DEGREE TWO RATIONAL MAPS WITH A PERIODIC CRITICAL POINT

Thesis submitted in accordance
with the requirements of the
University of Liverpool
for the degree of
Doctor in Philosophy
by

James Robert Pointer Stimson

July 1993

ABSTRACT

We are concerned with certain families of degree two rational maps of the Riemann sphere, each of which is defined as the set of degree two rational maps for which a critical point is of some fixed period, modulo conjugation by Möbius transformations. In general, we can take these to be maps of the form

$$z \mapsto \frac{z^2 + az + b}{z^2},$$

where the parameters a and b satisfy an algebraic equation. Thus each family can be realised as a (possibly) reducible algebraic variety in $P^2\mathbf{C}$. The motivation for this work was to understand as much as possible about the topology of these varieties, in particular with a view to deciding whether they are irreducible. (The importance of these families relates to hyperbolic maps and their position in the parameter space RM_2 of rational maps of degree two. [R1/2/3])

We show that the singularities of these varieties are contained in the boundary of the family and of RM_2, where the maps degenerate into finite order Möbius transformations. Our main result is to prove that the singularities of these varieties are made up of a number of smooth branches, each of which intersect two (or possibly one) hyperbolic components of polynomials in RM_2. To do this, we associate a critically finite branched covering of $\hat{\mathbf{C}}$ to each branch which is unique up to Thurston equivalence. We then use the combinatorial theory of laminations and their matings to establish which polynomials are associated to particular branches.

Again using combinatorial techniques, we produce results which go some way toward showing that the varieties are irreducible. We then calculate the genus of some examples.

ACKNOWLEDGEMENTS

Firstly I would like to thank my supervisor, Mary Rees, for all the help, support and encouragement she has given me over the past three years.

I thank the staff and students in the Departments of Pure Mathematics and Applied Mathematics in Liverpool for the friendly and stimulating environment they have provided; and especially to those with whom I shared an office, for the many interesting diversions and conversations we have shared, some of them even useful. Also a special mention to all those at SUNY Stony Brook for their hospitality and friendship.

I acknowledge the support of SERC, in particular for their financing of my stay in Long Island, and that of Boston University for the duration of the conference on Complex Dynamics in 1991.

CONTENTS

INTRODUCTION

The dynamics of holomorphic maps of the complex plane and the Riemann sphere have been subject to a revival of interest in the last fifteen years or so. New techniques have been introduced to the subject, for example quasi-conformal deformation theory from hyperbolic geometry, which along with the computer images of Mandelbrot and others, illustrating the geometric complexity generated by these iterated maps, have excited renewed interest in the subject. From the classical work of Fatou [F] and Julia [J] it is evident that the behaviour of critical points is of central importance to the dynamics of rational maps. Recent work has led to a more complete understanding of the dynamical behaviour of certain types of maps, and the variation of dynamics over appropriate parameter spaces. The work of Douady and Hubbard [DH1/2] has led to a good combinatorial description of the dynamical behaviour of polynomial maps, and in particular the quadratic case, where a combinatorial description of the Mandelbrot set, which lives in the natural parameter space for quadratic polynomial maps, has been established. Reinterpreted by Thurston [T] as Laminations, where the Mandelbrot set is described as the continuous image of a quotient of the unit disc, this forms an important part of the work that follows. Important open questions still remain, e.g., Is the Mandelbrot set locally connected?

In this thesis we are concerned with dynamically defined families of degree two rational maps of the Riemann sphere. Such maps have two critical points in general. The parameter space for degree two rational maps is essentially of complex dimension two, but not so simple to describe as for polynomial maps, so it is illuminating to take one-dimensional slices to help understand the variation in dynamics. For each n we define a family to be the set of degree two rational maps (up to conjugation by Möbius transformations) with a critical point of least period

n. We study these sets as algebraic varieties in the complex projective plane, so they are Riemann surfaces with singularities. If we take the periodic critical point to be 0, and such that $0 \mapsto \infty \mapsto 1$, for $n \geq 3$ we have maps in the family of the form

$$z \mapsto f_{a,b} = \frac{z^2 + az + b}{z^2}$$

for complex parameters a and b. For example, imposing the condition that $n = 3$ gives us the following for the orbit of 0: $0 \mapsto \infty \mapsto 1 \mapsto 1 + a + b$, giving the defining relation $1 + a + b = 0$ so that maps in this family are of the form

$$z \mapsto \frac{z^2 - (1 + b)z + b}{z^2}$$

for $b \neq 0$.

The initial motivation for this work was to understand more about the topology of these families, since they are an important part of the structure of parameter space: we describe maps for which the critical points converge to an attracting periodic orbit as hyperbolic, and in parameter space connected components of the set of hyperbolic maps, hyperbolic components, usually contain a central critically-finite map (i.e., such that the forward orbits of the critical points are finite sets). In fact, (see [R1]) these components have been classified, and it has been shown that they all contain a critically finite map, bar one exceptional component. Since the variation in dynamics over a single hyperbolic component is well understood, and since they are conjectured dense in the parameter space, an understanding of critically finite maps and the way that they fit together are important. It has been conjectured that the families described above are irreducible varieties, though this has only been known for $n \leq 4$. An effort to improve on this situation was the starting point for what follows.

The task of revealing the global topology of an irreducible variety, once its degree is known, is essentially one of analysing its singularities. Initially some direct calculations of local expansions for branches of some of the singularities (see Chapter 1) led to the conjecture that the singularities of these curves are made up of branches which in themselves are non-singular. This is equivalent to saying that the link associated to each singularity is a link of a certain number of trivial knots, one for each branch. The main result of this thesis is to prove this conjecture for all the singularities.

We study our families of rational maps in the larger setting of degree two branched coverings, making use of a natural homotopy-type equivalence for critically finite branched coverings due to Thurston. Points on the boundary of the sets we shall be considering correspond to Möbius transformations, whose k-fold (for some natural number k) self-compositions are the identity map. As an example, when $b = 0$ for the family we have described above we have the map $z \mapsto \dfrac{z-1}{z}$, whose three-fold composition is the identity map. However, it turns out that if we consider maps in the families as branched coverings, the boundary or limit points of them can be considered as particular types of non-rational (i.e., not equivalent to a rational map) critically finite branched coverings. Analysis of these non-rational maps is in fact essential to the understanding of the rational maps in the families. What we do is to associate these critically finite branched coverings to branches of the singularities, and we establish that this association is unique up to equivalence by carefully analysing the behaviour of the second critical point. We have an abstract combinatorial model of the branched coverings constructed from quadratic laminations via their matings, and this is used to sort out these models into their equivalence classes, and thus show how many models are associated to each branch. With some more work this establishes that each branch is smooth.

The last part of this thesis details a method which we attempt to use to prove the irreducibility of the varieties. Although this work is incomplete the calculations which are detailed establish the conjecture for the cases $n \leq 7$. Extending this to higher n is not difficult, but not likely to be very informative without a proof for all n since the complexity of the calculations increases exponentially with n. Many opportunities exist for extending this work, aside from the obvious irreducibility question: the interplay between the combinatorial description of critical point behaviour and the degrees of tangency of branches of singularities remains to be fleshed out more thoroughly; this also applies to the way different varieties intersect, a point only touched upon here. Given enough information about the singularities, and given that the variety is irreducible, the genus of each is computable. In this thesis we carry out this calculation for the cases $n \leq 7$, but it seems that an algorithm for computing the genus in general may be possible.

This thesis consists of six chapters. The first four form a self-contained unit and establish the main result of this thesis concerning the singularities of the families: in Chapter 1 we make explicit calculations of Puiseaux series around singularities of some of the varieties; in Chapter 2, we establish exactly where the singularities can occur (on two specific lines in the projective plane); in Chapter 3 we state our main theorem on the smoothness of the branches at the singularities, and reduce the problem to one in the context of abstract branched coverings, which is then solved in Chapter 4. Chapters 5 and 6 are fairly open-ended and suggest further avenues of research: in Chapter five we address the problem of whether the varieties are irreducible, making some calculations which suggest a general approach for the problem; in Chapter six, we make some genus calculations and discuss some further possibilities for research.

CHAPTER ONE

THE FAMILIES W_n

In this chapter we introduce the main objects of interest, which are families of degree two rational maps, and investigate some of their properties as algebraic varieties and Riemann surfaces. Included are explicit calculations of Puiseaux expansions at the singularities of a few of the families, motivating some of the conjectures that we will prove (or attempt to prove) in general.

§1.1 Degree two rational maps.

A rational map is the quotient of two polynomials, and rational maps are the holomorphic maps of the Riemann Sphere, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, to itself. The degree of a rational map, $d(R)$, where $R(z) = P(z)/Q(z)$ and $P(z)$ and $Q(z)$ have no common factors, is given by $\max\{d(P), d(Q)\}$, where $d(P)$ is given by the highest exponent of monomials in $P(z)$.

The objects of study in this thesis are degree two (quadratic) rational maps of the Riemann Sphere, i.e., maps of the form

$$f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$$

$$z \mapsto \frac{a_1 z^2 + a_2 z + a_3}{b_1 z^2 + b_2 z + b_3},$$

where $a_i, b_j \in \mathbf{C}$ are such that either a_1 or b_1 is non-zero and the numerator and denominator have no common factors. Thus, we can identify the space of all such maps with an open set of points $\{(a_1 : a_2 : a_3 : b_1 : b_2 : b_3)\}$ in $P^5\mathbf{C}$, subject to the above conditions (See [M4]). However, we are interested in the dynamical behaviour of such maps, i.e., the behaviour of points in $\hat{\mathbf{C}}$ under iteration by the function f, and for dynamical purposes, the natural parameter space is much smaller.

The bijective maps of $\hat{\mathbf{C}}$ are the Möbius Transformations, which have the form

$$\mu : z \mapsto \frac{c_1 z + c_2}{d_1 z + d_2}.$$

We will sometimes find it convenient to write the above in the usual matrix form

$$\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} z = \frac{c_1 z + c_2}{d_1 z + d_2}.$$

Composition of maps is then given by multiplication of matrices.

Now, if f and g are rational maps, we define the equivalence relation *conjugation*, $\sim$, by

$$f \sim g \iff f \circ \mu = \mu \circ g,$$

for μ some Möbius transformation. We say that f and g are conjugate via μ. Note that this implies that $f^n \circ \mu = \mu \circ g^n$, where f^n denotes the n-fold composition of f, so f and g have the same dynamical properties. So, when constructing a parameter space for any type of rational map, it is natural to factor out by equivalence classes of conjugate maps.

Central to the study of dynamics of rational maps is the iterative behaviour of critical points (See [F] and [J]).

Definition. z_0 is a *critical* point if $\dfrac{dR}{dz}(z_0) = 0$. The point $R(z_0)$ is a *critical value.*

We write (f, c_1, c_2) for a degree two rational map with critical points c_1 and c_2, and its equivalence class under the equivalence $(f, c_1, c_2) \sim (\tau \circ f \circ \tau^{-1}, \tau(c_1), \tau(c_2))$, (where τ is a Möbius transformation) is denoted $[f, c_1, c_2]$. So we define a parameter space for degree two rational maps (as in [R1]),

$$RM_2 \cong \{[f, c_1, c_2] \mid f \text{ a degree two rational map}\}.$$

Note that the critical points are "marked", so that the two critical points are distinguished – this has consequences for the parameter space. The space RM_2 is locally homeomorphic to $\mathbf{C}^2$, except at one singular point, so it is not a manifold. Consider the family of degree two rational maps,

$$\mathcal{F} = \{z \mapsto \frac{az^2+1}{z^2+b} \mid a, b \in \mathbf{C}\},$$

in which the critical points of functions are 0 and ∞. (To calculate the derivative at ∞ we conjugate by $z \mapsto 1/z$ and calculate the derivative at 0). At the point where $a = b = 0$ we have the map $z \mapsto 1/z^2$, where $c_1 = 0$ maps to $c_2 = \infty$ and vice-versa. Consider a fixed map in some neighbourhood of $z \mapsto 1/z^2$ in the above family. Then we can conjugate by $z \mapsto \sqrt{x}z$ to get the conjugate map

$$z \mapsto \frac{1}{\sqrt{x}}\left(\frac{axz^2+1}{xz^2+b}\right)$$

$$z \mapsto \frac{axz^2+1}{x\sqrt{x}z^2+b\sqrt{x}}.$$

But, if $x\sqrt{x} = 1$, then the above map is another map in the family, and so different maps are equivalent in a neighbourhood of $z \mapsto 1/z^2$. So this point has a punctured neighbourhood which has the homotopy type of $\{a, b \mid |a|^2 + |b|^2 < 1\}/(a, b) \sim (\omega a, \omega^2 b)$, where ω is a primitive third root of unity.

§1.2 Main definition: the families W_n.

Of natural importance to the study of iterative processes are the periodic points:

If $R^n(z_0) = z_0$, then z_0 is said to be *periodic* of period n. If $R^k(z_0) \neq z_0$ for $1 \leq k < n$, then z_0 is periodic of *least* period n. The set $\{R^j(z_0)\}$, for $1 \leq j \leq n$, is then known as a periodic orbit or cycle.

Definition. For $n \in \mathrm{N}$, define the family

$$W_n = \{[f, c_1, c_2] \mid f^n(c_1) = c_1,\ f^k(c_1) \neq c_1, 1 \leq k < n,\ [f, c_1, c_2] \in RM_2\}.$$

So W_n is the set of degree two rational maps with one (marked) critical point of least period n, modulo Möbius conjugation. Note that because the critical points are marked this means that the above families do *not* include all maps in RM_2 with a period n critical point. In particular, a member of the set W_n is not contained in the set W_m for $m \neq n$.

In the case $n = 1$ one of the critical points is fixed, and we can take this to be ∞. This gives us the set of quadratic polynomials:

$$W_1 \cong \{z \mapsto z^2 + c \,|\, c \in \mathbf{C}\}.$$

For $n = 2$, we note that

$$W_2 \cong \left\{\frac{1+az}{z^2} \,\Big|\, a \in \mathbf{C}\right\} \Big/ a \sim a\omega,$$

where $\omega \neq 1$ and ω is a primitive third root of unity. To see this, conjugate $z \mapsto \dfrac{1+az}{z^2}$ by $z \mapsto \omega z$, giving the map $z \mapsto \dfrac{1+a\omega z}{z^2}$.

Now, for $n \geq 3$, we can assume without loss of generality that $c_1 = 0$ and that under f, $0 \mapsto \infty \mapsto 1$ (because we can conjugate by a Möbius map which takes c_1 to 0, $f(c_1)$ to ∞, and $f^2(c_1)$ to 1) . So we can take f to be of the form:

$$f_{a,b} : \; z \mapsto \frac{(z^2 + az + b)}{z^2},$$

where $a, b \in \mathbf{C}$, and we shall do so from now on. Note that this family does not contain a map in the equivalence class $[z \mapsto 1/z^2]$, so a map $f_{a,b}$ corresponds uniquely to an element of RM_2. When $b = 0$ we get $f_{a,b}$ degenerating into the Möbius map $z \mapsto (z+a)/z$. It is clear that this is the only way in which a map of the above form can fail to be degree two.

Now under $f = f_{a,b}$, $c_1 = 0 \mapsto \infty \mapsto 1 \mapsto 1 + a + b$. (We shall write f for $f_{a,b}$, where no confusion can occur). So (an isomorphic copy of) W_3 is given by the equation $1 + a + b = 0$, and we have

$$W_3 \cong \{(a,b) \in \mathbf{C}^2 \,|\, 1 + a + b = 0, b \neq 0\}$$

$$\cong \mathbf{C} \setminus \{0\}.$$

(From now on we will abuse notation somewhat, writing W_n for a set isomorphic to W_n as strictly defined. i.e., we think of W_n as a set of maps of the form $f_{a,b}$, or equivalently a set of points of the form (a,b).)

Note that for $(a,b) = (-1,0)$ we have the Möbius map $\mu : z \mapsto (z-1)/z$, where $\mu^3 \equiv Id$. (We say that μ is of *order* three.) The two critical points have come together and cancelled each other out in some sense. As a (and thus b) tend to infinity we can conjugate by $z \mapsto \sqrt{a}z$ to get the limit map $z \mapsto 1/z$, of order two. Note that when the other critical point, $c_2 = (2+2a)/a$, is fixed this gives us three maps conjugate to the quadratic polynomials with a period three critical point.

Moving on, we observe that

$$1+a+b \mapsto \frac{(1+a+b)^2 + a(1+a+b) + b}{(1+a+b)^2},$$

so that for 0 to be of period 4 the numerator of the above expression must be equal to zero. Thus

$$W_4 \cong \{(a,b) \in \mathbf{C}^2 \mid 2a^2 + b^2 + 3ab + 3a + 3b + 1 = 0, 1+a+b \neq 0, b \neq 0\},$$

though the requirement that $1+a+b \neq 0$ is seen to be redundant (and indeed, for all n we can easily see that the condition $b \neq 0$ is sufficient).

We can think of this as a curve in $\mathbf{C}^2$, given by an irreducible quadratic equation, subject to some points being removed. Again at $(a,b) = (-1,0)$ we have the order three Möbius map, and at $(a,b) = (-1/2,0)$ we have $z \mapsto (z-1/2)/z$, which is of order four.

However, it is natural to study W_4 in the projective plane by homogenising its defining equation. We introduce the coordinate c so that points in $P^2\mathbf{C}$ are represented as $(a/c : b/c : c)$ noting that there is a canonical embedding of $\mathbf{C}^2$ into $P^2\mathbf{C}$ taking the point (a,b) to $(a:b:1)$. The equation for W_4 in $P^2\mathbf{C}$ becomes $2a^2 + b^2 + 3ab + 3ac + 3bc + c^2 = 0$. Now the Möbius maps are at the projective coordinates $(-1:0:1)$, $(-1/2:0:1)$ and also at $(1:-1:0)$ and $(1:-2:0)$, the latter two corresponding to the map $z \mapsto 1/z$.

Bearing these examples in mind we define a *puncture* of W_n to be a limit point of W_n, i.e., a point in the set $\overline{W_n} \setminus W_n$, where $\overline{W_n}$ denotes the closure of W_n in the usual topology on $P^2\mathbf{C}$. For example, W_3 has punctures at $(-1:0:1)$ and $(-1:1:0)$. It is clear that the punctures of W_n represent Möbius maps.

For any $n \geq 3$, the numerator of $f^n(0)$ is clearly a polynomial in the variables a and b, the solution of which gives values of a and b for which 0 is of a period which divides n. These values will therefore include all maps in W_n, (identifying (a,b) with the map $f_{a,b}$), as well as all maps in W_m, where m divides n. The denominator of $f^n(0)$ is zero exactly when 0 is of some period dividing $n-1$. For example, the numerator of $f^6(0)$, used to compute W_6, contains the factor $1+a+b$, corresponding to W_3.

For convenience we define a natural completion (and compactification) of the set W_n in the complex projective plane.

Definition. Let V_n be W_n together with its limit points or punctures. $V_n = \overline{W_n}$. So, for example, V_3 is isomorphic to the Riemann Sphere.

Thus, for $n \geq 3$, V_n is given by a polynomial equation in two variables, a and b, so V_n is an algebraic curve and can be considered as an *algebraic variety* in $P^2\mathbf{C}$. We consider this aspect in detail in §1.3.

§1.3 The families as varieties.

Let $\mathbf{C}[x_1, \ldots, x_j]$ be the ring of j-variable polynomials over $\mathbf{C}$. Then an *affine variety* is a subset of $\mathbf{C}^j$, which is the common zero set of a set of k polynomials in $\mathbf{C}[x_1, \ldots, x_j]$. i.e.,

$$V_A = \{x \in \mathbf{C}^j \mid p_1(x) = \cdots = p_k(x) = 0\}.$$

Let H be the ring of $j+1$ variable homogeneous polynomials over $\mathbf{C}$. An *algebraic variety* is a subset of $P^j\mathbf{C}$, which is the common zero of a set of k homogeneous polynomials in H. i.e,

$$V = \{x \in P^j\mathbf{C} \mid P_1(x) = \cdots = P_k(x) = 0\}.$$

A consequence of the fact that there is a canonical embedding of $\mathbf{C}^2$ into $P^2\mathbf{C}$ is the following:

Theorem 1.3.1. *An affine variety V_A in $\mathbf{C}^2$ uniquely determines an algebraic variety V in $P^2\mathbf{C}$, and vice-versa such that $V \cap \mathbf{C}^2 = V_A$.*

Proof. See [G] □

We are dealing exclusively with varieties (V_n) in $\mathbf{C}^2$ or $P^2\mathbf{C}$, defined by just one polynomial, i.e., algebraic curves. Such an algebraic variety or curve is *irreducible* if its defining polynomial is irreducible over $\mathbf{C}$. One result is of particular interest:

Theorem 1.3.2

(a) An algebraic curve in $P^2\mathbf{C}$ is always connected.

(b) An algebraic curve in $P^2\mathbf{C}$ is irreducible if and only if it cannot be disconnected by removing a finite number of points.

Proof. See [G] □

§1.4 Punctures of W_n: some elementary facts.

Knowing that the varieties V_n are defined by certain dynamically defined two-variable polynomials, we establish inductively their degree, obviously of importance to questions we may wish to ask about a variety. We also establish a simple result about the polynomials in a whose roots are the punctures of W_n.

Let us denote $f^n_{a,b}(0)$ by the quotient $p_n(a,b)/q_n(a,b)$, for $n > 2$. Then, writing p_n for $p_n(a,b)$ and q_n for $q_n(a,b)$, we have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n^2 + ap_nq_n + bq_n^2}{p_n^2}. \qquad (*)$$

We calculate the degree of p_n inductively, claiming:

Lemma 1.4.1.

$$d(p_n) = \begin{cases} (2^{n-1} - 1)/3, & \textit{if } n \textit{ is odd;} \\ (2^{n-1} - 2)/3, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. $p_3 = 1 + a + b$ and $q_3=1$, so true for $n = 3$. We will show that for n even $d(p_n) = d(q_n)$ and that for n odd $d(p_n) = d(q_n) + 1$. Inductive step: assume that $d(p_k) = d(q_k)$. Then by $(*)$ $d(p_{k+1}) = 2d(p_k) + 1$ and $d(q_{k+1}) = 2d(p_k)$, so that $d(p_{k+1}) = d(q_{k+1}) + 1$. The other case is similar: assume that $d(p_k) = d(q_k) + 1$. Then by $(*)$ $d(p_{k+1}) = 2d(p_k)$ and $d(q_{k+1}) = 2d(p_k)$, so that $d(p_{k+1}) = d(q_{k+1})$.

Checking the validity of the above formula is now a simple induction. □

We now establish a formula for the polynomials which give us the points at which Möbius maps occur. We have shown that some of these occur on the line $b = 0$. Let P_n and Q_n be respectively p_n and q_n with $b = 0$ substituted. So $P_3 = 1 + a$. For $n > 3$ $Q_n = P^2_{n-1}$. Then

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{((P_n/Q_n)^2 + aP_n/Q_n)}{(P_n/Q_n)^2} = (P_n^2 + aP_nQ_n)/(P_n)^2.$$

So $P_{n+1} = P_n(P_n + aP^2_{n-1})$.

We claim that there is a formula for P_n as follows:

Lemma 1.4.2.

$$P_n = \prod_{j=3}^{n} (P'_j)^{2^{max\{0,n-j-1\}}},$$

where

$$P'_j = \sum_{r=0}^{[(j-1)/2]} \binom{j-r-1}{r} a^r$$

and $[\cdot]$ *denotes the integral part of a number.*

Note that $P_3 = P'_3$.

Example: $P_5 = (1 + a)^2(1 + 2a)(1 + 3a + a^2) = P'^2_3 P'_4 P'_5$.

Proof. We re-write

$$P_{n+1} = P_n(P_n + aP^2_{n-1})$$

$$= P_n(P_{n-1}(P_{n-1} + aP^2_{n-2}) + aP^2_{n-1})$$

$$= P_nP_{n-1}(P_{n-1} + aP^2_{n-2} + aP_{n-1})$$

$$= P_n \dots P_3(P'_{n+1}).$$

But $P_n = P_{n-1} \dots P_3(P'_n)$ and $P_{n-1} = P_{n-2} \dots P_3$ etc., so that $P_{n+1} = P'_{n+1}P'_nP'^2_{n-1} \dots P'^{2^{n-4}}_3$.

We can prove $P'_j = P'_{j-1} + aP'_{j-2}$ inductively from $(*)$. Then the formula for P'_j follows by induction. □

This shows us that all the punctures in $P^2\mathbf{C} \setminus \mathbf{C}^2$ that occur on W_n also occur on W_m, for $m > n$, usually with a higher multiplicity.

Aside: Since $P'_j = P'_{j-1} + aP'_{j-2}$, the coefficients of P'_j can be read off from the $(j-2)$-nd diagonal row of Pascal's triangle: we show this for $3 \leq j \leq 7$.

P'_3	1 1	$1 + a$
P'_4	1 2	$1 + 2a$
P'_5	1 3 1	$1 + 3a + a^2$
P'_6	1 4 3	$1 + 4a + 3a^2$
P'_7	1 5 6 1	$1 + 5a + 6a^2 + a^3$

The intersections of V_n with the line at infinity are derived from the homogenised version of p_n, h_n For example $h_3 = a + b + c$. Then, to find the singularities at $c = 0$ we normalise so that $b = 1$ to get H_n. For example $H_3 = a + 1$. We obtain different recurrence relations for these polynomials, valid for $n > 3$ and putting $H_2 = 1$:

$$\text{For } n \text{ even} \quad H_{n+1} = H^2_{n-1}(aH_n + H^2_{n-1}),$$

$$\text{For } n \text{ odd} \quad H_{n+1} = H_n(H_n + aH^2_{n-1}).$$

We do not have the same relations for H_n as for P_n, the main consequence of which is that not all the points of $V_n \cap P^2\mathbf{C} \setminus \mathbf{C}^2$ occur on $V_m \cap P^2\mathbf{C} \setminus \mathbf{C}^2$ for $m > n$.

We now show that the roots of the polynomials P_n are all real.

Theorem 1.4.3. *The punctures in $\mathbf{C}^2$ of W_n are finite order Möbius maps, of the form $z \mapsto (z - a_0)/z$, where a_0 is positive real and the order of the Möbius map is less than or equal to n.*

Proof. We already know that the punctures correspond to Möbius maps of the above form, so we just have to show they are of finite order. It is sufficient to show that 0 is periodic, since any finite order Möbius map is conjugate to one with the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix},$$

and so if one point is periodic so are all points in $\hat{\mathbf{C}}$, because if a point is periodic then $\alpha^p = 1$, where p is the period.

V_n is given by the expression $f^n_{a,b}(0) = 0$, and by what we have said above this is equivalent to the statement that $p_n(a,b) = 0$. Then $p_n(a_0, 0) = 0$ corresponds to some Möbius map on V_n.

Consider the matrix for the Möbius map, normalised to have determinant -1,

$$\begin{pmatrix} 1/\sqrt{a_0} & -\sqrt{a_0} \\ 1/\sqrt{a_0} & 0 \end{pmatrix},$$

with eigenvalues given by the roots of the equation $-\lambda^2 + \lambda/\sqrt{a_0} - 1 = 0$, which are

$$(1/a_0 \pm \sqrt{(1/a_0 - 4)})/2.$$

Now, for the matrix to be of some order k, its eigenvalues must be k-th roots of unity, and so must be complex conjugates of each other. Clearly this can only happen for a_0 real and greater than $1/4$, because the quantity under the square root sign must be purely imaginary. □

In summary we have located the positions of the punctures in general, by fairly direct and elementary means. Deeper understanding of the singularities of V_n and aspects of their topology will require more subtle methods. In the remainder of this chapter we analyse some particular examples more thoroughly.

§1.5 Local theory of singularities of V_n.

Definition. Suppose that

$$V = \{x \in P^n\mathbf{C} | f_1(x) = \cdots = f_N(x) = 0\}$$

is an algebraic variety. Then a point $x_0 \in V$ is called a *smooth* point of V if there exists a neighbourhood W of x_0 and $\{f_{j_1}, \ldots, f_{j_l}\} \subset \{f_1, \ldots f_k\}$ satisfying:

(a) $V \cap W = \{x \in W | f_{j_1}(x) = \cdots = f_{j_l}(x) = 0\}$;

(b) rank $\left(\dfrac{\partial f_{i_k}(x)}{\partial x_j}\right)$ at x_0 is l.

If x_0 is not a smooth point, it is a *singularity*.

In the two-dimensional case, with one defining polynomial, this reduces to: A point $(a, b) \in V$ is smooth if either of the partial derivatives $\partial P/\partial a$ or $\partial P/\partial b$ is non-zero at (a, b). If not, it is *singular*.

The structure of a variety in a neighbourhood of a singularity will be of interest to us, and the following general theorem is highly relevant:

Theorem 1.5.1. *The Implicit Function Theorem. Let $f_1, \ldots, f_k$ be a set of functions which are holomorphic in a neighbourhood of the origin in $\mathbf{C}^n$, also satisfying*

$$det\left(\frac{\partial f_i(z)}{\partial z_j}|_{z=0}\right)_{1 \leq i,j \leq k} \neq 0.$$

Then there exist $w_1, \ldots, w_k$, (the implicit functions) holomorphic in a neighbourhood of 0 in $\mathbf{C}^n$, such that in this neighbourhood we have

$$f_1(z) = \cdots = f_k(z) = 0 \iff z_i = w_i(z_{k+1}, \ldots, z_n), i = 1, \ldots, k.$$

In the two-dimensional case, on a variety V, this reduces to: Let a point $(a, b) \in V$ be given, with either of the partial derivatives $\partial P/\partial a$ or $\partial P/\partial b$ non-zero at (a, b). (So we are at a smooth point of V.) Then there exists an implicit function w in a neighbourhood of (a, b) such that w is a function of a (or b) and $(a, w(a)) \in V$, so that locally we have a parametrisation of the variety.

However we can also consider whether some branch (or sheet) of variety is itself smooth or singular, and thus apply the last result in this case too, obtaining a local parametrisation of a sheet if it is non-singular. We explain this further later.

An irreducible variety in $P^2\mathbf{C}$ has a finite number of singularities. Let V be an irreducible variety with Σ its set of singularities. Then $V \setminus \Sigma$ is a Riemann surface by the Implicit Function Theorem, because Σ is a finite set of isolated points.

We now investigate individual singularities of V_n in detail:

We change coordinates so that the singularity we wish to study is at $(0,0)$. So p_n is now in the form

$$p_n = \sum x_{rs} a^s b^r.$$

Let

$$\mathbf{P}(b) = \bigcup_{n \in \mathbf{N}} \mathbf{C}[[b^{1/n}]]$$

be the ring of formal *Puiseaux series* in the variable x, that is, the ring of formal power series in $b^{1/n}$. In our examples we will calculate a few coefficients of a in terms of $\mathbf{P}(b)$.

