<u>AN ENTROPY-BASED APPROACH TO THE OPTIMUM DESIGN OF RELIABLE WATER DISTRIBUTION NETWORKS</u>

Tiku T. Tanyimboh

ABSTRACT

This thesis explores the applicability of Shannon's informational entropy and Jaynes' maximum entropy formalism in civil engineering inference. A well-established use of the maximum entropy formalism is in the estimation of probability distributions from statistical data. The present research, however, is concerned with generating solutions to wider, more general problems in which available data are incomplete and are not directly related to probabilities. In a nutshell, this research is an attempt to answer the following questions.

1. How can least biased flows be inferred in a looped network given only the nodal inflows and outflows and the direction of flow in each link?
2. Is there a correlation between entropy and reliability in water distribution networks?

The above questions are the focal points of this research, but it is instructive to consider the following auxiliary question. What is water distribution network reliability? In this research, by visualising a flow network as a continuous experiment, a multispace probability model is derived. Computational results suggest that entropy could be a useful surrogate reliability measure. The original and innovative aspects of this work are stated below.

1. A rigorous multiple-space probability model is derived for flow networks by using the relative frequency interpretation of probabilities. The conditional entropy formula is then used to derive a general formula for the entropy of network flows. Also, using the method of Lagrange multipliers, a convenient formula is derived for the maximum entropy of any parallel flow network in which each source node supplies all the demand nodes.

2. It is established that, for single-source networks, maximum entropy flows are such that each demand node receives equal proportions of the demand at that node from the paths supplying the node. The above property is used to develop an algorithm for calculating maximum entropy flows in single-source networks. Notable properties of the algorithm are its conceptual and practical simplicity, and computational efficiency.

3. Through entropy, it is demonstrated that there are reliability-related advantages in designing the pipes of a network to be similar in size. Moreover, it is seen that with similar-size pipes, some inference on the hydraulic behaviour of a network is quite possible. Also, entropy is shown to provide a means of controlling pipe diameter optimization. Furthermore, it is shown that flow entropy has the desirable computational properties and appears to possess the qualitative properties of a reliability measure. It is also found that a convenient way of making sure that a network has an acceptable amount of all-round resilience is to calculate the maximum entropy flows for the network and then size the pipes to carry those flows.

4. State reliability is defined as the ratio of the flow supplied to the flow required for that state. Network reliability is then obtained by summing the state reliabilities weighted by their respective probabilities of occurrence. Defined and calculated as suggested above, network reliability is more meaningful, mathematically correct, and pressure-dependency of flow and less-than-fully satisfactory service are accounted for.

AN ENTROPY-BASED APPROACH TO THE OPTIMUM DESIGN OF RELIABLE WATER DISTRIBUTION NETWORKS

Thesis submitted in accordance with the requirements of

the University of Liverpool

for the degree of

Doctor in Philosophy

by

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March 1993

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ACKNOWLEDGEMENT

I would like to thank my supervisor, Professor A. B. Templeman, for his encouragement, support, assistance and guidance throughout the course of this Ph.D. program. Professor Templeman has very generously given me an enormous amount of his time. Also, he has been instrumental in securing virtually all of the institutional financial assistance that I have received in the last two years of the program. I would also like to thank Dr. R. B. Hoole and Mrs. Sue Spencer for their help in obtaining the above-mentioned financial assistance.

Also, I would like to thank my brother-in-law and sister, Mr. and Mrs. Bate Agborsangaya, for their endless encouragement, and invaluable financial support particularly in the first two years of this program. They have very magnanimously provided the bulk of the finances in the said period.

Additionally, I would like to thank the following institutions for their financial support. First, the British Government for assistance under the Overseas Research Students' Award scheme. Second, The Department of Civil Engineering. Third, the following trusts: The Elmgrant Trust, The Humanitarian Trust, The Newby Trust, The Sir Richard Stapley Educational Trust and The Sidney Perry Foundation.

Finally, I would like to thank my friends, Charles, Greg and Martin, for helping me with the graphics, and Irene, Jasmine and Jasinta, for proof-reading the text.

Tiku T. Tanyimboh

To my mum M'Onege and sisters M'Etaka and M'Ebot.

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TABLE CAPTIONS

NOTATION

*	superscript denoting the optimum value of a variable.
α	dimensionless conversion factor for units.
β	Lagrange multiplier.
$\underline{\beta}$	vector of the βs.
Δq_l	equal change in the flows in loop l.
$\underline{\Delta q}$	change in $\underline{q}$.
$\underline{\Delta x}$	change in $\underline{x}$.
∇C	gradient of C.
γ	constant cost coefficient for pipes.
$\hat{\gamma}$	coefficient for piecemeal linearisation of cost per unit of $\hat{D}_{ij}$.
λ	Lagrange multiplier.
μ	Lagrange multiplier.
$\underline{\mu}$	vector of the μs.
ρ	density of water.
σ_{n-1}	sample standard deviation.
{ }	empty set.
$[\ ]_{max}$	the maximum value that the quantity shown in the brackets can attain.
a_{jn}	effective number of independent paths to node n through link jn.
a_n	a constant for node n.
A	a positive scaling constant.
b_n	a constant for node n.
B_{ij}	benefit parameter for link ij.
C	total cost of pipes.
C_i	ith minimal cut set.
C_{ij}	Hazen-Williams coefficient of pipe ij.
C_{ijm}	Hazen-Williams coefficient of segment m of pipe ij.
d_{1n}	dual variable for the minimum pressure constraint for node n.

d_{2n}	dual variable for the maximum pressure constraint for node n.
d_{ij}	dual variable for the length constraint for link ij.
d_k	number of paths in which link k is used.
d_l	dual variable for the lth loop constraint.
d_p	dual variable for the pth path constraint.
$\underline{d}$	vector representing all the dual variables of a problem.
D	set of all the demand nodes.
D_{ij}	diameter of pipe ij.
D_{ijm}	diameter of segment m of pipe ij.
D_{max}	maximum pipe diameter.
D_{min}	minimum pipe diameter.
D_U	set of commercially available discrete pipe diameters.
$\hat{D}_{ij}$	$= D_{ij}^{2.63}$.
e_1	exponent of the diameter in the cost function for pipes.
e_2	$= e_1/2.63$.
E	rate at which a network dissipates energy.
f_{ij}	friction factor of pipe ij.
F_0	$\equiv S/K$.
$\underline{F}(\underline{x})$	vector of the values of simultaneous equations at the point $\underline{x}$.
$<F_j>$	expected value of the jth expectation constraint.
g	acceleration due to gravity.
h_f	headloss across a fitting.
h_{ij}	headloss in pipe ij.
h_{ijm}	headloss in segment m of pipe ij.
h_{ijr}	headloss in pipe ij for the rth flow regime.
h_p	known headloss for path p.
h_{pr}	known headloss for path p for the rth flow regime.
$\hat{h}_{1n}$	$= \sum_{ij \in IJ_n} h_{ij}$ when referring to lower bound on node pressure.
$\hat{h}_{2n}$	$= \sum_{ij \in IJ_n} h_{ij}$ when referring to upper bound on node pressure.
$\hat{h}_l$	$= \sum_{ij \in IJ_l} h_{ij}$.
$\hat{h}_p$	$= \sum_{ij \in IJ_p} h_{ij}$.

H	maximum headloss between the source(s) and any node.
H^i	value of H for state i.
H^{ij}	value of H when link ij has failed.
H^m	value of H when link m has failed.
$H_{max,n}$	maximum desirable head at node n.
$H_{min,n}$	minimum desirable head at node n.
H_n	total head at node n.
$H_{n,r}$	total head at node n for the rth flow regime.
H_s	total head at source node s.
$\underline{H}$	vector of the H_n.
$\hat{H}_s$	source head required to satisfy all demands.
$H'_{min,n}$	head below which the supply at node n is zero.
$\tilde{H}_n$	a constant for node n.
I	set of all the nodes having an external inflow.
IJ	set of all the links in a network.
IJ_0	set of all the links in a network or its reduced configurations.
IJ_l	set of all the links in loop l.
IJ_n	set of all the links along a specified path from a selected source to node n.
IJ_{nr}	set of all the links along a specified path from a selected source to node n for the rth flow regime.
IJ_p	set of all the links in path p.
IJ_{pr}	set of all the links in path p for the rth flow regime.
J	Jacobian.
k	a general coefficient.
K	arbitrary positive constant.
K_f	coefficient of fitting.
l_{ij}	set of all the loops to which link ij belongs.
L_e	equivalent length of fitting.
L_{ij}	length of link ij.
L_{ijm}	length of segment m of link ij.

LP	linear programming.
LPG	linear programming gradient.
(n)	subscript or superscript denoting iteration number.
n_{ij}	Manning's coefficient for pipe ij.
$n1$	number of finite schemes.
nl_{jn}	total number of links in all the paths supplying node n using link jn.
np_{jn}	number of paths to node n using link jn.
N	number of outcomes or events; also, number of states.
N_e	number of equations.
N_i	number of events or outcomes in the ith finite scheme.
N_{ij}	number of candidate segments for link ij.
N_v	number of variables.
NB_{ij}	expected number of breaks per year per unit length of pipe ij.
NC	number of minimal cut sets.
ND_n	set of all the nodes (or links) immediately downstream of node n.
ND_{nr}	set of all the nodes (or links) immediately downstream of node n for the rth flow regime.
$\tilde{ND}_n$	set of all the link outflows at node n which are part of a loop.
NI	number of source nodes.
NIJ	number of links.
NJ	number of expectation constraints.
NL	number of loops.
NLP	non-linear programming.
NN	number of nodes.
NO	number of demand nodes.
NP	number of paths having a known headloss.
NP_n	number of paths serving node n.
NR	number of flow regimes.
NU_n	set of all the nodes (or links) immediately upstream of node n.

NU_{nr} set of all the nodes (or links) immediately upstream of node n for the rth flow regime.

o_i ith outcome or event.

o_{ij} outcome or event identified by ij.

$\underline{o}$ vector of outcomes or events.

O_i ith finite scheme.

p_{0n} fraction of T_n provided by q_{0n}; $\forall n \in I$.

p_{n0} fraction of T_n satisfying q_{n0}; $\forall n \in D$.

p_i probability of ith outcome or event.

p_{ij} probability of the outcome or event identified by ij .

$\underline{p}$ vector of probabilities.

$p(\ )$ probability of the event or outcome indicated in the parentheses.

$p(0)$ probability that no link is in the failure mode.

$p(m)$ probability that only link m has failed.

$\overline{p}$ notation reserved for probabilities which, as a set, do not constitute a finite scheme.

P_{0n} probability that a particle of flow in a network is from source node n.

P_{n0} probability that a particle of flow in a network leaves the network at demand node n.

q_{0n} external inflow at node n.

q_{0nr} external inflow at node n for the rth flow regime.

q_{ij} flow from node i to node j; flow in link ij.

q_{ijm} $(= q_{ij})$ flow in segment m of link ij.

q_{ijr} flow in link ij for the rth flow regime.

q_m flow in link m.

q_n external inflow or outflow at node n.

q_{n0} external outflow at node n.

q_{n0r} external outflow at node n for the rth flow regime.

$q_{n,r}$ external inflow or outflow at node n for the rth flow regime.

$\underline{q}$ vector representing the flows of a network.

$\hat{q}_i$ the ith independent flow in a looped network.

$\underline{\hat{q}}$ vector representing the independent flows.

q'_{n0} actual abstraction at node n, as opposed to the nominal demand q_{n0}.

$q'_{n0,i}$ actual abstraction at node n for state i .

$q'_{n0,ij}$ actual abstraction at node n when link ij has failed.

Q_0 sum of the link flows of a network.

Q_n sum of link inflows at node n.

$\bar{Q}_n$ sum of the link inflows and link outflows at node n.

r subscript to identify flow regimes.

R network reliability.

R^i network reliability for state i.

R_{ij} reliability of link ij.

R_m reliability of link m.

R_n reliability of node n.

R_n^i reliability of node n for state i.

$\overline{R}$ average reliability of the less-than-fully connected network configurations.

$\overline{R}_n$ average reliability of node n for the less-than-fully connected network configurations.

R' network unreliability.

R'_{ij} unreliability of link ij.

R'_m unreliability of link m.

S entropy of a probabilistic system.

S_0 entropy of the distribution of supplies or demands.

S_0^D entropy of the distribution of the demands.

S_0^S entropy of the distribution of the source supplies.

S_{max} maximum entropy value of a network.

S^I network entropy based on inflows at all nodes.

S_n entropy of node n.

S_n^i entropy of inflows at node n.

S^o	network entropy based on outflows at all nodes.
S^o_n	entropy of outflows at node n.
$\hat{S}$	minimum allowable value of network entropy.
S'_n	modified entropy of node n.
$\tilde{S}$	network entropy based on both inflows and outflows at all nodes.
$\tilde{S}_n$	entropy of node n based on both nodal inflows and outflows.
t	set consisting of all the terminal nodes.
t_{jn}	the ratio q_{jn}/Q_j.
T_0	total supply or demand.
T_n	total flow reaching or leaving node n.
$T'_{0,i}$	total flow suppliable when network is in state i .
$T'_{0,ij}$	total flow suppliable when link ij has failed.
U	the uniform probability distribution.
v_{ij}	flow velocity in pipe ij.
v_{max}	maximum flow velocity.
v_{min}	minimum flow velocity.
w_{ij}	a constant positive weight for link ij.
x	a general variable.
$\underline{x}$	a general vector.
y	a general constant.
z	a general constant.
Z^+	the set of positive integers.

CHAPTER 1 INTRODUCTION

1.1 ENTROPY-BASED INFERENCE ON CIVIL ENGINEERING SYSTEMS

Mathematical system models play an important role in the planning, design and operation of civil engineering projects. These models often require large amounts of data for their implementation and it is often the case that the required data are incomplete, unavailable or uncertain. It is not uncommon, when faced with incomplete data, to apply engineering judgement or a rule of thumb to estimate the missing data, but this approach is not always justifiable. Using ad-hoc procedures and informed guesswork in this way may introduce errors into designs and subsequent operational strategies.

Illuminating examples of the above phenomenon are found in Basu and Templeman (1985). Using carefully-chosen data, that paper shows, for example, that estimates of structural reliability can differ by factors measurable in orders of magnitude depending on the distributions used to represent the load and strength data. It is noted therein that load and strength data tend to cluster around their respective most likely values; it is intuitively obvious that data associated with rare occurrences are often scarce. With data clustered around the most likely value, there may be several analytical probability distributions which appear to fit the data equally well. However, these distributions have different tail characteristics and, as such, lead to different estimates of failure probability. Different tail characteristics result in different estimates of the probability of failure because the failure probability is calculated from the overlapping tail regions of the probability distributions of load and strength. It is therefore evident that it is inappropriate to use some known analytical distribution, for example, normal, log-normal, gamma, etc., to represent random data as such a distribution introduces extrinsic information and hence bias and errors into subsequent calculations. It can be seen from the foregoing discussion that there is a strong case for the development of consistent methodologies and a logical approach to data estimation.

In recent years, methods of logical inference based on the maximum entropy formalism (Jaynes, 1957) have been developed. Perhaps the most obvious use of the maximum entropy formalism is to infer least biased values for a set of probabilities subject to constraints on those probabilities. A somewhat similar application is in the estimation of least biased probability distributions from experimental data. As an example, experimentally-determined values of a continuously varying property such as the yield stress of mild steel, for example, can be used to infer the least biased probability distribution of that property. (The reader who is interested in the generation of maximum entropy probability distributions may consult Basu and Templeman, 1984) However, Templeman (1992) suggests that the maximum entropy formalism can be used to generate solutions to wider, more general problems which are not directly concerned with probabilities and in which the available information is incomplete. In the belief that the maximum entropy formalism has considerable potential for civil engineering data estimation, the present research was originally conceived to examine and develop information theoretic entropy-based methods for a range of problems. These problems include the estimation of structural parameters and behaviour, and the estimation of pipe parameters and the hydraulic performance of inaccessible water supply networks. It was also thought that existing entropy-based methods of estimating traffic flows from limited data would be examined so as to address these problems in a unified fashion and to develop a rigorous method capable of tackling them all. However, the above objective is presently regarded as a distant goal.

The material presented in this thesis is but part of ongoing research into possible civil engineering applications of information theoretic entropy and

the maximum entropy formalism. The innovative aspects of the research reported in this thesis can roughly be divided into two main parts. The first part is concerned with methods of calculating maximum entropy flows in networks with incomplete information; the second part is an exploratory investigation into possible uses of flow entropy in the design and reliability analysis of water distribution networks.

1.2 CALCULATING MAXIMUM ENTROPY FLOWS IN NETWORKS

In suggesting that the applicability of the maximum entropy formalism extends to wider problem beyond the realm of probabilities, Templeman (1992) raised the following question. How can least biased flows be inferred in a looped water distribution network in which the only available information is the values of the nodal inflows and outflows and the direction of flow in each pipe? There are in general very many flow patterns which satisfy the principle of conservation of mass in any looped flow network. As such, flow equilibrium alone will not lead to a unique solution to the problem posed above. On the other hand, the physical laws governing flow in pipe networks cannot be applied to the problem because there is no information on the pipe lengths, diameters and roughness coefficients. Templeman (1992) put forward some ideas on how the Shannon entropy function (Shannon, 1948) can be used to infer least biased flows in networks with incomplete data. However, he realised that the results obtained do not agree with those given by the method of superposition of equal path flows described herein. By intuition, it was concluded that the flows given by the method of superposition of equal path flows appear more reasonable.

One of the objectives of this research is to examine the problem of inferring least biased flows in networks with incomplete data, determine ways of introducing probabilities to flow networks in a rigorous manner, establish what form the entropy function should take for flow networks and find out how the maximum entropy formalism may be used to infer least biased flows in networks with incomplete data.

1.3 ENTROPY AND THE RELIABILITY AND DESIGN OF WATER DISTRIBUTION NETWORKS

The use of entropy and entropy maximization in water distribution networks represents the second phase of this research, and it may be viewed as an application of the ability to calculate the entropy of network flows and maximum entropy flows in networks. The ultimate aim of this phase is to use flow entropy in a water distribution network layout optimization framework. The idea that entropy can be used in this way is inspired in part by the work of Erlander (1977) in which the use of entropy as a simplistic measure of accessibility in a transportation system is advocated. Thus, Erlander (1977) states that a transportation system having a low value of entropy has a low level of accessibility whereas a high value of entropy corresponds to an even distribution of journeys throughout the network.

There is a very close relationship between the layout and hydraulic behaviour of a water distribution network. Therefore, a precondition to the use of entropy for layout optimization is a demonstration that entropy can somehow reflect certain performance-related properties like resilience and reliability. This demonstration is the central theme of much of the rest of the new material covered in this thesis. More specifically, the objective is to show that, in general, the entropy of a water distribution network increases as the resilience/reliability of the network increases. For water distribution networks, properties like performance, resilience and reliability are not straightforward to define and quantify, and there are no universally agreed definitions. Therefore, several measures including headloss, energy dissipation and the probability that no pipe is broken are used herein to rank water distribution

networks. These simplistic measures are supplemented by a new, more comprehensive reliability measure developed herein.

1.4 OBJECTIVES OF THE PRESENT RESEARCH

The objectives of this research are as set out below.

1. To develop a rigorous probability model for flow networks. The probabilities in this model should satisfy flow equilibrium and the mathematical properties of probabilities.
2. To determine the form of Shannon's entropy function that is appropriate for flow networks.
3. To use the maximum entropy formalism to infer least biased flows in networks with incomplete data.
4. To show that flow entropy has the necessary properties of a reasonable reliability measure for water distribution networks.

1.4.1 LAYOUT OF THESIS

In an attempt to compare like and like, an entropy-constrained least cost water distribution network model has been formulated. By varying the value of the entropy constraint, different designs are obtained and any differences in the designs are attributed to the value of entropy specified for each design. The background material in this thesis is therefore arranged as follows. In Chapter 2, the problem of minimizing the cost of the pipes of a water distribution network is reviewed. In Chapter 3, aspects of water distribution networks relating to reliability are reviewed. Chapter 4 is a brief introduction to the maximum entropy formalism.

The remainder of this thesis is arranged as follows. Chapter 5 is concerned with the development of a rigorous framework for inference on flow networks with incomplete data. By visualising flow through a network as a continuous experiment, the relative frequency interpretation of probabilities is used to formulate a multiple-space probability model for the flows on the links of a general flow network. Some of the shortcomings of a single-space probability model, from a flow network perspective, are highlighted. By generalising the formula for the entropy of multiple-space probability schemes and showing equivalents in the probability model for network flows, a formula is set up for the entropy of network flows. The problem of inferring least biased flows given incomplete information is then examined.

Chapters 6 and 7 are an attempt to show that flow entropy reflects the resilience of a water distribution network. In Chapter 6, numerical examples and simulation are used to investigate the relationship between the hydraulic behaviour of a network and flow entropy. The influence of flow entropy on the sizing of pipes is examined. The effect of flow entropy on flow rerouting is assessed. For a more rigorous assessment of the qualities of entropy, a new way of defining and calculating reliability is used in Chapter 7 to compare different least cost entropy-constrained designs of the same network. By basing the reliability calculations on head-driven simulation and the actual flow delivered, the new reliability measure reflects pressure dependency of supply and the way in which water users are affected. Finally, the main ideas and conclusions of Chapters 5 to 7 are summarised in Chapter 8. Some suggestions for further research are discussed. A possible way of using entropy to solve the joint pipe diameter and layout optimization problem is noted.

CHAPTER 2 OPTIMUM DESIGN OF WATER DISTRIBUTION NETWORKS

2.1 INTRODUCTION

In recent years, interest in the cost optimization of looped water distribution networks has increased. The need to replace systems of increasing age and to use public funds efficiently and economically has focused attention on both the cost and reliability aspects of water supply systems. A heavily redundant, highly looped network with large pipes provides a very reliable system with in-built resilience under exceptional conditions. Some exceptional conditions include increased demands for fire fighting and the need to maintain supply even when some pipes are temporarily taken out of service so that repairs can be done. This chapter is concerned with the optimum design of water distribution networks of prespecified layout. The constraints of the design problem consist of the constitutive equations, and additional constraints which are described in this chapter. As seen in Chapter 3, certain design strategies require a separate analysis of the constitutive equations; this approach is sometimes used to simplify the computational solution of the optimum design problem. In Chapter 3 the reliability and layout aspects of the optimum design problem are considered.

In this chapter, the constitutive equations for water distribution networks are presented. Three possible formulations of these equations are given; it is seen later in this chapter and subsequently in this thesis that the methods of analysis and design of pipe networks are dependent upon the way that the constitutive equations are set up. The analysis of flow in pipe networks is described in brief. Also, the optimum design problem, i.e., the least capital cost design of a pipe network, is described. Two methods of solving this problem that are widely held to be among the best are described. The method developed by Alperovits and Shamir (1977) is described in detail. The formulation of Quindry, Brill and Liebman (1981) is then presented, but only briefly because of the similarities in its overall approach to the Alperovits and Shamir method.

2.2 CONSTITUTIVE EQUATIONS

The flow of water in pipes is subject to a loss of energy. The energy loss per unit weight is called a *head loss*. Energy loss is caused by frictional resistance along the wall of the pipe, which opposes fluid motion. Also, energy loss occurs at points along a pipe where the cross-section changes, there is a bend, or there are fittings, such as flow measuring devices, valves, etc. Head losses other than that caused by frictional resistance are called *separation* or *minor losses*. In a pipe network, the constitutive equations are the pipe head loss or energy equations, the equations for flow equilibrium at each node and the equations for the conservation of energy, which are the loop and path equations.

Note: notation

Double subscripts are used throughout herein to identify flows, lengths, friction factors, head loss and diameters, etc. Flow and head loss *from node i to node j* are denoted by q_{ij} and h_{ij} respectively. Also, flow and head loss will be positive in the direction of flow, and negative otherwise. For diameters, lengths and friction factors, the order of the subscripts has no positional significance. Thus, for example, L_{ij}, C_{ij} and D_{ij} are, respectively, the length, friction coefficient and diameter of the pipe between nodes i and j. Also, q_{0i} and q_{i0} denote the external inflow and outflow respectively at node i. In other words, node zero is taken to be the origin or destination of external flows. However, q_i is also used for the external inflow or outflow at node i. In such a case, q_i will be either positive or negative, depending on whether it is an inflow or outflow, but the actual sign depends on the sign convention adopted.

2.2.1 HEAD LOSS EQUATIONS

2.2.1.1 FRICTION HEAD LOSS

The foremost head loss equation for pipes flowing full is the Darcy-Weisbach equation.

$$h_{ij} = 4f_{ij}\frac{L_{ij}}{D_{ij}}\frac{v_{ij}^2}{2g} \tag{2.1}$$

in which L_{ij} and D_{ij} are, respectively, the length and internal diameter of pipe ij; h_{ij} is the head loss, which is positive in the direction of flow; v_{ij} is the mean flow velocity in the pipe; g is the acceleration due to gravity; f_{ij} is the friction factor, which is dimensionless and depends on the flow rate and the roughness of the pipe. Jeppson (1976, pp. 30) has tabulated several equations for f_{ij} for various flow conditions.

In general, f_{ij} cannot be written explicitly in terms of the flow rate and the roughness of the pipe. Consequently, some iterative scheme is usually needed for its determination. Empirical approximate equations, which are easier to use than the Darcy-Weisbach equation, are available. The most frequently met approximation is the Hazen-Williams equation:

$$h_{ij} = \alpha L_{ij}(q_{ij}/C_{ij})^{1.852}/D_{ij}^{4.87} \tag{2.2}$$

where α is a dimensionless conversion factor for units (α = 10.67 in S.I. units); C_{ij} is the Hazen-Williams coefficient and q_{ij} is the flow rate, which is positive in the direction of flow.

Another empirical approximate head loss equation is Manning's equation.

$$h_{ij} = \alpha L_{ij}n_{ij}^2q_{ij}^2/D_{ij}^{5.333} \tag{2.3}$$

in which n_{ij} is Manning's coefficient and α, here, equals 10.29 in S.I. units.

The Hazen-Williams equation, Eq. (2.2), is used throughout herein. However, it will be understood that any alternative head loss equation may be used, instead, if it is considered appropriate.

2.2.1.2 MINOR LOSSES

In water supply networks, friction is usually the predominant cause of head loss. However, minor losses are not always negligible. Usually, the concept of *equivalent length* is used to account for the effect of minor losses. The equivalent length is the length of pipe of the same diameter, as the pipe with the fitting, that would cause the same head loss as the fitting. In general, except for example valves, the head loss across any fitting depends on the flow rate in the pipe with the fitting. Using the equivalent length, Eq. (2.1) may be written as

$$h_f = 4f_{ij}\frac{L_e}{D_{ij}}\frac{v_{ij}^2}{2g} = K_f\frac{v_{ij}^2}{2g} \qquad (2.4)$$

where h_f is the head loss across the fitting; L_e is the equivalent length of the fitting, for the given flow conditions; K_f is a coefficient for the fitting. From Eq. (2.4),

$$L_e = \frac{K_f D_{ij}}{4f_{ij}} \qquad (2.5)$$

See, for example, Douglas, Gasiorek and Swaffield (1979, pp. 274) for some typical values of K_f.

Effective length

The effective length of a pipe is the sum of its physical length and the equivalent length of all its fittings. Throughout herein, it is assumed that all pipe lengths are effective lengths.

2.2.2 CONTINUITY EQUATIONS

At each node, the inflows must balance the outflows. Therefore, the nodal flow equilibrium or continuity equations are

$$\sum_{j \in NU_n} q_{jn} - \sum_{k \in ND_n} q_{nk} = q_n \quad n = 1,\ldots,NN \qquad (2.6)$$

in which q_n is the external inflow or outflow at node n; the set NU_n consists of the upstream nodes of all internal inflows at node n; the set ND_n consists of the downstream nodes of all internal outflows at node n; NN is the number of nodes.

If all external inflows and outflows are known, Eqs. (2.6) are not linearly independent. Continuity at the NNth node will automatically be satisfied if it is satisfied at all the other $(NN-1)$ nodes, assuming that the inflows balance the outflows. Therefore, when all external flows are known, only $(NN-1)$ equations will be necessary in Eqs. (2.6). However, if at least two reservoirs and pumps are present, the flow from (to) these sources will depend on the flows and pressures throughout the system. Consequently, these external flows will not be known. Therefore all the NN Eqs. (2.6) will be linearly independent and will be required.

2.2.3 EQUATIONS FOR CONSERVATION OF ENERGY

2.2.3.1 LOOP EQUATIONS

In a pipe network, conservation of energy requires the net loss of energy around a closed loop to be zero. Therefore, an equation may be written for each loop.

$$\sum_{ij\in IJ_l} h_{ij} = 0 \quad l = 1,\ldots,NL \tag{2.7}$$

in which IJ_l consists of all links in loop l and NL is the number of loops. The number of loops NL refers to the number of "basic" or "natural" loops as depicted in Figure 2.1a. However, as shown in Figure 2.1, there are many ways in which links can be selected to define the loops.

2.2.3.2 PATH EQUATIONS

The total head loss along any path must equal the difference in head between the end points of that path. An equation may therefore be written for each path along which the head loss is known.

$$\sum_{ij\in IJ_p} h_{ij} = h_p \quad p = 1,\ldots,NP \tag{2.8}$$

in which IJ_p consists of all links *specified* for path p, which connects the end points of that path; h_p is the known head loss for path p; NP is the number of paths having a known head loss. In general, NP will be, at most, one less than the number of constant head nodes. Furthermore, to ensure linear independence of the path equations, the NP paths must be specified such that none of the paths duplicates information contained in any other path. Finally, an equation may be written for each link.

$$h_{ij} = H_i - H_j \quad \forall ij \in IJ \tag{2.9}$$

in which H_i and H_j are the total heads at nodes i and j respectively. The total head at a node is the sum of the elevation and the pressure head at that node.

2.2.4 VALVES AND PUMPS

Pumps and valves require special treatment. Jeppson (1976, pp. 80-113 and 129-144) has described ways of modelling pumps and valves. His treatment is quite extensive and includes many examples, with detailed solutions. This thesis, however, is not particularly concerned with pumps and valves.

2.3 ANALYSIS PROBLEM

Usually, analysis consists of determining the pipe flow rates q_{ij}, the nodal heads H_i and the pipe head losses h_{ij}, given all the pipe lengths, diameters and roughness characteristics. Also, the external flows are usually specified. Thus, the analysis problem typically has three kinds of variable. However, it is sometimes desirable to set up and solve the constitutive equations in terms of one kind of variable, which may then be used in explicit equations for the other variables. Also, the performance of solution strategies generally depends on the formulation of the system of equations (see e.g. Jeppson, 1976, pp. 69). Three formulations are therefore presented next. The system of equations may be based either on the pipe flow rates, the nodal heads, or corrective loop flow rates, which are described shortly.

2.3.1 SYSTEMS OF EQUATIONS

2.3.1.1 PIPE FLOW RATES AS UNKNOWNS

The equations for head loss and continuity, Eqs. (2.2) and (2.6) respectively, have been written with the flow rates, q_{ij}, as the independent variables or unknowns. It follows that the unknowns in the loop and path equations, Eqs. (2.7) and (2.8), are the q_{ij}. Following Jeppson (1976), these equations, which are based on the (unknown) pipe flow rates, will be called the q-equations or q-system of equations. For a network in which the number of links is NIJ, the following identity has been proven.

$$NIJ = NN + NL - 1 \quad (2.10)$$

Therefore there will be as many equations as unknown pipe flow rates in Eqs. (2.6), (2.7) and (2.8). (If NN rather than $(NN-1)$ continuity equations are required, the flow from/to the unknown external source will be the extra variable, i.e., as many variables as equations. Without loss of generality, subsequent equations are written for $(NN-1)$ continuity equations only.)

2.3.1.2 NODAL HEADS AS UNKNOWNS

The Hazen-Williams equation, Eq. (2.2), may be written with the head loss as the independent variable. On substituting h_{ij} with the nodal heads from Eqs. (2.9), Eqs. (2.2) may be written as

$$q_{ij} = \alpha C_{ij} D_{ij}^{2.63} \frac{(H_i - H_j)}{L_{ij}^{0.54} |H_i - H_j|^{0.46}} \quad \forall ij \in IJ \quad (2.11a)$$

in which, here, $\alpha = 0.2785$ in S.I. units. Also, Eqs. (2.11a) have been written such that the q_{ij} will always have the correct signs. However, to avoid overflow when the heads at adjacent nodes are nearly equal, it is perhaps better to determine the sign of q_{ij} according to whether H_i is greater or less than H_j, in which case Eqs. (2.11b) below may be used instead. The q_{ij} will be positive if H_i is greater than H_j and negative otherwise.

$$q_{ij} = \frac{\alpha C_{ij} D_{ij}^{2.63} sign(H_i - H_j) |H_i - H_j|^{0.54}}{L_{ij}^{0.54}} \quad \forall ij \in IJ \quad (2.11b)$$

Using Eqs. (2.11b), the continuity equations, Eqs. (2.6), may be written in terms of the nodal heads as

$$\sum_{j \in (NU_n \cup ND_n)} \frac{C_{jn} D_{jn}^{2.63} sign(H_j - H_n) |H_j - H_n|^{0.54}}{L_{jn}^{0.54}} = q_n \quad n = 1,\ldots,NN-1 \quad (2.12)$$

These equations, which are based on the (unknown) nodal heads, will be called the H-equations. Also, as the head loss in every pipe is considered explicitly, the H-equations completely describe the flow in a pipe network without recourse to the loop or path equations. Finally, there will be as many continuity equations as unknown nodal heads; usually, one nodal head will be constant and known.

2.3.1.3 CORRECTIVE LOOP FLOW RATES AS UNKNOWNS

It is possible to express all pipe flow rates in terms of a corrective flow rate around each loop. Thus

$$q_{ij}^{(n)} = q_{ij}^{(n-1)} + \sum_{l \in l_{ij}} \Delta q_l^{(n)} \quad \forall ij \in IJ \quad (2.13)$$

in which $q_{ij}^{(n-1)}$ is an estimated flow rate; $\Delta q_l^{(n)}$ is a correction to be applied, with regard to flow direction, to all flows in loop l; $q_{ij}^{(n)}$ is the corrected flow rate. More appropriately, bracketed superscripts (or subscripts) indicate values that apply in successive iterations in any iterative scheme. This notation is maintained throughout herein. Finally, l_{ij} consists of all loops sharing link ij. In other words, l_{ij}, $\forall ij \in IJ$, is the set which consists of all loops IJ_l, $l = 1,\ldots,NL$, such that link $ij \in IJ_l$. The unknowns in Eqs. (2.13) are the Δq_l. Eqs. (2.13) may be inserted in the head loss equations to give a complete set of equations, based on the Δq_l. These equations are called the Δq system of equations.

$$h_{ij}^{(n)} = \alpha L_{ij} \frac{\left[q_{ij}^{(n-1)} + \sum_{l \in l_{ij}} \Delta q_l^{(n)} \right]^{1.852}}{C_{ij}^{1.852} D_{ij}^{4.87}} \qquad \forall ij \in IJ \tag{2.14}$$

This system contains NL equations, one for each loop, in the NL Δq_l's, each of which corresponds to a loop. This system, like the q-system, requires the loop and path equations. Unlike the latter, however, it does not directly involve the continuity equations. To set up the Δq equations, the initial flow estimates, $q_{ij}^{(0)}$ must satisfy continuity. Thereafter, successive iterates will also satisfy continuity.

2.3.2 NUMERICAL SOLUTION

The equations for flow in pipe networks are highly non-linear. The analysis problem involves the solution of a suitable system of equations. There are three main numerical approaches, which are presented next.

2.3.2.1 LINEAR THEORY METHOD

The Hazen-Williams equation may be linearised as

$$h_{ij}^{(n)} = \frac{\alpha L_{ij} q_{ij}^{(n)} \left[q_{ij}^{(n-1)} \right]^{0.852}}{C_{ij}^{1.852} D_{ij}^{4.87}} \qquad \forall ij,\ n = 1,\ 2 \tag{2.15a}$$

in which the $q_{ij}^{(0)}$, $\forall ij$, are usually set to unity.

$$h_{ij}^{(n)} = \frac{\alpha L_{ij} q_{ij}^{(n)} \left[\frac{q_{ij}^{(n-1)} + q_{ij}^{(n-2)}}{2} \right]^{0.852}}{C_{ij}^{1.852} D_{ij}^{4.87}} \qquad \forall ij,\ n = 3,\ 4,\ 5,\ \ldots \tag{2.15b}$$

Eqs. (2.15) are linear in the $q_{ij}^{(n)}$ and may be used to set up the path and loop equations which are, consequently, also linear. Together with the continuity

equations, which are also linear in the $q_{ij}^{(n)}$, they form a system of linear equations which can be solved by any suitable algorithm, for example, Gaussian elimination. The resulting $q_{ij}^{(n)}$, however, will not necessarily satisfy the continuity, loop, path and head loss equations, because the $q_{ij}^{(n-1)}$ are estimates. However, the above procedure may be used in an iterative scheme, such as the following.

1. Set n to 1. Set all $q_{ij}^{(0)}$ to 1.
2. Set up the q-system of equations.
3. Solve the equations by a suitable means.
4. Test for convergence.
5. If convergence criteria are satisfied, exit. Otherwise, continue.
6. Increase n by 1 and go to step 2.

The linear theory method was developed by Wood and Charles (1972), who suggested the use of Eqs. (2.15b) for faster convergence. They also observed that all $q_{ij}^{(0)}$ may be set to unity, as in the above algorithm. Also, Isaacs and Mills (1980) used the H-equations in a similar way, but concluded that the H-equations are better suited to a network with some heads known, whereas the q-equations will work better if external flows are known.

2.3.2.2 NEWTON-RAPHSON METHOD

Martin and Peters (1963) used the Newton-Raphson iterative scheme for the solution of a system of non-linear equations. The Newton-Raphson formula for a single function, $F(x) = 0$, in one variable, x, is

$$x^{(n+1)} = x^{(n)} - \frac{F(x)}{dF(x)/dx} \tag{2.16}$$

The equivalent formula for a system of equations is (see e.g. Burden and Faires, 1985)

$$\underline{x}^{(n+1)} = \underline{x}^{(n)} - J_{(n)}^{-1} \underline{F}(\underline{x}^{(n)}) \tag{2.17}$$

in which $\underline{F}$ is the vector of the values of the simultaneous equations at the point $\underline{x}$; $\underline{x}$ is the vector of the variables; J is the Jacobian, which is the matrix of the first partial derivatives of each F with respect to each of the x's. Usually, inversion of J is computationally expensive and is avoided by using the following updating scheme. First, set

$$\underline{\Delta x}^{(n+1)} = J_{(n)}^{-1}\underline{F}(\underline{x}^{(n)}) \tag{2.18}$$

in which $\underline{\Delta x}$ is the vector of the change in each x. Premultiplying both sides of Eq. (2.18) by $J_{(n)}$ gives

$$J_{(n)}\underline{\Delta x}^{(n+1)} = \underline{F}(\underline{x}^{(n)}) \tag{2.19}$$

The linear Eqs. (2.19) are solved for $\underline{\Delta x}^{(n+1)}$, which is then used in the following updating formula.

$$\underline{x}^{(n+1)} = \underline{x}^{(n)} - \underline{\Delta x}^{(n+1)} \tag{2.20}$$

Obviously, the symmetry of J may be exploited for greater computational efficiency. The Newton-Raphson method, which has just been described, may be used to solve the H-, q- or Δq-system of equations for pipe networks. Also, Shamir and Howard (1968) have described a generalised formulation of the Newton-Raphson method. Under certain conditions, their approach enables the analysis of networks with mixed variables, including diameters, external flows, etc. See, for example, Shamir and Howard (1968) for further details on the Newton-Raphson method. Additionally, see Wood and Rayes (1981), for example, for a comparative study of various pipe network solution strategies.

2.3.2.3 HARDY-CROSS METHOD

In the Hardy-Cross method the constitutive equations are solved sequentially, rather than simultaneously, in each iteration. Also, each equation is solved for a single variable only, while keeping all the other variables fixed. The method is demonstrated here for the Δq-equations.

First, Eqs. (2.13) are simplified as

$$q_{ij}^{(n)} = q_{ij}^{(n-1)} + \Delta q_l^{(n)} \quad \forall l, \quad \forall ij \in IJ_l \tag{2.21}$$

That is, when considering the links in loop l, the corrections due to other loops sharing these links are neglected. The resulting head loss equations are

$$h_{ij}^{(n)} = \frac{\alpha L_{ij}\left[q_{ij}^{(n-1)} + \Delta q_l^{(n)}\right]^{1.852}}{C_{ij}^{1.852} D_{ij}^{4.87}} \quad \forall l, \quad \forall ij \in IJ_l \tag{2.22}$$

Thus the loop equations are

$$\sum_{ij \in IJ_l} h_{ij}^{(n)} = 0 \quad \forall l \tag{2.23}$$

in which the lth equation has only one variable which is $\Delta q_l^{(n)}$. Each of these equations may be solved separately for its $\Delta q^{(n)}$ to any desired accuracy, using the univariate Newton-Raphson iterative formula, Eq. (2.16). The resulting Δq_l are then used in Eqs. (2.13) or (2.21) to correct the pipe flow rates. Thus ends one iterative cycle.

A new cycle can be started by setting up and solving Eqs. (2.23) again, etc. This process can be continued until, for example, the changes in all the pipe flow rates in successive cycles become insignificant, and the loop and path equations are satisfied. See, for example, Jeppson (1976, pp. 147) for a sample step-by-step implementation of the Hardy-Cross method.

2.4 LEAST COST DESIGN OF PIPES

This section describes ways of cheaply designing a water distribution network after the demand at each node of the network has been specified. Also, it is assumed herein that the layout has been specified along with the direction of flow in each link. Sometimes, a network may have different flow patterns and these flow patterns need to be considered in the design process. As an example, a network which has a storage tank has one flow pattern when the tank is filling up and a different pattern when the tank is emptying. Also, as explained by Templeman (1982b), reliability considerations may make it necessary to consider a multiplicity of flow patterns when designing a distribution system. Various aspects of reliability are considered in Chapter 3.

In the remainder of this chapter, some of the ways of formulating and solving the problem of minimizing the cost of constructing a water distribution network are described. This problem of minimizing the cost of a network can take different forms including those of Chapter 3. The version of this problem which is considered in this chapter is for a general network in which the layout and flow patterns including the direction of flow in each link are specified. If the layout of a network is specified, then, the length of each link is known. Therefore, the problem of minimizing the cost of the network consists of determining the cheapest set of diameters for the pipes of the network, including the pipe flow rates.

2.4.1 OBJECTIVE FUNCTION AND CONSTRAINTS

Cost objective function

It is customary to minimize the capital cost of the pipe network. Usually, the cost per unit length of pipeline is given by a function of the form

$$F(D_{ij}) = \gamma D_{ij}^{e_1} \quad \forall ij \in IJ \qquad (2.24)$$

where the coefficients γ and e_1 depend on the units of D_{ij}. In general, e_1 is greater than unity and, typically, lies between about 1.25 and 2.0. The total cost of pipes is therefore

$$C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e_1} \qquad (2.25)$$

where C is the total cost of pipes. Eq. (2.25) is the objective function which is minimized over the diameters D_{ij}; γ and e_1 will be known and, for a fixed layout, the length L_{ij} will be constant and known.

When minimizing Eq. (2.25), the constraints of the network, which are described below, have to be satisfied. The constraints of the problem of minimizing the cost of the network are the constitutive equations plus constraints due to practical considerations, which restrict the velocity of flow in each pipe, the pipe diameters and the nodal heads. Also, the flow rates cannot be negative. These additional constraints may be written as:-

Maximum and minimum flow velocity constraints

$$v_{\min} \le v_{ij} \le v_{\max} \quad \forall ij \in IJ \qquad (2.26)$$

in which v_{ij} is the flow velocity in pipe ij; $v_{\min}$ and $v_{\max}$ are lower and upper bounds, respectively, on the velocities. The velocity v_{ij} is given by

$$v_{ij} = \frac{4q_{ij}}{\pi D_{ij}^2} \quad \forall ij \in IJ \qquad (2.27)$$

Substituting for v_{ij} in Eqs. (2.26) and rearranging,

$$\frac{\pi v_{\min}}{4} \le \frac{q_{ij}}{D_{ij}^2} \le \frac{\pi v_{\max}}{4} \quad \forall ij \in IJ \qquad (2.28)$$

Maximum and minimum nodal pressure constraints

$$H_{min,n} \le H_n \le H_{max,n} \quad \forall n \qquad (2.29)$$

in which the $H_{min,n}$ and $H_{max,n}$ are, respectively, the lower and upper bounds on the nodal heads H_n. According to the path equations, Eqs. (2.8), the head loss along any path must equal the difference in head between the end points of that path. Thus for any source s, $s \in I$, where the set I consists of all source nodes,

$$\sum_{ij \in IJ_n} h_{ij} = H_s - H_n \quad \forall n \tag{2.30}$$

where the set IJ_n consists of all links along a *specified* path from a *selected* source to node n. Eqs. (2.30) may be written as

$$H_n = H_s - \sum_{ij \in IJ_n} h_{ij} \quad \forall n \tag{2.31}$$

Therefore, substituting for H_n in Eqs. (2.29),

$$H_s - H_{max,n} \le \sum_{ij \in IJ_n} h_{ij} \le H_s - H_{min,n} \quad \forall n \tag{2.32}$$

Maximum and minimum pipe diameter constraints

$$D_{\min} \le D_{ij} \in D_D \le D_{\max} \quad \forall ij \in IJ \tag{2.33}$$

in which D_D is the set of available discrete pipe diameters; D_{max} and D_{min} are the upper and lower bounds, respectively, on the diameters.

Non-negativity of flows

$$q_{ij} \ge 0 \quad \forall ij \in IJ \tag{2.34}$$

The objective function and all the constraints, including the constitutive equations, of the problem of minimizing the cost of pipes are now brought together as Problem 1.

Problem 1

$$\underset{\forall D_{ij}}{Minimize} \quad C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e_1} \tag{2.25}$$

subject to:

$$h_{ij} = \alpha L_{ij} (q_{ij}/C_{ij})^{1.852} / D_{ij}^{4.87} \quad \forall ij \in IJ \tag{2.2}$$

$$\sum_{j \in NU_n} q_{jn} - \sum_{k \in ND_n} q_{kn} = q_n \quad n = 1,\dots,NN-1 \tag{2.6}$$

$$\sum_{ij \in IJ_l} h_{ij} = 0 \quad l = 1,\dots,NL \tag{2.7}$$

$$\sum_{ij \in IJ_p} h_{ij} = h_p \quad p = 1,\dots,NP \tag{2.8}$$

$$\frac{\pi v_{\min}}{4} \le \frac{q_{ij}}{D_{ij}^2} \le \frac{\pi v_{\max}}{4} \quad \forall ij \in IJ \tag{2.28}$$

$$H_s - H_{max,n} \le \sum_{ij \in IJ_n} h_{ij} \le H_s - H_{min,n} \quad \forall n \tag{2.32}$$

$$D_{\min} \le D_{ij} \in D_D \le D_{\max} \quad \forall ij \in IJ \tag{2.33}$$

$$q_{ij} \ge 0 \quad \forall ij \in IJ \tag{2.34}$$

In Problem 1, the L_{ij} are constant and will be known. Also, the Hazen-Williams coefficient will be known, and, if all pipes are made from the same material, the C_{ij} will be equal. The variables of Problem 1 are therefore the D_{ij}, q_{ij} and h_{ij}, $\forall ij \in IJ$. Also, the objective function of Problem 1 includes only the capital cost of pipes. However, if it is considered appropriate, the objective function may be modified to include other costs, such as operating costs. See, for example, Awumah and Goulter (1992), Cheng and Ma (1989) for some alternative cost functions. In the present research, however, only the capital cost of pipes is considered.

The above representation of the problem is for a single design load. However, if there are multiple demand patterns, it may be necessary to consider each flow regime explicitly in the constraints. Thus, corresponding to each demand pattern is a set of nodal heads, pipe flow rates and head losses. Consequently, for multiple demand patterns, all the constraints should be stated for each demand pattern. However, the diameter constraints do not depend on the flows and should be included only once. Also, an additional subscript r is used to identify the variables in the rth flow regime. Thus q_{ijr} and h_{ijr} are, respectively, the pipe flow rate and head loss in link ij for the rth flow regime, $\forall ij \in IJ,\ r = 1,...,NR$, where NR is the number of demand patterns. Also, $H_{nr},\ \forall n,\ \forall r$, is the head at node n for the rth load case, etc. Note that the external inflow or outflow at node n is $q_{n,r}$, q_{0nr} or q_{n0r} for the rth regime. Finally, the sets NU_n, ND_n and IJ_n also take the subscript r to identify the appropriate flow regime. Problem 1 for multiple load cases is stated below as Problem 2.

Problem 2

$$\underset{\forall D_{ij}}{Minimize} \cdot C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e_1} \qquad (2.25)$$

subject to:

$$h_{ijr} = \alpha L_{ij}(q_{ijr}/C_{ij})^{1.852}/D_{ij}^{4.87} \quad \forall ij \in IJ,\ \forall r \qquad (2.35)$$

$$\sum_{j \in NU_{nr}} q_{jnr} - \sum_{k \in ND_{nr}} q_{knr} = q_{n,r} \quad n = 1,...,NN-1,\ \forall r \qquad (2.36)$$

$$\sum_{ij \in IJ_l} h_{ijr} = 0 \quad l = 1,...,NL,\ \forall r \qquad (2.37)$$

$$\sum_{ij \in IJ_{pr}} h_{ijr} = h_{pr} \quad p = 1,...,NP,\ \forall r \qquad (2.38)$$

$$\frac{\pi v_{min}}{4} \le \frac{q_{ijr}}{D_{ij}^2} \le \frac{\pi v_{max}}{4} \quad \forall ij \in IJ,\ \forall r \qquad (2.39)$$

$$H_s - H_{max,n} \le \sum_{ij \in IJ_{nr}} h_{ijr} \le H_s - H_{min,n} \quad \forall n,\ \forall r \qquad (2.40)$$

$$D_{min} \le D_{ij} \in D_D \le D_{max} \quad \forall ij \in IJ \qquad (2.33)$$

$$q_{ijr} \ge 0 \quad \forall ij \in IJ,\ \forall r \qquad (2.41)$$

2.4.2 SOLUTION OF PROBLEM 1 (OR 2)

Problem 1 (or 2) is highly non-linear and extremely difficult to solve (Yates, Templeman and Boffey, 1984). The problem may be simplified slightly if continuous, as opposed to discrete pipe diameters, are used in the optimization. If continuous pipe diameters are used, some optimum diameters may not be available. Such pipes can be replaced in the actual network by two pipes in series which are equivalent to the pipe that is being replaced. To replace such pipes, first, the head loss in the equivalent pipes should be the same as in the pipe being replaced. Also, in order for the actual network to remain at least near optimal, the replacement pipes are usually selected from D_D such that, in magnitude, the diameters of these pipes lie either side of, and as close as possible to, the continuous D_{ij}^*. The superscript * is used for the optimum value of a variable. Furthermore, the sum of the lengths of the replacement pipes must equal the length of link ij.

Thus to determine the length of the replacement pipes, the following simultaneous equations can be set up and solved.

$$L_{ij} = L_{ij1} + L_{ij2} \qquad (2.41)$$

in which L_{ij1}, L_{ij2} are the lengths of the first and second replacement pipes, respectively, for link ij.

$$h_{ij}^* = h_{ij1} + h_{ij2} = \alpha\left[\frac{L_{ij1}}{D_{ij1}^{4.87}} + \frac{L_{ij2}}{D_{ij2}^{4.87}}\right]\left[\frac{q_{ij}^*}{C_{ij}}\right]^{1.852} \tag{2.42}$$

where, for example, h_{ij1} and D_{ij1} are respectively, the head loss and diameter for the first replacement pipe for link ij. The selection of the diameters D_{ijm}, $m = 1, 2$, from D_D has already been explained. They are therefore known. Also, all quantities in Eqs. (2.41) and (2.42) are known, except L_{ij1} and L_{ij2}. Furthermore, the equations are linear in the unknown lengths and can be solved to give

$$L_{ij1} = \frac{h_{ij}^* - k_2 L_{ij}}{k_1 - k_2} \tag{2.43}$$

$$L_{ij2} = -\frac{h_{ij}^* - k_1 L_{ij}}{k_1 - k_2} \tag{2.44}$$

in which the coefficients k_1 and k_2 are

$$k_1 = \frac{\alpha(q_{ij}^*/C_{ij})^{1.852}}{D_{ij1}^{4.87}} \tag{2.45}$$

$$k_2 = \frac{\alpha(q_{ij}^*/C_{ij})^{1.852}}{D_{ij2}^{4.87}} \tag{2.46}$$

In Eqs. (2.42) to (2.46) it is assumed that $C_{ij1} = C_{ij2} = C_{ij}$. However, different values may be used for these coefficients, if necessary, without any difficulty.

Returning to Problem 1, it is therefore possible to solve it, with the discreteness constraint relaxed, by any suitable algorithm for constrained non-linear programming. Also, any optimum diameter D_{ij}^* such that $D_{ij}^* \notin D_D$ can be replaced in the actual network, as described above, by two commercially available pipes connected in series. Furthermore, there will be in general many local minima. Consequently, it may be useful to solve the problem a few

times with different starting points to select the solution with the smallest objective function value. However, much research has gone into developing solution methods for Problem 1 (or 2) and two widely-used approaches are described next.

2.4.2.1 LINEAR PROGRAMMING GRADIENT METHOD

The Linear Programming Gradient (LPG) method (Alperovits and Shamir, 1977) is based on the Δq-equations. The method solves Problem 1 (or 2) as two subproblems which are linked in an iterative scheme. The details of each subproblem are presented in turn immediately following the present outline of the overall strategy. The first subproblem minimizes the cost of the network for a complete set of *specified* pipe flow rates. In this phase a linear programming problem is set up and solved. The results of this linear program are then used in the second subproblem, the gradient phase, to modify the pipe flow rates to further reduce the total cost of pipes. The resulting flow rates are then used to set up another linear programming problem. The new linear programming problem is solved, new flow rates are determined, etc. This cycle is repeated until there is no significant reduction in cost between successive cycles.

Details of the linear programming phase are given next. For simplicity, reference will be made to Problem 1 only. For a given flow rate, the head loss equation, Eq. (2.2), is non-linear in the diameter only. It follows that for a complete set of specified pipe flow rates, Problem 1 is non-linear in the diameters only. However, the problem is linear in the known pipe lengths and this property is exploited in the LPG method. For a complete set of specified pipe flow rates, it is desired to convert Problem 1 into a linear programming (LP) problem. This is achieved by reformulating the problem in such a way that the lengths, in which the problem is linear, rather than the diameters, in which the problem is non-linear, become the unknowns.

Thus suppose each pipe is made up of several segments of known diameter but unknown length. Figure 2.2a shows a conventional pipe and Figure 2.2b shows an equivalent three-segment pipe. For the given segments, the transformed, but equivalent, optimization problem is concerned with finding the optimum length of each segment of known diameter. Thus let L_{ijm} denote the length of segment m of link ij. As the length of each *link* is constant, the sum of the lengths of the segments of each link must be equal to the length of that link. That is,

$$\sum_{m=1}^{N_{ij}} L_{ijm} = L_{ij} \quad \forall ij \in IJ \tag{2.47}$$

$$L_{ijm} \geq 0 \quad \forall ijm \tag{2.48}$$

in which N_{ij} is the number of segments specified for link ij. N_{ij} may vary from link to link, as seen later herein. These equations, Eqs. (2.47), are the length (compatibility) equations and are linear in the L_{ijm}. They are an extra set of constraints in the LPG formulation of Problem 1. Also, the objective function is given by

$$C = \gamma \sum_{ij \in IJ} \sum_{m=1}^{N_{ij}} L_{ijm} D_{ijm}^{e_1} \tag{2.49}$$

in which D_{ijm} is the diameter of segment m of link ij. Thus C is linear in the segmental lengths, given the segmental diameters. Furthermore, from continuity, the flow rate in each segment of link ij equals the pipe flow rate q_{ij} of that link. Finally, the head loss in link ij is the sum of the head loss in each segment of that link, i.e.,

$$h_{ij} = \sum_{m=1}^{N_{ij}} h_{ijm} \quad \forall ij \in IJ \tag{2.50}$$

in which h_{ijm} is the head loss in segment m of link ij. The head loss in a segment is given by the head loss equation, i.e.,

$$h_{ijm} = L_{ijm} \frac{\alpha (q_{ij}/C_{ij})^{1.852}}{D_{ijm}^{4.87}} \quad \forall ijm \tag{2.51}$$

which is linear in the L_{ijm} if q_{ij} and D_{ijm} are specified.

On the other hand, the velocity constraints Eqs. (2.28) of Problem 1 are not a function of the L_{ij}. As such, they do not lend themselves to the variable transformation of the LPG formulation. However, knowing q_{ij}, Eqs. (2.28) can be used to obtain lower and upper bounds for the D_{ij}. The resulting values can be used with the pipe diameter constraints, Eqs. (2.33), to obtain the segmental diameters to be specified for each link. As the selection of segmental diameters is based on pipe flow rates, both the number of segments and their diameters will in general be different for each link.

After carrying out the variable transformation described above, this transformed, but equivalent, problem is then linear in the L_{ijm} and is stated below as Problem 3. In Problem 3, the nodal flow continuity equations, Eqs. (2.6) have been omitted because they are used when specifying the pipe flow rates for the linear program. Also, Problem 3 is the LP formulation corresponding to Problem 1 which is for a single flow regime. The LP formulation for multiple load cases follows from Problem 3 in the same way that Problem 2 is derived from Problem 1. This, however, is not done herein.

Problem 3

$$\underset{\forall L_{ijm}}{Minimize} \; C = \gamma \sum_{ij \in IJ} \sum_{m=1}^{N_{ij}} D_{ijm}^{e_1} L_{ijm} \tag{2.49}$$

subject to:

$$\sum_{ij\in IJ_l}\sum_{m=1}^{N_{ij}} \frac{\alpha(q_{ij}/C_{ij})^{1.852}}{D_{ijm}^{4.87}} L_{ijm} = 0 \quad l=1,\ldots,NL \tag{2.52}$$

$$\sum_{ij\in IJ_p}\sum_{m=1}^{N_{ij}} \frac{\alpha(q_{ij}/C_{ij})^{1.852}}{D_{ijm}^{4.87}} L_{ijm} = h_p \quad p=1,\ldots,NP \tag{2.53}$$

$$\sum_{ij\in IJ_n}\sum_{m=1}^{N_{ij}} \frac{\alpha(q_{ij}/C_{ij})^{1.852}}{D_{ijm}^{4.87}} L_{ijm} \le H_s - H_{min,n} \quad \forall n \tag{2.54}$$

$$\sum_{ij\in IJ_n}\sum_{m=1}^{N_{ij}} \frac{\alpha(q_{ij}/C_{ij})^{1.852}}{D_{ijm}^{4.87}} L_{ijm} \ge H_s - H_{max,n} \quad \forall n \tag{2.55}$$

$$\sum_{m=1}^{N_{ij}} L_{ijm} = L_{ij} \quad \forall ij \in IJ \tag{2.47}$$

$$L_{ijm} \ge 0 \quad \forall ijm \tag{2.48}$$

Problem 3 is an LP problem with continuous variables, the segmental lengths, and can be solved by the Simplex method (see e.g. Winston, 1987). At the solution of Problem 3, C^* and the L^*_{ijm} will be available. Also available will be the optimum value of the dual variable for each constraint (see e.g. Winston, 1987). Referring to Problem 3, there is a dual variable d_l, $l=1,\ldots,NL$, for the lth loop constraint; a dual variable d_p, $p=1,\ldots,NP$, for the pth path constraint; similarly, d_{1n}, $\forall n$, for the minimum service pressure constraint at node n; d_{2n}, $\forall n$, for the maximum service pressure constraint at node n; d_{ij}, $\forall ij \in IJ$, for the length constraint for link ij. Thus, at the solution of the LP problem, the d^*_l, d^*_p, d^*_{1n}, d^*_{2n} and d^*_{ij} will be available. Denote by $\underline{d}$ the vector of all the dual variables. In the gradient phase of the LPG method, $\underline{d}^*$ is used to modify the pipe flow rates so that if Problem 3 is set up and solved

for the new pipe flow rates, the new value of C^* will be lower than the current value of C^*. The gradient phase is described next.

The gradient phase of the LPG method (Alperovits and Shamir, 1977; Quindry, Brill, Liebman and Robinson, 1979; Fujiwara, Jenchaimahakoon and Edirisinghe, 1987) is concerned with adjusting the pipe flows so that the optimum design for the new flows is lower in cost than the optimum design for the old flows. That is, the vector of flow changes $\Delta\underline{q}$ is sought so that

$$C^*(\underline{q} + \Delta\underline{q}) \le C^*(\underline{q}) \tag{2.56}$$

in which the elements of $\underline{q}$ are the pipe flow rates. Using the Δq-equations gives $\underline{q}$ as

$$q_{ij}^{(n)} = q_{ij}^{(n-1)} + \sum_{l\in l_{ij}} \Delta q_l^{(n)} \quad \forall ij \in IJ \tag{2.13}$$

in which the superscript $(n-1)$ denotes values on entering the gradient phase and (n), values found in the gradient phase; n is the cycle or iteration number.

The problem of determining the Δq_l^* at each iteration is a non-linear programming problem that may be approached in different ways. It will usually be necessary to find the gradient of C (cost) with respect to each Δq_l, i.e., $\nabla C(\Delta\underline{q})$. However, C is related to the Δq_l only indirectly, through the loop constraints; Δq_l is the correction for all pipes in loop l. On the other hand, any path constraint with a path IJ_p, $p=1,\ldots,NP$, using any links contained in loop l, $l=1,\ldots,NL$, will be affected whenever Eqs. (2.13) are applied. Also, the nodal pressure constraints will be affected in a similar way. Thus C in Problem 3 is related to the Δq_l through all the problem constraints, except the length compatibility conditions, Eqs. (2.47). It is therefore possible to write: $C = f(\hat{h}_p, \hat{h}_l, \hat{h}_{1n}, \hat{h}_{2n})$ in which

$$\hat{h}_p = \sum_{ij \in IJ_p} \sum_{m=1}^{N_{ij}} h_{ijm} \quad p = 1,...NP \tag{2.57}$$

is the head loss in path p;

$$\hat{h}_l = \sum_{ij \in IJ_l} \sum_{m=1}^{N_{ij}} h_{ijm} \quad l = 1,...NL \tag{2.58}$$

is the head loss around loop l;

$$\hat{h}_{1n} = \sum_{ij \in IJ_n} \sum_{m=1}^{N_{ij}} h_{ijm} \quad \forall n \tag{2.59}$$

is the head loss in the path specified for node n;

$$\hat{h}_{2n} = \hat{h}_{1n} = \sum_{ij \in IJ_n} \sum_{m=1}^{N_{ij}} h_{ijm} \quad \forall n \tag{2.60}$$

Also, $\hat{h}_{1n}$ and $\hat{h}_{2n}$ are equal but, respectively, they correspond to the minimum and maximum nodal pressure constraints which are different from each other. Finally, Eqs. (2.57) to (2.60) are all functions of the pipe flow rates and, hence, functions of Δq_l, $l = 1,...NL$. That is: $\hat{h}_p = f(\Delta q_1,..., \Delta q_{NL})$; $\hat{h}_l = f(\Delta q_1,..., \Delta q_{NL})$; $\hat{h}_{1n} = f(\Delta q_1,..., \Delta q_{NL})$; $\hat{h}_{2n} = f(\Delta q_1,..., \Delta q_{NL})$, $\forall p$, $\forall l$, $\forall n$. Using the chain rule (see e.g. Kreyszig, 1983), the components of $\nabla C(\underline{\Delta q})$ are

$$\frac{\partial C}{\partial \Delta q_l} = \sum_{p=1}^{NP} \frac{\partial C}{\partial \hat{h}_p} \frac{\partial \hat{h}_p}{\partial \Delta q_l} + \sum_{\tilde{l}} \frac{\partial C}{\partial \hat{h}_{\tilde{l}}} \frac{\partial \hat{h}_{\tilde{l}}}{\partial \Delta q_l}$$

$$+ \sum_{n=1}^{NN} \frac{\partial C}{\partial \hat{h}_{1n}} \frac{\partial \hat{h}_{1n}}{\partial \Delta q_l} + \sum_{n=1}^{NN} \frac{\partial C}{\partial \hat{h}_{2n}} \frac{\partial \hat{h}_{2n}}{\partial \Delta q_l} \quad \forall l \tag{2.61}$$

The squiggle ~ is used merely to set apart the l's pertaining only to the second summation in the above equation. Also, the first term in each of the products is the dual variable for the constraint in question, i.e., $\partial C/\partial \hat{h}_p = d_p$, $\forall p$; $\partial C/\partial \hat{h}_l = d_l$, $\forall l$; $\partial C/\partial \hat{h}_{1n} = d_{1n}$, $\forall n$; $\partial C/\partial \hat{h}_{2n} = d_{2n}$, $\forall n$. At the solution of Problem 3, the LP phase, the optimum values of these dual variables will be available. It remains to find expressions for the gradients of each constraint with respect to each Δq_l, i.e., the second term in each product in Eqs. (2.61). This is done next.

The head loss equation for segment m of link ij, Eq. (2.51), is written here in terms of the flow changes Δq_l:

$$h_{ijm} = \frac{\alpha L_{ijm} \left[q_{ij} + \sum_{l \in l_{ij}} \Delta q_l \right]^{1.852}}{C_{ij}^{1.852} D_{ijm}^{4.87}} \tag{2.62}$$

The equations for the $\hat{h}$, Eqs. (2.57) to (2.60), are all sums of various subsets of the h_{ijm}. The required gradients can be found by summing the gradients of the appropriate h_{ijm}. Thus,

$$\begin{aligned} \frac{\partial h_{ijm}}{\partial \Delta q_l} &= \pm 1.852\, h_{ijm}/q_{ij} \quad l \in l_{ij} \\ &= 0 \quad l \notin l_{ij} \end{aligned} \tag{2.63}$$

where the plus sign applies if q_{ijm} and Δq_l are both clockwise or both anticlockwise in loop l. Otherwise, the negative sign applies. From Eq. (2.63), for example,

$$\frac{\partial \hat{h}_p}{\partial \Delta q_l} = \pm 1.852 \sum_{\substack{ij \in IJ_p \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.64}$$

Therefore, finally, the first summation of Eq. (2.61) is

$$\sum_{p=1}^{NP} \frac{\partial C}{\partial \hat{h}_p} \frac{\partial \hat{h}_p}{\partial \Delta q_l} = \pm 1.852 \sum_{p=1}^{NP} d_p \sum_{\substack{ij \in IJ_p \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.65}$$

in which the substitution $\frac{\partial C}{\partial \hat{h}_p} = d_p$, $p = 1,\ldots,NP$, has been used.

Corresponding expressions for the other terms in Eq. (2.61) can be obtained in a similar way. These expressions are:

$$\sum_{\tilde{i}} \frac{\partial C}{\partial \hat{h}_{\tilde{i}}} \frac{\partial \hat{h}_{\tilde{i}}}{\partial \Delta q_l} = \pm 1.852 \sum_{\tilde{i}} d_{\tilde{i}} \sum_{\substack{ij \in IJ_{\tilde{i}} \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.66}$$

where the squiggle ~ is used again merely for contrast.

$$\sum_{n=1}^{NN} \frac{\partial C}{\partial \hat{h}_{1n}} \frac{\partial \hat{h}_{1n}}{\partial \Delta q_l} = \pm 1.852 \sum_{n=1}^{NN} d_{1n} \sum_{\substack{ij \in IJ_n \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.67}$$

$$\sum_{n=1}^{NN} \frac{\partial C}{\partial \hat{h}_{2n}} \frac{\partial \hat{h}_{2n}}{\partial \Delta q_l} = \pm 1.852 \sum_{n=1}^{NN} d_{2n} \sum_{\substack{ij \in IJ_n \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.68}$$

Inserting the expressions of Eqs. (2.65) to (2.68) in Eq. (2.61) gives the equation for the components of $\nabla C(\underline{\Delta q})$:

$$\nabla C_l \equiv (1/1.852) \frac{\partial C}{\partial \Delta q_l} = \pm \sum_{p=1}^{NP} d_p \sum_{\substack{ij \in IJ_p \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm}$$

$$\pm \sum_{\tilde{i}} d_{\tilde{i}} \sum_{\substack{ij \in IJ_{\tilde{i}} \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \pm \sum_{n=1}^{N} (d_{1n} + d_{2n}) \sum_{\substack{ij \in IJ_n \\ l \in l_{ij}}} (1/q_{ij}) \sum_{m=1}^{N_{ij}} h_{ijm} \qquad l = 1,\ldots,NL \tag{2.69}$$

Eq. (2.69) gives the gradient of the cost C with respect to the flow change around each loop and is the expression sought. It may be used in the determination of the loop flow adjustments. On entering each gradient phase of the LPG iterative scheme, all the quantities in Eq. (2.69) will be available. The ∇C_l contain information that may be used in a suitable optimization algorithm. Such an algorithm can be used to find $\underline{\Delta q}$, which, when used in Eqs. (2.13), gives the flow rates for the next linear program. In this document, however, no actual algorithm for calculating $\underline{\Delta q}^{(n)}$ or $\underline{q}^{(n)}$ is given. It is perhaps more appropriate to refer to the original publications and, for example, Gill, Murray and Wright (1981), for methods of optimization. Alperovits and Shamir (1977) used a steepest gradient-based heuristic whereas Fujiwara, Jenchaimahakoon and Edirisinghe (1987) used a quasi-Newton method with a BFGS update (see e.g. Gill, Murray and Wright, 1981).

