{Z}^+$, $n > m$ and $(b/3) \geqslant 1$.

<u>Case(3): If $\gamma^c \simeq L$ is a link of three components</u>:

Then both n and m are even. Hence

$$(1+t+t^2)(\nabla_L(t))^{\sim}$$
$$= 1 + t^{6k+n+m} - t^{3k}\{t^n+t^m+t[1-t(\sum_{i=0}^{n-2}(-t)^i][1-t\sum_{i=0}^{m-2}(-t)^i]\}$$
$$= 1 - t^{3k}\{t^n+t^m+t[1-t((t^{n-1}+1)/(t+1))][1-t((t^{m-1}+1)/(t+1))]\}$$
$$+ t^{6k+n+m}$$

Then

$$(1+t)^2(1+t+t^2)(\nabla_L(t))^{\sim}$$
$$= (1+t)^2(1+t^{6k+n+m}) - t^{3k+1}(t^{n+m}+1) - t^{3k}(1+t+t^2)(t^n+t^m)$$

Therefore given a Lorenz link L of trip number 3 with 3 components and with reduced Alexander polynomial $(\nabla_L(t))^{\sim}$, then find the polynomial

$$f_L(t) = (1+t)^2(1+t+t^2)(\nabla_L(t))^{\sim}$$
$$= (1+t)^2(1+t^a) - t^b(1+t+t^2)(t^n+t^m) - t^{b+1}(t^{a-2b}+1)$$

where $(a+2)$ is the largest exponent of t in $f_L(t)$ and $n \geqslant m \geqslant 0$. Then $a = 2b+n+m$, $b = 3k$ and L has the braid representative,

$$[(\Delta_3)^{2(b/3)}(\sigma_1)^n(\sigma_2)^m]\in\Omega$$

where n, m and $b \in \mathbb{Z}^+$, $n \geq m$ and $(b/3) \geq 1$. Hence every Lorenz link $L \simeq \gamma^c$ of trip number 3 has a canonical form γ for its 3-braid representatives as in equation (4.1.1) and its reduced Alexander polynomial is given by the equation:

$$(1+t)^2(1+t+t^2)(\nabla_L(t))^{\sim} = (1+t)^2[1+(-t)^{6k+n+m}] - t^{3k+1}[1+(-t)^{n+m}]$$
$$- t^{3k}(1+t+t^2)[(-t)^n+(-t)^m] \qquad (4.1.2)$$

for every $\gamma \in \Omega$ □

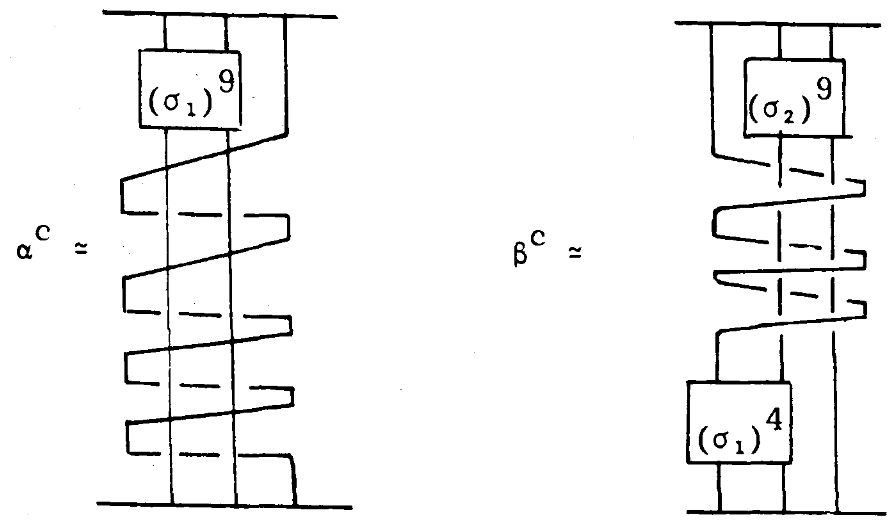
The following example shows that Lorenz links can not be determined (in general) by the isotopy type of each component and the associate linking pattern of its components. In fact this is an example of links with some knotted components. Consequently Morton.H gave his conjecture, in (4.0.1), about Lorenz links with unknotted components.

(4.1.3) Example:

Given the two braids,

$$\alpha = (\Delta_3)^8\sigma_1 \text{ and } \beta = (\Delta_3)^6(\sigma_1)^4(\sigma_2)^3 \in \Omega$$

Then theorem (4.1.2) tells us that the two links α^c and β^c are not isotopic, because α and β are not conjugate. But the two components in α^c and β^c have the same isotopy type of the unknot and the (2,9) torus knot, as in figure (4-1) and the two components in each link have linking number equals 8 □



Figure(4-1)

The following result gives an affirmative answer for Morton's conjecture cited in (4.0.1), in case of 3-braids.

(4.1.4) Theorem:

The linking pattern of L, for a Lorenz link L of trip number 3 with 3 components, determines the unique braid representative of L in Ω (as in lemma (4.1.1)) and so determines L.

Proof:

Let $\{a,b,c\}$ be the set of linking numbers of L and let $a \leqslant b \leqslant c$. Then a is the number of full twists in α, for $\alpha \epsilon \Omega$. So c-a and b-a are the powers of σ_1 and σ_2 in α, respectively. Hence L is isotopic to α^c where,

$$\alpha = [(\Delta_3)^{2a}(\sigma_1)^{2(c-a)}(\sigma_2)^{2(b-a)}] \epsilon \Omega$$

But theorem (4.1.2) tells us that α is unique in Ω, which complets the proof □

§4.2. ON LORENZ LINKS OF TRIP NUMBER 4 WITH 4 COMPONENTS

Theorem (3.1.22) tells us that a Lorenz link of trip number 4 has a minimal braid representative

$$\gamma = (\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^{2p_2}(\alpha)(\Delta_{3,\leftarrow})^{2q_1}(\sigma_3)^{2q_2}(\beta) \qquad (4.2.1)$$

where α, $\beta \in SB_4$ and p, p_i, $q_i \in \mathbb{Z}^+$ for $i = 1, 2$ such that $p \geq 1$. Then the braid words α and β are the positive permutation braids (the 24 braids of SB_4, as illustrated in figure (1-8b)),