It is important to note that there is one power series expansion for each branch of the singularity, and locally the variety can be expressed (see [Bk]) in the following way, for k branches:

$$v(a,b) = \prod_{j=1}^{k} \left(a - \alpha_j(b)\right),$$

where the α_j are given by the method detailed below.

Consider the set of (r,s) such that $x_{rs} \neq 0$, thought of as a set of points on the rectilinear grid with coordinate axes r and s. Form the convex hull of $\{(r,s) + (u,v) \mid x_{rs} \neq 0, u \geq 0, v \geq 0\}$. The set of edges of finite length of the boundary of this region is called the *Newton polygon*, $\Delta(P_n)$. Let the equation of the line which contains the line segment of $\Delta(P_n)$ through the s axis be $r + \mu_0 s = \nu_0$. We make the substitution $a = t_0 b^{\mu_0}$: Looking at the lowest order terms in b, which have order $m = \nu_0$, which has coefficient a polynomial in t_0, which we solve to give us the first approximation(s) for the solution of $p_n = 0$, also the first term of the Puiseaux series. For a given solution, say t', we substitute back for a, setting $a = b^{\mu_0}(t' + a_1)$, and factor out by the appropriate power of b (which is b^{ν_0}), and repeat the process with a_1 to calculate the next term(s).

The series thus obtained is in general an infinite series. However, the exponents have an important property (for more details refer to [Bk]): The denominators of the exponents in the series have a lowest common denominator. Thus, all terms in the series are expressible as integral powers of $b^{1/N}$ for some natural number N. So for any solution of this form a and b are expressible as functions of some parameter $t = b^{1/N}$, which we shall call the *sheet parameter*.

Puiseaux series tell us something about the topology of a singularity. In particular if a certain local solution has a Puiseaux series containing only integral powers, then one parameter is a function of the other. Hence the branch in question is non-singular. More information about the topology of a singularity, as detailed in [M1] and [Bk], is to compute its associated *link*, or *knot* for a single branch. A variety near a singularity at the origin will intersect a sufficiently small 3-sphere in a link, one knot for each branch. If a branch is non-singular its associated knot will be the trivial knot, but in general will be an iterated torus knot. These knots can be computed from the Puiseaux expansions: if this series has entirely integral powers this gives rise to a loop on the torus which is closed on winding once round, and is therefore just a simple loop. However, when considering more than one branch, two such loops will in general be linked. Their linking number is equal to the intersection number (pairwise) of the branches.

§1.6 Examples.

In this section we use the method for calculating local coordinates for any branch of a plane curve singularity, as detailed in the last section, and do this for all the singularities of V_k for $k \leq 7$, which lie on the projective lines $b = 0$ and $c = 0$. Later (§2.7) we will show that these are all the singularities of these varieties. For the calculations on V_6 and V_7 the computer algebra package Maple was used. We obtain expansions in local coordinates (a_j, b) or (a_j, c), where j indexes the branches/sheets, and such that the singularities are at $(0,0)$.

Singularities of V_5.

V_5 is given by the quintic equation p_5 in a and b. Setting $b = 0$ this reduces to $P_5 = (1+a)^2(1+2a)(1+3a+a^2) = 0$. This has a double root at $a = -1$ and three single roots, at $a = -1/2$ and $a = \dfrac{-3 \pm \sqrt{5}}{2}$. These four values correspond to the Möbius tranformations

$$z \mapsto \frac{z-1}{z}, \qquad z \mapsto \frac{z-1/2}{z}, \qquad z \mapsto \frac{z - \frac{3+\sqrt{5}}{2}}{z}, \qquad z \mapsto \frac{z - \frac{3-\sqrt{5}}{2}}{z},$$

which are maps of order 3, 4, 5 and 5 respectively.

At the double root there are two branches of the variety, and we have our first singularity. As in [Bk] We can calculate local expansions here for a in terms of b. Substituting $(a-1)$ for a, so that the singularity is at the origin we get a polynomial expression with lower order terms

$$a^2 - 3a^3 - 4a^2b - ab^2 + b^5.$$

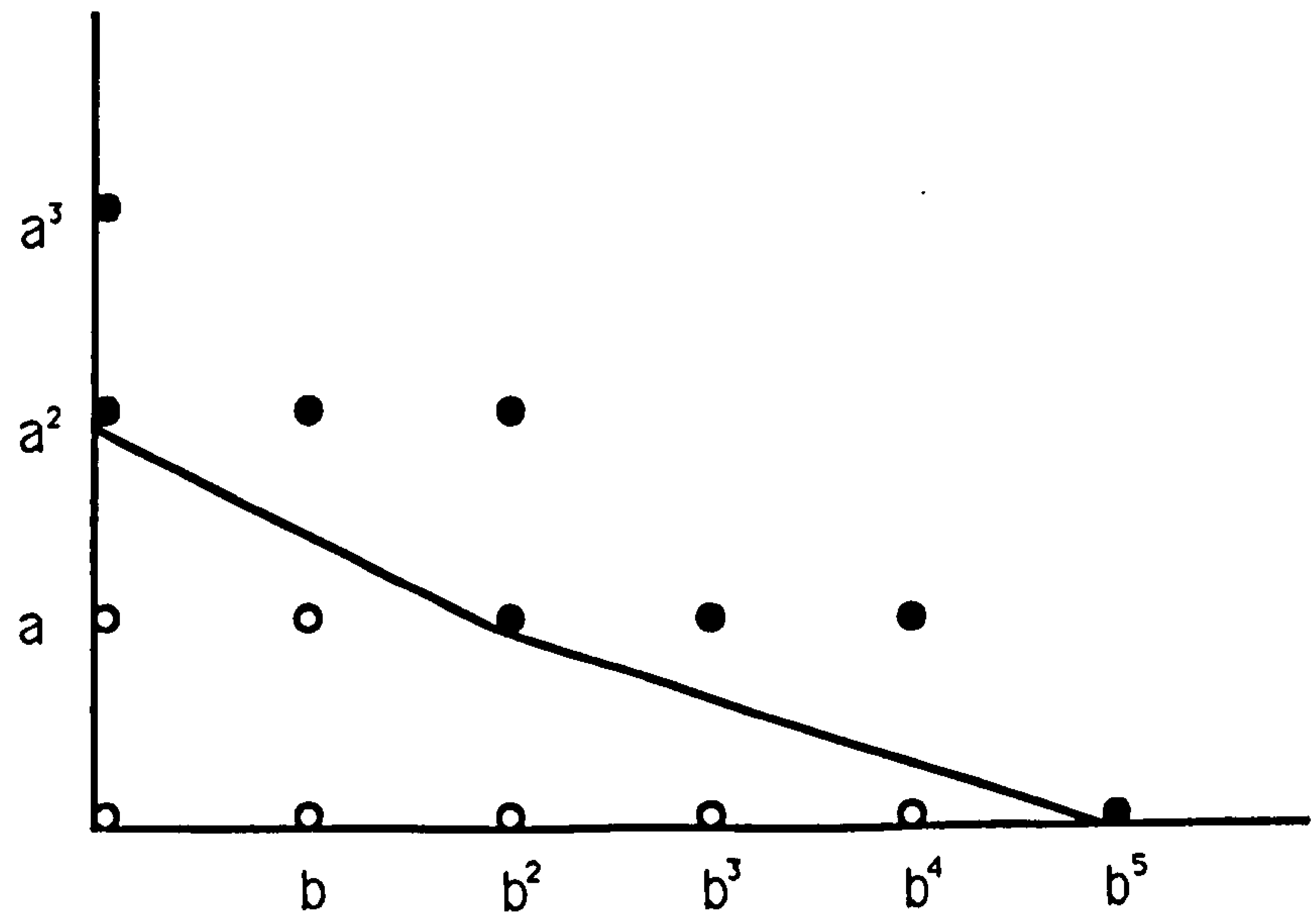


Figure 1.6a

Thus the Newton polygon shown above has slope $-1/2$ where it joins the a-axis and we make the substitution $a = t_0 b^2$. Gathering together the lowest order terms in b, we get that the coefficient of b^4 is $t_0^2 - t_0$. Putting this equal to zero we get the roots 1 and 0. The root 1 tells us that one approximate solution is given by $a = b^2$. To follow up the other solution we substitute $a = b^2 a_1$ giving the following lowest order terms: $b^4 a_1^2 - b^4 a_1 + b^5$. Factoring out by b^4 we get

$a_1^2 - a_1 + b$. This gives a Newton polygon with slope -1, so we substitute $a_1 = t_1 b$, and setting the term in b to zero gives $t_1 = 1$. So an approximate solution for a_1 is $a_1 = b$, so $a = b^2 a_1 = b^3$ is another approximate solution. Thus we have

$$a_1 = b^2 + O(b^3)$$
$$\text{and} \quad a_2 = b^3 + O(b^4)$$

as the two branches.

We will show later that each branch is in fact smooth, so that all the terms in the above expansions are integral powers of b. This means that the associated link to this singularity consists of two un-knots, linked with linking number 2, as shown below.

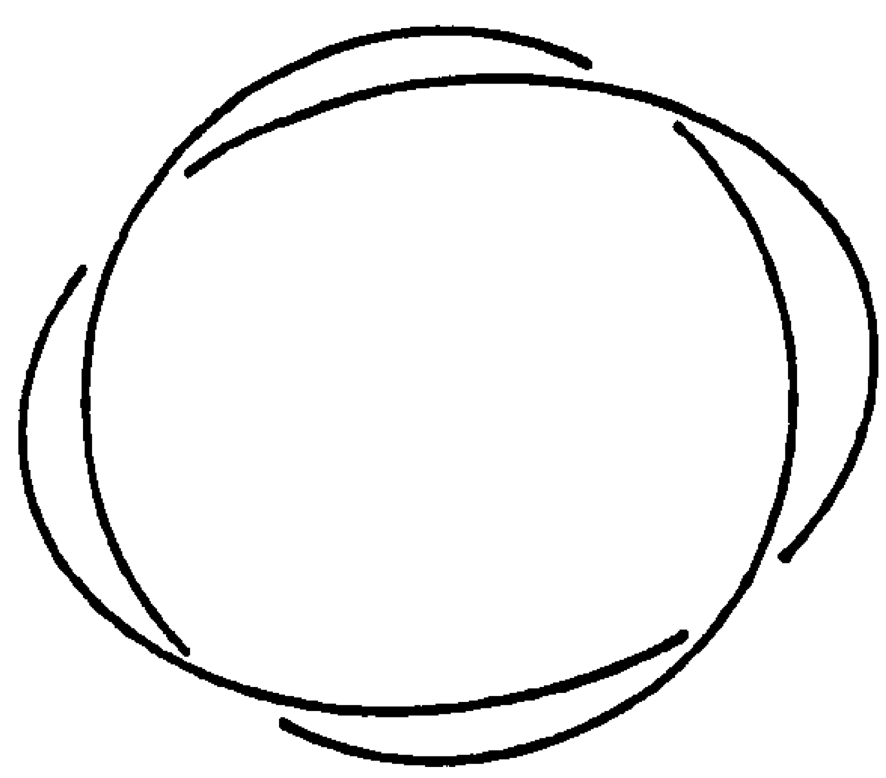


Figure 1.6b

To study possible branches near the $z \mapsto 1/z$ we must go over to projective coordinates. Homogenising the defining equation and setting $c = 0$, we get that $(a+b)^3(2a^2+2ab+b^2) = 0$. So there is a triple root at $(-1 : 1 : 0)$ and two single roots at $(\frac{-1+i}{2} : 1 : 0)$ and $(\frac{-1-i}{2} : 1 : 0)$. The local expansions around $(-1 : 1 : 0)$ give

$$a_1 = 1/2(3+\sqrt{5})c + O(c^2),$$
$$a_2 = 1/2(3-\sqrt{5})c + O(c^2),$$
$$\text{and} \quad a_3 = c^3 + O(c^4)$$

as the three branches.

So V_5 is a curve with (at least) two singularities. Notice that the branches are perhaps smooth — we prove that they are later. We calculate the genus of W_5 in Chapter 6.

Singularities of V_6.

We list the local expansions for singularities of V_6, given by the degree nine equation $P_6/P_3 = 0$.

$$\text{At } (-1/2:0:1): \quad a_1 = -b - b^2 + O(b^3), \quad \text{and } a_2 = -b + 2b^3 + O(b^4).$$

$$\begin{aligned} \text{At } (-1:0:1): \quad a_1 &= -b/2 + O(b^2), \\ a_2 &= b^2 + 2b^3 + 5b^4 + O(b^5), \\ a_3 &= b^2 + 2b^3 + 6b^4 + O(b^5), \\ \text{and } a_4 &= 2b^3 + O(b^4). \end{aligned}$$

$$\begin{aligned} \text{At } (-1:1:0): \quad a_1 &= -c + c^2 + O(c^3), \\ a_2 &= c^4 + O(c^5), \\ a_3 &= 1/2(3+\sqrt{5})c^2 + O(c^3), \\ \text{and } a_4 &= 1/2(3-\sqrt{5})c^2 + O(c^3). \end{aligned}$$

The Möbius maps at $a = -1/3$ and $(-3\pm\sqrt{5})/2$ all correspond to non-singular points of V_6.

Apart from the singularity at $(-1 : 1 : 0)$ there are five other points in $P^2\mathbf{C} \setminus C^2 \cap V_6$, all non-singular. They are at the coordinates $(\frac{-1+i}{2} : 1 : 0)$, $(\frac{-1-i}{2} : 1 : 0)$ and $(\alpha_k : 1 : 0)$ for $1 \leq k \leq 3$, where α_k are the distinct roots of the cubic equation $6a^3 + 6a^2 + 4a + 1$.

Singularities of V_7.

We detail the results of calculations made with the aid of a computer:

$$\begin{aligned} \text{At } (-1/2:0:1): \quad a_1 &= -b + O(b^3), \\ a_2 &= -b - b^2 + O(b^3), \\ a_3 &= -b + b^2 + 6b^3 + 30b^4 + O(b^5), \\ \text{and } a_4 &= -b + b^2 + 6b^3 + 38b^4 + O(b^5). \end{aligned}$$

$$
\begin{aligned}
\text{At } (-1:0:1): \quad a_1 &= -\,1/2(1+i)b + O(b^2),\\
a_2 &= -\,1/2(1-i)b + O(b^2),\\
a_3 &= b^2 + 1/2(7+\sqrt{5})b^3 + O(b^4),\\
a_4 &= b^2 + 1/2(7-\sqrt{5})b^3 + O(b^4),\\
a_5 &= b^2 + 2b^3 + 5b^4 + 14b^5 + 41b^6 + O(b^7),\\
a_6 &= b^2 + 2b^3 + 5b^4 + 14b^5 + 42b^6 + O(b^7),\\
a_7 &= b^2 + 2b^3 + 6b^4 + O(b^5),\\
a_8 &= -\,b^4 + O(b^5),\\
\text{and } a_9 &= -\,b^5 + O(b^6).
\end{aligned}
$$

$$
\begin{aligned}
\text{At } (-1:1:0): \quad a_1 &= -\,1/4(7+\sqrt{5}+(19-7\sqrt{5})^{1/2}\sqrt{2})c + O(c^2),\\
a_2 &= -\,1/4(7-\sqrt{5}+(19+7\sqrt{5})^{1/2}\sqrt{2})c + O(c^2),\\
a_3 &= -\,1/4(7+\sqrt{5}-(19-7\sqrt{5})^{1/2}\sqrt{2})c + O(c^2),\\
a_4 &= -\,1/4(7-\sqrt{5}-(19+7\sqrt{5})^{1/2}\sqrt{2})c + O(c^2),\\
a_5 &= -\,c - 9c^2 + O(c^3),\\
a_6 &= -\,c + O(c^3),\\
a_7 &= c^2 + (1+i\sqrt{13})c^3 + O(c^4),\\
a_8 &= c^2 + (1-i\sqrt{13})c^3 + O(c^4),\\
a_9 &= -\,1/2(3+\sqrt{5})c^3 + O(c^4),\\
a_{10} &= -\,1/2(3-\sqrt{5})c^3 + O(c^4),\\
\text{and } a_{11} &= c^5 + O(c^6).
\end{aligned}
$$

There are four more singularities on V_7. At both $(\frac{-1-i}{2} : 1 : 0)$ and $(\frac{-1+i}{2} : 1 : 0)$ there are three branches, all of which are non-tangent, so that the expansions differ at the b term. We do not write these down as the coefficients of the b terms are rather large algebraic expressions.

At the points $(\frac{-3-\sqrt{5}}{2} : 0 : 1)$ and $(\frac{-3+\sqrt{5}}{2} : 0 : 1)$ there are two branches which share a common tangent: for example at $(\frac{-3-\sqrt{5}}{2} : 0 : 1)$

$$a = \frac{1}{2}\Big(\frac{2299702\sqrt{5} - 5142290}{4651245 - 2080100\sqrt{5}}\Big)b + O(b^2)$$

for both branches — they differ at the b^2 term, but again the coefficients are large.

§1.7 Plan of action.

In the light of these examples we state two conjectures.

(1) The branches of all the singularities of V_n, for any n, are smooth.

The following two statements are also equivalent to (1) by §1.5:

(1a) The link associated to each singularity is made up of un-knots.

(1b) a is a (single-valued) function of b on each branch in the neighbourhood of each singularity or vice-versa.

The proof of this conjecture constitutes a major part of this thesis, and is contained in Chapters 2, 3 and 4.

The following conjecture, though still unproven, was the initial motivation for this work:

(2) The set V_n is an irreducible variety.

We will show that this is equivalent to the following statement, which holds by Theorem 1.3.2 in combination with Theorem 2.7.1:

(2a) The set W_n is connected.

Theorem 2.7.1 implies that all the singular points of V_n lie in $V_n \setminus W_n$. Thus removing any further set of points from W_n cannot affect its connectedness because it is made up of smooth components. The above equivalence then follows by Theorem 1.3.2.

We will also show in Chapter 4 that (1) is true if and only if adjacent to each singularity, on each branch of the singularity, there are two hyperbolic components (see Chapter 2) of polynomials, except those in $P^2\mathbf{C} \setminus \mathbf{C}^2$, where there is one. A start toward proving (2) is contained in Chapter 5.

CHAPTER TWO

DYNAMICS

In this chapter we introduce the relevant background material in the area of complex dynamics. We then use Laminations and their matings to give combinatorial models for quadratic polynomials and some quadratic rational maps, regarded as branched coverings. Using some quasi-conformal deformation theory we prove an important theorem about the location of singularities in the varieties V_n, a result which depends largely on the dynamical behaviour of rational maps in the variety. This will enable us to then look at these singularities in detail, which we do in Chapters 3 and 4.

The theme of this chapter is thus the *dynamical* behaviour of maps in the families V_n, in contrast to the mainly static description made so far. Indeed, the interplay between these two viewpoints is a major concern of this thesis, and of complex dynamics in general.

§2.1 Dynamics of rational maps.

We will just state the basics here. Most of the material in this section dates back to the work of Fatou [F] and Julia [J]. For a full introduction the reader is referred to [M2] or [Bl].

For a rational map, R, we denote its n-fold composition by R^n. The *forward orbit* of a point z, $\mathcal{O}^+(z)$, is the set $\{f^i(z)\,|\,i \geq 0\}$. The *(full) orbit*, $\mathcal{O}(z) = \mathcal{O}^+(z)$, together with the set $\{f^{-i}(z)\,|\,i > 0\}$, where $f^{-1}(z)$ is the set of pre-images of z under f.

The primary feature of the dynamics of the complex plane or Riemann sphere is the division of $\mathbf{C}$ or $\hat{\mathbf{C}}$ into two complementary fully-invariant sets. (A fully-invariant set, I, under a function f has the property that $f(I) = I$ and $f^{-1}(I) = I$.) We need the following:

Definition. A set of functions $\mathcal{F}$, defined on a subset $U \subset \hat{\mathbf{C}}$, is a *normal family* if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of U.

There is an often more useful way of characterising normal families: A set of functions $\{f_I\}$ is *(uniformly) equicontinuous* if, given $\epsilon > 0$, there exists $\delta > 0$ such that $|z_1 - z_2| < \delta$ implies $|f_i(z_1) - f_i(z_2)| < \epsilon$ for all $i \in I$. Then a family $\mathcal{F}$ is normal on some open set U if and only if it is equicontinuous on every compact subset of U (see [Af]).

Then, the *Fatou set* $F(R)$ is the set of points $z \in \hat{\mathbf{C}}$ for which there exists a neighbourhood U_z such that $\{f^j|U_z \,|\, j \geq 1\}$ is a normal family. This is an open set, made up of a number of connected components, on which the dynamics are relatively tame. (Note that it is possibly the empty set). By Sullivan ([S]) , it is known that the components of the Fatou set converge to stable periodic cycles — they are *eventually periodic*. We also refer to a point as eventually periodic if a point in its forward orbit is periodic.

The Julia set, $J(f)$, is the set of points for which the family $\{f^j(U_z)\}$ is not normal. i.e., $J(R) = \hat{\mathbf{C}} \setminus F(R)$. Note that $J(R) \neq \emptyset$. This is where the chaotic dynamics occur.

Definition. At a period n point, z_0, the *multiplier*, $\lambda(z_0) = \dfrac{d}{dz}R^n(z_0)$.

These fall into four types:

(1) $0 < |\lambda| < 1$ Attracting

(2) $\lambda = 0$ Super-attracting

(3) $|\lambda| > 1$ Repelling

(4) $|\lambda| = 1$ Neutral

The Julia set can also be characterised as the closure of the set of periodic repelling points of R.

If z_0 is of period n under R, then it is fixed under R^n. Thus, the theorems which we state for fixed points also hold for periodic points.

Theorem 2.1.1. *For an attracting or repelling fixed point, z_0, there is a an holomorphic bijection ϕ on a neighbourhood, $U(z_0)$, of z_0 such that on U, $\phi(R(z)) = \lambda\phi(z)$.*

Theorem 2.1.2. *For a super-attracting fixed point, z_0, there is a an holomorphic bijection ϕ on a neighbourhood, $U(z_0)$, of z_0 such that on U, $\phi(R(z)) = (\phi(z))^2$.*

Definition. The *attractive basin*, $\Omega(z_0)$, of a (super-)attracting point z_0 is the (open) set of points which tend to the orbit under iteration. i.e $\Omega(z_0) = \{z \mid \lim_{n\to\infty} R^n(z) = z_0\}$. The *immediate* attractive basin is the connected component of the attractive basin containing the fixed point. Note that both these sets are contained in $F(R)$.

The following result (due to Fatou and Julia) is of central importance:

Theorem 2.1.3. *An attracting fixed point attracts a critical point. i.e., The attractive basin of a fixed point contains a critical point.*

For proofs of 2.1.3 and 2.1.4 see [M2].

A neutral fixed point, with multiplier such that $\lambda^p = 1$ for the integer p, is called a *parabolic* point. For such points, there is a result analagous to 2.1.3 (Leau-Fatou Flower Theorem.), which we quote in a form specific to degree two polynomials, which is all we shall need (we also assume that the fixed point is zero and that $\lambda = 1$ for convenience). An *attracting petal* is an open connected set U, with compact closure $\overline{U}$ such that

$$f(\overline{U}) \subset (U \cup \{0\}) \quad and \quad \bigcap_{j\geq 0} f^j(\overline{U}) = \{0\}.$$

Note that $0 \in J(f)$.

Theorem 2.1.4. *Suppose that $f(z) = z + a_1 z^2$. Then there exist two attracting petals at 0.*

Each of the petals gives rise to an attractive parabolic basin of 0, consisting of points which eventually land in the petal and thus converge to 0. Again, these are contained in the Fatou set, and importantly a critical point is attracted to 0.

§2.2 Quadratic polynomials and external rays.

After an affine change of coordinates you can get any quadratic polynomial $a_1 z^2 + a_2 z + a_3$ into the form $z \mapsto z^2 + c$, for $c \in \mathbf{C}$. Note that any map in this form has a critical point at zero, and that the point at infinity is fixed and super-attracting. So the complex plane is a parameter space for the quadratic polynomials. We can make a crude partition of this into two sets dependent on the behaviour of the critical point ar zero. Hence:

Definition. The *Mandelbrot set*, M, is the set of quadratic polynomials of the form $z \mapsto z^2 + c$ for which the critical point at zero does not escape to infinity.

M is compact and connected.

Conjecture. The Mandelbrot set is locally connected.

The following two theorems are classical:

Theorem 2.2.1 *Riemann mapping theorem. Let U be an open, simply-connected subset of $\hat{\mathbf{C}}$ with boundary containing at least two distinct points. There exists a conformal bijection from U to the open unit disc.*

Theorem 2.2.2 *Riemann extension. The above map extends continuously to ∂U if and only if ∂U is locally connected.*

Note that for a polynomial the point at infinity is fixed and super-attracting. Then we have:

Theorem 2.2.3 *Uniformisation. Let P be a quadratic polynomial, $P \in M$. Then there exists a unique analytic bijection, ϕ, from $\Omega(\infty)$ to $\{z \in \hat{\mathbb{C}} : |z| > 1$, such that $\phi \circ P = (\phi(z))^2$ and $\phi(\infty) = \infty$.*

For a quadratic polynomial in M this neighbourhood is the entire attractive basin of infinity, of which the boundary is the Julia set. Thus the conjugation extends continuously to the boundary of the basin if and only if the Julia set is locally connected. (This is true for most quadratic polynomials).

Definition. *External Rays* (see [DH1]) are the image under the inverse uniformising map, ϕ^{-1}, of the rays of constant argument in $\hat{\mathbb{C}} \setminus \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$. External rays are labelled by this argument.

Theorem 2.2.4 *An external ray has unique endpoint in the Julia set if and only if the Julia set is locally connected.*

Proof. This is a direct consequence of theorems 2.2.2 and 2.2.3. □

For polynomials with a locally connected Julia set this gives us a *combinatorial* description of the behaviour of points in the Julia set (see [DH2]). An important class of polynomials satisfy this condition:

We say that a rational map is *hyperbolic* if the critical points converge to stable orbits. Hyperbolic rational maps form an open subset of the appropriate parameter space, and each connected component of such a set is called a *hyperbolic component*. The following is proved in [M2]:

Proposition 2.2.5. *The Julia set of a hyperbolic quadratic polynomial is locally connected.*

Similar results apply to the parameter space:

Theorem 2.2.6 *([DH1]) $\hat{\mathbb{C}} \setminus M$ is conformally equivalent to the unit disc.*

So we can define external rays to the Mandelbrot set as well. However, the question of whether every ray extends uniquely to the boundary of M remains open. It is known (at least) that all rays of rational argument have limit points which lie on the boundary of the Mandelbrot set.

Section 2.4 details how this combinatorial description can be taken further.

§2.3 Branched coverings of $\hat{\mathbf{C}}$.

Definition. A *Branched covering* of the Riemann sphere, $g : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a continuous orientation-preserving map, which is locally a homeomorphism, except at a number of *branch* points, where g is locally of the form $z \mapsto z^k, k \in \mathbf{N}$. The degree of g is the topological degree that it maps $\hat{\mathbf{C}}$ to itself. i.e, the degree is the number of pre-images a typical point has under g.

Rational maps are branched coverings and the critical points are branch points.

Definition. Let f be a branched covering of $\hat{\mathbf{C}}$. Then $X(f)$, the post-critical set, is defined as the set of forward iterates of the critical (or branch-) points of f. Then a *critically finite* branched covering (of $\hat{\mathbf{C}}$) is a branched covering for which the post-critical set is a finite number of points.

We introduce the following homotopy-type equivalence, due to Thurston, which is the natural one to take (see [T] and [R2]).

Definition. (A) Let f and g be degree two critically finite branched coverings of $\hat{\mathbf{C}}$. Then f and g are said to be equivalent (we write $f \simeq g$) if there exists an orientation-preserving homeomorphism $\phi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that there exists a path $\{f_t : t \in [0,1]\}$ through critically finite branched coverings from $f_0 = \phi \circ f \circ \phi^{-1}$ to $f_1 = g$, with $X(f_t) = X(g)$ for all $t \in [0,1]$. The operation $\simeq$ is an equivalence relation, but we sometimes write this as $(f, X(f)) \simeq_\phi (g, X(g))$, although $\simeq_\phi$ is not an equivalence relation.

Note: The definition of equivalence implies that $\phi(X(f)) = X(g)$, since $X(f_0) = X(g)$. Furthermore the cardinality of $X(f)$ is the same as the cardinality of $X(g)$.

Sometimes it is more convenient to have this definition in a different equivalent form:

(B): $f \simeq g$ if there exists a path $\{f_t \mid t \in [0,1]\}$ through critically finite branched coverings, from $f = f_0$ to $g = f_1$, such that $X(f_t)$ varies isotopically.

Thurston stated a condition for a critically finite branched covering to be equivalent to a rational map, proved in [DH3]. An equivalent condition due to Levy, Rees, Tan Lei (see [L] and [TL]) is suited to our purposes:

Definition. Levy cycle Let f be a degree two critically finite branched covering of $\hat{\mathbf{C}}$. Let $\mathcal{G} = \bigcup_i \gamma_i$ be a set of disjoint simple non-peripheral loops in $\hat{\mathbf{C}}$. (A non-peripheral loop is one that partitions $X(f)$ into two sets, both of which contain at least two points of $X(f)$.). Then $\mathcal{G}$ is a Levy cycle if there exist non-peripheral loops $\gamma_i'(1 \leq i \leq r)$ such that γ_i and γ_i' are isotopic in $\hat{\mathbf{C}} \setminus X(f)$, γ_i' is a component of $f^{-1}(\gamma_{i+1})$ and $f|\gamma_i'$ is a homeomorphism.

The following theorem appears in [L] and, improved in [TL].

Theorem 2.3.1. *An orientation-preserving critically finite degree two branched covering of the Riemann Sphere is equivalent to a rational map of degree two if and only if it admits no Levy cycle.*

§2.4 Quadratic Laminations.

The combinatorial information contained in the External Rays was re- interpreted in [T] as Laminations. The idea is that a lamination is a *topological model* for the dynamical behaviour of a quadratic polynomial, and the Julia set is homeomorphic to the unit circle modulo identifications given by a lamination. Thus the theory of laminations applies to quadratic polynomials with connected Julia sets (And locally connected by 2.2.4). Components of the Fatou set are modelled by certain *gaps*. We can choose the Euclidean or Poincaré metric on $\{z \in \hat{\mathbf{C}} \mid |z| \leq 1\}$.

Definition. A *Lamination*, $\mathcal{L}$, on $\{z \in \hat{\mathbf{C}} \mid |z| \leq 1\}$ is the closure of a set of geodesics, which are called *leaves*, non-intersecting on the interior of the unit disc, with endpoints on the unit circle. Note that the endpoints of a leaf can be coincident, so that the leaf is just one point on the unit circle. The components of the complement of $\bigcup_{\ell\in\mathcal{L}}$ in the open unit disc are called gaps. We refer to a leaf by its endpoints, which we measure in units of radians$/2\pi$, from 0 to 1.