Once $\underline{\Delta q}$ has been found, the pipe flow rates are updated using Eq. (2.13). The resulting $\underline{q}$ is used in the next LP phase. The LP results are then used in another gradient phase. And thus the cycle continues, with successive reductions in cost, until some suitable convergence criteria are satisfied. The description of the linear programming gradient approach to the solution of Problem 1 is now complete. There follows a brief presentation of an alternative, but quite similar approach, to the LPG method.

2.4.2.2 GRADIENT FORMULATION OF QUINDRY, BRILL AND LIEBMAN

The number of variables, usually, at least three per link per load case, in the LPG formulation grows very rapidly as the size of the network increases. Quindry, Brill and Liebman (1981) have described a method that is similar in strategy to the LPG method. However, it uses continuous variables for the diameters and assumes that each link consists of a single pipe of uniform diameter. In consequence, the number of variables is much smaller than in the LPG method.

The Quindry, Brill and Liebman formulation differs from the LPG approach mainly in the way that the linearisation is brought about. However, like LPG, a linear programming phase and a gradient step are linked in an iterative cycle such that the design at the end of each cycle is cheaper than the previous cycle's. The method is based on the H-equations, Eqs. (2.12). Thus, if a complete set of nodal heads are specified, these equations are non-linear in the diameters only; the $L_{ij}^{0.54}$ are known constants. These equations may then be linearised by defining a new variable (Lai and Schaake, 1969) as

$$\hat{D}_{ij} = D_{ij}^{2.63} \quad \forall ij \in IJ \tag{2.70}$$

This gives Eqs. (2.12) as

$$\sum_{j \in (NU_n \cup ND_n)} \frac{\alpha C_{jn} \hat{D}_{jn} sign(H_j - H_n) | H_j - H_n |^{0.54}}{L_{jn}^{0.54}} = q_n \quad n = 1, \ldots, NN-1 \tag{2.71}$$

in which q_n is the external inflow or outflow at node n. Eqs. (2.71) are linear in the $\hat{D}_{ij}$.

Also, using Eq. (2.70), the cost function Eq. (2.18) becomes

$$F(\hat{D}_{ij}) = \gamma \hat{D}_{ij}^{e_2} \tag{2.72}$$

in which $e_2 = e_1/2.63$ and is typically about 0.50 to 0.80. This cost function is non-linear in $\hat{D}_{ij}$. However, if several cost functions are used for the $\hat{D}_{ij}$ as shown in Figure 2.3, then, the cost may be linearised as

$$F(\hat{D}_{ij}) = \hat{\gamma} \hat{D}_{ij} \quad \forall ij \in IJ \tag{2.73}$$

where the cost coefficient $\hat{\gamma}$ takes on different values for each cost band. The objective function is then

$$C = \hat{\gamma} \sum_{ij \in IJ} L_{ij} \hat{D}_{ij} \tag{2.74}$$

With the objective function and constraints linear in $\hat{D}_{ij}$, the problem can be solved by linear programming. However, the cost coefficient $\hat{\gamma}$ for each $\hat{D}_{ij}$ depends on the value of that D_{ij} at the starting point of the computational solution of the LP problem. On the other hand, $\hat{D}_{ij}^{*}$ may not lie in the same cost band as at the start of the computational solution of the LP problem. In such a case, the initial value of $\hat{y}$ will be incorrect. It is therefore necessary, at the solution of each linear program, to reassign values to the $\hat{y}$ based on the $\hat{D}_{ij}^{*}$ and resolve the problem. This process should continue until the values of the $\hat{y}$ are the correct values for the $\hat{D}_{ij}^{*}$.

Like in LPG, the results of the linear program can be used to calculate adjustments to the nodal heads. The updated heads may then be used to set up another linear program, and so on, until there is no significant difference in the cost of successive LP designs, or there is convergence of the nodal heads. Quindry, Brill and Liebman (1981) have described a gradient step in which the changes to the nodal heads are calculated. They have also presented a derivation of $\nabla C(\underline{H})$. Alternatively, steps similar to those used herein in the derivation of $\nabla C(\underline{\Delta q})$ may be used to obtain $\nabla C(\underline{H})$. For more on the Quindry, Brill and Liebman (1981) formulation, the interested reader should consult the original publication and a far ranging discussion by Templeman (1982b).

To conclude this section on the solution of Problem 1, it may be noted that the Alperovits and Shamir (1977) LPG method can handle network components other than pipes. The interested reader may consult their paper. Also, Lansey and Mays (1989) have presented a formulation that uses both simulation and non-linear programming. In this approach, in each iteration of the optimization, a network solver is used to evaluate the continuity and energy equations. Thus, by implicitly solving the constitutive equations, the constraint size is reduced. This enables distribution networks with many components under various loading conditions to be handled. However, the method is computationally expensive and requires a large amount of computer time.

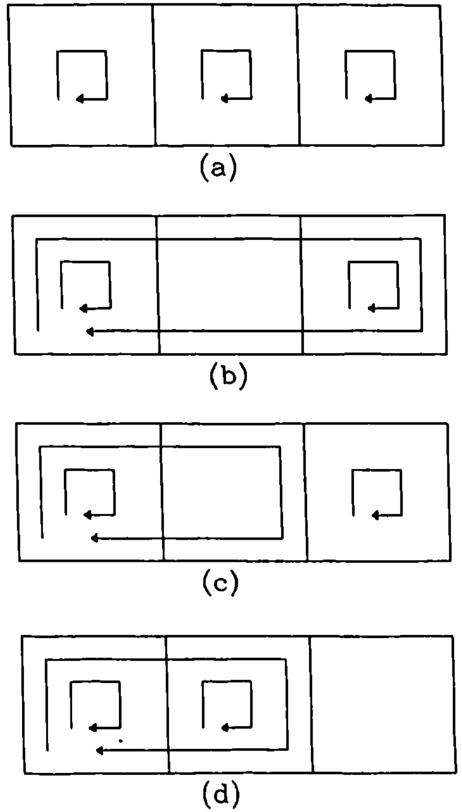


Figure 2.1 Choosing the loops of a network: (a) to (c) are some, but not all, of the possible choices. (d) is an example of a poor choice. (d) is not a good choice because the three right hand side links are unaccounted for and the three resulting loop equations are not linearly independent; for example, every link in the outer loop is included in one of the inner loops.

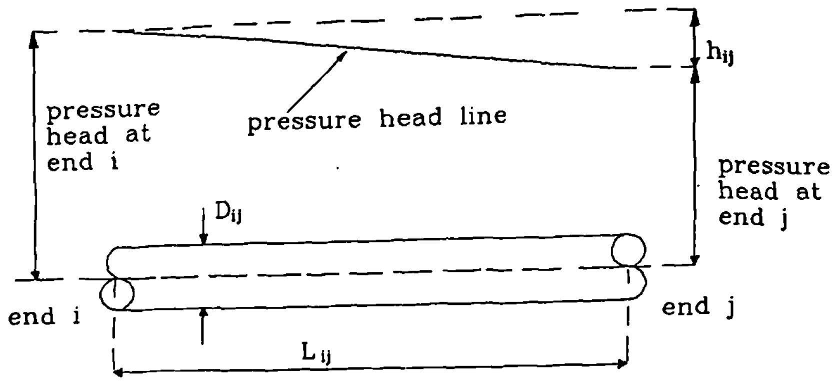


Figure 2.2a Headloss in a constant-diameter pipe

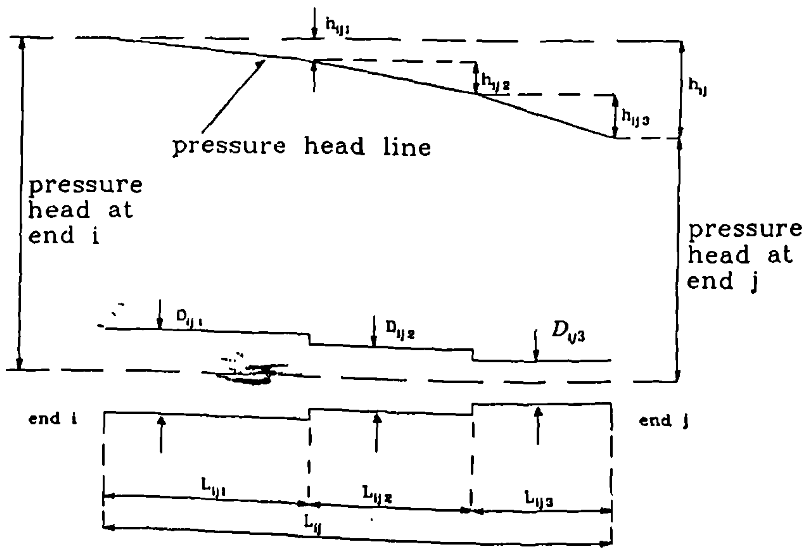


Figure 2.2b Headloss in a segmental-diameter pipe

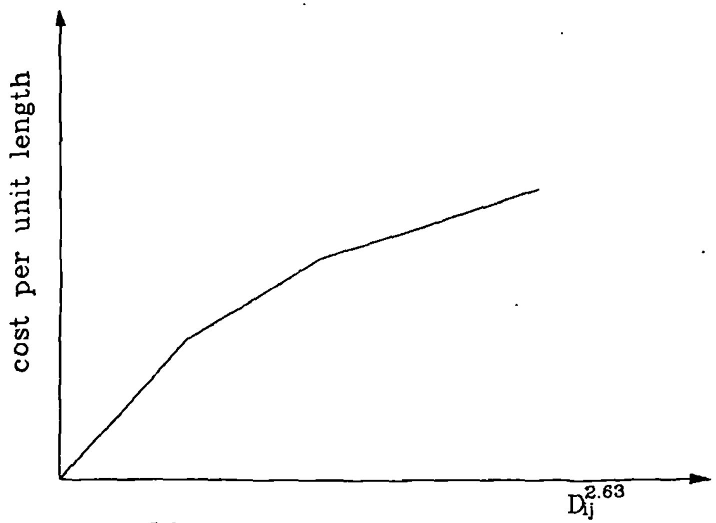


Figure 2.3 Schematic of piecemeal linearisation of pipe cost per unit of $D_{ij}^{2.63}$

CHAPTER 3 WATER DISTRIBUTION NETWORK RELIABILITY AND ITS OPTIMIZATION

3.1 INTRODUCTION

It is widely accepted that the supply of water in urban areas should be reliable and available on demand. However, a water supply system is a network with many components which are subject to random failures. To lessen the impact of failures within the distribution system, pipe networks for urban water supply are usually designed with loops. These loops reduce the possibility of some demand points being completely cut off from the rest of the network.

The effect and severity of each component failure will be different. Also, following a failure in a looped network, the behaviour of the reduced network will be different from its normal mode of operation and, in general, will not be predictable without hydraulic simulation of the reduced network. For these and other reasons which are discussed later in this chapter, it is difficult to define reliability in the context of urban water supply. In a moment, some definitions and measures of reliability from the literature are presented. Later, separate sections of this chapter consider how some of these measures can be calculated and optimized. Also, surrogate measures are very relevant to the present work and various aspects of these measures are considered separately in this chapter.

The issue of reliability is slightly different for tree-type networks. Usually, there will be no uncertainty regarding the consequence of a pipe failure in a tree-type network. Obviously, there is some uncertainty about the actual effects of failure on consumers, but that is outside the scope of the present research. Although some of the material in this chapter may be applicable to tree-type water distribution networks, the focus of the present research is on looped networks.

Two types of failure may be identified within a water distribution system. These include mechanical failure of the components, for example, pipes, valves, etc. Also, hydraulic failure may be said to have occurred if the system is incapable of delivering the right quantity of water at the right pressure. Hydraulic failure may ensue from a mechanical failure. Also, hydraulic failure at a node may arise because of excessive abstraction elsewhere in the network, for example, for fire fighting, leading to a reduction in pressures throughout the network. It may be noted that extraneous factors such as power supply and availability of water are also involved in the wider reliability issue. However, the work reported in this thesis is concerned only with failures due to an increase in demand or a pipe failure.

When a pipe breaks, there is loss of water and pressure at the point of breakage. This continues until the flow of water to the broken pipe is stopped. However, the pressure may not be fully restored because it will usually be necessary to take certain pipes out of service to isolate the broken pipe. This may lead to increased pipe flows and hence extra head loss in the reduced network. It is this extra loss of pressure that may lead to hydraulic failure if the shortfall in pressure reaches a critical level.

It has been explained above that a distribution network can suffer from both mechanical and hydraulic failure. Therefore, it is sometimes necessary to distinguish between mechanical and hydraulic reliability. Mechanical reliability measures the probability that the component or system being considered is operational at any time. The mechanical reliability of a network depends on the arrangement or layout of its components and the mechanical reliability of the individual components. On the other hand, hydraulic reliability is a measure of the probability that the system can supply the right amount of water at the right pressure. Although hydraulic reliability depends on mechanical reliability, it is largely governed by the hydraulic performance of the network. In turn, the performance depends on the layout and capacities of the pipes, the number, locations and capacities of storage facilities and pumps, the spatial and temporal variations in supply and demand, the number and positions of valves and other appurtenances, etc.

The above characterization of mechanical and hydraulic reliability appears straightforward. In practice, however, the issue is very complicated. In fact, there is no comprehensive or universally accepted practical definition of reliability in the context of water distribution. By "practical definition" is meant any definition or measure that is both sufficiently realistic and easy to calculate. Furthermore, the complexity of the optimum design problem is reduced considerably if the reliability measure can be optimized directly.

Walski (1987) has pointed out that a reliability measure should reflect the way that water users are affected. He therefore asserts that the indicator of reliability should include the length of time and number of users out of service, with units of "user-days of outage per unit time" or gallons per year of shortfall. Similarly, Wagner, Shamir and Marks (1988b) have stated that the amount of required flow not supplied is a good overall indicator of network reliability. However, they warn that: a low shortfall could be obtained by disconnecting, at any sign of emergency, one node of moderate demand in order to supply the others; and so the percentage of time spent in subnormal conditions should be noted for each node. On the other hand, Goulter *(1987) has stressed that a true measure of reliability* must somehow recognize both the probabilistic nature of failures and the severity of such failures.

Also, with regard to hydraulic failure, it is increasingly recognized that supply at a node does not stop immediately the head falls below the desired minimum level (Wagner, Shamir and Marks, 1988b; Cullinane, Lansey and Mays, 1992). It is therefore inappropriate to use a zero-one relationship to describe the changes in nodal supply in response to changes in nodal pressure near the desired minimum head. Between the desired minimum pressure and the

absolute minimum pressure below which service is unacceptable, there is reduced but perhaps acceptable service. Cullinane, Lansey and Mays (1992) have observed that the transition from full service through reduced service to failure is better represented by a continuous fuzzy function. As an example, Wagner, Shamir and Marks (1988b) have assumed that the reduction of supply from normal supply to no flow is related to the square root of the actual nodal head. This assumption is motivated by the consideration that hydraulic laws for flow through devices show that flow is proportional to the square root of head. Thus the actual flow supplied is given as follows.

$$q'_0 = \left[\frac{H_{min,n} - H_n}{H_{min,n} - H'_{min,n}} \right]^{0.5} q_{n0} \qquad \forall n \tag{3.1}$$

in which q'_{n0} is the actual abstraction at node n, $\forall n$; q_{n0} is the demand at node n, $\forall n$; $H_{min,n}$ is the desirable minimum head at node n, $\forall n$; $H'_{min,n}$ is the irreducible minimum head at node n, $\forall n$; H_n , $H'_{min,n} \leq H_n \leq H_{min,n}$, is the actual head at node n, $\forall n$.

To sum up, the following properties are desirable for a realistic measure of reliability.

1. It should be a time-based probabilistic measure.
2. It should reflect the shortfall in flow.
3. It should recognize the pressure dependency of demand. Furthermore, reduced service should be recognized and accounted for somehow, rather than be counted as failure.
4. It should be sufficiently easy to calculate. Furthermore, it would be most useful if it lends itself to direct optimization.

3.2 SOME RELIABILITY MEASURES

There are probably many possible indicators of reliability for water distribution networks. These may be divided roughly into two groups. One group consists of measures for properties that are inherently related to reliability. Usually, the exact relationship between these *surrogate* measures and reliability is unclear. The entropy measure proposed herein is one such measure. Surrogate measures have an important place in the reliability and design of water distribution networks. As such, they are covered separately in Section 3.5. A second group of reliability indices consists of actual reliability measures and these are considered next.

On their own, measures of mechanical reliability may be used to check for unreliability due to inadequate network connectivity or unreliable components. It has been shown (e.g. Provan and Ball, 1983) that the problem of calculating reliability exactly for a general network on the basis of connectivity only is extremely difficult to solve. Tung (1985) has reviewed some of the methods for calculating the mechanical reliability of water distribution networks. He concluded that the cut set approach with first order approximation (see e.g. Billington and Allan, 1983) is the most efficient from a computational viewpoint. A *cut set* is a set of system components which, if it fails, causes the system to fail. A summary of the cut set method is given in appendix A.

There are many indices of mechanical reliability. They measure the expectation of the level of connectivity in a network at any time. Some useful measures from the literature are now presented. Tung (1985) has defined network reliability as the probability that flow can reach all demand points, and unreliability as the probability that any demand point cannot be reached. Let R and R' stand for network reliability and unreliability respectively. Tung's definition may be written as:

$$R = p(\text{all demand nodes are reachable}) \tag{3.2}$$

$$R' = p(\text{at least one demand node is isolated}) \tag{3.3}$$

Also, Wagner, Shamir and Marks (1988a) have used a similar definition:

$$R = p(\text{all demand nodes are connected to a source}) \tag{3.4}$$

Also, they have defined nodal reliability R_n, $\forall n$, as:

$$R_n = p(\text{node } n \text{ is connected to a source}) \quad \forall n \tag{3.5}$$

Turning to hydraulic reliability, it has been explained herein that it is very dependent upon mechanical reliability. Since the latter is difficult to calculate, it might be expected that the former is difficult to calculate too. In fact, the problem of finding the probability that each node in a distribution network will receive sufficient supply is extremely difficult to solve (Valliant, 1979). Also, it has been seen that the constitutive equations for a distribution network are a highly non-linear system of equations whose analysis is computationally expensive. Furthermore, it is not possible to deduce the performance of a network with a failed component from the normal mode of operation of the network. This is due to the complex ways in which flows in a looped network are rerouted. It is therefore necessary in general to analyse a reduced network to know whether it is likely to fail hydraulically. In a network subject to random component failures, it will be necessary to analyse many reduced network configurations to find the exact probability that the network will perform satisfactorily. In general, the amount of computer time required for such calculations is very large, as there are up to 2^{NIJ} configurations, where NIJ is the number of links, to consider. It is therefore common to make some approximations when calculating hydraulic reliability.

Several useful measures of hydraulic reliability from the literature are presented next. Wagner, Shamir and Marks (1988a) have defined network *reliability* as *the* probability that all *specified* nodal demands can be supplied. Also, Bao and Mays (1990) have defined reliability as the probability that the system can provide the demanded flow rate at the required pressure head. Finally, following Carey and Hendrickson (1984), Fujiwara and de Silva (1990) have defined reliability in terms of the expected minimum total shortfall. That is, the unreliability is taken as the ratio of the expected minimum total shortfall to the total demand. The reliability is then defined as the complement of the unreliability. Thus,

$$R' = \frac{\text{expected minimum total shortfall}}{\text{total demand}} \tag{3.6}$$

$$R = 1 - R' \tag{3.7}$$

An outline of how the above measures may be calculated is given next. To obtain the probability of sufficient supply, Wagner, Shamir and Marks (1988a) assigned a capacity to each link, as explained shortly. They then determined whether each reduced network configuration can provide enough flow by modelling it as a maximum flow network problem. The problem of determining the maximum flow of a network subject to maximum link capacities is well known and efficient LP solution methods have been developed for it (see e.g. Bazaraa and Jarvis, 1977). Knowing the probability that the network will be in each reduced configuration, it is therefore theoretically possible to calculate the probability that the network will satisfy the demands. This probability is given by the complement of the joint probability of the configurations that cannot supply the required flow. That is, the reliability is the complement of the probability that the network will be in any of the reduced configurations which cannot supply the required flow.

The capacity of each link was set based on an assumed maximum hydraulic gradient of 0.01. It may be noted that pipes in a distribution network do not actually have a capacity; the flow through a pipe is in general determined by the amount of available pressure in the network.

Also, the definition of Bao and Mays (1990) is conceptually similar to the probability of sufficient supply. It is the probability that the system can provide the demanded flow rate at the required pressure head. However, Bao and Mays (1990) used a completely different approach in quantifying this measure. They were not concerned with hydraulic failure ensuing from mechanical failure. Rather, they were interested in the hydraulic reliability with regard to some given probability distributions for the nodal demands or pressure head requirements. A demand-based hydraulic network simulation in which all nodal demands are met, irrespective of actual pressure levels, was used. For each node, the hydraulic reliability was defined as the probability that the actual nodal pressure is equal to or greater than the minimum required. Based on this definition, nodal reliability may be obtained using the following equation.

$$R_n = p(H_n \geq H_{\min,n}) = \int_{H_{\min,n}}^{\infty} F(H_n)\, dH_n \quad \forall n \tag{3.8}$$

where $F(H_n)$ is the probability density function of the pressure head of the supplied flow at node n, $\forall n$. The interested reader may consult Bao and Mays (1990) for the distributions used. For the system reliability, Bao and Mays used three complementary measures: the reliability of the most unreliable node; the average of the nodal reliabilities; the sum of the nodal reliabilities weighted according to the respective nodal demands.

Finally, the definition of Fujiwara and de Silva (1990), Eqs. (3.6) and (3.7), is addressed next. To reduce the number of failed configurations to be considered (out of a total of 2^{NIJ} where NIJ is the number of links), Fujiwara and de Silva (1990) assumed that the probabilities of configurations with multiple failed links are negligible. Thus Eqs. (3.6) and (3.7) are based on states with no more than one failed link. For each configuration, the maximum flow delivered is approximated using a maximum flow network model with capacities assigned to each link. How these capacities are decided is discussed in the next section which deals with the optimization of reliability. Knowing the maximum flow delivered, the minimum shortfall for each state is given by the difference between the total demand and the maximum flow delivered. The expected minimum total shortfall for the system is then given by the sum of the shortfall for each state, weighted according to the respective state probabilities. Finally, the reliability is given by Eqs. (3.6) and (3.7).

3.3 LAYOUT AND RELIABILITY OPTIMIZATION

The reliability of a water distribution network is inherently linked to its layout. Most of the material in the literature is concerned with optimizing reliability for prespecified layouts and these are considered shortly. However, some effort has gone towards layout design which is dealt with immediately.

3.3.1 LAYOUT AND PIPE DIAMETER OPTIMIZATION

The problem of optimizing the layout of a distribution network can take different forms. For example, the optimum layout for the network of Figure 3.1a could be a subset of the links shown in the figure, but it could also involve layouts with extra nodes such as Figure 3.1b. There appears to be no published material in which the possibility of introducing extra nodes is considered. All subsequent references herein to layout optimization will be limited to the selection of a subset from all the possible links that connect the existing nodes.

Layout optimization is extremely difficult, partly because it involves Problem 1 which has been considered in Chapter 2. Also involved are other subproblems that are very difficult to solve and these are pointed out in the course of this review. Besides, reliability considerations significantly increase the complexity of the problem of simultaneously optimizing the layout and components of a

water distribution network; layout optimization is intimately related to the size of the components. The following review of layout optimization is structured in such a way that the main strategies that have been used are highlighted. To this end, it will be necessary to explain some, but not all, of the approaches in some detail.

Layout optimization models may be classified as follows. Those that start with all the potential links of the network and then go on to delete some of these links in an attempt to reduce cost while satisfying some reliability criteria. Some of the formulations in this category are those by Morgan and Goulter (1985), and Awumah, Bhatt and Goulter (1989). The above models are considered shortly. A second category includes models that begin with a spanning tree and then go on to add redundant links in an attempt to meet some reliability criteria while keeping the cost of the network as low as possible. The models in this category that are reviewed below are those by Rowell and Barnes (1982), and Loganathan, Sherali and Shah (1990).

Rowell and Barnes (1982) used two main steps. In step 1, a spanning-tree layout is simultaneously found and designed by solving a non-linear minimum cost flow problem. In step 2 redundant links are selected from the non-tree links to provide an alternative supply path to each demand node. This is a very difficult problem to solve (see e.g. Rowell and Barnes, 1982; Loganathan, Sherali and Shah, 1990). Rowell and Barnes used a 0-1 integer programming formulation which also determines the diameter of the added redundant links.

However, there are some weaknesses in the method. First, the formulation of the non-linear minimum cost flow model is based on the assumption that all the links in the initial tree will have the same hydraulic gradient. Goulter and Morgan (1984) have pointed out that this assumption is invalid. Perhaps more serious is the absence of any mechanism to ensure that any loop and path constraints are not violated following the addition of redundant links.

On the other hand, Loganathan, Sherali and Shah (1990) assured hydraulic consistency of the network by including a phase in which the network is redesigned following the addition of redundant links, but with minimal adjustment to the initial near optimal core tree design. Also, the approach has other important differences from Rowell and Barnes. Firstly, neither in the design of the core tree nor the final looped network are diameters determined at the same time as flows. Flows are first determined and then diameters are found using the LP step of the LPG method.

The design of the spanning tree is as follows. An initial tree-type design is required to initiate the procedure. For single-source networks a minimum spanning tree (see e.g. Templeman, 1982a) is specified and its flows determined from continuity. For multiple-source networks, a linear minimum cost flow model is used to determine the flow direction in each link and to obtain an initial spanning tree, including link flows. The rest of the procedure is the same for both single- and multiple-source networks. The initial tree with known flows is designed by LP. The next step is to sequentially add each potential link not already in the current tree and remove a link in the resulting loop to obtain a less expensive spanning tree design. The process ends with a near optimal core tree in which no further substitutions are possible.

Secondly, Loganathan, Sherali and Shah formulated the problem of finding redundant links based on connectivity only. The resulting 0-1 integer programming problem is thus solvable by an LP-based heuristic (see original publication and reference therein). The ensuing looped network is then redesigned by LP while keeping any changes to the core tree to a minimum. This phase needs some judgement by the designer and the interested reader may consult the original publication.

The method appears able to find good solutions (see demonstration examples in original publication). It decomposes a complex problem into efficiently solvable subproblems. In particular, it may be noted that the problem of finding redundant links is not solved with an integer programming algorithm. However, the cost of constructing the looped network is kept down by using mostly minimum diameter pipes for the redundant links. Such use of minimum diameter pipes is questionable (Wagner, Shamir and Marks, 1988a). This issue is discussed further in Chapter 6.

The models just described first select a core tree and add loop-completing links to it. In contrast, the models considered next start with all the potential links and eliminate those that are excess to requirements. Awumah, Bhatt and Goulter (1989) were able to formulate a 0-1 integer program by using assumed *initial nodal heads*. The main constraint of the formulation, from a reliability point of view, is the requirement that each node be connected by at least two links. Furthermore, network specific constraints are used to ensure that no nodes can be cut off by the failure of a single link. The need for such constraints in the present formulation is illustrated in Figure 3.2. However, the method requires several candidate diameters for each link and this leads to very many variables. In particular, there is also a 0-1 integer variable for each link and for each candidate diameter.

On the other hand, the method of Morgan and Goulter (1985) is an LP-based heuristic. A key feature of the formulation is that, for a given network, the number of variables and constraints remain constant no matter how many demand patterns are considered. This is the result of a clever choice of decision variables. For each link, two variables are defined. One of these represents the length of pipe of the *current* diameter to be replaced by the next smallest diameter. The other represents the length of pipe of the *current* diameter to be replaced by the next largest diameter. Obviously, at most one of the two will be non-zero at the solution, so the number of non-zero variables will be at most equal to the number of links.

The method addresses the reliability issue by ensuring that the design is sufficiently resilient. To this end, a multiplicity of flow patterns based on potential link failures and fire fighting demands (Templeman, 1982b) are considered. For multiple flow patterns, each pattern is simulated. Nodal pressure constraints are specified for the worst cases only, so that there are as many constraints as there are links, i.e., the same as the maximum number of non-zero decision variables. The rationale is that the maximum number of non-zero decision variables must be equal to the number of constraints. It may be noted that by using a network solver, the loop and nodal flow equilibrium equations are satisfied implicitly.

However, the formulation does not have an in-built capability for removing links. Instead, at the end of each iteration a link is removed manually, based on a heuristic link-weighting scheme. Also, link removal is sequential and may lead to results that are quite suboptimal. Finally, the analysis of all flow regimes for multiple load patterns is computationally expensive.

Thus far, reliability and cost optimization have been considered in the context of joint layout and component design. Also, no components other than pipes have been mentioned even though some of these formulations may be used for systems with other components. It will be recalled that the present research deals with pipes only. The remainder of this section is devoted to reliability optimization for fixed layouts.

3.3.2 PIPE DIAMETER OPTIMIZATION

The problem of minimizing the cost of constructing a water distribution pipe network, without explicit regard to reliability, has been described in Chapter 2. In this subsection, reliability considerations are integral to the optimum

design problem. For a fixed layout, the reliability-based optimum design problem consists of sizing the prespecified components of the water distribution network to some reliability specifications, at the least possible cost. Reliability can be assured in different ways. For example, a multiplicity of demand patterns, including very adverse cases, could be added to the constraints. Thus many of the methods for basic pipe diameter optimization could be used for some form of reliability-based pipe diameter optimization. However, the difficulty of solving the problem increases with each additional load case. In this section, only models that optimize the reliability of pipes directly and explicitly are reviewed. In these models, reliability is defined on the interval [0, 1]. They include Su, Mays, Duan and Lansey (1987); Cullinane, Lansey and Mays (1992); Fujiwara and de Silva (1990). Some reliability definitions and methods of evaluation have been presented in Section 3.2. The aim of this subsection is primarily to show how some of these measures can be optimized directly.

First, the model of Su, Mays, Duan and Lansey (1987) is an NLP formulation in which continuity and loop equations are satisfied implicitly by a simulation model. Only the cost objective function and the other usual constraint functions are specified explicitly. The value of the reliability constraint is calculated by the minimum cut set method. The minimum cut sets are determined by simulating link failures. A failure that results in any nodal heads being too low is a minimum cut set for those nodes and the system. Reliability is defined as the probability of sufficient flow/pressure.

The main strength of the model is that reliability is calculated quite accurately and is optimized directly. However, the use of simulation for system optimization and, in particular, in the determination of reliability requires a large amount of computer time. Also, the gradients of the reliability constraint cannot be calculated analytically. Finite difference approximations are used. In this model, the computational expense of calculating these gradients is very considerable because evaluation of the reliability function itself requires simulation. Finally, all non-zero shortfalls in flow/pressure are not included in the definition of reliability.

Second, the model of Cullinane, Lansey and Mays (1992) is an NLP formulation that may be used to directly optimize a simulation-based reliability measure. The method is conceptually similar to the approach of Su, Mays, Duan and Lansey (1987). The main disadvantage of the model is that it requires a large amount of computer time.

Last, in Fujiwara and de Silva (1990) reliability is optimized using an LP-based heuristic. Thus a benefit parameter B_{ij} is calculated for link ij, $\forall ij$, using the following equation.

$$B_{ij} = \left[w_{ij} \frac{\partial R}{\partial q_{ij}} + 1 \right] \exp\left[-\frac{\partial C}{\partial q_{ij}} / A \right] \quad \forall ij \tag{3.9}$$

in which B_{ij} is the benefit parameter for link ij, $\forall ij$: the full meaning of this parameter will be clear in a moment; w_{ij} is a constant positive weight for link ij, $\forall ij$; A is a positive scaling constant; C is the total system cost; R is the system reliability and q_{ij} is the flow in link ij, $\forall ij$. Also, it may be noted that the benefit parameter B_{ij}, $\forall ij$, was originally termed "length", as opposed to the terminology herein.

The rationale for Eq. (3.9) is that:

a) The benefit increases if $\frac{\partial R}{\partial q_{ij}}$, $\forall ij$, increases, and vice versa.

b) The benefit increases if $\frac{\partial C}{\partial q_{ij}}$, $\forall ij$, decreases, and vice versa.

c) The B_{ij}, $\forall ij$, are always strictly positive. This is assured by the $(+1)$ term; the $\frac{\partial R}{\partial q_{ij}}$, $\forall ij$, are always non-negative (see Fujiwara and de Silva, 1990).

Thus B_{ij} is an indicator of how beneficial, in terms of both cost and reliability, it is to increase flow in link ij, $\forall ij$. The interested reader may consult Fujiwara and de Silva for more details.

To determine which flows should be increased, the network is transformed to an equivalent supersource-supersink version as shown in Figure 3.3. Then, the path, from the supersource to the supersink, having the highest total benefit $\sum B_{ij}$, where the summation is taken over all links in the path, is found using any longest path algorithm (see e.g. Bazaraa and Jarvis, 1977). The flow in each link on this path is increased by a small predetermined amount (same increment for all links) and other link flows adjusted accordingly.

The above phase provides the linkage between an LP pipe-sizing phase and the previously described Fujiwara and de Silva (1990) model (Section 3.2) for calculating reliability. In the iterative scheme consisting of the above phases, the flows found using the benefit parameter are used as both link *flows* and *capacities* in the reliability model.

The strength of this formulation is its computational efficiency; it uses only LP. Also, it has a mechanism that attempts to increase reliability at a minimal cost. However, the definition of link capacities is blurred. The maximum throughput obtained for the capacitated network used in the reliability model may be an underestimate. Also, it is assumed that this underestimation should, to some extent, compensate for the neglect of pressure dependency of flows in the reliability calculations.

3.4 SURROGATE RELIABILITY APPROACHES

In has been seen in the preceding sections of this chapter that the definition, quantification and optimization of reliability in looped water distribution networks are fraught with difficulty. In an attempt to bypass some of these difficulties, graph and information theory principles have been applied. In the graph theory approaches, a degree of invulnerability is assured by emulating some of the properties of less vulnerable graphs. In this section, two graph theory-based papers are reviewed. They include Jacobs and Goulter (1989), and

Kessler, Ormsbee and Shamir (1990). This review follows shortly. On the other hand, Awumah, Bhatt and Goulter (1990, 1991, 1992) have used information-theoretic entropy to define a quantifiable surrogate for redundancy/reliability. The above research is closely related to the work reported herein and is discussed in some detail after the following paragraphs on the use of graph theory.

3.4.1 GRAPH THEORY-BASED APPROACHES

Jacobs and Goulter (1989) have presented a layout-only model. The model is essentially an application of the following graph theory result: damage-resistant optimal graphs are regular in degree at all nodes. The degree of a node is the number of links connected to it. As for the method, integer goal programming is used to generate a regular network, by minimizing the differences between the degrees of the nodes of the network. This network is then examined for weaknesses. The weaknesses may include, for example, a subnetwork not connected to the rest of the network. Network-specific constraints are then added and the model rerun. This sequence is repeated until all weaknesses have been removed. However, no hydraulic considerations are involved in the formulation. The only involvement of flows is by a heuristic weighting of each node by the inverse of its demand. However, this turned out to be quite ineffectual. Also, no costs enter the formulation and neither do link capacities or lengths.

However, the approach of Kessler, Ormsbee and Shamir (1990) is very different. The concept of *two trees* is used to design a network that is invulnerable to a single failure. In the layout phase of this method, two *spanning trees* are selected. The trees are selected such that they overlap and, together, they ensure the existence of an alternative path to each demand node following a single link or node failure. The trees are found with graph theory algorithms (see original paper for details). After finding the two trees, the pipes are sized

by LP. Invulnerability is conferred to the network by designing each tree so that, on its own, it can supply all the demands of the entire network at adequate pressure. This is done by specifying a lower bound on the nodal pressure for each node of each tree.

A useful feature of the two trees approach is the joint, although sequential, treatment of both layout/reliability and component optimization. Also, both paths to each demand node are actually designed. As such, there is a good idea of the extent to which each of the paths to each node can be relied on. However, the method is only applicable to single-source networks. Also, there is no means of determining the best pair of trees prior to a full design and evaluation of each pair. Furthermore, the invulnerability cannot be optimized directly or quantified.

Also, there is a slight theoretical weakness in the formulation. There are no loop equations in the constraint set which consists of length constraints (Chapter 2) and a lower bound on the head at each node of each tree. There is no guarantee that conservation of energy around the loops of the network will not be violated, as the nodal pressure constraints are inequalities. For example, consider any node with two non-overlapping supply paths which, obviously, start from the single source. It appears that the head loss around the loop defined by these paths will be zero only if the allowable head loss is the same for both paths, and the nodal pressure constraint of each tree is active or the two slacks are the same at the solution. There is no guarantee that these conditions will be met in general.

3.4.2 PREVIOUS ENTROPY-BASED RESEARCH

This subsection is mainly a review of independent work on entropy in the context of water distribution network reliability by Awumah et al. (1990, 1991, 1992). They have proposed several entropy-based functions which are all considered here. Also, Awumah and Goulter (1992) have used entropy to formulate an optimum layout/reliability problem. For convenience the arbitrary constant K in Shannon's entropy function is set to unity.

Before reviewing Awumah et al. it may be noted that Jowitt and Xu (1993) have proposed an entropy-related model for calculating the shortfall at the nodes of a water network following the failure of a pipe. The data for the model are the pipe flow rates, demands and source supplies under normal operating conditions. Based on the premise that the flows arriving at each node of the network mix perfectly, equations are set up for the destinations of the flow at each source, the origins of the flow at each node, and the origins and destinations of the flow in each pipe. Knowing the quantity of flow reaching each node from each pipe, and neglecting flow rerouting, it is possible to estimate the shortfall suffered at each node subsequent to the failure of each pipe. Jowitt and Xu have shown that the equations of the model for determining the composition of pipe flows by source and by destination turn out to be the same as those given by the gravity model; the gravity model can be derived using the maximum entropy formalism.

3.4.2.1 BASIC ENTROPY-BASED FUNCTION

The following function has been proposed by Awumah, Goulter and Bhatt (1991) as a measure of the *redundancy* in a water distribution network.

$$S = -\sum_{ij \in IJ} (q_{ij}/Q_0) \ln(q_{ij}/Q_0) \tag{3.10}$$

in which S is the entropy (or redundancy) of the network; Q_0 is the sum of the *link* flows, i.e.,

$$Q_0 = \sum_{ij \in IJ} q_{ij} \tag{3.11}$$

in which q_{ij}, $\forall ij$, is the flow in link ij and IJ consists of all the links of the network.

It is seen in the next chapter that Shannon's entropy (Shannon, 1948), on which Eq. (3.10) is based, is defined only for mutually exclusive probabilities or events. However, the probability-like quantities, q_{ij}/Q_0, $\forall ij$, are not mutually exclusive. This can be seen by considering any flow network in which some links are connected in series, for example, Figure 3.4a. For any pair of links (ij, jk), $\forall$(ij, jk), at least part of the flow in jk comes from ij. In other words the flow in jk is dependent upon the flow in ij. These flows are clearly not independent and as such are not mutually exclusive. In consequence, Eq. (3.10) is not consistent with the requirements of Shannon's entropy. A more rigorous formulation is developed in Chapter 5.

3.4.2.2 MODIFIED ENTROPY FUNCTION

The following substitution may be used to transform Eq. (3.10), but without changing it:

$$\frac{q_{jn}}{Q_0} = \frac{q_{jn}}{Q_n}\frac{Q_n}{Q_0}$$

in which Q_n, $\forall n$, is the sum of the link flows entering node n, i.e.,

$$Q_n = \sum_{j \in NU_n} q_{jn} \quad \forall n \tag{3.12}$$

in which NU_n, $\forall n$, represents the upstream nodes of link inflows at node n. The transformed but equivalent equation is therefore

$$S = \sum_{n=1}^{NN} (Q_n/Q_0) S_n - \sum_{n=1}^{NN} (Q_n/Q_0) \ln(Q_n/Q_0) \tag{3.13}$$

in which

$$S_n = -\sum_{j \in NU_n} (q_{jn}/Q_n)\ln(q_{jn}/Q_n) \quad \forall n \tag{3.14}$$

is the entropy (or redundancy) of node n, $\forall n$. A different and more complete definition of nodal entropy is presented in Chapter 5.

In an attempt to account for the interactions between adjacent nodes in a network, Awumah, Goulter and Bhatt (1991) used the following equation.

$$S'_n = S_n + \sum_{j \in NU_n} t_{jn} S'_j \quad \forall n \tag{3.15}$$

in which t_{jn}, $0 \le t_{jn} \le 1$, $\forall n$, is the fraction of the modified entropy S_j' of node j, $j \in NU_n$, that is passed on to node n and is:

$$t_{jn} = q_{jn}/Q_j \quad \forall n, \ \forall j \in NU_n \tag{3.16}$$

Also, to calculate the modified entropy, Eq. (3.15), of any node, it is necessary to first calculate the modified entropy of its predecessor nodes. If S'_n is used instead of S_n in Eq. (3.13) the modified network entropy is obtained and is:

$$S = \sum_{n=1}^{NN} (Q_n/Q_0) S'_n - \sum_{n=1}^{NN} (Q_n/Q_0)\ln(Q_n/Q_0) \tag{3.17}$$

Awumah et al (1991) observed that the S'_n, $\forall n$, give higher values of entropy for the network than the S_n, $\forall n$. However, there is little evidence that the former relate better to the conditions in a water distribution network.

3.4.2.3 INTERDEPENDENCIES BETWEEN PATHS

In general, the paths supplying a node may have some links in common. To account for such path dependencies, Awumah et al. (1990) proposed the following function for nodal entropy.

$$S_n = -\sum_{j \in NU_n} (q_{jn}/Q_n)\ln(q_{jn}/a_{jn}Q_n) \quad \forall n \tag{3.18}$$

in which a_{jn}, $\forall j \in NU_n$, $\forall n$, is the effective number of independent paths to node n through link jn. The value of a_{jn} is given by

$$a_{jn} = np_{jn}\left[1 - \frac{\sum_{k=1}^{nl_{jn}} (d_k - 1)}{\sum_{k=1}^{nl_{jn}} d_k}\right] \quad \forall n, \ \forall j \in NU_n \tag{3.19}$$

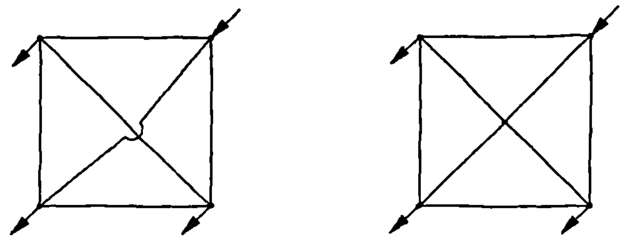
in which np_{jn}, $\forall n$, $\forall j \in NU_n$, is the number of (dependent or independent) paths to node n that use link jn; nl_{jn}, $\forall n$, $\forall j \in NU_n$, is the total number of links in all the paths supplying node n, that use link jn; d_k is the number of paths in which link k is used. (If the links of a network are numbered, they may be identified by a single subscript as in the above equation) The interested reader may refer to Awumah et al. (1990) for more on Eqs. (3.18) and (3.19). However, there is little evidence that Eqs. (3.18) are efficacious. Also, it may be noted that the calculation of the path parameter a_{jn} is computationally very expensive as it relies on path enumeration.

3.4.2.4 FLOW REVERSAL

It will be recalled that so far, nodal entropy has been defined in terms of link inflows only. In pipe networks, the direction of flow in a link may change

Joint layout and pipe diameter optimization with regard to reliability has been seen to be extremely complex. Also, there is no completely satisfactory model for solving this problem. The two independent paths approach is both difficult to formulate and solve. The two trees method is inapplicable to multiple-source networks. The entropy-based formulation of Awumah and Goulter (1992) fails to preserve loops. Furthermore, it may be noted that all these layout models do not define reliability on the [0, 1] interval.

In conclusion, the optimization of layout/reliability in water distribution networks remains a challenge. There is no doubt that there is a need for a layout/reliability design model. The excessive computational requirements of "exact" reliability models give added impetus to the search for a reasonable and quantifiable surrogate for reliability. Entropy apart, there appear to be no such measures. Already, there is some limited evidence from the work of Awumah, Goulter and Bhatt (1990, 1991, 1992) that further research into entropy has potential for success.



(a) Possible links in a simple network

(b) A layout for the network of (a) with an extra node

Figure 3.1 Two layouts for a simple network

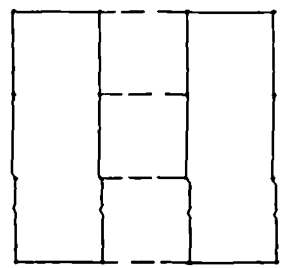
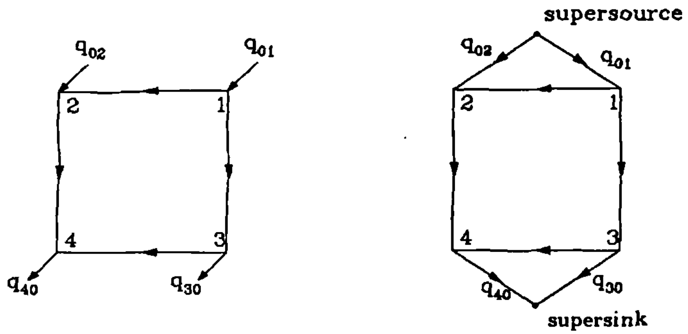


Figure 3.2 Illustration of global and local looping: at least two of the links depicted by broken lines must be included to ensure network-wide looping.



(a) A simple network

(b) Equivalent supersource–supersink representation of the network of (a)

Figure 3.3 A network and its supersource–supersink equivalent

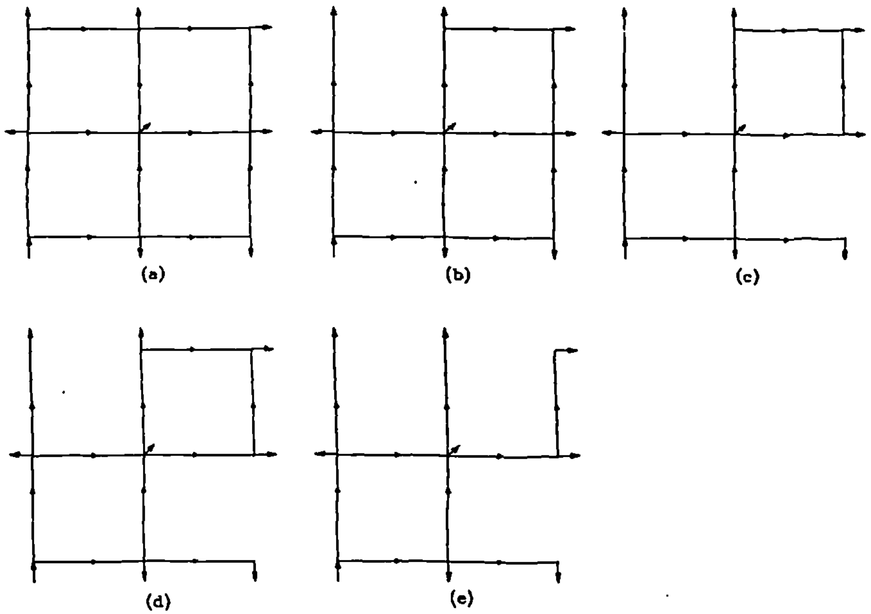
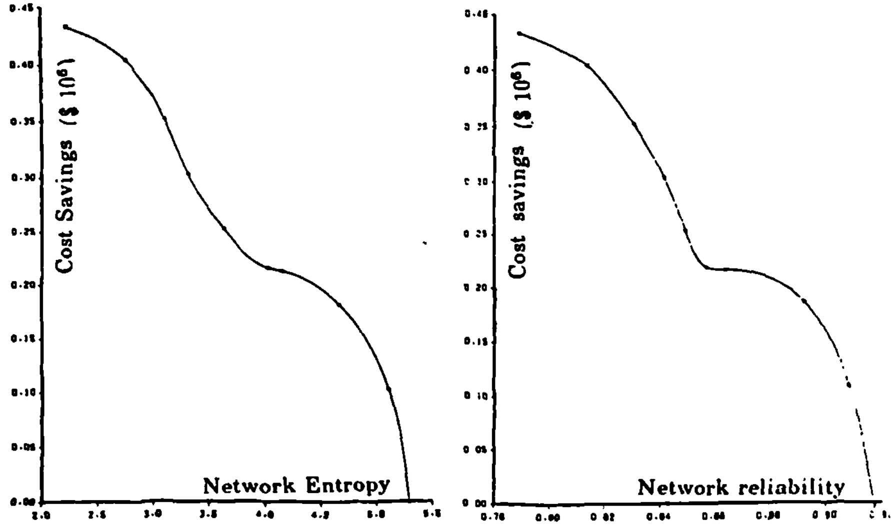


Figure 3.4 Some layouts for the network shown in (a): (a), (b), (c), (d) and (e) represent stages in the layout model of Awumah and Goulter (1992).



(a) Cost vs Entropy

(b) Cost vs Reliability

Figure 3.5 Graphical representation of the cost-defined similarity between entropy and reliability: the cost savings are the differences between the costs of the maximum entropy design and the sub-maximum entropy designs; the reliability is the average source node-demand node mechanical reliability; the network entropy is calculated from Eq. (3.13) using Eqs. (3.18) and (3.19); it may be noted that both graphs are plotted to the same cost scale; the graphs used in this figure are taken from Awumah and Goulter (1992) and are based on the layouts in Figure 3.4.

CHAPTER 4 INTRODUCTION TO THE MAXIMUM ENTROPY FORMALISM

4.1 INTRODUCTION

The concept of *entropy* first arose in classical thermodynamics which is concerned with the macroscopic properties of matter. Entropy evolved further with statistical mechanics which is concerned with predicting microscopic phenomena on the basis of some known macroscopic parameters such as temperature and pressure. Shannon (1948) introduced entropy in the context of information theory as a quantitative measure of the amount of information or uncertainty in a probability distribution. He demonstrated that the expression for entropy has a deeper meaning quite independent of thermodynamics. Only Shannon's information-theoretic entropy is directly relevant to the present research and is presented shortly.

Shannon's measure enabled the information content of different probability distributions to be compared quantitatively. Before Jaynes' (1957) work, it was common to determine the probability distribution of a phenomenon by augmenting some known information about the phenomenon with assumptions such as "equal *a priori* probabilities." Then, and only then, would the entropy of the distribution be calculated. Jaynes (1957) suggested that Shannon's measure could be used to make probabilistic inference without recourse to "additional assumptions not contained in the laws of mechanics." Furthermore, he stated:

> In freeing the theory from ... hypotheses of the above type, we make it possible to see statistical mechanics in a much more general light. Its principles and mathematical methods become available for treatment of many new physical problems.

Thus Jaynes proposed that, for a given phenomenon, Shannon's measure could be used to generate a probability distribution that would have the greatest amount of information or entropy. He concluded that:

> In the problem of prediction, the maximization of entropy is not an application of a law of physics, but merely a *method of reasoning which ensures that no unconscious arbitrary assumptions have been introduced.*

This approach is known as the *maximum entropy formalism* and is described in this chapter.

A major component of the present research is to identify the right formulation of Shannon's informational entropy for a general flow network. But first, it is necessary to describe Shannon's entropy. Also, the present work is concerned with probabilistic inference on water distribution networks based on partial information. This aspect stems from Jaynes' maximum entropy formalism. In this chapter, Shannon's entropy is presented. Also, its properties are given, with emphasis on the *joint entropy* of separate finite probability schemes because of the relevance to the present work. The maximum entropy formalism is then described. Finally, the relevance to the present work is highlighted.

4.2 INFORMATIONAL ENTROPY

Imagine an experiment with more than one possible outcomes, for example, the tossing of a coin. The result of each trial in such an experiment cannot be predicted with complete certainty. Also, the degree of uncertainty will vary, depending on the experiment. It would be said for instance that there is, say, a 50% chance or a probability of 0.5 that each trial in a coin tossing experiment will result in a head. Furthermore, if for some reason, the chance of a head resulting is 99%, then, the outcome of each trial is very predictable. It is therefore seen that the probabilities associated with each outcome of a probabilistic experiment convey some information about that experiment. In this section, Shannon's measure of the information content of a probability scheme is presented. But, first, the meaning of a *finite probability scheme* is given.

4.2.1 DEFINITION OF FINITE PROBABILITY SCHEMES

If a set of *events* or *outcomes* are such that one, and only one, of them can occur at each trial, then, the events or outcomes are *mutually exclusive*. Furthermore, if one of these events must occur at each trial, it means that the set is *exhaustive* and it represents a *complete system*. Also, the events of a complete system together with their corresponding probabilities form a *finite scheme*. Finally, the events or outcomes of a finite scheme are denoted by o_i, $i = 1,\ldots,N$, where N is the number of outcomes or events. The corresponding probabilities are denoted by p_i, $i = 1,\ldots,N$. Thus the finite scheme O is given by

$$O = \begin{pmatrix} \underline{o} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} o_1\, o_2 \ldots o_N \\ p_1\, p_2 \ldots p_N \end{pmatrix} \tag{4.1}$$

By definition, the probabilities of a finite scheme are non-negative, i.e.,

$$p_i \geq 0 \quad \forall i \tag{4.2}$$

and satisfy the normality condition, i.e.,

$$\sum_{i=1}^{N} p_i = 1 \tag{4.3}$$

4.2.2 ENTROPY OF FINITE SCHEMES

There is some uncertainty associated with every probabilistic scheme. Also, the degree of uncertainty is different for different schemes. For example, there is more uncertainty about a scheme with probabilities (0.5, 0.5) than a scheme with probabilities (0.01, 0.99). Shannon (1948) put forward the following measure for the uncertainty represented by a finite probability distribution.

$$S = -K\sum_{i=1}^{N} p_i \log p_i \tag{4.4}$$

in which S is the entropy or amount of uncertainty; K is an arbitrary positive constant; logarithms may be taken to any suitable base (see, for example, Jones, 1979): natural logarithms are used herein. It is axiomatic that the p_i, $i = 1,\ldots,N$, represent a finite scheme, i.e., the probabilities are non-negative, exhaustive, mutually exclusive and satisfy the normality condition Eq. (4.3). Also, $0 \log 0 = 0$ by definition. Shannon's entropy, Eq. (4.4), is a measure of uncertainty, or conversely, a measure of information as uncertainty and information are complementary. Thus the information gained as a result of an event happening is equivalent to the uncertainty removed.

4.2.3 SOME PROPERTIES OF SHANNON'S ENTROPY

Some of the more obvious properties of Shannon's entropy are presented next. These are the properties that might be expected of a reasonable measure of uncertainty. There are many other properties which are mostly mathematically-derived and for these, the interested reader may consult, for example, Guiasu (1977), Jones (1979), and Kapur and Kesavan (1987). The following properties, some of which are merely obvious statements, are stated without proof. The proofs are straightforward and are not repeated here but the interested reader may refer to Guiasu (1977), for example. However, the entropy of a scheme composed of two finite schemes is derived here.

1. $S \geq 0$

   The equality applies if, and only if, any one probability is unity and all the rest are zero. Obviously there is no uncertainty in such a scheme.

2. $S(p_1, \ldots, p_N) = S(p_1, \ldots, p_N, 0)$

in which there are $(N+1)$ entries on the right hand side. This property ensures that the uncertainty stays the same if an impossible outcome is included in the scheme.

3. $S \leq S(U)$

   where U is the uniform distribution in which $p_i = 1/N$, $\forall i$. The equality holds if, and only if, all the probabilities are equal. Thus S assumes its maximum value for the uniform distribution U and this agrees with one's expectation.

4. The maximum value of S increases with the number of outcomes, as expected. From property 3, if U is substituted in Eq. (4.4), the maximum value of S/K is $\ln(N)$.

5. The entropy function S is continuous. Furthermore, it is invariant with respect to positional changes in the p_i , $\forall i$.

6. S is a concave function. This property is important in the statement of the maximum entropy formalism as seen in Section 4.3.

7. **Entropy of Compound Probability Schemes**

The following property which is the definition of the entropy of a scheme composed of two separate schemes has direct relevance to the present research. The relevance is seen in Chapter 5. Suppose there are two finite schemes $O_1 = (\underline{\hat{o}}\underline{\hat{p}})^T$ and $O_2 = (\underline{\tilde{o}}\underline{\tilde{p}})^T$; the hat and squiggle are being used for contrast only. Suppose also that O_1 and O_2 are merged to form a compound finite scheme O_1O_2. The probability distribution that describes O_1O_2 is given by $p_{ij} = p(\hat{o}_i \cap \tilde{o}_j)$, $i = 1, ..., N_1$; $j = 1, ..., N_2$, where N_i , $i = 1, 2$, is the number of outcomes in scheme i, $i = 1, 2$; $p(\hat{o}_i \cap \tilde{o}_j)$, $\forall ij$, is the probability of the simultaneous occurrence of both $\hat{o}_i$ and $\tilde{o}_j$. Also, the sets of probabilities $\underline{\hat{p}}$, $\underline{\tilde{p}}$ and $\underline{p}$ are related by the following formula for conditional events (see Billington and Allan, 1983, for example):

$$p(\hat{o}_i \cap \tilde{o}_j) = p(\tilde{o}_j | \hat{o}_i) p(\hat{o}_i) \tag{4.5}$$

where $p(\tilde{o}_j | \hat{o}_i)$, $\forall ij$, is the (conditional) probability of $\tilde{o}_j$ given that $\hat{o}_i$ has happened. Also, $\{\hat{o}_i \cap \tilde{o}_j\} \equiv \{\tilde{o}_j \cap \hat{o}_i\}$. Therefore

$$p_{ij} = p_{ji} = p(\hat{o}_i | \tilde{o}_j) p(\tilde{o}_j) \tag{4.6}$$

In the special case where O_1 and O_2 are *mutually independent*,

$$p(\tilde{o}_j | \hat{o}_i) = p(\tilde{o}_j) \quad \forall ij \tag{4.7}$$

$$p(\hat{o}_i | \tilde{o}_j) = p(\hat{o}_i) \quad \forall ij \tag{4.8}$$

It follows from Eq. (4.6) that, for mutually independent schemes,

$$p_{ij} = \hat{p}_i \tilde{p}_j \quad \forall ij \tag{4.9}$$

Following the above procedure, these formulae may be extended if more than two schemes are involved.

The entropy of a compound scheme can easily be deduced now by using Eq. (4.5) in (4.4).

$$\begin{aligned} S(O_1O_2) &= -K \sum_i \sum_j p_{ij} \ln p_{ij} \\ &= -K \sum_i \sum_j p(\tilde{o}_j | \hat{o}_i) p(\hat{o}_i) \ln [p(\tilde{o}_j | \hat{o}_i) p(\hat{o}_i)] \\ &= -K \sum_i p(\hat{o}_i) \ln p(\hat{o}_i) \sum_j p(\tilde{o}_j | \hat{o}_i) - K \sum_i \sum_j p(\hat{o}_i) p(\tilde{o}_j | \hat{o}_i) \ln [p(\tilde{o}_j | \hat{o}_i)] \end{aligned}$$

Also, $\sum_j p(\tilde{o}_j | \hat{o}_i) = 1$, $\forall i$, and

$$S(O_2 | O_1) = -K \sum_i \sum_j p(\hat{o}_i) p(\tilde{o}_j | \hat{o}_i) \ln p(\tilde{o}_j | \hat{o}_i) \tag{4.10}$$

is the *conditional entropy* of O_2 given that O_1 first occurs. Therefore

$$S(O_1O_2) = S(O_1) + S(O_2 | O_1) \tag{4.11}$$

Also,

$$S(O_1) + S(O_2 | O_1) = S(O_2) + S(O_1 | O_2) \tag{4.12}$$

The proof of Eq. (4.12) merely involves the use of (4.6) instead of (4.5) in the above derivation of (4.11). The identity (4.12) means that the entropy of a compound scheme does not change if the positions of its constituent schemes are interchanged.

If O_1 and O_2 are mutually independent, the substitution of $p(\tilde{o}_j | \hat{o}_i) = p(\tilde{o}_j)$, $\forall ij$, in Eq. (4.10) gives

$$S(O_2 | O_1) = -K\left[\sum_i p(\hat{o}_i)\right]\sum_j p(\tilde{o}_j) \ln p(\tilde{o}_j)$$

That is, for independent schemes O_1 and O_2,

$$S(O_2 | O_1) = S(O_2) \tag{4.13}$$

Similarly, or by applying Eq. (4.13) in (4.12),

$$S(O_1 | O_2) = S(O_1) \tag{4.14}$$

It follows from Eq. (4.11) that for *mutually independent probability schemes* O_1 and O_2,

$$S(O_1O_2) = S(O_1) + S(O_2) \tag{4.15}$$

The foregoing results may be summarised as follows:

1. *The joint entropy or uncertainty of two independent schemes is the sum of their separate uncertainties.*
2. *In general, the joint entropy or uncertainty of two schemes is the entropy of one scheme plus the conditional entropy of the other.*
3. *The entropy or uncertainty of a compound scheme is invariant with respect to changes in the relative positions of its constituent schemes.*

The above definitions, Eqs. (4.11) and (4.15), for the entropy of a compound scheme may be extended if more than two finite probability schemes are involved by applying the procedure set out above. Property 2 is a key requirement of consistency for any reasonable measure of uncertainty (Khinchin, 1953; Jaynes, 1957). Also, it is central to the definition of the entropy of a flow network. All these issues are considered further in Chapter 5 in which the appropriate form of the entropy function for a flow network is developed.

Finally, the uniqueness of Shannon's entropy as a measure of uncertainty or information is stated next as a theorem. The statement of the theorem is taken from Khinchin (1953). The proof is not repeated here but the interested reader may consult Khinchin.

4.2.3.1 THE UNIQUENESS THEOREM

Among the properties of entropy, the following two can be considered to be basic.

1. For given N and for $\sum_{i=1}^{N} p_i = 1$, the function takes its largest value for $p_i = 1/N$, $\forall i$.
2. $S(O_1O_2) = S(O_1) + S(O_2 | O_1)$.
3. A third property which must obviously be satisfied by any reasonable definition of entropy is $S(p_1, p_2, \ldots, p_N) = S(p_1p_2, \ldots, p_N, 0)$, where there are $(N+1)$ entries on the right hand side.

Theorem

Let S be a function defined for any integer N and for all values of p_i, $\forall i$, such that $p_i \geq 0$, $\forall i$, and $\sum_{i=1}^{N} p_i = 1$. If for any N this function is continuous with respect to all its arguments, and if it has the properties 1, 2 and 3 above, then

$$S = -K \sum_{i=1}^{N} p_i \ln p_i \tag{4.4}$$

where K is an arbitrary positive constant.

This well-established theorem shows that Shannon's entropy for a finite scheme is the only one possible if it is to have certain general properties which seem necessary in view of the actual meaning of the concept (as a measure of uncertainty or as an amount of information). The proof of this theorem is somewhat outside the scope of the present outline of Shannon's informational entropy. The interested reader could refer to Khinchin (1953), Jones (1979), etc.

4.3 THE MAXIMUM ENTROPY FORMALISM

Consider the following problem. In a situation where little or no information is available about a probabilistic system, which probability distribution best describes the system and how can it be found? According to Laplace's *principle of insufficient reason*, all the outcomes of a finite probability scheme should be considered to be equally likely if there is no reason to think otherwise. Thus the uniform distribution U should be adopted whenever the selection of any other distribution cannot be upheld or there is no information based on which a different distribution may be selected. This criterion seems intuitively objective: U is maximally noncommittal to unavailable information and is therefore unbiased. However, in many instances some information may be available. A serious shortcoming of the principle of insufficient reason is that it has no means of dealing with such eventualities. It is seen shortly that

the so-called principle of insufficient reason is a special case of the *maximum entropy formalism*, which is a logical method of *probabilistic inference*. The maximum entropy formalism is described next.

Suppose a random variable x takes on discrete values x_i, $i = 1, \ldots, N$, with probabilities $p(x = x_i) = p_i$, $\forall i$. Suppose also that the $p_i, \forall i$, cannot be determined from observations on x but are known to satisfy relationships of the form:

$$\sum_{i=1}^{N} p_i F_{ji}(x) = \langle F_j \rangle \quad j = 1, \ldots, NJ \tag{4.16}$$

in which $\langle F_j \rangle$, $\forall j$, is the (known) expected value of the F_{ji}, $\forall i$. For example, if for some j, $F_{ji}(x) = x_i$, $\forall i$, then, $\langle F_j \rangle$ is the mean of the x_i, $\forall i$. What can logically be inferred about x if $(NJ + 1) < N$, i.e., the NJ constraints Eq. (4.16) together with the normality condition Eq. (4.3) are less than the $p_i, i = 1, \ldots, N$? Clearly, many distributions will satisfy Eqs. (4.3) and (4.16). Which of these should be chosen and on what basis?

Shannon's entropy may be used to measure the uncertainty in a probability distribution provided the distribution is known *a priori*. Also, it has been seen in Section 3.2 that the distribution with which is associated the most uncertainty is the uniform distribution U. Furthermore, U is maximally noncommittal to unavailable information. This suggests that the distribution which has the highest value of entropy and is compatible with the available information, i.e., Eqs. (4.16), is maximally noncommittal, unbiased and best describes the finite scheme within the limitations of the available information. Jaynes (1957) postulated that Shannon's entropy could be used for logical probabilistic inference rather than merely as a measure of uncertainty. Thus Jaynes stated:

> In making inference on the basis of partial information we must use that probability distribution which has maximum entropy subject to whatever is

known. This is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have.

The above method of inference is known as the *maximum entropy formalism*. It is equivalent to solving the following problem.

Problem 4

$$\underset{\forall p_i}{Maximize} \quad S/K = -\sum_{i=1}^{N} p_i \ln p_i \tag{4.4}$$

Subject to:

$$\sum_{i=1}^{N} p_i = 1 \tag{4.3}$$

$$\sum_{i=1}^{N} p_i F_{ji}(x) = \langle F_j \rangle \quad \forall j \tag{4.16}$$

$$p_i \geq 0 \quad \forall i \tag{4.2}$$

Problem 4 is a convex programming problem (Templeman and Li, 1985). As such there is a unique global maximum point p_i^*, $\forall i$, which is called the *maximum entropy distribution*. There is an analytical solution to Problem 4 and it can be found by examining the stationarity of its Lagrangean. In Chapter 5 it is shown that the problem of maximizing the entropy of the flows of a parallel network has the format of Problem 4. By setting up and solving Problem 4 for a general parallel network, it is shown how expressions may be derived for the p_i. The solution to Problem 4 is

$$p_i = \frac{\exp\left[\sum_{j=1}^{NJ} \mu_j F_{ji}\right]}{\sum_{i=1}^{N} \exp\left[\sum_{j=1}^{NJ} \mu_j F_{ji}\right]} \qquad i = 1, ..., N \tag{4.17}$$

in which the μ_j, $j = 1, ..., NJ$, are Lagrange multipliers. Templeman and Li (1985) have shown how the values of the Lagrange multipliers may be calculated conveniently using unconstrained non-linear programming. It should be noted that Problem 4 is the classical maximum entropy problem. It may not always be possible or easy to formulate the constraints or S in the problem of inferring least biased probabilities as in Problem 4.

If the entropy is maximized subject to the normality condition only, the result is $\underline{p}^* = U$. This may be seen by examining the stationarity of the Lagrangean of this special case. This result concurs with the principle of insufficient reason. Also, for a given system, the maximum entropy distribution has the property of being the most uniform distribution that satisfies the constraints of the system. In other words, of all the distributions satisfying Eqs. (4.16), the maximum entropy distribution is the least different from U.

The relevance of the maximum entropy formalism to the present work has been touched upon briefly in the introduction to this chapter. If the problem of inferring network flows can be couched in probability-like quantities, then, in theory, it should reduce to the problem of inferring the maximum entropy distribution which can be tackled by the maximum entropy formalism. This is taken up in the next chapter. Secondly, it has been seen in Chapter 3 that there are complex issues surrounding reliability in the context of looped water distribution networks. It appears, from the maximum entropy formalism, that it would be safe to size the pipes to carry flows that are maximally noncommittal to factors that cannot easily be predicted; subject, to the extent that is practicable, to whatever is known.