α, $\beta \in SB_4 = \{e, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_2, \sigma_1\sigma_3, \sigma_3\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_3, \sigma_1\sigma_3\sigma_2, \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_3, \sigma_2\sigma_3\sigma_2, \sigma_1\sigma_2\sigma_3\sigma_2, \sigma_1\sigma_2\sigma_1\sigma_3, \sigma_2\sigma_1\sigma_3\sigma_2, \sigma_3\sigma_2\sigma_3\sigma_1, \sigma_1\sigma_3\sigma_2\sigma_1, \sigma_3\sigma_2\sigma_3\sigma_1\sigma_2, \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2, \sigma_1\sigma_3\sigma_2\sigma_3\sigma_1, \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\}$.

(4.2.1) Proposition:

If L is a Lorenz link of trip number 4 with 4 components, then L is isotopic to γ^c, for some γ in the following classes:

$\Omega_1 = \{(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^{2p_2}(\Delta_{3,\leftarrow})^{2q_1}(\sigma_3)^{2q_2} \mid \text{either } p_1>q_1 \text{ or } p_1=q_1, p_2 \geq q_2\}$

$\Omega_2 = \{(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^{2p_2}\sigma_2(\Delta_{3,\leftarrow})^{2q_1}(\sigma_3)^{2q_2}\sigma_2 \mid \text{either } p_1>q_1, \text{ or } p_1=q_1, p_2 \geq q_2 \text{ and } (p_1,p_2,q_2) \neq (0,0,0) \text{ or } (p_2, q_1, q_2) \neq (0,0,0)\}$

$\Omega_3 = \{(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\Delta_{3,\leftarrow})^{2q_1}(\sigma_3)^{2q_2}[\sigma_1(\sigma_2)^2\sigma_1] \mid (q_1,q_2) \neq (0,0)\}$

$\Omega_4 = \{(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\Delta_{3,\leftarrow})^{2q_1}[\sigma_3(\sigma_2)^2\sigma_3\sigma_1(\sigma_2)^2\sigma_1] \mid p_1 \geq q_1\}$

$\Omega_5 = \{(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^{2p_2}(\sigma_2)^{2q_1} \mid p_2 \geq q_1 \geq 2\}$

$\Omega_6 = \{(\Delta_4)^{2p}(\sigma_1)^{2p_1}(\sigma_2)^{2q_1}[\sigma_3\sigma_2(\sigma_1)^2\sigma_2\sigma_3)] \mid p_1 \geq q_1 \geq 2\}$

Proof:

The proof will be given through a sequence of remarks:

(a): If γ is a pure braid as in equation (4.2.1), then $\beta = \rho[\alpha]$ (the reverse of α). So p_2 is increased by 1 (up to conjugacy) when α started with σ_1 and q_2 is also increased by 1 when α ended with σ_3. Hence

$$\gamma = (\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^{2p_2}(\alpha)(\Delta_{3,\leftarrow})^{2q_1}(\sigma_3)^{2q_2}\rho[\alpha] \qquad (4.2.2)$$

where

$$\alpha \in \{e,\ \sigma_2,\ \sigma_2\sigma_1,\ \sigma_3\sigma_2,\ \sigma_3\sigma_2\sigma_1,\ \sigma_2\sigma_1\sigma_3\sigma_2,\ \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\}$$

and

$$p,\ p_i,\ q_i \in \mathbb{Z}^+,\ \text{for } i = 1,\ 2 \text{ and } p \geqslant 1$$

A sketch of these possible seven cases are illustrated in figure (4-2). We can define some order on the powers p_i and q_i, i=1,2 such that no braid in the list is the result of turning over the other. i.e. γ and $\tau[\gamma]$ do not both appear separately in the list.

(b): If $\alpha = \sigma_2$ and either $p_2 = q_1 = q_2 = 0$ or $p_1 = p_2 = q_2 = 0$, then either