Definition. Let $\mathcal{L}$ be a lamination on the unit disc. Then an *invariant quadratic* lamination satisfies the following:

(a) If $\ell \in \mathcal{L}$, with endpoints a and b then $\ell^2 \in \mathcal{L}$, where ℓ^2 has endpoints $2a \pmod 1$ and $2b \pmod 1$.

(b) $-\ell \in \mathcal{L}$, where $-\ell$ is the leaf with endpoints $a + 1/2 \pmod 1$ and $b + 1/2 \pmod 1$.

(c) There is at least one leaf ℓ_1 such that $\ell_1^2 = \ell$.

The map $z \mapsto z^2$ extends to a map of $S^1 \cup \mathcal{L}$ to itself, by mapping each leaf affinely to the leaf whose endpoints are the images of the original leaf's endpoints under $z \mapsto z^2$. The longest leaf or leaves (as measured as the difference in angles between its two endpoints) of the lamination is/are called the *Major leaf/leaves* and its/their image is the *Minor leaf*. For periodic minor leaves we will sometimes refer to the periodic pre-image of the Minor leaf as the Major leaf.

This map can be extended further to a map including all the gaps as well, and thus the whole of $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$. Then a lamination $\mathcal{L}$ has a unique map, $s_{\mathcal{L}}$, such that $s_{\mathcal{L}}(0) = 0$, and $\mathcal{L}$ is invariant under $s_{\mathcal{L}}$. This can be extended to the whole sphere so that on $\{z : |z| > 1\}$, $s_{\mathcal{L}} \equiv z \mapsto z^2$. $s_{\mathcal{L}}$ is then a degree two branched cover of $\hat{\mathbb{C}}$.

We see how a lamination is related to a polynomial map: Given a quadratic polynomial P, with connected and locally connected Julia set, we proceed as follows: By 2.2.3 (uniformisation) we have a map ψ on $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$, such that ψ^{-1} extends to S^1 and maps it continuously onto $J(P)$. So ψ^{-1} defines equivalence classes of points in S^1 by $x \sim y \iff \psi^{-1}(x) = \psi^{-1}(y)$. If $x \sim y$ we join x and y by leaf ($\ell \in \mathcal{L}$).

Now $\hat{\mathbb{C}}/\sim$ is homeomorphic to $\hat{\mathbb{C}}$, so $s_{\mathcal{L}}$ induces a branched cover $\overline{s_{\mathcal{L}}} : \hat{\mathbb{C}}/\sim \to \hat{\mathbb{C}}/\sim$. So $\bar{s_{\mathcal{L}}}$ is conjugate to P if we define $s_{\mathcal{L}}$ inside the disc so that the conjugacy holds. Note that by this process it is true that for a hyperbolic polynomial P_h, the Lamination map obtained from any other polynomial in the same hyperbolic component is the same one as for P_h.

Since all gaps with polygonal boundaries collapse to points under this identification, the components of the Fatou set are given by infinite-sided gaps.

From now on we will denote a leaf with endpoints p and q by $[\![p,q]\!]$. Furthermore the Lamination with minor leaf μ will be denoted $\mathcal{L}_\mu$.The associated lamination map will be denoted by s_μ, but when dealing with periodic minor leaves, which are uniquely determined by just one of their endpoints, we will usually abbreviate this notation: for the minor leaf $[\![\frac{1}{3},\frac{2}{3}]\!]$ the associated Lamination map will be denoted $s_{1/3}$ or $s_{2/3}$.

As with External rays, this idea of lamination carries over to parameter space as well. The set of all minor leaves of quadratic invariant laminations forms a lamination, called the *Quadratic Minor Lamination*, or QML for short. There is a partial ordering on QML, defined thus: Let μ_1 and μ_2 be leaves in QML. Then $\mu_1 < \mu_2$ if and only if μ_1 separates μ_2 from zero. So that, for instance $[\![\frac{1}{3},\frac{2}{3}]\!] < [\![\frac{2}{5},\frac{3}{5}]\!]$. (Note that the conjecture that M is locally connected is equivalent to the statement that $\mathbf{C}/\sim_{QML}$ is homeomorphic to M.)

We will be particularly interested in laminations with periodic minor leaves; that is, minor leaves for which $\mu^{2^n} = \mu$ for some n. The endpoints of a periodic minor leaf necessarily are odd-denominator rationals.

The following observation will be useful later:

Useful Fact: (see [T]) Let $\mu_1 > \mu_2$. Then $\mathcal{L}_{\mu_1}$ contains the leaf μ_2. e.g., $\mathcal{L}_{[2/5,3/5]}$ contains the leaf $[\![\frac{1}{3},\frac{2}{3}]\!]$.

We can use this fact to define a ***minimal minor*** leaf, which we make much use of in future. A minimal minor leaf is a minor leaf μ such there does not exist any $\ell \in QML$ with $\ell < \mu$. Furthermore, given some periodic minor leaf μ', we say that the minimal leaf such that $\mu < \mu'$ is the minimal leaf of μ'. We can characterise minimal minors in the following useful way:

Lemma 2.4.1. *μ is a minimal periodic minor leaf if and only if $\mathcal{O}^+(\mu)$ bounds a region $P \subset \{z \in \hat{\mathbf{C}} \mid |z| \leq 1\}$ with polygonal boundary ∂P and μ is the shortest leaf in $\mathcal{O}^+(\mu)$.*

Proof. We can choose a homeomorphism $\phi : \partial P \to \partial P$ such that $\phi(\ell) = \ell^2$ for all $\ell \in \partial P$. We note that for any homeomorphism (i.e., degree one map) $\phi : S^1 \to S^1$ there is a well-defined *rotation number*, which is defined as $\lim\limits_{n\to\infty} \dfrac{\tilde{\phi}^n(x) - x}{n}$ for any lift $\tilde{\phi} : \mathbf{R} \to \mathbf{R}$ of ϕ and any $x \in \mathbf{R}$. Then the rotation number of $\phi : \partial P \to \partial P$ is a rational number determined by the relative positions of μ and μ^2 on ∂P. Thus it is independent of the particular choice of ϕ and we call it the rotation number of μ.

It is a fact that μ is determined uniquely by its rotation number and all rational rotation numbers occur for minor leaves. To see this, consider the component M_0 of M which consists of maps $z \mapsto z^2 + c$ having a fixed point whose modulus is less than 1. Then ∂M_0 consists of those maps for which the multiplier is of modulus 1, and each such number, of the form $e^{2\pi i\alpha}$, occurs for precisely one polynomial f (see [DH2]). The fixed point is parabolic if and only if α is rational, and then $J(f)$ is locally connected (see [M2]). Then the periodic minimal minors are precisely the minor leaves of laminations corresponding to f with a fixed parabolic point. Moreover, the rotation number of the minor leaf is α, where $e^{2\pi i\alpha}$ is the multiplier of the fixed parabolic point. $\square$

Example: $[\![\frac{1}{7}, \frac{2}{7}]\!]$ is a minimal minor leaf, whose forward orbit bounds the triangle with sides $[\![\frac{1}{7}, \frac{2}{7}]\!]$, $[\![\frac{2}{7}, \frac{4}{7}]\!]$ and $[\![\frac{4}{7}, \frac{1}{7}]\!]$.

We now discuss a way to use the presence of an invariant arc set to determine something about the minor leaf of a lamination map.

Lemma 2.4.2. *Let g be a degree two orientation-preserving branched covering of $\hat{\mathbf{C}}$ with a fixed critical point, which we can take to be at ∞, and let the other critical point be periodic. Suppose also that there is a set Γ of disjoint simple arcs in $\{z \in \hat{\mathbf{C}} \mid |z| \leq 1\}$ with endpoints in S^1 such that each arc is non-trivial in $\{z \in \hat{\mathbf{C}} \mid |z| \leq 1\} \setminus X(g)$ and $\Gamma \subset g^{-1}(\Gamma)$ up to isotopy. It then follows that if there* *is a homeomorphism from $S^1 \cup \Gamma \cup X(g)$ to $S^1 \cup \{\mu_q^{2^j} \mid j \geq 0\} \cup X(s_q)$ for some periodic minor leaf μ_q, then $\mu_q \geq \mu_p$.*

Proof. For simplicity we can assume that up to equivalence (in the sense of §2.3) that S^1 is invariant under g. Now it is not hard to show that g admits no Levy cycle and hence that g is equivalent to a quadratic polynomial. Thus g is also equivalent to a lamination map s_p for some odd-denominator rational p. We wish to obtain information about s_p. Let Γ' be the set of arcs of s_p corresponding to the set Γ for g. Then $\Gamma' \subset \mathcal{L}_p$ up to isotopy in $\hat{\mathbf{C}} \setminus X(s_p)$. Then the homeomorphism from $S^1 \cup \Gamma \cup X(g)$ to $S^1 \cup \{\mu_q^{2^j} \mid j \geq 0\} \cup X(s_q)$ means that $\mu_q \geq \mu_p$. □

We now introduce another way of determining a lamination map, equivalent to a given type of branched covering. Let us assume that we have a degree two branched covering f satisfying the same conditions as g above (so $f \simeq s_q$ for some odd-denominator rational q), but instead of having an arc set Γ, we have the following situation:

Lemma 2.4.3. *Suppose there are arcs β_j, for $1 \leq j \leq n$, such that β_j connects $f^j(0)$ to some point $x_j \in S^1$, and that there exists an isotopy χ such that: χ preserves S^1 and $X(f)$; $\chi(\beta_{j-1})$ is a component of $f^{-1}(\beta_j)$ for $j > 1$ (thus determined by x_{j-1}) and $\chi(\beta_n)$ is the sub-arc of $f^{-1}(\beta_1)$ which joins 0 to one of the (two) points $f^{-1}(x_1)$. Let ψ be an orientation-preserving homeomorphism of S^1 for which $\psi(x_j) = 2^{j-1}q$ and such that any component I of $S^1 \setminus \{x_j\}$ has length greater than half the circumference of S^1 if and only if $\psi(I)$ has length greater than half the circumference of S^1. Then $f \simeq_\psi s_q$.*

Proof. Let q and q' be the two endpoints of μ_q (recall that μ_q is associated uniquely to s_q). Note that in general ψ will only exist for at most one of q or q'. Then there are two natural ways to choose arcs β_j for the map s_q: we can join $s_q^j(0)$ to $2^{j-1}q$ by a straight line arc, or to $2^{j-1}q'$ by a straight line arc. In this case ψ is taken to be the identity. Let $\beta_j' = \beta_j(s_q)$ and $\beta_j = \beta_j(f)$. Then we extend ψ to the rest $\hat{\mathbf{C}}$ so that $\psi(\beta_j) = \beta_j'$, $\psi(X(f)) = X(s_q) \cup \{\infty\}$, where in particular $\psi(\infty) = \infty$; then we can extend ψ to the rest of $\hat{\mathbf{C}}$, keeping ψ in the same isotopy class relative to $X(f)$. Then it follows that $f \simeq_\psi s_q$. $\square$

2.4.4. Tunings.

Tuning is a way of modifying a polynomial map on part of its domain to obtain another with particular desired properties (see [R2]) — we shall use it to create maps with higher period critical points, as follows: Let f_1 and f_2 be quadratic polynomials, where f_2 has a period m critical point, 0. Then there is a disc D containing the critical point 0, such that $f^m(D)$ is isotopic to D relative to $X(f_1)$. Take $\overline{D} \subset D_1 \cap f_2^m(D_1)$, so that the sets $f_2^j(D_1)$ are disjoint for $1 \leq j \leq m$. Then the tuning of f_1 by f_2 is defined as:

$$f_1 \vdash f_2 = f_2 \text{ on } \hat{\mathbf{C}} \setminus \bigcup_{j=1}^{m} f_2^j(D_1),$$

$$(f_1 \vdash f_2)|f_2^j(D_1) \text{ is a homeomorphism for } 1 \leq j < m,$$

$$(f_1 \vdash f_2)^m = \lambda f_1 \text{ on } D,$$

for some suitable scaling function λ. Then extend $(f_1 \vdash f_2)$ continuously into $D_1 \setminus D$, so that $(f_1 \vdash f_2)$ is a polynomial map on $\hat{\mathbf{C}}$.

If f_1 has 0 of period m', then it is clear for that $(f_1 \vdash f_2)$, 0 is of period mm'.

For lamination maps, we define tuning in a similar way. For minor leaves μ_a and μ_b, with respective lamination maps s_a and s_b, the map $s_a \vdash s_b$ is the lamination map for the minor leaf $\mu_a \vdash \mu_b$.

For example, we write $[\![\frac{1}{3}, \frac{2}{3}]\!] \vdash [\![\frac{1}{3}, \frac{2}{3}]\!] = [\![\frac{2}{5}, \frac{3}{5}]\!]$ and $s_{\frac{1}{3}} \vdash s_{\frac{1}{3}} \simeq s_{\frac{2}{5}}$.

We can also say something about the position of leaves which are tunings in *QML*: all tunings μ_t of some minimal minor leaf μ are such that, if $\ell < \mu_t$, then ℓ is also a tuning of μ (including the empty tuning $\ell = \mu$).

§2.5 Matings of quadratic polynomials.

After making the simple observation that the sphere is two discs glued together, we could naïvely suppose that a union of two quadratic laminations on two discs might give us a model for some degree two rational maps.

A method of constructing rational maps from polynomial maps was devised by Douady and Hubbard (1982). For quadratic polynomials the process is as follows. Let K_c be the filled-in Julia set of a quadratic polynomial $P_c = z^2 + c$, that is

$$K_c = \hat{\mathbb{C}} \setminus \{z : P_c^n(z) \to \infty, \text{as } n \to \infty\},$$

where $c \in M$. For K_c connected and locally connected (true if 0 is periodic, which we are particularly interested in), K_c is a topological (closed) disc with a continuous map ψ_c from S^1 to ∂K_c such that

$$P_c\big(\psi_c(e^{2\pi i t})\big) = \psi_c(e^{2\pi i(2t)}).$$

Let P_{c_1} and P_{c_2} be two such polynomials. Then the idea is to form a sphere by *pasting* together K_{c_1} and K_{c_2}, identified along their boundaries using $Im(\psi_{c_1}) \sim Im(\psi_{c_2})$. Now P_{c_1} and P_{c_2} define a map on the sphere.

According to Douady and Hubbard the mating of two polynomials in complex conjugate limbs of the Mandelbrot set produces a branched covering, which is not equivalent to a rational map. Tan Lei's thesis contains a proof that this is sufficient.

We can give a more formal definition in terms of laminations. Given two laminations on the closed unit disc with periodic minor leaves, say $\mathcal{L}_a$ and $\mathcal{L}_b$, we can define the *mating* of $s_a(z)$ and $s_b(z)$, as

$$s_a \sqcup s_b(z) = \begin{cases} s_a(z), & \text{if } z \in \{z \in \hat{\mathbb{C}} \mid |z| \le 1\}; \\ s_b^{-1}(1/z), & \text{if } z \in \{z \in \hat{\mathbb{C}} \mid |z| \ge 1\}. \end{cases}$$

This is a degree two branched covering of the Riemann Sphere. What we are interested in is whether this is equivalent to a rational map. Fortunately the criterion 2.3.1 is possible to check in our case, because we can use the lamination theory to find or discount the presence of Levy cycles. This is summed up in the following theorem of Tan Lei.

Theorem 2.5.1. *Suppose that a and b are periodic. Then $s_a \sqcup s_b$ is equivalent to a rational map of degree two if and only if there does not exist $\ell \in QML$ such that $\ell \le a$ and $\ell \le b$.*

This is the lamination analogue of the conjugate-limbs statement.

Note that it is only a special class of rational maps that can be constructed via matings (but bigger than detailed here). Given a degree two rational map with two periodic critical points we can recover the associated lamination, and the two polynomials from which it was mated. We can find a closed simple loop $\gamma \subset \hat{\mathbf{C}} \backslash X(f)$ such that γ is isotopic to $f^{-1}(\gamma)$ relative to $X(f)$, with $f|f^{-1}\gamma$ degree two, and preserving the orientation of γ. Let D_1 and D_2 be the components of $\hat{\mathbf{C}} \setminus f^{-1}\gamma$. Let $\phi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a homeo., preserving $X(f)$ pointwise, which maps γ to $f^{-1}\gamma$.

Then let $f_1 = \phi \circ f$ in D_1, and let f_1 have a fixed branch point in D_2. Then f_1 is a degree two branched covering which is equivalent to a rational map. This is a polynomial since we can take the fixed critical point to be at infinity. (The same argument applies to f_2 on D_2.)

The material covered in the last two sections will primarily be of use in Chapters 3 and 5.

§2.6 Quasi-conformal surgery.

The following two sections are concerned with establishing the location of singularities on V_n. We start by introducing the necessary background material.

Recalling that a conformal transformation preserves angles, a *quasi-conformal* homeomorphism is a map on a subset of the complex plane which has the property that there is a universal bound on the distortion of angles, almost everywhere in the sense of Lebesgue measure (see [Af]).

We recall that a Riemannian metric σ on $\mathbf{C} = \{x + iy \mid x, y \in \mathbf{R}\}$ is given by a form of the following type on the tangent spaces to $\mathbf{C}$:

$$\sigma : \quad a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2,$$

where a, b and c are real-valued functions, the corresponding matrix is positive-definite, and σ is unique up to multiplication by a strictly positive real-valued function $\nu(x,y)$. This can be written in matrix form as

$$(x,y) \mapsto \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}.$$

Then the matrix of σ always has strictly positive eigenvalues.

A *complex structure* on $\mathbf{C}$ is given by a choice of equivalence classes of Riemannian metrics on $\mathbf{C}$, where $\sigma_1 \sim \sigma_2 \iff \sigma_2 = \nu\sigma_1$ for a scalar function $\nu > 0$.

The standard structure, denoted σ_0, corresponds to the identity matrix, where $a \equiv c \equiv 1$ and $b \equiv 0$.

If the functions a, b and c are measurable we have a measurable Riemannian metric. Furthermore, we say that σ is a *bounded* measurable structure if there exists a real number $K > 0$ such that

$$\frac{1}{K} \leq \frac{\lambda(x,y)}{\mu(x,y)} \leq K \quad \forall (x,y),$$

where λ and μ are the eigenvalues of the matrix of σ.

Given a C^1 differentiable map $\Phi(x,y) = u(x,y) + iv(x,y)$ of the complex plane, and remembering that all the elements are defined almost everywhere, it transforms the complex structure in the following way:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \to (D\Phi)^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} D\Phi,$$

where

$$D\Phi = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Thus holomorphic maps preserve the standard structure. Conversely, by [Af], if a map preserves σ_0 almost everywhere on an open connected subset of $\mathbf{C}$, then it is holomorphic there.

We write $\Phi_*\sigma$ for the transform of σ by Φ as just described. Note that we have the identity $\Phi^* = (\Phi^{-1})_*$.

If Φ is a quasi-conformal mapping of $\hat{\mathbf{C}}$, then $\Phi^*\sigma_0$ is a bounded measurable structure, as described above (see [Af]).

The following theorem (see [AB]) is essential to the idea of quasi-conformal surgery: (cf. theorem 2.2.1)

Theorem 2.6.1. *Measurable Riemann Mapping Theorem. Given a bounded measurable structure σ on $\mathbf{C}$, there exists a quasi-conformal homeomorphism Φ such that*

$$\Phi^*\sigma = \sigma_0$$

$$\text{and } \; (\Phi^{-1})^*\sigma_0 = \sigma.$$

Φ is unique up to composition on the right by conformal transformations of the complex plane.

This theorem is the basis for Douady's proof that the hyperbolic components in the Mandelbrot set are topological discs, Sullivan's proof of the non-existence of wandering domains for rational maps (see [S]) and also for Milnor's proof (in [M3]) that hyperbolic components for polynomial maps are homeomorphic to n-balls for appropriate n. We adapt Douady's technique for our purposes in the next section, as well as using Milnor's result.

The idea of quasi-conformal surgery is to alter the standard structure on certain domains of $\mathbf{C}$, on which we have some function defined, and then use the quasi-conformal homeomorphisms of 2.6.1 to produce new functions, which preserve the altered structure, on these domains with some required properties.

§2.7 Location of singularities of V_n

We identify the possible occurrences of the singularities of V_n, an essential first step to allow their further investigation. We adapt the ideas used in Douady's proof, which constructs a homeomorphism from a hyperbolic component to a standard model, by using a similar argument but showing that the map is continuously differentiable, using some of the results of Ahlfors-Bers [AB].

Theorem 2.7.1. *The singularities of V_n occur only at the punctures of W_n.*

Proof.

Recall that the defining equation of the affine variety V_n is given by the polynomial $p_n(a, b)$, for which we have established the location of roots on the line $b = 0$.

Let $V = V_n \cap \{(a,b) \in \mathbf{C}^2 \mid b \neq 0\}$. The theorem is then equivalent to the statement that V is a non-singular sub-manifold of the space $R = \{(a,b) \in \mathbf{C}^2 \mid b \neq 0\}$, where we are identifying R with the set of maps $f_{a,b}$ which are degree two. This is because we know that the punctures of W_n are all on the line $b = 0$, or on the line at infinity. By what has been said in §1.5 it suffices to show that for all $(a,b) \in V \cap R$, $Dp_n(a,b) \neq 0$.

Take $(a_0, b_0) \in V$ and let $f = f_{a_0,b_0}$. We consider two cases:

Case 1: $f^j(0)$ is not a critical point of f for $0 < j < n$.

Case 2: $f^j(0)$ is a critical point of f for some $0 < j < n$.

Proof of case 1

In some appropriately small neighbourhood, Ω, of (a_0, b_0) define $G(a,b,z) = f^n_{a,b}(z) - z$. By definition $G(a_0, b_0, 0) = 0$ and
$\dfrac{\partial G}{\partial z}(a_0, b_0, 0) = 0 - 1 \neq 0$. Now, by the Implicit Function Theorem (1.5.1) there is a unique holomorphic function $z(a,b)$ defined on Ω such that $z(a_0, b_0) = 0$ and $G(a,b,z(a,b)) = 0$.

Let us denote the multiplier of $f^n_{a,b}$ at $z(a,b)$ by $\lambda(a,b)$, so we have that

$$\lambda(a,b) = (f^n_{a,b})'(z(a,b)) = \prod_{j=0}^{n-1} f'_{a,b}\Big(f^j_{a,b}(z(a,b))\Big).$$

Then λ is a holomorphic function of z, and defined in a neighbourhood of (a_0, b_0) in R. Furthermore, because we are in case 1 so that $f^j(0)$ is not a critical point for $1 \leq j < n$, on a sufficiently small neighbourhood of (a_0, b_0), $f^j_{a,b}(z(a,b))$ is bounded away from a zero of $f'_{a,b}$ for $0 < j < n$. Thus, on this neighbourhood, $\lambda(a,b) = 0$ if and only if $f'_{a,b}(z(a,b)) = 0$. Hence for (a,b) sufficiently near (a_0, b_0), $(a,b) \in V$ if and only if $\lambda(a,b) = 0$ (i.e., 0 is a period n critical point).

By what we have said above we must show that $D\lambda \neq 0$ at the point (a_0, b_0).

Following the method of Douady ([D]) we will construct a C^1 function ϕ (Douady's analogous function is just C^0) for some $\delta > 0$,

$$\phi : \{\gamma \mid |\gamma| < \delta\} \to R,$$

$$\text{such that} \quad \lambda \circ \phi(\gamma) = \gamma, \tag{†}$$

and $\phi(0) = (a_0, b_0)$. Then

$$D\lambda \circ D\phi(0) = \frac{d}{d\gamma}(\lambda \circ \phi)(0) = 1,$$

which implies that $D\lambda \neq 0$ at (a_0, b_0). (We need the function to be C^1 so that the above expression is well defined.)

In order that the maps $\gamma \mapsto a(\gamma)$ and $\gamma \mapsto b(\gamma)$ are C^1 in the family $f_{a,b}$ it suffices to show that $\gamma \mapsto f_{a,b}$ is C^1 in γ — which we do in some neighbourhood of f_{a_0,b_0}, expressing nearby functions on V as g_γ for $|\gamma| < \delta$, where $g_0 = f_{a_0,b_0}$. Then, we will have shown that $\gamma \mapsto p_n(a,b)$ is C^1 on some neighbourhood of 0, and that $Dp_n(a_0, b_0) \neq 0$.

Note on Douady result:

In [D] Douady constructs a map

$$p_c \mapsto \text{ multiplier of periodic orbit,}$$

which is holomorphic in c, where p_c is the quadratic polynomial $z \mapsto z^2 + c$. This extends to the whole hyperbolic component of p_c, so he shows that the hyperbolic components in M are holomorphically equivalent to the open unit disc. We shall construct an analogous map, as outlined above, but we are only interested in $f_{a,b}$ in a small neighbourhood of f_{a_0,b_0}. We shall follow his arguments in roughly the same order.

We consider the following family for $|\gamma| < \delta$:

Let $h_\gamma : \{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\} \to \{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\}$ be given by

$$h_\gamma(z) = \frac{z(z+\gamma)}{1+\overline{\gamma}z}.$$

The derivative of the fixed point, 0, is γ, and there is a critical point in $\{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\}$. The critical point is located at approximately $-\gamma/2$, with critical value at approximately $\gamma^2/4$.

Let W be the immediate attractive basin of 0 for f^n. Then, by 2.1.2, there is a holomorphic conjugacy $\chi : W \to \{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\}$ such that

$$\chi \circ f^n = h_0 \circ \chi \text{ on } W$$

and $\chi(0) = 0$.

For sufficiently small δ, there is a fixed number $r_0 > 0$ such that for all $|\gamma| < \delta$ the set $h_\gamma^{-1}\{z \,|\, |z| = r_0\}$ is a simple loop which bounds a finite disc containing $\{z \,|\, |z| \le r_0\}$. (We can choose r_0 so that the critical value lies in the disc $\{z \,|\, |z| \le r_0\}$. Then the pre-image is a simply-connected region which maps with degree two over this disc.)

We now make some definitions exactly as in [D]: let A_γ denote the (topological) annulus bounded between $\{z \,|\, |z| = r_0\}$ and $h_\gamma^{-1}\{z \,|\, |z| = r_0\}$. Let D_γ be the (finite) disc bounded by $h_\gamma^{-1}\{z \,|\, |z| = r_0\}$. Let $D_{r_0} = \{z \,|\, |z| \le r_0\}$.

We illustrate the above:

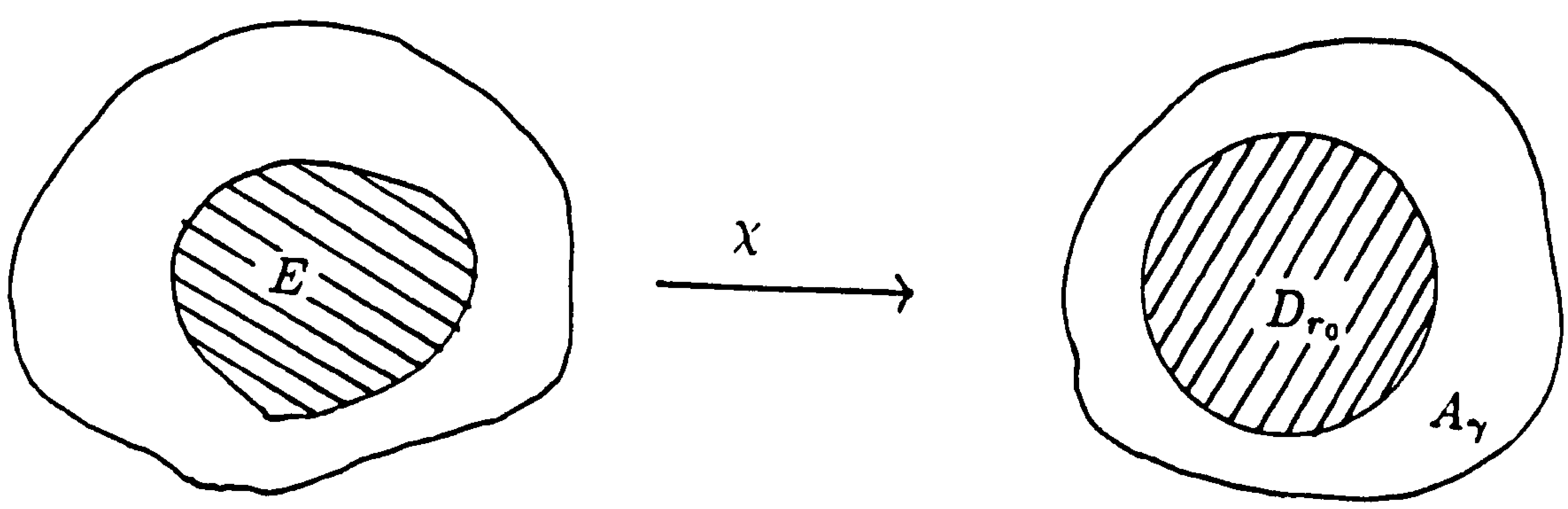


Figure 2.7a

Now we alter the conformal structure on $\hat{\mathbb{C}}$:

Let $E = \chi^{-1}(D_{r_0})$. We define a function f_γ on $\hat{\mathbb{C}}$ by

$$f_\gamma = S_f \circ \chi^{-1} \circ h_\gamma \circ \chi \quad \text{on} \quad E,$$

$$f_\gamma = f \quad \text{on} \quad \hat{\mathbb{C}} \setminus \chi^{-1}(D_\gamma),$$

where S_f is the branch of $f^{-(n-1)}$ which maps W to $f(W)$. Note that $f^{-(n-1)}$ is univalent on $f(W)$, since the only critical point on $f^j(W)$ for $0 \le j < n$ is in W. This means that f_0 coincides with its previous definition (as f_{a_0,b_0}) where so far defined.

Extend f_γ into the region $\chi^{-1}(D_\gamma) \setminus E$ to be C^2 in (z, γ) and a degree two branched covering, again keeping f_0 as defined before. Both critical points lie outside this region, so this is not a problem. Let r_1 be such that $D_{r_1} = \{z \mid |z| \leq r_1\}$ satisfies $D_{r_0} \subset D_\gamma \subset D_{r_1}$. We define a function $\overline{f_\gamma}$ by $\chi \circ f_\gamma \circ S_f^{-1} \circ \chi^{-1}$ on the image of E under ξ. On the image of this annulus $(\chi^{-1}(D_\gamma) \setminus E)$ under χ, which is A_γ, define $\overline{f_\gamma}(re^{i\theta}) = (1 - \nu(t))h_\gamma(re^{i\theta}) + \nu(t)h_0(re^{i\theta})$ for $t \in [0,1]$, $\theta \in [0, 2\pi)$ and where $r = (1-t)r_1 + tr_0$ and $\nu : [0,1] \to [0,1]$ is monotone, with first and second derivatives vanishing at 0 and 1. Then, for $t = 0$ we have $\overline{f_\gamma} = h_\gamma(re^{i\theta})$ on $\{z \mid |z| = r_1\}$ and for $t = 1$, we have $\overline{f_\gamma} = h_0(re^{i\theta})$ on $\{z \mid |z| = r_0\}$, agreeing with the definition of $f = f_\gamma$ on E when we define $f_\gamma = S_f \circ \chi^{-1} \circ \overline{f_\gamma} \circ \xi$ on $\chi^{-1}(D_\gamma) \setminus E$. Then, it follows that f_γ is C^2, because $\overline{f_\gamma}$ is C^2.