As explained earlier, the maximum entropy formalism makes it possible to find the most unbiased probability distribution for a system. Thus if $p(x = x_i)$, $\forall i$, is interpreted as the probability of x being in state i, $\forall i$, then the maximum entropy distribution is the only one which is such that every possibility or state, however remote, that is not excluded by the available information is ascribed a non-zero probability. This property would appear to give a sense of flexibility in the context of reliability in looped water distribution networks. An entropy-based approach to the problem of designing a reliable water distribution network is presented in Chapter 6. Other motivations for the approach are given there.

4.4 THE CONTINUOUS CASE

Where a random process is continuous the maximum entropy formalism still applies. In general, the above theory remains unaltered, but integrals over the continuous domain replace the summations and probability density functions replace the discrete probabilities. The corresponding maximum entropy problem may therefore be stated as follows.

$$\underset{f(x)}{Maximize} \quad S/K = -\int_a^b f(x)\ln(f(x))\,dx \tag{4.18}$$

subject to:

$$\int_a^b f(x)\,dx = 1 \tag{4.19}$$

$$\int_a^b F_j(x)f(x)\,dx = \langle F_j \rangle \quad j = 1, ..., NJ \tag{4.20}$$

in which x is a continuous random variable; F_j is a function of x; $\langle F_j \rangle$ is the expectation of F_j; the range $[a, b]$ of the integrals may extend to $[-\infty, +\infty]$; NJ is the number of expectation constraints; $f(x)$ is a probability density function. It is worth noting that the above integrals may not exist, and the entropy can be negative because $f(x)$ can be greater than unity. Also, because the entropy is defined in terms of a probability density rather than a probability, the entropy may not be invariant to a change of variables. For the above reasons, some results for discrete distributions may not be applicable to

certain continuous distributions. This research is concerned with discrete probabilities only.

4.5 THE MAXIMUM ENTROPY FORMALISM IN CIVIL ENGINEERING

Use of the maximum entropy formalism is widespread and it might be somewhat misleading to mention here any particular applications outside civil engineering. As a starting point, the interested reader could refer to Jones (1979), Guiasu (1977), Walsh and Webber (1977), Levine and Tribus (1979), Kapur and Kesavan (1987), Li (1987), Templeman and Li (1987, 1989), Kapur (1989). In addition to Awumah et al. (1990, 1991, 1992) and Jowitt and Xu (1993) which have been mentioned in Chapter 3, some civil engineering applications are stated below.

Basu and Templeman (1985) and Siddall and Diab (1971) have used the maximum entropy formalism in structural reliability analysis and probabilistic design. Basu and Templeman (1985) argue that by fitting a maximum entropy probability distribution to available data, a more logical and rigorous approach to structural reliability analysis results (see Chapter 1, Section 1.1). Also, Munro and Jowitt (1978) have used the maximum entropy formalism in decision analysis in the ready-mixed concrete industry. The principal problem is concerned with making optimal decisions under uncertainty about future orders. They argue that the evaluation of prior probabilities for the order states should be made objectively and not be affected by any personal bias. The maximum entropy formalism was therefore used to estimate the least biased probability distribution associated with the orders for each mix.

Traffic Engineering is an area of entropy-based research which is concerned with estimating trip matrices from limited data. The fundamental approach is to seek the trip matrix which can be realised in the most number of ways. To determine this matrix, a combinatorial formula for the number of possible arrangements of trips is set up. A monotonic increasing function of the number of trips is then maximized. The problem of inferring trip matrices has also been approached from an information-minimizing viewpoint in which the objective is to minimize the information content of a limited number of observations on the road network. The interested reader may refer, for example, to van Zuylen and Willumsen (1980), Willumsen (1981) and Bell (1983). Entropy is also used for inference on traffic streams at road junctions; see, for example, Mountain et al. (1983a, 1983b, 1986a, 1986b). A closed-form solution is derived herein for the above problem of estimating turning flows at road junctions. In other words, it is shown that the problem is solved by the gravity model.

Recently, entropy has been applied to open channel flow by Chiu (1987, 1988, 1989, 1991). The approach hinges on a probabilistic interpretation of the velocity distribution across the channel. Thus, the maximum entropy probability density function of the velocity is found by maximizing the entropy of the velocity distribution subject to constraints. These constraints include the normality condition, and first, second and third statistical moments which represent the hydrodynamic transport of mass, momentum and energy respectively.

Although the above sample is not exhaustive, it serves to highlight the observation that at the present time entropy is mostly used where there is a more-or-less obvious and natural probabilistic interpretation.

CHAPTER 5 CALCULATING MAXIMUM ENTROPY FLOWS IN NETWORKS

5.1 INTRODUCTION

Much of this research grew from the simplified problem shown in Figure 5.1a. Suppose that the figure represents a water distribution network. There is one source with a known inflow and 3 demand points, each with a known demand. Suppose further that the layout of the network and the flow directions in all the pipes are known but no other information of any kind is available. Under these conditions how can values be estimated for the flow rate in each pipe, assuming that the total demand equals the supply?

Any solution to the above problem must be such that at each pipe junction or node, the inflows including any external input balance the outflows including any node outflow. Node flow equilibrium or continuity therefore provides 4 linear equations in 4 variables. The variables are the unknown volumetric pipe flow rates. Thus

$$q_{12} + q_{13} = q_{01}$$

$$q_{12} - q_{24} = q_{20}$$

$$q_{13} - q_{34} = q_{30}$$

$$q_{24} + q_{34} = q_{40}$$

However, these equations are not linearly independent as any one is a linear combination of the others. For example, the last equation can be obtained by subtracting the second and third from the first. Therefore, one equation is redundant as it does not contain any information that is not deducible from the rest. In effect there are only 3 usable equations in 4 variables. The continuity equations are therefore not enough to uniquely determine the pipe flow rates.

The problem of not having enough equations from continuity does not arise when dealing with tree-type networks such as the one depicted in Figure 5.1b. For such systems, knowledge of the external inflows and outflows is the only information needed to determine the pipe flow rates from the continuity equations. The problem of Figure 5.1a, however, is different and progress seems impossible without extra information. If the value of any one of the flows were known there would be no difficulty, as the problem of calculating the link flows would effectively be converted to the problem of calculating link flows for a tree-type network. It would therefore be reduced to a straightforward exercise in backsubstitution, using any 3 of the 4 continuity equations to obtain the other flows.

An important distinction between a tree-type distribution or collection network and the network of Figure 5.1a is that the latter has a loop. In general, the presence of loops ensures that at least one node in a network is reachable by at least two routes. For the problem under consideration, node 4 can be supplied in three ways but using only two routes: via node 2 only, via node 3 only or via nodes 2 and 3 simultaneously. It is this possibility of choice of route for supplying node 4 that provides the difficulty in calculating flows. Nodes 2 and 3 can each be supplied by one route only, as dictated by the direction of the arrows. The problem of Figure 5.1a may therefore be posed in a slightly different way. Should the flow to node 4 be split between the two available routes? If the answer is yes, what is the best ratio? Otherwise, which route should or should not be used and why? These are fundamental questions which encapsulate the issues to be addressed.

The context in which the simplified problem of Figure 5.1a may arise is considered here for a water distribution network only. It is not uncommon for

the general plan layout of a distribution network to contain no information on pipe lengths, diameters, friction coefficients and other data. Sometimes some of the necessary data are found in various detailed drawings or tables. For very old networks, much of the information may be missing or may have changed over time. Such water distribution systems are usually buried. For various reasons, it may not be possible to obtain the missing information for every pipe. The most obvious restrictions on obtaining additional information are usually time and cost. A possible alternative is to physically measure the pipe flow rates. This can be time consuming and requires equipment which can be expensive. Faced with these difficulties, a method of quickly estimating the pipe flow rates would be most useful.

The basic flow inference problem is now stated. There is a set of sources and a set of demand nodes. The value of each source supply is known and each demand node has a known abstraction. There is some network linking all the sources and demand nodes. Also, there may be any combination of all the possible node interconnections: source node to source node, demand node to demand node and source node to demand node. It is desired to estimate the flows within the connecting network. This, in essence, is the flow inference problem and it arises *if there are more unknown link flow values than the* available flow equilibrium equations.

To see how the notion of uncertainty is related to the flow inference problem consider, for example, a buried water distribution network. When the inflow and outflow rates are known, the system is operating in a unique way. This means that each pipe has a unique flow rate and hence there is no randomness in the system. In fact, there are known physical laws governing fluid flow in pipe networks and these laws have been presented in Chapter 2 for water. Given sufficient further information, it would therefore be possible to do a rigorous analysis of the pipe network. The analysis would yield the exact value of the flow rate in each pipe, along with other parameters. It follows that there is no uncertainty in the physical system itself. But there is uncertainty about the behaviour of the system on the part of the *observer* in view of the observer's inability, in the absence of further information, to determine unique values for the system parameters.

The work reported in this chapter is concerned with maximum entropy flows in any general flow network. However, networks in which there are only source node-demand node links have special properties which are highlighted in this chapter. Moreover, analytical results exist for these networks and so they are treated separately throughout this chapter. In this chapter, the maximum entropy formalism is used for inferring the least biased flow estimates for the links of a flow network. Conceptually, there are two main hurdles to surmount. First, how can flows be described in a completely probabilistic way as required by Shannon's entropy? Second, what is the definition of entropy for a flow network? These issues are addressed in this chapter. In particular:

1. The relative frequency interpretation of probability provides a link between the purely mathematical concept of probability and the empirical concept of regularity in behaviour in repeated or *continuous* experiments.
2. The conditional entropy definition of the entropy of a compound probability scheme provides a basis for defining the entropy of the flows of any general network.
3. The problem of inferring the least biased flows is cast as an entropy maximization problem subject to flow equilibrium. This problem is examined and shown to be a convex programming problem and, as such, has a unique global maximum point.
4. The principle of insufficient reason is used to develop an efficient algorithm for calculating maximum entropy flows for *single-source networks.*

The work in this chapter leads naturally to the problem of designing an optimally reliable water distribution system. In this regard, some of the questions to be answered are the following. Is reliability enhanced if the distribution system is designed to carry maximum entropy flows? Is entropy maximization subject to all the constraints of a distribution network, including cost, a possible method of reliability optimization? More fundamentally, is flow entropy a reasonable surrogate for reliability? These and other related matters are the purview of Chapter 6. However, a prerequisite to those issues is the material in this chapter.

5.2 PROBABILITY SCHEMES FOR NETWORK FLOWS

The properties of a finite probability scheme have been enumerated in Chapter 4. In this section, a means of representing network flows in probabilistic terms is presented. Several sets of probabilities are described and relationships between the different sets are established. The probability definitions used herein are based on the relative frequency interpretation of probability in which the probability of an event is the limiting value of the ratio of its frequency to the sum of the frequencies of the individual events. The probability schemes defined here are used subsequently in this chapter to define the entropy of a flow network and infer network flows given insufficient data and, in Chapter 6, to optimize reliability in water distribution networks.

It is useful to consider first the definitions of *series* and *parallel connections* in the context of flow networks:

1. *Two or more links in a flow network are said to be in parallel if the operation of each is independent of the rest.*
2. *Two or more links in a flow network are said to be in series if at least one is incapable of operating independently of one or more of the rest.*

As an example, Figure 5.1a has both series and parallel connections. Link 2-4 is fed by 1-2 and, as such, the former is dependent on the latter. These two links are therefore in series. In contrast, links 1-2 and 1-3 would each still be able to carry flow if the other failed. They are therefore in parallel. Also, a network such as Figure 5.1a in which there are both series and parallel connections is called a series-parallel network. It must be emphasised that the above definitions have been adapted for flow networks and may not *necessarily apply to other networks.*

One way of defining probabilities for flow networks applies to parallel networks only and is described now. A more general method is considered shortly.

5.2.1 PARALLEL NETWORKS

Consider the network of Figure 5.2 in which there are no series connections. Several finite probability schemes may be defined for this network and these are taken in turn. First, the set consisting of the source inflows q_{0n}, $n = 1, 2, 3$. Corresponding to these flows, events o_{0n}, $n = 1, 2, 3$, may be defined as *the event that flow enters the network at node n*, $n = 1, 2, 3$. The corresponding probabilities P_{0n} may be obtained by normalising the q_{0n}, $\forall n$, as

$$P_{0n} = p(o_{0n}) = \frac{q_{0n}}{T_0} \quad n = 1, 2, 3 \tag{5.1}$$

in which T_0 is the total supply or demand, i.e.,

$$T_0 = \sum_{n=1}^{3} q_{0n} = \sum_{n=4}^{6} q_{n0} \tag{5.2}$$

Also, P_{0n} is the fraction of T_0 provided by source n, $n = 1, 2, 3$. To see that the probabilities represented by Eqs. (5.1) are a finite scheme, firstly, Eqs. (5.1) automatically satisfy non-negativity and normality, i.e.,

$$\sum_{n=1}^{3} P_{0n} = 1; \quad P_{0n} \geq 0 \quad \forall n \tag{5.3}$$

Secondly, it follows from Eq. (5.3) that the set is exhaustive as the sum of the probabilities would be less than unity otherwise. Finally, the o_{0n}, $n = 1, 2, 3$, are mutually *exclusive in the sense that* an element of flow can enter the network by any, but only one, of the source nodes. Thus

$$\{o_{0n} \cap o_{0m}\} = \{\ \} \quad \forall (n, m), \quad n, m = 1, 2, 3 \tag{5.4}$$

in which $\{\ \}$ represents the empty set.

There are similar definitions for the demands q_{n0}, $n = 4, 5, 6$.

$$P_{n0} = p(o_{n0}) = \frac{q_{n0}}{T_0} \quad n = 4, 5, 6 \tag{5.5}$$

where the o_{n0} may be considered as the event that flow leaves the network at node n, $n = 4, 5, 6$, and P_{n0} the probability that flow leaves the network at node n, $n = 4, 5, 6$. Also,

$$\sum_{n=4}^{6} P_{n0} = 1 \tag{5.6}$$

and

$$\{o_{n0} \cap o_{m0}\} = \{\ \} \quad n, m = 4, 5, 6 \tag{5.7}$$

The above definitions are valid for any parallel network in which there are any number of source nodes and demand nodes. Eqs. (5.1) to 5.7) may therefore be generalised as follows.

$$T_0 = \sum_{n\in I} q_{0n} = \sum_{n\in D} q_{n0} \qquad (5.8)$$

in which I and D represent the sets of source and demand nodes respectively.

$$P_{0n} = \frac{q_{0n}}{T_0} \quad \forall n \in I \qquad (5.9)$$

$$P_{n0} = \frac{q_{n0}}{T_0} \quad \forall n \in D \qquad (5.10)$$

$$\{o_{0n} \cap o_{0m}\} = \{\ \} \quad \forall n, m \in I \qquad (5.11)$$

$$\{o_{n0} \cap o_{m0}\} = \{\ \} \quad \forall n, m \in D \qquad (5.12)$$

$$\sum_{n\in I} P_{0n} = 1 \qquad (5.13)$$

$$\sum_{n\in D} P_{n0} = 1 \qquad (5.14)$$

Finally, referring again to Figure 5.2, probabilities are now defined for the link flows q_{ij}, $i = 1, 2, 3$, $j = 4, 5, 6$.

5.2.1.1 SINGLE-SPACE PROBABILITY MODEL FOR LINK FLOWS

One approach for converting link flows to probabilities considers all the link flows as elements of a single probability space. Let o_{ij}, $i = 1, 2, 3$, $j = 4, 5, 6$, represent the event that flow enters the network at node i and leaves at node

j. Probabilities may therefore be defined as for the supplies and demands. Proceeding straight to the general definitions,

$$T_0 = \sum_{ij\in IJ} q_{ij} = \sum_{i\in I} q_{0i} = \sum_{j\in D} q_{j0} \qquad (5.15)$$

in which IJ represents all links of the network.

$$p_{ij} = p(o_{ij}) = \frac{q_{ij}}{T_0} \quad \forall ij \in IJ \qquad (5.16)$$

$$\sum_{ij\in IJ} p_{ij} = 1 \qquad (5.17)$$

Similar equations to (5.11) or (5.12) have not been written, but it is obvious that the o_{ij}, $\forall ij \in IJ$, are mutually exclusive. If this were not so, the left hand side equality of Eqs. (5.15) would not hold as $\sum_{ij\in IJ} q_{ij}$ would be greater than T_0 because of double counting.

5.2.1.2 MULTIPLE-SPACE PROBABILITY MODEL FOR LINK FLOWS

A second approach for introducing probabilities in a parallel flow network defines separate finite schemes for each node of the network, giving rise to multiple probability spaces. Thus, for each source node n, $n \in I$, the link flows q_{nk}, $k \in ND_n$, are normalised as

$$p_{nk} = \frac{q_{nk}}{q_{0n}} \quad \forall n \in I, \ \forall k \in ND_n \qquad (5.18)$$

in which ND_n represents the set of all demand nodes connected to source node n; p_{nk} is the probability that a particle at node n, $\forall n \in I$, proceeds to node k, $\forall k \in ND_n$. It follows from Eqs. (5.18) that

$$\sum_{k\in ND_n} p_{nk} = 1 \quad \forall n \in I \qquad (5.19)$$

$$q_{0n} = \sum_{k \in ND_n} q_{nk} \tag{5.20a}$$

Similarly, for each demand node n, $\forall n \in D$,

$$p_{jn} = \frac{q_{jn}}{q_{n0}} \quad \forall n \in D, \ \forall j \in NU_n \tag{5.21}$$

in which NU_n represents the set of all source nodes connected to demand node n; p_{jn} is the fraction of q_{n0}, $\forall n \in D$, carried by link jn, $\forall j \in NU_n$. Also,

$$\sum_{j \in NU_n} p_{jn} = 1 \quad \forall n \in D \tag{5.22}$$

$$q_{n0} = \sum_{j \in NU_n} q_{jn} \quad \forall n \in D \tag{5.23a}$$

The multiple probability space formulation represented by Eqs. (5.18) to (5.23) considers each node in isolation, without regard to conditions elsewhere in the network. However, in general, conditions at the nodes of a network are not independent of other nodes. Therefore, the multiple-space probabilities are in fact conditional probabilities which are dependent on conditions at the source or demand nodes respectively. Thus, given that there is flow at node n, p_{nk} is the conditional probability that a particle at node n, $\forall n \in I$, proceeds to node k, $\forall k \in ND_n$. Similarly, given that flow reaches node n, $\forall n \in D$, p_{jn} is the conditional probability that a particle of flow at node n arrives there via link jn, $\forall j \in NU_n$. The node-based probabilities are therefore related by the conditional probability formula as follows:

$$p(o_{0n}o_{nk}) = p(o_{nk} | o_{0n})p(o_{0n}) = P_{0n}p_{nk} \quad \forall n \in I, \ \forall k \in ND_n \tag{5.24}$$

$$p(o_{jn}o_{n0}) = p(o_{jn} | o_{n0})p(o_{n0}) = p_{jn}P_{n0} \quad \forall n \in D, \ \forall j \in NU_n \tag{5.25}$$

in which $p(o_{0n}o_{nk})$ is the true or absolute probability that an element of flow follows link nk, $\forall n \in I$, $\forall k \in ND_n$; $p(o_{jn}o_{n0})$ is the true or absolute probability that an element of flow follows link jn, $\forall n \in D$, $\forall j \in NU_n$. Eqs. (5.24) and (5.25) are essentially the same and both give the absolute probability of flow being in a link.

It may be noted that each of the foregoing probability schemes is associated with an equation for flow equilibrium. These include Eqs. (5.2), (5.8), (5.15), (5.20a) and (5.23a). However, depending on the way they are defined, the probabilities themselves as depicted in Figure 5.2b may or may not satisfy flow equilibrium at each node. This is readily seen by observing that in Eqs. (5.22), for example, $\sum_{j \in NU_n} p_{jn}$ equals unity rather than P_{n0}, $\forall n \in D$. It must therefore be realised that although the p_{ij}, $i, j = 0, 1, \ldots$, may sometimes be equivalent to scaled flows, they are in fact probabilities which may not be used interchangeably with flows.

The problem of defining and maximizing entropy for a flow network centres mainly around the modelling of flows as probabilities and then finding a suitable formulation of entropy for these probabilities. The required probabilities have been defined for a general network in which there are no series connections. These probabilities satisfy all the requirements of a finite scheme. However, the flows must be defined so that they are always positive to ensure that the probabilities remain non-negative. How these probabilities are used to define flow entropy is described in Section 5.3.

5.2.2 GENERAL NETWORKS

Two ways of modelling flows as probabilities have been defined for a parallel network. These include a node-based approach which leads to multiple probability spaces. This is the more general approach as it is possible to define probabilities on the nodes of any general network including those having series connections and it is seen shortly how this can be done. An alternative way of defining the flows of a parallel network in probabilistic terms is to define a single scheme in which the elements are the normalised link flows. It is not possible to do this for a network in which there are series connections. In this subsection, the properties peculiar to parallel networks from the viewpoint of probabilistic flow modelling are highlighted. It is then shown that globally normalising link flows in a general network does not produce a finite probability scheme. Finally, a method of defining probabilities that applies to any flow network is suggested.

Some of the restrictions inherent in a parallel network can be seen by comparing Figures 5.1a and 5.2a. Each source node in Figure 5.2a directly supplies each demand node connected to it, without routing the flow through other nodes as in Figure 5.1a. It follows that each demand node in Figure 5.2a receives flow from its supply nodes directly. Also, it follows that each link in the network must be a direct connection between a source node and a demand node. This means that there cannot be any source node-source node or demand node-demand node connections. Before describing a scheme which applies to any network, the shortcoming of the single-space model is explained.

Two ways of normalising the link flows are described next and it is shown that neither represents a finite scheme. In a parallel network, the sum of the link flows equals the total demand or supply T_0. This is not the case for more general networks. To see this, reconsider Figure 5.1a in which $T_0 = q_{01} = q_{12} + q_{13}$. It is therefore obvious that the sum of the link flows Q_0 which also includes q_{24} and q_{34} must be greater than T_0. In general,

$$Q_0 \geq T_0 \qquad (5.26)$$

in which the equality applies if, and only if, all links are in parallel. The property represented by Eq. (5.26) is due to double counting in Q_0 for networks

having series connections. For example, in Figure 5.1a, the flow in link 2-4 is included both as q_{24} and in q_{12}. In other words, for non-parallel networks,

$$\{o_{ij} \cap o_{jk}\} \neq \{\} \quad \forall ij \in IJ, \ \ \forall jk \in IJ \qquad (5.27)$$

By virtue of Eqs. (5.27) the o_{ij}, $\forall ij \in IJ$, associated with the links of a flow network are in general not mutually exclusive. It follows that neither of the schemes obtained by normalising link flows on T_0 or Q_0 constitutes a finite probability scheme. However, it should be noted that the q_{ij}/T_0, $\forall ij \in IJ$, may be considered as the probability $\bar{p}_{ij}$ that an element of flow entering the network will pass through link ij; the bar is used herein to denote probabilities that, as a set, do not represent a finite scheme. It is permissible to treat the $\bar{p}_{ij}$ as probabilities from the frequency interpretation of probability in which the probability of an event is the limiting value of its relative frequency. The relative frequency of an event is the frequency or number of occurrences of it divided by the sum of the frequencies of all the events in the set. On the other hand, the probability-like quantities q_{ij}/Q_0, $\forall ij \in IJ$, even though they sum to unity, may not be interpreted as probabilities. They do not conform to the definition of relative frequency in which each elementary entity must be included only once. The significance of this observation in the present context is highlighted later in the appraisal of the entropy functions proposed by Awumah, Goulter and Bhatt (1990, 1991, 1992).

In many respects, the model to be described in a moment for general networks is similar to the multiple-space model for parallel networks. However, the former allows for the possibility of internal and/or external inflows and/or outflows being present at any node. It is assumed that each node may have either an external inflow or an external outflow but not both. Also, the existence of source node-source node or demand node-demand node connections in general networks means that the internal inflows at a node will not in general equal the nodal demand. Similarly, the sum of the internal outflows at a source node will not in general equal the external inflow.

Therefore, unlike parallel networks, the probability of flow reaching a node is in general unknown, unless the values of all link flows are known. It will be recalled that the main reason herein for modelling network flows as probabilities is to enable probabilistic inference on networks on the basis of partial information. Under these circumstances the values of the link flows will be unknown in advance. As such, the total amount of flow reaching a general node will not be known.

A multiple-space probability model for general networks is described next. Let T_n represent the total flow reaching or leaving node n, $n = 1, ..., NN$, i.e.,

$$T_n = \sum_{j \in NU_n} q_{jn} = \sum_{k \in ND_n} q_{nk} \quad n = 1,...,NN \tag{5.28}$$

in which NU_n and ND_n respectively include any external inflow and outflow. Thus T_n is the sum of all inflows, including any source supply, or the sum of all outflows, including any demand, at node n, $\forall n$. It has been explained with regard to parallel networks that the o_{jn}, $\forall n$, $\forall j \in NU_n$, and the o_{nk}, $\forall n$, $\forall k \in ND_n$, are two sets each having mutually exclusive elements. This property holds for any network. Also, each set of events is obviously exhaustive. Thus the sets $\{o_{jn} : j \in NU_n\}$ and $\{o_{nk} : k \in ND_n\}$, $\forall n$, represent finite schemes. Following the relative frequency viewpoint, the following probabilities may be defined.

$$\bar{p}_n = \frac{T_n}{T_0} \quad \forall n \tag{5.29}$$

in which $\bar{p}_n$, $\forall n$, is the probability that flow reaches node n;

$$T_0 = \sum_{n \in I} q_{0n} = \sum_{n \in D} q_{n0} \tag{5.8}$$

It may be noted that

$$\sum_{n=1}^{NN} \bar{p}_n \geq 1 \tag{5.30}$$

where the equality holds for parallel networks only and therefore the $\bar{p}_n$, $\forall n$, do not in general represent a finite scheme. It has already been seen from Eqs. (5.26) and (5.27) that the $\bar{p}_{ij}$, $\forall ij \in IJ$ are not mutually exclusive:

$$\bar{p}_{ij} = \frac{q_{ij}}{T_0} \quad \forall ij \in IJ \tag{5.31}$$

in which $\bar{p}_{ij}$ is the probability that there is flow in link ij, $\forall ij \in IJ$. Eqs. (5.31) can be related to Eq. (5.30) by substituting (5.28) in (5.29) to give

$$\bar{p}_n = \sum_{ij \in NU_n} \bar{p}_{ij} + P_{0n} \quad \forall n \in I \tag{5.32a}$$

$$\bar{p}_n = \sum_{ij \in ND_n} \bar{p}_{ij} + P_{n0} \quad \forall n \in D \tag{5.32b}$$

in which the sets NU_n and ND_n, $\forall n$, consist of *links* rather than nodes and

$$P_{0n} = \frac{q_{0n}}{T_0} \quad \forall n \in I \tag{5.9}$$

$$P_{n0} = \frac{q_{n0}}{T_0} \quad \forall n \in D \tag{5.10}$$

The probabilities represented by Eqs. (5.9) and (5.10) are, respectively, the fraction of the total supply provided by source node n, $\forall n \in I$, and the fraction of the total demand consumed at node n, $\forall n \in D$. The probabilities introduced so far are needed in the definition of network flow entropy as shown in Section 5.3. Furthermore, a rigorous probability model is a basic necessity for

Shannon's entropy and the maximum entropy formalism used herein for reliability optimization. It remains to define conditional probabilities for all the inflows and outflows at all nodes of the network and this is addressed immediately.

Two finite schemes may be defined for each node:

$$p_{jn} = \frac{q_{jn}}{T_n} \quad \forall n, \ \forall j \in NU_n \tag{5.33}$$

$$p_{nk} = \frac{q_{nk}}{T_n} \quad \forall n, \ \forall k \in ND_n \tag{5.34}$$

in which p_{jn} is the conditional probability that flow which is destined to reach node n, $\forall n$, uses link jn, $\forall j \in NU_n$; p_{nk} is the conditional probability that flow which is destined to pass through node n, $\forall n$, is included in q_{nk}, $\forall k \in ND_n$. It should be noted that the finite schemes described by Eqs. (5.33) and (5.34) respectively include:

$$p_{0n} = \frac{q_{0n}}{T_n} \quad \forall n \in I \tag{5.35}$$

$$p_{n0} = \frac{q_{n0}}{T_n} \quad \forall n \in D \tag{5.36}$$

in which p_{0n}, $\forall n \in I$, is the probability that a source node receives its total inflow T_n from its external inflow q_{0n}; p_{n0}, $\forall n \in D$, is the probability that a demand node uses its total inflow T_n to satisfy its demand q_{n0}.

The p_{0n}, $\forall n \in I$, and p_{n0}, $\forall n \in D$, are needed to make the sets represented by Eqs. (5.33) and (5.34) exhaustive. More important, there is uncertainty about the proportion of T_n, $\forall n \in D$, satisfying consumption at node n, or the proportion of T_n, $\forall n \in I$, that q_{0n} represents. This uncertainty stems from the uncertainty surrounding the T_n, $\forall n$, themselves which, in general, are unknown and must be inferred along with link flows in the flow inference

problem. The p_{0n}, $\forall n \in I$, and p_{n0}, $\forall n \in D$, are therefore unknown expectations the values of which are to be predicted by the maximum entropy formalism. Of course, the values of the T_n, $\forall n$, are known in a *parallel network* in which:

$$T_n = q_{0n} \quad \forall n \in I \tag{5.37}$$

$$T_n = q_{n0} \quad \forall n \in D \tag{5.38}$$

The conditional probabilities of Eqs. (5.33) and (5.34) may be related to absolute probabilities as follows.

$$p(o_n \cap o_{nk}) = p(o_n)p(o_{nk} \mid o_n) = \bar{p}_n p_{nk} \quad \forall n, \ \forall k \in ND_n$$

Using Eqs. (5.29) and (5.34) for $\bar{p}_n$ and p_{nk}, $\forall n$, $\forall k \in ND_n$, respectively gives

$$\bar{p}_n p_{nk} = \frac{T_n}{T_0}\frac{q_{nk}}{T_n} = \frac{q_{nk}}{T_0}$$

Substituting Eqs. (5.31),

$$\bar{p}_{nk} = \bar{p}_n p_{nk} \quad \forall n, \ \forall k \in ND_n \tag{5.39}$$

There is a similar expression relating the $\bar{p}_{jn}$ and p_{jn}, $\forall n$, $\forall j \in NU_n$, and it is

$$\bar{p}_{jn} = p_{jn}\bar{p}_n \quad \forall n, \ \forall j \in NU_n \tag{5.40}$$

It may be noted that all ps (but not the $\bar{p}$s) have been defined so that they automatically satisfy normality. Also, all probabilities remain non-negative if non-negativity of flows is enforced. Furthermore, all flows must always be defined in the direction in which they are positive. Finally, any node having only one (internal or external) inflow has only one element, unity, in the set represented by Eqs. (5.33). Similarly, the set represented by Eqs. (5.34) reduces to the singleton {1} for any node having only one (internal or external) outflow.

Armed with the above probabilistic model, the problem of defining and interpreting entropy in the context of flow networks can now be tackled and this is done in Section 5.3. To end this section, some numerical examples are provided next. These examples are rather trivial, but they may help clarify some of the definitions and observations that have been presented.

5.2.3 NUMERICAL EXAMPLES

The two examples below are based on the networks of Figures 5.3a and 5.3b. The flows have been selected arbitrarily, but they satisfy continuity. Also, to avoid lengthy explanations in the following calculations, the values of the flows in each network are arranged so that no two flows have the same numerical value. For example, regarding Figure 5.3a, 1.5 (units) must refer to q_{13}.

Example 1: Parallel Networks (Figure 5.3a)

$T_0 = 4 + 6 = 6.5 + 3.5 = 1.5 + 2.5 + 5 + 1 = 10$

$T_1 = 4 = 1.5 + 2.5$

$T_2 = 6 = 5 + 1$

$T_3 = 6.5 = 1.5 + 5$

$T_4 = 3.5 = 2.5 + 1$

$P_{01} = p_1 = 4/10 = (1.5 + 2.5)/10 = 0.4$

$P_{02} = p_2 = 6/10 = (5 + 1)/10 = 0.6$

$P_{30} = p_3 = 6.5/10 = (1.5 + 5)/10 = 0.65$

$P_{40} = p_4 = 3.5/10 = (2.5 + 1)/10 = 0.35$

Single-space link probabilities

$p_{13} = 1.5/10 = 0.15$

$p_{14} = 2.5/10 = 0.25$

$p_{23} = 5/10 = 0.5$

$p_{24} = 1/10 = 0.1$

Multiple-space link probabilities
outflow probabilities

$p_{13} = 1.5/4 = 0.375$

$p_{14} = 2.5/4 = 0.625$

$p_{23} = 5/6 = 0.833$

$p_{24} = 1/6 = 0.167$

inflow probabilities

$p_{13} = 1.5/6.5 = 0.231$

$p_{23} = 5/6.5 = 0.769$

$p_{14} = 2.5/3.5 = 0.714$

$p_{24} = 1/3.5 = 0.286$

Example 2: General Networks (Figure 5.3b)

$T_0 = 8 + 2 = 4.5 + 5.5 = 10$

$T_1 = 8 = 1 + 7;\ \bar{p}_1 = 8/10 = 0.8$

$T_2 = 2 + 1 = 3;\ \bar{p}_2 = 3/10 = 0.3$

$T_3 = 7 = 2.5 + 4.5;\ \bar{p}_3 = 7/10 = 0.7$

$T_4 = 3 + 2.5 = 5.5;\ \bar{p}_4 = 5.5/10 = 0.55$

$P_{01} = 8/10 = 0.8$

$P_{02} = 2/10 = 0.2$

$P_{30} = 4.5/10 = 0.45$

$P_{40} = 5.5/10 = 0.55$

node 1:

$p_{01} = 8/8 = 1.0$

$p_{12} = 1/8 = 0.125$; $p_{13} = 7/8 = 0.875$

node 2:

$p_{02} = 2/3 = 0.667$; $p_{12} = 1/3 = 0.333$

$p_{2.} = 3/3 = 1.0$

node 3:

$p_{13} = 7/7 = 1.0$

$p_{30} = 4.5/7 = 0.643$; $p_{34} = 2.5/7 = 0.357$

node 4:

$p_{24} = 3/5.5 = 0.545$; $p_{34} = 2.5/5.5 = 0.455$

$p_{40} = 5.5/5.5 = 1.0$

The interested reader may use the figures in these examples to verify any properties given in this section. For example, for Figure 5.3b, $\sum_{n=1}^{4} \bar{p}_n = 2.35 > 1$.

5.3 ENTROPY OF NETWORK FLOWS

In this section a formula is developed for the entropy of the flows of a general network. Informational entropy is a measure of the amount of uncertainty in a probabilistic system and is couched in probabilities. On the other hand, network flows do not directly involve probabilities but are capable of being interpreted in a probabilistic fashion as in Section 5.2. The probability

framework of the last section is used here to define the flow entropy of a network. The question that needs an answer is the following. What form does Shannon's entropy take for a general flow network? It has been seen in Section 5.2 that the flows of a network can in general be described by a multispace probability scheme. Also, the entropy of a compound scheme has been defined in Chapter 4. It is shown here how that definition may be applied to a flow network. The structure of the entropy function for parallel networks is quite straightforward and is addressed first.

5.3.1 PARALLEL NETWORKS

It has been seen that it is possible to describe link flows in a parallel network with a single-space probability scheme. Shannon's function may therefore be applied directly. Shannon's entropy for a finite scheme is

$$S = -K \sum_{i=1}^{N} p_i \ln p_i \qquad (5.41)$$

in which S is the entropy; K is an arbitrary positive constant; N is the number of outcomes; p_i is the probability of the ith outcome, $\forall i$. For a parallel network S/K is given by

$$S = -K \sum_{ij \in IJ} p_{ij} \ln p_{ij} \qquad (5.42)$$

in which IJ represents the set of all links of the network; p_{ij} represents the ratio of the flow in link ij, $\forall ij \in IJ$, to the total supply T_0 and is:

$$p_{ij} = \frac{q_{ij}}{T_0} \quad \forall ij \in IJ \qquad (5.16)$$

$$T_0 = \sum_{ij \in IJ} q_{ij} = \sum_{n \in I} q_{0n} = \sum_{n \in D} q_{n0} \tag{5.15}$$

in which q_{ij}, $\forall ij \in IJ$, is the flow in link ij; I represents the set of all source nodes; D represents the set of all demand nodes; q_{0n} is the source supply at node n, $\forall n \in I$; q_{n0} is the demand at node n, $\forall n \in D$. It will be recalled that there are no restrictions on what the q_{ij}, $\forall ij \in IJ$, might be. The only condition is that they be arranged in parallel or be capable of being represented as a parallel network. A practical example is considered in Section 5.4. However, the example below is a trivial demonstration.

Example 3

In this example the value of S/K is calculated for the flows of Figure 5.3a. The probabilities are given by Eqs. (5.16) and have been calculated in Example 1. Substituting those probabilities in Eq. (5.42) gives $S/K = 1.208$. It may be noted that a value has not been assigned to K. There is no real need to do so and any value of entropy mentioned herein is deemed to represent S/K rather than S, even when the K is omitted (or implicitly set to unity). In fact, K is assumed to be unity in the rest of this thesis and, as such, does not appear in most of the equations which follow.

It must be emphasised that Eq. (5.42) applies to parallel networks only. A more general definition is given in the next section. The value of the entropy of the network of Figure 5.3a used in Example 3 is calculated again but in accordance with the general definition and the same value of 1.208 is obtained as seen shortly.

5.3.2 GENERAL NETWORKS

In probability terms, a general flow network may be regarded as a compound scheme made up of many finite schemes which are not independent. Also, it

has been seen in Chapter 4 that the entropy of a compound scheme may be defined by using the notion of conditional entropy. It is now explained how conditional entropy can be used to define the entropy of the flows of a general network.

It will be recalled from Chapter 4 that

$$S(O_1O_2) = S(O_2O_1) = S(O_1) + S(O_2 \mid O_1) \equiv S(O_2) + S(O_1 \mid O_2) \tag{5.43}$$

in which O_1O_2 is a compound scheme composed of the schemes O_1 and O_2; $S(O_1O_2)$ is the joint entropy of O_1 and O_2; $S(O_2 \mid O_1)$ is the conditional entropy of O_2 on the assumption that O_1 has occurred and is:

$$S(O_2 \mid O_1) = -\sum_i \sum_j p(\hat{o}_i) p(\tilde{o}_j \mid \hat{o}_i) \ln p(\tilde{o}_j \mid \hat{o}_i) \tag{5.44}$$

in which the hat and the squiggle respectively identify elementary events in O_1 and O_2. Also, $S(O_1 \mid O_2)$ follows similarly, but with the roles of O_1 and O_2 interchanged, i.e.,

$$S(O_1 \mid O_2) = -\sum_j \sum_i p(\tilde{o}_j) p(\hat{o}_i \mid \tilde{o}_j) \ln p(\hat{o}_i \mid \tilde{o}_j) \tag{5.45}$$

In fact, Eqs. (5.44) and (5.45) represent the same thing as whichever scheme occurs first can always be represented by O_1 and the other by O_2.

Relating Eq. (5.44) to flow entropy, the $p(\tilde{o}_j \mid \hat{o}_i)$ or $p(\hat{o}_i \mid \tilde{o}_j)$, $\forall ij$, in Eqs. (5.44) and (5.45) are conditional probabilities which correspond to the p_{jn} and p_{nk}, $\forall n$, $\forall j \in NU_n$, $\forall k \in ND_n$, of a flow network. It will be recalled that p_{jn} is defined for flow that is destined for node n while p_{nk} is defined for flow that is destined to pass node n. In both cases it is assumed that o_n does occur. Also, the $p(\hat{o}_i)$ and $p(\tilde{o}_j)$, $\forall i$, $\forall j$, are absolute probabilities which correspond to the

$\bar{p}_{jn}$ and $\bar{p}_{nk}$, $\forall n$, $\forall j \in NU_n$, $\forall k \in ND_n$. Thus the $\hat{o}_i$ and the $\tilde{o}_j$ are the various ways in which O_1 and O_2 respectively happen. Similarly, o_{jn} and o_{nk} are respectively the various ways in which flow can reach or leave node n. Therefore, in accordance with Eq. (5.44), the respective conditional entropies can be written for the inflows and outflows at any node.

Let S_n^i be the conditional entropy of the *inflows* at node n, $\forall n$. Also, let S_n^o be the conditional entropy of the *outflows* at node n, $\forall n$. Using Eq. (5.44) and the above established correspondence between the probabilities, the following equations are obtained.

$$S_n^o = -\sum_{j\in NU_n}\sum_{k\in ND_n} \bar{p}_{jn} p_{nk} \ln p_{nk}$$
$$= -\sum_{k\in ND_n} p_{nk} \ln p_{nk} \sum_{j\in NU_n} \bar{p}_{jn}$$

Also, from Eqs. (5.23) and (5.29), $\bar{p}_n = \sum_{j\in NU_n} \bar{p}_{jn} = \dfrac{T_n}{T_0}$, $\forall n$. It follows that

$$S_n^o = -\bar{p}_n \sum_{k\in ND_n} p_{nk} \ln p_{nk} \qquad n = 1,\ldots,NN \tag{5.46a}$$

where:

$$p_{nk} = \frac{q_{nk}}{T_n} \qquad \forall n,\ \forall k \in ND_n \tag{5.34}$$

$$T_n = \sum_{j\in NU_n} q_{jn} = \sum_{k\in ND_n} q_{nk} \qquad n = 1,\ldots,NN \tag{5.28}$$

Similarly, the conditional entropy of the inflows at node n, $\forall n$, is:

$$S_n^i = -\bar{p}_n \sum_{j\in NU_n} p_{jn} \ln p_{jn} \qquad n = 1,\ldots,NN \tag{5.47}$$

$$p_{jn} = \frac{q_{jn}}{T_n} \qquad \forall n,\ \forall j \in NU_n \tag{5.33}$$

It should be noted that the ND_n and NU_n in Eqs. (5.46a) and (5.47) respectively include any demand or supply at node n, $\forall n$.

Having defined conditional entropies for the nodes of a flow network, it remains to establish how these nodal entropies can be combined to give the entropy of the network. The key lies in Eq. (5.43). Suppose there are three schemes, instead of two as in Eq. (5.43). What is the joint entropy $S(O_1O_2O_3)$ of O_1, O_2 and O_3? To answer this question, let $\bar{O}_1 = O_1O_3$, so that $O_1O_2O_3 = \bar{O}_1O_2$. Applying Eq. (5.43) to $\bar{O}_1O_2$ gives

$$S(\bar{O}_1O_2) = S(\bar{O}_1) + S(O_2 | \bar{O}_1) \tag{5.48}$$

Repeating the operation on $\bar{O}_1$ gives

$$S(\bar{O}_1) = S(O_1O_3) = S(O_1) + S(O_3 | O_1) \tag{5.49}$$

Substituting Eq. (5.49) in (5.48) and using $\bar{O}_1 = O_1O_3$,

$$S(O_1O_2O_3) = S(O_1) + S(O_3 | O_1) + S(O_2 | O_1O_3) \tag{5.50}$$

in which $S(O_2 | O_1O_3)$ is the conditional entropy of O_2 on the assumption that the joint scheme O_1O_3 has occurred. However, if O_3 is considered to be the second rather than the third scheme, and O_2 the third rather than the second, then Eq. (5.50) can be written as

$$S(O_1O_2O_3) = S(O_1) + S(O_2 | O_1) + S(O_3 | O_1O_2) \tag{5.51}$$

The entropy of a compound scheme is invariant with respect to positional interchanges amongst the constituent schemes as has been seen in Chapter 4. In the above manner an equation can be obtained for any number of constituent schemes. The general structure of that equation is easy to see from Eq. (5.51) and hence it can be deduced for any number of schemes. Thus,

$$S(O_1O_2\ldots O_{n1}) = S(O_1) + S(O_2 | O_1) + \ldots + S(O_m | O_1O_2\ldots O_{m-1}) + \ldots$$
$$+ S(O_{n1} | O_1O_2\ldots O_{n1-1}) \qquad n1 = 2, 3, \ldots;\ 2 < m \in Z^+ < n1 \tag{5.52}$$

in which the number of finite schemes is $n1$; Z^+ represents the set $\{0, 1, 2, 3, \ldots\}$.

The link between $S(O_{n1}|O_1O_2...O_{n1-1})$ and S_n^o and S_n^i, $n = 1, ..., NN$, in the context of a flow network is the following. Any assumption that flow has reached a node entails an inherent assumption that flow has reached all its predecessor nodes. Therefore, for a flow network, $S(O_{n1}|O_1O_2...O_{n1-1})$ represents S_n^i or S_n^o for inflows and outflows respectively. In $S(O_{n1}|O_1O_2...O_{n1-1})$ the position of individual schemes does not matter, the expression being a representation of the conditional entropy of any scheme on the assumption that all the rest have occurred.

A final issue concerns the identification in a flow network of a finite scheme in which the probabilities can be described absolutely as opposed to conditionally. That is, a scheme the entropy of which corresponds to the first term of Eq. (5.52). There are two such schemes which are respectively the P_{0n}, $n \in I$, and P_{n0}, $n \in D$.

In accordance with Eq. (5.52) the entropy of the flows of a network can be written either for inflows or for outflows as follows.

$$S^i = S_0^d + \sum_{n=1}^{NN} S_n^i \tag{5.53}$$

in which S^i is the network entropy based on the inflows; S_n^i is the conditional entropy of inflows, including any source supply, at node n, $\forall n$, as given by Eqs. (5.47); S_0^d is the entropy of the distribution of T_0 amongst the demand nodes, i.e.,

$$S_0^d = -\sum_{n \in D} P_{n0} \ln P_{n0} \tag{5.54}$$

Also,

$$S^o = S_0^s + \sum_{n=1}^{NN} S_n^o \tag{5.55}$$

is the network entropy based on the outflows, in which S_n^o is the conditional entropy of outflows, including any demand, at node n, $\forall n$, as given by Eq. (5.46a);

$$S_0^s = -\sum_{n \in I} P_{0n} \ln P_{0n} \tag{5.56}$$

is the entropy of the distribution of T_0 amongst the sources. For Eqs. (5.53) and (5.55) respectively, the q_{n0}, $\forall n \in D$, and q_{0n}, $\forall n \in I$, may be regarded as inflows into a supersink and outflows from a supersource. Furthermore, S^o is the entropy of the outflows given, tacitly, the inflows. It follows from the identity of (5.43) that the entropy of the outflows must equal the entropy of the inflows, i.e.,

$$S^i \equiv S^o \tag{5.57}$$

However, it may be noted that in general $S_n^i \neq S_n^o$, $\forall n$, just as $S(O_2|O_1) \neq S(O_1|O_2)$ in general. The numerical examples considered below demonstrate the use of the above equations.

5.3.3 EXAMPLES AND COMMENTS

The identity (5.57) and other observations are highlighted with the examples which follow.

Example 4

This example is trivial but it shows that the multispace formulations of Eqs. (5.53) and (5.55) give the same value of entropy as the well-established and incontrovertible single-space formulation of Eq. (5.42) for parallel networks.

Also, the identity (5.57) is verified. The calculations are for the network of Figure 5.3a whose value of entropy has been found by the single-space approach in Example 3 to be 1.208. All the probabilities for this network have been calculated in Example 1. First, S^o is calculated.

$S_0^s = 0.6730116$ *[Eq. (5.56)]*

$\frac{S_1^o}{\bar{P}_1} = 0.6615632; \ \frac{S_2^o}{\bar{P}_2} = 0.4505612; \ \frac{S_3^o}{\bar{P}_3} = 0; \ \frac{S_4^o}{\bar{P}_4} = 0$ *[Eqs. (5.46a)]*

$S_1^o = 0.2646252; \ S_2^o = 0.2703367; \ S_3^o = 0; \ S_4^o = 0$

$S^o = 1.208$ *[Eq. (5.55)]*

The above value of 1.208 is the desired result.

Next, S^i is calculated.

$S_0^d = 0.6474466$ *[Eq. (5.54)]*

$\frac{S_1^i}{\bar{P}_1} = 0; \ \frac{S_2^i}{\bar{P}_2} = 0; \ \frac{S_3^i}{\bar{P}_3} = 0.5402041; \ \frac{S_4^i}{\bar{P}_4} = 0.5982695$ *[Eqs. (5.47)]*

$S_1^i = 0; \ S_2^i = 0; \ S_3^i = 0.3511326; \ S_4^i = 0.2093943$

$S^i = 1.208$ *[Eq. (5.53)]*

This value of 1.208 is the desired result, i.e., the entropy of the outflows equals the entropy of the inflows (equals the entropy of the single-space scheme for a parallel network). It may be noted that, being a parallel network, $\bar{p}_n = p_n$, $\forall n$, in this example.

Example 5

In this example, S^i and S^o are calculated for the flows of the looped network

of Figure 5.3b the probabilities of which have been computed in Example 2. Proceeding as in Example 4, $S^i = S^o = 1.258$, and it is seen that Eq. (5.57) holds.

Example 6

In this example, an easy way of visualising Eqs. (5.55) and (5.57) is described. Eq. (5.55) for the entropy of the outflows is defined in a way which seems intuitively natural as it follows the progress of the flows. However, suppose the directions of all flows in Figure 5.3b are reversed as shown in Figure 5.3c. The value of the entropy of the resulting outflows can be found using Eq. (5.55) and is 1.258, which equals the value calculated in Example 5. In fact, it can be seen that all the "outflows" of Figure 5.3c are the inflows of Figure 5.3b. In other words, all the entries in Eq. (5.55) for Figure 5.3c are identical to the corresponding entries in Eq. (5.53) for Figure 5.3b.

Example 7

This example is based on the network of Figure 5.3d. It seems intuitively sensible that the entropy of a tree-type *single-source* network should be zero because the uncertainty associated with the *link flows* is zero. The extent to which Eqs. (5.53) and (5.55) agree with this notion is examined here. For the flows of the network of Figure 5.3d, $S^i = S^o = 0.997228$. Also, the entropy of the demands S_0^d is 0.997228. That is,

$$S^o = S^i = S_0^d \tag{5.58}$$

The above property holds for any tree-type single-source network. It can be seen, for example in Figure 5.3d, that all terms in Eq. (5.53) except S_0^d have a value of zero; $1 \ln 1 = 0$. Eqs. (5.58) follow from the identity of (5.57). Also, it follows from Eqs. (5.58) that the value of S given by Eq. (5.53) or (5.55) is the entropy of the *demands* and, therefore, the entropy of the *link flows* must be zero as required. Similarly, for a tree-type multiple-source network having a single demand node,

$$S^o = S^i = S_0^s \tag{5.59}$$

One way of seeing that Eqs. (5.59) hold is by reversing the flows of Figure 5.3d as explained earlier in Example 6 for Figures 5.3b and 5.3c.

Also, it follows from Eqs. (5.58) that, for the same demands, all tree-type layouts of a network consisting of demand nodes and a single source node have the same value of entropy S_0^o. Similarly, the entropy of the flows of a multiple-source network having only one demand node is the same for all tree-type layouts if the percentage contributions of the source supplies remain unchanged, and the value of the entropy is given by S_0^i, from Eqs. (5.59). For a looped network having tree-type portions, any such portions may be omitted for entropy purposes. However, since a tree-type configuration has a constant value of entropy, any tree-type branches in a looped network merely add a constant value to the real value of interest.

Henceforth, the superscripts o and i on S are used only when it is strictly necessary. Also, only one of Eqs. (5.53) and (5.55) is referred to unless it is strictly necessary to mention both.

5.3.4 OTHER DEFINITIONS OF NETWORK ENTROPY

In Section 5.4 it is shown how flow entropy can be used to infer network flows but first, this section on the definition of network entropy is concluded with some comments relating to the definitions of Awumah, Goulter and Bhatt (1990, 1991, 1992). Firstly, it has been seen in Section 5.2 that normalising link flows on their sum gives ratios which in general cannot be regarded as probabilities. It follows that the entropy-motivated definition of Eq. (3.10) is questionable. Also, Eq. (3.13) is questionable as it is equivalent to (3.10). However, it is necessary to examine Eq. (3.13) itself because Awumah, Bhatt and Goulter have given several different equations for the S_n, $\forall n$, which are

then substituted in Eq. (3.13). The Q_n, $\forall n$, are the sums of the link inflows at node n. Normalising the Q_n, $\forall n$, on the sum of the link flows Q_0 gives ratios which are not probabilities as there is double counting in Q_0. Therefore, from an entropy viewpoint, the second term of Eq. (3.13) is theoretically incorrect. Furthermore, the first term is questionable because the Q_n/Q_0, $\forall n$, are not proper weights. It follows that there is some uncertainty about the soundness of all the derivatives of Eq. (3.13): Eq. (3.17); Eq. (3.13) based on (3.18); Eq. (3.18) based on (3.20).

Secondly, it will be recalled from Section 5.2 that there is uncertainty associated with the external inflows or outflows at each node of a flow network. Although the values of supplies and demands may be known, there is uncertainty in the sense that it cannot be stated in percentage terms how much the source supply at each node contributes to the total flow reaching that node or, for a demand node, how much of the total inflow satisfies abstraction at that node. However, the definition of nodal entropy of Eq. (3.14) is based on the uncertainty associated with link flows only and, as such, is incomplete. This weakness has been addressed herein in Eq. (5.55).

A third observation is concerned with the possibility of defining network entropy in such a way that it is based on both the inflows and outflows at the nodes. This is different from the approach of Eqs. (5.53) and (5.55) in which the inflows and outflows respectively are used separately. From a purely information-theoretic viewpoint, Eqs. (5.53) and (5.55) appear to be correct definitions of network entropy as they agree with the single-space approach for parallel networks and are rigorously derived. However, if the q_{0n}, $\forall n \in I$, and q_{n0}, $\forall n \in D$, are regarded as constants, then, it may not be necessary in certain cases to include S_0^i or S_0^o which, consequently, are also constants. Also, considering that $S_0^i \neq S_0^o$, and hence $S_n^i \neq S_n^o$ in general, it may sometimes be desirable to define nodal and network entropy respectively as follows:

$$\tilde{S}_n = S_n^i + S_n^o \quad \forall n \tag{5.60}$$

$$\tilde{S} = \sum_{n=1}^{NN} \tilde{S}_n \tag{5.61}$$

Eqs. (5.60) are considered superior to (3.20) of Awumah, Bhatt and Goulter as the latter does not account for the uncertainty due to any nodal abstraction or external inflow. This is additional to the previously mentioned fundamental weaknesses of the Awumah, Bhatt and Goulter (1990, 1991, 1992) equations.

A· an example, the value of entropy given by Eq. (5.61) for the · etwork of Example 4, Figure 5.3a, is calculated from the figures of that example as $\tilde{S} = 1.095$. However, it is probably easier to calculate $\tilde{S}$ from Eqs. (5.62) below than (5.61).

$$\tilde{S} = 2S^i - S_0^s - S_0^d = 2S^o - S_0^s - S_0^d \tag{5.62}$$

5.4 CALCULATING MAXIMUM ENTROPY FLOWS IN NETWORKS

Having established what the appropriate entropy function for network flows is, the problem of determining the least biased flows can now be addressed. The problem consists of finding the least biased flows for the links of a looped network given only the supplies and demands, and the flow directions in the links. It has been seen in Section 5.1 that knowledge of the nodal supplies and demands is not sufficient for unique determination of the link flows in a network that is not tree-like. This problem of inferring least biased link flows for looped networks therefore requires inference based on partial information. In this section, it is explained how Jaynes' maximum entropy formalism can be used for probabilistic inference on the link flows of a non-tree-type network. The maximum entropy formalism states that in making inference based on

partial information, the probability distribution which has maximum entropy subject to whatever is known must be used. For a flow network, the entropy is maximized subject to the nodal flow equilibrium equations.

First, it is shown that in the problem of inferring network flows, even if $\tilde{S}$ is preferred over S, it is sufficient to maximize either S^i or S^o only. $\tilde{S}$ is the network entropy based on both the nodal inflows and outflows.

$$\tilde{S} = \sum_{n=1}^{NN} \tilde{S}_n \tag{5.61}$$

An upper bound on $\tilde{S}$ is

$$[\tilde{S}]_{max} = \left[\sum_{n=1}^{NN} S_n^i\right]_{max} + \left[\sum_{n=1}^{NN} S_n^o\right]_{max}$$

in which $[\,]_{max}$ denotes the maximum value that can be attained. Constants may be added to both sides of the above equation without destroying the equality. Therefore

$$[\tilde{S} + S_0^s + S_0^d]_{max} = \left[S_0^s + \left[\sum_{n=1}^{NN} S_n^i\right]_{max}\right] + \left[S_0^d + \left[\sum_{n=1}^{NN} S_n^o\right]_{max}\right]$$

$$= [S^o]_{max} + [S^i]_{max} = 2[S]_{max}$$

This result means that maximizing $\tilde{S}$ is the same as maximizing S.

5.4.1 PARALLEL NETWORKS

Two approaches for introducing probabilities into flow networks having only parallel connections have been considered in Section 5.2. One approach leads to a single probability space. The other leads to a multiple-space probability scheme. There is an appropriate entropy function for each approach and each function may be maximized subject to flow equilibrium at each node. Both approaches give identical results as shown next.

5.4.1.1 SINGLE PROBABILITY SPACE MODEL

The appropriate entropy function for the single-space scheme is:

$$S = -\sum_{ij \in IJ} p_{ij} \ln p_{ij} \tag{5.42}$$

$$p_{ij} = \frac{q_{ij}}{T_0} \quad \forall ij \in IJ \tag{5.16}$$

The flow equilibrium equations are:

$$q_{0n} = \sum_{k \in ND_n} q_{nk} \quad \forall n \in I \tag{5.20a}$$

or, dividing through by T_0,

$$\frac{q_{0n}}{T_0} = \sum_{k \in ND_n} p_{nk} = P_{0n} \quad \forall n \in I \tag{5.20b}$$

$$q_{n0} = \sum_{j \in NU_n} q_{jn} \quad \forall n \in D \tag{5.23a}$$

or, dividing through by T_0,

$$\frac{q_{n0}}{T_0} = \sum_{j \in NU_n} p_{jn} = P_{n0} \quad \forall n \in D \tag{5.23b}$$

The maximum entropy problem for parallel flow networks is therefore:

Problem 5

$$\underset{\forall p_{ij}}{Maximize}\ S = -\sum_{ij \in IJ} p_{ij} \ln p_{ij} \tag{5.42}$$

subject to:

$$\sum_{ij \in IJ} p_{ij} = 1 \tag{5.17}$$

$$\sum_{k \in ND_n} p_{nk} = P_{0n} \quad \forall n \in I \tag{5.20b}$$

$$\sum_{j \in NU_n} p_{jn} = P_{n0} \quad \forall n \in D;\ (NO-1) \text{ equations} \tag{5.23b}$$

in which the number of links NIJ is greater than $(NI + NO)$, the number of available equations; NI is the number of source nodes; NO is the number of demand nodes.

Problem 5 fits the format of Problem 4 and, as such, it is a convex programming problem with a unique maximum point. There is an analytical solution to Problem 5 and it can be found by examining the stationarity of its Lagrangean which may be written as:

$$L(\underline{p}, \lambda, \underline{\beta}, \underline{\mu}) = -\sum_{ij \in IJ} p_{ij} \ln p_{ij} + (1+\lambda)\left[\sum_{ij \in IJ} p_{ij} - 1\right] + \sum_{n \in I} \beta_n \left[\sum_{k \in ND_n} p_{nk} - P_{0n}\right]$$

$$+ \sum_{n \in D} \mu_n \left[\sum_{j \in NU_n} p_{jn} - P_{n0}\right] \tag{5.63}$$

in which λ, β_n, $\forall n \in I$, μ_n, $\forall n \in D$, are Lagrange multipliers and it is assumed that all the demand constraints but one are included in D. Stationarity of the Lagrangean with respect to a typical probability $p_{i'j'}$ yields:

$$-\ln p_{i'j'} - 1 + (1+\lambda) + \beta_{i'} + \mu_{j'} = 0 \tag{5.64a}$$

$$p_{i'j'} = \exp(\lambda)\exp(\beta_{i'} + \mu_{j'}) \tag{5.64b}$$

Stationarity of the Lagrangean with respect to λ yields the normality condition Eq. (5.17). Substituting (5.64b) in (5.17) gives:

$$\exp(\lambda)\sum_{ij \in IJ}\exp(\beta_i + \mu_j) = 1 \tag{5.65a}$$

$$\exp(\lambda) = \frac{1}{\sum_{ij \in IJ}\exp(\beta_i + \mu_j)} \tag{5.65b}$$

Substituting (5.65b) in (5.64b) gives:

$$p_{i'j'} = \frac{\exp(\beta_{i'})\exp(\mu_{j'})}{\sum_{i \in I}\exp(\beta_i)\sum_{j \in D}\exp(\mu_j)} \tag{5.66}$$

Stationarity of the Lagrangean with respect to a typical multiplier $\beta_{i'}$ yields the supply constraint for node i':

$$\sum_{j \in ND_{i'}} p_{i'j} = P_{0i'} \tag{5.20c}$$

Substituting the result of (5.66), this becomes:

$$\frac{\exp(\beta i')\sum_{j \in D}\exp(\mu_j)}{\sum_{i \in I}\exp(\beta_i)\sum_{j \in D}\exp(\mu_j)} = P_{0i'} \tag{5.67a}$$

$$\frac{\exp(\beta_{i'})}{\sum_{i \in I}\exp(\beta_i)} = P_{0i'} \tag{5.67b}$$

Stationarity of the Lagrangean with respect to a typical multiplier $\mu_{j'}$ yields the demand constraint for node j:

$$\sum_{i \in NU_{j'}} p_{ij'} = P_{j'0} \tag{5.23c}$$

Substituting the result of (5.66) and proceeding as for $\beta_{i'}$ gives

$$\frac{\exp(\mu_{j'})}{\sum_{j \in D}\exp(\mu_j)} = P_{j'0} \tag{5.68}$$

Substituting (5.67b) and (5.68) in (5.66) gives

$$p^*_{i'j'} = \left[\frac{\exp(\beta_{i'})}{\sum_{i \in I}\exp(\beta i)}\right]\left[\frac{\exp(\mu_{j'})}{\sum_{j \in D}\exp(\mu_j)}\right] = P_{0i'}P_{j'0}$$

When generalised for all i and j, the above result gives

$$p^*_{ij} = P_{0i}P_{j0} \quad \forall ij \in IJ \tag{5.69}$$

Reverting to flows,

$$\frac{q^*_{ij}}{T_0} = \frac{q_{0i}}{T_0}\frac{q_{j0}}{T_0} \quad \forall ij \in IJ$$

$$q^*_{ij} = \frac{q_{0i}q_{j0}}{T_0} \quad \forall ij \in IJ \tag{5.70}$$

The result of Eqs. (5.70) is the well-known *gravity model* of Transportation Engineering (Erlander, 1977; Kapur and Kesavan, 1987). Also, substituting the probabilities of Eqs. (5.69) in the entropy function (5.42) gives the maximum entropy value for parallel networks as

$$S^* = -\sum_{ij \in IJ} (P_{0i}P_{j0}) \ln(P_{0i}P_{j0}) \tag{5.71}$$

For a *fully-connected parallel network* in which each source node supplies all the demand nodes,

$$S^* = -\sum_{n \in I} P_{0n} \ln P_{0n} - \sum_{n \in D} P_{n0} \ln P_{n0} \tag{5.72}$$

in which D includes all demand nodes. It is recommended that Eq. (5.72) be used for calculating the maximum entropy value of any fully-connected parallel network because it has fewer terms than Eq. (5.71). Eqs. (5.71) and (5.72) have $(NI \times NO)$ and $(NI + NO)$ $p \ln p$ terms respectively. It should be noted that the value of $(NI \times NO)$ for the number of $p \log p$ terms in Eq. (5.71) applies only if the network is fully connected.

Example 8: Junction Turning Flows

Eqs. (5.70) give the value of the maximum entropy estimates of the link flows of any parallel flow network. Essentially, this example shows how junction turning flows can be represented as a parallel network which, in consequence, is solved by Eqs. (5.70). Referring to Figure 5.4, the q_{0n}, q_{n0}, $\forall n$, respectively represent the inflows and outflows on the arms of the junction. There are 12 traffic streams, as a driver approaching the junction from any of the four arms can proceed along any of the other three arms. However, there are only 8 flow equilibrium equations one of which is redundant; there is one equilibrium equation for each node of Figure 5.4b. Given no additional information, the problem of estimating the link flows requires inference based on partial information. Since the network representation of the flows is a parallel network, the solution is given by Eqs. (5.70).

5.4.1.2 MULTIPLE PROBABILITY SPACE MODEL

An alternative approach to the above single-space formulation is now described. The appropriate entropy function for the multispace scheme for parallel networks is:

$$S = S_0 + \sum_{n=1}^{NN} S_n \tag{5.73}$$

Either Eq. (5.53) or (5.55), their equivalence having been shown, may be used for calculating the conditional entropy in Eq. (5.73). Adopting Eq. (5.55), S_0 is the entropy of the distribution of the source supplies, and, for a *parallel network:*

$$S_0^s = -\sum_{n \in I} P_{0n} \ln P_{0n} \tag{5.56}$$

$$P_{0n} = p_n = \frac{q_{0n}}{T_0} \quad \forall n \in I \tag{5.9}$$

$$S_n = -p_n \sum_{k \in ND_n} p_{nk} \ln p_{nk} \quad \forall n \in I;\ S_n = 0 \quad \forall n \in D \tag{5.46b}$$

$$p_{nk} = \frac{q_{nk}}{q_{0n}} \quad \forall n \in I,\ \forall k \in ND_n \tag{5.18}$$

in which the bar on the $p_n, \forall n$, has been dropped because, for parallel networks, these probabilities are a finite scheme.

With probabilities defined at the supply nodes in Eqs. (5.18), flow equilibrium constraints are needed at the demand nodes and are

$$q_{n0} = \sum_{j \in NU_n} q_{jn} \quad \forall n \in D;\ (NO-1) \text{ equations.} \tag{5.23a}$$

From Eqs. (5.18),

$$q_{nk} = q_{0n} p_{nk} \quad \forall nk \in IJ \tag{5.74}$$

Eqs. (5.23a) and (5.74) give demand constraints as

$$\sum_{n \in NU_k} q_{0n} p_{nk} = q_{k0} \quad \forall k \in D; \; (NO-1) \text{ equations,}$$

which, when divided through by T_0 become

$$\sum_{n \in NU_k} P_{0n} p_{nk} = P_{k0} \quad \forall k \in D; \; (NO-1) \text{ equations.} \tag{5.23b}$$

The multispace maximum entropy problem for parallel flow networks is therefore:

Problem 6

$$\underset{\underline{p}}{Maximize} \; S = -\sum_{n \in I} P_{0n} \ln P_{0n} - \sum_{n \in I} P_{0n} \sum_{k \in ND_n} p_{nk} \ln p_{nk} \tag{5.75}$$

subject to:

$$\sum_{nk \in ND_n} p_{nk} = 1 \quad \forall n \in I \tag{5.19}$$

$$\sum_{n \in NU_k} P_{0n} p_{nk} = P_{k0} \quad \forall k \in D; \; (NO-1) \text{ equations.} \tag{5.23b}$$

In this formulation the variables are the p_{nk}, $\forall n \in I$, $\forall k \in ND_n$, and Eqs. (5.19) are the normality constraints. If, however, either the q_{0n}, $\forall n \in I$, or the q_{n0}, $\forall n \in D$, are unknown, then an extra normality constraint would be required for the P_{0n}, $\forall n \in I$, or P_{n0}, $\forall n \in D$, which would also be variables.