$$\gamma = (\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_2)^2$$

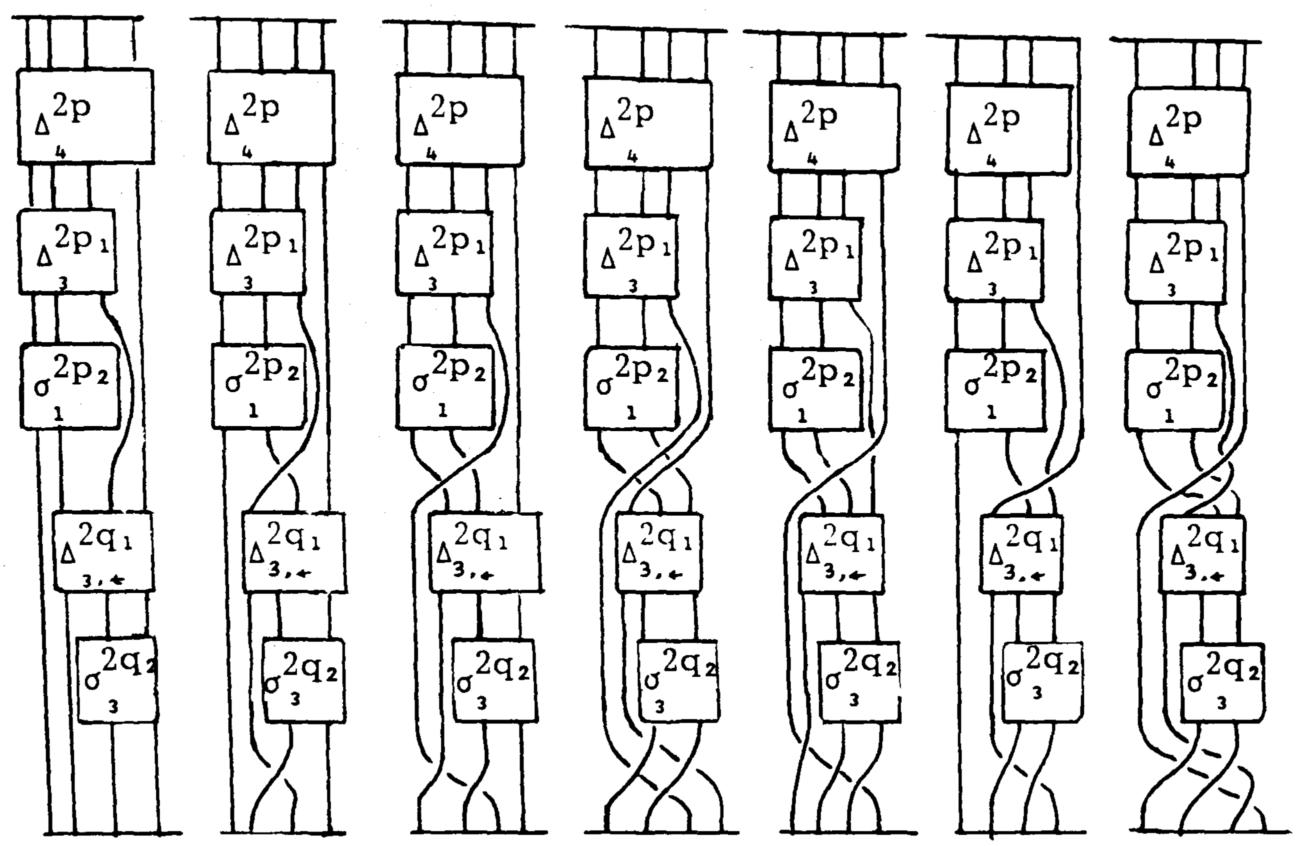
or

$$\gamma = (\Delta_4)^{2p}(\Delta_{3,\leftarrow})^{2q_1}(\sigma_2)^2$$

which are conjugate to

$$(\Delta_4)^{2p}(\Delta_3)^{2q}(\sigma_1)^2$$

for some integer q. Hence let $(p_1,p_2,q_2) \neq (0,0,0)$ or $(p_2,q_1,q_2) \neq (0,0,0)$, otherwise the case is included in case $\alpha = e$.



α: e σ_2 $\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_3\sigma_2$ $\sigma_3\sigma_2\sigma_1$ $\sigma_3\sigma_2$ $\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2$

Figure (4-2)

(c): If $\alpha = \sigma_2\sigma_1$ and $p_2 \neq 0$, then we have 1 more full twist in the first 3 strands, which (up to conjugation) is included in case $\alpha = e$. But if $p_2 = q_1 = q_2 = 0$, then

$$\gamma = (\Delta_4)^{2p}(\Delta_3)^{2p_1}\sigma_2(\sigma_1)^2\sigma_2$$

which is conjugate to

$$(\Delta_4)^{2p}(\Delta_3)^{2p_1}(\sigma_1)^2(\sigma_2)^2$$

Hence in case $\alpha = \sigma_2\sigma_1$, let $p_2 = 0$ and $(q_1,q_2) \neq (0,0)$, otherwise the case is included in case $\alpha = \sigma_2$.

(d): If $\alpha = \sigma_2\sigma_1\sigma_3\sigma_2$ and $p_2 \neq 0$ ($q_2 \neq 0$), then we have 1 more full twist in the first (last) 3 strands. Then the case (up to conjugacy) is included in case $\alpha = e$, when $p_2 = q_2 = 0$ and it is also included in case $\alpha = \sigma_2\sigma_1$ when either $p_2 = 0$, $q_2 \neq 0$ or $p_2 \neq 0$, $q_2 = 0$. Hence let $p_2 = q_2 = 0$, when $\alpha = \sigma_2\sigma_1\sigma_3\sigma_2$.

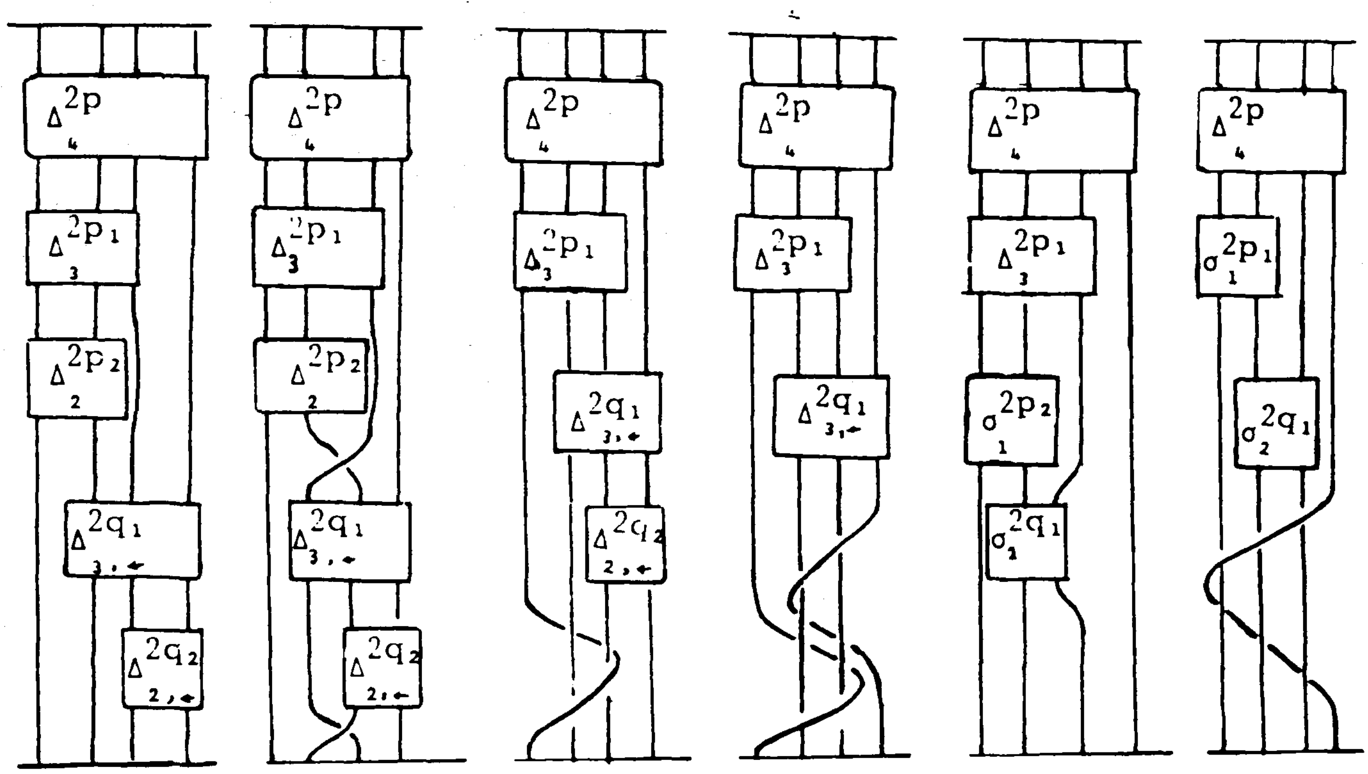
(e): If $\alpha = \sigma_3\sigma_2\sigma_1$ and $p_1+q_1 \neq 0$, then we have 1 more full twist in 4 strands which is included in case $\alpha = e$, when either q_2 or p_2 or both are zeros. But if $p_2 = 1$ or $q_2 = 1$, then the case is included in case $\alpha = \sigma_2$. Now for $p_1 + q_1 = 0$, the case is included in case $\alpha = \sigma_2\sigma_1$, when $q_2 = 1$ or $p_2 = 1$ or both are equal 1. Hence when $\alpha = \sigma_3\sigma_2\sigma_1$, let p_2, $q_2 \geqslant 2$ and consider the two cases $p_1+q_1 \neq 0$ and $p_1+q_1 = 0$.