Now we define a modified bounded measurable structure on $\hat{\mathbf{C}}$:

Let $\sigma_\gamma = \sigma_0$, the standard structure, on $\chi^{-1}(D_{r_0}) = E$ and outside $\bigcup_{j\geq 0} f^{-j}(W)$. Then on $f^n(E)$, $\sigma_\gamma = (f^n)_*\sigma_0 = (f_\gamma^n)_*\sigma_0$. Pulling back , we define σ_γ on the sets $f_\gamma^{-j}(E)$ inductively by $\sigma_\gamma = (f_\gamma^{-j})^*\sigma_\gamma$. Then σ_γ is well defined on the whole of $\hat{\mathbf{C}}$. (In particular, on the boundaries of the sets $f^{-j}(E)$ as $j \to \infty$, there is no problem as we are pulling back σ_0 by a C^1 (at least) function).

Note that f_γ^n has the multiplier γ, since $f_\gamma^n = \chi^{-1} \circ h_\gamma \circ \chi$ on a neighbourhood of $\chi(0) = 0$, which means that the multiplier is equal to the derivative of h_γ at 0.

So, on the domain $|\gamma| < \delta$, we have constructed a map f_γ such that $(\gamma, z) \mapsto f_\gamma(z)$ is holomorphic in z near $\{f^j(0) \mid 0 \leq j \leq n\}$: f_γ preserves the standard structure on some neighbourhood of the periodic critical orbit (on $f^j(E)$, where $1 \leq j \leq n$, in fact), so is holomorphic by §2.6. Furthermore, this unique attractive cycle has multiplier equal to γ.

We investigate the dependence of σ_γ on γ shortly — notes on σ_γ:

1. σ_γ is the standard conformal structure near to $\{f^j(0) \mid 0 \leq j < n\}$.

2. $(f_\gamma)_*\sigma_\gamma = \sigma_\gamma$. i.e., the structure σ_γ is preserved by f_γ.

Now the Measurable Riemann Mapping Theorem (as stated as 2.6.1) implies the existence of a family ψ_γ of quasi-conformal homeomorphisms of $\hat{\mathbf{C}}$ such that $(\psi_\gamma)_*\sigma_\gamma = \sigma_0$ and such that $\psi_0 \equiv Id$.

So $\psi_\gamma \circ f_\gamma \circ \psi_\gamma^{-1}$ preserves the standard structure almost everywhere in $\hat{\mathbf{C}}$.

Thus, by §2.6, $\psi_\gamma \circ f_\gamma \circ \psi_\gamma^{-1}$ is a holomorphic map on $\hat{\mathbf{C}}$. Since it topologically a degree two mapping, it is a degree two rational map g_γ with $g_0 = f_0 = f$.

We have therefore constructed the map $\phi : \gamma \to g_\gamma$, such that the multiplier at 0 is γ because g_γ is conjugate to f_γ. This then satisfies (†) as required. However, to prove the theorem we must show that ϕ is continuously differentiable in γ. i.e., it remains to show that the map $\dfrac{\partial g_\gamma}{\partial \gamma}$ exists and is continuous.

The first step is to show that $\gamma \mapsto \sigma_\gamma$ is differentiable, with derivative continuous with norm defined below:

To satisfy the conditions of the Ahlfors-Bers theorem (Theorem 10 in [AB]) we need to show that for at least one map F such that $F(\gamma)$ is represented by σ_γ, $F : \{|\gamma| < \delta\} \to \{\text{complex structures}\}$ is such that for $\gamma_0 \in \{|\gamma| < \delta\}$,

$$F(\gamma_0 + h) = hF_1(\gamma_0) + hR(\gamma_0, h), \tag{$*$}$$

where $F_1(\gamma_0) \in L_\infty$, $R(\gamma_0, h) \in L_\infty$ and $R(\gamma_0, h) \to 0$ as $h \to 0$ in L_∞ norm.

(Note: Their theorem is stated in terms of a function $\mu(z)$, where the Riemannian metric is expressed by $ds = \nu(x,y)|dz + \mu d\bar{z}|$, where $|\mu(z) < 1|$ and ν is as stated in §2.6 — one can check that the above condition holds for μ if it holds for the function F.)

Lemma 2.7.2. *$\gamma \mapsto \sigma_\gamma$ satisfies $(*)$ on $\hat{\mathbf{C}}$.*

Proof.

First, we show that $\gamma \mapsto \sigma_\gamma$ satisfies $(*)$ on $f_\gamma^{-1}(E)$.

As explained in §2.6, we must consider the transformation of σ_0, given by the identity matrix I_2, by the C^1 function f_γ as:

$$F : \gamma \mapsto (Df_\gamma)^T Df_\gamma.$$

This describes σ_γ on the set $f_\gamma^{-1}(E)$. Note that on the set E this gives a matrix which is a multiple of the identity, and thus is equivalent to the standard structure.

Since f_γ is C^1 in z, the matrix Df_γ has entries which are continuous functions of z. Thus, for fixed γ these entries taken on values in bounded subsets of $\mathbf{C}$

because they are defined on a compact set. They are also bounded functions of γ: again the domain is a compact set. Therefore the ratio of the eigenvalues of the above matrix is bounded, which is what we need (see §2.6). We will also need that on $f^{-2}(E)$, the matrix given by $(Df_\gamma^2)^T Df_\gamma^2$ satisfies $(*)$. This follows in the same way as for $(Df)^T Df$.

We extend this result to the rest of $\hat{\mathbf{C}}$:

We have $\sigma_\gamma = (Df_\gamma^3)^T Df_\gamma^3$ on $f_\gamma^{-3}(E)$.

Then on $f_\gamma^{-j}(f_\gamma^{-1}(E))$, we have σ_γ defined inductively as $(Df_\gamma^{j+1})^T Df_\gamma^{j+1}$ for $j \geq 1$.

We must take care with the domain on which the map $\gamma \mapsto \sigma_\gamma$ is defined, because the sets $\chi^{-1}(D_\gamma)$ vary as γ varies. Therefore, we consider the set $f^{-2}(E)$ which is defined independently of γ: now $f_\gamma^{-1}(E) \subset f^{-2}(E)$ for all $|\gamma| < \delta$ (Because $f^{-1}(E)$ is approximate to $\chi^{-1}(D_\gamma)$). Thus, outside of the disc $f^{-2}(E)$ we have that $f_\gamma = f_0 = f$. Likewise, $f_\gamma^{-1}(E) \subset f_\gamma^{-2}(E)$ and $f = f_\gamma$ on $\hat{\mathbf{C}} \setminus f_\gamma^{-2}(E)$.

So $f^{-1}f_\gamma^{-2}(E) = f_\gamma^{-3}(E)$ and $f^{-1}f^{-2}(E) = f_\gamma^{-1}f^{-2}(E)$. Thus for $j \geq 1$, $f^{-j}f_\gamma^{-2}(E) = f_\gamma^{-j-2}(E)$ and $f^{-j}f^{-2}(E) = f_\gamma^{-j}f^{-2}(E)$. Thus $f^j = f_\gamma^j$ outside $f^{-j-2}(E) = f_\gamma^{-j}f^{-2}(E)$.

So $f_\gamma^2 f^j = f_\gamma^{j+2}$ outside $f^{-j-2}(E) = f_\gamma^{-j}f^{-2}(E)$.

In matrix form we can write

$$(Df^j)_z = \nu_j U_j$$

for a scalar $\nu_j > 0$ and orthogonal matrix U_j, which are both dependent on z. Then, for $j \geq 1$, $F(\gamma)$ is given by

$$\frac{1}{\nu_j^2}(Df_\gamma^{j+3})^T Df_\gamma^{j+3} \quad \text{on} \quad f^{-j-3}(E) \setminus f^{-j-2}(E)$$

$$= \frac{1}{\nu_j^2}(Df^j)^T (Df_\gamma^3)^T Df_\gamma^3 Df^j$$

$$= U_j^T (Df_\gamma^3)^T Df_\gamma^3 U_j \quad \text{on} \quad f^{-j-3}(E) \setminus f^{-j-2}(E).$$

The derivative of $F(\gamma)$ with respect to γ is of the form

$$U_j^T G(\gamma, w) U_j$$

for $w \in f^{-4}(E)$, and where G is boundedly C^1 in γ and w. So $(*)$ holds.

Everywhere else $\sigma_\gamma = \sigma_0$, so the theorem is true here trivially. □

Therefore, by ([AB]) ψ_γ is differentiable in γ, with continuous derivative in a suitable Banach space, which we define in the next proof.

So to finish off the proof of 2.7.1 we need:

Proposition 2.7.3. *The map $\phi : \gamma \mapsto g_\gamma$ is C^1 in the usual topology.*

Proof. Recalling that $g_\gamma = \psi_\gamma^{-1} \circ f_\gamma \circ \psi_\gamma$ we must show that $\frac{\partial}{\partial \gamma} g_\gamma$ is well-defined and continuous in γ.

We know that ψ_γ is holomorphic in z in some neighbourhood of 0 because it preserves the standard structure there.

ψ_γ is normalised to fix the points 0, 1 and ∞. We introduce the Banach space $B_{R,p}$ of such functions, defined on the set $\{|z| \leq R\}$:

By [A-B] $(\gamma, z) \mapsto \psi_\gamma(z)$ vanishes at $(\gamma, 0)$ and is continuous and continuously differentiable in γ in the the Banach space $B_{R,p}$, with the norm

$$||w||_{B_{R,p}} = sup_{|z_1|,|z_2| \leq R} \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{1-2/p}} + \Big(\int\int_{|z| \leq R} |w_z|^p dx dy \Big)^{1/p} \qquad (\ddagger),$$

where w_z is the generalised derivative, $w_z = 1/2 (\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y})$.

We start by establishing continuity of the derivatives of $\psi_\gamma(z)$.

Lemma 2.7.4. *$(\gamma, z) \mapsto \frac{\partial}{\partial z} \psi_\gamma(z)$ is jointly continuous and $(\gamma, z) \mapsto \frac{\partial}{\partial \gamma} \psi_\gamma(z)$ is jointly continuous, and the pointwise γ-derivative of ψ_γ is the same as the derivative in $B_{R,p}$.*

Proof. We must show that continuity in the above sense implies continuity in the usual topology. Given $\delta > 0$ small, let $U = \{|z| \leq \delta\}$. Let $H_\gamma(z) = \frac{\partial}{\partial z} \psi_\gamma(z)$.

Then for z_1 such that $|z_1| < \delta/2$,

$$H_\gamma(z_1) = \frac{1}{2\pi i}\int_{|z|=\delta'} \frac{H_\gamma(z)}{z - z_1}dz$$

by the Cauchy integral formula, because H_γ is holomorphic in the annulus $\{\delta' \mid \delta \leq \delta' \leq \delta\}$.

$$= \frac{1}{2\pi i(\delta_1 - \delta_2)}\int_{\delta_1 \leq \delta' \leq \delta_2}\int_{|z|=\delta'} \frac{H_\gamma(z)}{z - z_1}dzd\delta'.$$

Changing to polar coordinates with the substitution $z = re^{i\theta}$, so that $dz = ire^{i\theta}d\theta$, we have

$$H_\gamma(z_1) = \frac{1}{2\pi i(\delta_1 - \delta_2)}\int_{\delta \leq r \leq \delta_2}\int_{0 \leq \theta \leq 2\pi} \frac{H_\gamma(re^{i\theta} + t_0)}{re^{i\theta} - (z_1)} ire^{i\theta}d\theta dr$$

$$= K\int\int_A \frac{H_\gamma(re^{i\theta} + z_0)e^{i\theta}}{re^{i\theta} - (z_1 - z_0)} rdrd\theta. \qquad (*)$$

$$\Rightarrow |H_{\gamma_2}(z_1) - H_{\gamma_1}(z_1)| = \left|K\int\int_A F(r,\theta)(H_{\gamma_2}(X) - H_{\gamma_1}(X))rdrd\theta\right|$$

where $F(r,\theta)$ is bounded above and below on A and $X = re^{i\theta}$. Thus

$$|H_{\gamma_2}(z_1) - H_{\gamma_1}(z_1)| = \left|\int\int_A F(r,\theta)\chi_A(H_{\gamma_2}(X) - H_{\gamma_1}(X)dxdy\right|.$$

By (‡) $||H_{\gamma_2} - H_{\gamma_1}||_p$ is bounded for some p. Now $||F\chi_A||_\infty$ is bounded on A, so $||F\chi_A||_q$ is also bounded on A where $1/p + 1/q = 1$.

By the Hölder inequality

$$|H_{\gamma_2}(z_1) - H_{\gamma_1}(z_1)| \leq ||H_{\gamma_2} - H_{\gamma_1}||_p ||F\chi_A||_q.$$

Thus H is continuous in γ in the usual topology. i.e., $\psi_\gamma(z)$ is differentiable in z with continuous γ-derivative for fixed z.

We claim that H is also continuous in t: but this follows from $(*)$, which gives us $|H_{\gamma_1}(z_2) - H_{\gamma_1}(z_1)|$

$$= K\left|\int\int_A H_{\gamma_1}(re^{i\theta} + z_0)e^{i\theta}\left(\frac{z_2 - z_1}{(re^{i\theta} - (z_2 - z_0))(re^{i\theta} - (z_1 - z_0))}\right)rdrd\theta\right|.$$

Continuity follows because the above denominator does not vanish and y the Hölder inequality. (Uniform in γ) Actually we show that we have joint-continuity in (γ, z). Consider

$$|H_{\gamma_2}(z_2) - H_{\gamma_1}(z_1)| = |H_{\gamma_2}(z_2) - H_{\gamma_1}(z_2) + H_{\gamma_1}(z_2) - H_{\gamma_1}(z_1)|$$

$$\leq |H_{\gamma_2}(z_2) - H_{\gamma_1}(z_2)| + |H_{\gamma_1}(z_2) - H_{\gamma_1}(z_1)|,$$

by the triangle inequality. Joint continuity therefore follows from z and γ equicontinuity.

Now, we consider the γ-derivative. We know that ψ_γ is differentiable with respect to γ in $B_{R,p}$. Therefore, the following real partial derivatives, given by L_γ and M_γ are continuous. i.e., there exist L and M, for h real, $L = L(\gamma) \in B_{R,p}$ satisfying

$$\frac{||\psi_{\gamma+h} - \psi_\gamma - L_\gamma h||}{|h|} \to 0 \text{ as } ||h|| \to 0$$

and

$$\frac{||\psi_{\gamma+ih}(z) - \psi_\gamma(z) - M_z h||}{h} \to 0 \text{ as } h \to 0.$$

(We want to show that $L_z(\gamma)$ is continuous in (z, γ).)

By (‡),

$$\frac{|\psi_{\gamma+h}(z) - \psi_\gamma(z) - hL_\gamma(z)|}{|h|} \to 0 \text{ as } |h| \to 0$$

locally uniformly in z and similarly for $M_\gamma(z)$.

So $\gamma \mapsto \psi_\gamma(z)$ has derivative $\big(L_\gamma(z), M_\gamma(z)\big)$, and L_γ and M_γ are γ-continuous in $B_{R,p}$.

Now, for $w \in B_{R,p}$,

$$\frac{|w(z_1)|}{|z_1|} \leq \frac{|w(z_1)|}{|z_1|^{1-2/p}} \leq ||w||,$$

where $||\cdot||$ is the norm in $B_{R,p}$ (see above).

So L_{z_1} is a pointwise derivative, so that $\dfrac{\partial}{\partial\gamma}\psi_\gamma(z)$ exists in the usual sense and is in the space $B_{R,p}$. It is therefore a Hölder function with exponent $1 - 2/p$

and hence continuous in z. Furthermore, it is locally equicontinuous in γ by the definition of the $B_{R,p}$ norm. Hence, we have joint continuity because $|L_{\gamma_2}(z_2) - L_{\gamma_1}(z_1)|$

$$\leq |L_{\gamma_2}(z_2) - L_{\gamma_2}(z_1)| + |L_{\gamma_2}(z_1) - L_{\gamma_1}(z_1)|$$
$$\leq k_1 ||L_{\gamma_1}|| \, |z_2 - z_1|^{1-2/p} + k_2 ||L_{\gamma_2} - L_{\gamma_1}||,$$

for constants k_1 and k_2. □

Since $\frac{\partial \psi}{\partial z}(\gamma, z) \neq 0$ near 0 (since ψ is holomorphic near 0), the Inverse Function Theorem implies the existence of a C^1 inverse function $\xi_\gamma : \psi_\gamma(z) \mapsto z$ on $\psi_\gamma(\{|\gamma| < \delta\})$ which is C^1 in z. Then ξ is also differentiable in z: let $\Psi(\gamma, z) = (\gamma, \psi(\gamma, z))$. Then Ψ is C^1 in (γ, z), with Jacobian

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial \psi}{\partial \gamma} & \frac{\partial \psi}{\partial z} \end{pmatrix}.$$

By the Inverse Function Theorem Ψ^{-1} exists and is also C^1 in (γ, z). But Ψ^{-1} must be of the form $\Psi^{-1}(\gamma, z) = (\gamma, \xi(\gamma, z))$. Hence $\frac{\partial \xi}{\partial z}$ is continuous in (γ, z).

Now, we can evaluate $\frac{\partial}{\partial \gamma} g_\gamma$ near 0. If this is C^1 in γ, then so is $\gamma \mapsto g_\gamma$.

$$\begin{aligned} \frac{\partial}{\partial \gamma}(\xi_\gamma \circ f_\gamma \circ \psi_\gamma) &= \frac{\partial \xi}{\partial z} \frac{\partial}{\partial \gamma}(f \circ \psi) + \frac{\partial \xi}{\partial \gamma} \\ &= \frac{\partial \xi}{\partial z}\left(\frac{\partial f}{\partial z}\frac{\partial \psi}{\partial \gamma} + \frac{\partial f}{\partial \gamma}\right) + \frac{\partial \xi}{\partial \gamma}, \end{aligned}$$

where all the above terms exist and are well-defined, and with each derivative evaluated at the appropriate point.

Thus, since $\frac{\partial f}{\partial \gamma}$ and $\frac{\partial f}{\partial z}$ are continuous in (γ, z), it suffices to show that $\frac{\partial \psi}{\partial z}$, $\frac{\partial \xi}{\partial z}$, $\frac{\partial \psi}{\partial \gamma}$ and $\frac{\partial \xi}{\partial \gamma}$ are continuous to prove the proposition. This we have done. □

Proof of case 2.

In this case some neighbourhood of (a_0, b_0) in R lies in a single hyperbolic component of RM_2 because f_{a_0,b_0} is critically finite.

In [M3] Milnor extends Douady's theorem ([D]) by showing that the hyperbolic components of polynomials of any degree d are canonically biholomorphic to a complex $(d-1)$ dimensional ball. In particular, for degree three polynomials (which have two finite critical points), he identifies hyperbolic components with sets $\{(\gamma,\tau) \,|\, |\gamma| < 1, |\tau| < 1\}$, where (γ,τ) identifies with the Blaschke product $h_\gamma \circ h_\tau$ for h_τ defined as for h_γ above. His results transfer over to our situation, since his arguments (for degree three polynomials) are based on considering finite orbits of the two critical points not at infinity — the critical point at infinity does not enter into the argument, because he considers hyperbolic components in the connectedness locus, the set of functions for which the orbits of the finite critical points are bounded.

Again, we wish to apply a similar argument, as in Case 1, requiring the family of quasiconformal homeomorphisms to be C^1 in the parameters γ and τ in some neighbourhood of f_{a_0,b_0}, (Since both its critical points are periodic, f_{a_0,b_0} is at the centre of a hyperbolic component.)

The construction of f_γ and f_τ proceeds analagously to Case1. In neighbourhoods of each of the immediate attractive basins of c_1 and c_2, there are holomorphic conjugacies to the maps h_γ and h_τ respectively. Then by [M3] these really do describe nearby maps in the hyperbolic component of f_{a_0,b_0}. The arguments used above in Case1. show that the altered functions g_γ and g_τ are C^1 in γ and τ respectively.

The intersection of the hyperbolic component with V then identifies with $\{0\} \times \{\tau \,|\, |\tau| < 1\}$, which is a smooth submanifold of $\mathbf{C}^2$. □

We have now established in principle exactly where the singularities of V_n are located, for any n. To prove the statement (1) in §1.7, therefore, we will analyse V_n close to the punctures of W_n, since this is where the singularities must be located. The next step we take is to investigate the nature of the branches of V_n near to the punctures of W_n, and hence establish results about the singularities.

CHAPTER THREE

ANALYSIS OF THE PUNCTURES

On a sheet near a puncture of W_n we have local coordinates a and b, where we have expressed a as a Puiseaux series in b, so that a is a (possibly multi-valued) function of b. However, the situation is simpler than this, as hinted at by the examples calculated in Chapter 1. Thus, in the course of this chapter and the next we shall prove the main theorem of this thesis, which is:

Theorem 3.0.1. *The following equivalent statements are true for any branch of a variety V_n in the vicinity of a puncture:*

(1) One of the Puiseaux series $\mathbf{P}(b)$ *or* $\mathbf{P}(a)$ *has terms only with integral powers.*

(2) a is a function of b or vice-versa.

(3) The knot associated to the branch is the un-knot.

Note that this theorem applies to punctures at which there is no singularity of V_n. Also note that though in our examples we have expansions $a = \mathbf{P}(b)$ we only prove the slightly weaker statement ((1) above).

The proof of this Theorem is dependent on viewing degree two rational maps as degree two branched coverings of the Riemann Sphere. So far we have been considering the maps of the set V_n as lying in an algebraic variety, which is the compactification of W_n by a set of degenerate degree one maps. As an alternative, we can think of the sets W_n as sets of degree two branched coverings, the limit points of which are also degree two branched coverings. It turns out that this interpretation of the limiting maps is what is necessary to understand the nature of V_n near to the punctures, and hence the local geometry.

§3.1 Outline of proof of Theorem 3.

At the puncture the Möbius map $f_{a_0,0}$ (which we assume to be of order k) has two fixed points, namely $1/2(1 \pm \surd(1+4a_0))$, which form a complex conjugate pair if $k > 2$. Their multipliers are given by $-4a_0/\left(1 \pm \surd(1+4a_0)\right)^2$, which are necessarily k-th roots of unity, meaning that the fixed points are parabolic. We choose one of these to be $\lambda_{a_0,0}$ and define $\lambda_{a,b}$ to be the multiplier of the fixed point of $f_{a,b}$ such that λ is a continuous function of a and b in some neighbourhood of $(a_0, 0)$. (Note that there are three fixed points for $f_{a,b}$, one of which tends to the same limit as both critical points as (a, b) tends to $(a_0, 0)$, and which is always repelling in a suitable small neighbourhood of $(a_0, 0)$.) In the case $k = 2$ we have to consider the limit map $z \mapsto 1/z$, for which the fixed points are ± 1, which have the same multiplier, -1. For definiteness we shall choose, in the case $k \geq 3$, λ such that the corresponding fixed point at $(a_0, 0)$ lies in the upper half plane of $\mathbf{C}$, with $\lambda^*(a_0, 0)$ being the multiplier of the complex-conjugate fixed point. In the case $k = 2$ we choose $\lambda(0,0) = \lambda^*(0,0) = -1$ in the (a', b') coordinates (see Chapter 1). It is important to note, as remarked earlier, that the modulus of the multiplier of the third fixed point is bounded away from and above 1 in a neighbourhood of the puncture, whereas λ and λ^* are roots of unity at the puncture. What we will do in the course of this and the next chapter is show that the quantity $\lambda_{a,b}$ is a smooth function of the sheet parameter t (see §1.5), so that at the puncture at $f_{a_0,0}$, where $t = 0$, the derivative $\dfrac{d\lambda}{dt}(0) \neq 0$.

Theorem 3.1.1. *The multiplier λ is a function of the sheet-parameter t, such that $\dfrac{d\lambda}{dt}(0) \neq 0$.*

Proof. Proved later in this section. □

Then, given the result of 3.1.1, $\dfrac{d\lambda}{dt}$ can be written as

$$\frac{d\lambda}{dt} = \frac{\partial\lambda}{\partial a}\frac{da}{dt} + \frac{\partial\lambda}{\partial b}\frac{db}{dt}.$$

Since $\frac{d\lambda}{dt} \neq 0$, both $\frac{da}{dt}$ and $\frac{db}{dt}$ cannot simultaneously be zero. Thus we have shown that either a or b is a smooth function of t. Hence (see §1.5) we have proved Theorem 3. □

Proving that $\frac{d\lambda}{dt} \neq 0$ breaks down into two stages. The first is to associate a unique (up to equivalence) critically finite branched covering to a sheet of the variety in a neighbourhood of the puncture. This is done in the rest of this chapter. To do this we must make an analysis of the behaviour of the critical point c_2 of maps near to a puncture of W_n (see section 3.4). By 2.7.1 the set of singularities of V_n is contained in the set of punctures of W_n, so we will have shown that all the singularities are made of smooth components. It turns out that the maps associated to each branch are not rational maps : after all, at the singularity we have a degenerate map. The non-equivalent models reflect the different way in which maps on different branches tend to the limit Möbius map.

Proof of theorem 3.1.1.

We start with the following observation:

Lemma 3.1.2. *λ is a holomorphic function of a and b, and hence of t, in a $\mathbf{C}^2$ neighbourhood of a puncture.*

Proof. Consider the map $F_{a,b} = f_{a,b}(z) - z$, where we have made a change of coordinates so that 0 is the fixed point in question of f_0. Then $F_{0,0}(0) = 0$ and $F'_{0,0}(0) \neq 0$, because we have already seen that it is a root of unity. By the Implicit Function Theorem, there exists a holomorphic function $z(a,b)$ in a neighbourhood of $(0,0)$, with $F_{a,b}(z(a,b)) = 0$ and $z(0,0) = 0$. Thus $f_{a,b}(z(a,b)) = z$. Noticing that $z(a,b)$ is the fixed point of $f_{a,b}$, since z is holomorphic, so is its derivative λ.
□

Since c_1 is periodic, by 2.1.2 it follows that at most one of the fixed points is attractive, and so λ and λ^* cannot simultaneously be less than 1 in modulus. Let us consider the sheet parameter t as it takes on values in a small circle C about 0. For some values of $t \in C$ $|\lambda(t)| < 1$, and for others $|\lambda^*(t)| < 1$.

Let $\lambda(0) = \zeta$, where $\zeta^k = 1$. Then we have the following:

Lemma 3.1.3. *$\frac{d\lambda}{dt}(0) \neq 0$ if and only if, for a particular value of r which is sufficiently near 1, there is exactly one value t near 0 with $\lambda(t) = r\zeta$.*

Proof. Let m be the least positive integer with $\frac{d^m}{dt^m}\lambda(0) \neq 0$. Then writing $a_l = \frac{1}{l!}\frac{d^l}{dt^l}\lambda(0)$, we have

$$\lambda(t) - \lambda(0) = \sum_{l \geq m} a_l t^l$$

$$= t^m \left(\sum_{l \geq m} a_l t^{l-m} \right) = t^m \left(\phi(t)\right)^m,$$

for a holomorphic function ϕ with $\phi(0) \neq 0$. So $\psi(t) = t\phi(t)$ is a local bijection with $\psi(0) = 0$. Also $\lambda(t) = r\zeta$ if and only if $\psi(t)$ is an m-th root of $1 - r\zeta$. So there are precisely m solutions of the equation $\lambda(t) = r\zeta$ near $t = 0$, for r near 1. $\square$

In order to prove 3.1.1., therefore, we must satisfy the latter condition in the statement of 3.1.3. Now the existence of such a line $\lambda(t) = r\zeta$ for $0 \leq r < 1$ is equivalent to the existence of a hyperbolic component of a polynomial, of which this line is a subset. Then, if there is exactly one hyperbolic component corresponding to $|\lambda < 1|$ with the puncture on its boundary, the above condition is satisfied. At the same time Lemma 3.1.3 applies equally to the function λ^*, again for which one hyperbolic component of a polynomial will satisfy the condition. By the definition of λ and λ^* these components are mutually exclusive. So we have shown that Theorem 3.1.1 is equivalent to the following statement:

Proposition 3.1.4. *In any sufficiently small neighbourhood of a Möbius map of order k in V_n, a sheet of the variety has non-vacuous intersection with*

(a): Two hyperbolic components of polynomials if $k > 2$.

(b): One hyperbolic component of a polynomial if $k = 2$.