There is an analytical solution to Problem 6 which can be found by examining the stationarity of its Lagrangean. Its Lagrangean may be written as:

$$L(\underline{p}, \underline{\beta}, \underline{\mu}) = S + \sum_{n \in I} \beta_n \left[\sum_{k \in ND_n} p_{nk} - 1 \right] + \sum_{k \in D} \mu_k \left[\sum_{n \in NU_k} P_{0n} p_{nk} - P_{k0} \right] \tag{5.76}$$

Proceeding in a similar way to the solution of Problem 5, the solution is:

$$p_{nk}^* = P_{k0} \quad \forall n \in I, \; \forall k \in ND_n \tag{5.77}$$

Using Eqs. (5.74), the link flows are given by

$$q_{nk}^* = q_{0n} p_{nk}^* = q_{0n} P_{k0} = \frac{q_{0n} q_{k0}}{T_0} \quad \forall nk \in IJ \tag{5.78}$$

These flows are identical to the link flows of Eqs. (5.70) found for the single-space formulation of Problem 5. Also, it may be noted that if the value of the source supplies are known, the first term of the objective function is constant and is therefore not strictly necessary for the the optimization. Finally, the above results can also be arrived at by using Eq. (5.53), but with *supply constraints.*

5.4.2 GENERAL NETWORKS

5.4.2.1 PROBLEM FORMULATION AND SOLUTION

The maximum entropy problem for the flows of a general network is similar to Problem 6 as might be expected. This problem is formulated next. The entropy function for general networks is

$$S = S_0^s + \sum_{n=1}^{NN} S_n^o \tag{5.55}$$

In a general network, the constraints are the flow equilibrium equations. Using outflow probabilities,

$$\sum_{k \in ND_n} p_{nk} = 1 \quad \forall n \tag{5.79}$$

The continuity equations are:

$$\sum_{j \in NU_n} q_{jn} = \sum_{k \in ND_n} q_{nk} \quad n = 1, ..., NN \tag{5.28}$$

From Eqs. (5.34), $q_{ij} = T_i p_{ij}$, $\forall ij \in IJ$, giving in (5.28),

$$\sum_{j \in NU_n} T_j p_{jn} = \sum_{k \in ND_n} T_n p_{nk} = T_n \left[\sum_{k \in ND_n} p_{nk} \right] \quad \forall n \tag{5.80a}$$

Substituting Eqs. (5.79) and dividing through by T_0 gives

$$\sum_{j \in NU_n} \bar{p}_j p_{jn} = \bar{p}_n \quad \forall n \tag{5.80b}$$

The maximum entropy problem is therefore:

Problem 7

$$\underset{p}{Maximize}\ S = -\sum_{n \in I} P_{0n} \ln P_{0n} - \sum_{n=1}^{NN} \bar{p}_n \sum_{k \in ND_n} p_{nk} \ln p_{nk} \tag{5.81}$$

subject to:

$$\sum_{k \in ND_n} p_{nk} = 1 \quad \forall n \tag{5.79}$$

$$\sum_{j \in NU_n} \bar{p}_j p_{jn} = \bar{p}_n \quad n = 1, ..., NN-1 \tag{5.80b}$$

$$\bar{p}_n \geq 0 \ \ \forall n; \ \ p_{nk} \geq 0 \ \ \forall n, \forall k \in ND_n \tag{5.82}$$

The variables of Problem 7 are all the ps and $\bar{p}$s. However, if the q_{0n}, $\forall n \in I$, are unknown, then, the P_{0n}, $\forall n \in I$, are variables too. Also, it may be noted that Problem 7 does not have the same format as the classical maximum entropy problem of Problem 4. Furthermore, comparing Problems 6 and 7, it can be seen that the former is a special case of the latter. As such, the solution to Problem 7 also solves Proble.n 6, but not vice versa. Finally, a similar and equivalent problem can be set up for the inflow probabilities.

Problem 7 is a constrained non-linear programming (NLP) problem which may be solved with any suitable algorithm. The resulting p^*s can be used to obtain the T^*s from Eqs. (5.29) as

$$T_n^* = T_0 \bar{p}_n^* \quad \forall n; \tag{5.83}$$

the q^*s from Eqs. (5.34) as

$$q_{ij}^* = T_i^* p_{ij}^* \quad \forall ij \in IJ \tag{5.84}$$

Some numerical examples are considered shortly, but first, Problem 7 is reformulated in terms of flows. From this reformulation it is deduced that the problem of determining maximum entropy flows is a convex programming problem. Essentially, Problem 7 consists of maximizing S subject to non-negativity of the flows and flow equilibrium at all nodes. The variables of the actual problem are the link flows. The problem may therefore be written as:

Problem 8

$$\underset{\underline{q}}{Maximize}\ S = F_0(\underline{q}) \tag{5.85a}$$

subject to:

$$F_n(\underline{q}) = 0 \quad n = 1, ..., NN-1 \tag{5.86}$$

$$q_{ij} \geq 0 \quad \forall ij \in IJ \tag{5.87a}$$

in which constraints (5.86) are the flow equilibrium equations (5.28) and $F_0(\underline{q})$ is S with the link flows substituted for probabilities, using Eqs. (5.28), (5.29), (5.9), (5.34) and (5.36).

Problem 8 has a unique global maximum point. To establish that this is the case, it is sufficient to show that Problem 8 is a convex programming problem. The necessary and sufficient conditions are that the constraints represent a convex set and $F_0(\underline{q})$ be concave. Both conditions are satisfied in Problem 8. Firstly, constraints (5.86) are linear in the problem variables and, as such, represent a convex set. Secondly, to show that $F_0(\underline{q})$ is concave, it is shown that $S(\underline{p})$ of Eq. (5.81), being equivalent to $F_0(\underline{q})$, is concave. Concavity is a well-known property of $-\sum_i p_i \ln p_i$ given that the p_i, $\forall i$, are a finite scheme (Templeman and Li, 1985). Also, the sum of any number of concave functions is concave. It follows that $S(\underline{p})$ is concave since the $\bar{p}_n$, $\forall n$, are non-negative.

This convexity result for Problem 8 can be exploited in its numerical solution. Firstly, the flow equilibrium constraints are equalities which may be used to eliminate some unknown link flows and so reduce the size of the problem. The number of unknown link flows that can be eliminated is given by the following equation which has been introduced in Chapter 2 as Eq. (2.10).

$$NIJ - NL = NN - 1 \tag{5.88a}$$

$$NL = NIJ - (NN - 1) \tag{5.88b}$$

in which NIJ is the number of links; NL is the number of loops; NN is the number of nodes. Actually, NL corresponds to the difference between the number of flow equilibrium equations and the number of links, or the number by which the number of unknown link flows exceeds the number of available flow equilibrium equations. The right hand side of Eq. (5.88a) therefore gives the number of unknown link flows which may be eliminated while NL from Eq. (5.88b) gives the number of independent link flow variables. All the problem variables and functions can therefore be expressed in terms of the independent variables.

By using the flow equilibrium constraints to eliminate some variables, these constraints are satisfied implicitly and are therefore eliminated from the problem. Also, the number of variables eliminated is generally considerable as demonstrated by Eq. (5.88a). Denoting the independent flows by $\hat{q}_i$, $i = 1, ..., NL$, Problem 8 can be written as:

Problem 9

$$\underset{\hat{\underline{q}}}{Maximize}\ S = F_0(\hat{\underline{q}}) \tag{5.85b}$$

Subject to:

$$q_m = F_m(\hat{\underline{q}}) \geq 0 \quad m = 1, ..., NIJ \tag{5.87b}$$

in which $q_m = F_m$ is the flow in link m, $m = 1, ..., NIJ$.

Secondly, Eqs. (5.87b) are a system of linear inequalities which define the feasible region of the problem. Outside this region, at least one flow is negative and so S is undefined in the infeasible region. Therefore the minimum point of S is guaranteed not to violate Eqs. (5.87b). It is therefore possible to solve Problem 9 with standard algorithms for unconstrained optimization. This approach has been adopted in the present research as it provides a simple and quick solution procedure. For the numerical examples presented shortly, $S = F_0(\hat{q})$ has been maximized using the NAG library routine E04JAF which is a routine for unconstrained optimization.

The number of independent flows is equal to the number of loops and the redundant links should be selected so that there is one for each loop. Even parallel networks can be thought of in terms of loops as demonstrated in Figure 5.5. More practical details on the solution of Problem 9 are given below in Example 9 and later in this subsection. Also, an easy means of calculating maximum entropy flows for single-source networks has been devised and is described in Subsection 5.4.3. The rest of this subsection consists of practical details, examples and comments. The comments mostly highlight the apparent reasonableness of the present formulation of the maximum entropy flows problem.

Example 9

The calculation of probabilities and S have previously been demonstrated. In this example the emphasis is on the constraints of Problem 9. A formalised approach for reducing the number of constraints and variables is given later in this subsection, but the following is a simple demonstration. The analysis is for the network of Figure 5.6a.

The flow equilibrium equations are:

$T_1 = q_{01} = q_{12} + q_{13}$

$T_2 = q_{12} = q_{20} + q_{24}$

$T_3 = q_{13} = q_{30} + q_{34} + q_{35}$

$T_4 = q_{24} + q_{34} = q_{40} + q_{46}$

$T_5 = q_{35} = q_{50} + q_{56}$

There are 7 links of which 2 are redundant. Let q_{12} and q_{34} be the independent flows. The other 5 flows can be expressed in terms of the independent flows using the above equations. Thus:

$q_{12} = q_{12}$

$q_{34} = q_{34}$

$q_{13} = q_{01} - q_{12}$

$q_{24} = -q_{20} + q_{12}$

$q_{35} = q_{01} - q_{30} - q_{12} - q_{34}$

$q_{46} = -q_{20} - q_{40} + q_{12} + q_{34}$

$q_{56} = q_{01} - q_{30} - q_{50} - q_{12} - q_{34}$

The ultimate constraints are the non-negativities of all the above flows. If the numerical values of Figure 5.6a for the nodal inflows and outflows are used, they give the following inequalities of which those that are binding are numbered with asterisks.

$q_{12} \geq 0$

$q_{34} \geq 0$ (*)

$q_{13} = 0.284 - q_{12} \geq 0 \equiv q_{12} \leq 0.284$

$q_{24} = -0.028 + q_{12} \geq 0 \equiv q_{12} \geq 0.028$ (*)

$q_{35} = 0.251 - q_{12} - q_{34} \geq 0 \equiv q_{12} + q_{34} \leq 0.251$

$q_{46} = -0.103 + q_{12} + q_{34} \geq 0 \equiv q_{12} + q_{34} \geq 0.103$ (*)

$q_{56} = 0.159 - q_{12} - q_{34} \geq 0 \equiv q_{12} + q_{34} \leq 0.159$ (*)

It can be seen from the graphical representation of the binding constraints in Figure 5.6b that the feasible region is a trapezium the vertices of which are (q_{34}, q_{12}): (0, 0.103), (0.075, 0.028), (0.131, 0.028), (0, 0.159). With all flows defined in terms of the independent flows, all the probabilities and S can be calculated for any feasible point (q_{34}, q_{12}), as demonstrated in Figure 5.6c by the topographical representation of the values of the entropy function. In practice it is quite easy to find a feasible (starting) point as feasibility merely consists of satisfying continuity at all nodes. At the optimum, $(S, q_{12}, q_{34})^*$ = (1.9151, 0.0841, 0.0562), from which $(q_{12}, q_{34}, q_{56}, q_{13}, q_{24}, q_{35}, q_{46})^*$ = (0.0841, 0.0562, 0.0187, 0.1998, 0.0562, 0.1107, 0.0373). The optimum point is shown in Figure 5.6b while the maximum entropy flows are shown in Figure 5.6d.

Commenting on these results, firstly, it can be seen in Figure 5.6d that node 4 receives exactly half of its total inflow from each of its supply paths 1-2-4 and 1-3-4. Also, node 6 is supplied by three paths, 1-2-4-6, 1-3-4-6 and 1-3-5-6, each of which contributes one third of the required flow. Nodes 2, 3 and 5 have only one supply path each, on which, obviously, they are entirely dependent for their respective flows. It can be seen that these results agree with Laplace's principle of insufficient reason in the sense that, on a node-by-node basis, these flows correspond to the uniform distribution U. A similar but slightly different interpretation is the following. In formulating the problem constraints, only two things are specified, namely, flow equilibrium at each node and the links or paths supplying each node. It would therefore be absurd if the maximum entropy flows did not correspond to U. There is more on this issue in Subsection 5.4.3. Secondly, it is reassuring to note that, based on the above observations, the maximum entropy inflow probabilities seem reasonable even though the nodal entropy used is more explicitly related to the nodal outflows than inflows. The results of this example provide further evidence that the present formulation of network entropy is probably right.

Thirdly, it is useful to re-examine Figure 5.6b from a layout/reliability perspective. In Chapter 6, many reasons are given to suggest that entropy can be used as a surrogate measure of reliability for water distribution networks. The following observations are repeated there, but with more detailed explanations as those details are not very appropriate here. Referring to Figure 5.6b, each edge of the trapezium corresponds to a layout in which one link carries no flow. For example, the edge given by $q_{12} = 0.028$ corresponds to $q_{24} = 0$. Obviously, the vertices, being the point of intersection of adjacent edges, correspond to layouts in which two links carry no flow. For example, the point ($q_{34} = 0.075$, $q_{12} = 0.028$) corresponds to $q_{24} = q_{46} = 0$. Although more explicit arguments are given in the next chapter, it seems intuitively sensible that the most reliable layout or flow distribution should be as far away as possible from each of the four edges because these edges correspond to layouts with less redundancy than the layout of Figure 5.6a. It is therefore reasonable to expect $(q_{34}, q_{12})^*$ to be around the centre of the trapezium of Figure 5.6b. In fact, $\hat{q}^* = (q_{34}, q_{12})^*$ bisects the line in the feasible region that passes through $\hat{q}^*$ and is parallel to the parallel sides of the trapezium. In other words, $\hat{q}^*$ is midway between the edges $q_{34} = 0$ and $q_{12} = 0.028$. However, $\hat{q}^*$ is slightly nearer the edge $q_{34} + q_{12} = 0.159$ than $q_{34} + q_{12} = 0.103$. The fact that the optimum point is not midway between the parallel edges may be slightly disappointing, but, as explained in Chapter 6, the edge $q_{34} + q_{12} = 0.103$ is slightly more critical than the opposite edge. As such, it is reasonable for the optimum point to be nearer the "safer" of the two parallel edges. Nevertheless, $\hat{q}^*$ is roughly in the central region of the trapezium as expected. The above discussion demonstrates the fact that the maximum entropy flows for the network of Figure 5.6a are maximally noncommittal and the most neutral with respect to the possibility of any link flows being zero. Therefore, this example provides further evidence that entropy can be used for layout/reliability optimization.

Finally, an observation concerning the convexity result for the problem of determining maximum entropy flows in networks may be made. It can be seen from the topographical representation of the entropy function in Figure 5.6c that, as expected, the surface depicted is smooth and bounded by "vertical planes," and there is a single peak, without any saddle or local maximum points. This figure confirms the convexity result for the case of a two-loop network.

5.4.2.2 DEPENDENT AND INDEPENDENT FLOWS

In view of the advantages of solving Problem 9 rather than Problem 7, it is useful to be able to select redundant links and calculate dependent flows in a formal way.

Selection of Redundant Links

To calculate the dependent flows as described shortly, the redundant links, which carry the independent flows, have to be identified first and this is straightforward. There is one redundant link per loop and any of the links in a loop may be chosen as the redundant link for that loop. There are therefore NL, i.e., $NIJ-(NN-1)$, redundant links. To conform to the requirements of the scheme for calculating the dependent flows which is described below, the redundant links should be numbered with NN, $NN+1$, ..., NIJ. The rest of the links should be numbered with 1, 2, 3, ..., $NN-1$. As an example, the links of the 3-loop, 14-link network of Figure 5.7 are numbered such that the 3 selected redundant links are numbered 12 to 14 with the other links numbered 1 to 11. It may be noted that the positions of the redundant links are interchangeable and so too are the non-redundant links.

Calculation of Dependent Flows

Essentially, the following is a scheme for calculating the dependent variables of a linear system of equations in which there are more variables than equations. However, the values of enough variables are needed so that the values of the rest of the variables can be found from the equations. The

variables with specified values are the independent variables and the number of these variables is equal to the difference between the total number of variables and the number of equations. It can therefore be seen that there will be enough equations to enable the values of the dependent flows to be calculated by any standard algorithm for solving systems of linear equations. As such, what is described below is merely a means of ordering the variables of a linear system with more variables than equations such that an algorithm such as Gaussian elimination, for example, can be used to resolve the system. The following explanation is done on the basis of any general linear system of equations. Since the constraints of the problem of calculating maximum entropy flows are the continuity equations which are linear, a scheme such as the one described here can be applied to those constraints. Thus, suppose there are 3 linear equations in 5 variables x_i, $i=1, ..., 5$, of the form

$$\sum_{i=1}^{5} k_{ji}x_i = y_j \quad j=1,\ 2,\ 3 \tag{5.89}$$

in which the k_{ji}, $\forall ji$, are constant coefficients and the values of the right hand sides y_j, $\forall j$, are known. Also, suppose the values of 2 (= 5 - 3) of the variables can be specified. Assuming that the specified variables are x_4 and x_5, Eqs. (5.89) can be written as

$$\sum_{i=1}^{3} k_{ji}x_i = y_j - \sum_{i=4}^{5} k_{ji}x_i = z_j \quad j=1,\ 2,\ 3 \tag{5.90}$$

Eqs. (5.90) are a system of linear equations which are solvable by any suitable method, for example, Gaussian elimination. The linear system of (5.90) can be generalised for any number of variables N_v and any number of equations N_e such that $N_v > N_e$ with the last $(N_v - N_e)$ variables specified. The $(N_v - N_e)$ specified variables are the independent variables while the remaining N_e variables are the dependent variables. Thus:

$$\sum_{i=1}^{N_e} k_{ji} x_i = z_j \quad j = 1, ..., N_e \tag{5.91}$$

$$z_j = y_j - \sum_{i=N_e+1}^{N_v} k_{ji} x_i \quad j = 1, ..., N_e \tag{5.92}$$

The above approach may be used as the basis of a routine for calculating the dependent flows if values are specified for the independent flows. Relating the above equations to the continuity equations, Eqs. (2.6), of a distribution network, the y_j, $j = 1, ..., N_e$, correspond to the external inflows or outflows at the nodes, i.e., the q_n, $n = 1, ..., NN - 1$. Also, the x_i, $i = 1, ..., N_v$, correspond to the link flows q_m, $m = 1, ..., NIJ$. Thus the dependent variables x_i, $i = 1, ..., N_e$, correspond to the dependent flows q_m, $m = 1, ..., NN - 1$, while the independent variables x_i, $i = N_e + 1, ..., N_v$, correspond to the independent flows q_m, $m = NN, ..., NIJ$.

Finally, the k_{ji}, $\forall ji$, correspond to the elements of the *continuity matrix* which describes both the connectivity of the network and the direction of flow in each link. Thus the subscripts j and i correspond to n and m respectively which identify the nodes and links respectively. Values for the k_{nm} (k_{ji}) are obtained as follows. If the sign convention that q_n is negative if q_n represents a source supply and positive if q_n represents a demand, then:

$k_{nm} = 0$ if link m is not connected to node n, $\forall nm$

$k_{nm} = +1$ if node n is the downstream node of link m, $\forall nm$

$k_{nm} = -1$ if node n is the upstream node of link m, $\forall nm$

Alternatively, the sign convention could be reversed by reversing the above signs of the q_n and k_{nm}. Also, in the above discussion it is assumed that the nodes are numbered consecutively from unity, i.e., 1, 2, 3, ..., NN, while the non-redundant links are numbered 1, 2, 3, ..., $(NN - 1)$, and the redundant

links, NN, $NN + 1$, ..., NIJ. Proceeding as described above, it is possible to set up the continuity constraints of Problem 8 in a routine way for any general network. Furthermore, as Problem 9 can be solved as an unconstrained optimization problem, the problem of calculating maximum entropy flows for any general network can be solved easily and efficiently using standard routines for unconstrained optimization in conjunction with standard routines for the solution of systems of linear equations. However, for single-source networks, a very simple method of calculating maximum entropy flows more efficiently has been developed and this approach is described in the next subsection.

It will be recalled that one of the ultimate aims of the present research is to use entropy to *simplify the problem of optimally designing reliable water* distribution systems. The following example provides another demonstration that the present extension of the entropy of finite probability schemes to network flows gives consistent results. In particular, the results of this example highlight some implications for the design and reliability of water distribution systems.

Example 10

This example is based on the network of Figure 5.8a. Numerical values for the inflows and outflows at nodes 1 and 2 are shown in Figures 5.8c and 5.8d. In Figure 5.8d the inflows at nodes 1 and 2 of Figure 5.8c are interchanged. Although the network of Figure 5.8a is not a parallel network, the subnetwork consisting of links 1-3, 1-4, 2-3 and 2-4 has only parallel connections, as shown in Figure 5.8b. Therefore, it seems intuitively sensible that at the solution of Problem 7 or 9 for the network of Figure 5.8a, the flows of the parallel subnetwork of Figure 5.8b should agree with those given by the gravity model or the single-space maximum entropy formulation for parallel networks. Obviously, unlike parallel networks in which T_n equals the external inflow or outflow at node n, the values of T_3 and T_4 are unknown for the network of

Figure 5.8a. However, it is shown in this example that at the solution of Problem 7 for the network of Figure 5.8a, the flows of the links shown in Figure 5.8b satisfy the gravity model result of Eqs. (5.70). In other words, it is verified that the following equations, the origin of which is explained shortly, hold at the solution.

$$q_{13}^{*} = q_{01}(q_{13}^{*} + q_{23}^{*})/T_0 = q_{01}T_3^{*}/T_0$$

$$q_{23}^{*} = q_{02}(q_{13}^{*} + q_{23}^{*})/T_0 = q_{02}T_3^{*}/T_0$$

$$q_{14}^{*} = q_{01}(q_{14}^{*} + q_{24}^{*})/T_0 = q_{01}(T_4^{*} - q_{34}^{*})/T_0$$

$$q_{24}^{*} = q_{02}(q_{14}^{*} + q_{24}^{*})/T_0 = q_{02}(T_4^{*} - q_{34}^{*})/T_0$$

The above expressions are adapted from Eqs. (5.70). $(q_{13} + q_{23})$ and $(q_{14} + q_{24})$ are respectively the total flow supplied directly from the sources to nodes 3 and 4. Also, from Figure 5.8a, $T_4 = q_{14} + q_{24} + q_{34}$. This gives $q_{14} + q_{24} = T_4 - q_{34}$. Furthermore, from Eqs. (5.70), the four relationships stated above should hold for *any combination of nodal inflows and outflows*, as long as these flows balance. Solving Problem 9 computationally for the inflows of Figures 5.8c and 5.8d gives the maximum entropy link flows of Figures 5.8c and 5.8d respectively, and these flows agree with the above expressions.

Another observation on the results of Figures 5.8c and 5.8d is that the link flows in the lower half of the network are the same in these figures. Further study reveals that they remain the same for any combination of source flows at nodes 1 and 2 totalling 55 units, for demands as in Figure 5.8a. The reason for this is that nodes 1 and 2 are both connected to the rest of the network in exactly the same way and so the network is unable to distinguish topologically between the two source nodes. Therefore, intuitively, this invariance result for the flows of the lower half of the network appears consistent. Furthermore, this invariance result is desirable from both design and reliability viewpoints. Designing the pipes of a water distribution network having the connectivity of Figure 5.8a to carry maximum entropy flows would

appear to confer a considerable degree of invulnerability upon the lower half of the network to possible variations in the source flows. How far this is borne out is investigated in Chapter 6 in a more general context.

In both Examples 9 and 10 there are no counterintuitive results, and there are positive indications that the definitions of probabilities and entropy and the formulation of Problem 7 are correct.

5.4.3 PATH FLOWS IN SINGLE-SOURCE NETWORKS

Instead of solving Problem 7 or 9 a different approach can be used for single-source networks. The approach which is described next considerably simplifies the problem of calculating maximum entropy flows, but is in general applicable to single-source networks only. In a single-source network, all paths start at the *source*. Consider any demand node served by more than one path. Given no further information about the paths, the maximum entropy formalism dictates that the uniform distribution U be used for the probability of each path supplying the node. Therefore, all the paths supplying a node should have the same probability of doing so. This means that flow to a node should be distributed equally amongst all the paths supplying that node. Therefore, to obtain the maximum entropy flows for a single-source network, each node should be taken in turn and its demand divided equally amongst all paths supplying it. The final network flows are then obtained by superposition of these path flows. That is, for each link, the flow for all paths through that link should be summed to obtain the maximum entropy flow for that link. The maximum value of the flow entropy for the network may then be calculated from these flows.

The network of Figure 5.9 is used to demonstrate the above procedure. The demand at each node is treated separately, as shown in Figure 5.10. Thus, for example, node 3 is served by two paths 1-3 and 1-2-3, each of which must carry

5 units of flow (half the demand of node 3). In Figure 5.11, the maximum entropy flow in each link is obtained by summing the flows in all paths using that link. Calculating the entropy for these flows gives $S^* = 2.159$. To check that the above values are correct, Problem 9 has been solved computationally for the network of Figure 5.9. The results are shown below and are identical to those obtained above.

$$(S, q_{14}, q_{13}, q_{23}, q_{35}, q_{12}, q_{25}, q_{34})^*$$

$$= (2.159, 5.000, 18.000, 18.000, 16.000, 36.000, 8.000, 10.000)$$

5.4.3.1 ALGORITHMS FOR CALCULATING MAXIMUM ENTROPY FLOWS

The present method of superposition of equal path flows has been formalised as described next. The description is fairly general but is based on the network of Figure 5.9 for clarity. Following the description, algorithms are presented for the proposed method. It is assumed for the time being that the number of paths NP_n serving node n, $n = 1, ..., NN$, is known, but a method for calculating NP_n is described shortly.

Consider Figure 5.12. The number of paths to each node is enclosed in a box next to the node. Nodes 4 and 5 are *terminal nodes*, which do not have any link outflows. The procedure starts with any terminal node; say, node 4. The total outflow at that node is divided by 3, this being the number of paths to it. The quotient is then multiplied by 1 and 2 respectively, these being the respective number of paths to nodes 1 and 3 which are the immediate upstream supply nodes to node 4. The products, respectively, are the flows in links 1-4 and 3-4.

The next step is to choose any node immediately upstream of node 4 whose link outflows have all been calculated. The procedure explained above for node 4 is then repeated for the chosen node. Returning to Figure 5.12, both nodes 1 and 3 have unknown link outflows and so they cannot be treated yet.

At this point, the procedure stops and restarts at any terminal node that has not yet been dealt with. That is, node 5 in the present example. If node 5 is processed as explained for node 4, the flow in link 2-5 is 8 units and for link 3-5, 16 units.

At this stage, the only unprocessed node with all its outflows known is node 3. Its total outflow is 36 units. This flow is partitioned according to the aforementioned procedure by dividing it by 2 and multiplying the result by 1 for each of the two incoming links. The flow in link 1-2 can now be found. It is the sum of the outflows from node 2, including the demand at node 2. The process ends here. The flows obtained by the procedure just described are identical to those found by superposing equal path flows for each node.

A further refinement to the method concerns path enumeration which is not an attractive proposition, even for networks of modest size. (See, for example, Aggarwal, Gupta and Mishra, 1973) However, there is a way round this difficulty. It can be observed, for example, in Figure 5.12, that each NP_n is the sum of all the NP_n's of the immediate upstream nodes. In other words, the number of paths to each node is the sum of the number of paths to all nodes upstream of, and directly supplying, the node being considered. This is a fact which can be exploited to weight the nodes and thus avoid explicit path enumeration. The steps involved are as follows.

1. Assign 1 to the source.
2. Select any node whose upstream nodes have all been processed. Sum the numbers assigned to all nodes immediately upstream of the chosen node. Assign the total to the present node.
3. Repeat step 2 until all nodes have been processed.

Throughout, the assumption that the direction of flow in each link is known continues to apply. Also, it should be noted that this method of calculating the number of paths to each node applies to single-source networks only.

A final detail is concerned with the order in which nodes can be processed when weighting the nodes or calculating link flows. Node weighting depends on conditions immediately upstream of a node whereas the calculation of link flows depends on conditions immediately downstream of a node. Therefore, if nodes are numbered such that the order matches the node weighting sequence, as in Figure 5.12, for example, then, link flows can be calculated by taking the nodes in reverse order. Thus, a possible sequence is obtained if each node is numbered as soon as all nodes immediately upstream of it have been numbered. The algorithms presented subsequently herein, for node weighting and for calculating maximum entropy flows, assume that the nodes of the network have been numbered according to this convention. The nodes may therefore be numbered with the following algorithm.

Node numbering algorithm

1. Number the source with 1. Set n to 1.
2. Increase n by 1. Select any node whose immediate upstream nodes have all been numbered and number it with n.
3. If $n = NN$, exit. Otherwise, go to step 2.

Simple algorithms are now presented for node weighting and flow distribution respectively. Before applying these algorithms, the nodes must first be numbered with the node numbering algorithm. As defined previously, NP_n, $n = 1, ..., NN$, is the number of paths from the source to node $n, n = 1, ..., NN$.

Node weighting algorithm

1. Set n to the source number, 1. Set NP_n to 1.
2. Increase n by 1 and calculate NP_n:

$$NP_n = \sum_{j \in NU_n} NP_j$$

in which NU_n represents the set of upstream nodes of inflow links at node n.

3. If $n = NN$, exit. Otherwise, go to step 2.

Flow distribution algorithm

1. Set n to the number of nodes NN.
2. Calculate T_n:

$$T_n = \sum_{k \in ND_n} q_{nk}$$

in which any demand at node n must be included.

3. Calculate q_{jn}, $\forall jn \in NU_n$:

$$q_{jn} = T_n \frac{NP_j}{NP_n}$$

4. If $n = 1$, go to step 5. Otherwise, reduce n by 1 and go to step 2.
5. Calculate S^* if necessary. Exit.

These algorithms are rigorous for single source networks. However, they are in general inapplicable to multiple-source networks for the following reasons. The proposed method is a direct application of the result that maximization of Shannon's entropy function subject only to normality of the probabilities leads to the uniform distribution U. The corresponding result for a general network with multiple sources requires all the sources to contribute the same quantity of flow to the total supply as S_0 attains its highest possible value if all the P_{0n} are equal. In general, this condition will not be met as the flows from different sources will usually be unequal. Furthermore, the flow directions in a multiple-source network will be such that, for each node, the path flows will be unequal in general.

However, in any network, if the flow directions and the distribution of the source flows are such that all paths serving a node can carry the same amount of flow, then, the proposed method will give the right result. This also applies to any multiple-source network that is effectively operating as a single-source network. An example is provided next to illustrate some of the above points, but it must be stressed that the proposed method is intended as an alternative to numerical optimization for single-source networks only.

Example 11

A sample two-source network in which all the conditions for uniformity of the path flows are satisfied is shown in Figure 5.13a. Link 1-2 is a direct connection between the sources. The node weighting algorithm may be applied to multiple-source networks with source-source connection. However, each source is given a weight of unity in step 1. To see why, suppose the sources are replaced by a supersource numbered 0 with 55 units. Suppose further that link 1-2 is replaced by a direct link from the supersource to nodes 1 and 2 respectively as shown in Figure 5.13b. If the node weighting algorithm is carried out on this transformed, but equivalent network, both nodes 1 and 2 are assigned a weight of 1. This provides confirmation that each source in a multiple-source network, with all sources interconnected, should have a weight of unity. It may be noted that there need not be a direct link for every source-source combination. It is sufficient that each source be directly connected to at least one other source.

To obtain the maximum entropy flows for multiple-source networks with source-source connections, the flow algorithm may be applied as described for single-source networks, but with a slight modification. In step 2, $(T_n - q_{0n})$ is

found, and used in step 3, instead of T_n. The external inflow q_{0n} will be zero for all nodes other than source nodes. Also, this modified version of the algorithm may be used for networks having a single source. Finally, the value of S^* may be calculated once the maximum entropy flows are available. The problem of Figure 5.13a has been solved by both numerical optimization using the NAG library routine E04JAF and the present method. Both methods give the same results, of which the flows are shown in Figure 5.13a.

However, in a general network, if at least one of the requirements for equality of path flows is not satisfied, the single-source method cannot be used. For example, in Figure 5.14, the flow direction of $2 \rightarrow 1$ in the source-connecting link is the reverse of the direction in Figure 5.13a. Figures 5.13a and 5.14 are identical in every other respect. The problem of determining the maximum entropy flows for the network of Figure 5.14 cannot be solved by the present single-source method as demonstrated by the optimum point which is

$$(S, q_{23}, q_{13}, q_{34}, q_{35}, q_{21}, q_{14}, q_{25})^*$$

$$= (1.947, 12.917, 28.871, 8.871, 22.917, 0.000, 6.129, 7.083)$$

These maximum entropy flows are shown in Figure 5.14. The fact that the optimum point contains a zero element is an indication that the single-source method cannot solve this problem. This example shows that the present method is in general inapplicable to multiple-source networks.

The material in this subsection is, obviously, also applicable to multisource networks with a single demand point. Flow reversal, as used in Figures 5.3b and 5.3c obviously holds true. If the flows of single-source network are reversed in this way, the resulting network has multiple sources and a single demand node. Moreover, it will be recalled from Eq. (5.57) that the entropy of inflows equals the entropy of outflows.

With regard to the reliability and design of water distribution systems, several interesting comments may be made on the maximum entropy flows of Figures 5.13a and 5.14. Figure 5.14 has a lower value of $S^* = 1.947$ than Figure 5.13a whose value is 2.020. This bodes well for the possible use of flow entropy in layout and reliability optimization for several reasons. At the optimum, the network of Figure 5.13a with three loops has more redundancy than the network of Figure 5.14 with two loops. It is therefore fitting that Figure 5.13a, with a better layout and higher level of redundancy, should have a higher value of entropy. Also, by correctly setting certain link flows to zero, entropy maximization has the capability of identifying links which are either superfluous or are having an inappropriate direction of flow. This property is highly desirable in the context of layout optimization. Furthermore, in both Figures 5.13a and 5.14, the flow from node 3 to node 4 is greater than the direct supply from source node 1 to node 4. Similarly, the flow from node 3 to node 5 is greater than the direct supply from source node 2 to node 5 in both figures. This seems desirable from a resilience/flexibility standpoint if there is variation in the source supplies and/or the demands. Node 3 has a direct connection to both sources and both demand nodes. The flows in links 3-4 and 3-5 may vary considerably if the source supplies or the demands vary. Therefore, designing links 3-4 and 3-5 to have larger capacities would enhance the network's flexibility.

There are many computational advantages of using the algorithms of this subsection. These advantages can be summed up by saying that the single-source method is non-iterative and does not require optimization. Even the explicit solution of the system of equations for flow equilibrium at the nodes of the network is avoided. In other words, the algorithm ensures that continuity is satisfied without explicitly solving the continuity equations. The above advantages have associated benefits including minimal computer memory requirements. It is evident from the above properties that the proposed method is computationally very efficient. Other advantages of the

method are that it is simple both conceptually and in its application. This simplicity makes it easy to use the present approach for hand calculations on small networks. Also, computer implementation of the algorithms for weighting nodes and calculating link flows is straightforward.

5.5 SUMMARY AND CONCLUSION

In this chapter, it has been shown how the concept of entropy for finite probability schemes may be applied to general flow networks. Using the relative frequency interpretation of probabilities, various ratios of the flows in a general network have been cast in a probabilistic light. Probabilities obtained in the above way do not always constitute a finite scheme. Given properly defined probabilities, if the sum of these probabilities is greater than unity, then, the probabilities do not represent a finite scheme.

Two kinds of flow network have been defined including parallel and non-parallel networks. Parallel networks are characterised by their having source node to demand node connections only. Also, the sum of the link flows of a parallel network equals the total supply or demand. This property makes it possible to convert the link flows of a parallel network into a finite probability scheme by normalising the link flows on their sum. Well-established results of entropy maximization including the gravity model are therefore directly applicable to the resulting single-space probability scheme. On the other hand, more general networks require a conditional probability model because the flows in series-connected links are not mutually exclusive. Such a conditional probability model is based on multiple probability spaces which are obtained by normalising the outflows or inflows at each node on the total flow arriving at that node. The concept of entropy can be applied to the resulting multiple-space probabilities by means of the conditional entropy formula for compound probability schemes.

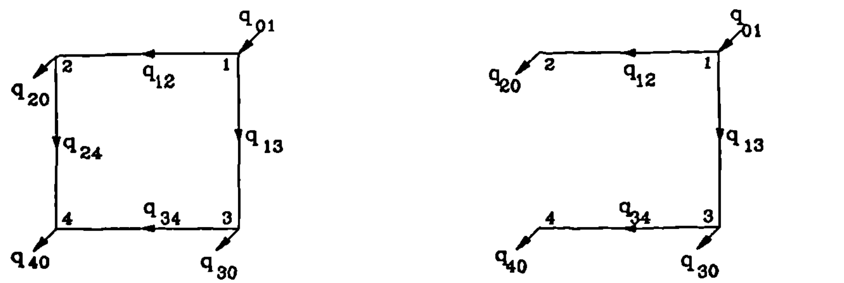
With network flows successfully modelled as probabilities and an equation derived for the entropy of network flows, the problem of inferring least biased estimates for the values of the link flows of a general network has been cast as an entropy maximization problem subject to flow equilibrium. It has been shown herein that this problem is a convex programming problem and, as such, has a single maximum point. Also, it has been shown how this convexity result, together with the linearity of the constraint set, can be exploited to simplify the numerical solution of the problem of inferring maximum entropy flows for any general network.

Furthermore, it has been shown in this chapter that, on a node-by-node basis, maximum entropy flows for single-source networks correspond to the uniform distribution U in the sense that each demand node receives equal proportions of the demand at that node from each of the paths serving the node. This property has been used herein to develop a simple but rigorous and efficient algorithm for calculating maximum entropy flows in single-source networks. Although the said algorithm is path based, a simple node weighting technique has been developed herein to circumvent explicit path enumeration. The above path-based approach has many advantages of which the main ones are that: it is a simple method which is suitable for use either by hand or on the computer; there is no need for linear or non-linear programming; the method is non-iterative; the system of equations for flow equilibrium are not solved explicitly.

A potential benefit of the present path-based method for calculating maximum entropy flows is that, in conjunction with the LP phase of the LPG design method, it opens up the possibility of using standard routines for linear programming as a quick and easy way of designing reliable single-source water distribution systems. The above conjecture is supported by some evidence from three of the examples of this chapter. In Example 9, it is stated that maximum entropy flows would appear to be maximally noncommittal with respect to the

possibility of any link flows being zero, these zero link flows being associated with less redundant layouts. In Example 10, it is suggested that designing the pipes of a water distribution network to carry maximum entropy flows would appear to make the network considerably invulnerable to possible variations in the flows of the network. In Example 11, several potential benefits of using entropy in a layout optimization framework are mentioned. It is also observed in that example that, with respect to a sample network, the links with larger maximum entropy flows turn out to be those links which are more likely to experience a considerable amount of variation in their flows. Therefore, designing these links to have larger capacities would appear to enhance the flexibility of the network.

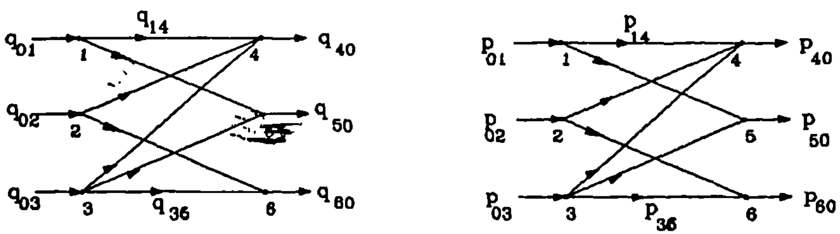
The observations of the previous paragraph provide further justification for the present attempt to use flow entropy as a surrogate for reliability. Further evidence that entropy can be used as a surrogate reliability measure is reviewed in Chapter 6. Entropy is then used in that chapter as a constraint in a cost-minimizing model for designing water distribution systems. Numerical examples are presented and discussed.



(a) A looped network: flow equilibrium alone does not permit the unique determination of the link flows.

(b) A tree-type network: link flows can be found by using flow equilibrium at each node.

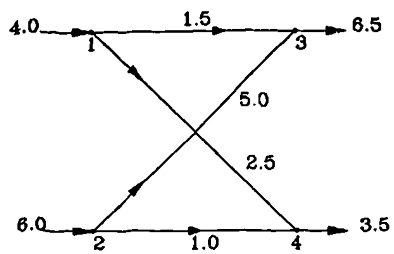
Figure 5.1 Looped and tree-type networks



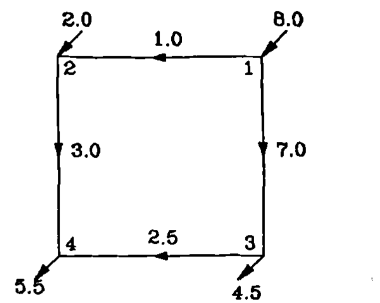
(a) flows

(b) probabilities

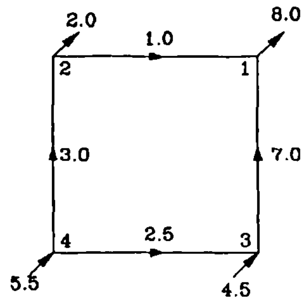
Figure 5.2 A parallel network: every link connects a source node to a demand node directly.



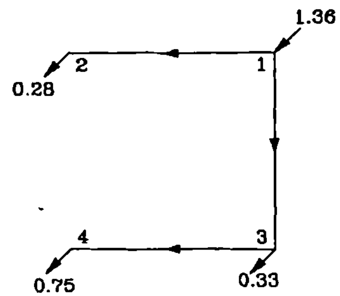
(a) Network of Examples 1, 3 and 4



(b) Network of Examples 2 and 5

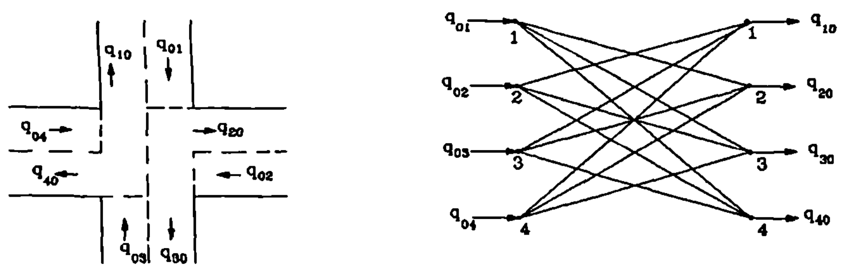


(c) Network of (b) with all flows reversed (Example 6)



(d) Network of Example 7

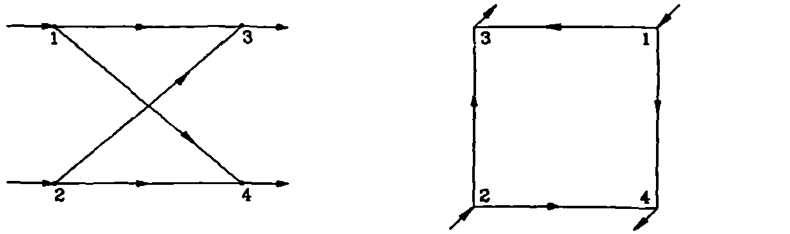
Figure 5.3 Sample networks with assumed flows



(a) A four-arm road junction

(b) Network representation of a four-arm road junction

Figure 5.4 Network representation of a road junction



(a) A parallel network

(b) Loop–emphasising representation of the network of (a)

Figure 5.5 Loop-emphasising representation of a parallel network

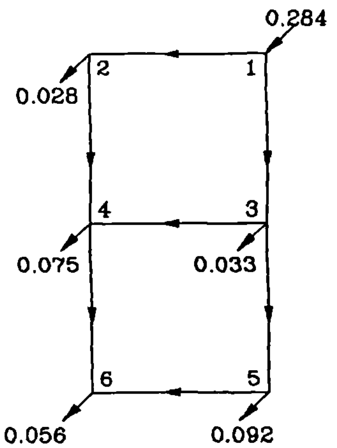


Figure 5.6a Network to demonstrate the constraints of Problem 9

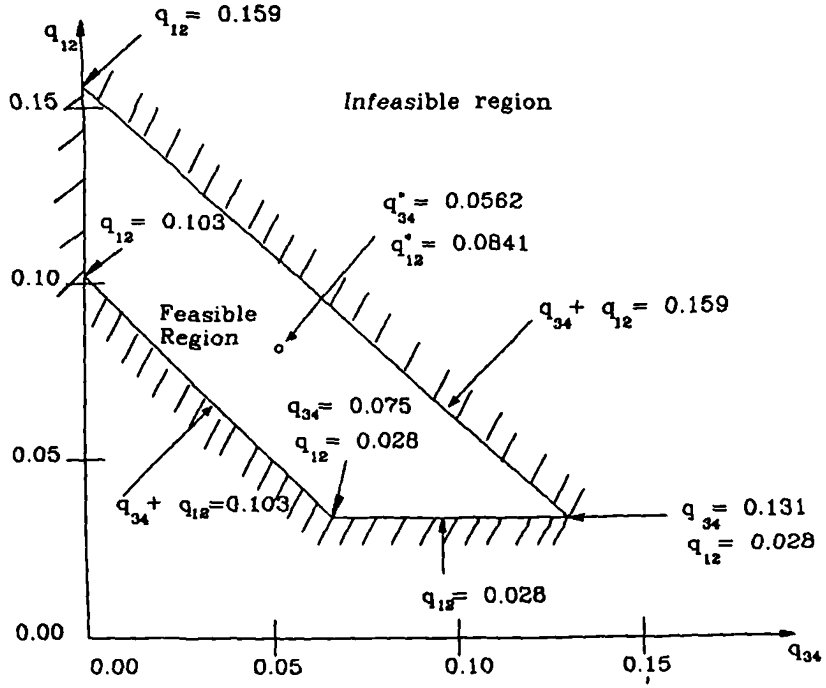
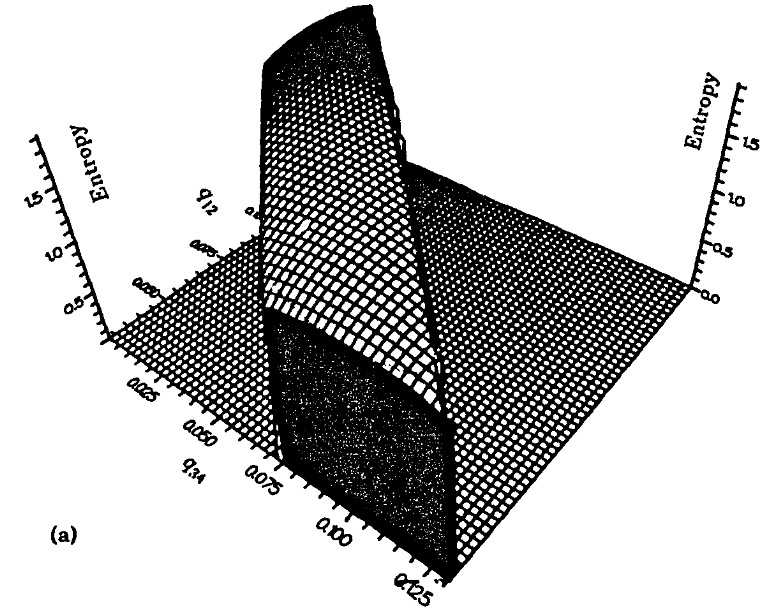
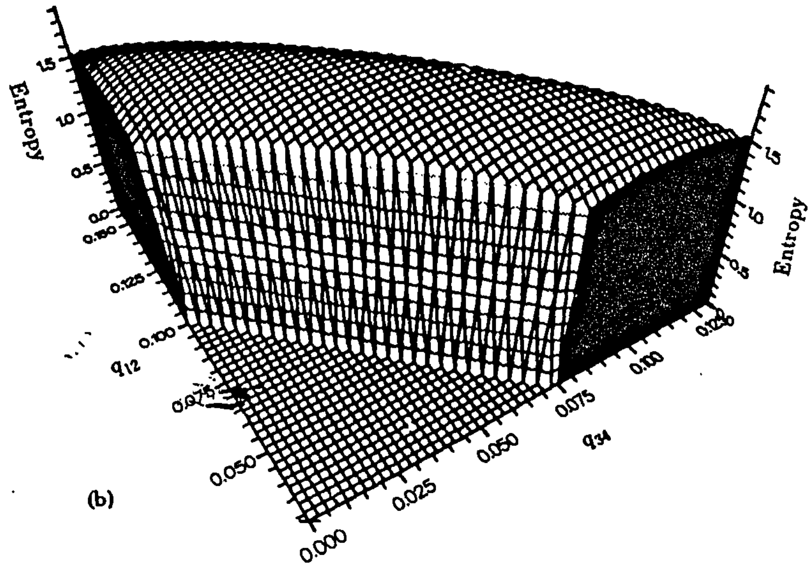


Figure 5.6b Graphical representation of the constraints and optimum point of Problem 9 for the network of Fig. 5.6a



(a)



(b)

Figure 5.6c Topographical representation of the entropy function for the network of Fig. 5.6a: (a) shows a view from the edge (face) given by $q_{12} = 0.028$; (b) shows a view from the edge (face) given by $q_{34} + q_{12} = 0.103$.

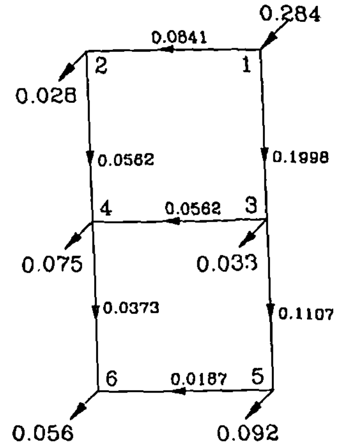


Figure 5.6d Maximum entropy flows for the network of Fig 5.6a

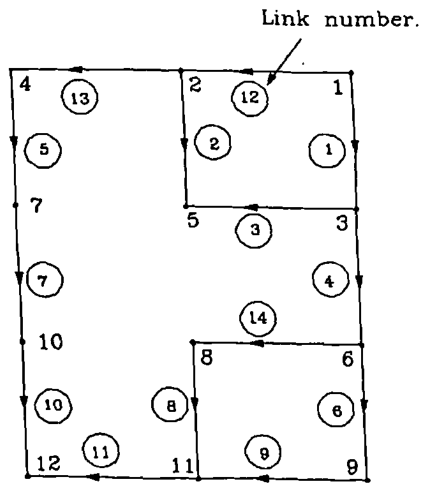


Figure 5.7 Network to demonstrate the selection of independent flows

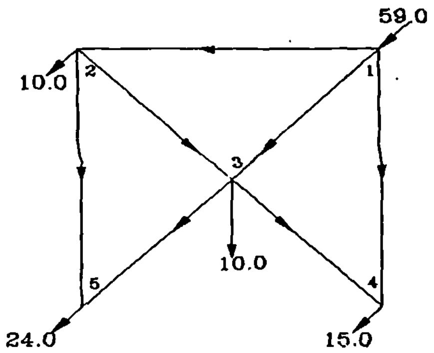


Figure 5.9 A single-source network used to demonstrate the calculation of equal path flows

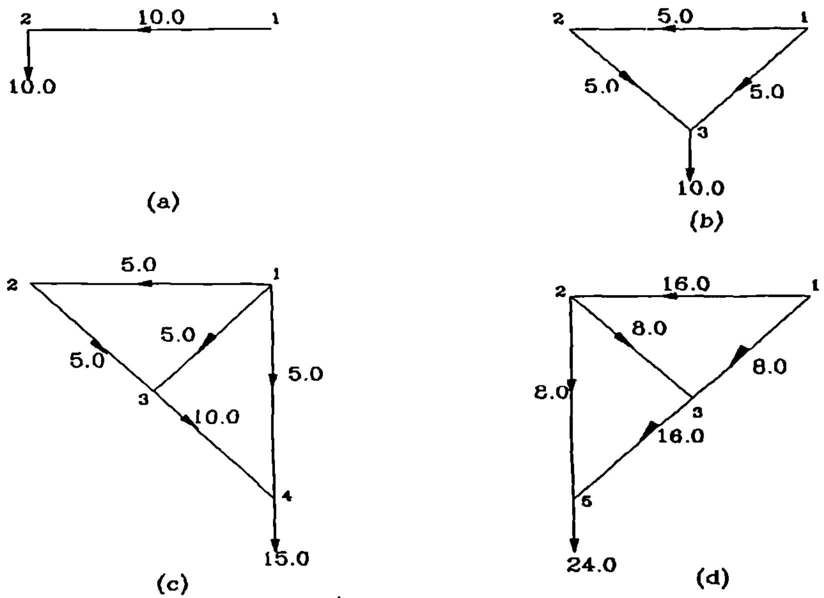


Figure 5.10 Equal path flows from the source (node 1) to each of the demand nodes 2, 3, 4 and 5

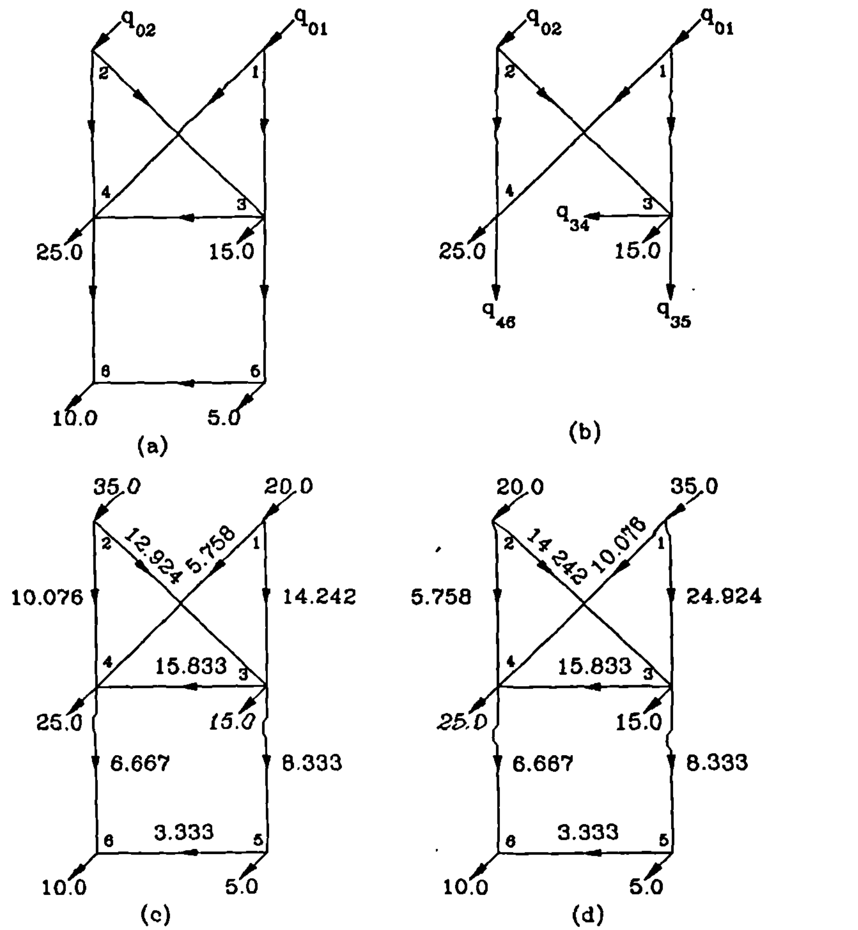


Figure 5.8 Networks and results for Example 10: the demands are shown in (a); two instances of the inflows at nodes 1 and 2 are shown in (c) and (d), along with the corresponding maximum entropy flows; in (d), the inflows of (c) at nodes 1 and 2 are interchanged; in (b), (a) is reduced to emphasise the parallel subnetwork consisting of the links shown in (b).

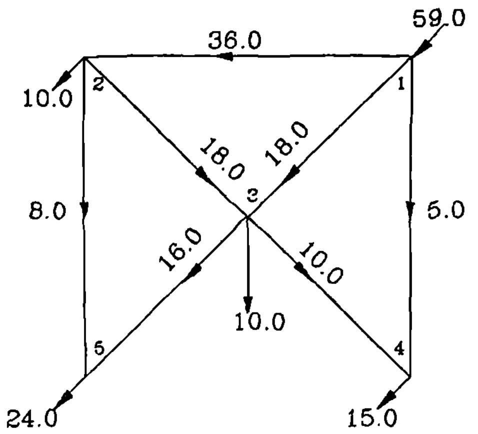


Figure 5.11 Maximum entropy flows for the network of Fig. 5.9: the flows are found by superposing the path flows of Fig. 5.10.

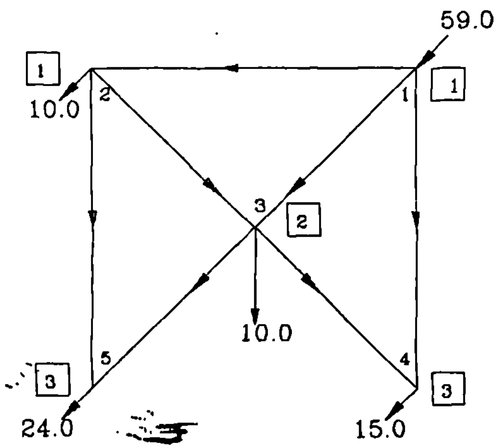


Figure 5.12 Number of paths to each node for the network of Fig. 5.9: the respective number of paths to the nodes are shown in the boxes next to the nodes.

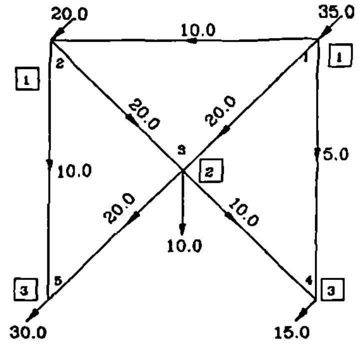


Figure 5.13a Maximum entropy flows for a two-source network which has equal path flows to each demand node

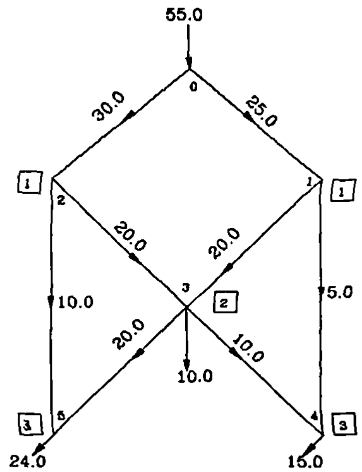


Figure 5.13b Supersource representation of the network and flows of Fig. 5.13a

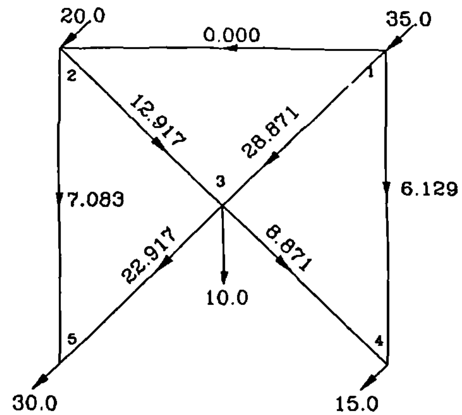


Figure 5.14 Maximum entropy flows for a two-source network with unequal path flows to each demand node

CHAPTER 6 ENTROPY-BASED APPROACH TO THE DESIGN OF WATER NETWORKS

6.1 INTRODUCTION

It is well known that the conventional manual method of designing water distribution networks leads inevitably to very expensive designs. In essence, the approach consists of using intuition and experience to specify a highly redundant layout with many loops and then conservatively designing the network so that the pipes have excess capacity. Together with the extensive looping, the excess capacity of the pipes is relied on in times of emergency including greatly increased demands to supply the necessary flow. An integral part of the above design process is to verify that the network is sufficiently resilient by simulating a few extreme load cases. By virtue of the existence of both spare capacity in the pipes and multiple supply paths to the nodes of the network, the resulting design is very reliable because it is extremely resilient and damage tolerant. Obviously, a design such as the one described above has a very high capital cost.

For more effective control of the cost of constructing water distribution systems, cost minimization methods are used. There is a large amount of material in the literature on the problem of minimizing the (capital) cost of water distribution systems, and the main approaches to the solution of this problem have been described in Chapter 2. It is a well-known fact that this problem of minimizing the capital cost of a water distribution system is extremely difficult to solve: the problem has many variables, many constraints, is highly non-linear, and is non-convex. Also, the size of the problem is nearly linearly proportional to the number of demand patterns as there is a set of pipe flow rate and head loss variables and, hence, a set of constraints for each demand pattern. In fact, Yates, Templeman and Boffey (1984) have shown that

the problem is NP-hard. Moreover, Templeman (1982b) has stated that research into the solution of this least cost design problem should be focused on the development of quick heuristic methods which can find an approximate optimum.

From a reliability perspective, there are several shortcomings of existing least cost design methods (Templeman, 1982b). For a single demand pattern, the cheapest design has a tree-like configuration. In such a design/layout, each node has only one supply path and, consequently, the system is not sufficiently damage tolerant as some nodes are isolated from the rest of the system if any link breaks. To prevent the optimization process from producing a tree-like design, a looped layout is prespecified and a minimum diameter constraint added to the constraint set. Using a minimum diameter constraint in this way does not really resolve the tree-type branchedness problem as the design which results from the optimization process is essentially branched, with the diameter of the loop-creating links set to the minimum allowable pipe size.

However, from a reliability standpoint, there is some doubt about the value of having minimum diameter loop-creating pipes in an essentially branched network. Not being explicitly designed to carry some predetermined flow, the assumed alternative paths which rely on the minimum diameter loop-completing links may not be usable if there is insufficient pressure in the system. Head loss is very sensitive to pipe diameter, and forcing a large amount of flow through a small diameter pipe may require a prohibitive amount of pressure. There is a demonstration of this phenomenon in the numerical examples provided in this chapter. Furthermore, as observed by Templeman (1982b), optimization tends to remove redundancy and so any spare capacity which is not required by the design demand pattern is removed. The removal of redundancy seriously curtails the ability of a network to cope with other demand patterns than that for which the network was specifically designed. One way of lessening this loss of resilience is to incorporate a multiplicity of demand patterns into the cost optimization process. However, this approach is not very satisfactory because, as explained earlier in this chapter, doubling the number of demand patterns, for example, almost doubles the size of the optimization problem and, as such, the problem becomes much more time-consuming to solve.

Nevertheless, in the absence of a satisfactory and quickly calculable quantitative measure of resilience or reliability, it seems inevitable that a multiplicity of demand patterns should be explicitly considered in the design process to ensure that the resulting design has the flexibility to cope with such eventualities as the failure of a link or fire fighting demands at each node of the network. However, it has been seen above that a serious impediment to the inclusion of multiple demand patterns in the optimization process is the size of the problem that would result. It is therefore obvious that it would be advantageous to reduce the size of the basic design problem which has been presented in Chapter 2 as Problem 1. For example, if the number of constraints can be reduced by 50%, then, two demand patterns may be considered explicitly without significantly increasing the difficulty of solving the problem. It is shown in this chapter how the number of constraints of Problem 1 can be reduced in a very straightforward and routine way. In fact, apart from the constitutive equations, very many of the constraints are almost always slack at the optimum.

Still on the question of making sure that a water network is sufficiently resilient/reliable, an alternative to the explicit consideration of multiple demand patterns is to explicitly consider reliability in the design problem. Various strategies that are based on this approach have been described in Chapter 3 in which it has been seen that the ensuing optimization problems are generally even more difficult to solve than the basic pipe diameter optimization problem. Any candidate accurate measure of reliability which, when used in the optimum design problem, can achieve the necessary

resilience needs to be based on the simulation/analysis of multiple flow patterns, and calculating such measures is very difficult and time consuming.

It is evident from the above discussion and Chapters 2 and 3 that the problem of designing a reliable water network as cheaply as possible is formidable. It is therefore obvious that any formalised cost-effective method of solving this problem should be acceptable. Thus, in the spirit of Templeman (1982b), perhaps the emphasis should move away from locating an optimum in the strict mathematical programming sense. It is therefore reasonable to place emphasis, instead, on the ability of a design procedure to quickly strike an acceptable compromise between reliability and cost. As demonstrated throughout the remainder of this chapter, the above philosophy is central to the present research.

This chapter describes a new approach to water network design and reliability. This approach uses pipe flow entropy as a surrogate measure of reliability. Flow entropy does not measure reliability directly, but, as demonstrated in this chapter, appears in general to have all the characteristics of an accurate quantitative and qualitative reliability measure. Also, flow entropy has the advantage of being easy to calculate for existing systems and easy to incorporate into cost optimization procedures.

In this chapter, firstly, the reasons previously mentioned herein for supposing that entropy maximization can be useful in a water network optimization framework are restated, followed by additional justification including some evidence from the literature. Secondly, it is shown that most of the constraints of Problem 1 which are additional to the constitutive equations can safely be omitted from the constraint set. Thirdly, an entropy-constrained mathematical programming problem for the least cost design of water distribution networks is presented. Fourthly, numerical examples are presented, the results of which are discussed. In particular, it is shown that designing the pipes of a water network to carry maximum entropy flows does confer a considerable amount of flexibility on the network and that flow entropy can be used to significantly reduce the tendency of cost minimization models to produce essentially branched designs. Lastly, conclusions which may have far-reaching implications for the optimum network design process are drawn.

6.2 BACKGROUND TO THE ENTROPY-BASED APPROACH

Apart from Awumah et al. (1990, 1991, 1992) and Jowitt and Xu (1993) which have been reviewed in Chapter 3, there is little published material on the potential uses of entropy in the context of water supply network design and reliability. However, there is some evidence in the literature that a degree of uniformity in the pipe diameters and flow rates is desirable. Unfortunately, there is no well-developed method of bringing forth such uniformity. Although the said evidence of the need for uniformity is not directly related to entropy, it has been seen in Chapter 4 that if uniformity is expressed in the form of a probability distribution, then it can be related to the maximum entropy formalism. It is therefore reasonable to suppose that the maximum entropy formalism can somehow be used to bring about uniformity in a water distribution system. In this section, previous research elsewhere in which it has been suggested that it is useful to design water networks such that the pipe diameters or flows are uniform is cited. The relevant points from some of the numerical examples of Chapter 5 are restated from a reliability viewpoint. The maximum entropy flow distribution is then characterised.

Evidence regarding the desirability of uniformity in pipe diameters and flows is presented next. To this end, an aspect of the traditional method of pipe network design is considered first, followed by a very brief literature review. Traditionally, velocities are usually confined between about $0.25m^3/s$ and about $5m^3/s$. This condition forces large pipes with small flows to be replaced by smaller pipes and small pipes with large flows by larger pipes. This process

results in a collection of pipes that are not very dissimilar in diameter. In a similar vein, Rowell and Barnes (1982) have stated that when designing a network, pipes with extremely high hydraulic gradients are inefficient and should be replaced by larger ones, while pipes with extremely low gradients should be replaced by smaller pipes. The outcome of the application of this philosophy is a network in which all pipe diameters are fairly similar in magnitude.

Also, Goulter and Coals (1986) have reported that minimizing the differences in the reliabilities of pipes connected to each node seemed an effective way of improving overall network reliability. It is quite common practice to base pipe failure rates on diameters (Su, Mays, Duan and Lansey, 1987) and so minimizing the differences in the reliabilities of pipes is similar in effect to minimizing the differences in diameters. In a looped distribution network with relatively few tree-type branches, there is a node at each end of most links $ij, ij \in IJ$, where IJ is the set of all links of the network. In other words, most nodes are the meeting points of at least two pipes. However, to simplify the present explanation, it is assumed that this is the case for all nodes. Therefore, any attempt to minimize the differences in the reliabilities or diameters of the pipes meeting at any node n, $\forall n$, has an effect on the pipe(s) at the other end of each link $jn, jn \in NU_n$, and each link $nk, nk \in ND_n$. NU_n and ND_n respectively represent the set of all the links carrying flow to and away from node $n, n = 1,...,NN$, where NN is the number of nodes in the network. Therefore, imposition of this requirement results in the diameter of each pipe in the network being as close as possible to that of all the other pipes meeting at each end of the pipe. This condition leads to uniformity of all diameters of the network.

Finally, Walters (1988) has suggested that reliability could be improved by ensuring an even division of flow between the pipes converging at each node. Also, Awumah, Goulter and Bhatt (1991) have stated that it is desirable to have links of equal capacities incident at each node. They argued that as the head loss in a pipe is roughly proportional to the square of its flow, a smaller pipe suffers a disproportionately high increase in head loss because of a small increase in its flow.

Taken to their logical conclusions, the above observations imply a need for uniformity in diameters and flows throughout a water distribution network. It can therefore be seen that through uniformity, entropy and reliability are somehow related. However, for a deeper insight into why entropy can be used to improve the reliability of a water network, it is necessary to directly relate reliability to the essence of entropy which is uncertainty, and this is easy to do. Water distribution network reliability centres around uncertainty. There is uncertainty about component failures, pipe capacities and/or sufficiency of pressure, flow rerouting, durations of failures and repairs, impact of inadequate supply on consumers, variations in demands and supplies, etc. There is even some uncertainty about the meaning of reliability itself in the context of water supply.

It seems, therefore, that entropy has a part to play in the reliability of water networks. As explained in Chapter 4, it appears on the basis of the maximum entropy formalism that it would be safe to size the pipes of a network to carry flows which are maximally noncommittal to factors which cannot easily be predicted, and, to the extent that is practicable, subject to whatever is known. Referring once more to Chapter 4, it will be recalled that the maximum entropy distribution is the only one which is such that any possible occurrence which is not excluded by the available information is ascribed a non-zero probability, however unlikely such an occurrence may appear. This suggests that a "maximum entropy design" of a network would possess an in-built flexibility to cope with flows which the network was not specifically designed to carry.

It has been shown in Chapter 5 how to calculate the entropy of the flows of a network and how to calculate maximum entropy flows for a looped network given the nodal inflows and outflows and the direction of flow in each link. Flow entropy is used in Section 6.4 as a surrogate for reliability. It is therefore useful to briefly characterise maximum entropy flows from a reliability/flexibility viewpoint. It has been stated in Chapter 5 that for a single-source network, the maximum entropy flow distribution corresponds to the uniform probability distribution U in the sense that, for each node, all supply paths to the node carry equal proportions of the demand at that node.