(f): If $\alpha = \sigma_3\sigma_2$, then the case is the result of turning over the case $\alpha = \sigma_2\sigma_1$ with replacing q_i, p_i by p_i, q_i respectively, for $i = 1, 2$, hence they are conjugate.

(g): If $\alpha = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2$ and $p_2+q_2 \neq 0$, then we have 1 more full twist in the first 3 strands and then have 1 more full twist in the 4 strands. So the case is included in case $\alpha = e$. But if $p_2+q_2 = 0$ and $p_1+q_1 \neq 0$, then we have 1 more full twist in the 4 strands and the case is included in case $\alpha = \sigma_2\sigma_1$. Therefore consider the case when $p_1+q_1 = 0$, so γ is conjugate to

$$(\Delta_4)^{2p}(\Delta_3)^2\sigma_3(\sigma_2)^2\sigma_3$$

which is included in case $\alpha = \sigma_3\sigma_2$. This completes the proof of the proposition, where sketches of the diagrams of these six classes are illustrated in figure (4-3) □



Ω_1 Ω_2 Ω_3 Ω_4 Ω_5 Ω_6

Figure (4-3)

	lk_{12}	lk_{13}	lk_{14}	lk_{23}	lk_{24}	lk_{34}	conditions	notes
η_1	p_1+p_2	p_1	0	p_1+q_1	q_1	q_1+q_2	either $p_1>q_1$ or $p_1=q_1, p_2\geq q_2$	$lk_{13}+lk_{24} = lk_{23}$
η_2	p_1+p_2	p_1	0	p_1+q_1+1	q_1+q_2	q_1	either $p_1>q_1$ or $p_1=q_1, p_2\geq q_2$ and either $(p_1, p_2, q_2)\neq 0$ or $(p_2, q_1, q_2)\neq 0$	$lk_{13}+lk_{34}+1=lk_{23}$
η_3	p_1+1	p_1+1	0	p_1+q_1	q_1	q_1+q_2	$0\neq(q_1, q_2)$ and $p_1\geq q_1$	$lk_{13}+lk_{24}-1=lk_{23}$
η_4	p_1+1	p_1+1	0	p_1+q_1	q_1+1	q_1+1	$p_1\geq q_1$	$lk_{13}+lk_{24}-2=lk_{23}$
η_5	p_1+p_2	p_1	0	p_1+q_1	0	0	$p_2\geq q_1\geq 2$	$lk_{14}=lk_{24}=lk_{34}=0$
η_6	p_1	0	1	q_1	1	1	$p_1\geq q_1\geq 2$	$lk_{14}=lk_{24}=lk_{33}=1$

Table (4-1)

(4.2.2) Theorem:

The linking pattern of a Lorenz link L of trip number 4 with 4 components, determines a canonical 4-braid representative for L and so determines L. More precisely:
There is a unique representative $\gamma\in\{\Omega_i|\ 1 \leqslant i \leqslant 6\}$ and γ is uniquely determined by the pattern of the linking numbers of L, where Ω_i as in proposition (4.2.1).

The proof will be started with the following two remarks:

(4.2.3) Remark:

The smallest linking number in the components of γ^c is the maximum number of full twists in 4 strands for all $\gamma\in\{\Omega_i|1 \leqslant i \leqslant 6\}$. Then let $\gamma = (\Delta_4)^{2p}Q$, where Q is a positive prime to $(\Delta_4)^2$. Hence there are at least two arcs in Q with zero linking number. Therefore the maximum number of full twists in γ is invariant for the link type γ^c, otherwise we have two different sets of linking numbers to the same link, which is impossible. So given $\gamma = (\Delta_4)^{2p}Q\in\Omega_i$ and $\gamma' = (\Delta_4)^{2p'}Q'\in\Omega_j$, where Q and Q' are prime to $(\Delta_4)^2$, $p \neq p'$ and $1 \leqslant i, j \leqslant 6$, then γ^c and $(\gamma')^c$ represent two different link types. Hence it is enough to study conjecture (4.0.1) for a fixed number of full twists p and for a prime (to $(\Delta_4)^2$) positive braid Q.