For $k \geq 3$ we have two polynomial hyperbolic components incident at the puncture (if $\frac{d\lambda}{dt}(0) \neq 0$), one of which corresponds to values of $|\lambda(t)| < 1$, and the other to $|\lambda^*(t)| < 1$. Thus λ and λ^* are smooth functions of t and we have the following picture for a branch at the puncture (drawn in the t-plane):

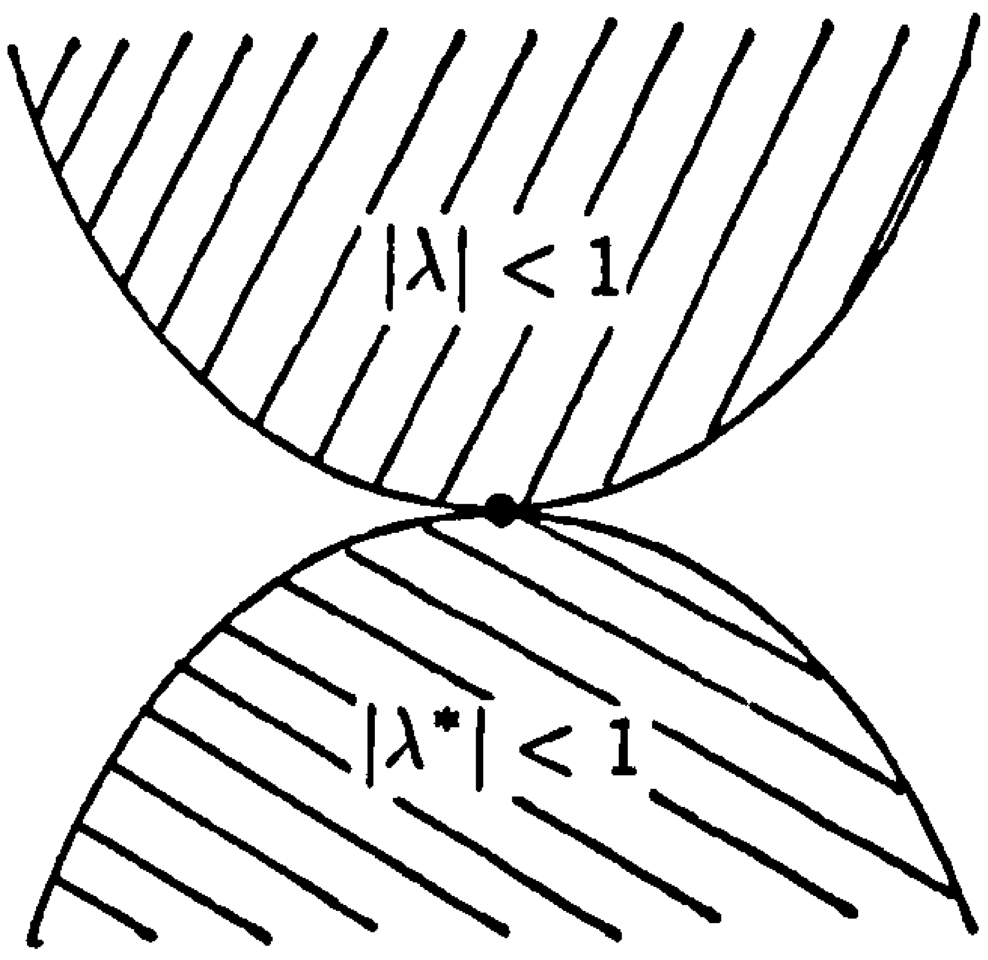


Figure 3.1a

For $k = 2$ we also have two polynomial hyperbolic components incident, when we are considering (a', b') coordinates. However, consider a hyperbolic component in the (a, b) coordinates near the limit where $k = 2$: In transforming the coordinates to (a', b') we have changed the parameter a by $a' = 1/(\sqrt{a})$. This has the effect of producing two copies of this component, since the operation $\sqrt{}$ is two-valued. Thus, in the (a, b) coordinates two hyperbolic components correspond to just one hyperbolic component (for $|\lambda| < 1$) in the (a', b') coordinates.

To prove 3.1.4, we associate a canonical critically finite degree two branched covering to each sheet in the neighbourhood of the puncture. Note that we do not finish the proof of 3.1.4 (and thus 3.1.1 and Theorem 3.0.1) until the end of Chapter 4 Section 3.

§3.2 Construction of the branched covering.

We construct a certain branched covering for any map in the vicinity of a puncture. The idea is that this map represents the limiting behaviour of functions $f_{a,b}$ on a particular sheet, when they are thought of as branched coverings. As rational maps, we have seen that the limiting map (in the family $\{f_{a,b}\}$) is some Möbius map, with only one critical point (the two critical points have come together in the limit), and that this is the same for any sheet through this puncture. In contrast the limiting branched covering has two critical points, and this extra flexibility will enable us to distinguish between different sheets: each sheet will be characterised by a canonical branched covering, reflecting the different ways (to be explained later) that the orbit of $c_1(f_{a,b})$ converges in the limit.

We construct the branched covering for maps in the hyperbolic components of polynomials first of all, showing that within a component near the puncture, that these are equivalent (in the sense described in §2.3). We then extend this to maps all the way round the puncture on a given sheet, again showing equivalence. i.e., we associate a unique (up to equivalence) branched covering to to all maps on the sheet near a puncture, and this branched covering is not equivalent to a rational map. So we will construct an *invariant* of the branch, which will distinguish between different branches.

We now start to set up an explicit definition of the branched covering, in which the following definition is central: Let $\alpha : [0,1] \to \hat{\mathbf{C}}$ be a simple path. Then define (see [R2]) a homeomorphism σ_α to be the identity outside a small neighbourhood of the image of $\alpha[0,1]$, and such that it maps $\alpha(0)$ to $\alpha(1)$.

Let f be in the set V_n, near to a puncture where the Möbius map is of order k. Then let $\alpha(f_{a,b})$ be a path such that $\alpha(0) = c_2$ and $\alpha(1) = f^k_{a,b}(c_2)$. Thus $\sigma_\alpha^{-1} \circ f^k_{a,b}(c_2) = c_2$. The exact way that α is to be defined will be made explicit during the prcof of Theorem 3.1.1, but we will choose α so that the forward orbit of c_2 under $f_{a,b}$ avoids the forward orbit of c_1, so that $(\sigma_\alpha^{-1} \circ f_{a,b})^k$ is a map of $\hat{\mathbf{C}}$ with one period n critical point and one fixed critical point. We illustrate, with the understanding that α is not yet defined precisely:

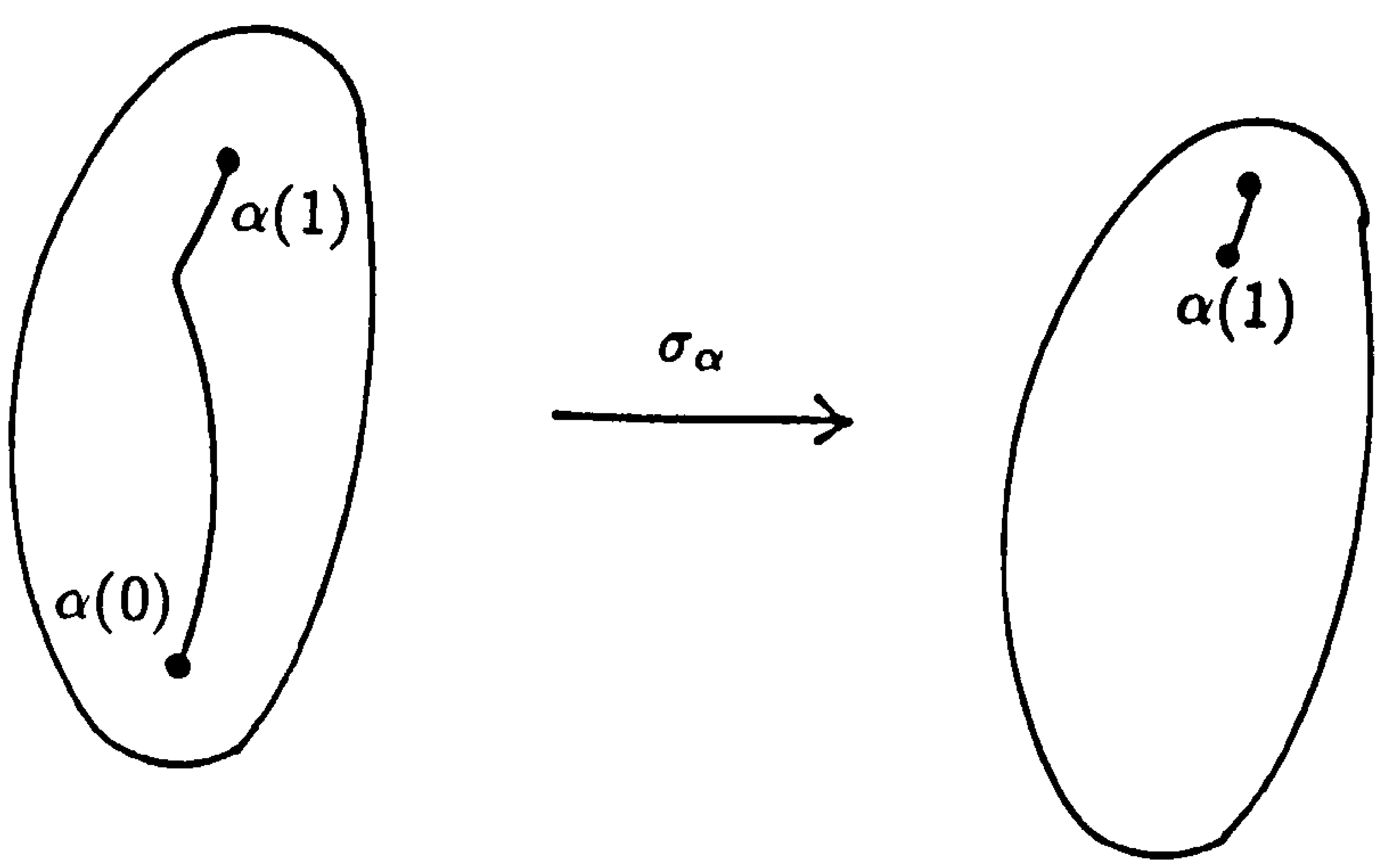


Figure 3.2a

Note that as $f_{a,b} \to f_{a_0,0}$, the path α shrinks to a point and thus $\sigma_\alpha \to Id$.

The following theorem, proved in the rest of this chapter, is the first important stage toward proving Theorem 3.

Theorem 3.2.1. *Let $(a_0, 0)$ be a puncture of W_n, with corresponding Möbius map of order k. Then for functions $f_{a,b}$ on one sheet of the variety V_n in a neighbourhood U of $(a_0, 0)$, we can find paths $\alpha = \alpha(a,b)$ such that the maps given by $\sigma_\alpha^{-1} \circ f_{a,b}$ are critically finite branched coverings of $\hat{\mathbf{C}}$. Moreover, all these maps are equivalent as branched coverings.*

We will construct the path α indirectly, starting with the following:

Proposition 3.2.2. *There is an integer $r_0 > 0$, such that for any map $f_{a,b} \in U$, the straight line arc joining $f_{a,b}^{kr_0}(c_2)$ and $f_{a,b}^{k(r_0+1)}(c_2)$, denoted $\tau_{r_0}(a,b)$, is such that $\tau_{r_0}(a,b) \cap Y = \emptyset$, where $Y = \{f_{a,b}^j(c_1) \,|\, j \geq 0\} \cup \{f_{a,b}^j(c_2) \,|\, 0 \leq j < k(r_0+1), j \neq kr_0\}$.*

Remark: It then follows that $\tau_{r_0}(a,b)$ is a continuous function of a and b.

Proof. It is necessary to show the set $X(f)$ (see §2.3 for definition) is bounded away from τ_{r_0}.

So we need to make an analysis of the possible behaviour of c_2 under iteration by $f_{a,b}$, especially in relation to $\mathcal{O}^+(c_1)$. In order to do this, it will be necessary to change the coordinate system for the parameters to a more convenient form.

§3.3 A change of coordinates.

First of all, let us change coordinates so that 0 and ∞ are fixed points, and so that the multiplier at 0 is λ. So we have a map of the form

$$g:\ z \mapsto \frac{\lambda z - \mu z^2}{1 + \nu z}.$$

Solving for critical points gives us that $\lambda - 2\mu z - \mu\nu z^2 = 0$, so imposing the condition that 1 is critical means $\lambda = -2\mu - \mu\nu$. Furthermore, we note that the map becomes degree one when $\nu = -1$, because $g(z)$ becomes

$$\frac{\mu((2+\mu)z - z^2)}{1+\nu z} = \frac{\mu(z-z^2)}{1-z} = \mu z,$$

so we also make the substitution $\nu = -1 + \rho$, so that for small ρ we are near a puncture. Indeed, for the rest of this chapter we will be implicitly assuming that ρ is sufficiently small, and so $g = g(\rho)$ is near to a puncture in some family, V_j. We now have our map in the form

$$g_{\mu,\rho} : \; z \mapsto \frac{\mu((1+\rho)z - z^2)}{1+(\rho-1)z}.$$

Note that all the punctures are on the unit circle $\{|\mu| = 1, \rho = 0\}$, since the map $z \mapsto \mu z$ must be of finite order and this is true precisely when $\mu^k = 1$.

We have critical points $c_1 = 1$ and $c_2 = \dfrac{1+\rho}{1-\rho}$.

Solving $g(z) = z$ we find that there is a fixed point z_0 at

$$\frac{\mu+\mu\rho+1}{\mu-\rho+1} = \frac{1+\dfrac{\mu\rho}{1+\mu}}{1-\dfrac{\rho}{1+\mu}} = (1+\frac{\mu\rho}{1+\mu}) + \sum_{j=1}^{\infty}(1(-\frac{\rho}{1+\mu})^j)$$

$$= 1+\rho+\sum_{j=1}^{\infty}\frac{\rho^{j+1}}{(1+\mu)^j} = 1+\rho+O(\rho^2).$$

Working out the derivative here,

$$\frac{dg_{\mu,\rho}}{dz} = \mu\frac{(\rho^2+\rho^3/(1+\mu)+O(\rho^4))}{(\mu\rho^2/(1+\mu)+\rho^3/(1+\mu)^2+O(\rho^4))}$$

$$= \frac{\mu\rho^2+O(\rho^3)}{\mu^2\rho^4/(1+\mu)^2+O(\rho^5)} \approx \frac{(1+\mu)^2}{\mu}\rho^{-2}.$$

For ρ small, therefore, this fixed point is repelling. (However, in the limit, this fixed point disappears.)

From now on we will assume that the map $g_{\mu,\rho}$ is on a variety V_n, so that 1 is of period n, and near to a puncture where the Möbius map is of order k. Note that on a sheet near a puncture μ is a (possibly multi-valued) function of ρ, and that

μ is approximately a k-th root of unity, because we are close to a k order Möbius map on V_n. At the puncture $\rho = 0$ and $\mu = \mu_0$, with ${\mu_0}^k = 1$, and $z \mapsto \mu_0 z$ is the order k Möbius map.

Note that with the new parameters μ and ρ we do not have a natural representation of the sets V_n. This is because in the (μ, ρ) coordinates we have maps in V_n represented more than once. In particular the points of intersection with the line at infinity in the (a, b) coordinates are now represented by a single point. However, we shall see that the local geometry of branches near the punctures is not affected:

Let us consider $g_{\mu,\rho}$. Conjugating by $z \mapsto 1/z$, we get the map

$$\begin{aligned} z \mapsto & \frac{1+(\rho-1)/z}{(\mu/z)(1+\rho-1/z)} \\ &= \frac{z^2+(\rho-1)z}{\mu((1+\rho)z-1)} \\ &= \frac{(1/\mu)(1-\rho)z-z^2}{1+(-\rho-1)z}. \end{aligned}$$

Thus, the maps $g_{\mu,\rho}$ and $g_{1/\mu,-\rho}$ are conjugate.

Consequently, maps $g_{\mu,\rho}$ in a neighbourhood of a Möbius map $z \mapsto \mu_0 z$, are equivalent to maps in a neighbourhood of $z \mapsto \overline{\mu}_0 z$. So there are, in essence, two copies of each finite puncture of V_n. When μ is approximately -1, we have equivalences between pairs of maps, both of which lie in a neighbourhood of $\mu = -1$.

We justify the validity of the coordinate change:

Proposition 3.3.1. *The branches of V_n (in the (a, b) coordinates) near a puncture are smooth if and only if the corresponding branches in the (μ, ρ) coordinates are smooth.*

Proof. It is sufficient to show that there is a local change of coordinates in the neighbourhood of a puncture which is a diffeomorphism of the appropriate neighbourhoods. Consider the case when $k \geq 3$. The change of coordinates takes a neighbourhood of some Möbius map $z \mapsto (z - a_0)/z$ to a neighbourhood of

$z \mapsto \mu_0 z$. Consider the points $c_1, f(c_1), f^2(c_1)$: for $f_{a,b}$ this is the set $\{0, \infty, 1\}$, and for $g_{\mu,\rho}$ we have $\{1, \mu, \mu^+\}$, where $\mu^+ = \dfrac{\mu^2(1+\rho-\mu)}{1+\rho\mu-\mu}$. There is a Möbius transformation ξ, for which $\xi(1) = 0, \xi(\mu) = \infty$ and $\xi(\mu^+) = 1$, and which therefore induces an equivalence between $f_{a,b}$ and $g_{\mu,\rho}$.

$$\xi(z) = \left(\frac{\mu^+ - \mu}{\mu^+ - 1}\right) \cdot \frac{(z-1)}{(z-\mu)} \quad \text{and} \quad \xi^{-1}(z) = \mu \frac{z + 1/\mu}{z - 1 + \left(\frac{\mu+1}{\mu^+}\right)}.$$

Now $f_{a,b} = \xi \circ g_{\mu,\rho} \circ \xi^{-1}$, a rational expression in terms of μ, ρ and z. So a and b are smooth functions of μ and ρ. So if μ is a function of ρ (or vice-versa), then a is a function of b (or vice-versa).

Conversely, consider the Möbius transformation χ, such that $\chi(F_1(a,b)) = 0$, $\chi(F_2(a,b)) = \infty$ and $\chi(0) = 1$, where F_1 and F_2 are the two fixed points of $f_{a,b}$ with multipliers λ and $\overline{\lambda}$. Then

$$g_{\mu,\rho} = \chi \circ f_{a,b} \circ \chi^{-1}.$$

Since $F_1(a,b)$ and $F_2(a,b)$ are smooth functions of the variables a and b, the above shows that μ and ρ are also.

The case $k = 2$: as in §1.2 we conjugate by the map $z \mapsto \sqrt{a}z$, and make a change of variables $a' = 1/(\sqrt{a})$, $b' = 1/b$, so that we are considering maps

$$f_{a',b'}(z) = \frac{a'z^2 + z + a'^3/b'}{z^2},$$

where the puncture is at $(a', b') = (0, 0)$. (The term a'^3/b' vanishes as $a \to 0$ by the comment in §1.2.) So for $n \geq 3$ the change of coordinates is from a neighbourhood of $z \mapsto 1/z$ to a neighbourhood of $z \mapsto -z$. Otherwise, this case is the same as for $k \geq 3$. □

Note: in the rest of this chapter we shall write $f = g_{\mu,\rho}$.

§3.4 Consequences for the orbits of the critical points.

In this section we show that, near a puncture, the forward orbit of $c_2(f)$ settles down to a well behaved pattern, eventually moving away from 1 in a definite direction under f^k.

Let us consider the forward orbit of $c_1 = 1$:

$$1 \mapsto \frac{\mu(1+\rho)-\mu}{1+\rho-1} = \mu.$$

Now, we can approximate f as follows:

$$f(z) = \frac{(1+\rho)\ \mu\ z(1-z(1+\rho)^{-1})}{1-z(1-\rho)}$$

$$= \frac{(1+\rho)\ \mu\ z\Big(1-z(1-\rho+\rho^2+O(\rho^3))\Big)}{1-z(1-\rho)}$$

$$= (1+\rho)\ \mu\ z\left(1-\frac{\rho^2 z+O(\rho^3)}{(1-z(1-\rho))}\right). \qquad (*)$$

Note that if $z = 1+\rho+o(\rho)$, then the denominator in the above expression is $\rho^2+o(\rho^2)$ and the equation $(*)$ is no longer a valid approximation. So near the fixed point z_0, we have not constrained the location of future iterates.

However, the equation $(*)$ is certainly valid well away from 1, so ignoring higher than first order terms of ρ, we can estimate the first k iterates of $c_1 = 1$:

$$1 \mapsto \mu \mapsto \mu^2(1+\rho) \mapsto \cdots \mapsto \mu^k(1+\rho)^{k-1}.$$

Indeed, away from 1, and thus away from a neighbourhood of both critical points $(*)$ shows us that the map f is approximate to the Möbius map $z \mapsto \mu_0 z$. (Note that for ρ small c_2 is near to 1 — in fact, $c_2 = 1+2\rho+O(\rho^2)$.)

So near a singularity $f^k(1)$ is approximately 1, except, of course, when $n = k$ and $f^k(1) = 1$ by definition. Let us assume for now that under f^k, $1 \mapsto 1+T$, where $T = T(\mu,\rho)$ (standing for translation) is some small quantity (equal to zero where n=k).

Example: In the case $n = 4$ and $k = 2$ we have two local solutions for μ in terms of ρ, one of which is given by $\mu = -1+1/4\rho+o(\rho)$. To get this solution we solve the equation $f^4(1) = 1$ in the same way as for a in terms of b. In this case $1 \mapsto -1+1/4\rho+o(\rho) \mapsto \big(-1+1/4\rho+o(\rho)\big)^2(1+\rho)+o(\rho) = 1+1/2\rho+o(\rho)$. So $T = 1/2\rho+O(\rho^2)$.

Lemma 3.4.1. $T = O(\rho)$. *Hence* $\mu = \mu_0 + O(\rho)$.

Proof.

Let $f^k(1) = 1 + a\rho^s + o(\rho^s)$, where $0 < s < 1$ and $a \neq 1$. This means that $\mu = \mu_0 + (a/k)\rho^s + o(\rho^s)$.

Then under f,

$$1 + a\rho^s \mapsto \frac{\mu((1+\rho)(1+a\rho^s) - (1+a\rho^s)^2)}{1 + (\rho-1)(1+a\rho^s)}$$

$$= \mu\frac{(-\rho + a\rho^s - a\rho^{s+1} + a^2\rho^{2s}}{a\rho^s(1 - \rho - \rho^{1-s}/a)}$$

$$= \mu(1 - \rho + a\rho^s - \frac{1}{a}\rho^{1-s})(1 + \rho + \frac{1}{a}\rho^{1-s} + \rho^2 + \frac{1}{a^2}\rho^{2-2s} + o(\rho^2))$$

$$= \mu\big(1 + a\rho^s + o(\rho^s)\big).$$

So, using the approximation $(*)$, valid because $s \neq 1$,

$$f^k\big(1 + a\rho^s + o(\rho^s)\big) = \mu^k\big(1 + a\rho^s + o(\rho^s)\big) = 1 + 2a\rho^s + o(\rho^s) = f^{2k}(1).$$

Similarly, $f^{kj}(1)$ is approximately $1 + ja\rho^s$, contradicting the fact that 1 is periodic. Therefore $a = 0$.

Then $1 + T = \mu^k(1+\rho)^{k-1}$, so that $\mu^k = 1 + O(\rho)$ and thus $\mu = \mu_0 + O(\rho)$. □

In the light of this, we rewrite $1 + T$ as $1 + A\rho + o(\rho)$, so that under f^k, $1 \mapsto 1 + A\rho + o(\rho)$, where A can be zero. Note that A is an invariant of the sheet in a neighbourhood of the puncture. Recalling that $c_2 = 1 + 2\rho + o(\rho)$ let us study the future iterates of a general point $1 + t\rho$, $t \in \mathbf{C}$.

$$1 + t\rho \mapsto \mu\frac{((1+\rho)(1+t\rho) - (1+t\rho)^2)}{1 + (\rho-1)(1+t\rho)}$$

$$= \mu\left(\frac{1 - t + t\rho - t^2\rho}{1 - t + t\rho}\right)$$

$$= \frac{\mu}{1-t}\left(\frac{-t + 1 + t\rho - t^2\rho}{1 + t/(1-t)\rho}\right)$$

$$= \frac{\mu}{1-t}(-t + 1 + t\rho - t^2\rho)(1 - \frac{t}{1-t}\rho + \left(\frac{t}{1-t}\right)^2\rho^2 + O(\rho^3))$$

$$= \mu\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right). \tag{$\dagger$}$$

Lemma 3.4.2. *The k-th iterate of a point $1 + t\rho$, where $t \neq 1$, is given by the expression*

$$f^k(1 + t\rho) = 1 + A\rho + \frac{t^2}{t-1}\rho + o(\rho). \tag{$**$}$$

Proof. Using the approximation ($\dagger$) we have that

$$1 + t\rho \mapsto \mu\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right)$$

$$\mapsto (1+\rho)\mu^2\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right)\left(1 - \frac{\rho^2 z + O(\rho^3)}{1 - z(1-\rho)}\right) \quad \text{by } (*)$$

$$= (1+\rho)\mu^2\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right).$$

Repeated application of $(*)$ yields the fact that the k-th iterate of $1 + t\rho$ is given by

$$(1+\rho)^{k-1}\mu^k\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right).$$

But since $(1+\rho)^{k-1}\mu^k = 1 + A\rho + o(\rho)$, the above is equal to

$$(1 + A\rho + o(\rho))\left(1 + \frac{t^2}{t-1}\rho + O(\rho^2)\right) = 1 + A\rho + \frac{t^2}{t-1}\rho + o(\rho).$$

□

Note that the above equation does not hold at $t = 1$, near the repelling fixed point. If a point in $\mathcal{O}^+(1)$ lands near here (which is certainly possible) then our previous approximations do not hold either.

Returning to our example, we can compute the third and fourth iterates of 1 using Lemma 3.4.2. In this case $A = 1/2$. Under f^2, $1 + 1/2\rho + o(\rho) \mapsto 1 + 1/2\rho - 1/2\rho + o(\rho) = 1 + o(\rho)$, in agreement with the fact that $f^4(1) = 1$.

Note the multiplier of f^k at 0 is $\mu^k(1+\rho)^k$, but $T = A\rho + o(\rho) = \mu^k(1 + \rho)^{k-1} - 1$, so the multiplier is $T(1+\rho) = (A\rho + 1)(1+\rho) = 1 + (1+A)\rho + o(\rho)$.

We use this last result to examine the behaviour under iteration of the second critical point, $c_2 \approx 1+2\rho$. If we write a point near 1 in the form $1+t\rho+o(\rho)$, then its k-th iterate is given by the last proposition in terms of a number $1+s\rho+o(\rho)$, for some $s \in \mathbf{C}$. On some neighbourhood of 1, f^k is a return map, and we can simplify the analysis by making the affine change of coordinates $w(z) = \dfrac{z-1}{\rho}$. In particular $w(1) = 0$ and $w(c_2) = 2 + o(\rho)$. So we can consider the return map in a neighbourhood of 1 as being approximated by the map $R: \ w \mapsto A + \dfrac{w^2}{w-1}$, for which 0 and 2 are the critical points, and 0 is of period n. More precisely, $f^k(z) = w^{-1} \circ R \circ w(z) + o(\rho)$ on a bounded set of z-values in a neighbourhood of $z = 1$.

For the map R there is a fixed point at infinity, which we can examine directly by conjugating by $w \mapsto 1/w$. We have

$$R(1/w) = A + \frac{1/w^2}{1/w - 1} = \frac{Aw - Aw^2 + 1}{w - w^2}.$$

$$\left(R(w)\right)^{-1} = \frac{w - w^2}{Aw - Aw^2 + 1}$$

$$= (w - w^2)(1 - Aw + Aw^2 + O(w^3)) = w - (1+A)w^2 + O(w^3).$$

The derivative at zero is 1, so infinity is parabolic under R, and attracts a critical point (and possibly both—when $A = -1$). Furthermore we know by 2.1.3 that the attractive direction at infinity is the argument of $(1+A)\rho$. Since 0 is periodic its orbit is bounded away from infinity, so 2 must be attracted out along this direction. We check that this direction is well-defined:

Lemma 3.4.3. $A \neq -1$.

Proof. Let $A = -1$. Then consider the iteration of 0 under R. $0 \mapsto -1 \mapsto -3/2$. It is clear that for real $w < -1$, $-1 + w^2/(w-1) < -1$, because $w^2/(w-1) < 0$. Thus the forward orbit of 0 is contained in the negative real axis, contradicting the periodicity of 0. □

Corollary 3.4.4. *The multiplier of the fixed point of f^k at 0, λ is such that $\lambda = 1 + c_0\rho + o(\rho)$, where $c_0 \neq 0$.*

Proof. Recalling that the multiplier of f^k at 0 is $\lambda = 1 + (1+A)\rho + o(\rho)$, we have by Lemma 3.4.3 that $A \neq -1$, so $(1+A) \neq 0$. □

By Lemma 3.4.3, there is always a well-defined argument along which points are attracted to infinity under R. In particular this is true for 2: there exists a number r_0 such that the set of iterates of 2, $\{R^j(2) \mid 1 \leq j \leq r_0\}$ is bounded away from the straight line joining $R^{r_0}(2)$ and $R^{r_0+1}(2)$.

Since $w^{-1}(z)$ is an affine transformation, it maps the straight line between $R^{r_0}(2)$ and $R^{r_0+1}(2)$, say τ'_{r_0}, to the straight line τ_{r_0}. Since the first $(r_0 - 1)$ iterates of 2 under R are bounded away from τ'_{r_0}, conjugating by w we see that $(r_0 - 1)$ iterates of c_2 under f^k are bounded away from $X(f)$: Since r_0 is finite we can choose ρ sufficiently small so that the error term (of order ρ) is as small as we please. The number r_0 clearly depends only on the function R, which in turn depends solely on A, an invariant of the sheet. So r_0 is just dependent on the sheet and applies to an open neighbourhood (determined by our choice of ρ), which we denote U, of functions on this sheet.

Since the points $f^{kr_0}(c_2)$ and $f^{k(r_0+1)}(c_2)$ are continuously dependent on ρ, so is $\tau_{r_0}(\mu, \rho)$ and hence $\tau_{r_0}(a, b)$. Thus we have proved Proposition 3.2.2. □

We aim to use the branched covering defined by $\sigma_{\tau_{r_0}}^{-1} \circ f$, which by Proposition 3.2.2 has certain desirable properties, to construct the path α, and hence the branched covering $\sigma_\alpha^{-1} \circ f$, so that we can prove Theorem 3.2.1. For this to be true we need the following:

Proposition 3.4.5. *There is an inverse image of the path $\tau_r(\mu, \rho)$ under r iterates of f^k, which has endpoints c_2 and $f^k(c_2)$, and is isotopic relative to Y' to a simple path from c_2 to $f^k(c_2)$, where $Y' = \{f^j(c_1) \mid j \geq 0\} \cup \{f^j(c_2) \mid 0 \leq j \leq k\}$.*

We approach this problem by first considering the case when f is in the hyperbolic component of a polynomial. The above proposition is proved at the end of §3.5.

§3.5 Construction of branched covering in a hyperbolic component

So let us consider $f = g_{\mu,\rho}$, which has an attractive fixed point at zero (so that f is in the hyperbolic component of a polynomial). For convenience let $z_0 = f^{kr_0}(c_2)$. Then $\sigma_\tau^{-1} \circ f^k$ fixes z_0. We know that under f, τ is attracted in toward 0, but not how. However, it turns out that we can control its behaviour quite well.

By §2.1 there is a holomorphic bijection ψ such that ψ conjugates f in a neighbourhood the fixed point to the map $z \mapsto \lambda z$ in a disc neighbourhood of 0, $\{z \mid |z| \leq 1\}$, such that $\psi(c_2) = 1$. We can assume that λ is real and positive , so that $\mathcal{O}^+(\psi(c_2))$ is attracted along constant argument rays toward the origin. (the rays have argument $|\lambda|e^{2\pi i/k}$.) Let $\psi(z_0) = \zeta_0$ and let π be the path of constant argument from ζ_0 to $\lambda^k\zeta_0$. Thus the whole path π is attracted toward the origin. Some of the inverse images of π under the map $z \mapsto \lambda z$ lie on a straight line from 0 to 1. Let π_0 be the straight-line path from 1 to λ^k, so that $\lambda^{kr_0}(\pi_0) = \pi$.