For a more general interpretation, reconsider the network of Example 9 (Figure 5.6a) which is shown here as Figure 6.1a. If the possibility of flow reversal is ignored, then, Figures 6.1b to 6.1i represent the possible layouts for supplying the demands of Figure 6.1a. Figure 5.6b, which is the graphical representation of the constraints and optimum point of the maximum entropy flows problem of the network of Figure 5.6a (Figure 6.1a), is shown here as Figure 6.1j. The four edges of the trapezium which represents the feasible region each correspond to one of the four layouts having six links, Figures 6.1b to 6.1e. For example, Figure 6.1b corresponds to the edge $q_{34} = 0$. Also, the four vertices each correspond to one of the layouts having five links, Figures 6.1f to 6.1i. As an example, Figure 6.1f corresponds to the vertex $(q_{34},\ q_{12}) = (0,\ 0.159)$.

The layouts of Figures 6.1a to 6.1i are now examined from a resilience/flexibility perspective. Each of the four tree-type layouts has only one flow distribution which satisfies all the demands of the network. On the other hand, each of the four single-loop layouts can meet the demands in very many ways. The two-loop layout has even more ways of supplying the demands. Also, it is easy to see that if appropriate link flows of the two-loop layout of Figure 6.1a are set to zero, then, this layout can take on the respective flow characteristics of all the other layouts, but the reverse is not true. As such, the two-loop layout is the most flexible. In a similar way, the single-loop layout of Figure 6.1b can take on the respective flow characteristics of the tree-type layouts of Figures 6.1f and 6.1h, but not vice versa. In this way, one can see that the tree-type layouts are the least flexible. It can therefore be expected that to minimize inflexibility, the network should be designed to carry flows which are as far away as possible from each of the distributions of the tree-type layouts. Recalling the comments of Example 9 relating to the position of the optimum point and, referring to Figure 6.1j, the point representing the maximum entropy flow distribution lies exactly midway between the edges $q_{34}=0$ and $q_{12}=0.028$. Putting it another way, the feasible region is symmetrical as shown in Figure 6.1j, and the point representing the maximum entropy flow distribution lies on the axis of symmetry. As such, the optimum point is not unduly close to any of the vertices, which have been shown to represent the most inflexible layouts. Therefore, lengthways (i.e. in the direction perpendicular to the axis of symmetry), the optimum point lies in the expected position from a flexibility viewpoint.

With flexibility still in mind, it remains to justify the location of the optimum point along the axis of symmetry. Obviously, $\hat{\underline{q}}^* = (q_{34},\ q_{12})^*$ is nearer the boundary $f = q_{34} + q_{12} = 0.159$ than the boundary $f = q_{34} + q_{12} = 0.103$. However, $f = 0.159$ corresponds to the layout of Figure 6.1c while $f = 0.103$ corresponds to Figure 6.1d. Each of these layouts has two links the removal of any one of which would result in node isolation. The links in question are 3-5 and 4-6 for Figure 6.1c, and 3-5 and 5-6 for Figure 6.1d. For Figure 6.1c the worst case failure is that of link 3-5 which would result in a node with a demand of 0.092 being isolated. For Figure 6.1d the worst case failure is also that of link 3-5, but the total demand at the isolated nodes is 0.148, a much higher value than 0.092 for Figure 6.1c. For this reason, the boundary $f = 0.159$ (Figure 6.1c) may be said to be safer than the opposite boundary $f = 0.103$ (Figure 6.1d). Therefore, the optimum point being closer to the boundary $f = 0.159$ is not counterintuitive; exactly how close to this boundary the optimum point should

be depends on the value of the nodal abstractions. Finally, it is obvious that, as desired from the point of view of flexibility, the maximum entropy flow distribution is based on the two-loop layout of Figure 6.1a; $\hat{\underline{q}}^*$ does not lie on any of the boundaries of the feasible region.

From the above discussion, it can be concluded that the maximum entropy flow distribution is the most central of all the distributions capable of satisfying the demands of the network. Although the above conclusion is not totally unexpected, it suggests that the magnitude of the possible changes to the values of the link flows of a network can be minimized by designing the network to carry maximum entropy flows. From the point of view of flexibility in water supply, as observed near the beginning of this section, the headloss in a pipe is approximately proportional to the square of the flow rate. Thus, for example, if the pipe flow rate is doubled, the headloss is almost quadrupled, resulting in an increase in headloss of almost 300%; if the flow rate is trebled, the corresponding increase in headloss is approximately 800%. It is therefore self-evident that if the increases in pipe flow rates can be minimized throughout the network, then, the additional headloss in the system due to any increases in pipe flow rates will also be minimized in consequence. Also, if the increase in headloss is minimized, the resilience of the network should be considerably enhanced as the network is then able to carry flows for which it was not specifically designed without suffering exceedingly high headlosses. This is the essence of the flow entropy-based approach to the design of reliable water networks, and this design method is described in Section 6.4.

6.3 REDUCING THE SIZE OF PROBLEM 1

Before adding an entropy constraint to Problem 1 in the next section, some rather obvious ways of reducing the size of this problem are mentioned here. It has been observed herein that Problem 1 has many variables and many constraints. For large networks the computational expense of solving this problem can be very large. Any simplification of the problem is therefore worthwhile. For convenience, Problem 1 is restated below. The symbols used in Problem 1 have previously been defined and those definitions are maintained here.

Problem 1

$$\underset{\forall D_{ij}}{Minimize} \quad C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e} \tag{6.1}$$

subject to:

$$h_{ij} = \alpha L_{ij} (q_{ij}/C_{ij})^{1.852} / D_{ij}^{4.87} \quad \forall ij \in IJ \tag{6.2}$$

$$\sum_{j \in NU_n} q_{jn} - \sum_{k \in ND_n} q_{kn} = q_n \quad n = 1,\dots,NN-1 \tag{6.3}$$

$$\sum_{ij \in IJ_l} h_{ij} = 0 \quad l = 1,\dots,NL \tag{6.4}$$

$$\sum_{ij \in IJ_p} h_{ij} = h_p \quad p = 1,\dots,NP \tag{6.5}$$

$$\frac{\pi v_{\min}}{4} \le \frac{q_{ij}}{D_{ij}^2} \le \frac{\pi v_{\max}}{4} \quad \forall ij \in IJ \tag{6.6}$$

$$H_s - H_{max,n} \le \sum_{ij \in IJ_n} h_{ij} \le H_s - H_{min,n} \quad \forall n \tag{6.7a}$$

$$D_{\min} \le D_{ij} \in D_D \le D_{\max} \quad \forall ij \in IJ \tag{6.8a}$$

$$q_{ij} \ge 0 \quad \forall ij \in IJ \tag{6.9}$$

This section is not primarily concerned with the constitutive equations, which are fundamental to the optimization problem. However, it is obvious that the number of pipe flow rate variables can be reduced considerably if the problem is solved in terms of selected independent flows. This approach also removes the flow equilibrium constraints from the actual optimization process. Details

of how independent flows may be selected and dependent flows calculated have been given in Chapter 5, Section 5.4. However, the decision to use the above simplification or not will generally depend on the solution strategy used to solve the optimization problem. In any case, it is useful to take into consideration the fact that it is more straightforward to obtain general expressions for the partial derivatives of the constraint functions if each link flow is a variable, i.e., if the dependent-and-independent flows approach is not used. It is also obvious that the pipe headloss equations can, for example, be reduced to the function

$$h_{ij} = f(L_{ij}, D_{ij}, q_{ij}, C_{ij}) = \alpha L_{ij} (q_{ij}/C_{ij})^{1.852} / D_{ij}^{4.87}$$

whose value is h_{ij} for specified values of L_{ij} D_{ij}, q_{ij}, C_{ij}. In this way, the h_{ij} variables and constraints can be eliminated.

In the following discussion, reference is made to the network of Figure 6.2 for illustrative purposes, but the material in this discussion is generally applicable. In addition to the constitutive equations, for the network of Figure 6.2, there are: 7 non-negativity constraints on the flows; 10 node pressure constraints, i.e., an upper and a lower limit for each of the 5 demand nodes; similarly, 14 flow velocity constraints; 14 diameter constraints. There are, therefore, 24 non-linear velocity and node pressure constraints in addition to the constitutive equations for the network of Figure 6.2. It is now shown that for this network, only the minimum diameter constraints and the minimum node pressure constraint for node 6, which is a terminal node, are strictly essential to the solution of the problem.

Firstly, the exponential increase in cost with diameter ensures that a large diameter pipe is not used if a smaller one will do. The D_{max} constraints may therefore be removed. However, it is vital to ascertain from the final results that no diameter exceeds D_{max}. Any such violated constraints should be reinstated and the program rerun. This process of reinstating violated constraints and rerunning the program should be repeated as necessary. The cost function also makes each v_{max} constraint more likely to be in the active set than the corresponding v_{min} constraint because of the preference for small diameter pipes. The v_{min} constraints may therefore be eliminated in favour of post-optimization verification as described above. Furthermore, it is probably easier to solve the problem without the upper bounds on the flow velocities, check the velocities at the solution and resolve the problem with any violated upper bounds reinstated.

Secondly, except perhaps for some critical nodes, for example near a reservoir, high residual pressures are not very likely to be a problem in a cost minimization formulation. As explained earlier, small diameter pipes are preferred, and any available pressure tends to be completely used. It follows that each $H_{min,n}$ rather than the corresponding $H_{max,n}$ is more likely to be critical. Therefore it seems that explicit consideration of the upper limits on the nodal pressures is not generally necessary. Of course, it must be confirmed at the solution that no node has a residual pressure that is too high.

Thirdly, closer examination of the $H_{min,n}$ constraints reveals that these constraints need not be specified for every node. For a fixed layout and set of flow directions, nodes furthest from the sources tend to have the lowest pressures. Thus, reconsidering Figure 6.2, the following equations can be written for IJ_4 and IJ_6 respectively.

$$\sum_{ij\in IJ_4} h_{ij} = h_{12} + h_{24} \le H_1 - H_{\min,4} \qquad (6.10)$$

$$\sum_{ij\in IJ_6} h_{ij} = h_{12} + h_{24} + h_{46} = \sum_{ij\in IJ_4} h_{ij} + h_{46} \le H_1 - H_{\min,6} \qquad (6.11)$$

Equations similar to Eqs. (6.10) and (6.11) can be written for all pairs of directly-connected nodes, but the following discussion which is based on nodes 4 and 6 is generalised shortly for the whole network. There are three cases to consider for the values of $H_1 - H_{min,4} = f_4$, say, and $H_1 - H_{min,6} = f_6$, say, respectively. Either $f_4 = f_6$, or $f_4 > f_6$, or $f_4 < f_6$. If $f_4 \ge f_6$, then, it follows from Eqs. (6.10) and (6.11) that the minimum head constraint will be binding for node 6 and slack for node 4. In such a case, there is no need to explicitly consider a minimum head constraint for node 4. However, if $f_4 < f_6$, then, the status of the minimum head constraint for node 4 at the solution of the optimization problem cannot be predicted in advance. In this case, it is necessary to include a minimum head constraint for node 4 in the problem. To generalise the above results, let the values of f_4 and f_6 be called the maximum allowable headlosses for nodes 4 and 6 respectively; where there are multiple sources, these terms still apply, but node 1 (or H_1) in the expressions for f_4 and f_6 respectively is replaced by whichever source nodes are specified for the respective paths selected for the nodes. It can therefore be said that there is no need to specify a minimum pressure constraint for a non-terminal node, if the maximum allowable headloss for that node is not less than the maximum allowable headloss for the terminal node downstream of the non-terminal node being considered. For this reason, it will usually be necessary to explicitly consider minimum head constraints for terminal nodes only. It is therefore safe to omit minimum head constraints for all non-terminal nodes and check at the solution that these omitted constraints are not violated. If any omitted constraint is violated, it should be considered explicitly.

One more observation concerns reintroduction of bounds initially assumed slack into the problem. As both the D_{min} and D_{max} constraints cannot simultaneously be binding for unequal values of D_{max} and D_{min}, each D_{max} constraint that is violated should replace the corresponding D_{min} constraint and vice versa. The above observations regarding the statuses of upper and lower bounds at the solution also apply to the velocity and pressure head constraints.

In the above manner, the size of Problem 1 and, in consequence, the computational effort needed for its solution can be reduced considerably. As an example, for the network of Figure 6.2, if the above simplifications are applied, only the 7 minimum diameter constraints and the minimum head constraint for node 6 are initially explicitly considered. However, post-solution verification that no omitted constraints have been violated is essential. Rather than a manual check, this verification process can be included in the optimization program. A (sub)routine which, when a solution is found, checks the constraints of Problem 1 to identify those that are violated is very easy to write and may be included in the optimization program.

6.4 ENTROPY-BASED OPTIMUM DESIGN OF RESILIENT WATER NETWORKS

It has been concluded in Section 6.2 that the maximum entropy flow distribution is the most central of all the flow distributions capable of satisfying the demands of a flow network. Also, it has been suggested that the magnitudes of the potential changes to the values of the link flows of a network can be minimized by designing the network to carry maximum entropy flows. As explained in Section 6.2, if the increases in pipe flow rates can be minimized throughout the network, then, the additional headloss due to any increases in pipe flow rates will also be minimized in consequence. If a water distribution network is designed in such a way that it can accommodate moderate changes to its flow distribution without the headloss increasing excessively, then, the network should be considerably resilient. In this section and the next, the effects of incorporating flow entropy into Problem 1 are studied from a flexibility standpoint. A mathematical

programming problem is formulated in this section followed by numerical examples and discussions in Section 6.5.

6.4.1 FORMULATING THE ENTROPY-BASED OPTIMUM DESIGN PROBLEM

The present research is exploratory in nature and this fact has largely dictated the way in which the entropy-based optimum design problem has been approached. Three possible formulations can be identified including a cost objective function with an entropy constraint, an entropy objective function with a cost constraint, and a multiple-objective approach in which both entropy and cost are included in the objective function. The entropy-constrained cost minimization approach is used herein because the cost minimization model and its shortcomings are well known. As such, it should be easy to appreciate the effects of adding an entropy constraint to Problem 1. Various ways of simplifying Problem 1 have been discussed in Section 6.3. The resulting problem, with an entropy constraint added, is stated below as Problem 10. In Problem 10, S_{max} is the the maximum value of the flow entropy of the network which is obtained by solving Problem 9; $\hat{S}$ is a specified minimum allowable value of entropy for the network; S^* is reserved for the actual value of the entropy constraint at the optimum.

Problem 10

$$\underset{\forall D_{ij}}{Minimize} \quad C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e} \tag{6.1}$$

subject to:

$$h_{ij} = \alpha L_{ij} (q_{ij} / C_{ij})^{1.852} / D_{ij}^{4.87} \quad \forall ij \in IJ \tag{6.2}$$

$$\sum_{j \in NU_n} q_{jn} - \sum_{k \in ND_n} q_{kn} = q_n \quad n = 1, \ldots, NN - 1 \tag{6.3}$$

$$\sum_{ij \in IJ_l} h_{ij} = 0 \quad l = 1,...,NL \tag{6.4}$$

$$\sum_{ij \in IJ_p} h_{ij} = h_p \quad p = 1,...,NP \tag{6.5}$$

$$\sum_{ij \in IJ_n} h_{ij} \le H_s - H_{min,n} \quad \forall n \in t \tag{6.7b}$$

$$D_{ij} \ge D_{\min} \quad \forall ij \in IJ \tag{6.8b}$$

$$S - S_0 = \sum_{n=1}^{NN} S_n = -\sum_{n=1}^{NN} \bar{p}_n \sum_{k \in ND_n} p_{nk} \ln p_{nk} \ge \hat{S} - S_0 ; \quad 0 \le \hat{S} \le S_{\max} \tag{6.12}$$

$$q_{ij} \ge 0 \quad \forall ij \in IJ \tag{6.9}$$

in which t is the set of all terminal nodes; all other symbols have previously been defined and those definitions are unchanged. The present research is exploratory and no attempt has been made to develop a special algorithm for Problem 10, which is non-linear in both the objective and constraint functions. Furthermore, Problem 10 may be non-convex and, as such, may have local minima which are not necessarily global. Any suitable algorithm for constrained non-linear programming may be used to find a local minimum of Problem 10. The numerical examples of Section 6.5 have been solved computationally using the NAG library routines E04VDF and E04UCF for non-linear constrained optimization.

6.5 MAJOR NUMERICAL EXAMPLES AND DISCUSSION

The main conclusions of the present research are based on the results of Examples 12 and 13 which follow shortly. In each example, Problem 10 is solved computationally for different values of the minimum allowable value of flow entropy $\hat{S}$ for a network with a predetermined fixed layout. The solution of Problem 10 for a specified value of $\hat{S}$ represents a possible design of the network being considered. Thus, for each sample network, different designs are generated in this way, from the minimum to the maximum entropy

value of the sample network being considered. It will be recalled from Chapter 5 that for a single source network, the minimum value of entropy is the entropy value for the case in which each demand node is supplied by one path only. The loop and minimum diameter constraints in Problem 10, however, force the entropy constraint to become slack at a value which is slightly higher than the minimum entropy value of the network being considered, and this happens if a sufficiently low value of $\hat{S}$ is used. In other words, the minimum diameter constraints prevent the optimization process from completely eliminating any pipes and, as such, these constraints ensure that all loops are retained. With the loops retained, various factors combine to make sure that there is flow in the loop-completing links. For example, the loop constraints have to be satisfied, and whenever a pipe is present, it is probably cheaper to use it to carry some flow than not to use it at all. Finally, with the loop-completing links carrying some flow, any of the spanning-tree configurations with which the minimum value of entropy for the network is associated cannot be attained. This explains why the entropy constraint is generally slack at a value slightly higher than the minimum value for the looped network being considered.

The aim of Examples 12 and 13 is to demonstrate that flow entropy is a good surrogate for reliability and that designing water distribution networks to carry maximum entropy flows provides a very convenient means of sizing the pipes of the network so that the distribution system is considerably resilient and cost effective. To this end, in Examples 12 and 13, it is shown by various measures notably headloss and energy that, generally, as the entropy of a network increases, the flexibility/reliability of the network increases.

It is convenient at this point to briefly explain the philosophy behind, and some of the practicalities of, the use of headloss and energy to assess the performance/flexibility of a water distribution network. The energy and headloss measures described below are general trend indicators only. A more *rigorous analysis* is presented in Chapter 7. The hydraulic performance of a looped network design can be *assessed* by demand-driven simulation, an approach which assumes that all network demands *are satisfied*. A possible implementation of such a demand-driven simulation consists of analysing the pipe network by the Hardy-Cross method with initial pipe flow rates for the Hardy-Cross iterations which satisfy both continuity and the network supplies and demands. In other words, the initial pipe flow rates must be such that in addition to nodal flow equilibrium being satisfied at each node, any external inflow or outflow at a node must equal the supply or demand at that node. The values of the external flows are unchanged at the solution of the Hardy-Cross algorithm and, as such, all the demands are satisfied no matter what the pressures throughout the network are.

A demand-driven simulation can therefore be used to consider the following two emergency situations: single link failures with the design or normal demands and "fire fighting" demands which, as explained shortly in Example 12, mimic fire fighting requirements at each node in turn. The procedure for calculating the headloss measure for a single-source network is described next. For each pipe failure (with normal demands) or each "fire fighting" load (on the full network), the notional head required at the source to satisfy all demands is found as follows. First, a demand-driven simulation of the (full or reduced, i.e., with one link closed off) network is carried out. Second, the head loss in any specified path from the source to the most pressure-critical node is found by summing the head losses in all links in the path. Third, the total head loss is added to the minimum desirable head at the most pressure-critical node. Strictly speaking, however, it is sufficient to calculate the notional usable head H required to satisfy all demands *as follows*.

$$H = \max < \sum_{ij \in IJ_n} h_{ij} \quad \forall n \in t > \tag{6.12}$$

in which t represents the set of all terminal nodes, i.e., nodes not having any pipe outflow. It may be noted that in general t changes according to changes in the demands and layout of the network. This rather devious notional headloss parameter allows quite rigorous comparisons. Other simple measures such as percentage *demand satisfied at adequate pressure do not account for* non-zero shortfalls in pressure or supply.

Turning to energy, which is also used in this chapter for assessing designs, Rowell and Barnes (1982) have suggested that, for a given flow, the efficiency of a pipe can be gauged from the rate at which the pipe dissipates energy. Therefore, it would appear that a network can be assessed on the basis of the total amount of energy that the network dissipates per unit time. The total energy E dissipated by a pipe network can be calculated as

$$E = \rho g \sum_{ij \in IJ_0} q_{ij} h_{ij} \tag{6.13}$$

where IJ_0 is the full or reduced network as appropriate, ρ and g are the density of water and acceleration due to gravity respectively.

6.5.1 EXAMPLE 12

The layout and demands of the sample network used in this example are taken from Alperovits and Shamir (1977). The reduced version shown in Figure 6.3 does not include the link between the reservoir and the first demand node. Also, the inflow at node 1 is the net flow. All node and pipe data are as given in Figure 6.3 and Table 6.1. The cost of the pipes in £ per metre is taken as $\gamma D_{ij}^{2.4}$, $\forall ij$, in which the diameter is in metres, $\gamma = 900$. The value of D_{min} is taken as 0.1m. Problem 10 has been formulated and solved computationally for the network of Figure 6.3 for various values of minimum allowable entropy $\hat{S}$. The values of entropy used have been selected such that the range of possible values for the sample network is well covered. The minimum and maximum entropy values for the demands and flow directions shown in Figure 6.3 are 1.515 and 1.915 respectively.

The NAG library routine E04VDF has been used for the numerical optimization. The routine E04VDF is an easy-to-use version of the NAG library routine E04VCF which is a comprehensive routine for constrained non-linear programming. The routine uses sequential quadratic programming and requires the partial derivatives of the objective and constraint functions. Also, by treating bounds, linear constraints and non-linear constraints separately, the routine works in the region within which the optimization problem is "meaningful"; an upper and a lower bound are specified. or stated as non-existent, for each variable and each linear or non-linear constraint. Beginning with the user-provided initial point, once a point that is feasible with respect to the bounds and linear constraints is found, the problem functions are thereafter evaluated only at points which are feasible with respect to the bounds and linear constraints. As such, the routine is suitable for optimization in an entropy framework because the entropy function is undefined for flows that are infeasible with respect to the (linear) flow equilibrium constraints. It will be recalled from Problems 7, 8 and 9 that, because of the entropy function, Problem 10 is undefined outside the feasible region of Problem 7.

As stated earlier, the network of Figure 6.3 has been designed using the formulation of Problem 10 for different values of $\hat{S}$ with the design load of Table 6.1. The design results are summarised in Tables 6.2 and 6.3 which show the optimum diameters and flows respectively. As all links are of equal length, some statistical measures of spread, namely, the sample standard deviation and the coefficient of variation, are also shown for the diameters. The mean diameter and the above measures of spread are used herein in a somewhat qualitative manner as general trend indicators.

The hydraulic performance of the entropy-constrained designs has been assessed by simulation as described earlier using a consumption-based approach that assumes total demand satisfaction. Two kinds of emergency have been considered. The results in Tables 6.4 and 6.5 are for single link failures with normal demands. Tables 6.6 and 6.7 are for the "fire fighting" loads in Table 6.1 which mimic fire fighting requirements at each node in turn. The only difference between the "fire fighting" demands and the design demands is that an arbitrary fire fighting demand of of $0.25m^3/s$ replaces the design demand of each node in turn. Obviously, the inflow at node 1 is adjusted so that the inflow equals the sum of the outflows. The fire fighting demand of $0.25m^3/s$ is approximately 88% of the total design demand of 0.284 m^3/s. As an example, Case 1 in Table 6.1 corresponds to the case of a fire at node 2, and the demand there is $0.25m^3/s$ instead of $0.028m^3/s$ which is the normal demand at node 2. Also, additional information regarding the position of critical nodes and links is presented in Tables 6.8 to 6.11. Details on these tables are given the first time that the tables are referred to in the following discussion.

6.5.1.1 DISCUSSION

Much of interest can be said about the results of Tables 6.2 to 6.11, and these results are considered below under various headings.

Cost and pipe-diameter properties of the *entropy-constrained* designs

The diameters and flows of Tables 6.2 and 6.3 respectively clearly show the network becoming less and less implicitly branched as the entropy increases; a pictorial representation of the pipe diameters is provided in Figure 6.4. The above observation is supported by the sample standard deviations and coefficients of variation of the diameters. Furthermore, the averages of the diameters increase with entropy. To see how reliability and pipe diameter are related, some findings of a case study are quoted. Clark, Stafford and Goodrich

(1982) carried out a study involving over 6,854km of mains. The peak demand of the distribution network in question was over $11.39m^3/s$. It was concluded that:

1. Large diameter pipes tend to have a longer period before the first maintenance event than do smaller diameter pipes.
2. ... once a length of pipe begins to require maintenance, the maintenance rate increases exponentially.

It may therefore be said that large diameter pipes are more reliable than smaller diameter pipes and so the (mechanical) reliability of a network would be expected to increase if the diameters of the smaller pipes and the "mean diameter" increase. It follows from the results of Table 6.2 that (mechanical) reliability can be expected to increase as entropy increases.

Also, the efficiency of the entropy-constrained designs in terms of cost is extremely high. The increase in cost from a design with a slack entropy constraint of 1.578 to the design corresponding to the maximum entropy value of 1.915 is only 4.8% or, approximately, 5%. It becomes clear in a moment how much more resilient the design having an entropy value of 1.915 is than the design with an entropy value of 1.578. There are two complementary reasons for this low percentage increase in cost. The first is that the diameters of large pipes are *reduced while simultaneously increasing the diameters of small* pipes. The cost of making small pipes larger is thus, more or less, offset. The second factor is the exponential nature of the cost function. It ensures that, if all lengths are equal and the average diameter is constant, the most uniform set of diameters is the cheapest. This is easily demonstrated numerically for the case of two pipes.

The above observation would seem to contradict the experience gained from conventional cost minimization models. However, this apparent anomaly can be explained. Cost minimization proceeds with an overall reduction in the diameters, up to a point where further reductions would lead to constraint violation. Cost savings are therefore possible because smaller pipes are

considerably cheaper than large ones. Put in a slightly different way, it is cheaper to increase the diameter of a small pipe than it is to increase the diameter of a larger pipe by the same amount. In the present model, the entropy constraint indirectly limits the overall reduction in diameters. Once this limit is reached, further cost savings can only come by exploiting the fact that for a fixed number of pipes and a fixed "average diameter", the cheapest option is to avoid large pipes as far as possible. In consequence, the most uniform diameters are preferred.

In the present example, the 5% increase in cost is a small price to pay for the overall increase in mechanical reliability or diameters. This leads to the conjecture that even for more complex networks, any increase in cost is not very likely to be very much higher than this figure of 5%. This speculation is based on the fact that there will be many more non-minimum diameter pipes than minimum diameter pipes since there is only one loop-completing link per loop. Therefore, by reducing the diameters of the largest pipes slightly, it will be possible to considerably increase the diameters of the smallest pipes without substantially increasing the *total* cost of the pipe network. It would seem, therefore, that the opportunity to increase the diameters of minimum diameter pipes in this way will increase as the number of loops increases.

Network performance following the removal of one link

The link failure results in Table 6.4 show several noteworthy points. First, the maximum notional usable source pressure head required to satisfy all network demands decreases very rapidly as entropy increases. Of more importance is the accompanying reduction in the average value. It may therefore be inferred that problem areas are not being improved at the expense of other areas. In particular, the high rate of reduction underlines quite dramatic system-wide gains in performance. However, the highest value of usable source head for the maximum entropy design is slightly higher than the design having an entropy value of 1.900. On the other hand, this increase in the value of the maximum head for $\hat{S} = 1.915$ is outweighed by the large concurrent improvement in the next worst case, as demonstrated by the continued reduction in the average head. It may also be noted that, from $\hat{S} = 1.578$ to $\hat{S} = 1.915$, there is slight oscillation in general in the values in each row, but the amplitudes of these oscillations decrease as the entropy increases.

The improvement in hydraulic performance observed in Table 6.4 can be explained in terms of uniformity in diameters and flows respectively. For a pipe with given fixed values of length, friction coefficient and flow, the head loss is nearly inversely proportional to the fifth power of the diameter. The consequence of this relationship is that, for example, doubling the diameter of a pipe while keeping the flow in the pipe constant reduces the head loss in that pipe by a factor of approximately 2^5 or 32; trebling the diameter reduces the head loss in the pipe by a factor of approximately 3^5 or 243. The above characterisation of the relationship between headloss and pipe diameter is, however, a slight oversimplification because pipe flow rate and diameter generally tend to increase or decrease hand in hand. Nevertheless, it can be inferred that an implicit tree-type network is likely to experience much higher increases in head loss if a link fails than a network having pipes with (similar) average-size diameters. Additionally, the head loss in a pipe is nearly proportional to the square of the flow rate. By virtue of this quadratic head loss-flow rate relationship, a network in which the pipes are designed to carry uniform flows is less likely to suffer very high increases in head loss than a network in which the pipes are designed to carry flows that are very dissimilar in magnitude. The above statement can be justified on the grounds that the diameter of a pipe depends on the flow rate in that pipe; a pipe carrying a small flow generally has a small diameter whereas a pipe carrying a larger flow generally has a larger diameter. The above factors i.e. uniformity of diameters and flows, including the observation that pipes with the most uniform diameters are the cheapest if the lengths are equal, explain why it is

possible to considerably improve the implicitly branched network at a cost increase of only about 5%.

The values in Table 6.5 show a similar behaviour to those in Table 6.4 in that both the mean and maximum energy dissipated decrease as entropy increases. However, in assessing designs on the basis of energy dissipation it is useful to realise that the same flow will dissipate less energy in a large, more costly pipe than in a smaller one. Therefore, a design with a low rate of energy dissipation may not necessarily be cost effective. Nevertheless, for each design, a deeper interpretation is possible by comparing the reduced networks to the full network. For $\hat{S} = 1.6$, for example, failure of pipe 4-6 hardly increases the amount of energy dissipated. This suggests that arc 4-6 is being underused in this implicitly branched network design. A more important comparison is between the average rate of energy dissipation for the reduced networks and the rate for their corresponding full network. The difference between these values is a rough measure of how much, on average, the behaviours of the reduced networks deviate from the behaviour of the full network and, as such, this measure can be used to gauge the flexibility of a network. The values in the last row of Table 6.5 fall very rapidly as the entropy increases, suggesting that a design with a high value of entropy is much more resilient than a counterpart having a lower value of entropy.

Also, it can be inferred from Table 6.5 that up to an entropy value of approximately 1.7, links 2-4 and 4-6 would apparently be really useful only if a major link failed. However, the pressure requirements may be prohibitive, as demonstrated by Table 6.4. This suggests that the value of having minimum diameter loop-forming pipes in an essentially branched network is imaginary rather than real. Furthermore, from the point of view of stagnation, it is questionable how much these minimum diameter pipes contribute to circulation in the present example in view of the magnitudes of their flows.

Network performance under fire fighting loads

The fire fighting results in Tables 6.6 and 6.7 clearly show resilience/flexibility increasing with entropy. Despite a slight oscillation in the values for each node, the averages exhibit a steady downward trend as entropy increases. The rate of improvement is, however, somewhat lower than is the case for link failures. It can therefore be concluded that all the results of Tables 6.2 to 6.7 indicate that, for the sample network, flow entropy is a good measure of flexibility.

Flow entropy as a control parameter

It will be recalled from Section 6.2 that the fact that flows in pipe networks are rerouted in complex ways which are difficult to predict is one of the factors which make it difficult to calculate the reliability of water networks. However, a close examination of the simulated results for the entropy-constrained designs suggests that if $\hat{S}$ is sufficiently large, the behaviour of the network with regard to flow rerouting due to increased network flows or a single link failure may be less unpredictable. In other words, entropy can be used to control the flow rerouting characteristics of a network. The above proposition is discussed next with the help of Tables 6.8 to 6.11 and Figures 6.5 and 6.6.

Tables 6.8 and 6.9 are obtained as follows. For each design, the critical link is the link which, when removed, results in the reduced network which dissipates the highest amount of energy or has the highest value of the usable source head needed to satisfy all demands. The critical node is the terminal node having the highest value of $\sum_{ij \in IJ_n} h_{ij}$ following the failure of the critical link. Thus, for Table 6.8 for example, the critical link for $\hat{S} = 1.578$, for example, is found from Table 6.4 to be 1-3, this being the link the reduced network for which has the highest value of the net source head needed to satisfy all the demands. Having found the critical link or worst case failure according to the headloss or usable source head criterion for this design of

$\hat{S} = 1.578$, the critical node is then obtained from the simulated results of the above worst case failure. The data of Table 6.8 are obtained in this way by repeating the above procedure for the other designs. Table 6.9 is obtained in a similar way, the only difference being that the critical link is determined from the energy dissipation values of Table 6.5. Tables 6.8 and 6.9 happen to show identical results, but this may not generally hold true. How Tables 6.10 and 6.11 are obtained is explained shortly.

For the design demands, Figure 6.5a shows the worst case link failure and the flow directions for the reduced network as determined by simulation for the $\hat{S} = 1.578$, $\hat{S} = 1.600$ and $\hat{S} = 1.800$ designs. Figure 6.5b shows the above details for $\hat{S} = 1.900$ and $\hat{S} = 1.915$. Figure 6.5c is for $\hat{S} = 1.700$. The notion that entropy can be used to control the flow rerouting properties of a network is based on the flow directions of Figure 6.5. Looking at each diagram in turn, the flows are routed in Figure 6.5a in such a way that node 6 is the critical node, even though the demand of $0.092 m^3/s$ at node 5 is over 64% greater than the demand of $0.056 m^3/s$ at node 6. This rather unexpected way in which the flows are routed in Figure 6.5a is caused by the pipe connecting node 4 to node 6 having too small a diameter. This casts further doubt on the use of minimum diameter loop-completing pipes as a means of providing alternative supply routes to demand nodes. The flow directions of Figure 6.5b, however, accord with intuition as node 5, in this figure, is farther from the source than node 6 in terms of reachability. The fact that the flows of Figure 6.5b are routed in a more obvious way than those of Figure 6.5a is because the pipe diameters of the designs with an entropy value of 1.900 and 1.915 respectively are more uniform. Some implications of the ability to predict flow directions for reduced networks are stated shortly. (In Figure 6.5c, the critical link is 3-5 rather than 1-3 because, for the design with an entropy value of 1.700, $D_{46} = 100mm$ while $D_{24} = 165mm$. Therefore, when link 1-3 fails, link 2-4 can cope better with the resulting increase in its flow than link 4-6 can when link 3-5 fails. Further discussion of this figure is deferred. Results from Example 13 will help to bring out the significance of the figure.)

Tables 6.10 and 6.11 are based on the simulated performance of the entropy-constrained designs for the fire fighting flows. Both tables show identical results, and the critical node in these tables is the node at which a fire fighting demand causes the worst network performance as measured by headloss and energy dissipation respectively. Figure 6.6a shows the flow directions for the worst case fire fighting load for the designs with an entropy value of 1.578, 1.600 and 1.700 respectively, as determined by simulation. Figure 6.6b shows the same information for the designs with entropy values of 1.800, 1.900 and 1.915 respectively. The flow directions of Figure 6.6b are the same as the design flow directions and are, intuitively, more natural than those of Figure 6.6a; one would instinctively expect that as node 6 is the farthest node from the source, node 6 should be the most pressure-critical node. It may also be noted that in Figure 6.6a, the path 1-3-5-6-4-2 covers a distance of 5000m compared to a maximum distance of 3000m in Figure 6.6b. The flows of Figure 6.6b are routed in a more obvious way because the designs with an entropy value of 1.800, 1.900 and 1.915 respectively have more uniform pipe diameters.

On the basis of the above discussion, it may be conjectured that entropy can be used to reduce the amount of unpredictability in the way in which flows are rerouted due to increased network flows or the removal of a link. The ability to predict how flows will be rerouted in a network is most useful from both reliability analysis and design perspectives. As an example, if one could be certain that the critical node for fire fighting flows for the network of Figure 6.3 will be node 6, then, instead of explicitly considering fire fighting flows at each node in the (optimum) design problem, it is probably sufficient to explicitly consider the fire fighting demand of node 6 only. More generally, it appears that entropy can be used to force the critical nodes for fire fighting

flows to be at the design terminal nodes of a network and, in consequence, to reduce the need to explicitly consider fire fighting flows at all nodes of a network at the network design stage. It should be noted, however, that the above appraisal is based on equal fire fighting demands. More work is necessary to establish what happens when the fire fighting demands differ considerably from node to node.

6.5.2 EXAMPLE 13

In this example, the design and analysis of Example 12 are repeated, but on the slightly larger sample network depicted in Figure 6.7. Apart from properties such as the distribution of demands, for example, that are specific to the network of Figure 6.7, most of the details of this example are the same as those of Example 12; any differences between Examples 12 and 13 are stated here. The main aim of this example is to provide evidence that the conclusions of Example 12 appear to hold more generally. The layout and demands of the network of Figure 6.7 are taken from Fujiwara and de Silva (1990). The nodes are numbered as in Fujiwara and de Silva. The pipe and node data are given in Figure 6.7 and Table 6.12. As in Example 12, the cost of the pipes in £ per metre is taken as $\gamma D_{ij}^{2.4}$, $\forall ij \in IJ$, with $\gamma = 900$ and the diameter in metres. Also, it may be noted that the fire fighting nodal demand of $0.25 m^3/s$ is 120% of the total design demand of $0.2081 m^3/s$.

Problem 10 has been formulated and solved computationally for the network of Figure 6.7 using the NAG library routine E04UCF. The main difference between E04UCF and E04VCF, which has been described briefly in Example 12, is that the former computes those gradients which are not supplied by the user using finite difference approximations. In the present example, only the gradients of the entropy constraint with respect to the pipe flow rates have been computed in this way. All other gradients have been calculated from analytical expressions. The minimum and maximum entropy values for the network of Figure 6.7 are 1.973 and 2.800 respectively. The values of the minimum allowable entropy value $\hat{S}$ used for the entropy-constrained designs have been chosen so that there is a value at either end of the range, one near the middle, and two values near the upper limit. The three highest values are quite close in magnitude and it is instructive to compare the performance of the corresponding designs. Results of the numerical optimization and simulations are summarised in Tables 6.13 to 6.17 which are discussed next.

6.5.2.1 DISCUSSION

The results of this example are broadly similar to those of Example 12. This discussion is therefore brief, although details are given where necessary. The ability of flow entropy to control the position of the critical link is also discussed from a reliability-based design perspective. Tables 6.13 to 6.17 are now considered. Starting with Table 6.13, the design having an entropy value of 2.170 is implicitly branched, while there is a trend of increasing uniformity as the entropy value increases. Also, the mean diameters show that there is a trend of increasing mechanical reliability as entropy increases, and the maximum entropy design is 5.4% more expensive than the essentially branched design having an entropy value of 2.170.

A closer look at the last four rows of Table 6.13 shows the results for $\hat{S} = 2.750$ to be somewhat out of line. Also, quoting costs in £ 10^6 to six decimal places (all the variables, constants, functions, etc. of the computer program are in double precision), the costs for $\hat{S} = 2.750$ and $\hat{S} = 2.800$ are 0.292272 and 0.291767 respectively. It would appear, therefore, from a cost minimization viewpoint, that the results for $\hat{S} = 2.750$ correspond to a poor local optimum. It is important, however, to bear in mind the dual nature of the present demonstration: to show that entropy-constrained cost minimization is a possible practical design approach and that reliability generally increases as entropy increases. In other words, when interpreting the simulation results, it is useful to consider whether the values corresponding to the design with

$\hat{S} = 2.750$, which has the most uniform diameters, weaken the premise that flow entropy is a possible reliability measure. On the other hand, it must be stressed that, being, it would appear, a poor local minimum, the redundancy of the $\hat{S} = 2.750$ design has not been removed completely. As such, this design should prove to be quite resilient.

The maximum and, in particular, the mean headloss values in Table 6.14 show that the resilience of the designs generally increases as the entropy increases. It can also be seen in the same table that, for entropy values higher than approximately 2.5, the critical node is node 9, which is the design terminal node of the sample network; for lower values of entropy, the critical node can be any node including node 9. This is further evidence that entropy can be used to force a network to behave in a more predictable way. The usefulness of knowing the positions of the critical nodes in a network prior to a complete design and analysis of the network has been stated in Example 12. Furthermore, it can be seen in Table 6.14 that the position of the critical link is more predictable if the entropy is higher than approximately 2.750; links 1-2 and 1-4 are both connected to the source and, as such, one would intuitively expect that the critical link should be one of these links. Table 6.14 also shows the next critical link for each design. From the positions of the critical and next critical links, it can be seen that the critical link for lower values of entropy than approximately 2.750 can be anywhere in the network. This provides further evidence that entropy can be used as a control parameter in a network design framework. If one could be certain about which links in a network are critical, then, if the need to explicitly design the network to cope with a single link failure arises, it would probably be sufficient to explicitly consider the failure of the critical links only. For the present network, it would probably be necessary to explicitly consider the failure of links 1-2 and 1-4 only; without, for example, examining the spatial distribution of the demands, it is probably impossible to know which link, 1-2 or 1-4, will be critical prior to a complete design.

Table 6.15 is generally as expected and does not contradict Table 6.14 in any way. It may also be noted that Table 6.15 shows link 4-5 as the next critical link for an entropy value of 2.170, unlike Table 6.14 in which the next critical link is link 8-9. However, the above difference merely reinforces the notion that the critical link can be anywhere in the network if the entropy value is not sufficiently high. Furthermore, compared to Table 6.14, the positions of the critical and next critical links are swapped in Table 6.15 for values of entropy greater than approximately 2.750. This reversal of the statuses of links 1-2 and 1-4 is, however, not counterintuitive since, as observed above, either link can be the critical link. This is particularly true of this network because of the symmetry in the layout and demands.

Tables 6.16 and 6.17 also show the trend of increasing resilience as entropy increases. However, it is worth noting that the design having an entropy value of 2.750 seems slightly more resilient with respect to fire fighting than the maximum entropy design. Also, only the design with an entropy value of 2.750 has its critical node at node 9. However, referring again to Figure 6.6, it can be seen that nodes 3 and 7 are the most downstream nodes in a series of nodes which have only one supply path: nodes 1, 2 and 3; nodes 1, 4 and 7. These nodes tend to be few in a looped network. If, in designing a network, fire fighting flows are explicitly considered at terminal nodes only as suggested in Example 12, then simulations should be performed to verify that fire fighting demands can be satisfied at nodes such as nodes 3 and 7. As an alternative to post-optimization verification that the network can satisfy fire fighting demands at these critical single-path nodes, these demands could also be explicitly considered in the design problem in view of the fact that the critical single-path nodes are generally few in a looped network. In spite of the fact that the design with an entropy value of 2.750 is more resilient with respect to fire fighting demands, if all the results in Tables 6.14 to 6.17 are taken into consideration, it can be concluded that the resilience of the designs of the sample network generally increases as entropy increases.

At first sight, the critical nodes in Tables 6.16 and 6.17 appear not to be in line with the notion that entropy can control the behaviour of a water network. However, the sample network is symmetrical and so one would expect nodes 3 and 7 to have the same status. This expectation would naturally be reflected in any predictions about the behaviour of the network. Looking again at Tables 6.16 and 6.17, the fact that for an entropy value of 2.775 and 2.800 *both* nodes 3 and 7 are critical shows that, due to their having a sufficiently high value of entropy, these designs are behaving in a more regular and more predictable way. In contrast, for an entropy value of 2.170 and 2.500, only one of these two nodes is critical in each case. Regarding $\hat{S} = 2.750$, the critical node being node 9 is reasonable and, if $\hat{S} = 2.750$, $\hat{S} = 2.775$ and $\hat{S} = 2.800$ are taken together, then, it can be maintained that entropy increases regularity in the behaviour of water networks.

Very briefly going back to Example 12, Figure 6.5 has been examined regarding the positions of the critical nodes. In the light of what has been stated in Example 13 regarding the positions of critical links, Figure 6.5 is now reconsidered. Once again, it is seen that if the entropy value is high enough, the critical link is more likely to be where it is intuitively expected; Figure 6.5b. On the other hand, for lower values of entropy, the critical link can be anywhere; Figures 6.5a and 6.5c.

In summary, through Examples 12 and 13, it has been shown that generally, the higher the entropy of a network, the more resilient the network is and the more regular is the behaviour of the network. Therefore, in view of the fact that there is good correlation between entropy and resilience, it can be concluded that entropy is a good surrogate for reliability. It can also be concluded that entropy-constrained cost minimization is potentially a good practical design method.

6.5.3 GENERAL DISCUSSION

Taken together, the discussions of Examples 12 and 13 are fairly comprehensive. However, a few additional general remarks relating to the entropy-constrained approach to the optimum design of water distribution networks are made here. Firstly, it is important to recall that the mapping between entropy and network flow distribution is not one to one. This absence of one-to-one correspondence is due to the fact that many, perhaps very different flow distributions can have the same value of entropy as demonstrated by Figure 5.6c in Chapter 5. Fortunately, the maximum entropy flows are unique for any network, since the solution of Problem 7 is unique. However, the uniqueness or otherwise of the corresponding maximum entropy design remains to be established. This aspect needs further research.

Secondly, since completely different less-than-maximum entropy designs can have the same value of entropy, there is some uncertainty about the performance/resilience of networks which are not designed to carry maximum entropy flows. Also, as networks designed to carry maximum entropy flows have been shown to be resilient, the following conclusion can be drawn. To design a water distribution network so that the network has all-round resilience, the network should be designed to carry maximum entropy flows. The above conclusion provides the answer to the question raised by Awumah et al. (1990, 1991, 1992) as to what value of the minimum allowable value of entropy $\hat{S}$ should be used for a given network. Furthermore, it has been seen in Examples 12 and 13 that the maximum entropy design may be of the order of 5% more costly than a corresponding non-entropy constrained essentially branched design with loop-completing minimum diameter pipes. As such, the maximum entropy design is cost effective. Thus, if one wishes to design a network to specific reliability targets, then, the necessary explicit reliability constraints should be added to Problem 1 and the resulting problem solved computationally using a suitable non-linear programming algorithm. The

difficulties involved if this approach is adopted have been highlighted herein. However, if a network having good all-round resilience is acceptable, then, as suggested earlier, the maximum entropy design should be used.

Finally, it can be concluded that the *maximum* entropy design approach has the following two very important implications. First, because maximum entropy flows are unique and can be found separately by solving Problem 9, the pipe flow rates are no longer variables in Problem 10. As such, the flow equilibrium and entropy constraints in Problem 10 become redundant and should be omitted. The computational solution of the resulting reduced problem is considerably easier and faster. Second, for any network, the optimum design problem can be considerably simplified by using the following two-step procedure. The maximum entropy flows are found by solving Problem 9 using a suitable algorithm for unconstrained non-linear programming. The pipes are then sized using the linear programming phase of the Linear Programming Gradient (LPG) method. With linear programming, the global minimum cost design will usually be found, but it may be noted that linear programs can have multiple optima (see, for example, Winston, 1987). Furthermore, for single-source networks, the algorithm developed in Chapter 5 for calculating maximum entropy flows can be used and, therefore, a maximum entropy design can be accomplished using linear programming only.

6.6 SUMMARY AND CONCLUSION

Networks designed by the conventional manual method inevitably tend to be expensive, while the conventional cost minimization approach leads to implicit tree-type networks for a single dominant load. Using a multiplicity of load cases or an explicit reliability measure to prevent networks from being implicitly branched increases the difficulty of solving the least cost network design problem very considerably. There is therefore a need to develop ways by which the size of the design problem can be systematically reduced so that it becomes easier to explicitly consider more than one demand pattern. The formulation of Problem 10, but without the entropy constraint, represents a basic attempt to address this need.

It also follows from the above discussion that there is a need for a good and easily calculable surrogate reliability measure which can be incorporated into cost optimization procedures. Network flow entropy is perhaps the first such measure. Network reliability and entropy are related through uncertainty. Moreover, the higher the value of entropy for a set of nodal inflows and outflows, the more uniform the link flows of the network are. Also, the more central a flow distribution is with respect to all the distributions capable of satisfying flow equilibrium throughout the network, the smaller the potential changes in the link flows are, and, in consequence, the smaller the increase in headloss due to any changes in pipe flow rates. Since excessive headloss adversely affects network performance, the above properties are central to the ability of flow entropy to act as a good surrogate for water network reliability.

Sample entropy-constrained minimum cost designs have been obtained in this chapter. These designs show that the entropy constraint increases the resilience of a network by making the pipes larger and more uniform than they would otherwise be. A rather unexpected property of the designs is that the entropy constraint does not appear to make the designs much more expensive. The sample designs also show that the resilience of these designs increases as the value of the entropy of the flows increases. The above trend of increasing resilience with entropy shows that entropy is a good surrogate for reliability. Furthermore, it has been shown that as the entropy increases, the behaviours of the network designs become more predictable.

Regarding network design, if a network having good all-round resilience is acceptable, then, the network should be designed to carry maximum entropy flows. Such a design can be accomplished by first calculating the maximum

entropy flows and then sizing the pipes by linear programming. For single-source networks, the maximum entropy flows can be calculated using the simple algorithm developed in Chapter 5. Finally, the fact that entropy can be used to control the hydraulic properties of a water network may have profound design and/or reliability implications.

Although, independently, Awumah et al. (1990, 1991, 1992) have used flow entropy as a surrogate reliability measure, it is felt that the present study is more authoritative in several respects including the definitions of probabilities and network flow entropy, the like-and-like basis of the comparisons, better explanations of why the entropy approach seems to work, and the firmer conclusions drawn.

Table 6.1 Node data for the network of Fig. 6.3 (Ex. 12)

Node	Total Head (m)	Least Head (m)	Design Load (m^3/s)	"Fire Fighting" Loads (m^3/s) Case 1	Case 2	Case 3	Case 4	Case 5
1	20.00	20.00	-0.284	-0.506	-0.501	-0.459	-0.442	-0.478
2	0.00	0.00	0.028	0.250	0.028	0.028	0.028	0.028
3	0.00	0.00	0.033	0.033	0.250	0.033	0.033	0.033
4	0.00	0.00	0.075	0.075	0.075	0.250	0.075	0.075
5	0.00	0.00	0.092	0.092	0.092	0.092	0.250	0.092
6	0.00	0.00	0.056	0.056	0.056	0.056	0.056	0.250

The negative sign indicates an inflow. The fire fighting loads are the same as the design load, but with 0.25 m^3/s at each node in turn, with other nodes at their respective design demands.

Table 6.2 Optimum diameters (mm) for the network of Fig. 6.3 (Ex. 12)

Link	Network Entropy 1.578	1.600	1.700	1.800	1.900	1.915
1-3	401	401	390	384	365	367
2-4	100	100	165	191	238	235
3-5	338	337	337	329	281	294
4-6	100	100	100	151	250	234
5-6	263	262	262	249	152	185
1-2	157	165	203	224	263	261
3-4	237	237	213	215	247	234
Mean	228	229	239	249	257	258
σ_{n-1}	116	115	100	81	63	58
$\frac{\sigma_{n-1}}{mean}$	0.510	0.504	0.419	0.325	0.245	0.224
Cost	0.250	0.251	0.254	0.259	0.261	0.263

The entropy constraint is slack at a value of 1.578; costs are in £ 10^6.

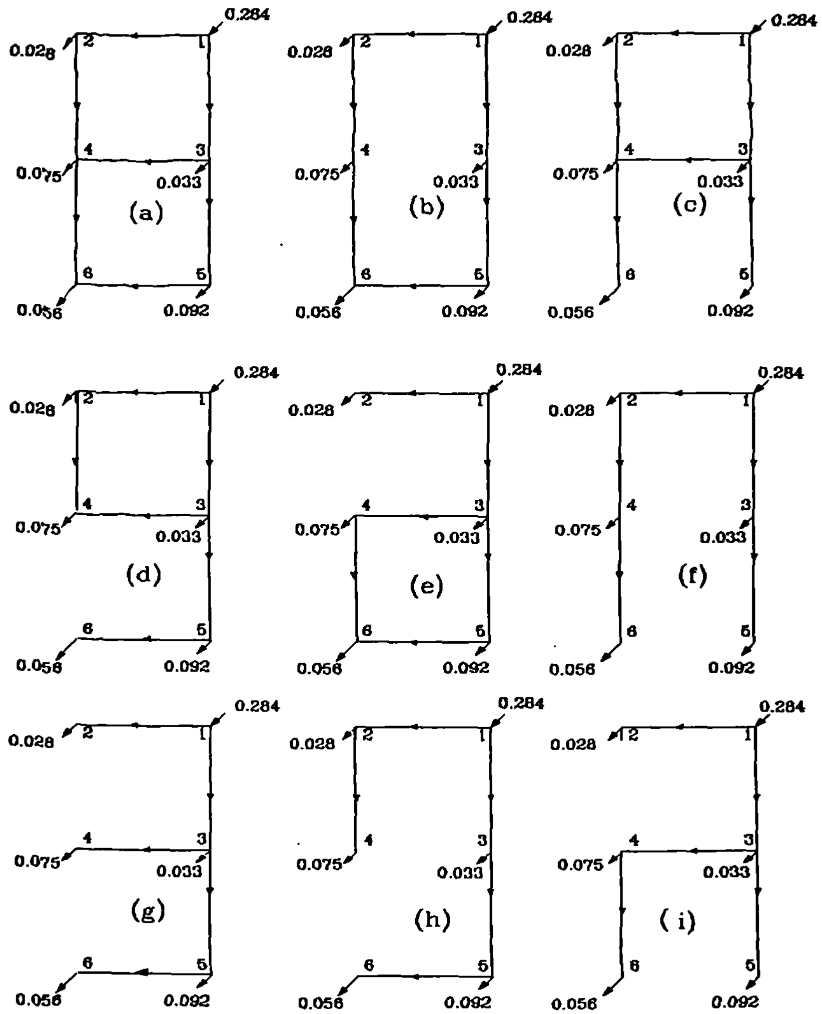


Figure 6.1 Layouts for the network of Example 9 (Fig. 5.6a shown here as Fig. 6.1a): only the flow directions of (a) have been considered in determining the layouts of (b) to (i).

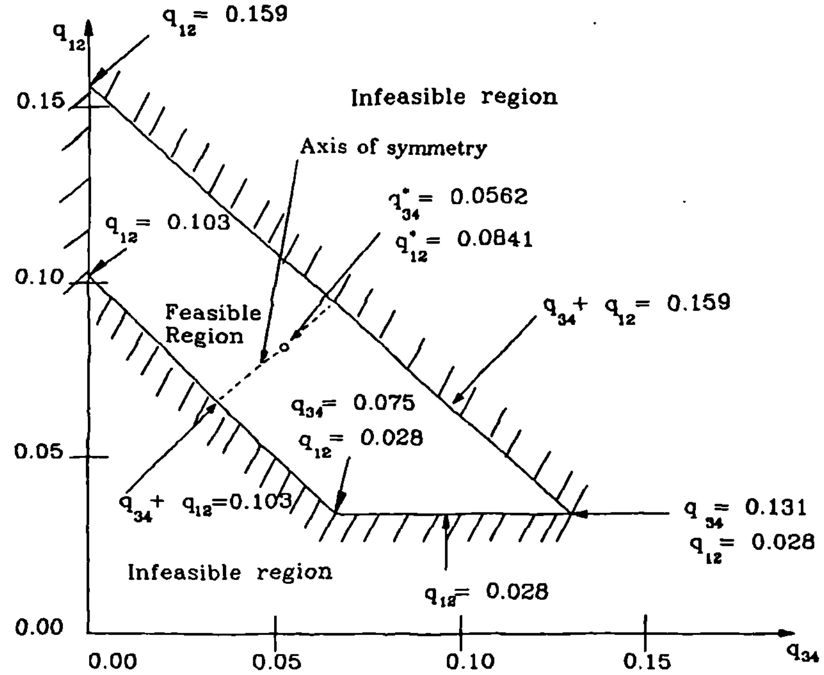


Figure 6.1j Graphical representation of the constraints and optimum point of Problem 9 for the network of Fig. 6.1a

Table 6.3 Optimum flows (m^3/s) for the network of Fig. 6.3 (Ex. 12)

Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	0.253	0.251	0.234	0.223	0.199	0.200
2-4	0.003	0.005	0.022	0.033	0.057	0.056
3-5	0.147	0.147	0.147	0.139	0.101	0.110
4-6	0.001	0.001	0.001	0.009	0.047	0.038
5-6	0.055	0.055	0.055	0.047	0.009	0.018
1-2	0.031	0.033	0.050	0.061	0.085	0.084
3-4	0.073	0.072	0.054	0.051	0.065	0.057

The entropy constraint is slack at a value of 1.578.

Table 6.4 Usable head (m) to satisfy all design demands (Fig. 6.3, Ex. 12)

Failed Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	8817.9	8454.6	1065.2	567.7	219.3	231.6
2-4	20.6	20.7	28.0	27.6	35.7	34.7
3-5	2896.2	2895.5	2899.2	453.9	215.5	141.9
4-6	20.2	20.4	20.4	22.6	87.7	47.5
5-6	500.8	500.3	501.4	92.2	22.7	26.6
1-2	158.3	155.9	54.1	43.9	46.3	46.2
3-4	267.9	259.5	75.4	37.2	39.8	33.5
Mean	1811.7	1758.1	663.4	177.9	95.3	80.3
Max	8817.9	8454.6	2899.2	567.7	219.3	231.6

The usable head values are those needed to meet the design load and minimum pressures of Table 6.1 with the failed link closed off.

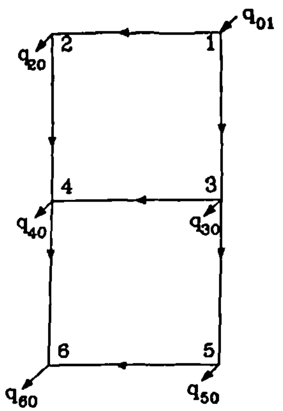


Figure 6.2 Network used to demonstrate how to reduce the number of constraints of Problem 1

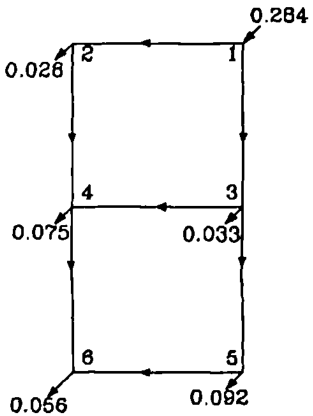


Figure 6.3 Layout and design demands for the network of Example 12: all flows are in m^3/s ; $L_{ij} = 1000m$; $C_{ij} = 130$; $\forall ij$.

Table 6.5 Total energy (MW) for design demands (Fig. 6.3, Ex. 12)

Failed Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	22.375	21.405	2.688	1.434	0.550	0.584
2-4	0.048	0.047	0.054	0.057	0.072	0.071
3-5	4.272	4.269	4.279	0.712	0.262	0.210
4-6	0.048	0.047	0.046	0.045	0.087	0.062
5-6	0.322	0.321	0.321	0.090	0.045	0.047
1-2	0.099	0.099	0.084	0.084	0.103	0.102
3-4	0.250	0.240	0.093	0.066	0.074	0.066
Mean	3.916	3.775	1.081	0.356	0.171	0.163
Max	22.375	21.405	4.279	1.434	0.550	0.584
Full network	0.0475	0.0467	0.0459	0.0444	0.0444	0.0439
Mean - Full network	3.8685	3.7283	1.0351	0.3116	0.1266	0.1191

The energy values are those that occur when the (full or reduced) network meets the design load and minimum pressures of Table 6.1 with any failed link closed off.

Table 6.6 Usable head (m) for fire fighting demands (Fig. 6.3, Ex. 12)

Node	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
2	509.2	424.7	130.4	83.3	50.2	51.5
3	37.8	38.0	38.4	38.1	37.9	37.8
4	102.3	101.6	102.2	73.2	56.0	55.8
5	50.7	51.0	51.3	49.8	60.8	53.5
6	111.9	113.2	113.6	106.1	106.9	102.8
Mean	162.4	145.7	87.2	70.1	62.4	60.3
Max	509.2	424.7	130.4	106.1	106.9	102.8

The usable head values are those needed to meet the fire fighting loads and minimum pressures of Table 6.1.

Table 6.7 Total energy (MW) for fire fighting demands (Fig. 6.3, Ex. 12)

Node	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
2	1.317	1.108	0.417	0.229	0.218	0.222
3	0.150	0.150	0.151	0.150	0.157	0.155
4	0.317	0.313	0.314	0.249	0.209	0.212
5	0.182	0.182	0.181	0.181	0.207	0.193
6	0.368	0.370	0.370	0.352	0.354	0.345
Mean	0.467	0.425	0.287	0.246	0.229	0.225
Max	1.317	1.108	0.417	0.352	0.354	0.345

The energy values are those that occur when the network meets the fire fighting loads and minimum pressures of Table 6.1.

Table 6.8 Headloss-based critical links and nodes for link failures (Fig. 6.3, Ex. 12)

	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1 915
Critical Link	1-3	1-3	3-5	1-3	1-3	1-3
Critical Node	6	6	5	6	5	5

The critical link is the link the removal of which causes the greatest extra source head requirement. The critical node is the terminal node having the greatest value of $\sum_{ij \in I J_n} h_{ij}$ following the failure of the critical link.

Table 6.9 Energy-based critical links and nodes for link failures (Fig. 6.3, Ex. 12)

	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
Critical Link	1-3	1-3	3-5	1-3	1-3	1-3
Critical Node	6	6	5	6	5	5

The critical link is the link the removal of which causes the greatest amount of energy to be dissipated. The critical node is the terminal node having the greatest value of $\sum_{ij \in I J_n} h_{ij}$ following the failure of the critical link.

Table 6.10 Headloss-based critical nodes for fire fighting (Fig. 6.3, Ex. 12)

	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
Critical Node	2	2	2	6	6	6

The critical node is the node at which the fire fighting demand causes the greatest extra source head requirement.

Table 6.11 Energy-based critical nodes for fire fighting (Fig. 6.3, Ex. 12)

	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
Critical Node	2	2	2	6	6	6

The critical node is the node at which the fire fighting demand causes the greatest amount of energy to be dissipated.

Table 6.12 Fire fighting loads (m^3/s) for the network of Fig. 6.7 (Ex. 13)

Node	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
1	-0.4373	-0.4373	-0.4373	-0.4373	-0.4373	-0.4373	-0.4373	-0.3956
2	0.2500	0.0208	0.0208	0.0208	0.0208	0.0208	0.0208	0.0208
3	0.0208	0.2500	0.0208	0.0208	0.0208	0.0208	0.0208	0.0208
4	0.0208	0.0208	0.2500	0.0208	0.0208	0.0208	0.0208	0.0208
5	0.0208	0.0208	0.0208	0.2500	0.0208	0.0208	0.0208	0.0208
6	0.0208	0.0208	0.0208	0.0208	0.2500	0.0208	0.0208	0.0208
7	0.0208	0.0208	0.0208	0.0208	0.0208	0.2500	0.0208	0.0208
8	0.0208	0.0208	0.0208	0.0208	0.0208	0.0208	0.2500	0.0208
9	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.2500

The negative sign indicates an inflow. The fire fighting loads are the same as the design load, but with $0.25 m^3/s$ at each node in turn, with other nodes at their respective design demands.

Table 6.13 Optimum designs for the network of Fig. 6.7 (Ex. 13)

Link	Network Entropy									
	2.170		2.500		2.750		2.775		2.800	
	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)
1-2	201	0.047	309	0.119	294	0.104	294	0.104	294	0.104
2-3	156	0.023	161	0.024	225	0.049	183	0.031	201	0.038
1-4	349	0.161	273	0.089	294	0.104	294	0.104	294	0.104
2-5	100	0.003	267	0.075	197	0.034	234	0.052	221	0.045
3-6	100	0.002	100	0.003	196	0.029	137	0.010	164	0.017
4-5	317	0.116	222	0.044	197	0.034	234	0.052	221	0.045
5-6	151	0.020	286	0.078	179	0.023	222	0.042	207	0.035
4-7	156	0.023	155	0.024	225	0.049	183	0.031	201	0.038
5-8	286	0.079	152	0.019	179	0.023	222	0.042	207	0.035
6-9	100	0.001	272	0.061	216	0.031	216	0.031	216	0.031
7-8	100	0.003	100	0.004	196	0.029	137	0.010	164	0.017
8-9	271	0.061	100	0.002	216	0.031	216	0.031	216	0.031
Mean	191		200		218		214		217	
σ_{n-1}	92		80		39		50		41	
$\frac{\sigma_{n-1}}{mean}$	0.484		0.401		0.178		0.232		0.187	
Cost	0.277		0.282		0.292		0.290		0.292	

The entropy constraint is slack at a value of 2.170; costs are in £ 10^6.