(4.2.4) Remark: [The key of the proof of theorem (4.2.2)]

Now let $\gamma = (\Delta_4)^{2p}Q\in\Omega_i$, $1 \leqslant i \leqslant 6$. Then order the arcs in top of Q from left to right and let lk_{ij} be the linking number of the i-th arc with the j-th arc in Q. Order the set $\{lk_{ij}|1 \leqslant i,j \leqslant 6\}$ in some pattern such as a matrix (lk_{ij}) or simply as a 6-tuple $\eta = (lk_{12},lk_{13},lk_{14},lk_{23},lk_{24},lk_{34})$. Let η_i be the corresponding class of patterns of linking numbers of the elements in Ω_i, for $1 \leqslant i \leqslant 6$. Then for $\beta \in \Omega_j$ with $\xi \in \eta_j$ (where the arcs labelled (1,2,3,4) on top and on bottom of β). Permute the components of ξ to have $\xi' \in \eta_i$ for some $\beta' \in \Omega_i$ with arcs labelled (1',2',3',4'). Then to show that conjecture (4.0.1) holds in B_4, it is enough to prove that $\beta = \beta'$ if and only if $\xi = \xi'$. Furthermore if there are no $\beta \in \Omega_i$ and $\beta' \in \Omega_j$, $i \neq j$ with the same linking pattern ξ, then $\Omega_i \cap \Omega_j = \Phi$, $1 \leqslant i, j \leqslant 6$, $i \neq j$. Hence each element γ, $\gamma \in \{\Omega_i | 1 \leqslant i \leqslant 6\}$, represents a different conjugacy class in B_4. In case when Q has an arc of linking numbers equal zeros with the others or equal ones, then (clearly from the diagrams in figure (4-3)) Q is conjugate to either Q' or $[Q'\sigma_3\sigma_2(\sigma_1)^2\sigma_2\sigma_3]$ respectively, where Q' is conjugate to a Lorenz braid of trip number 3. Hence by induction the conjecture (4.0.1) holds. The pattern of the linking numbers of Q is given for each Ω_i, as in table (4-1) □

<u>Proof of theorem (4.2.2) </u>:

We want to show that, if two braids β and $\gamma \in \Omega$ with the same pattern of linking numbers, then $\beta = \gamma$, where $\Omega = \{\Omega_i | 1 \leqslant i \leqslant 6\}$. We need also to prove that $\Omega_i \cap \Omega_j = \Phi$, for $1 \leqslant i, j \leqslant 6$, with $i \neq j$. Let us take $\eta = \{\eta_i | 1 \leqslant i \leqslant 6\}$. Now we are going to investigate that in each class Ω_i for $1 \leqslant i \leqslant 6$.

For Ω_1: Let $\beta \in \Omega_1$ with linking pattern $\xi \in \eta_1$, then consider the two cases:

<u>The special case: Ω_{1s}, when $q_1 = 0$</u>

(a): If $p_1 = q_1 = 0$, then $\xi = (p_2,0,0,0,0,q_2)$, $p_2 \geqslant q_2$. Hence let $q_2 \neq 0$, otherwise the case (by induction, as in remark (4.2.4)) is in B_3. Then $\xi' \in \eta_1$ only when $\xi' = \xi$ and so $\beta' = \beta$. But $\xi' \notin \eta_i$, for $i \neq 1$, otherwise either β' has an arc of zeros linking numbers with the others as in η_2, η_3 and η_5 or $\xi'\epsilon\eta_4$,η $\cdot_6$ with at least 4 non-zero components, which is impossible.

(b): If $p_1 \neq 0$ and $q_1 = 0$, then $\xi = (p_1+p_2,p_1,0,p_1,0,q_2)$. So let $q_2 \neq 0$, otherwise the case is in B_3 (by induction). Now if $\xi'\epsilon\eta_1$, then either $\{1,4\}\to\{1',4'\}$ and $\{2,4\}\to\{2',4'\}$ or $\{1,4\}\to\{2',4'\}$ and $\{2,4\}\to\{1',4'\}$. Then $(1,2,3,4)\to\{(1',2',3',4')$ or $(2',1',3',4')\}$. So $\xi' = \xi$ and $\beta' = \beta$. If $\xi'\epsilon\eta_2$, then $lk_{1'4'} = lk_{3'4'} = 0$ in ξ', otherwise $\xi'\notin\eta_2$. Hence either $\{1,4\}\to\{1',4'\}$ and $\{2,4\}\to\{3',4'\}$ or $\{1,4\}\to\{3',4'\}$ and $\{2,4\}\to\{1',4'\}$. So $(1,2,3,4)\to\{(1',3',2',4')$ or $(3',1',2',4')\}$. Then $\xi' = (p_1,p_1+p_2,0,p_1,q_2,0)$. So $\xi'\notin\eta_2$, see table (4-1). If $\xi'\epsilon\eta_3$, then to have only 2 zero components in $\xi'\epsilon\eta_3$, we must have $lk_{2'4'} = lk_{1'4'} = 0$, otherwise $\xi'\notin\eta_3$. Hence either $\{1,4\}\to\{1',4'\}$ and $\{2,4\}\to\{2',4'\}$ or $\{1,4\}\to\{2',4'\}$ and $\{2,4\}\to\{1',4'\}$. So $(1,2,3,4)\to\{(1',2',3',4')$ or $(2',1',3',4')\}$. Then $\xi' = \xi$ which contradicts the conditions in class η_3, so $\xi'\notin\eta_3$. If $\xi'\epsilon\eta_4$, then $lk_{1'4'} = lk_{2'3'} = 0$. Hence either $\{1,4\}\to\{1',4'\}$ and $\{2,4\}\to\{2',3'\}$ or $\{1,4\}\to\{2',3'\}$ and $\{2,4\}\to\{1',4'\}$, which is impossible, then $\xi'\notin\eta_4$. Finally since ξ in this case contains exactly two zeros, then ξ' neither in η_5 nor in η_6.