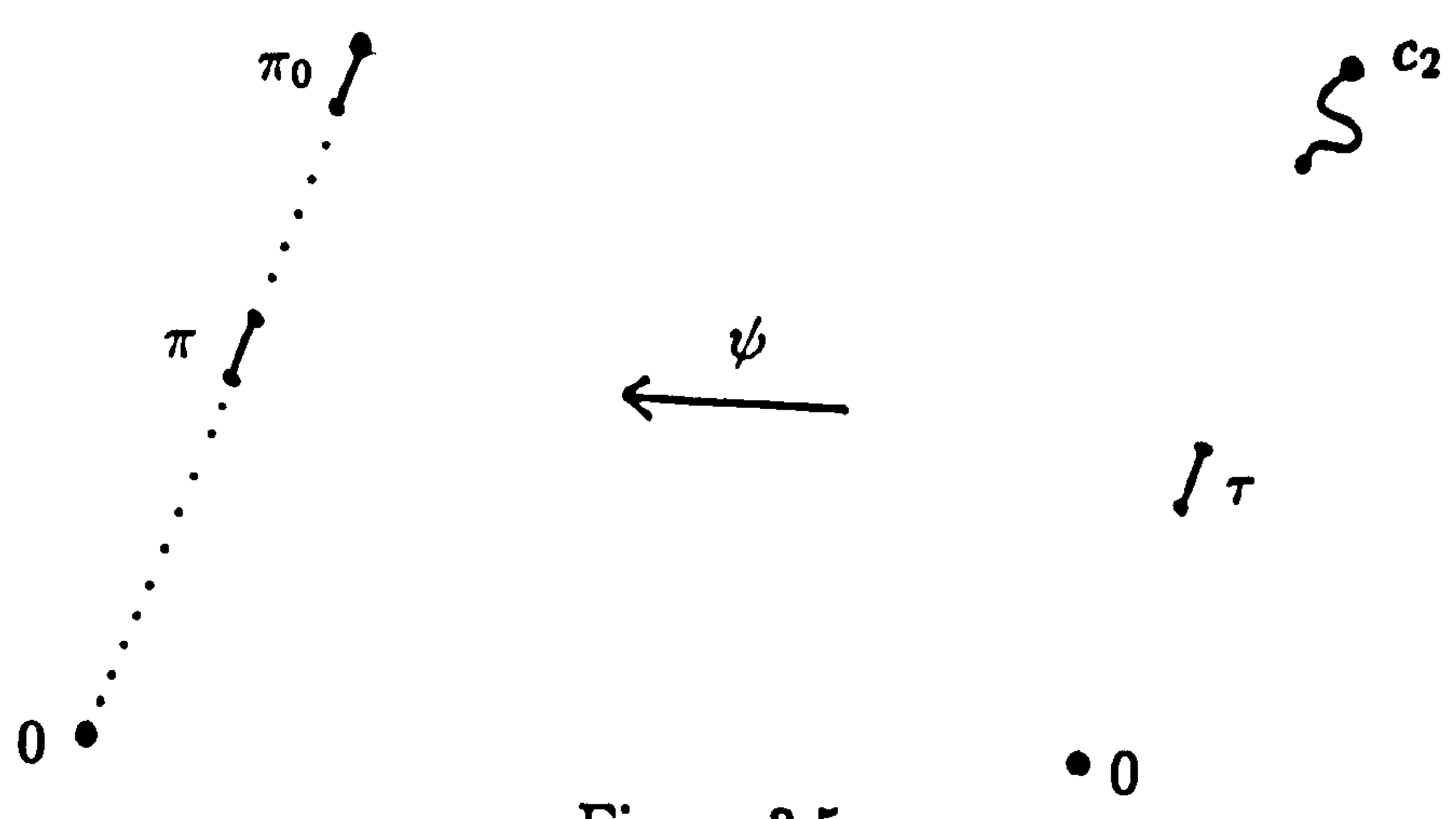


Figure 3.5a

The path $\psi^{-1}(\pi_0)$ joins c_2 to $f^k(c_2)$. So the path $f^{kr_0}(\psi^{-1}(\pi_0))$ has the same endpoints as τ, but are they isotopic? (‡)

Lemma 3.5.1. *Let N be such that $|f^j(c_2)| \geq 1/2$ for $j+k \leq N$. Let τ_j denote the straight line path joining $f^j(c_2)$ to $f^{j+k}(c_2)$ for $j \geq kr_0$. Then, for $kr_0 \leq j' \leq j$, a component of $f^{j'-j}(\tau_j)$ has endpoints at $f^{j'}(c_2)$ and $f^{j'+k}(c_2)$, and is isotopic to $\tau_{j'}$ relative to $Y_{j'} = \{f^i(c_1) \mid i \geq 0\} \cup \{f^l(c_2) \mid l \leq j' + k\}$. In particular, a component of $f^{kr_0-j}(\tau_j)$ is isotopic to τ relative to Y_{kr_0}.*

Proof. If $z = f^j(c_2)$, for $kr_0 \le j \le N$, then $1 - |z| \gg \rho$ (true for $j = kr_0$ by definition, and then for $kr_0 \le j \le N$ by induction on j). Hence by $(*)$

$$|f(z)| = |\lambda z|(1 - o(1)\rho + O(\rho^2)) \le |z|(1 - a_1\rho),$$

for some constant a_1. Hence $|f^{j+k}(c_2)| \le |f^j(c_2)|(1 - a_1\rho)$ for all $j \ge 0$. So a component of $f^{-1}(\tau_j)$ is approximate to the straight line path τ_{j-1} and has the same endpoints for all $kr_0 \le j \le N$. The result follows by induction. □

Next we show that some path τ_{rk} is isotopic to $\psi^{-1}(\lambda^{k(r-r_0)}\pi)$ for some number r. To do this we need to do the following:

We show that the uniformising map, ψ, for the map f in the hyperbolic component of a polynomial is close to the identity.

We write

$$f(z) = \lambda z(1 + g(z)),$$

where $\lambda = \mu(1+\rho)$ and $g(z) = -\rho^2 z/(1 - z(1-\rho)) + O(\rho^3)$, so $f(0) = g(0) = 0$ and $f'(0) = \lambda$.

By §2.1.1 there is a uniformising map, ψ, such that $f \circ \psi = \psi \circ (\lambda z)$. Furthermore by [M1],

$$\psi(z) = \lim_{n\to\infty} \psi_n(z) = \lim_{n\to\infty} \frac{f^n(z)}{\lambda^n}.$$

We use this expression in the following:

Proposition 3.5.2 *There is a constant $\beta > 0$, such that given $\epsilon > 0$ there exists δ such that on $|z| < 1-\beta$, and if $|\rho| \le \delta$, then $|\psi(z) - z| < \epsilon$ and $|\psi'(z) - 1| < \epsilon$.*

Proof.

Firstly, let us introduce the following:

Lemma 3.5.3. *Given $\beta'' > 0$, there exists a constant $M > 0$ and $\delta > 0$, such that if $|\rho| < \delta$ and $|z| < 1 - \beta''$, then $|g(z)| \le M\rho^2|z|$, where g is as described above, and $|f(z)| \le |\lambda||z|(1 + M\rho^2)$.*

Proof. We have
$$g(z) = \frac{-\rho^2 z}{1 - z(1-\rho)} + O(\rho^3).$$
If $|\rho| < \delta$ and $|z| < 1 - \beta''$, then the above denominator is bounded below by $\beta''(1-\delta)$, and the result for $|g(z)|$ follows on taking $M = 1/\big(\beta''(1-\delta)\big)$. The result for $|f(z)|$ follows directly because $f(z) = \lambda z\big(1 + g(z)\big)$. □

Consider
$$\begin{aligned}\psi_{n+1}(z) - \psi_n(z) &= \frac{f\big(f^n(z)\big)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \\ &= \frac{\lambda f^n(z) + \lambda f^n(z) g\big(f^n(z)\big)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \\ &= \frac{f^n(z) g\big(f^n(z)\big)}{\lambda^n}.\end{aligned}$$
Fixing z and ρ, consider $\Delta_j(z) = |\psi_{j+1}(z) - \psi_j(z)|$.
$$= \left|\frac{f^j(z) g\big(f^j(z)\big)}{\lambda^j}\right|.$$

We must show that there exists a constant β' such that for $|z| < 1 - \beta'$ and $|\rho| < \delta$
$$\sum_{j=0}^{\infty} |\Delta_j(z)| < \epsilon.$$

Now for $|z| < 1 - \beta''$,
$$|g\big(f^j(z)\big)| \leq M\rho^2 |f^j(z)|$$
by lemma 3.5.3. So
$$\begin{aligned}|\Delta_j(z)| &\leq \frac{M\rho^2 |f^j(z)|^2}{|\lambda^j|} \\ &\leq M\frac{\rho^2 |\lambda|^{2j}}{|\lambda|^j} |z| (1 + M\rho^2)^{2j} \quad \text{by3.5.3.} \\ &\leq M|z|\rho^2 \big(|\lambda|(1 + M\rho^2)^2\big)^j.\end{aligned}$$

Recall that (by 3.4.2) there is a constant c_0, such that $\lambda = 1 + c_0\rho + o(\rho)$. So there exists a constant a_1 such that $|\lambda|(1 + M\rho^2)^2 \leq 1 - a_1|\rho|$.
So $|\Delta_j(z)| \leq M\rho^2 |z| \big(1 - a_1|\rho|\big)^j$.

Thus

$$\sum_{j=1}^{\infty} \Delta_j(z) \leq \sum_{j=1}^{\infty} M|z|\rho^2(1 - a_1|\rho|)^j$$
$$\leq M|z|\left|\frac{\rho^2}{a_1\rho}\right|$$
$$\leq K|\rho||z|,$$

where K is a constant independent of z or ρ.

Choosing $\delta' < 1/K$, we get that for $|\rho| < \delta'$, $|\psi(z) - z| < \epsilon$.

The derivative, ψ', is given by the formula

$$|\psi'(z) - 1| = \frac{1}{2\pi i} \int_{|w|=1-\beta/2} \frac{\psi(w) - w}{(w - z)^2} dw.$$

Now for $|z| < 1 - \beta$ and $|w| = 1 - \beta/2$ the quantity $|w - z| > \beta/2$, so that $1/|w - z|^2 < 4/\beta^2$. Thus for $|\rho| < \delta'$,

$$\left|\frac{\psi(w) - w}{(w - z)^2}\right| < \frac{4\epsilon}{\beta^2}.$$

By the Estimation Lemma

$$|\psi'(z) - 1| < 1 - \beta/2.$$

Thus

$$|\psi'(z) - 1| < \left|\frac{4}{\beta^2}\right| K|\rho||z|.$$

Choosing $\delta < \dfrac{\beta^2}{4K}$ and $\delta < \delta'$, we get that for $|\rho| < \delta$, $|\psi(z) - z| < \epsilon$ and $|\psi'(z) - 1| < \epsilon$. $\square$

Choose least r so that $|f^{rk}(c_2)| \leq |1 - \beta|$. Then, $\psi|_{\left(f^{(r-r_0)k}(\tau)\right)}$ is C^1 close to the identity. So τ_{rk} and $\psi^{-1}(\lambda^{(r-r_0)k}\pi)$ are isotopic, and by the fact that $z \mapsto \lambda z$ commutes with f, τ_{rk} and $f^{rk}(\psi^{-1}(\pi_0)) = \psi^{-1}(\lambda^{(r-r_0)k\pi})$ are isotopic in $\mathbf{C} \setminus \{f^j(c_2) \,|\, 1 \leq j \leq rk\}$.

Then τ and $f^{kr_0}(\psi^{-1}\pi_0)$ are isotopic, by lemma 3.5.1. This answers the question (‡) asked earlier.

Lemma 3.5.4. *There are pre-images of τ under $f^{(r_0-s)k}$ with the same endpoints, and isotopic to $f^{sk}(\psi^{-1}(\pi_0))$ for $s \le r_0$ in $\hat{\mathbb{C}} \setminus \{f^m(c_2) \,|\, m \le sk\} \cup \mathcal{O}^+(c_1)$.*

Proof. This follows straightforwardly because τ and $f^{r_0 k}(\psi^{-1}(\pi_0))$ are isotopic in $\hat{\mathbb{C}} \setminus \{f^m(c_2) \,|\, m \le (r_0+1)k\} \cup \mathcal{O}^+(c_1)$. □

This means in particular that a component of $f^{-r_0 k}(\tau)$ has endpoints c_2 and $f^k(c_2)$. This we define to be the path α for f in the hyperbolic component.

Because τ does not intersect $\{f^m(c_2) \,|\, m \le (r_0+1)k\} \cup \mathcal{O}^+(c_1)$ except at its endpoints, for any $f \in U$ there is a component of $f^{-r_0 k}(\tau)$ with endpoints at c_2 and $f^k(c_2)$ for all f in U. So we define α to be this component.

We are now in a position to prove Theorem 3.2.1.

Take two maps, f_0 and f_1 in U, with associated paths α_0 and α_1 respectively. It is now straightforward to see that the branched coverings $\sigma_{\alpha_0}^{-1} \circ f_0$ and $\sigma_{\alpha_1}^{-1} \circ f_1$ are equivalent: Take a path $\{f_t \,|\, t \in [0,1]\}$ in U between f_0 and f_1. Then the associated path α_t varies continuously between α_0 and α_1 and $\sigma_{\alpha_t} \circ f_t$ is the required isotopy between $\sigma_{\alpha_0}^{-1} \circ f_0$ and $\sigma_{\alpha_1}^{-1} \circ f_1$. Thus Theorem 3.2.1 is proved. □

§3.6 Nature of the branched coverings.

We now decide exactly what these (critically finite) branched coverings are.

Proposition 3.6.1. *The map $\sigma_\alpha^{-1} \circ f$ has a Levy cycle.*

Proof. By §3.4 all of the points in $\mathcal{O}^+(c_1)$ and $\{f^j(c_2) \,|\, 0 \le j \le (r_0+1)k$ are approximately located at the k points in $\{\mu_0^j \,|\, 1 \le j \le k\}$. Since $\sigma_\alpha^{-1} \circ f$ is critically finite there exist constants M_j such that each of these points is contained in one of the open sets, defined as a ball of radius $M_j\rho$ about the point μ_0^j. For ρ sufficiently small these balls are disjoint. Let C_j be the boundary of the round disc M_j about the point μ_0^j containing $\{f^{kl+j}(c_0) \,|\, l \le r_0+1\}$ for $1 \le j \le k$. Then $\{\bigcup C_j \,|\, 1 \le j \le k\}$ is a Levy cycle for $\sigma_\alpha^{-1} \circ f$: that is, C_{j-1} is isotopic in $\hat{\mathbb{C}} \setminus X(\sigma_\alpha^{-1} \circ f)$ to a component of $(\sigma_\alpha^{-1} \circ f)^{-1}(C_j)$ with k replacing $j-1$ if $j = 1$.

Note also that C_j contains the arc τ_{kl+j} for $l \leq r_0 + 1$. Then $\bigcup_{1 \leq j \leq k} C_j$ forms a Levy cycle. □

Proposition 3.6.2. *There is an invariant circle separating $\mathcal{O}^+(c_1)$ and $\mathcal{O}^+(c_2)$ under $\sigma_\alpha^{-1} \circ f$.*

Proof. We define a circle which divides $\hat{\mathbb{C}}$ into two regions, one containing $\mathcal{O}^+(c_1)$, the other $f^j(c_2)$ for $j < r_0 k$.

Define τ_j to be the pre-image of τ_{kr_0} under $f^{kr_0 - j}$ joining $f^j(c_2)$ and $f^{j+k}(c_2)$, so that τ_0 and α are isotopic in $\hat{\mathbb{C}} \setminus (\{f^j(c_1) \,|\, j \geq 0\} \cup \{f^j(c_2) \,|\, 0 \leq j \leq k\})$. Choose a constant k_0 so that $\mathcal{O}^+(c_1) \cap \{|z| \leq 1 - k_0|\rho|\} = \emptyset$, but also satisfying $\tau_j \subset \{|z| \leq 1 - k_0\rho\}$ for $j \geq k(r_0 - 1)$. Let Γ be the boundary of the union of $\{|z| \leq 1 - k_0\rho\}$ together with appropriate small disc neighbourhoods of the k components of $\bigcup_{0 \leq j \leq kr_0} \tau_j$. Then Γ is separates the forward orbit (under $\sigma_\alpha^{-1} \circ f$) of c_1 from that of c_2 and Γ and $(\sigma_\alpha^{-1} \circ f)^{-1}\Gamma$ are isotopic in $\hat{\mathbb{C}} \setminus X(\sigma_\alpha^{-1} \circ f)$. □

We illustrate a general example arising from 3.6.1 and 3.6.2:

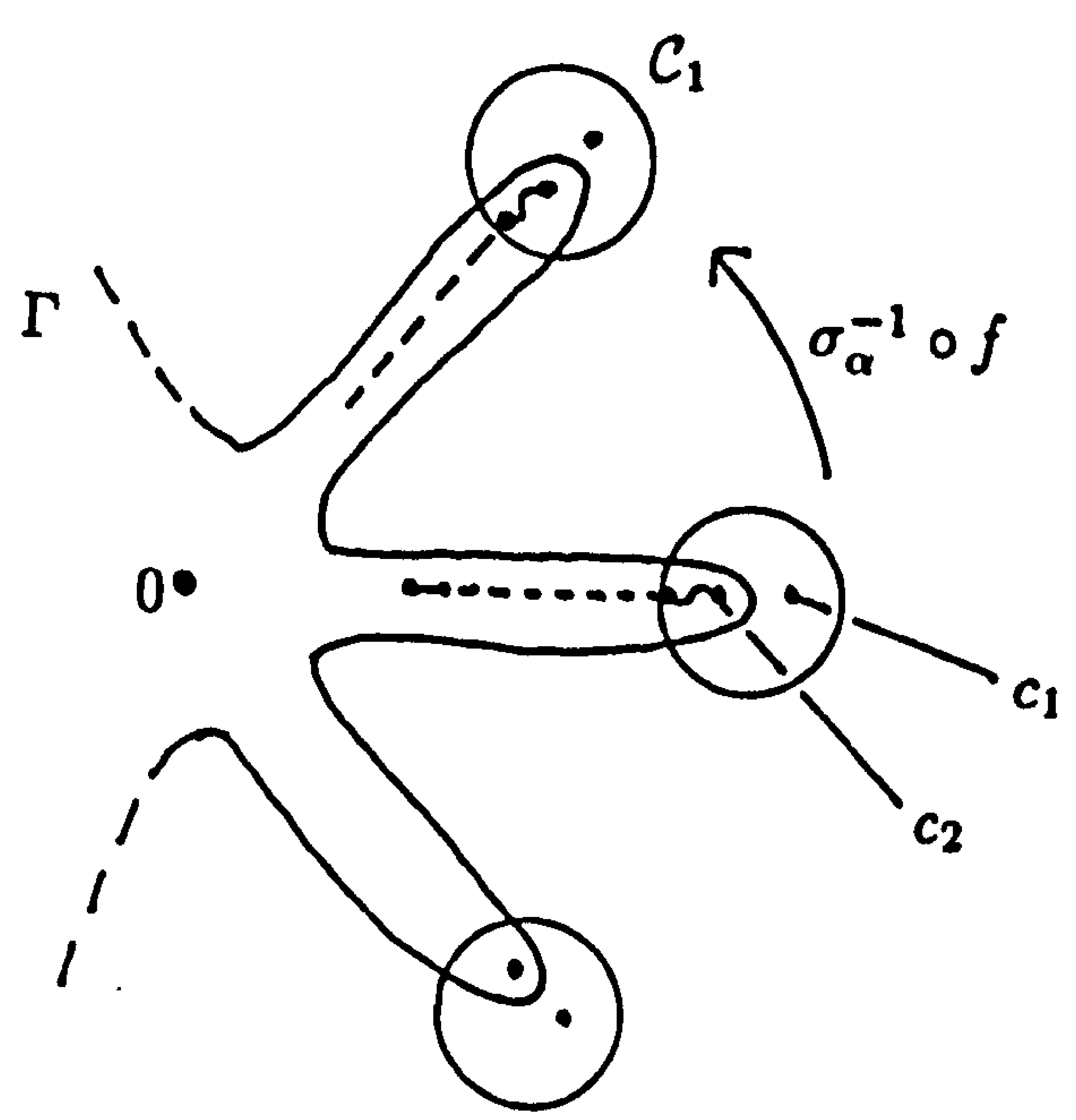


Figure 3.6a

We now have more information about the branched covering we have constructed. The extra information obtained by way of the uniformised coordinates, apart from showing that the branched covering is not equivalent to a rational map, also places the orbits of the critical points relative to the Levy cycle. Each loop of the Levy cycle contains one point in $\mathcal{O}^+(c_2)$ and at least one point in $\mathcal{O}^+(c_1)$.

Proposition 3.6.3. *The branched covering associated to a sheet in the neighbourhood of a k-fold Möbius map on the variety V_n is equivalent to a mating of quadratic laminations such that*

(a) On $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ $\mathcal{L}$ has minor leaf m of period n, with a minimal leaf ℓ such that $\ell < m$ and ℓ is of period k.

(b) On $\hat{\mathbb{C}} \setminus \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ $\mathcal{L}$ has minor leaf $(1/\ell)$.

Proof. We construct the equivalence directly, using the circle of 3.6.2. and the Levy loop of 3.6.1. Let $\phi(\Gamma) = S^1$ and let ϕ take the component of $\hat{\mathbb{C}} \setminus \Gamma$ containing 0 to $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$. Write $g = \phi \circ \sigma_\gamma^{-1} \circ f \circ \phi^{-1}$. Then the arcs of $\phi(\Gamma) \cap \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ are cyclically permuted (up to isotopy) by g. There is a unique minimal minor leaf ℓ (see §2.4) in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ such that $\overline{\ell}$ is cyclically permuted by $s_{\overline{\ell}}$ in the same order as $\phi(\Gamma) \cap \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ is permuted by g. We can assume that ϕ maps Γ close to the orbit of $\overline{\ell}$ under $s_{\overline{\ell}}$ and to the orbit of ℓ^{-1} under $i \circ s_\ell \circ i^{-1}$, where $i(z) = 1/z$. Then, since each component of $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\} \setminus \Gamma$ contains at most one point of the forward orbit of c_2 in $X(\sigma_\alpha^{-1} \circ f)$, it is clear that we can take $g = i \circ s_\ell \circ i^{-1}$ on $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$. So f is a mating of s_ℓ with some polynomial with critical point of period m. □

Knowing that we have a unique/canonical equivalence class of rational maps associated to a sheet, we need to establish what these equivalence classes are in order to know how many hyperbolic components of polynomials are incident (on a sheet) to the puncture. This we do in the following chapter.

CHAPTER FOUR

EQUIVALENCE OF BRANCHED COVERINGS

From Chapter 3 we have a unique equivalence class of branched coverings associated to each sheet of V_n in a neighbourhood of an order k Möbius map. In this chapter we organise the branched coverings that can arise this way into equivalence classes, so completing the proof of conjecture (1) stated in §1.7 and as Theorem 3. In order to prove this we develop the theory of equivalences between elements of $\mathcal{B}(n,k)$.

The starting point for what follows is Thurston's theory of laminations (see §2.4), together with the non-rationality theorem (2.5.1). Thurston's notion of equivalence of branched coverings is then applied to the matings of laminations, where the fact that the combinatorics are relatively easy to handle, makes this kind of equivalence possible to work with.

Let $\mathcal{B}(n,k)$ denote the set of matings of laminations such that: 0 is of period n, with minor leaf μ_q in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$, with a minimal leaf, $\mu_p < \mu_q$ of period k; ∞ is of period k, with minor leaf μ_p in $\hat{\mathbb{C}} \setminus \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$.

Theorem 4.0.1. *Let $f \in \mathcal{B}(n,k)$. Then $[f] = \{g \mid f \simeq g, g \in \mathcal{B}(n,k)\}$ contains one element when $k = 2$ and two elements when $k > 2$.*

§4.1 A non-trivial equivalence.

While investigating possible equivalences between branched coverings in the set $\mathcal{B}(n,k)$ we shall duplicate some results of Chapter 3 in the context of abstract branched coverings.

Let q be an odd-denominator rational, $q \in (0,1)$, such that it is of period n under the doubling map $z \mapsto 2z (\operatorname{mod} 1)$. Let μ_q be the unique minor leaf of QML such that $e^{2\pi i q}$ is an endpoint of μ_q. Let μ_p be the minimal leaf of QML such that $\mu_p \leq \mu_q$. Then μ_p is periodic of period k, where $k \leq n$. Thus the mating of lamination maps $s_q \sqcup s_{1-p} \in \mathcal{B}(n,k)$, and all maps in $\mathcal{B}(n,k)$ are matings of this type.

Theorem 4.1.1. *The branched covering $s_q \sqcup s_{1-p}$ is not equivalent to a rational map.*

Example: $s_{\frac{1}{5}} \sqcup s_{\frac{6}{7}}$ is not equivalent to a rational map. $\mu_q = [\![\frac{3}{15}, \frac{4}{15}]\!]$ and $\mu_p = [\![\frac{1}{7}, \frac{2}{7}]\!]$.

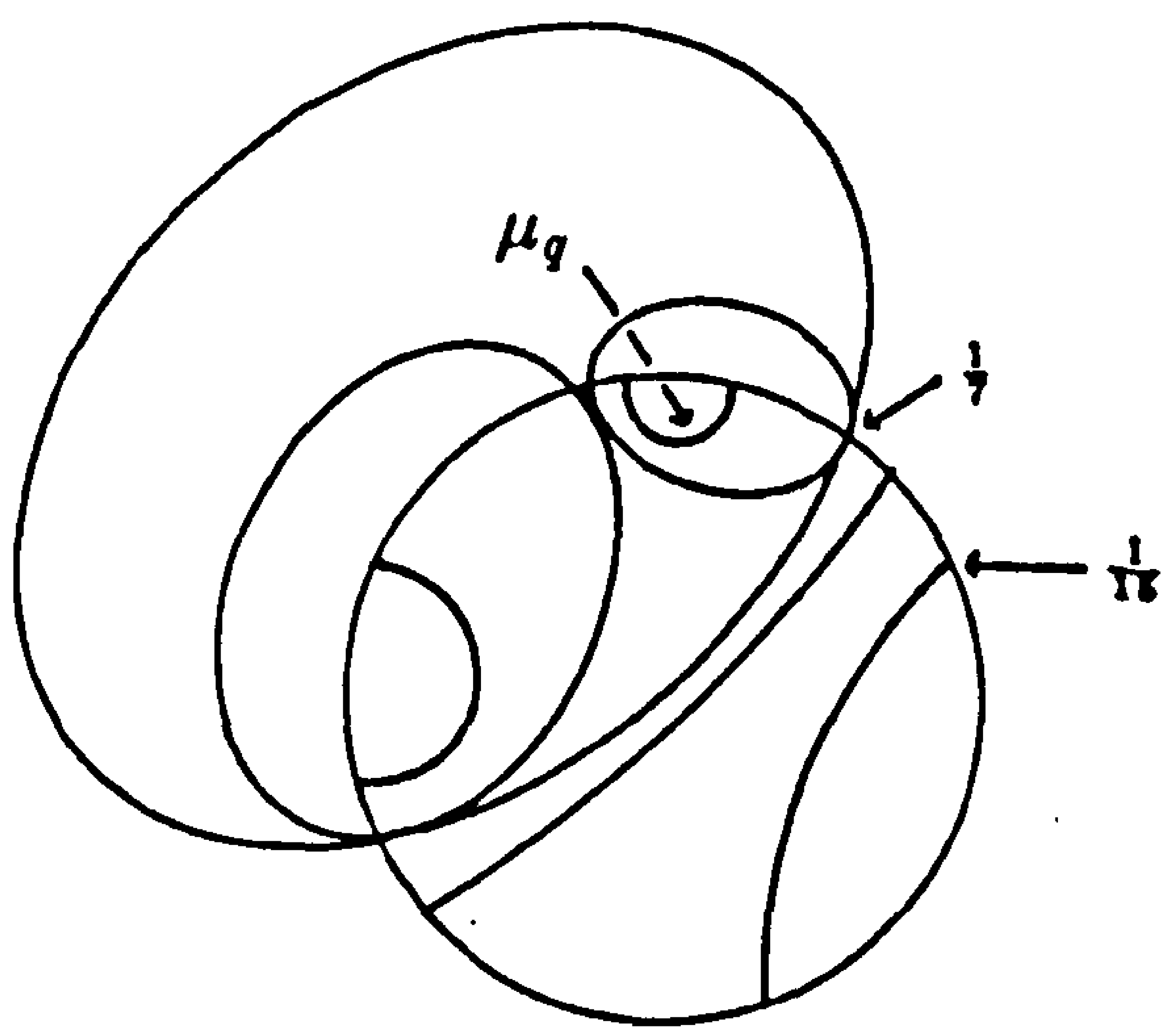


Figure 4.1a

Proof. It follows from [T] that an invariant lamination with minor leaf μ_q contains the leaf μ_p, which is of period k. So $s_q \sqcup s_{1-p}$ has a Levy cycle with a Levy loop approximately through μ_p: μ_p returns to itself after k iterations, where $k \leq n$, and by the definition of s_p there is the leaf $(\overline{\mu_p})^{-1}$ in $\{z \in \hat{\mathbb{C}} \,|\, |z| \geq 1\}$ with the same endpoints as μ_p, which, of course, also has period k ($(1-\ell)$ is the leaf with endpoints $\overline{z_1}$ and $\overline{z_2}$, where z_1 and z_2 are the endpoints of the leaf ℓ). These two leaves form a loop L_1 bounding both critical points, and its periodic backward images form k loops $L_1 \ldots L_k$, where each loop has a two points in common with other loops — these points being in the set $\{e^{2\pi i j p} \,|\, 1 \leq j \leq k\}$. If we alter L_1 by an arbitrarily small perturbation to L_1' so that L_1' is contained in the finite disc bounded by L_1, then the pre-images of L_1' which are near to L_j form an invariant cycle up to isotopy relative to $X(s_q \sqcup s_{1-p})$. L_1' bounds both critical points, and each loop L_j' bounds at least two points of $X(s_q \sqcup s_{1-p})$ (one in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$, one in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$) and the loops are disjoint. Thus $\bigcup_{1\leq j \leq k} L_j'$ forms a Levy cycle and $s_q \sqcup s_{1-p}$ is not equivalent to a rational map by 2.5.1. □

The rest of this chapter deals with branched coverings which are not equivalent to rational maps. We now show the existence of a non-trivial equivalence between maps in $\mathcal{B}(n, k)$.

Notation: Any mating $s_a \sqcup s_b$ we will deal with has one periodic critical orbit in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ and another in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$. We refer to these two sets as $X(s_a)$ and $X(s_b)$ respectively. So $X(s_a \sqcup s_b) = X(s_a) \cup X(s_b)$.