Table 6.14 Usable head (m) to satisfy all design demands (Fig. 6.7, Ex. 13)

Failed Link	Network Entropy				
	2.170	2.500	2.750	2.775	2.800
1-2	109.7	109.5	85.5	76.5	78.9
2-3	115.8	98.9	46.6	45.3	40.6
1-4	1248.6	56.0	79.8	74.8	76.2
2-5	23.9	59.1	30.0	38.3	34.3
3-6	24.1	24.1	34.0	25.8	28.2
4-5	360.2	35.2	30.0	38.3	34.3
5-6	48.7	329.3	28.7	39.2	33.9
4-7	98.5	114.4	46.6	45.4	40.6
5-8	333.9	46.9	28.7	39.2	33.9
6-9	28.3	660.3	42.9	41.1	41.7
7-8	24.0	24.3	34.0	25.8	28.2
8-9	664.1	24.0	42.9	41.1	41.7
Mean	256.3	131.8	44.16	44.23	42.7
Max	1248.6	660.3	85.5	76.5	78.9
Critical link	1-4	6-9	1-2	1-2	1-2
Next crit link	8-9	5-6	1-4	1-4	1-4
Critical node	7	9	9	9	9

Table 6.15 Total energy (MW) for design demands (Fig. 6.7, Ex. 13)

Failed Link	Network Entropy				
	2.170	2.500	2.750	2.775	2.800
1-2	0.094	0.194	0.151	0.135	0.140
2-3	0.066	0.059	0.065	0.050	0.054
1-4	2.125	0.114	0.155	0.142	0.146
2-5	0.038	0.087	0.044	0.056	0.051
3-6	0.038	0.037	0.045	0.037	0.039
4-5	0.491	0.052	0.044	0.056	0.051
5-6	0.047	0.305	0.040	0.051	0.046
4-7	0.060	0 064	0.065	0.050	0.054
5-8	0.312	0.045	0.040	0.051	0.046
6-9	0.038	0.443	0.048	0.046	0.047
7-8	0.038	0.037	0.045	0.037	0.039
8-9	0.448	0.037	0.048	0.046	0.047
Mean	0.316	0.123	0.066	0.0633	0.0631
Max	2.125	0.443	0.155	0.142	0.146
Full network	0.0377	0.0366	0.0345	0.0349	0.0346
Mean - Full network	0.2783	0.0864	0.0315	0.0284	0.0285
Critical link	1-4	6-9	1-4	1-4	1-4
Next crit link	4-5	5-6	1-2	1-2	1-2
Critical node	7	9	9	9	9

Table 6.16 Usable head (m) for fire fighting demands (Fig. 6.7, Ex. 13)

Node	Network Entropy				
	2.170	2.500	2.750	2.775	2.800
2	181.7	51.9	52.9	52.8	52.3
3	646.5	504.3	117.8	242.3	161.3
4	50.1	56.4	52.9	52.8	52.7
5	82.3	81.5	85.3	80.7	82.0
6	463.0	128.1	135.2	135.4	134.7
7	550.2	586.4	117.8	242.3	161.3
8	129.0	452.3	135.2	135.4	134.7
9	146.7	145.5	136.2	138.0	136.9
Mean	281.2	2F0.8	104.1	134.9	114.5
Max	646.5	586.4	136.2	242.3	161.3
Critical Node	3	7	9	3, 7	3, 7

Table 6.17 Total energy (MW) for fire fighting demands (Fig. 6.7, Ex. 13)

Node	Network Entropy				
	2.170	2.500	2.750	2.775	2.800
2	0.540	0.183	0.194	0.190	0.191
3	1.693	1.328	0.409	0.697	0.507
4	0.166	0.208	0.194	0.190	0.191
5	0.297	0.296	0.328	0.297	0.307
6	1.267	0.453	0.479	0.480	0.478
7	1.441	1.541	0.409	0.697	0.507
8	0.455	1.238	0.479	0.480	0.478
9	0.437	0.433	0.413	0.417	0.414
Mean	0.789	0.710	0.363	0.431	0.384
Max	1.693	1.541	0.479	0.697	0.506
Critical Node	3	7	9	3, 7	3, 7

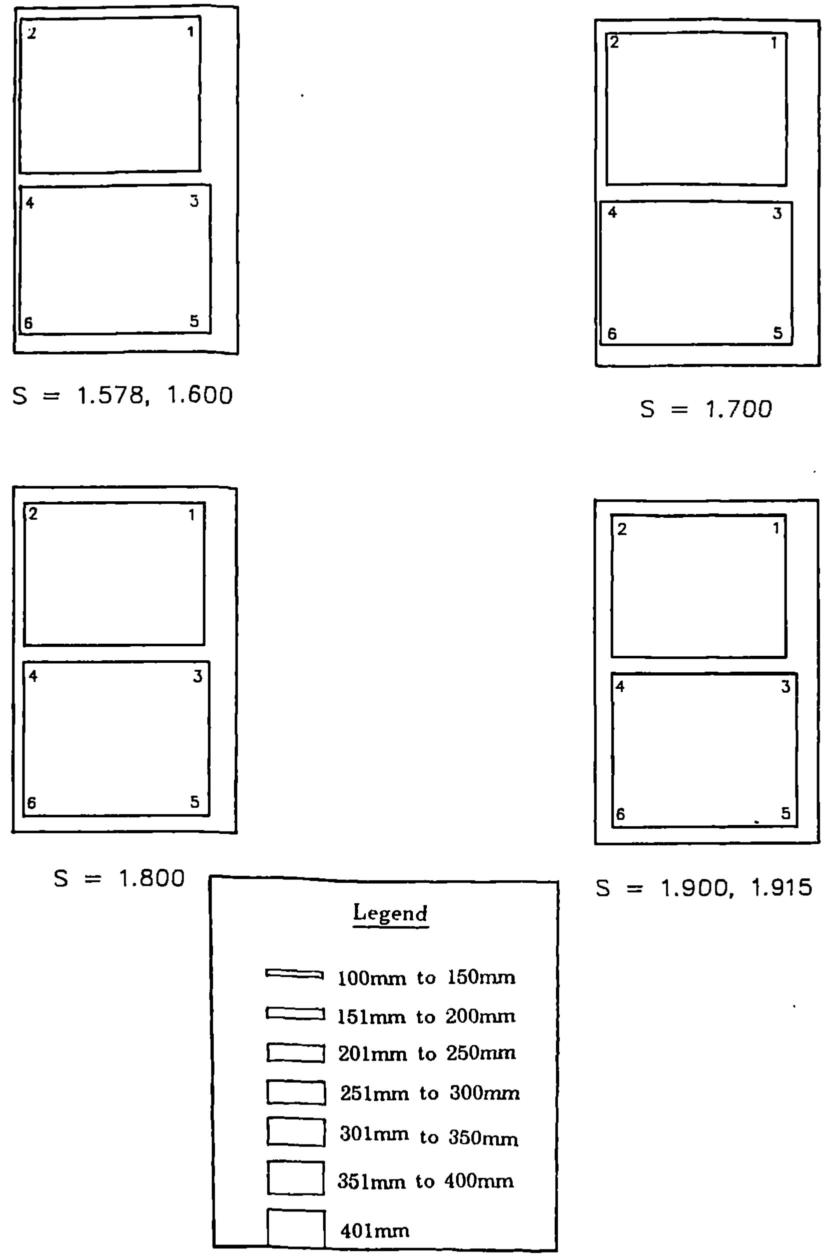
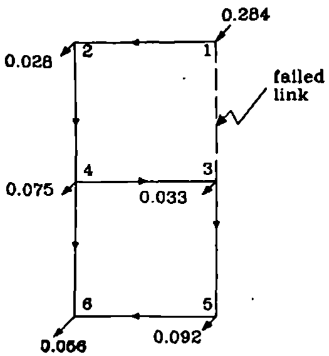
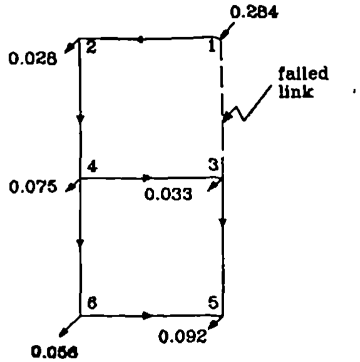


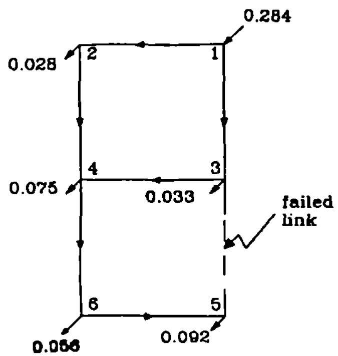
Figure 6.4 Pictorial representation of the diameters of the entropy-constrained minimum cost designs of the network of Fig. 6.3 (Example 12)



a) Flow directions for the reduced networks of the designs having an entropy value of 1.578, 1.600 and 1.800

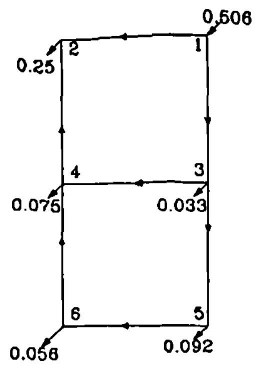


b) Flow directions for the reduced networks of the designs having an entropy value of 1.900 and 1.915

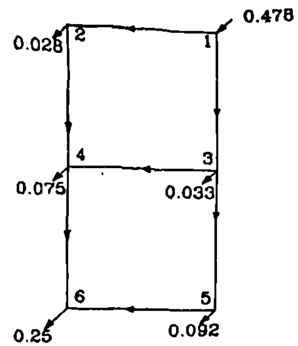


c) Flow directions for the reduced network of the design having an entropy value of 1.700

Figure 6.5 Worst case link failures of the entropy-constrained designs of the network of Fig. 6.3 (Example 12): all flows are in m^3/s.



a) Critical node and flow directions for the designs having an entropy value of 1.578, 1.600 and 1.700; the critical node is node 2.



b) Critical node and flow directions for the designs having an entropy value of 1.800, 1.900 and 1.915; the critical node is node 6.

Figure 6.6 Critical nodes for the fire fighting demands of the entropy-constrained designs of the network of Fig. 6.3 (Example 12): all flows are in m^3/s.

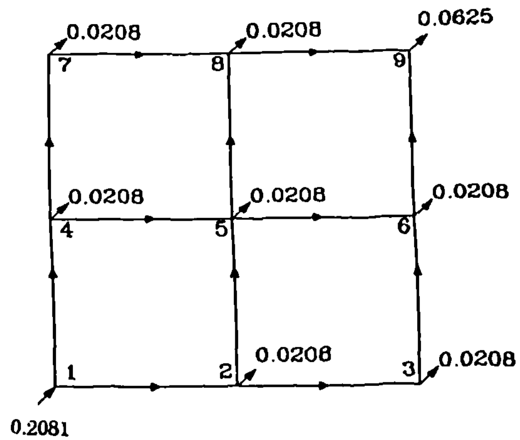


Figure 6.7 Layout and design demands for the network of Example 13: all flows are in m^3/s : $L_{ij} = 1000m$; $C_{ij} = 130$; $\forall ij$.

CHAPTER 7 NEW METHODS FOR CALCULATING THE RELIABILITY OF WATER NETWORKS

7.1 INTRODUCTION

The primary objective of this chapter is to suggest two conceptually simple ways of calculating the reliability of a water distribution network which incorporate most of the desirable properties laid down in Chapter 3. The proposed approaches are dependent on network simulation and, unlike other methods, can account for less-than-fully satisfactory service. One of the approaches uses head-driven simulation to obtain a probabilistic reliability measure on the interval [0, 1]. This is the more general approach and is described in Section 7.3. The other method also defines reliability on the interval [0, 1], is suitable for single-source networks and is based on pressures at the source node. These source node pressures are obtained by demand-driven simulations. This method is described in Section 7.2. Also, the reliability measures developed in this chapter are applied to the entropy-constrained minimum cost designs of Chapter 6, to show that these reliability measures are very sensible and that flow entropy is a good surrogate for network reliability.

In this chapter, network reliability is defined as the ratio of the expected equivalent flow delivered at adequate pressure to the total demand. The equivalent flow delivered at adequate pressure is defined as the flow that the actual network pressures can sustain without the residual pressure at any node falling below the desirable minimum value. Also, for a single-source network, the notional source head for complete demand satisfaction is defined as the theoretical source head required to satisfy all the system demands at adequate pressure, under specified conditions including component failure and increased nodal abstraction.

As seen in Chapter 6, the value of the source head can be found by analysing the network using either the q or Δq system of equations which yield the pipe flow rates. To calculate the source head, each terminal node, which is a node having no internal outflow, is assigned a nodal head equal to its minimum desirable value. The required source head $\hat{H}_s$ is then given by

$$\hat{H}_s = \max < H_{min,n} + \sum_{ij \in IJ_n} h_{ij} \quad \forall n \in t > \tag{7.1a}$$

in which t represents the set of all terminal nodes; $H_{min,n}$ is the minimum desirable head at demand node n; h_{ij} is the head loss in link ij; IJ_n is the set of all pipes in a specified path between the source and demand node n. However, the net or usable source head H is the real value of interest in the present calculations and is

$$H = \max < \sum_{ij \in IJ_n} h_{ij} \quad \forall n \in t > \tag{7.1b}$$

For a network having a broken pipe the analysis is as described above, but the link with the broken pipe section is omitted. That is, it is assumed that flow to the link in question can be stopped. Also, it may be noted that the elements of t, i.e., the position of the terminal nodes, may vary according to the amount of abstraction at each node and the layout of the (full or reduced) network. As such, the set t is in general different for each network configuration and flow pattern. It is seen in Section 7.2 how the notional source head may be used in the calculation of network reliability.

The definition of reliability given earlier is superficially similar to the definition of Eqs. (3.6) and (3.7) of Fujiwara and de Silva (1990) which is based on the expected maximum flow delivered. However, the latter does not involve pressure. Also, the present definition is superficially similar to the probability of sufficient supply (Wagner, Shamir and Marks, 1988a) or the probability of sufficient supply at adequate pressure (Bao and Mays, 1990). However, the present definition is sharper. Furthermore, the capacitated network approach of Wagner, Shamir and Marks (1988a) does not fully account for pressure dependency of flow. Some weaknesses of Bao and Mays (1990) are highlighted shortly.

The notional source head and the present definition of reliability are motivated by the following considerations. First, Wagner Shamir and Marks (1988b) have proposed a way of estimating the shortfall or total flow delivered for any simulated failure and the approach accounts for reduced service by using the following equation which was introduced in Chapter 3 as Eq. (3.1):

$$q'_{n0} = q_{n0}\left[\frac{H_{min,n} - H_n}{H_{min,n} - H'_{min,n}}\right]^{0.5} \quad ; \quad H'_{min,n} \le H_n \le H_{min,n} \quad \forall n \in D \qquad (7.2)$$

in which D represents the set of all demand nodes; the q'_{n0}, $\forall n \in D$, are the rates of abstraction at the demand nodes that the actual network pressures can sustain without the residual pressure at any node falling below the minimum desirable level; q_{n0} is the demand at node n; $H_{min,n}$ is the minimum desirable head at node n; $H'_{min,n}$ is the irreducible minimum desirable head at node n. Also, if $H_n \ge H_{min,n}$, Eqs. (7.2) do not apply in which case the flow supplied equals the demand. However, Wagner, Shamir and Marks (1988b) have not gone on to define network reliability even though they have stated that the shortfall is a good overall indicator of reliability, or rather, the unreliability. In Section 7.3 it is shown how Eqs. (7.2) may be used to calculate nodal and network reliability.

Also, Germanopoulos (1985), Jowitt and Xu (1993) have described a method of calculating system shortfall using head-driven simulation. The approach is based on the following equation.

$$q'_{n0} = \left[1 - b_n \exp\left[-a_n \frac{H_n}{\tilde{H}_n}\right]\right] q_{n0} \quad \forall n \in D \qquad (7.3)$$

in which a_n, b_n and $\tilde{H}_n$ are constants for node n. $\tilde{H}_n$ corresponds to the nodal pressure at which a given proportion of q_{n0} is known to be provided. The above constants have to be determined for a given network and this may need some calibration. Eqs. (7.3) may be used instead of (7.2), but the latter have been used herein.

Second, although Cullinane, Lansey and Mays (1992) have recognised the importance of reduced service, they have defined network reliability as the average of the demand node reliabilities However, this definition is theoretically questionable because the nodal reliabilities are not independent as conditions at any node are influenced by conditions at other nodes. In recognition of this difficulty, Bao and Mays (1990) have used the arithmetic mean of the nodal reliabilities as one of three complementary measures, the others being the reliability of the most unreliable node and the mean of the nodal reliabilities weighted according to their respective demands. For single-source networks, the source head approach is used herein to enable reliability calculations to be based on the source rather than at the individual demand nodes. As such, for single-source networks, the difficulties posed by the interdependencies between the demand node reliabilities are bypassed. The details are given shortly in Section 7.2. For general networks, the calculations herein for network reliability are based on the total flow supplied at adequate pressure rather than the nodal reliabilities. In other words, the reliability at a node is based on the flow supplied at adequate pressure at that node whereas network reliability is based on the sum of the abstractions at the demand nodes.

Third, the use of Eqs. (3.8) of Bao and Mays (1990) in calculating the probability of sufficient supply disregards reduced service. Also, Su, Mays, Duan and Lansey (1987) have defined network unreliability as the probability of one or more failure modes resulting in one or more nodal heads being lower than the minimum desirable level. However, this definition is incapable of distinguishing between shortfalls in head of, for example, 1mm and 1m. Furthermore, in calculating reliability, all residual pressures which are less than the desirable minimum are implicitly assigned a zero utility value irrespective of the magnitude of the shortfall. The approach used herein accounts for all non-zero shortfalls in flow or pressure.

Fourth, Bao and Mays (1990) have observed that it is difficult to derive mathematically and compute the probability that any node in a water distribution network receives sufficient flow at adequate pressure. Some of the reasons for this difficulty are that this probability is the joint probability that both flow and pressure are sufficient. Also, the flow and pressure at a node are not independent of each other. Furthermore, both are affected by conditions elsewhere in the network in complex ways which cannot be predicted without simulation. The above difficulty has somewhat been avoided in the approach used herein as follows. The flows that the actual network pressures can support are sought. These flows are then combined with the probabilities that the flows occur, to obtain an estimate of the reliability of the network, as explained in Section 7.2 for single-source networks and Section 7.3 for general networks.

7.2 SINGLE-SOURCE NETWORKS

7.2.1 ANALYSIS

In this section a method is developed for calculating the reliability of single-source networks. A more general approach is given in the next section, but the procedure described here for single-source networks provides a simpler and quicker means of doing the analysis. For single-source networks, the

approach is as follows. It is assumed that all nodal demands and pressures are satisfied. With this assumption, the problem of calculating the amount of flow suppliable at adequate pressure may be reinterpreted as follows. What is the notional usable source head that can satisfy all nodal demands at pressures no lower than the minimum desirable and how can it be found? Also, given the demands, the notional usable source head required and the actual usable source head, what is the demand that the actual usable head can support at adequate pressure?

To answer the first question, a demand-driven simulation of the (full or reduced) network is done and the net notional source head H is obtained from Eq. (7.1b), i.e.

$$H = \max < \sum_{ij \in IJ_n} h_{ij} \quad \forall n \in t> \tag{7.1b}$$

The approach used herein to answer the second question, that is, to determine the actual flow supplied, is based on Eqs. (7.2) but (7.3) could also be used. Finally, given the probability that the flow thus obtained occurs, it is possible to calculate network reliability as the ratio of the expected equivalent flow delivered at adequate pressure to the total demand. These ideas are explained in detail immediately and examples are given later in this section.

The proposed definition of reliability can be written as

$$\text{Reliability} = \frac{\text{Expected equivalent flow supplied at adequate pressure}}{\text{Demand}} \tag{7.4a}$$

and, for a given network configuration and nodal demands, i.e., for a given state,

$$\text{Reliability} = \frac{\text{Equivalent flow supplied at adequate pressure}}{\text{Demand}} \tag{7.4b}$$

Also, from Eqs. (7.2), the following equation may be written.

Flow supplied at adequate pressure$\simeq$

$$\text{Demand}\left[\frac{\text{Available residual head}}{\text{Minimum desirable residual head}}\right]^{0.5} \tag{7.5a}$$

in which the flow supplied equals the demand if the available residual head is greater than or equal to the minimum desirable value. Let H^i represent the minimum net source head needed to deliver all the required network flows at the right pressure for the ith network state. Then, assuming that Eq. (7.5a) is also applicable to the source, this equation may be adapted for the source as follows.

$$T'_{0,i} = T_0\left[\frac{H}{H^i}\right]^{0.5} \; ; \; H^i \geq H \quad i = 1, ..., N \tag{7.5b}$$

in which H and T_0 are respectively the net available source head and the total demand which equals the source supply when the network is operating under normal conditions; $T'_{0,i}$ is the equivalent total demand satisfied at adequate pressure or the actual total flow that H can deliver when the network is in state i and this flow equals the source supply for state i; N is the number of states. For each state i, the (full or reduced) network is analysed and the net required head H^i is obtained from Eq. (7.1b). For each state i, Eqs. (7.4b) and (7.5b) give the reliability R^i as

$$R^i = \frac{T'_{0,i}}{T_0} = \left[\frac{H}{H^i}\right]^{0.5} \; ; \; H^i \geq H \quad i = 1, ..., N \tag{7.6}$$

Furthermore, according to Eq. (7.4a), network reliability R is then given by

$$R = \sum_{i=1}^{N} R^i p_i \tag{7.7}$$

in which p_i is the probability of the network being in state i, $\forall i$. Substituting Eqs. (7.6), this becomes

$$R = \sqrt{H}\sum_{i=1}^{N}\frac{p_i}{\sqrt{H^i}} \; ; \; H^i \geq H \tag{7.8}$$

It remains to define the states and their probabilities p_i, $i = 1, ..., N$. The design demands are considered for different network configurations which include the fully connected network and the reduced networks corresponding to single link failures. For the design demands the states are therefore the above (full or reduced) networks. Also, pipes typically have high reliabilities; see for example, Su, Mays, Duan and Lansey (1987). It follows that the probability of multiple failures occurring may be negligible in general. The following equations for the p_i, $\forall i$, are based on the assumption that link failures are independent. The rationale for this assumption is that pipe breakage tends to be caused by factors external to the pipe network and, in general, does not depend on the number of already broken pipes. The probability $p(0)$ that all links are working is therefore

$$p(0) = \prod_{ij \in IJ} R_{ij} \tag{7.9}$$

in which R_{ij} is the mechanical reliability of link ij, $\forall ij \in IJ$; IJ is the set of all links of the network. Also, the probability $p(ij)$ that only link ij, $\forall ij \in IJ$, is not working is

$$p(ij) = R'_{ij}\prod_{\substack{\tilde{ij} \in IJ \\ \tilde{ij} \neq ij}} R_{\tilde{ij}} = \frac{R'_{ij}}{R_{ij}}\prod_{\tilde{ij} \in IJ} R_{\tilde{ij}} \quad \forall ij \in IJ \tag{7.10}$$

in which

$$R'_{ij} = 1 - R_{ij} \quad \forall ij \in IJ \tag{7.11}$$

is the unreliability, or the probability of failure, of link ij, $\forall ij$.

Eqs. (7.8) to (7.10) can be combined to give expressions for network reliability as follows. When the network is fully connected, the required head equals the available head, i.e, $H^i = H$, and so, according to Eqs. (7.6), the reliability is unity. On the other hand, the net source head required H^i, $\forall i$, may be less than H if the network is highly redundant. In such a case the reliability will also be unity. Therefore, in Eqs. (7.5b), (7.6) and (7.8), H^i is set to H whenever H^i is less than H. From Eqs. (7.8) to (7.10), network reliability is given by

$$R = 1\prod_{ij \in IJ} R_{ij} + \sqrt{H}\left[\prod_{ij \in IJ} R_{ij}\right]\sum_{ij \in IJ} \frac{R'_{ij}/R_{ij}}{\sqrt{H^{ij}}} \quad ; \quad H^{ij} = H \text{ if } H^{ij} < H \tag{7.12}$$

in which H^{ij} is the net source head required to satisfy all demands with link ij, $\forall ij$, not working. That is, network reliability is

$$R = \left[1 + \sqrt{H}\sum_{ij \in IJ} \frac{R'_{ij}/R_{ij}}{\sqrt{H^{ij}}}\right]\prod_{ij \in IJ} R_{ij} \tag{7.13}$$

Also, if the second term of Eq. (7.12) is normalised by dividing it by the probability $(1 - p(0))$ that the network is not fully connected, the average reliability $\overline{R}$ of the reduced networks is obtained as

$$\overline{R} = \frac{\sqrt{H}\left[\prod_{ij \in IJ} R_{ij}\right]\sum_{ij \in IJ} \frac{R'_{ij}/R_{ij}}{\sqrt{H^{ij}}}}{1 - \prod_{ij \in IJ} R_{ij}} \tag{7.14}$$

Substituting Eq. (7.14) in (7.13) gives

$$R = \prod_{ij \in IJ} R_{ij} + \overline{R}(1 - \prod_{ij \in IJ} R_{ij}) \tag{7.15}$$

which may be rearranged as

$$\overline{R} = \frac{R - \prod_{ij \in IJ} R_{ij}}{1 - \prod_{ij \in IJ} R_{ij}} \tag{7.16}$$

A very high value of $\overline{R}$ means that the network is quite redundant and so link failures do not significantly affect the performance of the network. . Also, as the terms for multiple link failures are not included in Eq. (7.12) and these terms are non-negative, R and $\overline{R}$ are lower bounds.

7.2.2 EXAMPLES

Several numerical examples are provided next to show that R and $\overline{R}$ are realistic and quite comprehensive. On one hand, Examples 14, 15 and 16 are based on the two-loop network of Figure 6.3, which, for convenience, is shown here as Figure 7.1. The required usable head values for the entropy-constrained designs of this network have been calculated in Chapter 6. For convenience, the relevant diameter, flow and headloss data from Tables 6.2 to 6.4 are repeated here in Tables 7.1 to 7.3. For simplicity in referring to the above tables, the entropy-constrained designs have been labelled Designs 1 to 6 as shown in each table. On the other hand, Example 17, which is considered shortly, is based on the four-loop network of Figure 6.7 which is shown here as Figure 7.2.

Example 14

This is a simple example to demonstrate how R and $\overline{R}$ are calculated. Also, it is shown that, for the same values of H^{ij}, if the mechanical reliabilities of the links increase, then, R increases. In other words, it is shown that the reliability measure can reflect the mechanical reliability of a network. The required usable head data used in this example are those of Design 2 in Table 7.3.

Let $R_m = 0.95$; $R'_m = 0.05$, $m = 1, ..., 7$

$p(0) = 0.95^7$; $p(m) = 0.05 \times 0.95^6 \quad m = 1, ..., 7$

$$\sum_{m=1}^{7} (H^m)^{-0.5} = 0.6575309$$

From Eq. (7.14),

$$\overline{R} = \frac{\sqrt{20} \times 0.05 \times 0.95^6 \times 0.6575309}{1 - 0.95^7} = 0.3582789 \simeq 0.358$$

and, from Eq. (7.15),

$$R = 0.95^7 + 0.3582789(1 - 0.95^7) = 0.806$$

If these calculations are repeated with $R_m = 0.99$, $\forall m$, the result is

$\overline{R} = 0.408$; $R = 0.960$

If $R_m = 0.999$, $\forall m$,

$\overline{R} = 0.419$; $R = 0.996$

Commenting on the values of R and $\overline{R}$, it can be seen that they agree with the subjective expectation that more reliable components should increase reliability. Also, the average reliability $\overline{R}$ of the reduced networks increases as the link reliabilities increase. $\overline{R}$ increases because the probabilities of multiple failures decrease as the mechanical reliabilities of the links increase. As the joint probability of multiple failures becomes smaller, $\overline{R}$, which is based on single failures only, approaches the true value of the average reliability of the less-than-fully connected networks. In other words, the magnitude of the error in the numerator of Eq. (7.14) becomes smaller as the mechanical reliabilities of the links increase.

Example 15

The aim of this example is to show that R and $\overline{R}$ are sensitive to slight variations to the design of a distribution network. This demonstration is based on the headloss values in Table 7.3 which are for Designs 1 to 6 in Tables 7.1 and 7.2. Designs 1 to 6 are based on the network of Figure 7.1 and they have been obtained as described in Chapter 6.

The reliability measures R and $\overline{R}$ have been calculated for Designs 1 to 6 assuming equal probabilities of link failure of 0.05 and 0.01 respectively; i.e. $R_m = 0.95$ and 0.99 respectively, $\forall m$. The results of the calculations are shown in Table 7.4, and these results are discussed next. It can be seen that the column entries increase from top to bottom which corresponds to an increase in reliability from Design 1 through to 6. Since the two sets of probabilities used are the same for all the designs, these values show that R and $\overline{R}$ seem capable of correctly reflecting the hydraulic performance of a network. Also, all R and $\overline{R}$ based on $R_m = 0.99$, $\forall m$, are higher than corresponding values based on $R_m = 0.95$, $\forall m$. As observed in Example 14, this shows that R and $\overline{R}$ appear to be able to correctly account for mechanical reliability. Finally, considering the conclusions of Chapter 6 to the effect that reliability increases from Design 1 through to 6 and the fact that the results of Table 7.4 show the expected trend of increasing reliability from Design 1 through to 6, it would appear that R and $\overline{R}$ are sensible measures of reliability.

Example 16

In this example, the calculations of Example 15 are repeated, but with link failure probabilities which are a function of the pipe diameters and lengths. That is, R and $\overline{R}$ are calculated for Designs 1 to 6 with more "realistic" probabilities. Two reliabilities are calculated for each link. One is based on Eq. (7.17) below which is taken from Cullinane, Lansey and Mays (1992).

$$R_{ij} = \frac{0.21218 D_{ij}^{1.462131}}{0.00074 D_{ij}^{0.285} + 0.21218 D_{ij}^{1.462131}} \qquad \forall ij \in IJ \qquad (7.17)$$

in which the diameter D_{ij} is in inches. It is not clear from Cullinane, Lansey and Mays (1992) whether the above equations are link length dependent, but that does not matter in the present example as all lengths are equal. The other set of reliabilities for the links is based on Eqs. (7.18) and (7.19) below which are taken from Su, Mays, Duan and Lansey (1987).

$$R_{ij} = \exp(-NB_{ij} L_{ij}) \qquad \forall ij \in IJ \qquad (7.18)$$

in which NB_{ij} is the expected number of breaks per year per unit length of pipe ij, $\forall ij$, and is given by Eqs. (7.19); L_{ij} is the length of link ij, $\forall ij$.

$$NB_{ij} = \frac{0.6858}{D_{ij}^{3.26}} + \frac{2.7158}{D_{ij}^{1.3131}} + \frac{2.7685}{D_{ij}^{3.5792}} + 0.042 \qquad \forall ij \in IJ \qquad (7.19)$$

in which the diameter is in inches. As can be seen in Table 7.5, Eqs. (7.17) tend to give quite high reliabilities whereas (7.18) tend to give quite low reliabilities.

The results of Table 7.6 including the probabilities of full network connectivity $p(0)$ support the conclusions of Example 15. Network reliability R increases steadily from Design 1 to Design 6. Also, the subsidiary measure $\overline{R}$ increases from Design 1 to Design 6. However, it may be noted that $\overline{R}$ is higher for Design 3 than Design 4. Further work is necessary to establish whether this is in fact a true reflection of the average reliabilities of the designs during the total period of partial connectivity. Finally, all R and $\overline{R}$ values based on the higher reliabilities of Eqs. (7.17) are higher than corresponding values based on the lower reliabilities of Eqs. (7.18) and this is as expected. These results therefore reinforce the conclusions of Example 15 regarding the apparent efficacy of the present source head approach to calculating the reliability of single-source networks. Moreover, it should be noted that the differences between the entropy values of Designs 1 and 2 on one hand and Designs 5 and 6 on the other hand are much smaller than for Designs 2 and 3, 3 and 4, and 4 and 5. It is therefore remarkable that R and $\overline{R}$ can correctly tell apart the two most unreliable designs which are Designs 1 and 2, and the two most reliable designs which are Designs 5 and 6. Looking again at Designs 1 and 2 in Table 7.1 (and Table 7.2), it can be seen that these two designs look extremely alike. Despite this likeness, R and $\overline{R}$ can correctly tell apart Designs 1 and 2. Such sensitivity to very subtle changes in a network is desirable and the fact that R and $\overline{R}$ appear to possess this capability is worth emphasising.

Example 17

In this example the calculations of Examples 15 and 16 are repeated for the entropy-constrained designs of the network of Figure 6.7, which, for convenience, is shown here as Figure 7.2. The designs and required head values are also repeated here as Tables 7.7 and 7.8. To facilitate referencing, these designs have been labelled Designs 7 to 11. This example enables the previously highlighted properties of R and $\overline{R}$ to be observed on this network which is different from Figure 7.1 in many ways including the number of links, nodes, loops, the total demand, symmetry, layout, etc. The results of the reliability calculations are summarised in Tables 7.9 to 7.11. These results are discussed next.

Starting with the network reliability values in Table 7.9 for equal link reliabilities, it can be seen that all the values are consistent with the observations of Chapter 6 regarding the reliability of these entropy-constrained designs: the reliabilities, R and $\overline{R}$, increase from Design 7 to Design 11 for both sets of reliabilities. In particular, since the network reliability values of Table 7.9 are based on equal link reliabilities, the network reliabilities should follow Table 7.8 very closely. The average headloss values of Table 7.8 lead to the conclusion that Design 11 is better than Design 9 which, in turn, is better than Design 10, etc. The above ranks are maintained in Table 7.9.

The network reliabilities of Table 7.11 are based on more "realistic" link reliabilities. Once again, it can be seen that the behaviours of both R and $\overline{R}$ are generally as expected: all network reliabilities calculated using the Su *et al* link reliabilities are lower than corresponding values based on the higher Cullinane et al. link reliabilities; network reliabilities increase from Design 7 to Design 11 for both sets of link reliabilities; as expected, Design 9 is consistently better than Design 10. However, comparing Designs 9 and 11 is not straightforward because both sets of reliability values do not agree with each other regarding the relative merits of Designs 9 and 11.

Still comparing Designs 9 and 11, it can be seen that both sets of $p(0)$ values show that Design 9 is mechanically more reliable and this is as expected, from Table 7.7. However, the respective differences in the mechanical reliabilities of Designs 9 and 11 are not the same for both sets of reliabilities in Table 7.11. Since the $p(0)$ values based on Cullinane et al. are almost equal, network reliabilities based on these values show Design 11 as being superior because of its superior mean required head; Table 7.8. The $p(0)$ values based on Su et al., however, are quite different and this tips the balance in favour of Design

9. Taking all the above observations into consideration, it can be said that the present reliability measures are exhibiting the properties expected of an accurate reliability measure.

7.2.3 DISCUSSION

There are two main themes in this discussion: the correlation between entropy and reliability which is considered shortly, and the following discussion of the reliability measures R and $\overline{R}$.

Network reliability analysis

The foregoing examples and comments provide ample evidence that, as comparative measures for a given layout, R and $\overline{R}$ are very realistic measures of water network reliability. However, what can one infer from these measures about a given network/design, independently of other networks? To put the question another way, is it possible to compare completely different networks or layouts on the basis of R and $\overline{R}$? This issue is briefly examined next with the help of the networks of Figures 7.1 and 7.2.

R and $\overline{R}$ are absolute as opposed to relative measures which, in theory, can be used to compare different designs/layouts for a given network or even completely different networks. However, to compare R and $\overline{R}$ across networks, the calculations for the different networks must be done to the same accuracy. For example, the errors in R and $\overline{R}$, i.e., the differences between the lower bounds represented by R and $\overline{R}$ and the corresponding exact reliability values, are greater for the 12-link network of Figure 7.2 than for the 7-link network of Figure 7.1. This is because the network of Figure 7.2 has more pipes than Figure 7.1 and the number of states which are not accounted for in Eq. (7.12) is an exponential function of the number of components or pipes. The number of states which are not accounted for is given by $2^{NIJ}-(NIJ+1)$, in which NIJ is the total number of pipes and 2^{NIJ} is the total number of states. For 7 pipes, this number amounts to $2^7 - 8 = 120$. For 12 pipes, the number is $2^{12} - 13 = 4083$. Therefore, if the individual components of a medium size to large network are not very reliable, the errors in R and $\overline{R}$ will be large. Also, a network having passive redundancy or spare capacity, i.e., rather large diameter pipes (considering the available source head), may still provide considerable, although perhaps much reduced, service with two links inoperative. Therefore, if the link failure probabilities are not sufficiently low, multiple-link failures may need consideration. However, the increase in computational effort would be very considerable and disproportionately high. For example with 12 links, there are 66 configurations in which only two links are inoperative. From the above discussion, it would seem that inter-network or -layout comparisons on the basis of R and $\overline{R}$ are practicable onlv if all the networks have a similar number of links and similar link reliabilities. Under these conditions, the errors in the reliability measures would be similar for all the networks.

Some of the advantages and disadvantages of the net source head approach are highlighted next. The strengths of the method are that it provides a quick and easy means of estimating the reliability of a water distribution network, and subnormal service is accounted for in a consistent way. However, the main weaknesses are that the method is not suitable for multiple-source networks and it is not applicable if any demand points are cut off from the source because a fundamental assumption in the calculation of the net source head is that all demands are satisfied. Also, no indication of the reliabilities of individual nodes is given and this could be considered a disadvantage. All the above weaknesses are addressed in Section 7.3 in which the basic ideas are the same as those of this section except that reliabilities are calculated from flows.

Entropy and reliability

Before looking at the correlation between entropy and reliability, it is useful

to recall that the reliability measurers of this chapter and those of Chapter 6 lead to essentially the same conclusion with regard to the reliability of the entropy-constrained designs. The comments of Examples 15, 16 and 17 are based on an implicit assumption that the relative merits regarding reliability of Designs 1 to 6 on one hand and Designs 7 to 11 on the other hand are known beforehand. This assumption is due primarily to the analyses of Chapter 6 in which a fairly good idea of the relative reliabilities of the designs has been obtained. In Chapter 6, a considerable amount of intuition has been used in the reliability assessment of the designs and, as stated in that chapter, the diameter-based statistical measures of mean, standard deviation and coefficient of variation are somewhat qualitative. A more reliable diameter-based measure is the probability $p(0)$ that the network is fully connected. Looking again at the reliability tables in this chapter, it can be seen that $p(0)$ is in line with the above statistical averages and the headloss and energy measures of Chapter 6.

In all, there are three related but rigorous (although not necessarily comprehensive) reliability measures, $p(0)$, R and $\overline{R}$, in this chapter which can be used to rank the entropy-constrained designs, in addition to the measures of Chapter 6. All these measures clearly show that the reliability of a network generally increases as the entropy increases and, as such, entropy is a good surrogate for pipe network reliability. The above conclusion has many important implications for water distribution network reliability and design and some of these have been set out in Chapter 6. Finally, recalling that less-than-maximum entropy designs are not unique, graphs of $p(0)$, R and $\overline{R}$ against entropy have not been shown herein. More points/designs are needed so that scatter diagrams showing the true nature of the above relationships can be obtained.

7.3 GENERAL NETWORKS

Some shortcomings of the demand-driven simulation approach to estimating network reliability of the last section have been noted. The calculations in that section are based on the assumption that the head needed to satisfy all demands can be found. However, an alternative and slightly more elaborate way of doing the calculations is based on the premise that the actual flow supplied at adequate pressure can be calculated and this approach is described next. As seen shortly, the main difference between the equations of this section and Section 7.2 is that the former are based on flows whereas the latter are based on heads. Also, the present equations are more generally applicable.

To determine the flows supplied at the nodes, a head-driven simulation of each state or reduced network is performed. Any suitable equation, for example (7.2) or (7.3), for calculating the q'_{n0}, $\forall n \in D$, which are the actual flows supplied at the nodes at adequate pressure could be used. For each state, the nodal reliabilities R^i_n, $i = 1, ..., N$, are defined as the ratio of the equivalent flow supplied at adequate pressure to the demand at node n, $\forall n \in D$, i.e.,

$$R^i_n = \frac{q'_{n0,i}}{q_{n0}} \quad \forall n \in D;\ i = 1, ..., N \tag{7.20}$$

Eqs. (7.20) are merely a restatement of (7.4b) and the left hand side equality of (7.6); $q'_{n0,i}$, $\forall n$, $\forall i$, is the flow supplied at adequate pressure at node n during state i. Also, following Eqs. (7.4a) and (7.7), the nodal reliabilities R_n are given by

$$R_n = \sum_{i=1}^{N} p_i R^i_n \quad \forall n \in D \tag{7.21}$$

Substituting Eqs. (7.20), this becomes

$$R_n = \frac{1}{q_{n0}} \sum_{i=1}^{N} p_i q'_{n0,i} \quad \forall n \in D \tag{7.22}$$

which is the expected equivalent flow supplied at adequate pressure divided by the demand at the node.

When the network is fully connected, $q'_{n0} = q_{n0}$, $\forall n \in D$, and so the corresponding state reliability is unity for all demand nodes, from Eqs. (7.20). Also, if states with multiple link failures are neglected, then, assuming that link failures are independent, Eqs. (7.22) become

$$R_n = 1\prod_{ij \in IJ} R_{ij} + \frac{1}{q_{n0}}\left[\prod_{ij \in IJ} R_{ij}\right] \sum_{ij \in IJ} (R'_{ij}/R_{ij}) q'_{n0,ij} \quad \forall n \in D \tag{7.23}$$

in which $q'_{n0,ij}$ is the flow supplied at node n, $\forall n \in D$, with link ij, $\forall ij \in IJ$, inoperative. Also, Eq. (7.9) for the probability $p(0)$ that all links are operational and Eqs. (7.10) for the probability $p(ij)$ that only a specified link ij, $\forall ij \in IJ$, is not working have been used in Eqs. (7.23). The nodal reliability is therefore

$$R_n = \left[\prod_{ij \in IJ} R_{ij}\right]\left[1 + \frac{1}{q_{n0}} \sum_{ij \in IJ} (R'_{ij}/R_{ij}) q'_{n0,ij}\right] \quad \forall n \in D \tag{7.24}$$

Also, if the second term of Eqs. (7.23) is normalised by dividing it by the probability that the network is not fully connected, the average nodal reliabilities of the less-than-fully connected configurations are obtained as

$$\overline{R}_n = \frac{\left[\prod_{ij \in IJ} R_{ij}\right] \sum_{ij \in IJ} (R'_{ij}/R_{ij}) q'_{n0,ij}}{q_{n0}(1 - \prod_{ij \in IJ} R_{ij})} \quad \forall n \in D \tag{7.25}$$

which, when substituted in Eqs. (7.24), give

$$R_n = \prod_{ij \in IJ} R_{ij} + \overline{R}_n(1 - \prod_{ij \in IJ} R_{ij}) \quad \forall n \in D \tag{7.26}$$

which, finally, may be rearranged as

$$\overline{R}_n = \frac{R_n - \prod_{ij \in IJ} R_{ij}}{1 - \prod_{ij \in IJ} R_{ij}} \quad \forall n \in D \tag{7.27}$$

It remains to develop similar equations for the reliability R of the network. The analysis for R is similar to R_n, $\forall n \in D$. However, T'_0 is used instead of q'_{n0} and is given by

$$T'_{0,i} = \sum_{n \in D} q'_{n0,i} \quad i = 1, ..., N \tag{7.28}$$

in which $T'_{0,i}$, $\forall i$, is the flow which the network can deliver at adequate pressure for state i. Following Eqs. (7.24), the network reliability is

$$R = \left[\prod_{ij \in IJ} R_{ij}\right]\left[1 + \frac{1}{T_0} \sum_{ij \in IJ} (R'_{ij}/R_{ij}) T'_{0,ij}\right] \tag{7.29}$$

in which $T'_{0,ij}$, $\forall ij \in IJ$, is the total flow that the network can supply at adequate pressure with link ij, $\forall ij \in IJ$, inoperative and is given by Eqs. (7.28). Finally, $\overline{R}$ is given by

$$\overline{R} = \frac{R - \prod_{ij \in IJ} R_{ij}}{1 - \prod_{ij \in IJ} R_{ij}} \tag{7.16}$$

As in Section 7.2, states having multiple inoperative links are not included in the derivations of R and $\overline{R}$ and so these reliability measures are lower bounds. Also, it is worth noting that unlike the last section, the present equations are applicable to multiple-source networks.

The amount of time remaining for the present research is not sufficient to write programs to carry out head-driven simulation and so there are no examples in this section. However, it is evident that provided the head-driven simulation is sound, then, the present flow-based equations are likely to be at least as good as the head-based equations of Section 7.2 which have been shown to lead to realistic measures of reliability. Nevertheless research into the methods proposed in this chapter for calculating reliability is necessary and will be continued.

7.4 SUMMARY AND CONCLUSION

In this chapter a new definition has been suggested for state reliability as the ratio of the flow supplied at adequate pressure to the demand. A state is defined as a specified set of nodal demands with a set of operative and/or inoperative network components. Reliability is obtained by summing the state reliabilities weighted by their respective probabilities of occurrence. In other words, reliability is defined as the ratio of the expected equivalent flow delivered at adequate pressure to the total demand. This new definition of reliability does away with the ambiguities of existing definitions in the literature. However, there is an implicit assumption that all the states during which some service is maintained can be identified and simulated. To this end, systematic implicit exhaustive enumeration can be used to simplify state enumeration and reduce the amount of simulation. What this means in essence is that, if a state involving certain failed components is found to be sufficiently improbable to be negligible, then, other states having the same (and additional) failed components need not be considered as these sates have even lower probabilities of occurrence. Also, if a state is found to have a negligible value of $p_i R^i$, then, other states having these (and other) failed components need not be simulated as their respective values of $p_i R^i$ are even lower. Such an implicit exhaustive enumeration scheme can be implemented without too much difficulty if it is carried out systematically.

Referring to the desirable properties that have been identified in Chapter 3 for a reasonable measure of reliability, first, the proposed measure reflects the actual flow supplied. Second, the probabilistic nature of component failure is incorporated in the measure. Third, pressure dependency of flow is recognised and, furthermore, less-than-fully satisfactory service is accounted for. Last, the present method is computationally expensive due to the necessity for simulation.

There are two notable features in the present approach. One of these features is that the difficulties posed by the interdependencies between the demand node reliabilities have successfully been dealt with. For single-source networks, these difficulties have been avoided by basing the reliability calculations on the pressures at the source node. For general networks, network reliability is computed using the sum of the flows supplied at the nodes. This is in contrast to existing approaches which determine network reliability by heuristically averaging the nodal reliabilities, and this is theoretically questionable.

The other important aspect of the present approach is its comprehensive treatment of subnormal service. The quantification of subnormal service in the context of reliability and design is useful as it permits different designs to be compared on the same basis. Unlike the present measure, such a comparison would be difficult if measures such as the percentage of flow supplied at adequate pressure or the probability of sufficient supply were to be used. These measures are based solely on the amount of time that the supplied flows equal or exceed the demands, or the nodal heads equal or exceed the minimum desirable service pressure (Su, Mays, Duan and Lansey, 1987; Wagner, Shamir and Marks, 1988a; Bao and Mays, 1990). In spite of the reservations expressed in the introduction to this chapter, the probability of sufficient supply is an indication of the percentage of time that the network can be expected to function satisfactorily. However, the present reliability measure R is an indication of the expected amount of flow that can be withdrawn from the network at any given time and, as such, would be expected to be qualitatively superior. Furthermore, $\overline{R}$ provides an estimate of the expected proportion of the total demand than can be met when there is a mechanical failure in the distribution system. Together, R and $\overline{R}$ provide a more accurate picture of water network reliability than hitherto.

It has been assumed herein that the reliabilities of pipes can somehow be determined. However, more serious questions have to be answered if flow-based states are to be included in the reliability calculations. As an example, the ability to meet fire fighting demands is important, but these demands may have low probabilities of occurrence. One might therefore wonder whether it would not be more informative to do additional reliability calculations based on fire fighting flows only. In other words, there would be perhaps two R values, one based on normal demands and the other based on fire fighting demands. Also, if necessary, a possible means of incorporating flow pattern-dependent states in the reliability measures would appear to be as follows. First, the major demand patterns would be identified. Second, these patterns could be weighted by some suitable means including their respective probabilities of occurrence. Third, the reliability could be calculated for each pattern as in Section 7.3. Last, the overall reliability could then be taken to be the sum of the reliabilities of the flow patterns weighted according to their respective weights.

Further work including verification that the conclusions of Examples 14 to 17 are not network specific is necessary. Also, it would be instructive to compare values of reliability based respectively on demand- and head-driven simulation. Essentially, the achievement of this chapter is that it has shown how the various aspects of reliability may be combined to give a realistic measure of reliability. Some refinement may be necessary, but the basic elements of the framework have been established.

Table 7.1 Optimum diameters (mm) for the network of Fig. 7.1 (Exs. 14 to 16)

Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	401	401	390	384	365	367
2-4	100	100	165	191	238	235
3-5	338	337	337	329	281	294
4-6	100	100	100	151	250	234
5-6	263	262	262	249	152	185
1-2	157	165	203	224	263	261
3-4	237	237	213	215	247	234
Mean	228	229	239	249	257	258
σ_{n-1}	116	115	100	81	63	58
$\frac{\sigma_{n-1}}{mean}$	0.510	0.504	0.419	0.325	0.245	0.224
Cost	0.250	0.251	0.254	0.259	0.261	0.263
Design	1	2	3	4	5	6

Costs are in £ 10^6.

Table 7.2 Optimum flows (m^3/s) for the network of Fig. 7.1 (Exs. 14 to 16)

Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	0.253	0.251	0.234	0.223	0.199	0.200
2-4	0.003	0.005	0.022	0.033	0.057	0.056
3-5	0.147	0.147	0.147	0.139	0.101	0.110
4-6	0.001	0.001	0.001	0.009	0.047	0.038
5-6	0.055	0.055	0.055	0.047	0.009	0.018
1-2	0.031	0.033	0.050	0.061	0.085	0.084
3-4	0.073	0.072	0.054	0.051	0.065	0.057
Design	1	2	3	4	5	6

Table 7.3 Usable head (m) to satisfy all design demands (Fig. 7.1, Exs. 14 to 16)

Failed Link	Network Entropy					
	1.578	1.600	1.700	1.800	1.900	1.915
1-3	8817.9	8454.6	1065.2	567.7	219.3	231.6
2-4	20.6	20.7	28.0	27.6	35.7	34.7
3-5	2896.2	2895.5	2899.2	453.9	215.5	141.9
4-6	20.2	20.4	20.4	22.6	87.7	47.5
5-6	500.8	500.3	501.4	92.2	22.7	26.6
1-2	158.3	155.9	54.1	43.9	46.3	46.2
3-4	267.9	259.5	75.4	37.2	39.8	33.5
Mean	1811.7	1758.1	663.4	177.9	95.3	80.3
Max	8817.9	8454.6	2899.2	567.7	219.3	231.6
Design	1	2	3	4	5	6

Table 7.4 Reliability measures for the network of Fig. 7.1 (Ex. 15)

Design	Entropy	$R_{ij} = 0.95 \quad \forall ij$		$R_{ij} = 0.99 \quad \forall ij$	
		R	$\overline{R}$	R	$\overline{R}$
1	1.578	0.80638	0.35816	0.95974	0.40739
2	1.600	0.80642	0.35828	0.95975	0.40752
3	1.700	0.822	0.412	0.964	0.468
4	1.800	0.848	0.495	0.970	0.563
5	1.900	0.851	0.506	0.971	0.573
6	1.915	0.859	0.533	0.973	0.606

Table 7.5 Pipe reliabilities for the network of Fig. 7.1 (Ex. 16)

Link	Design 1 (Entropy = 1.578)			Design 6 (Entropy = 1.915)		
	Dia (mm)	R (Eq. (5.17))	R (Eq. (5.18))	Dia (mm)	R (Eq. (5.17))	R (Eq. (5.18))
1-3	401	0.999865	0.930800	367	0.999850	0.925563
2-4	100	0.999306	0.722742	235	0.999746	0.888189
3-5	338	0.999834	0.920141	294	0.999805	0.909567
4-6	100	0.999306	0.722742	234	0.999745	0.887721
5-6	263	0.999777	0.899680	185	0.999663	0.857715
1-2	157	0.999592	0.830954	261	0.999775	0.898951
3-4	237	0.999748	0.889112	234	0.999745	0.887721

Table 7.6 Reliability measures for the network of Fig. 7.1 (Ex. 16)

Entropy	Based on Eq. (7.17)			Based on Eq. (7.18)		
	$p(0)$	R	$\overline{R}$	$p(0)$	R	$\overline{R}$
1.578	0.99743	0.999081	0.642277	0.29737	0.564747	0.380536
1.600	0.99745	0.999092	0.643524	0.30029	0.568503	0.383321
1.700	0.99781	0.999224	0.646511	0.35610	0.621047	0.411475
1.800	0.99815	0.999386	0.668312	0.41967	0.678602	0.446184
1.900	0.99827	0.999392	0.648580	0.44231	0.690947	0.445835
1.915	0.99833	0.999441	0.665402	0.45434	0.706100	0.461389

Table 7.7 Optimum designs for the network of Fig. 7.2 (Ex. 17)

Link	Network Entropy									
	2.170		2.500		2.750		2.775		2.800	
	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)
1-2	201	0.047	309	0.119	294	0.104	294	0.104	294	0.104
2-3	156	0.023	161	0.024	225	0.049	183	0.031	201	0.038
1-4	349	0.161	273	0.089	294	0.104	294	0.104	294	0.104
2-5	100	0.003	267	0.075	197	0.034	234	0.052	221	0.045
3-6	100	0.002	100	0.003	196	0.029	137	0.010	164	0.017
4-5	317	0.116	222	0.044	197	0.034	234	0.052	221	0.045
5-6	151	0.020	286	0.078	179	0.023	222	0.042	207	0.035
4-7	156	0.023	155	0.024	225	0.049	183	0.031	201	0.038
5-8	286	0.079	152	0.019	179	0.023	222	0.042	207	0.035
6-9	100	0.001	272	0.061	216	0.031	216	0.031	216	0.031
7-8	100	0.003	100	0.004	196	0.029	137	0.010	164	0.017
8-9	271	0.061	100	0.002	216	0.031	216	0.031	216	0.031
Mean	191		200		218		214		217	
σ_{n-1}	92		80		39		50		41	
$\frac{\sigma_{n-1}}{mean}$	0.484		0.401		0.178		0.232		0.187	
Cost	0.277		0.282		0.292		0.290		0.292	
	Design 7		Design 8		Design 9		Design 10		Design 11	

Costs are in £ 10^6.

Table 7.8 Usable head (m) to satisfy all design demands (Fig. 7.2, Ex. 17)

Failed Link	Network Entropy				
	2.170	2.500	2.750	2.775	2.800
1-2	109.7	109.5	85.5	76.5	78.9
2-3	115.8	98.9	46.6	45.3	40.6
1-4	1248.6	56.0	79.8	74.8	76.2
2-5	23.9	59.1	30.0	38.3	34.3
3-6	24.1	24.1	34.0	25.8	28.2
4-5	360.2	35.2	30.0	38.3	34.3
5-6	48.7	329.3	28.7	39.2	33.9
4-7	98.5	114.4	46.6	45.4	40.6
5-8	333.9	46.9	28.7	39.2	33.9
6-9	28.3	660.3	42.9	41.1	41.7
7-8	24.0	24.3	34.0	25.8	28.2
8-9	664.1	24.0	42.9	41.1	41.7
Mean	256.3	131.8	44.16	44.23	42.7
Max	1248.6	660.3	85.5	76.5	78.9
Design	7	8	9	10	11

Table 7.9 Reliability measures for the network of Fig. 7.2 (Ex. 17)

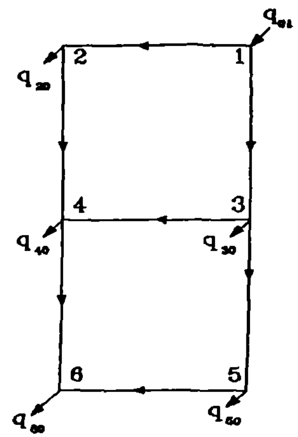
Design	Entropy	$R_{ij} = 0.95 \quad \forall ij$		$R_{ij} = 0.99 \quad \forall ij$	
		R	$\overline{R}$	R	$\overline{R}$
7	2.170	0.734	0.422	0.9474	0.5370
8	2.500	0.757	0.471	0.9546	0.6000
9	2.750	0.802	0.569	0.9687	0.7249
10	2.775	0.799	0.562	0.9677	0.7155
11	2.800	0.804	0.573	0.9693	0.7297

Table 7.10 Pipe reliabilities for the network of Fig. 7.2 (Ex. 17)

Link	Design 7 (Entropy = 2.170)			Design 11 (Entropy = 2.800)		
	Dia (mm)	R (Eq. 5.17))	R (Eq. 5.18))	Dia (mm)	R (Eq. 5.17))	R (Eq. 5.18))
1-2	201	0.999695	0.869318	294	0.999805	0.909567
2-3	156	0.999588	0.829799	201	0.999695	0.869318
1-4	349	0.999840	0.922318	294	0.999805	0.909567
2-5	100	0.999306	0.722742	221	0.999727	0.881206
3-6	100	0.999306	0.722742	164	0.999612	0.838602
4-5	317	0.999821	0.915508	221	0.999727	0.881206
5-6	151	0.999572	0.823768	207	0.999705	0.873155
4-7	156	0.999588	0.829799	201	0.999695	0.869318
5-8	286	0.999798	0.907246	207	0.999705	0.873155
6-9	100	0.999306	0.722742	216	0.999719	0.878464
7-8	100	0.999306	0.722742	164	0.999612	0.838602
8-9	271	0.999785	0.902476	216	0.999719	0.878464

Table 7.11 Reliability measures for the network of Fig. 7.2 (Ex. 17)

Entropy	Based on Eq. (7.17)			Based on Eq. (7.18)		
	$p(0)$	R	$\overline{R}$	$p(0)$	R	$\overline{R}$
2.170	0.994923	0.998583	0.720841	0.093018	0.277135	0.203000
2.500	0.995388	0.998784	0.736462	0.115770	0.325679	0.237391
2.750	0.996564	0.999266	0.786386	0.203985	0.477182	0.343206
2.775	0.996359	0.999215	0.784482	0.185436	0.450748	0.325711
2.800	0.996530	0.999271	0.790024	0.200874	0.474216	0.342052



L_{ij} = 1000m, C_{ij} = 130; for all ij.

Figure 7.1 Network on which Examples 14, 15 and 16 are based

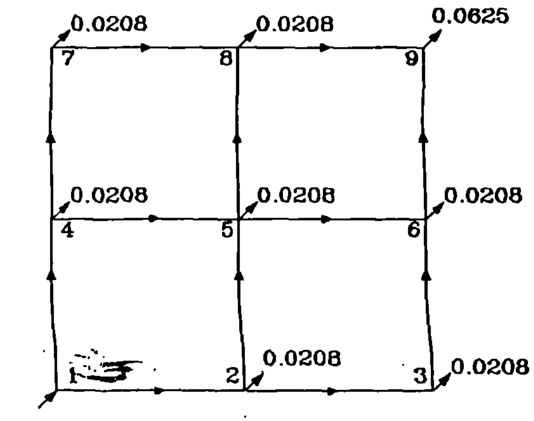


Figure 7.2 Network on which Example 17 is based:
all flows are in m^3/s ; $L_{ij} = 1000m$; $C_{ij} = 130$; $\forall ij$.

CHAPTER 8 CONCLUSION

8.1 INTRODUCTION

Data estimation forms an integral part of the planning, design and operation of civil engineering projects. This process of data estimation presently involves a great deal of guesswork. Recent research (Basu and Templeman, 1985) has demonstrated that it is inappropriate to base modern design on arbitrary assumptions and has *thereby highlighted the need for consistent methodologies* and a logical approach to data estimation. The present research has explored the possibility of using Shannon's informational entropy and Jaynes' maximum entropy formalism in the solution of general problems with incomplete data which are not directly related to probabilities. The problem of inferring least *biased flows in networks* has been considered. This has led to further investigations into possible uses of flow entropy and the maximum entropy formalism in the optimum design and reliability analysis of water distribution networks.

Urban water supply systems should be designed to be reliable in operation. This need for reliability has traditionally been satisfied by using heavily redundant pipe networks which have large diameter pipes and many loops. The loops ensure that there is in theory more than one supply route for each demand node. Large diameter pipes provide the network with the necessary resilience to cope with a wide range of flows which the network was not specifically designed to carry. Unfortunately, it is a reality that an over-redundant network is very expensive to construct. Moreover, in the traditional approach to the design of water systems, cost is not a primary consideration.

For better control over the capital cost of water distribution systems, mathematical programming techniques have been used to find the cheapest

design that can satisfy the constraints of the distribution system. However, it is becoming increasingly clear that a straightforward cost minimization approach is not suitable for water distribution systems. Optimization, by its very nature, tends to remove redundancy and so any spare capacity which is not required by the design demand pattern is removed. Complete removal of redundancy is not desirable as it reduces network resilience. Since reliability of water supply and the need to use public funds more economically are both important, a compromise between cost and reliability is necessary.

One way of making sure that some redundancy is retained is to explicitly consider a multiplicity of demand patterns in the cost minimization process. The main disadvantage of this approach is that each additional demand pattern considerably increases the computational complexity of solving the optimization problem. An alternative to explicitly considering a multiplicity of demand patterns is to design the network to meet explicit reliability requirements. However, reliability is not easy to define and the difficulty of calculating it increases as its definition becomes more realistic. The difficulty of calculating reliability is one reason that it is extremely difficult to optimize both the reliability and cost of a water supply network.

The approach adopted herein to strike a balance between cost and reliability is to use flow entropy as a surrogate for reliability. This novel approach enables the cost of the system to be minimized without the redundancy being completely removed. The following are some of the other advantages of the present approach. From a computational viewpoint, flow entropy is easy to calculate and easy to incorporate in optimization processes. A practical implication of the above properties is that entropy can be used in a design framework. Additionally, using an entropy constraint, as in Problem 10, increases the computational effort only marginally. As such, the entropy constraint, by itself, will not in general restrict the size of the network that can be handled. Moreover, evidence has been presented herein suggesting that with an entropy constraint, the size of the optimum design problem could be reduced significantly in a routine manner. From a reliability perspective, the essence of entropy is uncertainty, and reliability centres around uncertainty. While it is difficult to state precisely the nature of the relationship between entropy and reliability, the present research has shown that entropy generally has the desired properties of a reasonable reliability measure.

8.2 SUMMARY OF THE PRESENT RESEARCH

The fact that reliability is difficult to calculate has led to the search herein for an easily calculable surrogate reliability measure. Network flow entropy has most of the desirable computational properties and appears to possess the qualities of a useful reliability measure. Network reliability and entropy are related through uncertainty. Moreover, the higher the value of entropy for a set of nodal inflows and outflows, the more uniform the link flows of the network are. Also, the more uniform a flow distribution is with respect to all the distributions capable of satisfying flow equilibrium throughout the network, the smaller the potential changes in the link flows are, and, in consequence, the smaller the increase in headloss due to any changes in pipe flow rates. Since excessive headloss adversely affects network performance, the above properties are central to the ability of flow entropy to act as a good surrogate for water network reliability. Furthermore, from a design standpoint, the uniformity of pipe diameters is largely determined by the uniformity of the pipe flow rates, and this uniformity of pipe diameters has been shown herein to be vital in providing a network with resilience to carry flows other than those which the network was specifically designed to carry.

The maximum entropy formalism has been used herein to infer the least biased flows in a general network in which only the flow directions and nodal inflows and outflows are specified. One of the difficulties faced has been how to introduce in a rigorous way probabilities to water networks. This problem has

been overcome by using the relative frequency interpretation of probabilities. It is perhaps possible to visualise this relative frequency interpretation in the present context by imagining that routing flow through a water distribution network is a continuous experiment. Another difficulty is the fact that the maximum entropy formalism is based on the Shannon entropy function which was undefined for networks, prior to the present research. This problem has been addressed by using the conditional entropy formula for the entropy of compound probability schemes to derive rigorously a general formula for the flow entropy of networks in which the direction of flow is specified.

Equations have been derived for the maximum entropy value of parallel networks in general and fully-connected parallel networks in particular. The gravity model for calculating maximum entropy flows in parallel networks has been around for some time, but by deriving closed-form solutions for the maximum entropy value, the present research has gone one step forward. The advantage of using the result (5.72) for fully-connected parallel networks is that it involves fewer $p \ln p$ terms than Eq. (5.71). The former has $(NI + NO)$ terms whereas the latter has $(NI \times NO)$ terms for fully-connected networks, where NI and NO are the number of source and demand nodes respectively. Also, analytical results derived herein using the present multiple probability space entropy model are identical to known results based on the existing single probability space entropy model. In other words, all existing closed-form results for parallel networks have been rederived using the present multiple-space model.

Maximum entropy flows have been characterised as the most uniform flows with respect to all the sets of flows which can satisfy the demands of the network. Also, it has been shown that maximum entropy flows for single-source networks are such that each demand node receives equal proportions of the demand at that node from each of the paths serving the node. The above property of maximum entropy flows for single-source networks has been used herein to develop a simple but rigorous and efficient algorithm for calculating maximum entropy flows for single-source networks. The above-mentioned algorithm has several practical and computational advantages which are explained in Chapters 5 and 6. Some of the notable properties of the algorithm are that it is conceptually simple, computationally efficient, its computer implementation is straightforward, the calculations involved are not iterative and mathematical programming techniques are not required.

The entropy-constrained approach has been used herein to introduce entropy to the least cost pipe network design problem. It is shown in Chapters 6 and 7 that, in general, the resilience/reliability of a network increases as the flow entropy increases. As such, it is concluded that flow entropy is a good surrogate for water network reliability. Awumah et al. (1990, 1991, 1992) reached essentially the same conclusion, but the present work is more rigorous in several respects and establishes the entropy-based reliability research on a firmer base. Also, it is demonstrated in Chapters 6 and 7 that designing networks to carry maximum entropy flows does confer a considerable amount of all-round resilience on the network. Moreover, designing networks to carry maximum entropy flows considerably simplifies the optimum design problem by eliminating the flow variables and flow equilibrium constraints. In effect, the problem is reduced to the problem of sizing the pipes for prespecified flows, and this can be accomplished by linear programming. In this regard, the algorithm developed herein for calculating maximum entropy flows in single-source networks is particularly useful.

Another important outcome of the introduction of entropy to the optimum design problem is that the designer can exercise control over the optimization process. It is seen that the degree of implicit tree-type branchedness of network designs reduces as the value of the flow entropy increases. The entropy constraint increases the resilience/reliability of a network by making the pipes

more uniform and larger. Also, as entropy increases, it becomes easier to predict which nodes and links will be critical. Even the flow rerouting properties become more obvious. The above properties of entropy-constrained designs may have profound implications for both reliability analysis and the optimum design problem. Regarding design, the number of link failure and fire fighting flow patterns that need to be considered explicitly in the optimization problem can be reduced considerably if it is known which nodes and links are most likely to be critical. Regarding reliability analysis, the size of the combinatorial problem involved in an analytical approach in which simulation is not used, for example, Wagner, Shamir and Marks (1988a), can be reduced considerably if the flow rerouting behaviour of the network is known. There is therefore a need for further research into the behaviour of entropy-constrained least cost designs and the maximum entropy least cost design in particular.

To compare rigorously the entropy-constrained least cost designs generated herein, a new, sharper definition of reliability has been suggested. Reliability has been defined as the ratio of the expected equivalent flow delivered (at adequate pressure) to the total demand. Two ways of calculating reliability according to the above definition have been put forward. Two notable features of the present approach are that simple ways to resolve the difficulties posed by the interdependencies between demand node reliabilities have been found, and subnormal service is accounted for. The above aspects are best explained by considering single- and multiple-source networks in turn.

For single-source networks, network reliability calculations are based on the usable pressure head at the source node. A formula based on some of the ideas of Wagner, Shamir and Marks (1988b) has been developed herein for calculating how much flow can be supplied when service is subnormal. To determine the flow supplied when service is subnormal, the actual usable head at the source is divided by the notional usable head at the source that would enable all demands to be satisfied. The product of the total demand and the square root of the above ratio gives the actual flow supplied when service is not fully satisfactory. Thus, by basing the calculations on pressures at the source node, the difficulties posed by interdependencies between demand node reliabilities, which are discussed in Section 7.1, are avoided. Also, contrary to existing approaches, subnormal service is fully accounted for in the calculations. It is possible to quantify subnormal service in the above calculations because of a subtle reversal herein of the roles of normal and subnormal service.

For multiple-source networks, a more general approach has been used. The actual flow supplied when service is subnormal is determined using head-driven simulation. The total supply is the sum of the actual nodal abstractions, and it is this total supply that is used in the calculations for network reliability. Using the total supply in this way differs from current practice in which network reliability is obtained by averaging the reliabilities of the individual nodes. Therefore, it can be seen that the present approach avoids the difficulties posed by interdependencies between demand node reliabilities.

An allied definition of reliability is based on the notion of state reliability put forward herein. A state has been defined as a specified set of nodal demands with a set of operative and/or inoperative network components. State reliability has been defined as the ratio of the flow supplied at adequate pressure to the demand. Network reliability is obtained by summing the state reliabilities weighted by their respective probabilities of occurrence. One advantage of using state reliability to define network reliability is that it allows the freedom to chose the states to suit the circumstances. In the above way, some of the conceptual difficulties associated with the quantification of reliability are eliminated. Also, it could be said that a partial answer to the question of what reliability means has been provided. The problem of

calculating reliability is thus reduced to the practical problem of enumerating states, simulating these states and calculating their probabilities of occurrence. Lastly, an attractive feature of the present definition of reliability is that the definition is effectively a statement of the quantity of flow or the fraction of the demand that the network can be expected to supply at any given time.

In conclusion,

1. The present research has provided further evidence that Shannon's entropy and Jaynes' maximum entropy formalism may extend to much wider fields than traditional applications would suggest.
2. Flow entropy has the desirable computational properties and would appear to have the necessary qualitative properties of a reliability measure. As such, flow entropy provides a quick and easy means of estimating the level of reliability in a water network.
3. A convenient way of making sure that a network has a high level of all-round resilience is to calculate the maximum entropy flows for the network and then size the pipes to carry those flows.
4. For a water distribution network, state reliability has been defined as the ratio of the flow supplied to the flow required for that state. Network reliability can be obtained by summing the state reliabilities weighted by their respective probabilities of occurrence. Calculated in the above manner, network reliability can be made as comprehensive as desired.

8.3 SUGGESTIONS FOR FUTURE RESEARCH

The present research has provided answers to various questions relating to network flow entropy and water supply networks. However, many aspects have not been explored herein, and some of these are discussed next. As a result of this research, it may be easier to find solutions to some of the problems posed below.

Calculating maximum entropy flows in networks with pumps, valves, etc.

Flow entropy has been defined herein solely on the link flows, without regard to pressure. As such, it appears that there would be in principle no great difficulty in calculating maximum entropy flows in the presence of such network components as pumps, valves, reservoirs, storage tanks, etc. However, information regarding the statuses of some of these additional components would be needed. As an example, it would be necessary to know whether any valves are open or closed. Otherwise, the problem may take on a combinatorial nature. If this is the case, then, maximum entropy flows should be found for each possible state, the state having the highest value of maximum entropy selected as the most probable state and the maximum entropy flows for that state taken as the maximum entropy flows for the network.

Also, if there are at least two reservoirs or tanks and/or pumps, then, the inflow or outflow to or from some of these sources may not be known. In such a case, the value of the entropy of the distribution of the demands or source supplies may not be known. Consequently, the entropy of the sources (or demands), S_0^s (or S_0^d), should generally be included in the entropy function when inferring least biased flows for a network having two or more pumps and/or tanks. It may also be noted that there may actually be a series of maximum entropy flows for practical networks, due to variations in demands, the water levels in service reservoirs, etc., over a 24-hour period, for example. Whether it is desirable or not to somehow aggregate these maximum entropy values is one question that should be addressed. With components other than pipes, however, it is not known whether flow entropy would still be a good surrogate for reliability and this is an area for further research.

Energy entropy

The relative frequency interpretation of probabilities is a means of introducing probabilities to phenomena which are not directly related to probabilities. If

pipe flow rates can be cast in a probabilistic light, perhaps the same can be done with the rate of energy dissipation, for example. Possible uses of flow entropy have been demonstrated in this thesis in the context of water supply. There may be other applications of entropy in other networks. The definitions and uses of other entropies are worth investigating. In particular, the "entropy" of the rate at which energy is dissipated in a network may turn out to be an even better reliability measure and design tool for water networks than flow entropy. It has been seen in Chapters 6 and 7 that the network assessments based on the amount of energy dissipated have been broadly similar to those based on headloss. Also, it has been explained herein why the uniformity of pipe diameters and flows is desirable. Since energy is a function of both flows and diameters, perhaps "energy entropy" could be used in the optimum design problem to give even more uniform flows and diameters than flow entropy.