<u>The general case: Ω_{1g}, when $q_1 \neq 0$:</u>

In this case $\xi = (p_1+p_2,p_1,0,p_1+q_1,q_1,q_1+q_2)$ with only one zero component. If $\xi'\epsilon\eta_1$, then $\{1,4\}\to\{1',4'\}$, hence $\{2,3\}\to\{2',3'\}$. So either:

(i): $(1,2,3,4)\to(1',2',3',4')$, then $\xi = \xi'$ hence $\beta = \beta'$,

(ii): $(1,2,3,4)\to(1',3',2',4')$, then $\xi' = (p_1,p_1+p_2,0,p_1+q_1,q_1+q_2,q_1)$, hence $p_2 = q_2 = 0$, $\xi' = \xi$ and so $\beta = \beta'$,

(iii): $(1,2,3,4)\to(4',3',2',1')$, then $\xi' = (q_1+q_2,q_1,0,p_1+q_1,p_1,p_1+p_2)$, hence $\xi' = \xi$ with $p_i = q_i$ for i=1,2 and so $\beta = \beta'$ or (iv): $(1,2,3,4)\to(4',2',3',1')$, then $\xi' = (q_1,q_1+q_2,0,q_1+p_1,p_1+p_2,p_1)$, hence $\xi' = \xi$ with $p_2 = q_2 = 0$ and $\beta = \beta'$.

But in this case it is clearly that $\xi' \notin \eta_i$, $i \neq 1$, otherwise it contradicts the conditions in table (4-1). Then it is proved that $\Omega_1 \cap \Omega_i = \Phi$, for $2 \leqslant i \leqslant 6$ and the elements in Ω_1 are uniquely determined by the corresponding pattern of the linking numbers.

The other cases will be studied by following the previous technique.

For Ω_2: Let $\beta \in \Omega_2$ with linking pattern $\xi \in \eta_2$, then consider the two cases:

The special case: Ω_{2s}, when $q_1 = 0$:

(a): If $p_1 = q_1 = 0$, then $\xi = (p_2,0,0,1,q_2,0)$, $p_2 \geqslant q_2$. Hence let $q_2 \neq 0$, otherwise table (4-1) tells us that the case (by induction) is in B_3. Then the 2-nd arc in Q is the only arc with non-zero linking numbers. So $\{2\}\to\{2'\}$, hence $\{1,3,4\}\to\{1',3',4'\}$. Then either:

(i): $(1,2,3,4)\to(1',2',3',4')$, hence $\xi' = \xi$,

(ii): $(1,2,3,4)\to(3',2',4',1')$, then $\xi' = (q_2,0,0,p_2,1,0)$, so $p_2 = q_2 = 1$, hence $\xi' = \xi$,

(iii): $(1,2,3,4)\to(4',2',1',3')$, then $\xi' = (1,0,0,q_2,p_2,0)$, so $p_2 = q_2 = 1$, hence $\xi' = \xi$,

(iv): $(1,2,3,4)\to(3',2',1',4')$, then $\xi' = (1,0,0,p_2,q_2,0)$, so $p_2 = q_2 = 1$, hence $\xi' = \xi$,

(v): $(1,2,3,4)\to(4',2',3',1')$, then $\xi' = (q_2,0,0,1,p_2,0)$, so $p_2 = q_2$, hence $\xi' = \xi$ or

(vi): $(1,2,3,4)\to(1',2',4',3')$, then $\xi' = (p_2,0,0,q_2,1,0)$, so $q_2 = 1$ hence $\xi' = \xi$.

Therefore in all cases $\beta' = \beta\epsilon\Omega_2$. Now assume that $\xi'\epsilon\eta_3$, then $q_2 = 1$. But there is no arc in Q', as in Q, with non-zero linking numbers which gives a contradiction. Also since, see table (4-1). ξ has 3 zero components, then ξ' neither in η_4 nor in η_6. Finally Q has no arc with zero linking numbers, then $\xi'\notin\eta_5$.

(b): If $p_1 \neq 0$, $q_1 = 0$, then $\xi = (p_1+p_2, p_1, 0, p_1+1, q_2, 0)$. So let $q_2 \neq 0$, otherwise the case (by induction) is in B_3. Now if $\xi'\epsilon\eta_2$, then either $\{1,4\}\to\{1',4'\}$ and $\{3,4\}\to\{3',4'\}$ or $\{1,4\}\to\{3',4'\}$ and $\{3,4\}\to\{1',4'\}$. So either $(1,2,3,4)\to(1',2',3',4')$ with $\xi' = \xi$ or $(1,2,3,4)\to(3',2',1',4')$ with $\xi' = (p_1+1, p_1, 0, p_1+p_2, q_2, 0)$, $p_2 = 1$ then $\xi' = \xi$. Hence $\beta' = \beta$. Now assume that $\xi'\epsilon\eta_3$, then from table (4-1), $\{4\}\to\{4'\}$ and $\{1,3\}\to\{1',2'\}$. Hence either $(1,2,3,4)\to(1',3',2',4')$ so $\xi' = (p_1, p_1+p_2, 0, p_1+1, 0, q_2)$, then $p_2 = 0$ and $p_1-1 = p_1+1$, which gives a contradiction or $(1,2,3,4)\to(2',3',1',4')$ so $\xi' = (p_1, p_1+1, 0, p_1+p_2, 0, q_2)$, which implies that $\xi'\notin\eta_3$. Since $(1,1,0,0,1,1)$ is also the only element in η_4 with 2 zero components, then $\xi'\notin\eta_4$. Finally it is clear that Q' has never an arc of linking numbers equal zeros or equal ones, then ξ' neither in η_5 nor in η_6.