Theorem 4.1.2. *Let μ_q be defined as in Theorem 4.1.1. Then $s_q \sqcup s_{1-p} \simeq s_{q'} \sqcup s_p$, for some q' where $\mu_{q'} \geq \mu_{(1-p)}$. Equality holds when $\mu_q = \mu_p$.*

Proof. We can draw a circle, C, which is invariant up to isotopy (relative to $X(s_q \sqcup s_{1-p})$—unless stated otherwise, we shall always assume that this is the set that the isotopy is relative to) under $s_q \sqcup s_{1-p}$, as illustrated for the example (refer to Figure 4.1a). This idea first occurs in Wittner's thesis [W]. There are many non-isotopic ways of constructing such a circle, but we will make a standard choice, as follows. The periodic forward orbit of the minimal leaf μ_p forms the polygonal boundary of a region P in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ which is invariant up to isotopy.

Let us assume that p is the endpoint of μ_p which is clockwise from the other as measured from the inside of the unit disc for less than half the unit circle's circumference — we call this the clockwise endpoint of μ_p — and let p' be the other endpoint.

We now construct C: Let G denote the set of components of $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\} \setminus \bigcup \mathcal{L}_p^{-1}$ which intersect $X(s_{1-p})$. Let us rotate P by a small amount in an anti-clockwise direction to get a region P'. Now perturb each component of G, so that a component G_0 becomes G_0', and so that $\overline{G_0'} \cap S^1$ lies strictly inside a component of $S^1 \setminus \{\text{vertices of } P\}$. These intervals for the different components are then disjoint. Then connect each vertex of P' to the nearby component of G', which lies in a clockwise direction from the vertex, by a simple arc, thickened into a small tubular neighbourhood. The resulting construction is a connected set intersecting S^1 in a small arc for each vertex of P'. Let C be the boundary of this set. Then C is as illustrated in Figure 4.1b. Note that the circle C is clearly invariant up to isotopy and separates $X(s_{1-p})$ from $X(s_q)$.

Let ϕ be an orientation-preserving homeomorphism of $\hat{\mathbb{C}}$, which maps C to the unit circle, such that $0 \mapsto 0$ and $\infty \mapsto \infty$. Assuming that ϕ induces an equivalence of branched coverings, let g be the map which (we hope to show) is equivalent to $s_q \sqcup s_{1-p}$ via ϕ. So $g(z) = \phi \circ (s_q \sqcup s_{1-p}) \circ \phi^{-1}(z)$. $\phi(C) = S^1$ must be an invariant circle under g. Thus $X(s_q)$ is mapped into the unit disc, and this set of image points must be periodic of period n. Similarly $X(s_{1-p})$ is mapped outside of the unit disc to a period k orbit. Now, g is the mating of two quadratic polynomial functions, because of the invariant circle $\phi(C)$. (see §2.5.)

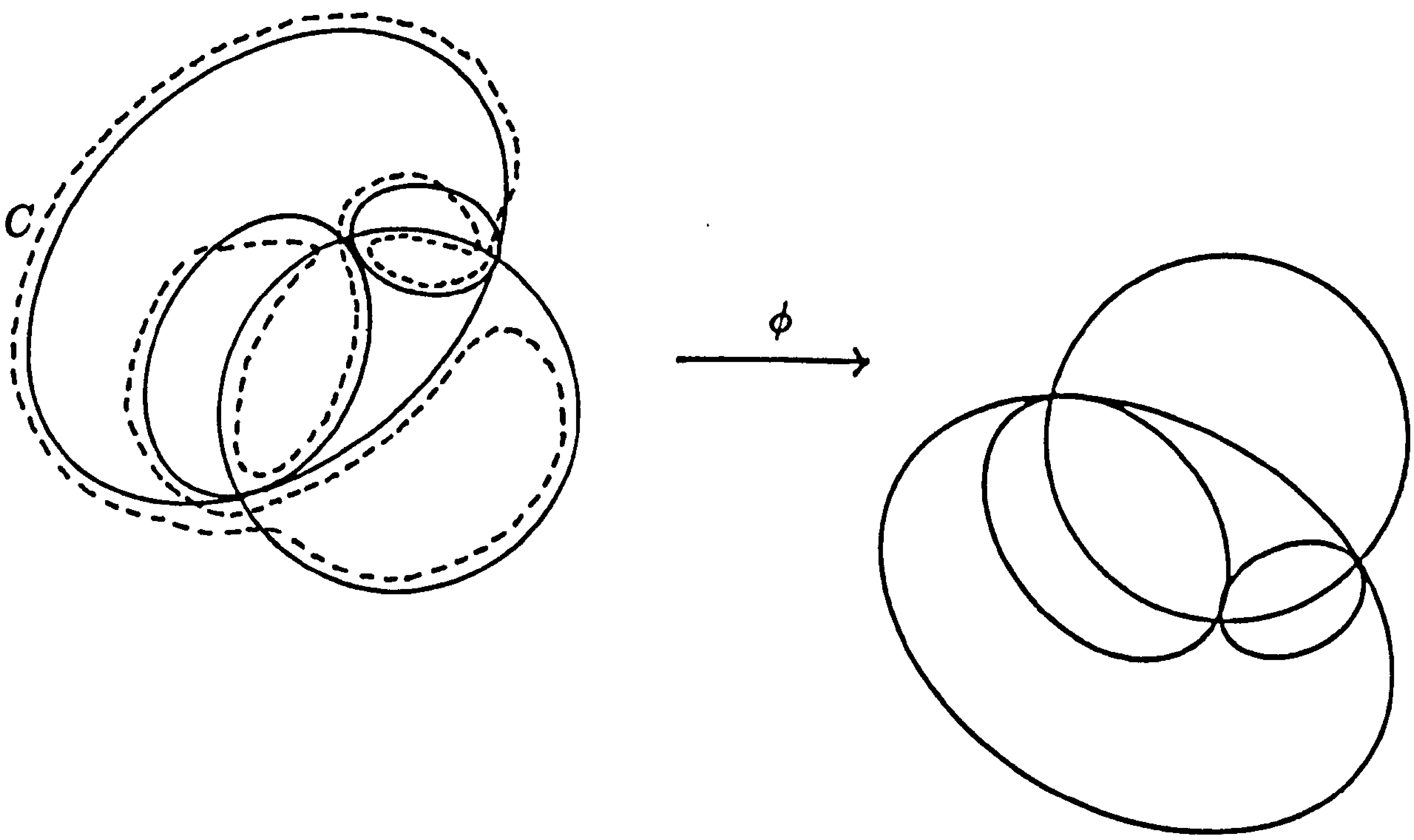


Figure 4.1b

So we have that $g \simeq g_1 \sqcup g_2$ for lamination maps g_1 and g_2 which correspond to polynomials, where g_2 has a critical point of period k and g_1 has a critical point of period n. Now, because each loop of the Levy cycle (see 4.1.1) intersects C precisely twice (at least after subjection to a small isotopy) and because we can assume that each pair (there are k pairs) occurs near to a vertex of P, on either side of $\phi(C) = S^1$ there is a collection of k disjoint simple arcs with endpoints in S^1 and invariant under g up to isotopy. Now g_1 and g_2 have corresponding arc sets in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$, again with endpoints in S^1. The rotation number associated the arc set of g_1 must be the same as the rotation number of $\bigcup_{1 \leq j \leq k} s^j_{1-p}(\mu_{1-p})$, and the rotation number of g_2 is the same as that of $\bigcup_{1 \leq j \leq k} s^j_p(\mu_p)$, because any homeomorphism preserves the rotation number. It follows by 2.4.1 that $g_1 \simeq s_{q'}$ for $\mu_{q'} \geq \mu_{1-p}$ and $g_2 \simeq s_p$. i.e., $\mathcal{L}_{q'}$ must contain the minimal leaf μ_p. □

§4.2 The main equivalence theorem.

Let μ_q and $\mu_{q'}$ be periodic minor leaves of QML such that q and q' are endpoints of μ_q and $\mu_{q'}$ respectively. Let μ_p and $\mu_{p'}$ be the minimal leaves of QML such that $\mu_p \leq \mu_q$ and $\mu_{p'} \leq \mu_{q'}$.

First, note that if the periods of μ_q and $\mu_{q'}$ are not the same, then it follows immediately from the definition of equivalence that $s_q \sqcup s_p \not\simeq s_{q'} \sqcup s_{p'}$, since the cardinalities of $X(s_q)$ and $X(s_{q'})$ are different (the same is true if the periods of μ_p and $\mu_{p'}$ are different). So from now on we assume that μ_q and $\mu_{q'}$ are of the same period, n, and that μ_p and $\mu_{p'}$ are of the same period, k.

Theorem 4.2.1. *Let the maps $s_q \sqcup s_{1-p}$ and $s_{q'} \sqcup s_{1-p'}$ be equivalent as branched coverings, with q and q' of period n and p and p' of period k. Then $\mu_p = \mu_{p'}$, or $\mu_p = 1 - \mu_{p'}$. If $\mu_p = \mu_{p'}$ then $\mu_q = \mu_{q'}$. If $\mu_p = 1 - \mu_{p'}$ then $\mu_{q'}$ is uniquely determined by μ_q.*

The latter case refers to the equivalence demonstrated in Theorem 4.1.2. The proof of theorem 4.2.1 therefore involves ruling out all other possible non-trivial equivalences: we will break this down into a few cases. We rule out equivalences where μ_p and $\mu_{p'}$ are of the same period, but not either identical or complex-conjugate.

Theorem 4.2.2. *The branched coverings $s_q \sqcup s_{1-p}$ and $s_{q'} \sqcup s_{1-p'}$ are equivalent only if $\mu_p = \mu'_p$ or $\mu_p = \mu_{1-p'}$.*

One further result we will need (see [R2]):

Theorem 4.2.3. *Let f be a degree two branched covering map of $\hat{\mathbb{C}}$ to itself, with a Levy cycle Γ. Then there exists a compact set P, where P is a component of $\hat{\mathbb{C}}\backslash(\cup\Gamma)$ such that P is isotopic (rel $X(f)$) in $\hat{\mathbb{C}}\backslash X(f)$ to a compact set $Q, Q \subset f^{-1}(P)$ and such that $f : Q \to P$ is a homeomorphism. If the isotopy class of P is irreducible (true for matings), then this set P is unique and so is the isotopy class of homeomorphisms $Q \to P$.*

Note: For a map in $\mathcal{B}(n,k)$ we have seen that (4.1.1) there is a Levy cycle made up from the periodic orbits of μ_p and $(1-\mu_p)^{-1}$. This region P is a multiply-connected Riemann surface: $P \cap \{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\}$ is a topological disc with a polygonal boundary, the sides of which are cyclically permuted by f. The polygon is isotopic relative to $X(f)$ to the periodic forward orbit of minimal minor leaves.

Proof of 4.2.2: We have invariant regions P for s_q and P' for $s_{q'}$, which by 4.2.3 are unique up to isotopy. Let $f_1 = s_q \sqcup s_{1-p}$ and $f_2 = s_{q'} \sqcup s_{1-p'}$. We know that P and P' have boundaries (boundary is a set of circles) which are cyclically permuted by f_1^{-1} and f_2^{-1} respectively. Let's consider a putative ϕ which induces an equivalence: ϕ must map $X(s_q)$ to $X(s_{q'})$. Then ϕ must map P to P' up to isotopy, (since these are both unique fixed regions up to isotopy) and the boundary components of P to those of P' in such a way that $f_1 \circ \phi = \phi \circ f_2$ always holds on ∂P. Let us suppose that such a ϕ exists.

Label the loops of the Levy cycle for f_1 by the numbers 1,2 ...such that 1 contains both the critical points, the loops are numbered consecutively anti-clockwise round S^1 and similarly $1', 2' \ldots$ for f_2. Then $\phi(1) = 1'$.

Let C be a set of arcs in P be such that the arcs of the circle between Levy loops are $C_1, C_2 \ldots C_k$, so that C_j has endpoints in ∂P, numbered anti-clockwise so that C_1 joins loops 1 and 2 etc. Let C' be defined similarly for f_2. So C and C' are invariant sets of arcs, cyclically permuted under f_1^{-1} and f_2^{-1} respectively (up to isotopy).

Claim: $\phi(C) = C'$ up to isotopy.

Proof. Suppose not. The loops j are arranged in ascending order anticlockwise round the unit circle so that C_j has endpoints in j and $j+1$. Then, for some i, $\phi(C_i)$ has endpoints in two non-adjacent loops L_1 and L_2. Let A be the open set got by taking the open unit disc and removing the Levy loops together with the discs they bound, and A' the analagous set in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$.

We can assume that $\phi(C_i)$ has the minimum number of intersections with S^1, after moving its endpoints in $\phi(i)$ and $\phi(i+1)$ isotopically. Consider the partition of $\phi(C_i)$ into arcs which are contained entirely within A or A'. Then $f_2|_A : A \to A$ and $f_2|'_A : A' \to A'$ are homeomorphisms. Then there exists some arc, α, contained in either A or A', which separates at least one Levy loop, say L, from the rest of A or A'. We can take this arc to be the one with an endpoint on $\phi(i)$. Its other endpoint is either on a loop not adjacent to $\phi(i)$ on S^1, or on some arc of C' not adjacent to $\phi(i)$.

Claim: The set of k forward iterates of α under f_2 is not isotopic to a disjoint set of arcs.

Proof. Consider the forward images of α under f_2. (see Figure 4.2a) These are determined by the loops $f_2^j(\phi(i))$ and $f_2^j(L)$ up to isotopy. Some forward image of α under f_2 must be such that it will intersect α. Let us assume that $\alpha \in A$, so α partitions A into two sets, A_L and A_L^c, where $L \in \partial A_L$. (Because $f_2|_A$ is a homeomorphism.) Then for some j', $f_2^{j'}(\alpha)$ has an endpoint at L and the other endpoint must be in ∂A_L^c. So this arc intersects α. $\square$

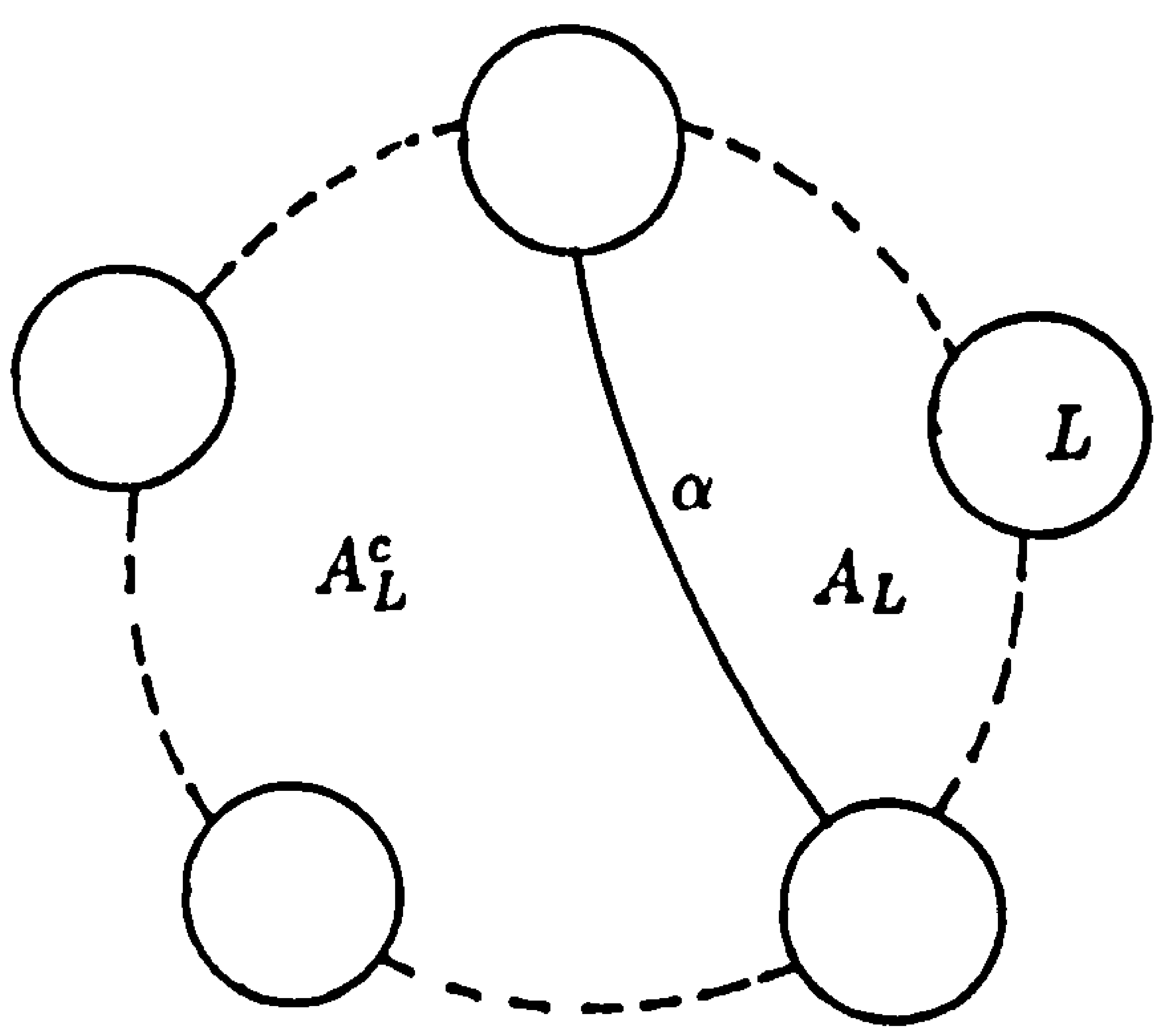


Figure 4.2a

So the set $\bigcup_{1\le j\le k}\phi(C_j)$ is not an invariant set, permuted by f_2. Contradiction.□

Thus, the image under ϕ of C_1 is either C_1' or C_k', because these are the arcs with endpoints on $1'$. In the former case this implies inductively that $\phi(C_j) = C_j'$ for $1 \le j \le k$, and in the latter case that $\phi(C_j) = C_{k+1-j}'$. Thus the only two possibilities are that $\phi(\mu_p) = \mu_p$ or $\phi(\mu_p) = 1 - \mu_p$, and we are done. □

§4.3 Equivalence within a limb.

Recall that in order to prove Theorem 3.0.1 we need to show that there are exactly two (or one) members in each equivalence class of $s_q \sqcup s_p$ (or $s_q \sqcup s_{1/3}$.) We have shown that there is a minor leaf $\mu_{q'}$ such that $s_q \sqcup s_{1-p} \simeq s_{q'} \sqcup s_p$, and that any other equivalent mating must be of the form $s_x \sqcup s_{1-p}$ or $s_y \sqcup s_p$ by 4.2.1. So it suffices to show that if $s_q \sqcup s_{1-p} \simeq s_{q'} \sqcup s_{1-p}$, then $\mu_q = \mu_{q'}$.

So we now consider the case where $\mu_p = \mu_{p'}$. Recall that q and q' are both of period n.

Theorem 4.3.1. *The branched coverings $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $s_q \sqcup s_{1-p}$ and $s_{q'} \sqcup s_{1-p}$ are equivalent as branched coverings if and only if $\mu_q = \mu_{q'}$.*

Firstly, let us make a few observations. We know (useful fact in §2.4) that the leaf μ_p is contained in both $\mathcal{L}_q$ and $\mathcal{L}_{q'}$. Also there is the leaf μ_p in the lamination $\mathcal{L}_p$, so the leaf $(1-\mu_p)^{-1}$ is contained in the mating $\mathcal{L}_q \cup \mathcal{L}_{1-p}^{-1}$. So there is a loop made up of μ_p and $(1-\mu_p)^{-1}$, which we call γ_1, such that $\gamma_1 \in \mathcal{L}_q \cup \mathcal{L}_{1-p}^{-1}$. It is clear that $\gamma_1 \in \mathcal{L}_{q'} \cup \mathcal{L}_{1-p}^{-1}$ also. In addition, we know that the k loops, which are forward images of γ_1 under $s_q \sqcup s_{1-p}$ are the same as the k forward iterates of γ_1 under $s_{q'} \sqcup s_{1-p}$ (they form the respective Levy cycles—see 4.1.1). Let γ_2 be the loop $f^{-k}(\gamma_1) \cap In(\gamma_1)$, where $f = s_q \sqcup s_{1-p}$, $\gamma_2 \neq \gamma_1$ and $In(\gamma_1)$ is the bounded component of $\hat{\mathbb{C}} \backslash \gamma_1$. γ_2 is well-defined because this k-th pre-image of μ_p is the same for any $\mu_q > \mu_p$. (We define similarly γ_2' for $f_2 = s_q' \sqcup s_{1-p}$. $\gamma_2' = \gamma_2$.) So we have $\gamma_2 \mapsto \gamma_1$ under f^k and under f_2^k.

Suppose that μ_q is not a tuning of μ_p. Let m be the least number such that $f(0)$ is separated from $f(\infty)$ by a component of $f^{-m}(\mu_q)$. Then we define $G = G(q)$ as the component of $\hat{\mathbb{C}} \setminus \bigcup_{j \leq m+1} f^{-j}(\gamma_1)$ containing $f(\infty)$. Note that G is contained in the region bounded between γ_1 and γ_2. ∂G is a set of loops, all of which are iterated pre-images of γ_1.

If μ_q is a tuning of μ_p we define G to be the annulus bounded between γ_1 and γ_2 and we define $G' = G(\mu_{q'})$ in the same way as $G(\mu_q)$. Observe that if $s_q \sqcup s_{1-p} \simeq s_{q'} \sqcup s_{1-p}$ then $G = G'$. So from now on we assume that $G = G'$. We divide the proof of Theorem 4.3.1 into three cases:

(1) Both μ_q and $\mu_{q'}$ are in the same component of $\hat{\mathbb{C}} \setminus G$.

(2) Both μ_q and $\mu_{q'}$ are in G. This is the case where both minor leaves are tunings of μ_p.

(3) μ_q and $\mu_{q'}$ are in different components of $\hat{\mathbb{C}} \setminus G$.

We illustrate an example: $\mu_q = [\![\frac{11}{31}, \frac{12}{31}]\!]$, $\mu_p = [\![\frac{1}{3}, \frac{2}{3}]\!]$ and $\gamma_2 \mapsto \gamma_1$ under $(s_{\frac{11}{31}} \sqcup s_{\frac{1}{3}})^2$.

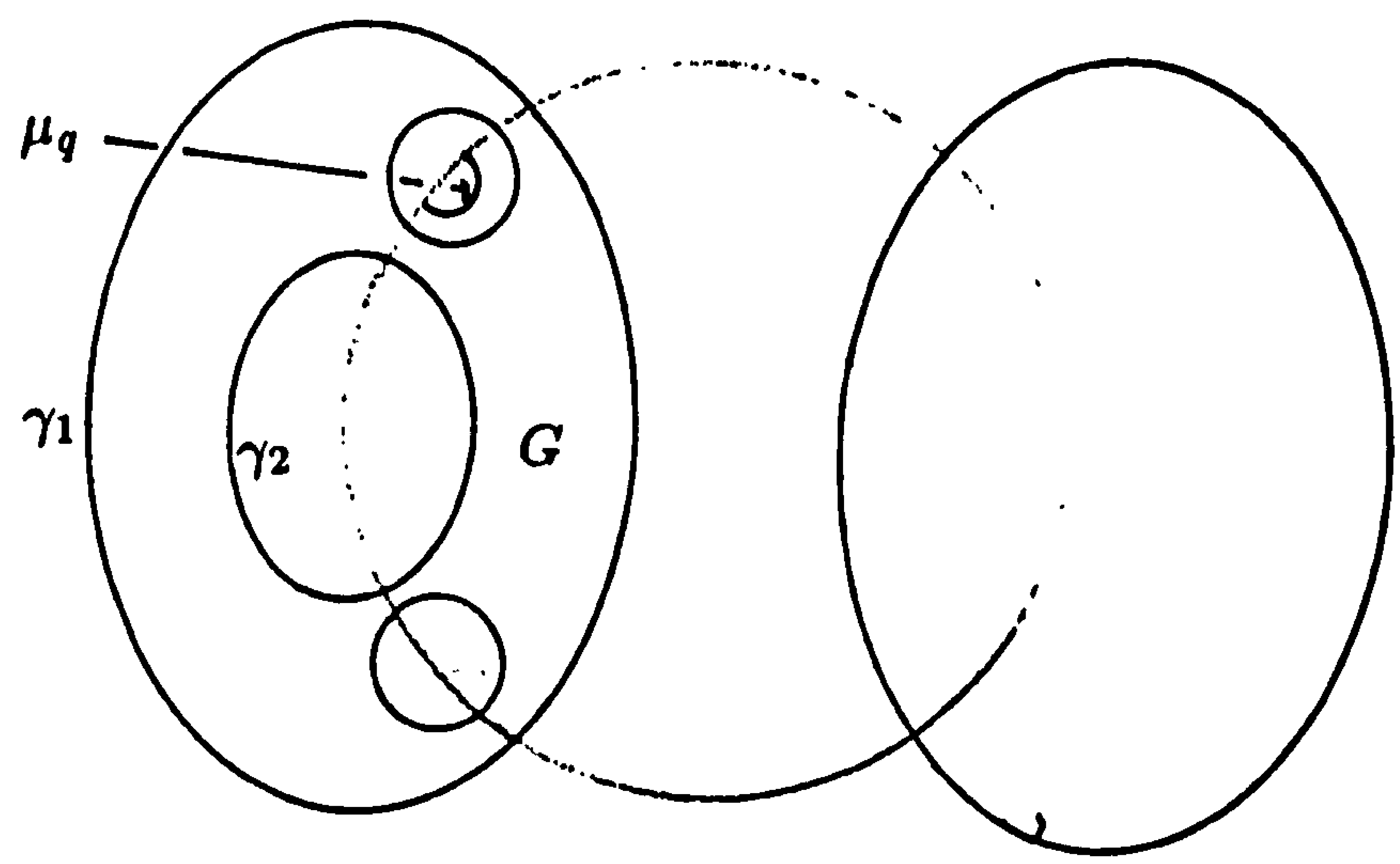


Figure 4.3a

Construction of critical branched covering.

We shall construct a critical branched covering (see [R3]), an invariant of maps in $\mathcal{B}(n,k)$, which will help us to distinguish between cases (2) and (3) above. In case (1) both matings under consideration will have the same critical branched covering.

So let us fix our ideas on $f = s_q \sqcup s_{1-p}$. We will consider a *critical branched* *covering*, $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, defined by f^k on G, so that $g : G \to G$ is a degree two branched covering map:

First suppose that s_q is a tuning of s_p. Then extend g to map components of G homeomorphically. Now suppose that s_q is not a tuning of s_p. We extend g so that it maps the components of $\hat{\mathbb{C}} \backslash G$ homeomorphically, except for one component containing the point 0, which maps with degree two. Note $f^k(\gamma_1) = \gamma_1 = g(\gamma_1)$. i.e., γ_1 is fixed set-wise (indeed pointwise) by g. $f(0)$ is a critical value of f and hence of f^k, so the pre-image of $f(0)$ under f^k which is contained in $In(\gamma_1)$ is a critical point of f^k and hence of g, $\star$. Also in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\} \cap G$ there is the critical value $f(\infty)$, which is a fixed critical value (and hence fixed critical point) of g. We choose g so that $g^{m+1}(\star) = g^m(\star)$ for least m with $g^m(\star)$ in the same component of $\hat{\mathbb{C}} \setminus G$ as 0.

So g has one fixed critical point $f(\infty)$ and the other one $\star$ is either periodic, in the tuning case, or strictly pre-periodic and eventually fixed in the non-tuning case. We will show directly that it is equivalent to a particular quadratic polynomial.

Let G_∞ be the gap of $\mathcal{L}_{1-p}^{-1}$ containing $f(\infty)$. Then $g \mid \partial G_\infty$ is semi-conjugate under a surjection ψ to the map $z \mapsto z^2$ on S^1. We can extend ψ to G_∞ so that ψ semi-conjugates g to $z \mapsto z^2$ on $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$.

Note that ψ collapses each arc of $\partial G \cap \{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ to a point on S^1: $\gamma_1 \cap \{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ consists of one arc fixed under g. Since the only fixed set on S^1 under $z \mapsto z^2$ is the point of argument 0, $\psi(C_1)$ is this point. All of its pre-images must also be sets of points.

In the case of a tuning, we have $s_q = s_p \vdash s_a$, and $g \simeq s_a$ from the definition of $s_p \vdash s_a$ (see 2.4.4).

Now suppose that s_q is not a tuning of s_p. Let γ be the component of ∂G which separates 0 from G. Then $\gamma \cap G_\infty$ has two components which map under ψ to $\pm a$, for some a which eventually maps under $z \mapsto z^2$ to the fixed point 1. Then we can find a homeomorphism ϕ which approximates ψ on ∂G_∞, so mapping ∂G_∞ approximately to S^1, and maps G_∞ approximately to $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$. γ is mapped to a loop bounding a small neighbourhood of the diagonal in $\{z \in \hat{\mathbb{C}} \mid |z| \leq$

1} which joins $\pm a$, and the loops $g^j(\gamma)$, for $m \geq j > 0$,are each mapped to a small loop around a^{2^j}. Thus $\phi \circ g \circ \phi^{-1}$ is uniquely determined up to equivalence by $\pm a$.

Now, each pair $\pm a$ determines a unique quadratic polynomial with a strictly pre-periodic eventually fixed critical point. Also for different pairs of endpoints of the form $\pm a$ we have different corresponding polynomials. And all quadratic polynomials with strictly pre-periodic eventually fixed critical points are of this form up to equivalence (refer to [DH2] for this kind of theory).

If $s_q \sqcup s_{1-p} \simeq s_{q'} \sqcup s_{1-p}$, then the critical branched coverings are also equivalent by the preceding construction. In case (3) the critical branched coverings are inequivalent, so that $s_q \sqcup s_{1-p} \not\simeq s_{q'} \sqcup s_{1-p}$.

We illustrate (4.3b) by the following example: $\mu_q = [\![\frac{11}{31}, \frac{12}{31}]\!]$ for which the polynomial associated to g has major leaf $[\![\frac{1}{8}, \frac{5}{8}]\!]$.