Calculating maximum entropy flows in networks

Problem 7 considerably simplifies the problem of calculating maximum entropy flows in networks. The formulation of Problem 7 is for a specified set of flow directions. For a network with *NIJ* links there are approximately 2^{NIJ} sets of flow directions. While the flows in most links can be in either direction, a few links, for example, those connected to a primary source, can carry flow in only one direction in practice. Suppose it is required to calculate maximum entropy flows in a network without the flow directions being known? This is a far more difficult problem to solve than Problem 7. Presently, perhaps the only obvious way of inferring least biased flows in networks in which the flow directions are not known is to solve Problem 7 for each possible set of flow directions and then select the set of maximum entropy flows having the greatest value of (maximum) entropy. However, with such an exhaustive enumeration approach, the size of the problem is prohibitive for networks of realistic size. To put the scale of the problem in perspective, a small network with 20 links has approximately $2^{20} = 1.04857 \times 10^6$ sets of flow directions. An ambitious research objective may therefore be, for example, to develop a general methodology and algorithm for calculating maximum entropy or least biased flows in general networks with arbitrary topologies.

A more modest goal, however, is the development of a general method for calculating maximum entropy flows, given a set of flow directions, which does not require mathematical programming. Such a method would be extremely useful. The simple algorithm developed herein for calculating maximum entropy flows is generally incapable of handling multiple-source networks. If such an algorithm could be developed for multiple-source network, then, it would be possible to use standard, robust linear programming routines to design distribution networks without relying on non-linear programming to determine the maximum entropy flows.

A prerequisite to the development of an alternative approach to non-linear programming for calculating maximum entropy flows in multisource networks is a detailed study of the general properties of maximum entropy flows. The present research has rendered possible such a study. This study would establish some criteria that could be used for terminating a possible iterative scheme for calculating maximum entropy flows in multiple-source networks. Presently, it is not obvious how it would be possible to know that the maximum entropy flows have been found without some quantifiable properties of the flows being known. In this regard, it may be noted that it would not be sufficient to test for convergence of both the flows and the entropy function. Premature convergence of the flows may lead to a false impression of convergence of the entropy function even though this apparent convergence of the entropy function may be solely attributable to the fact that the flows themselves are hardly changing. It can therefore be seen that the problem of developing an alternative method of calculating maximum entropy flows in general networks may not be as simple as it might appear at first sight.

Faced with the difficulties posed by possible premature convergence, another approach may be worth considering. It seems that some progress can be made by seeking a heuristic which is known to give flows which approximate to maximum entropy flows most of the time. Perhaps a possible line of attack is to somehow find the proportions of the total flow arriving at a node that originate from each source, route the flow from each source or the flow to each node, and then superpose these routed flows. Perhaps the microflow model of Jowitt and Xu (1993), which can also be given a maximum entropy interpretation, could be useful in this regard. An obvious snag, however, is that the above-mentioned model has been specifically developed for use when the values of the link flows under normal operating conditions are known. On the other hand, in the problem of inferring least biased flows in networks, no such information is available.

Joint pipe diameter, flow directions, layout and reliability optimization

Preliminary calculations during the present research show that flow entropy could also be used as a measure of "goodness" of layout. Pipe diameter, layout and reliability/performance are so intimately related that they cannot be separated without the resulting design being suboptimal. There is no method capable of solving the joint flow directions, layout, pipe diameter and reliability optimization problem, and entropy appears to offer a way forward. However, there are some computational difficulties associated with layout optimization. One of these difficulties is that in the loop-based pipe network equations, the loop constraints, together with the minimum diameter constraints, do not allow pipes to be eliminated during the optimization process. The minimum diameter constraints ensure that no pipe diameter is reduced to a value less than D_{min}. Omission of the minimum diameter constraints, however, allows some pipes to be eliminated completely, resulting in a network with a tree-like configuration. Completely eliminating links in this way creates difficulties for optimization algorithms as any loop which is opened ceases to exist, yet the loop constraint for the presently non-existent *loop* remains in the constraint set. As such, the said loop constraint is automatically violated. For this reason, the loop-based formulations are not suitable for layout optimization.

Luckily, the H-equations, which are the system of constitutive equations described by Eqs. (2.12) in which the variables are the nodal heads, are an alternative to the loop-based formulation of Problem 10. The H-equations, Eqs. (2.12), describe flow in pipe networks without recourse to loop or path equations. Therefore, using the H-equations to formulate the optimum design problem would bypass the optimization difficulties described above regarding the impossibility of satisfying loop constraints for loops that become non-existent as optimization proceeds. Moreover, looking at Eqs. (2.12), there appear to be no immediately obvious reasons for which either the diameters or the nodal heads cannot be zero. As such, it appears that there would be no problems arising from the complete removal of any link, or the elimination of any node if, for example, the node is neither a source nor demand node. It follows that the H-equations appear suitable for use in a layout optimization framework.

With regard to redundancy, optimization using the H-equations would still result in the optimum design being a branched network in the absence of reliability or minimum diameter constraints. However, it has been shown herein that entropy does assure the retention of some redundancy. The question of how to calculate flow entropy with the H-equations springs up, and a possible answer is as follows. Referring once more to Eqs. (2.12), it can be seen that each term in the summation is a pipe flow rate. Furthermore, it can be determined from the sign of each flow whether the flow is an inflow or an outflow. As such, network entropy can be calculated since all the link flows and the total flow reaching each node can be determined. It is evident from the above discussion that the H-equations, in conjunction with flow entropy, hold out the possibility of joint layout and pipe diameter optimization. Joint

layout, pipe diameter and surrogate reliability optimization along the lines suggested above is well worth investigating. There appear to be no satisfactory procedures for solving the layout aspects of distribution network design.

Accurate reliability measures

The reliability measure proposed in this thesis allows the inclusion of various network components including pipes, pumps, valves and reservoirs. However, there is an implicit assumption that the mechanical unreliability of each component can be calculated. There appears to be a small amount of uncertainty in general about how these unreliabilities should be calculated. More research would probably shed some light on this issue. It can be conjectured that the maximum entropy formalism could be used to infer least biased distributions for pipe unreliability using pipe breakage data. Being maximally noncommittal with respect to unavailable data, the above-mentioned maximum entropy distribution would probably be more dependable than a distribution based on regression analysis.

Another important matter is the way in which the expected equivalent flow measure proposed herein could be used to calculate the reliability of a network having multiple flow patterns. A related problem is the calculation of reliability using extended-period simulation. Some possibilities have been suggested in Chapter 7. The approach used by Cullinane, Lansey and Mays (1992) for calculating extended-period network reliability may also provide some useful ideas. Furthermore, it has been suggested in Chapter 7 that the ability of a network to meet fire fighting demands at each node should be quantified separately, in view of the low probabilities of occurrence of these demands. Further work is necessary to determine both the probabilities that should be used in these calculations and whether it is necessary to compute and report fire fighting reliabilities separately.

Two ways of calculating reliability have been suggested in Chapter 7, using demand- and head-driven simulation respectively. It would be instructive to compare values of reliability given by the two approaches. Both approaches should give similar results, but any discrepancies would probably imply the existence of gaps in the understanding of the behaviour of pipe networks regarding the pressure dependency of nodal abstraction in general and subnormal service in particular. Also, it has been difficult previously to compare reliability measures, partly because of the inability to quantify subnormal service. Ways of resolving the above difficulty have been suggested in Chapter 7 and, as a result, it may be easier to carry out comparative assessments of reliability measures. The results of such comparative studies may help to determine the direction which reliability analysis should follow in the future.

Algorithm development and practical design programs

The focus of this research has been the determination, and demonstration of the suitability, of a possible surrogate reliability measure, flow entropy. Many aspects regarding the development of algorithms for Problems 7, 8 or 9, and Problem 10 have not been explored. The development of special numerical algorithms for the above problems is equally important and more work is needed in this area.

It has been mentioned herein that the difference in cost between a minimum diameter-constrained least cost network design and a maximum entropy-constrained least cost network design may be generally of the order of 5%. It has also been suggested that this difference in cost may be less for larger networks. More extensive research is needed to confirm the above conjecture. The recommendation herein that networks should be designed to carry maximum entropy flows is based on the premise that the maximum entropy design will not generally be much more expensive than alternative designs.

Linear programming, together with the suggestions herein for reducing the size of Problem 10, could be useful in designing networks of realistic size to carry maximum entropy flows, provided those flows can be calculated. With pipe networks, some post-design analysis is generally necessary. However, there is some evidence herein that the hydraulic behaviour of networks designed to carry maximum entropy flows may be quite predictable. Verification of this property on a wider scale is needed, and, if confirmed to be generally true, the amount of post-design analysis of a network designed to carry maximum entropy flows would be reduced significantly.

In broad terms, there appear to be no great conceptual difficulties in implementing most of the ideas, concepts and conclusions of this research. With reference to the computations for the results presented herein, only the analytical derivatives of the entropy function for the two-loop network used in Example 12 are network specific. Although a general expression for the derivatives of the entropy function has not been presented, this expression can be derived and would further enhance the efficiency of the entropy-constrained least cost design approach. Except the analytical derivatives of the entropy function for the network of Example 12, all the computational results presented herein have been obtained using general programs which are not network specific. On the IBM 3081 mainframe, the amount of computer time needed to solve the optimization problems and calculate network reliability is trivial, for the sample networks considered in this thesis. Generally, it is felt that no serious problem-specific difficulties would be encountered in developing practical design programs, from a purely computer programming perspective. However, aspects such as the development of algorithms capable of performing the necessary optimization for networks of realistic size may be more problematic. Also, other important aspects of program development such as data input and manipulation, user-friendliness, the presentation of results, graphics, etc., may be difficult, but these concerns are not specific to the present research.

CHAPTER 8 CONCLUSION

8.1 INTRODUCTION

Data estimation forms an integral part of the planning, design and operation of civil engineering projects. This process of data estimation presently involves a great deal of guesswork. Recent research (Basu and Templeman, 1985) has demonstrated that it is inappropriate to base modern design on arbitrary assumptions and has thereby highlighted the need for consistent methodologies and a logical approach to data estimation. The present research has explored the possibility of using Shannon's informational entropy and Jaynes' maximum entropy formalism in the solution of general problems with incomplete data which are not directly related to probabilities. The problem of inferring least biased flows in networks has been considered. This has led to further investigations into possible uses of flow entropy and the maximum entropy formalism in the optimum design and reliability analysis of water distribution networks.

Urban water supply systems should be designed to be reliable in operation. This need for reliability has traditionally been satisfied by using heavily redundant pipe networks which have large diameter pipes and many loops. The loops ensure that there is in theory more than one supply route for each demand node. Large diameter pipes provide the network with the necessary resilience to cope with a wide range of flows which the network was not specifically designed to carry. Unfortunately, it is a reality that an over-redundant network is very expensive to construct. Moreover, in the traditional approach to the design of water systems, cost is not a primary consideration.

For better control over the capital cost of water distribution systems, mathematical programming techniques have been used to find the cheapest

design that can satisfy the constraints of the distribution system. However, it is becoming increasingly clear that a straightforward cost minimization approach is not suitable for water distribution systems. Optimization, by its very nature, tends to remove redundancy and so any spare capacity which is not required by the design demand pattern is removed. Complete removal of redundancy is not desirable as it reduces network resilience. Since reliability of water supply and the need to use public funds more economically are both important, a compromise between cost and reliability is necessary.

One way of making sure that some redundancy is retained is to explicitly consider a multiplicity of demand patterns in the cost minimization process. The main disadvantage of this approach is that each additional demand pattern considerably increases the computational complexity of solving the optimization problem. An alternative to explicitly considering a multiplicity of demand patterns is to design the network to meet explicit reliability requirements. However, reliability is not easy to define and the difficulty of calculating it increases as its definition becomes more realistic. The difficulty of calculating reliability is one reason that it is extremely difficult to optimize both the reliability and cost of a water supply network.

The approach adopted herein to strike a balance between cost and reliability is to use flow entropy as a surrogate for reliability. This novel approach enables the cost of the system to be minimized without the redundancy being completely removed. The following are some of the other advantages of the present approach. From a computational viewpoint, flow entropy is easy to calculate and easy to incorporate in optimization processes. A practical implication of the above properties is that entropy can be used in a design framework. Additionally, using an entropy constraint, as in Problem 10, increases the computational effort only marginally. As such, the entropy constraint, by itself, will not in general restrict the size of the network that can be handled. Moreover, evidence has been presented herein suggesting that

Maximum entropy traffic flows in road networks

Although the concept of maximum entropy has been used extensively to infer trip matrices in road networks, it would appear that much of the research is based on the single-space probability entropy model. Also, the available information tends to be the volume of traffic on a few links of the network, in contrast to the present research in which the available information is the nodal inflows and outflows. It would be interesting to investigate the applicability of the present multispace probability entropy model to the inference of trip matrices. Such research may, in turn, shed some light on how the present method of calculating maximum entropy flows may be adapted to cope with flow networks in which some link flows are known. Problem 7 is inapplicable to such a network.

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APPENDIX A CALCULATING SYSTEM RELIABILITY BY THE CUT SET METHOD

Definitions

A *cut set* is a set of system components the failure of which causes the system to fail.

A *minimal cut set* is a set of system components which, when failed, causes the system to fail, but does not cause the system to fail if at least one component of the set is working. In other words, all the components of a minimal cut set must fail for the system to fail. It follows that the probability of failure $p(C_i)$ of the ith minimal cut set C_i is

$$p(C_i) = \prod_m R'_m \quad i = 1,\ldots,NC \tag{A1.1}$$

in which R'_m is the unreliability of the mth component in the minimal cut set being considered; NC is the number of minimal cut sets.

Calculating System Reliability

Since the failure of each minimal cut set causes the system to fail, it follows that the failure of any combination of the minimal cut sets causes the system to fail. The system unreliability R' is therefore given by

$$R' = p\left[\bigcup_i C_i\right] \tag{A1.2}$$

The remainder of this short appendix is concerned with the evaluation of the probability given by Eq. (A1.2). For $NC = 2$,

$$p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2) \tag{A1.3}$$

$$p(C_1 \cap C_2) = p(C_1)p(C_2)$$

Eq. (A1.3) can be verified readily using a Venn diagram. For $NC = 3$, the desired result can be obtained by setting $C_2 \equiv C_2 \cup C_3$, substituting in Eq. (A1.3), and expanding the resulting right hand side. Thus:

$$p(C_1 \cup C_2 \cup C_3)$$

$$= p(C_1) + p(C_2 \cup C_3) - p(C_1 \cap (C_2 \cup C_3))$$

$$= p(C_1) + p(C_2 + C_3 - C_2 \cap C_3) - p((C_1 \cap C_2) \cup (C_1 \cap C_3))$$

$$= p(C_1) + p(C_2) + p(C_3) - p(C_2 \cap C_3) - p(C_1 \cap C_2 + C_1 \cap C_3 - C_1 \cap C_2 \cap C_3)$$

$$= p(C_1) + p(C_2) + p(C_3) - p(C_1 \cap C_2) - p(C_1 \cap C_3) - p(C_2 \cap C_3)$$

$$+ p(C_1 \cap C_2 \cap C_3) \tag{A1.4}$$

The above probability can be evaluated using

$$p\left[\bigcap_i C_i\right] = \prod_i p(C_i) \tag{A1.5}$$

Continuing as in Eq. (A1.4), the following rule can be deduced for any number of components NC.

To evaluate the unreliability of a system:

> *add the unreliabilities of all the minimal cut sets,*
> *subtract all the second order products of the unreliabilities,*
> *add all the third order products of the unreliabilities,*
> *subtract all the fourth order products of the unreliabilities,*
> *etc.*

More generally, add all the odd order products of the unreliabilities and subtract all the even order products of the unreliabilities.

The above rule can be used to calculate system unreliability, and the reliability is given by $R = 1 - R'$.

The above method of calculating reliability gives accurate results, but the number of terms involved in the calculations increases too rapidly as the number of minimal cut sets increases. To show the rate of increase, for 1, 2, 3, 4, 5, 6, ... minimal cut sets respectively, there are 1, 3, 7, 15, 31, 63, ... terms. For this reason, approximations are often used, particularly if the individual components have high reliabilities. With high component reliabilities, the unreliabilities of the minimal cut sets are low. In consequence, the values of the products in the above general rule for calculating system unreliability are small. One approximation is obtained by assuming that second and higher order terms are negligible. The above assumption leads to an upper bound on unreliability as

$$R' = \sum_{i=1}^{NC} p(C_i) \qquad (A1.6)$$

The network reliability corresponding to the above unreliability is a lower bound and, as such, is safe. The reliability is, obviously,

$$R = 1 - R' \qquad (A1.7)$$

The interested reader may consult Billington and Allan (1983) for more material on the evaluation of network reliability.

APPENDIX B PUBLICATIONS

The following papers are based on the present research and have been accepted for publication.

1. Calculating Maximum Entropy Flows in Networks; accepted for publication in *J. Operational Research Society.*
2. Optimum Design of Flexible Water Distribution Networks; accepted for publication in *Civil Engineering Systems.*
3. Maximum Entropy Flows for Single-Source Networks; accepted for publication in *Engineering Optimization.*

APPENDIX B1 CALCULATING MAXIMUM ENTROPY FLOWS IN NETWORKS

CALCULATING MAXIMUM ENTROPY FLOWS IN NETWORKS

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ABSTRACT

The paper describes methods for calculating most likely values of link flows in networks with incomplete data. The object is to present a thorough and rigorous treatment of maximum entropy flow estimation methods and to develop a methodological framework capable of handling different types of network problems. A multiple probability space constrained entropy approach is described for the general network problem. Results are presented and discussed for an example network intended for water supply.

KEY WORDS : networks, information theory, engineering, probability

INTRODUCTION

The work described in this paper is concerned with methods for probabilistic inference on networks with incomplete data. It was initially stimulated by the problem shown in Figure 1. Suppose that a pipe network transports water from several source nodes to several demand nodes and that volumetric supplies and demands at all nodes can be measured and are therefore known. Suppose also that the layout of the pipe network and the flow direction in each pipe element are known but no other data whatsoever are available. Under these conditions, how can 'most likely' flow rates in all the pipes of the network be estimated?

If the layout of pipes in Figure 1 had been a branched system with no loops the supply and demand data would have been sufficient to determine uniquely all the unknown internal flow rates in the pipes. Since the layout in Figure 1 is looped, however, there are more unknowns than equilibrium equations and more information is needed about all the pipes in order to carry out a full looped pipe network analysis. In the absence of this information there are very many possible pipe flow rate distributions which satisfy the equilibrium conditions. Which of these many possible solutions is in some sense most likely and how can it be found?

The problem has practical relevance. Sometimes, especially for old water supply systems, though plan layouts may be available, details of the pipe diameters, friction coefficients and other data may have been lost or may have changed over time. Such water supply networks are usually buried and it may be time consuming and expensive to obtain these data for every pipe element in order to determine the flow rates accurately by calculation. Physical measurements of pipe flow rates requires equipment and can be similarly expensive and time consuming. In these circumstances a method of quickly estimating most likely pipe flow rates would be most useful.

Reduced to its essentials, this problem can be viewed as a transportation problem. There is a set of sources and a set of destinations each supplying or demanding known quantities of flow. Between these two sets of nodes there is some network providing the means of flow transport. It is desired to estimate the most likely internal distribution of flows within this transportation network.

Other problems with these characteristics arise in different practical contexts. For example, consider the well known problem in traffic engineering of estimating the turning flows at a junction or roundabout. Figure 2a shows a roundabout with four arms, each being a two-way road. Suppose that traffic flow rates can be measured on each arm in each direction thus giving four inflows to and four outflows from the roundabout. On entering the roundabout on any arm each vehicle

either turns left and leaves by the first exit, or goes straight on and leaves by the second exit, or turns right and leaves by the third exit. The possibility that a driver may make a complete circle and leave by his entrance road is ignored here although this may be included if special circumstances make this significant. For a four arm roundabout and three possible directional choices for a vehicle on any arm, there are therefore twelve unknown turning flows but only seven independent inflow and outflow conditions which they must satisfy. The number of possible solutions is infinite. How can most likely values be estimated for all the twelve unknown turning flows?

Figure 2b shows a network representation of the turning flow problem in which the four inflows to the roundabout are shown on the left and the four outflows on the right. Each link of the network represents a possible turning flow whose most likely numerical value must be found. Although the network of Figure 2b is different from that of Figure 1 the nature of the two estimation problems is closely similar in both cases.

The practical problems outlined above, and others, are closely related. Methods already exist[1,2,3] for solving some of the problems but are incapable of solving more general network problems. The purpose of this paper is to address them in a unified fashion to try to determine a rigorous method capable of tackling all of them. Also there are different interpretations of the term 'most likely' in a flow distribution context. One purpose of this paper is to attempt to clarify that issue. It is shown that the Shannon/Jaynes maximum entropy formalism[4,5] provides both an interpretation of 'most likely' and a methodology for determining most likely flows.

UNCERTAINTY AND NETWORK FLOWS

The term 'most likely' used with reference to flow rate estimation reflects likelihood in the context of observer uncertainty. In the case of the buried water supply network, when the system is operating and exhibiting the known inflow and outflow rates, the individual pipes have unique water flow rates which obey known physical laws. Although those physical laws are known there is insufficient data to allow the behaviour of the system to be calculated from them. The uncertainty in the problem arises not from any randomness or uncertainty in the system itself but from the inability of the observer to deduce its deterministic behaviour. Given sufficient further data, that observer uncertainty could be entirely eliminated and the unique pattern of flows would be revealed. If a method exists whereby some estimate of that flow pattern can be made without extra data, it therefore follows that the method must in some way depend upon and manipulate the observer uncertainty about the physical problem rather than operate methodologically upon the physical processes of the network itself.

The roundabout problem is rather different. Here there are no physical laws governing turning flows. Individual drivers are free to make a choice of turning direction so there is inherent uncertainty in the turning flows themselves. At best, given any amount of extra data, a model of the roundabout system would only be able to estimate some statistical average values for turning flows. Additionally, as with the water supply system, there is observer uncertainty about what any average turning flow rate is. In this paper the 'most likely' estimation process operates upon the observer uncertainty rather than upon the uncertainties in the turning flows themselves.

Figure 3 shows a simplified network which incorporates some, but not all, the characteristic features of the examples described above. In Figure 3 the nature of the flow is not specified; it may be water, vehicles or any unspecified commodity. Let there be M source or supply nodes denoted by $i = 1,\dots,M$ and let I_i be the (known) inflow at source node i. Let there be N demand or destination nodes denoted by $j = M+1,\dots,M+N$ and let O_{M+j} be the (known) outflow rate at demand node j. Also let there be a total of L direct flow-transporting links ij between source nodes i and demand nodes j, and let t_{ij} denote the (unknown) flow rate on link ij. Note that the above definitions imply that a node in the network must be either a source or a destination node, there are no other possibilities. Also, the total number of links L may generally be smaller than, and cannot exceed, MN. The network can be described as

fully-connected if each source node is directly connected to every demand node, in which case the number of links L = MN. With these definitions the following equations represent flow equilibrium at all nodes of the network.

$$\sum_{j=M+1}^{M+N} t_{ij} = I_i \qquad i = 1,\dots,M \qquad (1)$$

$$\sum_{i=1}^{M} t_{ij} = O_j \qquad j = M+1,\dots,M+N \qquad (2)$$

$$\text{with} \quad t_{ij} \geq 0 \qquad \forall\, ij$$

For a balanced problem, in which the sum of all source flows is exactly equal to the sum of all flow demands, Eqs. (1) and (2) are not linearly independent; they contain one redundant equation. Without loss of generality it is assumed that the final demand equation at node j = M+N is omitted. Eqs. (1) and (2) therefore comprise (M + N - 1) equations in L unknown link flows. For the purposes of this paper it is assumed that L > (M + N - 1), thus a unique solution of Eqs. (1) and (2) does not exist; many different solutions are possible. Values are sought for the L link flows t_{ij} satisfying Eqs. (1) and (2) which are in some sense most likely.

One solution can be found easily if sufficient of the link flows are set to zero until Eqs. (1) and (2) become solveable uniquely for the remaining link flows. This type of result is typically achieved by linear programming if a linear total transportation cost function

$$\underset{t_{ij}}{\text{Minimize}} : \quad \text{Cost} = \sum_{ij} c_{ij} t_{ij} \qquad (3)$$

is added to Eqs. (1) and (2). In Eq. (3) c_{ij} represents the cost of transporting one unit of flow along link ij. Linear programming simply allocates zero flow along the more expensive links and uses only the cheaper links to carry non-zero flows. However, none of the problems described above is necessarily of this nature; there is no information to the effect that cost minimization is involved in them, and no data values for the unit cost coefficients. Consequently there are no grounds for expecting the most likely solution of Eqs. (1) and (2) to have this characteristic pattern.

Indeed, the LP-type solution is a very poor candidate for the title of 'most likely'. Given that possible links exist to carry flow it seems very unlikely that some of them should carry zero flow. The value zero is at one extreme end of the range of possible values for a link flow: it seems intuitively more likely that link flows should have values in the middle of the range rather than at either extreme. Another argument against zero being a most likely value for some link flows comes from the fact that only some, but not all, of the links may have this value. Which links should then be specifically selected to have this zero value rather than any other links? There is no reason to prefer any particular link to have this honour rather than any other link. Extrapolating this argument further, there is no reason to allocate different values to the flow rates in different links unless the equilibrium equations (1) and (2) dictate such a solution. By this reasoning, a most likely set of link flows should be as uniform in value as is permitted by Eqs. (1) and (2).

This characterization of most likely values as most uniform values, subject to satisfying the equilibrium equations, has been developed intuitively. It implicitly uses Laplace's principle of insufficient reason, which requires that in the absence of any good reason to allocate different values to unknown quantities the same value should be allocated to them all. The sufficient reason for a non-uniform choice in this case is that values have to satisfy Eqs. (1) and (2). Laplace's principle therefore leads to the idea that the most likely flows will satisfy Eqs. (1) and (2) and will be as uniform in value as possible.

Laplace's principle is generally recongnized not to be a fundamental principle. It is a consequence of the Shannon/Jaynes maximum entropy formalism[4,5] (MEF) which provides an ideal tool with which to tackle the most likely flow problem of Figure 3. Values must be assigned to the L link flows and each of those values has some observer uncertainty

associated with it. If the uncertainty associated with each link flow can be represented probabilistically (or as some probability-like quantity which satisfies all the conditions which are axiomatic to probabilities) then, by virtue of the MEF, the most likely flow rate assignment problem of Figure 3 can be posed as the problem of maximizing the Shannon entropy of the link probabilities subject to whatever is known about the system, i.e. Eqs. (1) and (2).

THE GRAVITY MODEL

The network flow problems described above are concerned with allocating flow values but the MEF is couched in terms of allocating probabilities. The question of how to introduce probabilities into Figure 3 and Eqs. (1) and (2) is now addressed. Two different ways of doing this will be examined in detail and will be shown to give the same most likely flow values.

The first approach considers the equilibrium equations (1) and (2) and denotes by T the sum of all link flows in the network. Thus:

$$T = \sum_{ij} t_{ij} = \sum_{i=1}^{M} I_i = \sum_{j=M+1}^{M+N} O_j \qquad (4)$$

Probability-like quantities p_{ij}, which satisfy non-negativity and normality conditions associated with probabilities, may be introduced as ratios of link flow t_{ij} to total flow T. Thus:

$$p_{ij} = t_{ij}/T \qquad \forall\ ij \qquad (5)$$

The most likely flow estimation problem now becomes that of maximizing the Shannon entropy of the link probabilities subject to Eqs. (1) and (2) with link flows substituted by Eq. (5). i.e.

$$\underset{p_{ij}}{\text{Maximize}} : \quad (S/K) = -\sum_{ij} p_{ij} \ln (p_{ij}) \qquad (6)$$

$$\text{Subject to : } \sum_{ij} p_{ij} = 1 \qquad (7)$$

$$\sum_{j=M+1}^{M+N} p_{ij} = I_i/T \qquad i = 1,\ldots,M \qquad (8)$$

$$\sum_{i=1}^{M} p_{ij} = O_j/T \qquad j = M+1,\ldots,M+N-1 \qquad (9)$$

Eq. (6) is the Shannon entropy function[4] in which S is the entropy and K is an arbitrary positive constant which is not required for maximization. The above problem represented by Eqs. (6) to (9) has a unique solution which may be determined by examining the stationarity of its Lagrangean. The solution may be derived as shown in the Appendix and is:

$$p_{ij} = I_i O_j/T^2 \qquad \forall\ ij \qquad (10)$$

and Eq. (5) immediately recovers the required estimates of the most likely link flow as:

$$t_{ij} = I_i O_j/T \qquad \forall\ ij \qquad (11)$$

which corresponds to the well-known *gravity model* of transportation engineering. Substituting the probabilities (10) into the entropy function (6) gives the maximum entropy value for a fully-connected network as:

$$(S/K)^* = -\sum_{i=1}^{M} (I_i/T)\ln(I_i/T) - \sum_{j=M+1}^{M+N} (O_j/T)\ln(O_j/T) \qquad (12)$$

In the case of a less-than-fully connected network (such as the roundabout turning flow problem of Figure 2b) result (11) still holds but yields a maximum entropy value $(S/K)^*$ which is smaller than that given by (12). This indicates that in a fully connected network the maximum value

of the internal entropy is determined by the external macroscopic boundary conditions, but removing links removes potential uncertainty from the internal system and the maximum internal entropy can no longer reach this absolute macroscopic value.

The emergence of the gravity model as representing most likely flow estimates is neither new nor unanticipated. Transportation engineers have been using it for many years as an estimator of traffic flows in a variety of circumstances[1]. Its validity is incontrovertible and is further strengthened by the fact that it may be derived, as here, from first principles as a direct consequence of the maximum entropy formalism. However, it is important to note the restrictions and limitations implicit in Figure 3 and Eqs. (1) and (2). Consequently, the gravity model is directly applicable to the roundabout turning flow problem of Figure 2 but not to the general network problem, for instance the water supply network of Figure 1 in which all links do not start at a source and end at a demand node. Also, in general, there may be several different paths between a source node and a demand node.

In order to handle general networks something more than the gravity model is required. The key to developing an alternative lies in defining probabilities in a different way from Eq. (5), and is described next.

MULTIPLE PROBABILITY SPACE MODELS

Shannon's entropy is defined for independent and exhaustive probabilities only. These conditions are satisfied by the probabilities in Eq. (5), provided all links of the network start at a source and end at a demand node, as depicted in Figure 3. Essentially, this proviso means that the probabilities in Eqs. (5) are suitable, in the context of entropy, only for networks in which there are no links connected in series. For example, in Figure 1, all link pairs (ij, jk), ∀i, j, k = 1, ...,N are connected in series. In any network with series connections, probability-like terms may still be defined but may not be independent. For any series-connected pair of links (ij, jk),, the flow in link jk is made up, at least in part, of the flow from link ij. As such, these flows are not independent. Consequently, if the corresponding probability-like quantities are defined as in Eqs. (5), those quantities would not be independent.

Refering to Figure 3 again, an alternative way of introducing probabilities is to define $\bar{p}_{ij}$ as the proportion of the total inflow at node i which is carried onwards by link ij (i.e. the probability that an element of the flow which enters node i is transported along link ij to node j). With this definition, if the inflow at node i is I_i, then the expected value of the flow on link ij will be $t_{ij} = \bar{p}_{ij} I_i$. Substituting this into Eqs. (1) and (2) gives:

$$\sum_{j=M+1}^{M+N} \bar{p}_{ij} = 1 \qquad i = 1,\ldots,M \qquad (13)$$

$$\sum_{i=1}^{M} \bar{p}_{ij} I_i = O_j \qquad j = M+1,\ldots,M+N-1 \qquad (14)$$

These equations are equivalent to Eqs. (8) and (9) with the earlier definition of probabilities. Whereas in the earlier formulation there was one normality condition (7) which embraced all the probabilities, there are now M normality conditions (13) and M sets of probabilities. Furthermore these probability sets are not independent: they are conditional upon the probabilities associated with the source node inflows I_i, $i = 1,\ldots,M$. The question which needs to be addressed is what form of entropy is the correct one to use with multiple dependent probability spaces?

This question has been addressed by Khinchin[6] who has given several forms of the Shannon entropy function for multiple probability spaces. Two general results are useful:

i) For two *independent* discrete probability distributions Q and R, the entropy of the joint distribution QR is the sum of the entropies of Q and R separately. Thus:

$$S(QR) = S(Q) + S(R) \qquad (15)$$

ii) For two *mutually dependent* probability distributions Q and R, the entropy of the joint distribution Q*R (* is used to denote that Q and R are mutually dependent) is given by

$$S(Q{*}R) = S(Q) + S(R|Q) \qquad (16)$$

where $$S(R|Q) = \sum_q p_q S_q(R) \qquad (17)$$

p_q is the probability of the q-th outcome in probability distribution Q, $S_q(R)$ is the entropy of probability distribution R conditional upon the q-th outcome in probability distribution Q occurring, and S(R|Q) is the entropy of probability distribution R conditional upon Q occurring.

Important features of the above results are that S(Q*R) is invariant with respect to changes in position of Q and R. S(R*Q) is therefore identical to S(Q*R) and is obtained from Eqs. (16) and (17) with the roles of Q and R interchanged. Also S(Q*R) reduces to S(QR) when Q and R are independent and further reduces to S(Q), the Shannon entropy function (6), when there is only one probability distribution. Extensions to more than two probability distributions follow the rules given above.

Returning to Eqs. (13) and (14), the M probability sets cannot be treated as independent. For a particular set, i, probabilities $\bar{p}_{ij}$ measure the likelihood that the inflow at i is transported to demand node j, j = M+1,...,M+N. These probabilities within set i are independent but set i itself is conditional upon inflow existing at node i. If p_i is defined to be the probability of inflow at node i then these probabilities can be determined from the network data: p_i is given by the proportion of the total inflows at all source nodes which exists at node i. Thus:

$$p_i = I_i / \sum_{i=1}^{M} I_i = I_i/T \qquad i = 1,\ldots,M \tag{18}$$

Eqs. (16) and (17) then give the form of the entropy function to be used with the mutually dependent probability sets defined by Eqs. (13) and (18):

$$S/K = - \sum_{i=1}^{M} p_i \ln (p_i) - \sum_{i=1}^{M} p_i \sum_{j=M+1}^{M+N} \bar{p}_{ij} \ln (\bar{p}_{ij}) \tag{19}$$

which, on substituting the known probabilities p_i given by Eq. (18) becomes

$$S/K = - \sum_{i=1}^{M} (I_i/T) \ln (I_i/T) - \sum_{i=1}^{M} (I_i/T) \sum_{j=M+1}^{M+N} \bar{p}_{ij} \ln (\bar{p}_{ij}) \tag{20}$$

Most likely flows in the multiple probability space formulation are then given by maximizing the entropy function (20) over probabilities $\bar{p}_{ij}$ for all ij, subject to constraints (13) and (14). There is an analytical solution to this problem which can be found by examining the stationarity of its Lagrangean in a similar way to result (10), as given in the Appendix. The solution is:

$$\bar{p}_{ij} = O_j/T \qquad \forall\ ij \tag{21}$$

The link flows are then given by

$$t_{ij} = \bar{p}_{ij} I_i = I_i O_j/T \qquad \forall\ ij \tag{22}$$

which are exactly the same as the flows (11) calculated earlier for the single probability space formulation. Substituting the probabilities (21) into the entropy function (20) and assuming a fully connected network yields the same maximum entropy value (12) given by the earlier formulation.

Two different formulations for determining most likely flows in a network similar to Figure 3 have now been developed. Both the single and

multiple probability space formulations give identical flow régimes and total uncertainty values. Both formulations are restricted to the conditions associated with Figure 3 and neither is directly applicable to the water supply network of Figure 1. The next section describes how the multiple probability space formulation can be extended to become applicable to more general networks and uses the Figure 1 example for demonstration purposes.

MULTIPLE PROBABILITY SPACE GENERAL NETWORKS

Because of difficulties of probabilistic independence, as discussed earlier, the single probability space model cannot easily be extended to a network such as Figure 1. Turning to the multiple probability space approach, the key elements in the model developed earlier for Figure 3 were as follows:

(i) A set of normalized probabilities $\bar{p}_{ij}$ was defined at each node i where the nodal inflow split into at least two outflows. The $\bar{p}_{ij}$ therefore represent the probabilities associated with flow splitting processes.

(ii) The probabilities $\bar{p}_{ij}$ of flows leaving node i were conditional upon probabilities associated with the arrival of inflow at node i, p_i.

(iii) The entropy function used was the conditional entropy function.

(iv) Constraints upon the entropy function maximization originated in the flow equilibrium equations at all nodes except the final demand node.

These four elements provide the basis for extending the multiple probability space model to more general networks such as that shown in Figure 1. Figure 4 shows the network example of Figure 1 in a more general form with numerical values of source flows and demands replaced by algebraic quantities. Figure 4a gives details of the node numbering and link connectivities, supplies and demands, and also identifies the link flow quantities, t_{ij}. Figure 4b gives details of all the probabilities associated with this particular problem. First, all the elements of the model are assembled according to requirements (i) to (iv) above.

In accordance with (i) there are flow splitting processes at the source nodes 1 and 2, and also at nodes 3, 4 and 5. At the source nodes probabilities are easily defined as ratios of link outflows to the total link outflows at each node. Thus, since the total of the link outflows is in this case equal to the known source inflow:

$$\bar{p}_{13} = t_{13}/I_1 \quad ; \quad \bar{p}_{14} = t_{14}/I_1 \qquad (22a)$$

$$\bar{p}_{23} = t_{23}/I_2 \quad ; \quad \bar{p}_{24} = t_{24}/I_2 \qquad (22b)$$

The flow splitting probabilities at nodes 3, 4 and 5 are also ratios of individual outflows to the sum of the outflows at a node and lead to the following definitions of probabilities. Note particularly that a probability must be assigned to the demand from a node. The reason for doing this is that, although the demand at a node is known, the total outflow from a node is not known. Hence the ratio of the demand to the total outflow is not known and a probability is required to represent this. The use of zero as a second suffix denotes these demand probabilities. Thus:

$$\bar{p}_{34} = t_{34}/T_3 \quad ; \quad \bar{p}_{35} = t_{35}/T_3 \quad ; \quad \bar{p}_{30} = O_3/T_3 \qquad (22c)$$

in which $T_3 = t_{34} + t_{35} + O_3$,

$$\bar{p}_{46} = t_{46}/T_4 \quad ; \quad \bar{p}_{40} = O_4/T_4 \qquad (22d)$$

with $T_4 = t_{46} + O_4$, and .

$$\bar{p}_{56} = t_{56}/T_5 \quad ; \quad \bar{p}_{50} = O_5/T_5 \qquad (22e)$$

with $T_5 = t_{56} + O_5$.

The way in which Eqs. (22) define the probability sets ensures that they satisfy normality without the need for separate normality constraints.

Non-negativity of the probabilities is also ensured provided that link flows are always in the direction defined and never become negative.

In accordance with (ii), all the above probabilities are conditional upon flow existing at the nodes at which these flow splitting probabilities are defined. In the case of the source nodes 1 and 2 the probabilities that flow exists there are known and are given by the first equality in Eq. (18).

$$p_1 = I_1/(I_1 + I_2) \quad ; \quad p_2 = I_2/(I_1 + I_2) \qquad (23)$$

In accordance with (iii) the conditional entropy function for the network of Figure 4 may be assembled systematically using the principles of conditional probability and the conditional entropy definitions (16) and (17). Starting with the two source nodes there is the entropy, S_0, of the probabilities p_1 and p_2 that flow exists at the source nodes. This is:

$$S_0 = -p_1 \ln (p_1) - p_2 \ln (p_2) \qquad (24a)$$

There are then the entropies of the probabilities associated with the link flows leaving the two source nodes. For node 1, probabilities $\bar{p}_{13}$ and $\bar{p}_{14}$ are conditional upon p_1, and the conditional entropy, S_1, is therefore:

$$S_1 = - p_1[\bar{p}_{13} \ln (\bar{p}_{13}) + \bar{p}_{14} \ln (\bar{p}_{14})] \qquad (24b)$$

For node 2, probabilities $\bar{p}_{23}$ and $\bar{p}_{24}$ are conditional upon p_2 and the entropy is therefore:

$$S_2 = - p_2[\bar{p}_{23} \ln (\bar{p}_{23}) + \bar{p}_{24} \ln (\bar{p}_{24})] \qquad (24c)$$

At node 3 there is entropy associated with the probabilities for link flows leaving node 3. These flows are conditional upon flow arriving at node 3. Flow can arrive at node three by two routes: from node 1 via link 1-3, and from node 2 via link 2-3. The probability of flow arriving at node 3 by the first of these routes is $p_1\bar{p}_{13}$ and by the second route is $p_2\bar{p}_{23}$. Thus the conditional entropy at node 3 is:

$$S_3 = - [p_1\bar{p}_{13} + p_2\bar{p}_{23}][\bar{p}_{34}\ln(\bar{p}_{34}) + \bar{p}_{35}\ln(\bar{p}_{35}) + \bar{p}_{30}\ln(\bar{p}_{30})] \quad (24d)$$

Similarly for node 4 the entropy of the flows leaving node 4 is conditional upon flow arriving at node 4 from the source nodes 1 and 2, and also from node 3. Thus the conditional entropy at node 4 is:

$$S_4 = - [p_1\bar{p}_{14} + p_2\bar{p}_{24} + \bar{p}_{34}(p_1\bar{p}_{13} + p_2\bar{p}_{23})] [\bar{p}_{46}\ln(\bar{p}_{46}) + \bar{p}_{40}\ln(\bar{p}_{40})] \quad (24e)$$

Similarly at node 5 the conditional entropy turns out to be:

$$S_5 = - [\bar{p}_{35}(p_1\bar{p}_{13} + p_2\bar{p}_{23})] [\bar{p}_{56}\ln(\bar{p}_{56}) + \bar{p}_{50}\ln(\bar{p}_{50})] \quad (24f)$$

The conditional entropy of the entire network of Figure 4 is then simply the sum of the separate entropies in Eqs. (24a) to (24f):

$$S/K = S_0 + S_1 + S_2 + S_3 + S_4 + S_5 \quad (25)$$

In accordance with (iv), constraints are generated by the flow equilibrium conditions at all nodes except the final demand node, node 6 in the network of Figure 4, and are:

$$t_{13} + t_{14} = I_1 \quad (26a)$$
$$t_{23} + t_{24} = I_2 \quad (26b)$$
$$t_{13} + t_{23} = t_{34} + t_{35} + O_3 \quad (26c)$$
$$t_{14} + t_{24} + t_{34} = t_{46} + O_4 \quad (26d)$$
$$t_{35} = t_{56} + O_5 \quad (26e)$$

The above model contains both link flow unknowns t_{ij} and probability unknowns $\bar{p}_{ij}$ which are connected through the probability definitions (22). The first step in solving the model consists of simplifying it to an easily solveable form. There are several ways in which this can be done; the way chosen here is to express everything in terms of a reduced number of link flow variables. The five flow equilibrium constraints (26) contain eight link flow unknowns but are equalities. They can therefore be used to express all link flows in terms of just three independent link flow variables. Choosing t_{14}, t_{24} and t_{34} as the independent variables yields:

$$t_{13} = I_1 - t_{14} \quad (27a)$$
$$t_{23} = I_2 - t_{24} \quad (27b)$$
$$t_{35} = I_1 + I_2 - O_3 - t_{14} - t_{24} - t_{34} \quad (27c)$$
$$t_{46} = - O_4 + t_{14} + t_{24} + t_{34} \quad (27d)$$
$$t_{56} = I_1 + I_2 - O_3 - O_5 - t_{14} - t_{24} - t_{34} \quad (27e)$$

Definitions (22a) to (22e) may then be used to express all the probabilities $\bar{p}_{ij}$ in terms of unknowns t_{14}, t_{24} and t_{34}, and the entropy function (25) then becomes a function only of these three variables.

The flow directions defined in Figure 4a require that all the link flow quantities t_{ij} should be non-negative. The final model should therefore consist of maximizing the entropy function (25) over the link flow variables t_{14}, t_{24} and t_{34} subject to constraints that all the right-hand side functions in Eqs. (27) and the link flow variables themselves must be non-negative. However, the final model was actually solved by NAG library routine E04JAF which is an unconstrained optimization routine, and the constraints were omitted whilst maximizing the entropy function. The optimum solution was then substituted into the constraints to check that they were all satisfied but inactive. The reasons for adopting this approach were that it provides a much simpler and quicker solution, and that, as has been argued earlier, it is expected that the maximum entropy solution will have flows which are as uniform as possible without any being equal to zero.

The entropy function (25) for the example of Figure 4 was constructed somewhat laboriously using the conditional entropy definitions (16) and (17). In fact its form is quite simple and has only two types of terms. The first type comprises the entropy of the source flow probabilities, S_0, given by Eq. (24a). In most network examples the source flows are known and form part of the available data, so these source flow entropy terms are actually known constants. They may be omitted from the objective function maximization and are not necessary to the solution process. The only case in which they are needed is that of a network in which source flows are not specified and must be estimated along with the pipe flows. The form of this first type of terms is always

$$S_0 = - \sum_{i=1}^{M} p_i \ln (p_i) \qquad i = 1,\ldots,M \qquad (28)$$

in which p_i is as defined in Eq. (18).

The second type of terms are S_1 to S_5 given by Eqs. (24b) to (24f). Each of these equations represents the conditional entropy at a node of the network where flow splitting probabilities are defined. The form of these terms is always the same and consists of the entropies of all the outflow probabilities, including that of any demand at the node, multiplied by the total probability of flow arriving at that node by all possible paths. The form of these terms at any node k in the network is therefore

$$S_k = -p_k \left[\sum_{kj \in N_k} \bar{p}_{kj} \ln (\bar{p}_{kj}) \right] \qquad (29)$$

in which N_k is the subset of outflows from node k including any demand. p_k is the total probability of flow arriving at node k by all possible paths.

All nodes of the network generate terms of the form of Eq. (29) and the entropy function sums them all. At a node which has only one outflow (either a demand or a link outflow) the single entropy term in the [ ] in Eq. (29) is zero by virtue of the way in which probabilities $\bar{p}_{kj}$ are defined. This accords with intuition since at such a node there should be no additional uncertainty to what already exists elsewhere in the network. The entropy function as defined by Eqs. (28) and (29) therefore has a conveniently structured form which permits it to be assembled easily for any network.

The network of Figure 1 is used as a numerical example of the calculation of maximum entropy flows in a general network. Table 1 gives numerical results for two instances of Figure 1. In case A the inflows and outflows are exactly as shown in Figure 1. In case B the inflows at nodes 1 and 2 are interchanged. In both cases the link flow directions are as in Figure 1.

DISCUSSION

The results in Table 1 show several interesting features. Firstly, the link flows in the lower half of the network are the same for both cases A and B. Further study reveals that they remain the same for any combination of source flows at nodes 1 and 2 totalling 55 units. The reason for this is that nodes 1 and 2 are each connected to the rest of the network in exactly the same fashion, and they provide flow to nodes 3 and 4 in exactly the same proportions. From Table 1 the proportions of source flow transmitted from each source to nodes 3 and 4 may be calculated as 0.7121 and 0.2879 respectively. These proportions of the total available flow are required at nodes 3 and 4 in order to serve the demands in the rest of the network using maximum entropy pipe flows. The network is unable to distinguish topologically between the two source nodes, and, for demands as in Figure 1, any combination of source flows at nodes 1 and 2 which totals 55 units will be distributed to nodes 3 and 4 in these same proportions.

Secondly, a means of partially checking the validity of the results of Table 1 is to consider the example of Figure 1 with all directions reversed. The two source flows become demands, the four demands become sources, and the flow directions along all links are reversed. A complete reversal of all directions in this fashion completely changes the definitions of flow splitting probabilities and leads to a different conditional entropy function from Eq. (25). However, it should not change the intrinsic total uncertainty (entropy) in the system. Solving the reversed model, therefore, should yield exactly the same results as those of Table 1. It does, and this provides reassurance that the calculation are correct.

A third comment which may be made on the results of Table 1 is that the invariance of the most likely flow values in the links in the lower half of the network is reassuring from both design and reliability viewpoints. In the case of a water supply network, designing the pipes to carry these flows would appear to confer a considerable degree of invulnerability upon the lower half of the network to possible variations in the source flows. The issue of network reliability and the importance of maximum entropy flows in this context is too large for detailed examination here but has been studied by Awumah, Goulter and Bhatt[2,3].

CONCLUSIONS

The problem of estimating from incomplete data most likely values of flows on the links of a general network has been studied. Most likely flows were first characterized as those flows which maximize an entropy function for the network subject to the available information. Possible forms of this entropy function were examined and it was shown that a multiple probability space conditional entropy model was most appropriate for general network problems. The detailed solution of a typical general network problem was presented and the results discussed.

A considerable amount of work still needs to be done. The paper has established the basic elements and outlined the structure of a computer method for the calculation of most likely flows in any general network. That computer program needs to be written and tested and the results examined. The nature of these maximum entropy flows needs to be studied and interpreted. Conjectures about their potential value in the design of engineering networks and about possible close relationships between entropy and reliability need to be critically-examined and tested. The ability to infer most likely values in networks has many potential uses which need to be explored. Essentially, this paper has shown how the calculations may be done: the value of the results remains to be established.

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APPENDIX Derivation of result (10)

The Lagrangean of problem (6)-(9) may be written as:

$$L(\underline{p},\lambda,\underline{\alpha},\underline{\beta}) = -\sum_{ij} p_{ij}\,\ell n(p_{ij}) + (1+\lambda)\left[\sum_{ij} p_{ij} - 1\right]$$

$$+\sum_{i=1}^{M}\alpha_i\left[\sum_{j=M+1}^{M+N} p_{ij} - I_i/T\right] + \sum_{j=M+1}^{M+N-1}\beta_j\left[\sum_{i=1}^{M} p_{ij} - O_j/T\right]$$

Stationarity of the Lagrangean with respect to a typical probability, $p_{i'j'}$, yields

$$-\ell n(p_{i'j'}) - 1 + (1+\lambda) + \alpha_{i'} + \beta_{j'} = 0$$

$$p_{i'j'} = \exp(\lambda)\exp(\alpha_{i'} + \beta_{j'}) \qquad \text{(A1)}$$

Stationarity of the Lagrangean with respect to λ yield the normality condition (7). Substituting (A1) in (7) gives

$$\exp(\lambda)\sum_{ij}\exp(\alpha_i + \beta_j) = 1$$

$$\exp(\lambda) = 1/\sum_{ij}\exp(\alpha_i + \beta_j) \qquad \text{(A2)}$$

Substituting (A2) in (A1) gives

$$p_{i'j'} = \exp(\alpha_{i'} + \beta_{j'})/\sum_{ij}\exp(\alpha_i + \beta_j)$$

$$= \exp(\alpha_{i'})\exp(\beta_{j'})/\left[\sum_{i=1}^{M}\exp(\alpha_i)\sum_{j=M+1}^{M+N-1}\exp(\beta_j)\right] \qquad \text{(A3)}$$

Stationarity of the Lagrangean with respect to a typical multiplier, $\alpha_{i'}$, yields the inflow constraint (8) for node i':

$$\sum_{j=M+1}^{M+N} p_{i'j} = I_{i'}/T$$

Substituting result (A3), this becomes

$$\exp(\alpha_{i'})\sum_{j=M+1}^{M+N-1}\exp(\beta_{j'})/\left[\sum_{i=1}^{M}\exp(\alpha_i)\sum_{j=M+1}^{M+N-1}\exp(\beta_j)\right]$$

$$= I_{i'}/T$$

$$\exp(\alpha_{i'})/\sum_{i=1}^{M}\exp(\alpha_i) = I_{i'}/T \qquad \text{(A4)}$$

Stationarity of the Lagrangean with respect to a typical multiplier, $\beta_{j'}$, yields the outflow constraint (9) for node j':

$$\sum_{i=1}^{M} p_{ij'} = O_{j'}/T$$

Substituting result (A3) and proceeding as for $\alpha_{i'}$ this gives

$$\exp(\beta_{j'})/\sum_{j=M+1}^{M+N-1}\exp(\beta_j) = O_j/T \qquad \text{(A5)}$$

Substituting results (A4) and (A5) into (A3) gives

$$p_{i'j'} = \left[\exp(\alpha_{i'})/\sum_{i=1}^{M}\exp(\alpha_i)\right]\left[\exp(\beta_{j'})/\sum_{j=M+1}^{M+N-1}\exp(\beta_j)\right]$$

$$= \left[I_{i'}/T\right]\left[O_{j'}/T\right]$$

$$p_{i'j'} = I_{i'}O_{j'}/T^2$$

which is result (10) when generalized for all i and j.

Result (21) for the multiple probability space model may be derived in a closely similar manner by examining the Lagrangean of the problem defined by Eqs. (20), (13), and (14).

Table 1 Maximum entropy flows for the network of Figure 1

link	flow rate	
	case A	case B
1 - 3	14.242	24.924
1 - 4	5.758	10.076
2 - 3	24.924	14.242
2 - 4	10.076	5.758
3 - 4	15.833	15.833
3 - 5	8.333	8.333
4 - 6	6.667	6.667
5 - 6	3.333	3.333
(S/K)	2.41098	2.41098

FIGURE CAPTIONS

Figure 1 Water supply network

Figure 2 a) Four-arm roundabout, and b) the turning flows represented as links of a network

Figure 3 Network notation

Figure 4 Water supply network example of Figure 1
a) Supply, demand and flow definitions
b) Probabilities

TABLE CAPTION

Table 1 Maximum entropy flows for the network of Figure 1

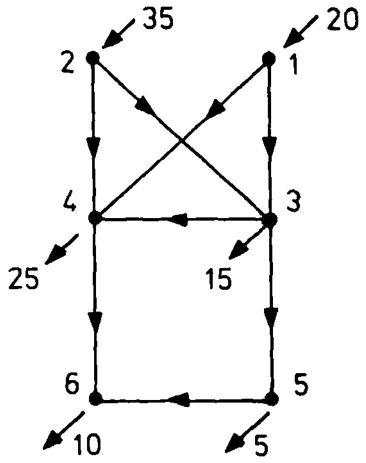

Fig. 1

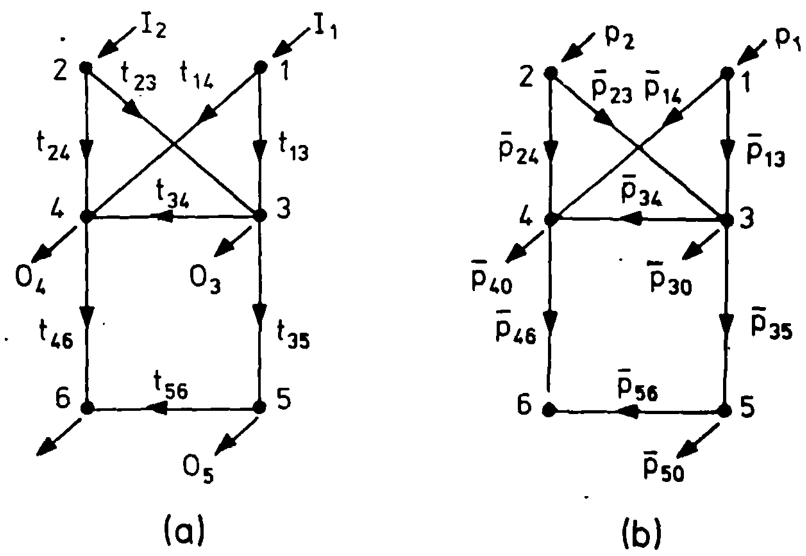

Fig. 4

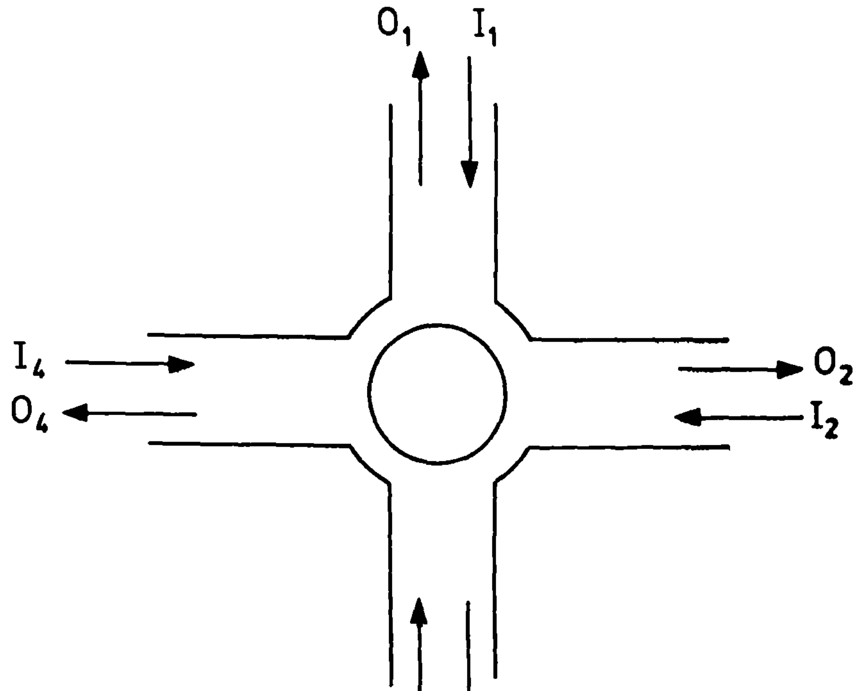

Fig. 2a

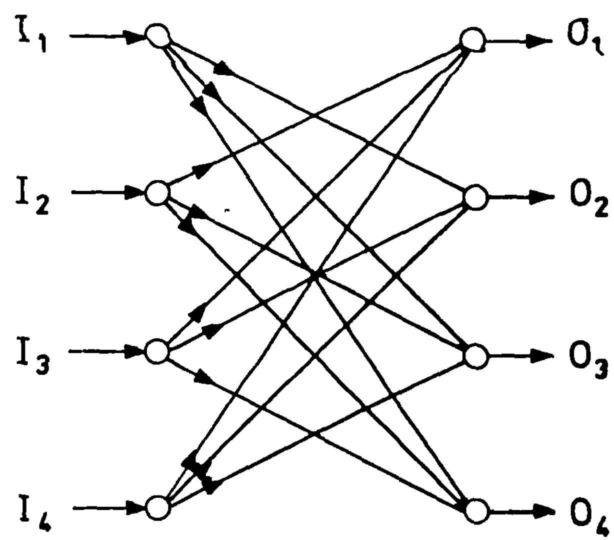

Fig. 2b

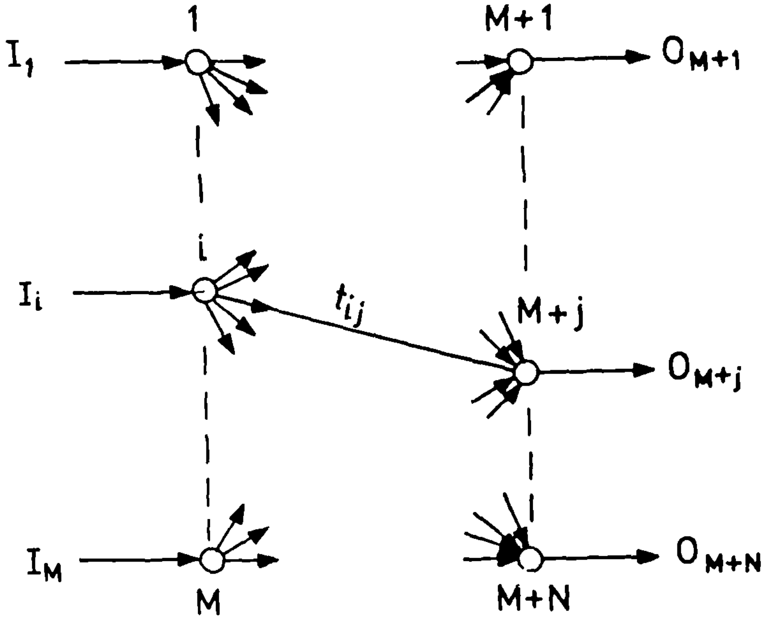

Fig. 3

APPENDIX B2 OPTIMUM DESIGN OF FLEXIBLE WATER DISTRIBUTION NETWORKS

OPTIMUM DESIGN OF FLEXIBLE WATER DISTRIBUTION NETWORKS

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ABSTRACT

A method for designing flexible water distribution networks is presented. Flexibility is the extent to and ease with which a distribution network can cope with eventualities for which it was not specifically designed. This paper shows that some flexibility can be achieved by maximizing the entropy of the flows. A sample network is considered and designs for various levels of entropy are examined. Several indices including energy and head loss are used to compare the designs. The results suggest that an entropy constraint can reduce the tendency towards implicitly branched configurations characteristic of cost minimization models. A striking feature of the proposed methodology is its apparent ability to produce resilient designs without a substantial increase in cost. The results further highlight some implications for connectivity-based reliability measures and core tree approaches to layout optimization.

1. INTRODUCTION

In recent years interest in the cost optimization of looped water distribution networks has increased. The need to replace systems of increasing age and to use public funds efficiently and economically has focused attention upon both the reliability and cost aspects of water supply systems. A heavily redundant, highly looped network with large pipes provides a very reliable system with in-built resilience under exceptional loading conditions such as fire fighting

and pipe breakage. Such a system is, unfortunately, also very expensive to construct. There is clearly a need for a design procedure which is able to strike an acceptable compromise between reliability/resilience and cost.

There is, however, no comprehensive definition of the reliability of a water distribution network although some key elements have been identified. These include the mechanical reliability of the entire network and its components, the shortfall suffered by consumers in the event that a component fails, and resilience regarding the network's performance under various adverse conditions. Also, the probabilistic nature of failures and consumption must be accounted for. Other wider factors such as availability of supply and operating policies are also involved. This paper, however, deals with distribution and concentrates on the pipes only.

Of particular concern is the difficulty of quantifying any sufficiently realistic measure of reliability. On one hand, simulation-based probabilistic measures can be far ranging. However, they are not easily transferable across networks and obtaining them can be time consuming. On the other hand, analytical indicators are difficult to calculate. In any case the majority are mere indices for various forms of connectivity. These include the probabilities of node isolation, source-node connection, availability of alternative paths, and, no more than a specified number of pipe failures. Fujiwara and De Silva (1990) have reviewed models incorporating these measures. For water distribution networks the existence of paths is only the most basic requirement. The paths must have adequate capacity to be usable. To be fully functional there must be sufficient pressure in the system to drive and deliver the flow at acceptable pressures.

More realistic measures of the reliability of water distribution networks have extremely high computational requirements. For example, Wagner, Shamir and

Marks (1988) proposed the probability of sufficient supply, defined as: "... the probability that a system can meet a specified level of flow at each demand point." Also, Fujiwara and De Silva (1990) made certain simplifying assumptions about the flow capacity of pipes and proceeded to define the system reliability in terms of the expected maximum flow delivered, as: "... the complement of the ratio of the expected minimum total shortfall in flow to total demand ..."

Regarding design, Templeman (1982) pointed out that cost optimization leads inevitably towards a branched network with no loops and thus with low reliability/resilience. Moreover, the addition of a few minimum diameter loop-completing pipes to such a branched system does very little towards providing alternative flow paths. Also, Yates, Templeman and Boffey (1984) concluded that (discrete) pipe size optimization for distribution networks is NP-hard. Furthermore, reliability requirements increase the complexity of the optimization problem very significantly as reliability analysis is also NP-hard (Wagner, Shamir and Marks, 1988, including references therein). Reliability-constrained cost optimization formulations consequently tend to comprise two separate models, one for cost minimization, the other for determining system reliability. The latter sometimes incorporates another model for hydraulic simulation or analysis. It follows that linking these models in an optimization framework can be exacting and the consequent iterative schemes usually suffer from their lack of a proper mechanism for achieving cost effective increases in reliability.

It is evident from the above discussion that designing cost- and reliability-optimal networks is a complex process. This study therefore approaches this optimum design problem from a different angle. An indirect method that operates on a surrogate reliability measure is proposed. Therefore, recognising the possible variations in demand and the fact that pipes are not

100% reliable, an indicator is sought that reflects the extent to, and ease with which, a distribution network can cope with eventualities for which it is not specifically designed. In other words, an index of versatility is required such that whatever the (unknown) reliability of the system is, when this surrogate measure is optimized the resulting design should have a reliability that at least approximates to the optimum.

This paper demonstrates on a sample network that the flow entropy is a possible measure of versatility or flexibility. Tanyimboh and Templeman (1992) have shown how to calculate the entropy of the flows of any network. Also, flow entropy can be optimized directly. As such, the difficulties of linkage between separate reliability, hydraulic simulation or analysis and cost minimization models can be avoided if entropy is used as a surrogate for reliability in the optimum network design problem. In this paper, an entropy-constrained least cost model is presented and applied to a sample network. The model is non-linear and it can be solved by any suitable constrained non-linear programming algorithm. Also, the results of the example are discussed and these results appear promising.

2. FLEXIBILITY OF WATER DISTRIBUTION NETWORKS

The versatility of a distribution network is governed by its layout, the diameter and reliability of the pipes, the number and location of valves, the number, location and capacity of reservoirs and pumps, etc. This investigation is, however, restricted to the pipes and this section looks at the qualities of pipes and flows that might contribute to flexibility in a looped network. Also, tree-type, branched networks are devoid of flexibility and are consequently unsuitable for urban water supply in general. It follows that for a fixed layout a "good" looped network should be rich in flexibility. Thus it is possible to glean an idea of what contributes to flexibility by examining some desired qualities of distribution networks.

Traditionally, velocities are restricted within fairly tight bounds. This condition forces large pipes with small flows to be replaced by smaller pipes and small pipes with large flows by larger pipes. This process results in a collection of pipes that are not very dissimilar in diameter. In a similar vein, Rowell and Barnes (1982) argued that pipes with extremely high hydraulic gradients are inefficient and should be replaced by larger ones, while pipes with extremely low gradients should be replaced by smaller pipes. The outcome of this philosophy is a network with all pipe diameters in a fairly narrow band.

Also, Goulter and Coals (1986) reported that minimizing the differences in the reliabilities of pipes connected to each node seemed an effective way of improving overall network reliability. It is quite common practice to base pipe failure rates on diameters (Su, Mays, Duan and Lansey, 1987) in which case minimizing the differences in the reliability of pipes is similar in effect to minimizing the differences in diameters. In a looped distribution network with relatively few tree-type branches, there is a node at each end of most links $ij, ij \in IJ$, where IJ is the set of all links of the network. In other words, most nodes will be the meeting point of at least two pipes. However, to simplify the present explanation, it is assumed that this is the case for all nodes. Therefore, any attempt to minimize the differences in the reliabilities or diameters of the pipes meeting at any node n, $\forall n$, will have an effect on the pipe(s) at the other end of each link $kn, kn \in NU_n$, and each link $nj, nj \in ND_n$. NU_n and ND_n represent the set of all the link and any external inflows and all the link and any external outflows respectively at node $n, n = 1,\ldots,N$, where N is the number of nodes in the network. Therefore, imposition of this requirement results in the diameter of each link in the network being as close as possible to that of

all the other links meeting at each end of it. This condition leads to uniformity of all diameters of the network.

Finally, Walters (1988) suggested that reliability could be improved by ensuring an even division of flow between the pipes converging at each junction. Also, Awumah, Goulter and Bhatt (1991) asserted that it is desirable to have links of equal capacity incident at each node. They argued that as the head loss in a pipe is roughly proportional to the square of its flow, a smaller pipe suffers a disproportionately high increase in head loss because of a small absolute increase in its flow.

Taken to their logical conclusion, the above observations imply a need or desire for some form of uniformity throughout a water distribution network. It would seem therefore that flexibility can be improved by increasing this uniformity. However, progress beyond the recognition of this possibility has hitherto been hampered by the inability to systematically generate such uniformity. Recently, however, Tanyimboh and Templeman (1992) have presented a rigorous method for finding the most uniform flows in a general looped distribution network. Furthermore, they concluded that for water supply, designing the pipes to carry those flows would appear to confer considerable invulnerability.

3. FLOW ENTROPY OF LOOPED NETWORKS

The focus of the preceding section was uniformity. It is seen shortly that uniformity can be related to the more fundamental concept of entropy maximization. The philosophy of entropy maximization is concerned with the determination of the most likely probability distribution in situations where many distributions agree equally well with the (little) information that is available about the probabilities that are sought. In other words, the question that is addressed by entropy maximization may be stated in the following way.

Given only some, but not all, of the information required for unique determination of a complete set of probabilities, what is the most likely value of each probability and how can these values be found? Jaynes (1957) established the principles behind probabilistic inference under uncertainty. His maximum entropy formalism maximizes the Shannon entropy of an unknown probability distribution subject to whatever is known *a priori* about the probabilities. Application of the method results in the most uniform probability distribution that is compatible with the available information. That is, this resultant maximum entropy distribution has the property of being as close as possible to the distribution in which all the probabilities are equal, while satisfying all the known or necessary properties of the required probability distribution.