The general case, Ω_{2g}, when $q_1 \neq 0$:

In this case $\xi = (p_1+p_2, p_1, 0, p_1+q_1+1, q_1+q_2, q_1)$ with only 1 zero component. then $\{1,4\}\to\{1',4'\}$, $\{2,3\}\to\{2',3'\}$. So either:

(i): $(1,2,3,4)\to(1',2',3',4')$, then $\xi = \xi'$,

(ii): $(1,2,3,4)\to(4',2',3',1')$, so $\xi' = (q_1+q_2, q_1, 0, p_1+q_1+1, p_1+p_2, p_1)$. Hence $\xi' = \xi$ with $p_i = q_i$ for $i = 1, 2$,

(iii): $(1,2,3,4)\to(1',3',2',4')$, so $\xi' = (p_1, p_1+p_2, 0, p_1+q_1+1, q_1, q_1+q_2)$, hence $\xi' = \xi$, with $p_2 = q_2 = 0$ or (iv): $(1,2,3,4)\to(4',3',2',1')$, so $\xi'=(q_1,q_1+q_2,0,q_1+p_1+1,p_1,p_1+p_2)$, hence $\xi' = \xi$ with $p_2 = q_2 = 0$ and $p_1 = q_1$. Therefore in all cases $\beta' = \beta\in\Omega_2$. Now assume that $\xi'\in\eta_3$, then $\{1,4\}\to\{1',4'\}$, hence $\{2,3\}\to\{2',3'\}$. So either:

(i): $(1,2,3,4)\to(1',2',3',4')$, so $\xi' = \xi$,

(ii): $(1,2,3,4)\to(1',3',2',4')$, so $\xi' = (p_1,p_1+p_2,0,p_1+q_1+1,q_1,q_1+q_2)$, then $\xi' = \xi$ with $p_2 = 0$,

(iii): $(1,2,3,4)\to(4',2',3',1')$, so $\xi' = (q_1+q_2,q_1,0,p_1+q_1+1,p_1+p_2,p_1)$, then $\xi' = \xi$ with $p_2 = q_2 = 0$ and $p_1 = q_1$ or

(iv): $(1,2,3,4)\to(4',3',2',1')$, so $\xi' = (q_1,q_1+q_2,0,p_1+q_1+1,p_1,p_1+p_2)$, then $\xi' = \xi$ with $p_2 = q_2 = 0$ and $p_1 = q_1$, so $\xi' = \xi$. But in all of these cases we have, $lk_{1'3'}+lk_{2'4'}-1 \neq lk_{2'3'}$, which contradicts the assumption that $\xi'\in\eta_3$. Similarly let $\xi'\in\eta_4$, then $\{1,4\}\to\{1',4'\}$, hence $\{2,3\}\to\{2',3'\}$. So either:

(i): $(1,2,3,4)\to(1',2',3',4')$, so $\xi' = \xi$,

(ii): $(1,2,3,4)\to(1',3',2',4')$, so $\xi'=(p_1,p_1+p_2,0,p_1+q_1+1,q_1,q_1+q_2)$, then $\xi' = \xi$ with $p_2 = q_2 = 0$,

(iii): $(1,2,3,4)\to(4',2',3',1')$, so $\xi'=(q_1+q_2,q_1,0,p_1+q_1+1,p_1+p_2,p_1)$, then $\xi' = \xi$ with $p_2 = q_2 = 0$ and $p_1 = q_1$ or

(iv): $(1,2,3,4)\to(4',3',2',1')$, so $\xi' = (q_1,q_1+q_2,0,p_1+q_1+1,p_1,p_1+p_2)$, then $\xi' = \xi$ with $p_2 = q_2 = 0$ and $p_1 = q_1$, hence $\xi' = \xi$. But in all of these cases we have, $lk_{1'3'}+lk_{2'4'}-2 \neq lk_{2'3'}$, which contradicts the assumption that $\xi'\in\eta_4$. It is also clear that $\xi'\notin\eta_5$. Finally let $\xi'\in\eta_6$, since $p_1+q_1+1 \geqslant 2$, then either $\{2,3\}\to\{2',3'\}$, or $\{2,3\}\to\{1',2'\}$, which is impossible because $\{1,4\}\to\{1',3'\}$, then $\xi'\notin\eta_6$. Therefore $\Omega_2 \cap \Omega_i = \Phi$, for $1 \leqslant i \leqslant 6$, $i \neq 2$.

Let $\beta\in\Omega_3$ with linking pattern $\xi\in\eta_3$, then consider the two cases:

the special case: Ω_{3s}, when $q_1 = 0$

(a): If $p_1 = q_1 = 0$, then $\xi = (1,1,0,0,0,q_2)$, $q_2 \neq 0$. Then let $q_2 = 1$. The arcs labelled 1 and 3 have 2 non zero components and 1 zero component, but the arcs labelled 2 and 4 have 2 zero components and 1 non zero component Hence $\{2,4\}\to\{2',4'\}$ and $\{1,3\}\to\{1',3'\}$. So $(1,2,3,4)\to\{(1',2',3',4'), (3',4',1',2'), (3',2',1',4'), (1',4',3',2')\}$, which implies that $\xi' = \xi$, hence $\beta = \beta'$. In this case (as shown in table (4-1)) $\xi' \not\in \eta_i$, for i = 4,5,6. But if $q_2 > 1$, then $\{3,4\}\to\{3',4'\}$, $\{1\}\to\{1'\}$ and $\{2,3\}\to\{2',3'\}$. So $(1,2,3,4)\to(1',2',3',4')$ which implies that $\xi' = \xi$, hence $\beta = \beta'$. Then table (4-1) tells us that $\xi' \not\in \eta_i$, for i = 4,5,6.

(b): If $p_1 \neq 0$, $q_1 = 0$, then $\xi = (p_1+1, p_1+1,0,p_1,0,q_2)$, $q_2>0$. So the 3-rd arc in Q is the only arc with non-zero linking numbers and the 4-th arc in Q is the only arc with two zero linking numbers, hence $\{3\}\to\{3'\}$ and $\{4\}\to\{4'\}$. But the 2-nd arc in Q has 3 different linking numbers, while the 1-st arc does not satisfy that. So $(1,2,3,4)\to(1',2',3',4')$, then $\xi' = \xi$, hence $\beta = \beta'$. Therefore in this case, table (4-1) tells us that $\xi' \not\in \eta_i$, for i = 4,5,6.

The general case: Ω_{3g}, when $q_1 \neq 0$

In this case $\xi = (p_1+1,p_1+1,0,p_1+q_1,q_1,q_1+q_2)$, with only one zero component, then $\{1,4\}\to\{1',4'\}$, so $\{2,3\}\to\{2',3'\}$, hence either:

(i): $(1,2,3,4)\to(1',2',3',4')$, hence $\xi = \xi'$,

(ii): $(1,2,3,4)\to(1',3',2',4')$, so $\xi' = (p_1+1,p_1+1,0,p_1+q_1,q_1+q_2,q_1)$, then $q_2 = 0$, hence $\xi = \xi'$,

(iii): $(1,2,3,4)\to(4',2',3',1')$, so $\xi' = (q_1,q_1+q_2,0,p_1+q_1,p_1+1,p_1+1,)$, then $q_2 = 0$, $q_1 = p_1+1$, hence $\xi = \xi'$ or (iv): $(1,2,3,4)\to(4',3',2',1')$, so $\xi' = (q_1+q_2, q_1, 0, p_1+q_1, p_1+1, p_1+1,)$, then $q_2 = 0$, $q_1 = p_1+1$, hence $\xi = \xi'$. Therefore in all of these cases we have, $\beta' = \beta\epsilon\Omega_3$. Now assume that $\xi'\epsilon\eta_4$, then $\{1,4\}\to\{1',4'\}$ and $\{2,3\}\to\{2',3'\}$, so either:

(i): $(1,2,3,4)\to(1',2',3',4')$, hence $\xi = \xi'$ or

(ii): $(1,2,3,4)\to(1',3',2',4')$, so $\xi' = (p_1+1, p_1+1, 0, p_1+q_1, q_1+q_2, q_1)$, $q_2 = 0$, hence $\xi = \xi'$,

then in these two cases we have $lk_{1'3'}+lk_{2'4'}-2 \neq lk_{2'3'}$, which contradicts the assumption that $\xi'\epsilon\eta_4$.

(iii): $(1,2,3,4)\to(4',2',3',1')$, so $\xi' = (q_1, q_1+q_2, 0, p_1+q_1, p_1+1, p_1+1)$, then $q_2 = 0$ or

(iv): $(1,2,3,4)\to(4',3',2',1')$, so $\xi' = (q_1+q_2, q_1, 0, p_1+q_1, p_1+1, p_1+1)$, then $q_2 = 0$,

then in these two cases $\xi'\notin\eta_4$, because $q_1 < p_1+1$.

But $\xi'\notin\eta_5$, because it has no three zero components for $q_1 > 0$. Similarly ξ does not have at least 3 components each equals 1, as in the elements of η_6, hence $\xi'\notin\eta_6$. Therefore $\Omega_3 \cap \Omega_i = \Phi$, for $i = 4,5,6$.

Ω_4 : Let $\beta\epsilon\Omega_4$ with linking pattern $\xi\epsilon\eta_4$, then consider the two cases:

<u>The special case: Ω_{4s}, when $q_1 = 0$</u>

(a): If $p_1 = q_1 = 0$, then $\xi = (1,1,0,0,1,1)$, hence either $\{1,4\}\to\{1',4'\}$ and $\{2,3\}\to\{2',3'\}$ or $\{1,4\}\to\{2',3'\}$ and $\{2,3\}\to\{1',4'\}$, which implies that $\xi' = \xi$, hence $\beta = \beta'$. In this and suing table (4-1), we can see that neither $\xi'\epsilon\eta_5$ nor $\xi'\epsilon\eta_6$.

(b): If $p_1 \neq 0$, $q_1 = 0$, then $\xi = (p_1+1, p_1+1, 0, p_1, 1, 1)$. So $\{1,4\}\to\{1',4'\}$, then $\{2,3\}\to\{2',3'\}$. But the 1-st arc in Q has the greatest two linking numbers, then either $(1,2,3,4)\to(1',2',3',4')$, or $(1',3',2',4')$, which implies that $\xi' = \xi$, hence $\beta' = \beta$. Note also that

$\xi' \notin \eta_5$, because ξ' does not has, at least, 3 zero components. But if $\xi' \in \eta_6$, then $p_1 = 1$, so $\xi = (2,2,0,1,1,1)$, hence no arc in β' with linking numbers equal 1 as in η_6, therefore $\xi' \notin \eta_6$.

The general case: Ω_{4g}, when $q_1 \neq 0$

In this case $\xi = (p_1+1, p_1+1, 0, p_1+q_1, q_1+1, q_1+1)$,

with only 1 zero component, then $\{1,4\} \to \{1',4'\}$ and $\{2,3\} \to \{2',3'\}$, hence either $(1,2,3,4) \to \{(1',2',3',4')$ and $(1',3',2',4')\}$, then $\xi' = \xi$ or $(1,2,3,4) \to \{(4',2',3',1')$ and $(4',3',2',1')\}$, which implies that $\xi' = \xi$ with $p_1 = q_1$, so $\beta' = \beta \in \Omega_4$. But using table (4-1), we can see that neither $\xi' \in \eta_5$ nor $\xi' \in \eta_6$. Therefore $\Omega_4 \cap \Omega_i = \Phi$, for i = 5,6.

For Ω_5 and Ω_6: Let $\beta_1 \in \Omega_5$ and $\beta_2 \in \Omega_6$, with linking patterns $\xi_1 \in \eta_5$ and $\xi_2 \in \eta_6$, respectively. Then the 4-th arcs in both Q_1 and Q_2 have linking numbers equal zeros and ones, respectively. Hence (by induction as shown in remark (4.2.4)) β_1 and β_2 are determined by their associated linking patterns. We can see also that $\Omega_5 \cap \Omega_6 = \Phi$, which completes the proof of the theorem □

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