Water distribution network reliability centres around uncertainty: uncertainty about component failures, pipe capacities and/or sufficiency of pressure, flow rerouting, duration of failure and repair, impact of inadequate supply on consumers, variations in demand, etc. The notion of entropy therefore seems appropriate for the reliability of water distribution networks provided appropriate probability-type quantities can be defined. Furthermore, these probabilities must satisfy the necessary conditions for a complete probability space including non-negativity, independence and normality. In the method of finding the most uniform flows of a looped network presented by Tanyimboh and Templeman (1992), appropriate probabilities are defined for the flows. These probabilities are then used to set up the entropy function for the flows of the network. The entropy function for flow networks is presented next.

For a network in which the direction of flow is specified, Shannon's entropy function takes the following form.

$$S/K = S_0 + \sum_{n=1}^{N} P_n S_n \tag{1}$$

where S is the entropy (Shannon, 1948), K an arbitrary positive constant, and P_n the probability of flow arriving at node n. The other symbols are defined shortly. In Tanyimboh and Templeman (1992) the node probability P_n, $n = 1,...,N$, is found by adding the probability of flow arriving at the node by each path. Algebraic manipulation of the above rule gives a convenient equation.

$$P_n = T_n / T_0 \,, \ \forall n \tag{2}$$

in which T_0 is the total supply or demand; T_n is the total inflow, including any external inflow, or the total outflow, including any demand, at node n, $n = 1,...,N$. Also, the entropy of the distribution of the total supply among the sources is S_0 and is obtained from

$$S_0 = -\sum_{i \in I} p_{0i} \ln p_{0i} \tag{3}$$

In the above equation, I is the set of all source nodes and p_{0i} is the fraction of the total flow supplied by source i. Its value is given by

$$p_{0i} = q_{0i} / \sum_{i \in I} q_{0i} = q_{0i} / T_0 \,, \quad \forall i \in I \tag{4}$$

in which q_{0i} is the external inflow at node i. In Eq. (1), S_n is the entropy of the outflows, including any demand, at node n and it is defined by Eqs. (5) and (6) below.

$$S_n = -\sum_{nj \in ND_n} p_{nj} \ln p_{nj}, \quad \forall n \tag{5}$$

$$p_{nj} = q_{nj} / \sum_{nj \in ND_n} q_{nj} = q_{nj}/T_n, \quad \forall n, \quad \forall nj \in ND_n \tag{6}$$

The flow notation q_{ij}, $i, j = 0,1,...,N$, is such that the same symbol q is used for both internal and external inflows and outflows. For an external inflow, the first subscript will be zero and the second, the source node number. Also, the second subscript for a demand will be zero while the first will be the number of the node where the demand occurs. Otherwise, q_{ij} is the pipe flow from node i to node j.

It must be stressed that Eqs. (1) to (6) apply to a looped network with defined flow directions in all pipes. It is therefore important that any non-looped portions of a network under consideration be omitted when setting up S/K. Also, Eq. (1) is slightly different from and more rigorous with respect to the requirements of Shannon's entropy (Shannon, 1948) than the alternatives put forward by Awumah, Goulter and Bhatt (1990, 1991). This and other aspects of network flow entropy are the subject of a forthcoming publication by the present authors.

4. ENTROPY-CONSTRAINED LEAST COST MODEL

Eq. (1) may be maximized subject to a budget constraint and the other constraints of a water distribution network which are presented shortly. Alternatively, an equivalent entropy-constrained cost minimization model in which network entropy is adopted as a measure of flexibility may be used. An entropy-constrained approach is used in this paper so that designs with

different specified levels of entropy may be compared. The entropy-constrained model is presented next.

Entropy apart, the objective and constraint functions have been well documented elsewhere. They are simply listed below for the case of a predetermined layout for completeness.

Cost:

$$\underset{D_{ij}}{\text{Minimize}} \quad C = \gamma \sum_{ij \in IJ} L_{ij} D_{ij}^{e_1} \tag{7}$$

where:

C = total cost of pipes.

γ = constant cost coefficient.

L_{ij} = *effective length of link ij.*

D_{ij} = internal diameter of pipe ij.

e_1 = exponent greater than 1.0.

Flow equilibrium at each node:

$$\sum_{kn \in NU_n} q_{kn} - \sum_{nj \in ND_n} q_{nj} = 0, \quad n = 1,..., N-1. \tag{8}$$

Pipe flow equation:

$$h_{ij} = \alpha L_{ij} (q_{ij}/C_{ij})^{1.852} / D_{ij}^{4.87}, \quad \forall ij \in IJ \tag{9}$$

In Eq. (9), h_{ij} is the head loss in pipe ij, α is a conversion factor for units which equals 10.67 for the S.I. system and C_{ij} is the Hazen Williams coefficient for pipe ij.

Loop energy equation:

$$\sum_{ij \in IJ_l} h_{ij} = 0, \quad l = 1,...,L \qquad (10)$$

in which IJ_l is the set of all pipes in loop l and L is the number of loops.

Energy equation for paths with a constant head loss:

$$\sum_{ij \in IJ_p} h_{ij} = h_p, \quad p = 1,...,P \qquad (11)$$

In the above equation, IJ_p and h_p are, respectively, the set of all the links in path p and the constant head loss along that path; P is the number of specified paths having a constant head loss.

Node pressure limits:
For any node $m, m \in I$, having a constant pressure head,

$$H_{max,n} \geq H_m - \sum_{ij \in IJ_n} h_{ij} \geq H_{min,n}, \quad n = 1,...,N \qquad (12)$$

The notation in Eq. (12) is such that H, H_{max} and H_{min} are respectively, the head and the maximum and minimum allowable head at the node identified by the accompanying node number. The set IJ_n is composed of all pipes in any specified path between any selected constant-head node and node n.

Entropy constraint:

$$S/K \geq S_{min}/K \qquad (13)$$

The parameter S_{min} is a specified minimum value of entropy.

Flow velocity limits:

$$v_{min} \leq 4q_{ij}/\pi D_{ij}^2 \leq v_{max}, \quad \forall ij \in IJ \qquad (14)$$

The parameters v_{max} and v_{min} are respectively the selected maximum and minimum allowable velocities for pipe flows.

Diameter availability:

$$D_{min} \leq D_{ij} \leq D_{max}, \quad \forall ij \in IJ \qquad (15)$$

where D_{max} and D_{min} represent the upper and lower limits, respectively, of the range of available diameters.

Non-negativity of flows:

$$q_{ij} \geq 0, \quad \forall ij \in IJ \qquad (16)$$

Eqs. (7) to (16) represent a non-convex constrained non-linear programming problem in many variables and constraints. The variables are the q_{ij}, D_{ij} and h_{ij}, $ij \in IJ$. For large networks the computational expense of solving this problem can be very large. Any simplification of the model is therefore worthwhile.

First, some simple observations. The exponential increase in cost with diameter, Eq. (7), ensures that a large diameter is not used if a smaller one will do. The D_{max} constraints in Eqs. (15) may therefore be removed. However, it is vital to ascertain from the final results that no diameter exceeds D_{max}. Any such violated constraints should be reinstated and the program re-run, a process to be repeated as necessary. The cost function also makes each v_{max} constraint more likely to be in the active set than the corresponding v_{min}. The v_{min} constraints may therefore be eliminated in favour of post-optimization verification as described above.

The flow velocities are further restricted by all the problem inequalities. The lower bounds on the node pressure heads may therefore be used to eliminate the v_{max} constraints as argued below. Assuming that the system is likely to have just enough pressure, then the $H_{min,n}$ constraints prevent the velocities from being extremely high. In the event that this assumption is invalid the post-optimization confirmation is a safeguard.

Except perhaps for some critical nodes, for example near a reservoir, high residual pressures are not very likely to be a problem in a cost minimization formulation. As already explained, smaller diameters are preferred. It follows that each $H_{min,n}$ rather than the corresponding $H_{max,n}$ is more likely to be critical. Therefore it seems that explicit consideration of the upper limits on the nodal pressures is not generally necessary. Of course this must be confirmed at the solution.

Regarding the minimum service pressure constraints, for a fixed layout and set of flow directions, nodes furthest from the sources tend to have the lowest pressures. The minimum service pressure constraints for these nodes are more likely to be binding than the constraints for most of the other nodes. Therefore, the most downstream nodes have to be considered explicitly when

specifying minimum service pressure constraints. The constraints for most of the other nodes may be omitted, but there is a possibility that some of these omitted constraints will be violated. Referring to the most downstream nodes as terminal nodes, if the minimum service pressure of some non-terminal node is greater than that of the terminal node downstream of it, then, the minimum service pressure constraint for this node may be violated.

The final observation concerns reintroduction of bounds initially assumed slack into the problem. As both D_{min} and D_{max} cannot simultaneously be binding for unequal D_{max} and D_{min}, each D_{max} constraint that is violated should replace the corresponding D_{min} constraint and vice versa. This also applies to the velocity and pressure head limits.

Obviously the pipe flow and node equilibrium constraints are equalities that may be used to eliminate some variables and constraints. Eqs. (9) therefore reduce to a function $f(L_{ij}, D_{ij}, q_{ij}, C_{ij})$ whose value is h_{ij} for specified $L_{ij}, D_{ij}, q_{ij}, C_{ij}$, thus eliminating the h_{ij} variables and constraints. Also, for any looped network it is possible to specify one independent flow variable for each loop. All other flows may then be expressed in terms of the independent flows using the node flow continuity equations which are therefore eliminated.

All these arguments may be used to reduce the size of the problem represented by Eqs. (7) to (16) and, in consequence, to reduce the computational effort needed for its solution. However, post-solution verification that no omitted constraints have been violated is essential.

5. NUMERICAL EXAMPLE

The layout and demands of the sample network are taken from Alperovits and Shamir (1977). The reduced version in this paper does not include the link between the reservoir and the first demand node. Also, the inflow at node 1 is the net flow. All node and pipe data are as given in Figure 1 and Table 1. The cost of pipes per metre is taken as $\gamma D^{2.4}$ in which the diameter D is in metres and $\gamma = £900$.

The model, simplified as explained in the preceding section, was solved for various values of entropy with the NAG library routine E04VDF. The values of entropy used were selected such that the range of possible values for the sample network was well covered. The results are summarised in Table 2. As all links are of equal length, some statistical measures of spread for the diameters are also shown.

The hydraulic performance of the 5 entropy-constrained designs was assessed by simulation using a consumption-based approach that assumes total demand satisfaction. Two kinds of emergency were considered. The results in Tables 3 and 4 are for single link failures with normal demands. Tables 5 and 6 are for the "fire fighting" loads in Table 1 which mimic fire fighting requirements at each node in turn.

For each pipe failure (with normal demands) or each "fire fighting" load (on the full network), the notional head required at the source to satisfy all demands is found as follows. First, a demand-driven simulation of the (full or reduced) network is carried out. Second, the head loss in any specified path from the source to the most pressure-critical node is found by adding the head loss in all links in the path. Third, the total head loss is added to the minimum desirable head at the most pressure-critical node. Strictly speaking, it is sufficient to calculate the notional usable head H required to satisfy all demands as follows.

$$H = \max < \sum_{ij \in IJ_n} h_{ij} \quad \forall n \in t > \tag{17}$$

in which t represents the set of all terminal nodes, i.e., nodes not having any internal outflow. It may be noted that in general t changes according to changes in the demands and layout of the network. Also, in a network in which there are multiple sources, H corresponds to the maximum head loss that occurs between any source-terminal node pair. This rather devious parameter allows quite rigorous comparisons. Other measures such as percentage demand satisfied at adequate pressure do not account for non-zero shortfalls in pressure or supply.

Also, Rowell and Barnes (1982) suggested that, for a given flow, the efficiency of a pipe can be gauged from the rate at which the pipe dissipates energy. Therefore, it would appear that a network can be assessed on the basis of the total amount of energy that the network dissipates per unit time. The total energy dissipated, E, is calculated from Eq. (18).

$$E = \rho g \sum_{ij \in IJ_0} q_{ij} h_{ij} \tag{18}$$

where IJ_0 is the full or reduced network as appropriate, ρ and g are the density of water and acceleration due to gravity respectively.

6. DISCUSSION

Much of interest can be said about the results of Tables 2 to 6. Each table will be considered in turn. Firstly, the diameters and flows of Table 2 clearly show the network becoming less and less implicitly branched as the entropy increases. This observation is backed by the sample standard deviation and coefficient of variation of the diameters. Furthermore, the average of the diameters increases with entropy. Mechanical reliability is thus enhanced, as several researchers (for example, Clark, Stafford and Goodrich, 1982) have reported that large pipes have lower rates of failure than smaller ones.

Also, the efficiency of the method in terms of cost seems extremely high. The increase in cost from a design with a slack entropy constraint of 1.578, the minimum value for the network of Figure 1, to the design corresponding to the maximum entropy value of 1.915 is only 5%. There are two complementary reasons for this low percentage increase in cost. The first is that larger diameters are reduced while simultaneously increasing small ones. The cost of making small pipes larger is thus, more or less, offset. The second factor is the exponential cost function. It ensures that, if all lengths are equal and the average diameter is constant, the most uniform set of diameters is the cheapest.

The above observation would seem to contradict the experience from conventional cost minimization models. This apparent anomaly can be explained. Cost minimization proceeds with an overall reduction in the diameters, up to a point where further reductions would lead to constraint violation. Cost savings are therefore possible because smaller pipes are considerably cheaper than large ones. In other words, it is cheaper to increase the diameter of a small pipe than it is to achieve the same increase in a larger pipe. In this model, the entropy constraint indirectly limits the overall reduction in diameters. Once this limit is reached, further cost savings can only come by exploiting the fact that for a fixed number of pipes and a fixed "average" diameter, the cheapest option is to avoid larger pipes as far as possible. In consequence, the most uniform diameters are preferred.

In the present example, the 5% increase in cost is a small price to pay for the overall increase in mechanical reliability or diameters. Also, it would seem that even for larger networks, any increase in cost is not very likely to be very much higher than this figure of 5%. This is because there will be many more non-minimum diameter pipes than minimum diameter pipes since there is only one "redundant" or loop-completing link per loop. Therefore, by reducing the

diameter of the largest pipes slightly, it will be possible to increase the diameter of the smallest pipes considerably without substantially increasing the total cost. It would seem, therefore, that the greater the number of loops, the more there will be room for manoeuvre.

Secondly, the link failure results in Table 3 show several noteworthy points. First, the maximum notional source pressure head required decreases very rapidly as entropy increases. Of more importance is the accompanying reduction in the average value. It may therefore be inferred that problem areas are not being improved at the expense of other areas. In particular, the high rate of reduction underlines quite dramatic system-wide gains in performance. There is, however, a slight increase in the maximum value in the last column. On the other hand, this increase is outweighed by the large concurrent improvement in the next worst case, as demonstrated by the continued reduction in the average head.

The improvement in hydraulic performance observed in Table 3 can be explained in terms of uniformity in diameters and flows respectively. For pipes with given fixed values of length, friction coefficient and flow, the head loss is nearly inversely proportional to the fifth power of the diameter. The consequence of this relationship is that, for example, doubling the diameter reduces the head loss by a factor of 2^5 or 32; trebling the diameter reduces the head loss by a factor of 3^5 or 243. It follows that an implicit tree-type network is likely to experience much higher increases in head loss if a link fails or network flows are increased than a network having pipes with (similar) medium diameters. Additionally, the head loss in a pipe is nearly proportional to the square of the flow rate. By virtue of this quadratic head loss-flow rate relationship, a network in which the pipes are designed to carry uniform flows is less likely to suffer very high increases in head loss than a network in which the pipes are designed to carry flows that are very dissimilar in magnitude. The above factors, including the observation that pipes with the most uniform diameters are the cheapest if the lengths are equal, explain why it is possible to considerably improve the implicitly branched network at a cost increase of only 5%.

Thirdly, the values in Table 4 show a similar behaviour to those in Table 3. However, in assessing designs on the basis of energy dissipation it is useful to realise that the same flow will dissipate less energy in a large more costly pipe than in a smaller one. Therefore a design with a low rate of energy dissipation may not necessarily be cost effective. Nevertheless, for each design, a deeper interpretation is possible by comparing the reduced networks to the full network. For S/K = 1.6, for example, failure of pipe 4-6 hardly increases the amount of energy dissipated. This suggests that arc 4-6 is being underused in this implicitly branched network design. A more important comparison is between the average rate of energy dissipation for the reduced networks and the rate for their corresponding full network. As demonstrated by the last row of Table 4, the difference between these values falls very rapidly as the entropy increases, suggesting that a design with a high value of entropy is much more resilient than a counterpart having a lower value of entropy.

Also, it can be inferred from Table 4 that up to an entropy value of approximately 1.7, links 2-4 and 4-6 would apparently be really useful only if a major link failed. However, the pressure requirements may be prohibitive, as demonstrated by Table 3. This casts doubt upon the value of having minimum diameter loop-forming pipes in an essentially branched network. Furthermore, from the point of view of stagnation, it is questionable how much these minimum diameter pipes contribute to circulation in the present example in view of the magnitude of their flows.

Fourthly, the fire scenario results in Tables 5 and 6 clearly show resilience increasing with entropy. Despite a slight oscillation in the values for each node, the averages exhibit a steady downward trend as entropy increases. The rate of improvement is, however, somewhat lower than is the case for link failures. Thus, all the results of Tables 2 to 6 indicate that, for the sample network, flow entropy is a reasonable measure of flexibility.

There remains the question of the correct entropy value to specify, or the true meaning of entropy in the context of water distribution network reliability and design. The importance of uncertainty has already been highlighted. Entropy as used herein is a measure of link flow uniformity that depends on the layout, spatial distribution of supply and demand, and the size of the network in terms of the number of nodes and links. For any fixed looped layout determination of its maximum flow entropy is relatively straightforward. For the purpose of looped distribution network reliability and design, the datum of zero is ascribed to all tree-type networks. The reason is that any link failure in a branched network results in complete loss of supply to at least one demand point. Such systems possess very little damage tolerance and, as such, will usually be unacceptable from a reliability perspective. More fundamentally, there is no uncertainty associated with the flows of the links of a tree-type branched network. The minimum entropy value for a looped network, however, will be considerably higher than zero.

The application of entropy as a convenient design tool does not eliminate the responsibility of ensuring that the design is of adequate reliability. Therefore some form of reliability assessment will still be necessary. The nature and scope of this assessment will depend on the circumstances. It is therefore envisaged that the entropy-based model will ultimately form part of a design and verification routine. To start the process, a low entropy value of zero is specified. For this initial entropy value, the entropy constraint will be slack at the solution with a value near the minimum for the network. This provides a possible starting point for the design and assessment cycles. The design and assessment process is terminated when the desired goals are achieved.

The final comment concerns the success of the optimization model which is largely due to the availability of analytical partial derivatives of the entropy function. Also, the structure of this function permits easy assembly of the function and its derivatives for general looped networks.

7. SUMMARY AND CONCLUSIONS

A method for indirectly optimizing the reliability of water distribution networks has been presented. The procedure has been shown to be effective on a sample network in several respects including cost, hydraulic performance, resilience and mechanical reliability. In the present paper, the complexities of direct reliability optimization have been bypassed by using flow entropy as a surrogate for reliability. Node-by-node assembly of the entropy function and its partial derivatives permit general application of the formalism without a disproportionate increase in computational effort. However, the method would apparently suffer from dimensionality for multiple load cases. On the other hand, the simulated results suggest that the entropy constraint is similar in effect to the inclusion of a multiplicity of load cases, provided a threshold value of entropy is at least equalled.

Also, some evidence from the literature foreshadowing entropy has been collated. The simulated results confirm that there is a distinct need for uniformity in a water distribution network including the uniformity of pipe diameters and flow rates. Moreover, it is found that, for the sample network, merely linking the end points of a core tree with minimum diameter pipes for the sake of looping does not necessarily enhance performance or reliability.

However, simulation on a larger scale is needed to establish that the conclusions are not network-specific. Also, the method as presented herein is incapable of handling other network components than pipes and this is clearly an area for further work. Furthermore, although the method goes a long way beyond simple assurance that alternative paths to demand centres exist, flow entropy operates on flows only. Perhaps further improvements would be possible if a method that directly recognizes the importance, regarding flexibility, of both diameters and flows can be found. Finally, the true relationship between entropy and network reliability remains to be explored and exploited.

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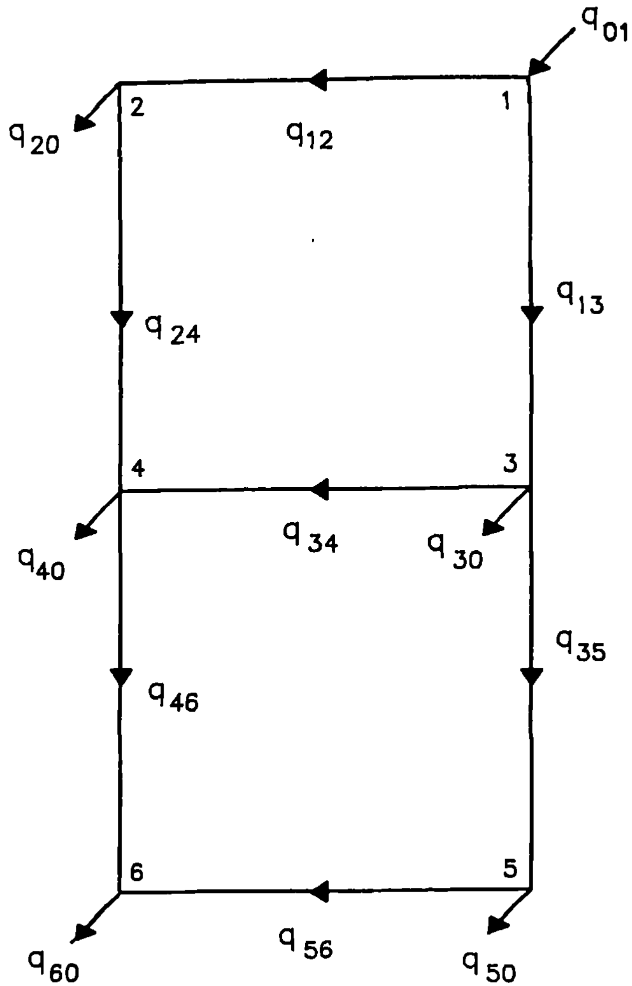

$L_{IJ} = 1000m, \quad C_{IJ} = 130, \quad \forall_{IJ}$

Fig. 1 Layout of Sample Network (with flow notataion)

NODE PRESSURES AND FLOWS

Node	Total Head (m)	Least Head (m)	Design Load (m^3/s)	"Fire Fighting" Loads (m^3/s)				
				Case1	Case2	Case3	Case4	Case5
1	20.00	20.00	-0.284	-0.506	-0.501	-0.459	-0.442	-0.478
2	0.00	0.00	0.028	0.250	0.028	0.028	0.028	0.028
3	0.00	0.00	0.033	0.033	0.250	0.033	0.033	0.033
4	0.00	0.00	0.075	0.075	0.075	0.250	0.075	0.075
5	0.00	0.00	0.092	0.092	0.092	0.092	0.250	0.092
6	0.00	0.00	0.056	0.056	0.056	0.056	0.056	0.250

Table 1. Node Data for Sample Network.

DIAMETERS AND FLOWS FOR RANGE OF ENTROPY VALUES

Link	S_{min}/K									
	1.600		1.700		1.800		1.900		1.915	
	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)	Dia. (mm)	Flow (m^3/s)
1-3	401	0.251	390	0.234	384	0.223	365	0.199	367	0.200
2-4	100	0.005	165	0.022	191	0.033	238	0.057	235	0.056
3-5	337	0.147	337	0.147	329	0.139	281	0.101	294	0.110
4-6	100	0.001	100	0.001	151	0.009	250	0.047	234	0.038
5-6	262	0.055	262	0.055	249	0.047	152	0.009	185	0.018
1-2	165	0.033	203	0.050	224	0.061	263	0.085	261	0.084
3-4	237	0.072	213	0.054	215	0.051	247	0.065	234	0.057
Mean	229		239		249		257		258	
σ_{n-1}	115		100		81		63		58	
$\frac{\sigma_{n-1}}{mean}$	0.504		0.419		0.325		0.245		0.224	
Cost	0.251		0.254		0.259		0.261		0.263	

Table 2. Optimum Design for Different Values of Entropy: N.B. The entropy constraint becomes slack at $S_{min}/K = 1.578$, and at a cost of 0.250. All costs are in units of £ 10^6.

SOURCE HEAD (m) FOR REDUCED NETWORKS

Failed Link	S_{min}/K				
	1.600	1.700	1.800	1.900	1.915
1-3	8454.6	1065.2	567.7	219.3	231.6
2-4	20.7	28.0	27.6	35.7	34.7
3-5	2895.5	2899.2	453.9	215.5	141.9
4-6	20.4	20.4	22.6	87.7	47.5
5-6	500.3	501.4	92.2	22.7	26.6
1-2	155.9	54.1	43.9	46.3	46.2
3-4	259.5	75.4	37.2	39.8	33.5
Mean	1758.1	663.4	177.9	95.3	80.3
Max	8454.6	2899.2	567.7	219.3	231.6

Table 3. **Notional Head Requirements for Link Failures:** The source head values are those needed to meet the design loads and minimum pressures of Table 1 with the failed link closed off.

TOTAL ENERGY (MW) FOR NORMAL FLOWS

Failed Link	S_{min}/K				
	1.600	1.700	1.800	1.900	1.915
1-3	21.405	2.688	1.434	0.550	0.584
2-4	0.047	0.054	0.057	0.072	0.071
3-5	4.269	4.279	0.712	0.262	0.210
4-6	0.047	0.046	0.045	0.087	0.062
5-6	0.321	0.321	0.090	0.045	0.047
1-2	0.099	0.084	0.084	0.103	0.102
3-4	0.240	0.093	0.066	0.074	0.066
Mean	3.775	1.081	0.356	0.171	0.163
Max	21.405	4.279	1.434	0.550	0.584
Full Network	0.0467	0.0459	0.0444	0.0444	0.0439
"Mean" − "Full Network"	3.7283	1.0351	0.3116	0.1266	0.1191

Table 4. **Network Energy Dissipation for Normal Flows:** The energy values are those that occur when the (full or reduced) network meets the design loads and minimum pressures of Table 1 with any failed link closed off.

SOURCE HEAD (m) FOR INCREASED FLOWS

Node	S_{min}/K				
	1.600	1.700	1.800	1.900	1.915
2	424.7	130.4	83.3	50.2	51.5
3	38.0	38.4	38.1	37.9	37.8
4	101.6	102.2	73.2	56.0	55.8
5	51.0	51.3	49.8	60.8	53.5
6	113.2	113.6	106.1	106.9	102.8
Mean	145.7	87.2	70.1	62.4	60.3
Max	424.7	130.4	106.1	106.9	102.8

Table 5. **Notional Head Requirements for "Fire Fighting":** The source head values are those needed to meet the design loads and minimum pressures of Table 1, except that at each node in turn 0.25 m^3/s replaces the design load.

TOTAL ENERGY (MW) FOR INCREASED FLOWS

Node	S_{min}/K				
	1.600	1.700	1.800	1.900	1.915
2	1.108	0.417	0.229	0.218	0.222
3	0.150	0.151	0.150	0.157	0.155
4	0.313	0.314	0.249	0.209	0.212
5	0.182	0.181	0.181	0.207	0.193
6	0.370	0.370	0.352	0.354	0.345
Mean	0.425	0.287	0.246	0.229	0.225
Max	1.108	0.417	0.352	0.354	0.345

Table 6. **Network Energy Dissipation for "Fire Fighting":** The energy values are those that occur when the network meets the design loads and minimum pressures of Table 1, except that the design load is replaced by 0.25 m^3/s at each node in turn.

ABBREVIATED TITLE

FLEXIBLE WATER NETWORKS

FIGURE CAPTION

Figure 1 Layout of sample network (with flow notation).

TABLE CAPTIONS

KEY WORDS

WATER NETWORKS, RELIABILITY, RESILIENCE/FLEXIBILITY, ENTROPY

APPENDIX B3 MAXIMUM ENTROPY FLOWS FOR SINGLE-SOURCE NETWORKS

MAXIMUM ENTROPY FLOWS FOR SINGLE-SOURCE NETWORKS

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ABSTRACT

This paper was prompted by growing evidence that Shannon's measure of uncertainty can be used as a surrogate reliability measure for water distribution networks. This applies to both reliability assessment and reliability-governed design. Shannon's measure, however, is a non-linear function of the network flows. Therefore, the calculation of maximum entropy flows requires non-linear programming. Hence, a simpler, more accessible method would be most useful. This paper presents an alternative and rigorous method for calculating maximum entropy flows for *single-source* networks. The proposed method does not involve linear or non-linear programming. Also, it is not iterative. Consequently, the method is very efficient. In this paper, the methodology is described, several examples are presented and an algorithm is suggested.

KEYWORDS: Networks, water supply, entropy, reliability

NOTATION

E_n, E_{kn} = event identified by subscript

I = set of all source nodes

K = arbitrary positive constant

N = number of nodes

ND_n = set of all links or nodes immediately downstream of node n

NP_n = number of paths from the source to node n

NU_n = set of all links or nodes immediately upstream of node n

$p_{0i} = q_{0i}/T_0$

$p_{nj} = q_{nj}/T_n$

$p(E_{kn})$ = probability of E_{kn}

$P_n \equiv p(E_n)$ = probability of E_n

q_{i0} = external outflow at node i

q_{nj} = flow from node n to node j

q_{0i} = external inflow at node i

S = Shannon's entropy

S_n = entropy of outflows at node n

S_0 = entropy of external inflows

T_n = total outflow from node n

T_0 = total supply or demand

x^* = optimum value of x

1. INTRODUCTION

Very recently, there has been growing interest in the potential applications of the maximum entropy formalism in water distribution networks. The areas of interest, so far, have been assessment of reliability, layout optimization and/or optimum design with reliability considerations. So far, the results have been encouraging.

Awumah, Goulter and Bhatt[3,4] used the Shannon entropy function[8] as the basis of some measures for network redundancy. The development of these measures involved some intuitive input. They presented evidence[4] that the measures could be used for the design of reliable water distribution networks. The minimum cost gradient formulation of Quindry, Brill and Liebman[7] was modified slightly by replacing all the minimum diameter constraints by minimum nodal entropy limits, one for each node. Awumah, Goulter and

Bhatt[4] observed that the modified, non-linear model, could be used for optimizing both the layout and diameters of the network.

Also, Awumah and Goulter[2] obtained trade-off curves for a sample network. These included a curve for cost versus reliability. The reliability measure used was node pair reliability. A second curve related cost to network entropy. The shape of these curves showed a remarkable degree of similarity. If this holds for water distribution networks generally, it could be interpreted as evidence of a close relationship between entropy and (mechanical) reliability.

Tanyimboh and Templeman[9] have rigorously established the appropriate entropy function for the flows of a looped transportation network. The approach relies on a multiple probability space model and the conditional entropy formula of Khinchin[6]. It applies to any network with known nodal inflows and outflows. Also, it requires a specified flow direction for each arc. Maximum entropy flows were calculated for a sample network and it was observed that, for the sample network, there was uniformity in the probability that certain key nodes receive their flow from each source. The importance of this uniformity in the context of reliability was highlighted.

Subsequently, Tanyimboh and Templeman[10] collated evidence of the need for uniformity of the flows and/or diameters of a distribution network. The multiple probability space conditional entropy measure was then incorporated as a constraint in a non-linear cost minimization model. They observed that as the specified lower bound upon the value of entropy was increased, so too did the resilience of the resulting design. Resilience represents the ability to handle load patterns which are different from those specifically designed for; in the example of Ref. [10] these different cases were pipe failures and/or large fire-fighting demands. Also, it was noted that, on average, the diameters increased. This was taken to be evidence of a correlation between entropy and (mechanical) reliability. Another related observation, and perhaps the

most important, was that implicit tree-type branchedness (Templeman[11]) decreased as the entropy increased.

The above research provides ample justification for more extensive research into the properties and other potential applications of maximum entropy flows. However, the entropy function is non-linear. Also, maximum entropy flows must satisfy flow equilibrium at each node. Consequently, the determination of maximum entropy flows needs constrained non-linear programming. This is rather restrictive. A simpler way of calculating maximum entropy flows would therefore be highly desirable. This paper presents a simple method for calculating maximum entropy flows for *single-source* networks. The proposed method is path based. However, explicit path enumeration is not used. This has been made possible through a simple, but efficient, algorithm for determining the number of paths from the source to each node. Also, neither linear nor non-linear programming is involved. Further, unlike non-linear programming, the procedure is not iterative. Thus the method has a very high computational efficiency.

In this paper, first, the multiple probability space entropy function is presented. Then, some of the results obtained by Awumah, Goulter and Bhatt[4] are interpreted on the basis of this function. This is followed by a description of the proposed method of calculating maximum entropy flows for single-source networks. Examples are solved and it is shown that the results are the same as those given by maximizing the entropy subject to continuity at each node. Also, an algorithm for determining the number of paths from the source to each node is presented. Finally, another algorithm, for calculating maximum entropy flows for a single-source network is presented.

2. ENTROPY FUNCTION FOR LOOPED FLOW NETWORKS

For a distribution network with loops and all the flow directions specified, Shannon's entropy may be written as in Eq. (1).

$$S/K = S_0 + \sum_{n=1}^{N} P_n S_n \tag{1}$$

where S is the entropy (Shannon[9]), K an arbitrary positive constant, and P_n, $n = 1,...,N$, the probability of flow arriving at node n, $n = 1,...,N$, where N is the total number of nodes in the network. The value of P_n may be obtained from Eq. (6), which will be derived shortly. The other terms are defined below. The entropy of the external inflows is S_0 where

$$S_0 = -\sum_{i \in I} p_{0i} \ln p_{0i} \tag{2}$$

In the above equation, I is the set of all source nodes and p_{0i} is the proportion of the total supply to the network that is provided by source i. Its value is given by

$$p_{0i} = \frac{q_{0i}}{\sum_{i \in I} q_{0i}} = \frac{q_{0i}}{T_0}, \quad \forall i \in I \tag{3}$$

where q_{0i} is the external inflow at node i and T_0 is the total supply or demand. In Eq. (1), S_n, $n = 1,...,N$, is the entropy of the outflows, including any demand, at node n, $n = 1,...,N$. It is defined by Eqs. (4) and (5) in which p_{nj} is the fraction of T_n carried by link nj, where T_n, $n = 1,...,N$, is the total outflow, including any demand, from node n, $n = 1,...,N$. Also, p_{nj}, $\forall n$, $j = 0$, represents the fraction of T_n that satisfies consumption at node n.

$$S_n = -\sum_{nj \in ND_n} p_{nj} \ln p_{nj}, \quad \forall n \tag{4}$$

The set ND_n, $n = 1,...N$, consists of all the outflows, including any demand, from node n, $n = 1,...,N$.

$$p_{nj} = \frac{q_{nj}}{\sum_{nj \in ND_n} q_{nj}} = \frac{q_{nj}}{T_n}, \quad \forall n, \forall nj \in ND_n \tag{5}$$

The symbol q is used for both internal and external inflows and outflows. For an external inflow, the first subscript will be zero and the second, the source node number. Also, the second subscript for a demand will be zero whereas the first will be the number of the corresponding node. Otherwise, q_{ij}, $i, j = 1,...,N$, is the pipe flow from node i to node j. Tanyimboh and Templeman[10] stated that Eq. (6) is a convenient formula for the node probabilities, P_n, $n = 1,...,N$.

$$P_n = T_n / T_0, \forall n \tag{6}$$

There follows a simple derivation of this equation. Define E_n, $n = 1,...N$, as the event that a particle in the network reaches node n, $n = 1,...,N$. Then, $P_n \equiv p(E_n)$, $n = 1,...N$, is the probability that the event E_n, $n = 1,...,N$, occurs. In general, each event is conditional upon the events upstream of it. These events are consequently not independent. The probability rule for conditional events may therefore be written, as in Eq. (7), for these events.

$$p(E_k \cap E_n) = p(E_n | E_k) p(E_k), \quad \forall n, \ \forall k \in NU_n \tag{7}$$

where NU_n, $n = 1,...N$, is the set of all link and external inflows at, and all nodes immediately upstream of, node n, $n = 1,...,N$. By virtue of Eq. (7), the entropy of a network cannot be a simple sum of the entropy at each node (Khinchin[6]).

The ratio q_{ij}/T_0, $i, j = 1,...,N$, is the flow carried in link ij, expressed as a proportion of the total supply. In other words, q_{ij}/T_0 is the probability that a particle entering the network will flow through link ij. For the set of all links converging on a node, these probabilities are mutually exclusive. A particular particle can arrive at the node through any, but only one, of the links. Also, as flow cannot arrive at the node other than by the links converging on that node, the set is exhaustive. For a node n, $n = 1,\dots,N$, E_n happens whenever there is flow in a link kn, $kn \in NU_n$. Let E_{kn}, $\forall n$, $kn \in NU_n$, be the event that there is flow in link kn. Thus

$$p(E_n \mid E_{kn}) = 1, \ \forall n, \ \ \forall kn \in NU_n \tag{8}$$

The probability of flow arriving at node n is the joint probability that flow reaches the node by all links supplying it, i.e.,

$$p(E_n) = p\Big(\bigcup_{kn \in NU_n} \{ E_{kn} \cap E_n \} \Big) \tag{9}$$

Therefore, applying Eq. (7) for $p(E_{kn} \cap E_n)$,

$$p(E_n) = p\Big(\bigcup_{kn \in NU_n} \{ E_{kn} \cap E_n \} \Big) = \sum_{kn \in NU_n} p(E_{kn} \cap E_n) = \sum_{kn \in NU_n} p(E_{kn}), \quad n = 1,\dots,N. \tag{10}$$

The second equality of Eq. (10) holds because the E_{kn}, $\forall n$, $\forall kn \in NU_n$, have been shown to be mutually exclusive. As explained, $p(E_{kn})$ is given by q_{kn}/T_0. Substituting for $p(E_{kn})$ in Eq. (10) gives the desired result, Eq. (6).

The Eqs. (1) to (6) apply to a network with loops, in which the flow direction in all pipes is defined. Therefore, to use Eq. (1), any non-looped portions of a network under consideration should be omitted when evaluating the network entropy.

The equations suggested by Awumah, Goulter and Bhatt[4] will be examined next. These equations appear superficially to resemble the current Eqs. (1) to (6) but are different in detail. Reasons for preferring the current equations will be presented. Eqs. (6) and (8) of Awumah, Goulter and Bhatt[4] are reproduced

here as Eqs. (A) and (B). In Eqs. (A) to (C), the original notation has been preserved, where possible.

$$\hat{S} = -\sum_{j=1}^{N}\sum_{i=1}^{n(j)} (q_{ij}/Q_0) \ln(q_{ij}/Q_0) \tag{A}$$

where $\hat{S}$ is the network entropy and Q_0 is the sum of all *link flows*, as opposed to T_0 which is the total supply or demand. Also, $n(j)$ is the number of *internal inflows* at node j.

$$\hat{S} = \sum_{j=1}^{N} (Q_j/Q_0)\bar{S}_j - \sum_{j=1}^{N} (Q_j/Q_0) \ln(Q_j/Q_0) \tag{B}$$

in which $\bar{S}_j$ is the entropy of the *internal inflows* at node j. It is the same as S_j in the original publication. Therefore,

$$\bar{S}_j = -\sum_{i=1}^{n(j)} (q_{ij}/Q_j) \ln(q_{ij}/Q_j), \quad j = 1,\dots,N \tag{C}$$

Also, Q_j is the sum of the *internal inflows* at node j, as opposed to T_j which is the sum of all inflows, including external inflows, at the node.

Comparing (A) to (C) with the current (1) to (6), the flows or elementary events, q_{ij}, overlap. This is readily seen from the consideration that $q_{ij} \cap q_{jk} \neq 0$, $\forall i, j, k = 1,\dots,N$. It follows that the probability-like terms q_{ij}/Q_0 in Eq. (A) are not independent. As such, Eq. (A) (or the equivalent Eq. (B)) is not appropriate for those terms (Shannon[9]).

Also, perhaps the most obvious difference between Eqs. (1) and (A) is that the latter does not (directly) account for the spatial distribution of the external inflows and outflows. In Eq. (1), the relative magnitudes of the sources is accounted for by S_0. Also, the abstraction at each node is accounted for. This is achieved by defining the nodal entropies S_n as the entropy of all the outflows, including consumption, at each node.

3. CALCULATING MAXIMUM ENTROPY FLOWS FOR SINGLE-SOURCE NETWORKS

In a single-source network, all paths start at the source. Consider any demand node served by more than one path. Given no further information about the paths, there is no reason for any path to be preferred over any other path to the demand node. This accords with Laplace's principle of insufficient reason. More appropriately, it is a direct consequence of the maximum entropy formalism[6]. Therefore, all the paths supplying a node should have the same probability of doing so. This means that flow to the node should be distributed equally amongst all the paths supplying the node.

Therefore, to obtain the maximum entropy flows, each node should be taken in turn and its demand divided equally amongst all paths supplying the node. The final network flows are then obtained by superposition of these path flows. That is, the flow for all paths through a link should be added to obtain the flow in that link. These are the link maximum entropy flows. The maximum value of the flow entropy for the network may then be calculated, using Eq. (1).

The network of Figure 1 will be used to demonstrate the above points. The demand at each node is treated separately, as shown in Figure 2. In Figure 3, the flow in each arc is obtained by adding the flow in all paths using that arc. These are the maximum entropy flows. Substituting these flows in Eq. (1) gives $(S/K)^* = 2.159$

To check that the above values are correct, Eq. (1) was maximized, subject to flow equilibrium at each node and non-negativity of all the link flows. First,

all the flows were expressed in terms of three selected flows using four of the nodal continuity equations. Equilibrium at the fifth node holds automatically because the inflows balance the outflows. The resulting objective function, with only three variables, was then maximized subject to lower bounds on these variables. The lower bounds were calculated from the non-negativity conditions. The NAG library routine E04JAF was used for the maximization. The results are shown below. They are identical to those obtained above.

$(S/K, q_{14}, q_{13}, q_{23}, q_{35}, q_{12}, q_{25}, q_{34})^*$

$= (2.159, 5.000, 18.000, 18.000, 16.000, 26.000, 8.000, 10.000)$

Maximum Entropy Flows Algorithm and Example

The method of superposition used in Figures 2 and 3 is not very practical. First, nodal flow routing, as in Figure 2, requires the tracing of all paths to each node. Also, a knowledge of the interdependencies between the paths serving each node is needed. For example, consider Figure 2d. Link 1-2 carries a flow of 16 units because it is known that the link is shared by two paths serving node 5. On the other hand, the flow in link 1-3 is 8 units because it is known that only one path serving node 5 uses that link. Furthermore, in a looped network, each link will be processed many times, as in Figure 2, for example. It would be more efficient if the flow in each link could be calculated in a single operation. In consequence, the approach of Figure 2 and 3 is quite laborious. Also, the effort needed increases very significantly as the number of nodes or links increases. The method to be described next is derived from the method of superposition of path flows which has been presented. However, it addresses all the above weaknesses, except path enumeration, which will be dealt with shortly. The description of the method is fairly general but is based on the network of Figure 1 for clarity. Following the description, algorithms are presented for the proposed method.

Consider Figure 4. The number of paths to each node is enclosed in a box next to the node. Nodes 4 and 5 are terminal nodes, which do not have any link outflows. The procedure starts with any terminal node; say, node 4. The total outflow at that node is divided by 3, this being the number of paths to it. The quotient is then multiplied by 1 and 2 respectively, these being the respective number of paths to nodes 1 and 3, the immediate upstream supply nodes to node 4. The products, respectively, are the flow in links 1-4 and 3-4.

The next step is to choose any node immediately upstream, whose link outflows have all been calculated. The procedure explained for node 4 is repeated with *all* the appropriate upstream nodes taking the place of nodes 1 and 3. Returning to Figure 4, both nodes 1 and 3 have unknown link outflows. In consequence, they cannot be treated yet.

At this point, the procedure stops and re-starts at any terminal node that has not yet been dealt with. That is, node 5 in the present example. If node 5 is processed as explained for node 4, the flow in link 2-5 is 8 units and for link 3-5, 16 units.

At this stage, the only unprocessed node with all its outflows known is node 3. Its total outflow is 36 units. This flow is partitioned according to the aforementioned procedure in the ratio 1:1 between the two incoming links. This is equivalent to dividing the total outflow from node 3 by 2 and multiplying the result by 1 in each case.

The flow in link 1-2 can now be found. It is the sum of the outflows from node 2, including the demand at node 2. The process ends here.

The flows obtained by the procedure just described are identical to those found by superposing equal path flows for each node.

A further refinement to the method concerns path enumeration. Path enumeration is not a realistic proposition, even for networks of only modest size. However, there is a way round this difficulty. It can be observed, for example in Figure 4, that each boxed number is the sum of all the boxed numbers immediately upstream. In other words, the number of paths to each node is the sum of the number of paths to all nodes upstream of, and directly supplying, the node being considered. This is a fact which can be exploited to weight the nodes and thus avoid explicit path enumeration. The steps involved are as follows.

1. Assign [1] to the source.

2. Select any node whose upstream nodes have all been processed. Add the numbers assigned to all nodes immediately upstream of the chosen node. Assign the total to the present node.

3. Repeat step 2 until all nodes have been processed.

Throughout, the assumption that the direction of flow in each link is known continues to apply. Also, it must be noted that this method of calculating the number of paths to each node applies to single-source networks only.

A final detail of the proposed method for calculating maximum entropy flow for single-source networks is concerned with the order in which nodes can be processed when weighting the nodes or calculating link maximum entropy flows. For both node weighting and flow calculation, it is desirable to know, at each stage, which nodes can be processed. When calculating flows, a node can be processed only if all its link outflows are known. Consider Figure 4 again. In describing how link flows are calculated, the nodes were selected in the order: 4,5,3,2. Another possible order is: 5,4,3,2. These two sequences are, respectively, the reverse of the two possible sequences for calculating the number of paths supplying each node. Consequently, a possible nodal sequence for flow distribution will be available if the nodes are numbered such that the order matches the node weighting sequence, as in Figure 4, for example. Thus the right sequence is obtained if each node is numbered only after all nodes upstream of it have been numbered. The algorithms presented subsequently herein, for node weighting and for calculating maximum entropy flows, assume that the nodes of the network have been numbered according to this convention. The nodes may therefore be numbered with the following algorithm.

Node Numbering Algorithm

1. Number the source with 1. Set n to 1.
2. Increase n by 1.
3. Select any node whose immediate upstream nodes have all been numbered. Number it with n.
4. If $n = N$, exit. Otherwise, continue.
5. Go to step 2.

Simple algorithms are now presented for node weighting and flow distribution, respectively. Before applying these algorithms, the nodes must first be numbered with the node numbering algorithm. Define NP_n, $n = 1,...N$, to be the number of paths from the source to node $n, n = 1,...,N$.

Node Weighting Algorithm

1. Set n to the source number, 1. Set NP_n to 1.
2. Increase n by 1.
3. Calculate NP_n:

$$NP_n = \sum_{k \in NU_n} NP_k$$

4. If $n = N$, exit. Otherwise, continue.
5. Go to step 2.

Flow Distribution Algorithm

1. Set n to the number of nodes, N.
2. Calculate T_n:

$$T_n = \sum_{nj \in ND_n} q_{nj}$$

3. Calculate q_{kn}, $\forall kn \in NU_n$:

$$q_{kn} = T_n \times \frac{NP_k}{NP_n}$$

4. If $n = 1$, go to step 7. Otherwise, continue.
5. Reduce n by 1.
6. Go to step 2.
7. Calculate S^*, if necessary. Exit.

4. GENERAL NETWORKS

The algorithms in this paper are rigorous for single source networks. However, they are in general inapplicable to multiple-source networks, for the following reasons. The proposed method is a direct application of the following result: maximization of Shannon's entropy function, subject only to normality of the probabilities, leads to the uniform probability distribution in which all the probabilities are equal. The corresponding result for a general network with multiple sources requires all the sources to contribute the same quantity of flow to the total supply; S_0 in Eq. (2) attains its highest possible value if all the p_{0i} are equal. In general, this condition will not be met as the flow from different sources will usually be unequal. Furthermore, the flow directions in a multiple-source network will be such that, for each node, the path flows will be unequal in general.

However, in any network, if the flow directions and the distribution of the source flows are such that all paths serving a node can carry the same amount of flow, the proposed method will give the right result. This also applies to any multiple-source network that is effectively operating as a single-source network. Two examples are next provided to illustrate some of the above points. However, it must be stressed that the proposed method is intended as an alternative to numerical optimization, for single-source networks only.

A sample two-source network in which all the conditions for uniformity of the path flows are satisfied is shown in Figure 5a. Link 1-2 is a direct connection between the sources. The node weighting algorithm may be applied to multiple-source networks with source-source connection. However, each source is given a weight of unity in step 1. Thus, suppose the sources were replaced by a supersource numbered 0 with 55 units. Suppose, further, that link 1-2 were replaced by a direct link from the supersource to nodes 1 and 2 respectively as shown in Figure 5b. If the node weighting algorithm is carried out on this transformed, but equivalent network, both nodes 1 and 2 would be assigned a weight of 1. This provides confirmation that each source in a multiple-source network, with *all* sources interconnected, should have a weight of unity. It may be noted that there need not be a direct link for every source-source combination. It is sufficient that each source be directly connected to at least one other source.

To obtain the maximum entropy flows for multiple-source networks with source-source connections, the flow algorithm may be applied as described for single-source networks, but with a slight modification. In step 2, $(T_n - q_{0n})$ is found, and used in step 3, instead of T_n. The external inflow q_{0n} will be zero for all nodes other than source nodes. Also, this modified version of the algorithm may be used for networks having a single source. Finally, using Eq. (1), the value of S^* may be calculated, once the maximum entropy flows are available. The problem of Figure 5a was solved by both numerical optimization using the NAG library routine E04JAF and the present method. Both methods gave the same result, of which the flows are shown in Figure 5a.

However, in a general network, if at least one of the requirements for equality of path flows is not satisfied, the single-source method cannot be used. For example, in Figure 6, the direction of flow in the source-connecting link is the reverse of the direction in Figure 5a. The problem of determining the maximum entropy flows for the network of Figure 6 cannot be solved by the present single-source method. This problem was solved using the NAG library routine E04JAF. The vector describing the optimum point is

$$(S/K, q_{23}, q_{13}, q_{34}, q_{35}, q_{21}, q_{14}, q_{25})^*$$

$$= (1.947, 12.917, 28.871, 8.871, 22.917, 0.000, 6.129, 7.083)$$

That the optimum point contains a zero element is an indication that the single-source method will not solve this problem. This example shows that the present method is in general inapplicable to multiple-source networks.

Comments on Figures 5a and 6

Several interesting comments may be made on the maximum entropy flows of Figure 5a and 6. Figure 6 has a lower value of $S^* = 1.947$ than Figure 5a, whose value is 2.020. This bodes well for the possible use of flow entropy in layout and reliability optimization for several reasons. First, q_{21}^*, but not q_{12}^*, being zero is desirable. It would be reasonable to augment a smaller source for even greater flexibility. On the other hand, because of the position of node 3, there is no need for flow to be transferred from source node 2 to source node 1. At the optimum, the network of Figure 5a, with three loops, has more redundancy than the network of Figure 6, with two loops. It is therefore fitting that Figure 5a, with a better layout and higher level of flexibility/redundancy, should have a higher value of entropy. Also, by correctly setting certain link flows to zero, entropy maximization has the capability of identifying superfluous links. This is highly desirable in the context of layout optimization. Furthermore, in both Figures 5a and 6, the flow from node 3 to node 4 is greater than the direct supply from source node 1 to node 4.

Similarly, the flow from node 3 to node 5 is greater than the direct supply from source node 2 to node 5, in both figures. This is desirable, from a resilience/flexibility standpoint, if there is variation in the source supplies. *Node 3 has a direct connection to both sources and the flow in links 3-4 and* 3-5 may vary considerably if the source supplies vary. Therefore, designing these links to have a larger capacity would enhance the network's flexibility. Similarly, a larger capacity for these links is desirable if the demands at nodes 4 and 5 vary. The same arguments as for varying source supplies apply.

5. CONCLUSIONS AND SUMMARY

A rigorous, simple, non-iterative algorithm for calculating maximum entropy flows for single source networks has been presented. Although the method is path-based, a simple node-weighting technique is used to avoid path enumeration. The above properties give the procedure a high computational efficiency. This can be very useful in a design or reliability framework, where very many function evaluations may be necessary. The routine lends itself to both manual computations, for small networks, and implementation on a computer, for large systems.

A suggested possible application of the proposed approach is in the design of flexible single-source water distribution networks by linear programming. In the Alperovits and Shamir[1] method, for example, the proposed routine would give the flows that the pipes should be designed to carry. The gradient step of the linear programming gradient method would not be needed. Also it would be useful to compare the reliability of such a design to other designs. Furthermore, there is some evidence (Tanyimboh and Templeman[10]) that the cost of a water distribution network designed to carry maximum entropy flows may not be much higher than if the network is designed with minimum-sized loop-completing links.

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FIGURE CAPTIONS

Figure 1 Single-source network.
Figure 2 Equal path flows from the source (node 1) to each of the demand nodes 2,3,4 and 5.
Figure 3 Maximum entropy flows for the network of Figure 1 found by superposing the path flows of Figure 2.
Figure 4 Number of paths to each node for the network of Figure 1.

Figure 5a Maximum entropy flows for a two-source network with equal path flows to each demand node.
Figure 5b Supersource representation of the network and flows of Figure 5a.
Figure 6 Maximum entropy flows for a two-source network with unequal path flows to each demand node.

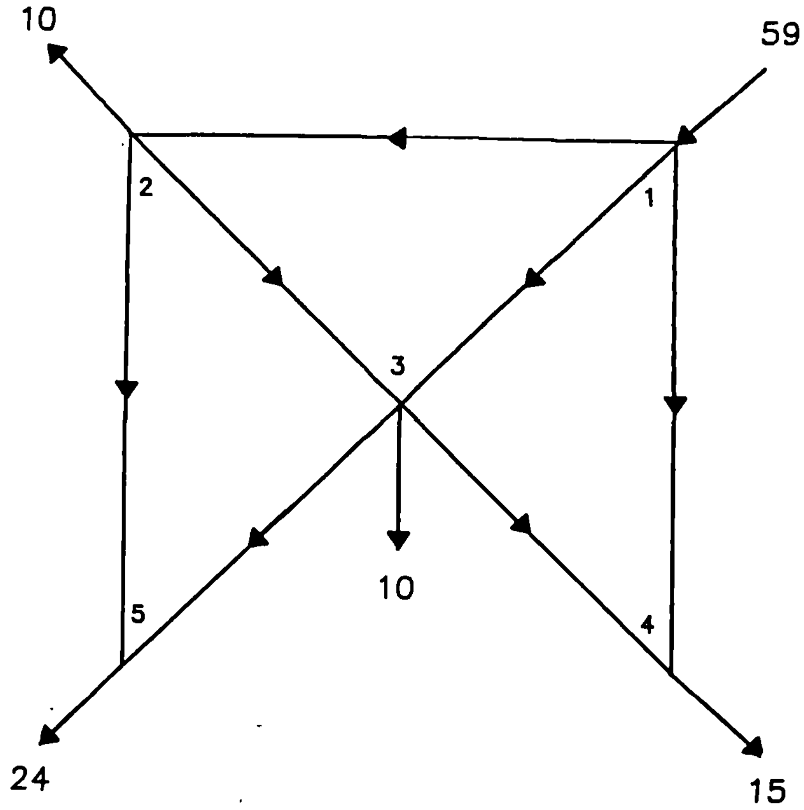

Figure 1 Single source network

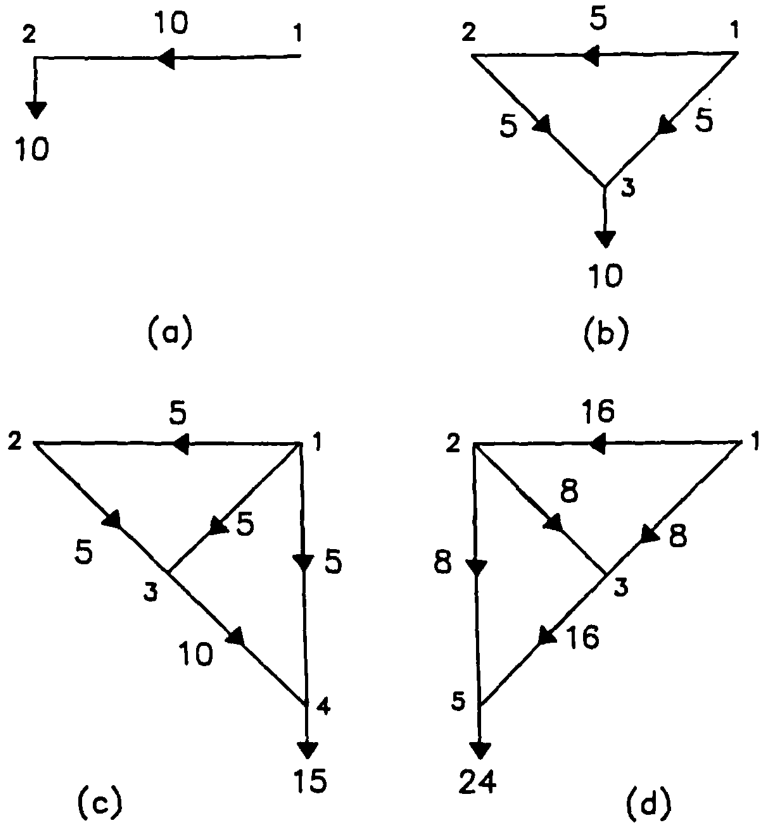

Figure 2 Equal path flows from the source (node 1) to each of the demand nodes 2,3,4 and 5

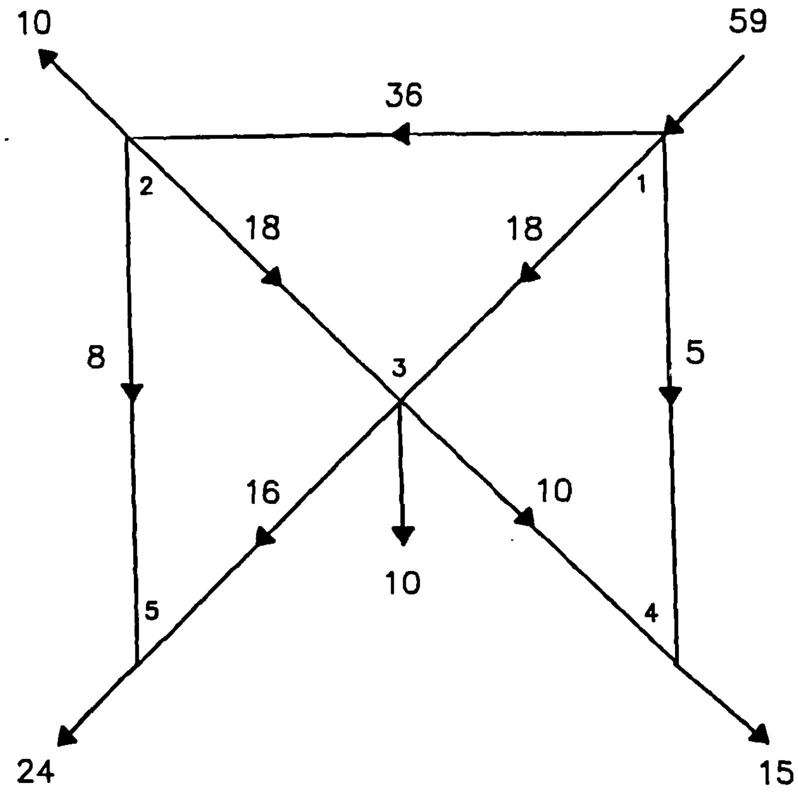

Figure 3. Maximum entropy flows for the network of Figure 1 found by superposing the path flows of Figure 2

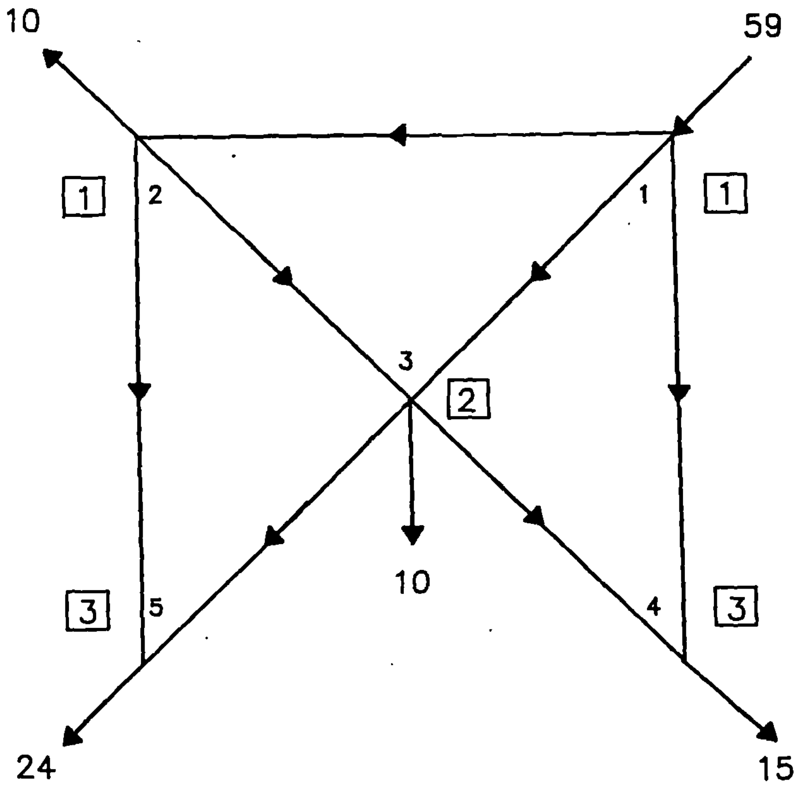

Figure 4 Number of paths to each node for the network of Figure 1

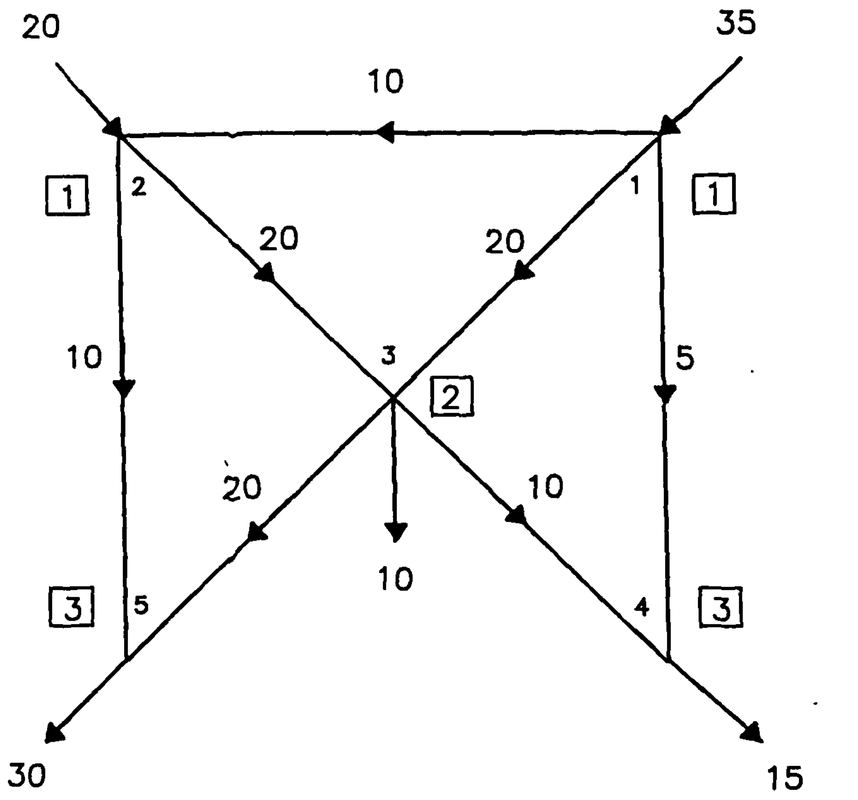

Figure 5a Maximum entropy flows for a two-source network with equal path flows to each demand node.

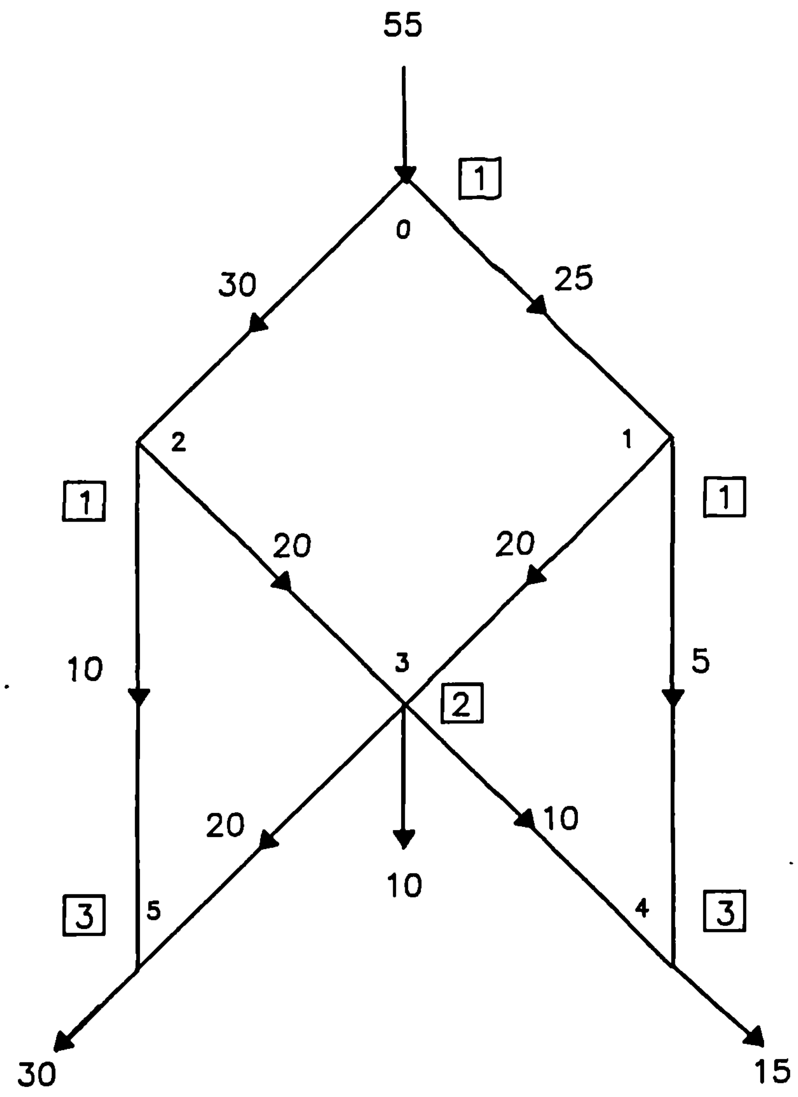

Figure 5b Supersource representation of the network and flows of Figure 5a.

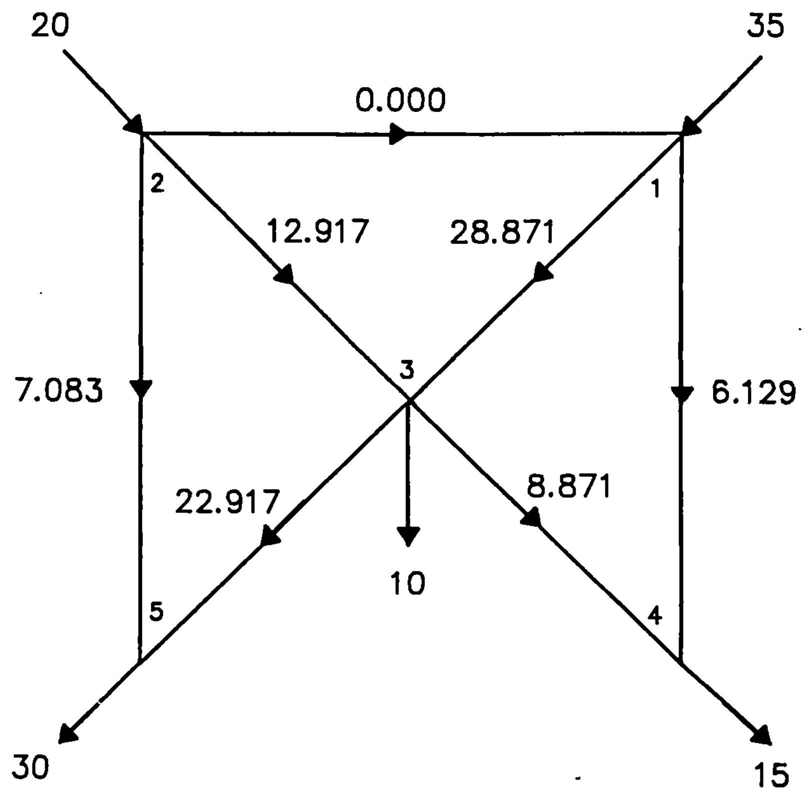

Figure 6 Maximum entropy flows for a two-source network with unequal path flows to each demand node