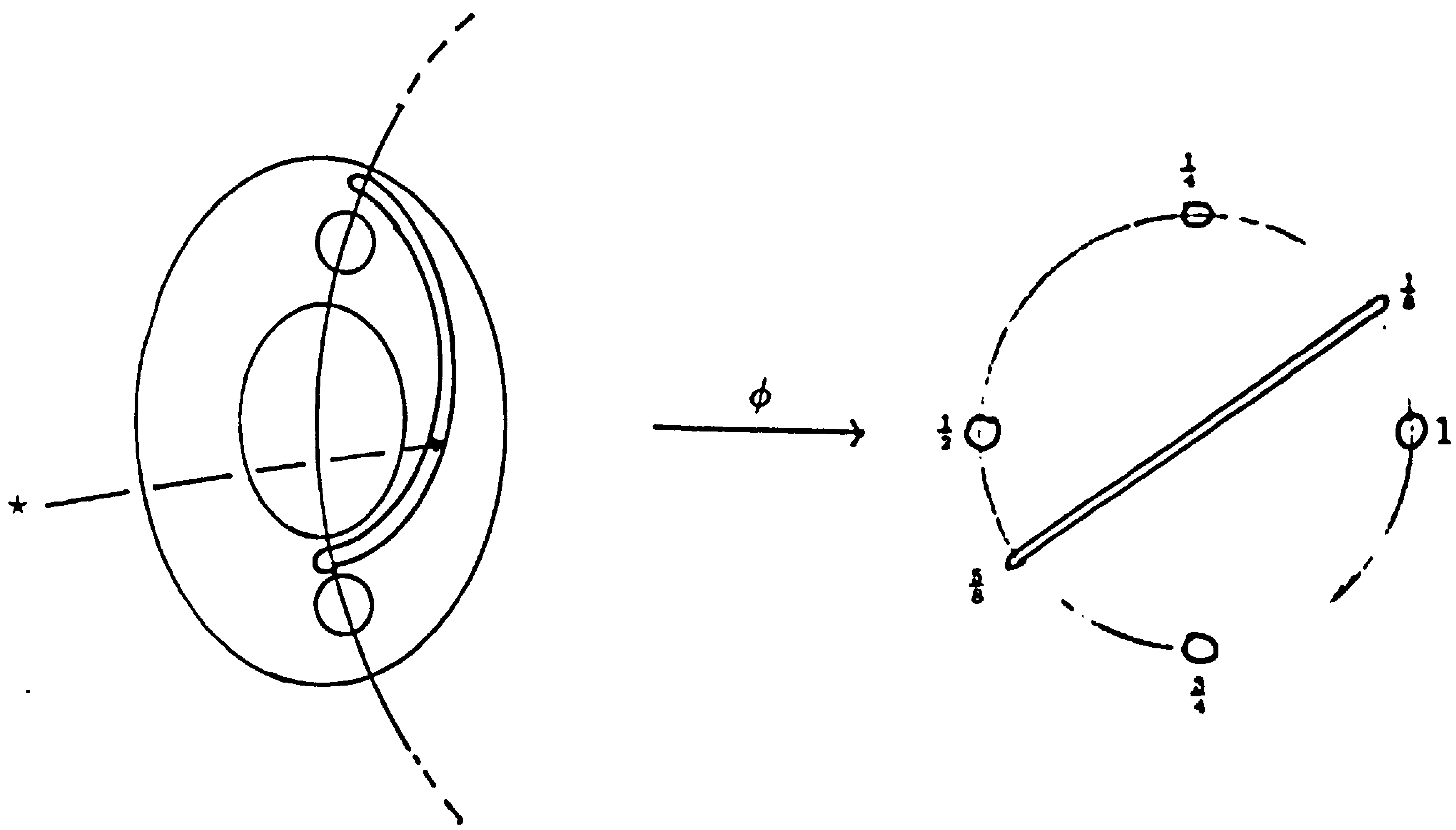


Figure 4.3b

Then $\mu_{q'} = [\![\frac{19}{31}, \frac{20}{31}]\!]$ — the associated polynomial in this case has major leaf $[\![\frac{3}{8}, \frac{7}{8}]\!]$.

In case (2) the critical branched coverings $s_q \sqcup s_{1-p}$ and $s_{q'} \sqcup s_{1-p}$, where $s_q = s_p \vdash s_a$ and $s_{q'} = s_p \vdash s_{a'}$, are equivalent if and only if $s_a \simeq s_{a'}$.

Example: $s_{\frac{22}{63}} \sqcup s_{\frac{1}{3}} = (s_{\frac{1}{7}} \vdash s_{\frac{1}{3}}) \sqcup s_{\frac{1}{3}} \not\simeq s_{\frac{27}{63}} \sqcup s_{\frac{1}{3}} = (s_{\frac{3}{7}} \vdash s_{\frac{1}{3}}) \sqcup s_{\frac{1}{3}}$.

Of course if the critical branched covers are the same, as they are in case (1), then we have more to do.

Proof of 4.3.1 case (1).

μ_q and $\mu_{q'}$ are separated from G by the same component $\gamma = \ell_0$ of ∂G. Remembering that n is the period of μ_q under s_q, we define ℓ_i to be the pre-image of ℓ_{i-1} under s_q such that ℓ_i separates $s_q^{n-i+1}(0)$ from G. Firstly note that $\ell_0 \neq \ell_n$. (Because the leaf ℓ_0 is strictly pre-periodic). Also ℓ_0 separates ℓ_n from G. Hence for all j, ℓ_{j-n} bounds ℓ_j from 0 by an induction on j.

We introduce a number which will allow us to distinguish maps which are tunings from those which are not. Let $m_b = \#\{0 \leq j < n \mid s_q^{j+1}(0)$ is in the same component of $\hat{\mathbb{C}} \setminus \ell_{nb}$ as $s_q(0)\}$. Then, for some $a \geq 1$, $m_a = m_{a-1}$ since m_b is a finite number which is non-increasing as b increases. ($m_a = 1$ for μ_q which are not tunings, $m_a \neq 1$ for non-trivial tunings). Let $m = m_a$, where a is minimal, and write n=mr. Then ℓ_{an} and $\ell_{(a-1)n}$ are isotopic in $\hat{\mathbb{C}} \setminus X(s_q \sqcup s_{1-p})$. In fact, ℓ_{an} and ℓ_{an-rj} are isotopic in $\hat{\mathbb{C}} \setminus X(s_q \sqcup s_{1-p})$ for $0 \leq rj \leq n$, because ℓ_{an} and ℓ_{an-rj} do not intersect and the components of $\hat{\mathbb{C}} \setminus \ell_{an}$ and $\hat{\mathbb{C}} \setminus \ell_{an-rj}$ which are disjoint from G contain the same elements of $X(s_q \sqcup s_{1-p})$. Write Q for the component of $\hat{\mathbb{C}} \setminus \ell_{an}$ which is disjoint from G. Then $(s_q \sqcup s_{1-p})^r$ maps $\hat{\mathbb{C}} \setminus \ell_{an}$ to $\hat{\mathbb{C}} \setminus \ell_{an-rj}$ with degree two and mapping Q with degree two over the component of $\hat{\mathbb{C}} \setminus \ell_{an-r}$ which is disjoint from G. It follows that $s_q \simeq s_{q_1} \vdash s_t$, where the critical point of s_t has period m and the critical point of s_{q_1} is of period r. Note that when s_q is not the tuning of another map, $m = 1$.

Now let ℓ'_i be defined for $\mu_{q'}$ in the same way that ℓ_i was defined for μ_q. Note that $\ell_0 = \ell'_0$. Recall that we have $f_1 \simeq_\phi f_2$, where $f_1 = s_q \sqcup s_{1-p}$ and $f_2 = s_{q'} \sqcup s_{1-p}$, and whose respective critical branched coverings are the same, g. Then, we have (as in [R2]) that if $(f_1, X(f_1)) \simeq_\phi (f_2, X(f_2))$, then $(f_1, f_1^{-j}X(f_1)) \simeq_{\phi_j} (f_2, f_2^{-j}X(f_2))$, where $\phi_j \simeq \phi$ relative to $X(f_1)$ and $\phi_j(f_1^{-j}X(f_1)) = f_2^{-j}X(f_2)$. That is, $\phi_j \circ f_1 \circ \phi_j^{-1}$ is homotopic to f_2 through branched coverings which map the set $f_2^{-j}X(f_2)$ to itself. We can also assume that $g \simeq_\phi g$. Since g is equivalent to a polynomial, it follows from Thurston's theorem (see §2.3) that ϕ is isotopic to the identity relative to $X(g)$. It follows that ϕ_j preserves loops in $\bigcup_{i \leq j} f_1^{-i}(\ell_0)$ up to isotopy relative to $f_1^{-j}X(f_1)$.

Take $j = an$ and assume without loss of generality that $\phi = \phi_{an}$. Then $\ell_{an} = \phi(\ell_{an}) = \ell'_{an}$. If s_q is a tuning, then so is $s_{q'}$ and $s_{q'} \simeq s_{q'_1} \vdash s_{t'}$ where the critical point of $s_{t'}$ is of period m (remember that $n = rm$). Now we can identify s_t with a branched covering h which has a single fixed critical point outside Q and is equal to f_1^r in Q. Similarly, we identify $s_{t'}$ with a branched covering h'. Then $h \simeq_\phi h'$ and hence $s_t \simeq s_{t'}$. Now ϕ induces an equivalence between $s_{q_1} \sqcup s_{1-p}$ and $s_{q'_1} \sqcup s_{1-p}$ by altering the definition of ϕ on the set $\bigcup_{0 \leq j < r} f_1^j(Q)$ — each component $f_1^j(Q)$ of this set contains one point of $X(s_{q_1} \sqcup s_{1-p})$, which we map by ϕ to the corresponding points of $X(s_{q'_1} \sqcup s_{1-p})$. So, in order to show that $\mu_q = \mu_{q'}$, it suffices to show that $\mu_{q_1} = \mu_{q'_1}$. Thus we can assume that ℓ_{an} separates $s_q(0) = f_1(0)$ from all the other points in $X(f_1)$. (i.e., we have dealt with the tunings)

Since $\ell_{an} = \ell'_{an}$ we have that $\ell_i = \ell'_i$ for $i \leq an$. Then $\ell_i = \ell'_i$ for all i: suppose that $\ell_{b_{n+i}} = \ell'_{b_{n+i}}$ for $0 \leq i < n$. Then ℓ_{bn+i+1} is the component of $s_q^{-1}\ell_{bn+i}$ which is bounded from G by $\ell_{bn-n+i+1}$ and separating $s_q^{n-i-1}(0)$ from G. However ℓ'_{bn+i+1} is the analogous component and must, therefore be equal to ℓ_{bn+i+1}. The result follows for all i by induction.

Finally, $\lim_{i\to\infty} \ell_{in}$ is in the boundary of the gap of $\mathcal{L}_q$ containing $s_q(0)$, and likewise q'. So $q = q'$.

This completes the proof of 4.3.1, and thus Theorem 3, and concludes the main body of work in this thesis. We shall start to consider some of the consequences of Theorem 3 and how it can be applied in Chapter 6.

§4.4 Symbolic dynamics.

In this section, we sharpen the statement of Theorem 4.1.2, while introducing some concepts we will use in later chapters. This section in not part of the main thread of this chapter, and is not necessary for the proof of Theorem 3.

Given a lamination $\mathcal{L}$ with a minimal periodic leaf μ_p, we assign to each leaf in $\mathcal{L}$ a number in the set $\{1, 2, \ldots, n = 0\}$, such that if a leaf ℓ is such that $\ell > \mu_p$ then it is coded 1, if $\ell > s_q^{j-1}(\mu_p)$ then it is coded j; and if ℓ is not greater than any of the leaves in $\mathcal{O}^+(\mu_q)$, with 0. So each period n minor leaf has a sequence of n symbols representing it, denoted $\sigma(\mu_q)$: Let σ_i for $1 \le i \le n$ be the i-th symbol, so that $\sigma = (\sigma_1 \ldots \sigma_n)$. Then σ_i is the code symbol for $s_q^{i-1}(\mu_q)$.

$\sigma(\mu_q)$ is subject to the following rule:

Lemma 4.4.1. *Any number $N \neq 0$ in $\sigma(\mu_q)$ must be followed by the (sub)sequence $((N+1)\ (N+2) \ldots (k-1)\ 0)$.*

Proof. This is forced by the dynamics of μ_q relative to μ_p. Let $\mathcal{R}(\ell)$ be the simply connected region bounded between ℓ and S^1. Then for $0 \le j < k-1$ the map s_q restricted to $\mathcal{R}_j = \mathcal{R}(s_q^j(\mu_p))$ is a homeomorphism onto its image, $\mathcal{R}_{j+1} = \mathcal{R}(s_q^{j+1}(\mu_p))$. So j is followed by $j+1$ (mod k). The region $\mathcal{R}_0$ coded by 0 maps with degree two over the unit disc, so there is no restriction of what symbol can follow a 0. □

In particular this means that sequences always start with the symbols $1, 2 \ldots (k-1), 0$ and end in a 0. We write $\sigma(\mu) = (1\ 2\ \ldots (k-1)\ 0 \ldots 0)$. The action of s_q on $\sigma(\mu)$ is a *sub-shift of finite type* with matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

where each entry of 1 in row i and column j denotes that the symbol i may be followed by the symbol j and each entry of 0 denies that possibility.

(Aside: In fact the set of such sequences of length n is isomorphic to the set of ordered partitions of n using the numbers $1, \ldots, k$. e.g., (1 2 3 0 0 3 0) is written as $4+1+2$ under this correspondence.)

Note that if μ_q is a tuning of a period a leaf by a period b leaf, then $\sigma(\mu_q)$ is given by a sequence of length a repeated b times. e.g., for $[\![\frac{22}{63}, \frac{25}{63}]\!] = [\frac{1}{7}, \frac{2}{7}] \vdash [\frac{1}{3}, \frac{2}{3}]$, $\sigma = (1\ 0\ 1\ 0\ 1\ 0)$.

The coding $\sigma(\mu)$ is not necessarily unique to a minor leaf μ. The following two results detail the precise relationship between minor leaves and their codings.

Lemma 4.4.2. *A length n sequence determines a unique minor of a period n leaf if and only if it contains only one symbol 1, or is a tuning of μ_p with $[\frac{1}{3}, \frac{2}{3}]$.*

Proof. The subsequence (0 1) corresponds to a degree two cover of $\mathcal{R}(\mu_p)$ by part of the central region containing 0, denoted $\mathcal{R}_c$ (Note that $\mathcal{R}_c \subset \mathcal{R}_0$). Thus there are two possible leaves which can map to a given leaf in $\mathcal{R}(\mu_p)$. Consequently, by pulling back under s_q, there are two possible pre-images in $\mathcal{R}_c$, and at least two minor leaves with this coded sequence. For any other subsequence $\big(i\ (i+1)\big)$ or $(0\ j)$ for $j \neq 1$, $\mathcal{R}_i \rightarrow \mathcal{R}_{i+1}$ or $\mathcal{R} \rightarrow \mathcal{R}_j$ (where $\mathcal{R}$ is the appropriate subset of $\mathcal{R}_0$) is a homeomorphism, so a leaf in $\mathcal{R}_{i+1}$ has one pre-image in $\mathcal{R}_i$ (and a leaf in $\mathcal{R}_j$ has one pre-image in $\mathcal{R} \subset \mathcal{R}_0$). So if (0 1) does not occur in $\sigma(\mu_q)$ (so 1 occurs only as the first entry) there is a unique minor leaf corresponding to σ.

$[\![\frac{1}{3}, \frac{2}{3}]\!] \vdash \mu_p$ is a minor leaf of period $2k$, with $\sigma = [0\ 1\ \ldots\ (k-1)\ 0\ 1\ \ldots (k-1)\ 0]$. There is one occurrence of (0 1), so there are two minor leaves with this sequence. However one of them is the leaf μ_p. $\square$

Let $\phi_e(n,k)$ be defined as the function: $\phi_e(n,k) =$ the number of divisors of n, which are multiples of k. For example $\phi_e(8,3) = 1$.

Corollary 4.4.3. *A length n sequence with j occurrences of* (0 1) *represents $2^j - \phi_e(n,k)$ minor leaves of period n.*

Proof. As we have shown, each (0 1) corresponds to a degree two map from $\mathcal{R}_c$ to $\mathcal{R}(\mu_q)$, so each occurrence doubles the number of minor leaves associated to σ. So there are 2^j minor leaves of some period corresponding to σ with j occurrences of (0 1). However, if σ is made up of a repeated sequence of m symbols, *one* of the minor leaves represented by σ is of period m. Clearly this happens for $m|n$ and when $k|m$ The number of these is given by $\phi_e(n,k)$. $\square$

Example: $\sigma = (1\ 0\ 1\ 0\ 1\ 0)$ corresponds to the three period six minor leaves $[\![\frac{22}{63}, \frac{25}{63}]\!]$, $[\![\frac{38}{63}, \frac{41}{63}]\!]$ and $[\![\frac{27}{63}, \frac{36}{63}]\!]$, and the period two minor leaf $[\![\frac{1}{3}, \frac{2}{3}]\!]$.

Definition. Primary polygon. The leaves $\{s_q^j(\mu_p)\,|\,1 \leq j \leq k\}$ form a k-sided polygon, Π_0. Then $\Pi_1 = \{s_q^{-1}(\Pi_0)\} \setminus \Pi_0$ is another k-sided polygon and $\Pi_1 \to \Pi_0$ homeomorphically. Let $\Pi_j = \{s_q^{-j}(\Pi_0)\} \setminus \bigcup_{i<j} \Pi_i$. Then Π_j is a set of 2^{j-1} distinct polygons. Given a polygon $\pi \in \Pi_j$ (for $j > 0$) we can encode its forward orbit in the same way as for minor leaves. (Indeed each side of a polygon can be thought of as a pre-periodic minor leaf). A polygon is ***primary*** if $\sigma(\pi)$ does not contain (0 1). Thus π is primary if none of its forward iterates (including π itself) is contained in $\mathcal{R}_c$. We can use the ordering of minor leaves to define an ordering on polygons (and minor leaves): we say that $\pi < \mu$ if $\ell < \mu$ for some $\ell \in \pi$ and $\pi < \pi'$ if $\ell < \ell'$ for some $\ell \in \pi$ and for any $\ell' \in \pi'$.

Let us call a polygon $\pi \in \Pi(\mu_q)$ ***standard*** if it is identical to a polygon of the lamination $\mathcal{L}_p$. The Π_m polygons in $\mathcal{R}_c$ are not necessarily the standard ones. Each can have endpoints on both arcs of $S^1 \cap \mathcal{R}_c$. We call these and their pre-images ***thin*** polygons. Note that primary polygons are always standard and thin ones never are. Given a leaf μ we can define its maximal standard polygon π as the polygon for which $\pi' < \pi$ for any standard polygon $\pi' < \mu$. Similarly the minimal standard polygon τ is the polygon for which $\tau' > \tau$ for any standard polygon $\tau' > \mu$.

Lemma 4.4.4. *For any minor leaf μ_q, there is a unique maximal standard polygon π_M such that $\pi_M < \mu_q$. If there exists a unique minimal standard polygon π_m such that $\pi_m > \mu_q$, and if $\pi_M \subset \Pi_j$, then $\pi_m \subset \Pi_{j'}$, where $j' > j$.*

Proof. π_M clearly exists because there cannot be an infinite increasing sequence of polygons $\{\pi_i\}$ such that $\pi_i < \mu_q$ for all i and such that all the elements of the sequence are standard: and the primary (and hence standard) polygon $\pi_0 < \pi_M$. Since π_M is standard, any polygon $\pi > \pi_M$ must be in Π_i, for $i > j$. In particular, this is true for π_m. □

We introduce the idea of a regular minor leaf — one that is closely approximated by primary polygons, in the following sense: a leaf is *regular* if the polygon π_M is primary and π_m exists. (in fact π_M primary implies π_m primary, if the latter exists). We illustrate in Figure 4.4a, where $\pi_0 = [\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}]$ and $\mu_q = [\![\frac{15}{127}, \frac{16}{127}]\!]$.

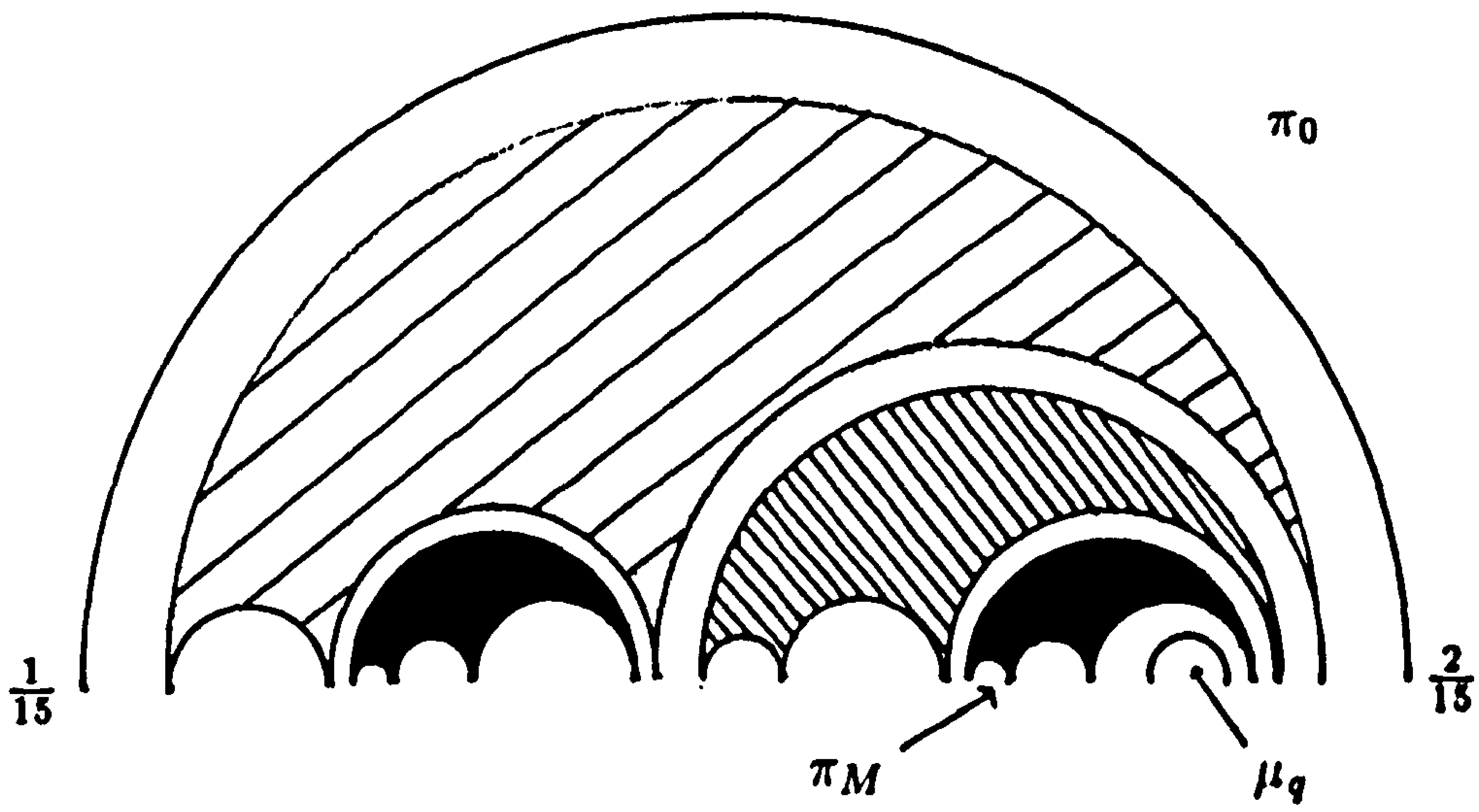


Figure 4.4a

Note that for $\pi = \pi_M$, $s_q^{-1}(\pi)$ consists of two thin polygons (which are not primary—their coding starts with (0 1)). All the pre-images of these polygons must also be thin, and the leaf μ_q is in the limit of some sub-sequences of these thin polygons.

Lemma 4.4.5. *A minor leaf is regular if and only if one of the following is true:*

(1) $\sigma(\mu_q)$ contains the symbol 1 exactly once.

(2) μ_q is not a tuning and $\sigma = \sigma_a + \sigma_b$, where σ_a is of type (1) and σ_b is the length k sequence $(1\ 2\ \ldots\ 0)$ repeated a number of times.

(3) μ_q is a tuning of μ_p by a symmetric leaf. (In other words, symmetric about real axis. So $\mu_s \geq [\![\frac{1}{3}, \frac{2}{3}]\!]$, where $\mu_q = \mu_s \vdash \mu_p$.)

Proof. This is a matter of placing the minor leaves relative to the relevant (non-thin) polygons, and we consider only these. For any minor leaf μ_q, we have that $s_q^{n-1}(\pi_M) = \pi_0$. Whether or not μ_q is regular depends on whether π_M is primary.

We characterise the three cases:

Case(1): For each leaf in the forward orbit of μ_q we have that $s_q^j(\mu_q) > s_q^j(\pi_M)$ and they are both in the same region $\mathcal{R}_i$ for some i where $0 \leq i < k$, unless possibly $s_q^j(\pi_M) = \pi_0$. Now let the second to last occurrence of 0 in $\sigma(\mu_q)$ occur at the n_0-th place; then $j = n_0$. To see this: the n-th symbol 0 corresponds to $s^{n-1}(\mu_q)$. Take the pre-images of π_0 and π_1 which sandwich $s^{n-2}(\mu_q)$. Then if $\sigma_{n-1} = k-1$, the pre-images are π_0 and an element of Π_2. i.e., $s_q^{n-2}(\pi_M) = \pi_0$. Taking further pre-images the occurrence of 0 in σ signifies that $s_q^j(\pi_M) = \pi_1$.

Case(2): Let $\sigma_a(\mu_q)$ be of length m. Then $\pi_M \subset \Pi_j$, where $m - k < j < m$: we use the fact that, for $m < i < n$, $s_q^i(\mu_q)$ is not separated from π_0 by any other standard polygons; The forward orbit of μ_q intersects $\mathcal{R}_0$ for the first time on the m-th iterate, and $s_q^m(\pi_M) = \pi_0$ by the above fact.

Case(3): Here $\pi_M = \pi_0$ and $\pi_m \subset \Pi_k$, so μ_q is primary. π_m exists exactly when μ_q is a tuning of μ_p by a symmetric leaf.

Each of the above cases describes a regular minor leaf μ_q, because in each case we have shown π_M must be primary. Con 'ersely, assume that π_M is primary, and that we can ignore tunings - we have already dealt with this case. Then the only possibilities for μ_q are cases (1) and (2). □

Examples of regular leaves:

(1) $[\![\frac{3}{15}, \frac{4}{15}]\!]$. $\sigma = (1\ 2\ 0\ 0)$.

(2) $[\![\frac{26}{127}, \frac{33}{127}]\!]$. $\sigma = (1\ 2\ 0\ 0\ 1\ 2\ 0)$.

(3) $[\![\frac{10}{63}, \frac{17}{63}]\!] = [\![\frac{1}{3}, \frac{2}{3}]\!] \vdash [\![\frac{1}{7}, \frac{2}{7}]\!]$. $\sigma = (1\ 2\ 0\ 1\ 2\ 0)$.

Examples of non-regular leaves:

(4) $[\![\frac{19}{127}, \frac{20}{127}]\!]$. $\sigma = (1\ 2\ 0\ 1\ 2\ 0\ 0)$.

(5) $[\![\frac{22}{63}, \frac{25}{63}]\!] = [\![\frac{1}{7}, \frac{2}{7}]\!] \vdash [\![\frac{1}{3}, \frac{2}{3}]\!]$. $\sigma = (1\ 0\ 1\ 0\ 1\ 0)$.

The following Theorem is a refinement of Theorem 4.1.2 in the case of regular minor leaves. In some sense it is the expected "nice" result because of its obvious symmetry. However, we only prove it in one particular case.

Theorem 4.4.6. *Let μ_q be a regular periodic minor leaf, with minimal periodic leaf μ_p. Then $s_q \sqcup s_{1-p} \simeq s_{1-q} \sqcup s_p$.*

Tuning case: We deal with the case of type (3) regular minor leaves.

If μ_q is a tuning of its minimal leaf μ_p, then a copy of the full lamination $\mathcal{L}_p$ is contained in $\mathcal{L}_q$. In particular all the pre-periodic polygons in $\mathcal{L}_q$ are the standard ones — there are no thin (pre-periodic) polygons in the sets Π_j. However, there are period n polygons: Let T_0 be the set of period n polygons and T_j be the set of their j-th pre-images. Then we have that $\phi_j(T_j) \subset \{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$.

The following lemma deals with all the tunings we are considering.

Lemma 4.4.7. *Let μ_a be a periodic minor leaf. Then $(s_a \vdash s_p) \sqcup s_{1-p} \simeq (s_a \vdash s_{1-p}) \sqcup s_p$.*

Proof. By 4.1.2 we know that $(s_a \vdash s_p) \sqcup s_{1-p} \simeq s_{q'} \sqcup s_p$ for some $\mu_{q'} > (1 - \mu_p)$. However, $\mu_{q'}$ must be a tuning of μ_{1-p}, because $\phi(T_0)$ must be of period n, with $\phi^k(\tau) \subset \phi(T_0)$ for $\tau \subset \phi(T_0)$ and ϕ is defined as in 4.1.2. So we have $(s_a \vdash s_p) \sqcup s_{1-p} \simeq (s_{a'} \vdash s_{1-p}) \sqcup s_p$, where $\mu_{a'}$ is of the same period as μ_a.

But because ϕ is orientation-preserving $\mu_a = \mu_{a'}$: $\phi(T_0)$ must have the same orientation as T_0, with its leaves permuted in the same way under f^k — this is true exactly when $\mu_a = \mu_{a'}$. □

Case (3) of 4.4.6 proved. □

It remains to establish what the analogous result for non-regular minor leaves is, where the symmetry is more involved than with the regular case. Theorem 4.4.8 offers this result without proof.

Consider the region $\mathcal{R}_0$, which is bounded by two arcs of the unit circle, and arcs of π_0 and π_1. We say that two leaves, ℓ_1 and ℓ_2, in $\mathcal{R}_0$ are in ***opposite halves*** of $\mathcal{R}_0$ if either

(a) ℓ_1 has both endpoints on the same arc of $S^1 \cap \mathcal{R}_0$, while ℓ_2 has both endpoints on the other arc of $S^1 \cap \mathcal{R}_0$.

(b) Both ℓ_1 and ℓ_2 have endpoints on both arcs of $S^1 \cap \mathcal{R}_0$ and are separated from each other by 0.

Definition. Let μ_q be a non-regular minor leaf, with the first occurrence of $(0\ 1)$ in $\sigma(\mu_q)$ at the j-th and $(j+1)$-th positions. Then μ_{q^*}, the ***partner*** of μ_q, is the unique minor leaf such that $\sigma(\mu_{q^*}) = \sigma(\mu_q)$ (there are at least two such leaves) and such that $s_q^j(\mu_q)$ and $\left(s_{q^*}^j(\mu_{q^*})\right)$ are in opposite halves of $\mathcal{R}_0$.

Theorem 4.4.8. *If μ_q is a non-regular minor leaf, then $s_q \sqcup s_{1-p} \simeq s_{1-q^*} \sqcup s_p$, where μ_{q^*} is the partner of μ_q.*

Example: $s_{\frac{19}{127}} \sqcup s_{\frac{6}{7}} \simeq s_{\frac{91}{127}} \sqcup s_{\frac{1}{7}}$, because $[\![\frac{19}{127}, \frac{20}{127}]\!]^* = [\![\frac{35}{127}, \frac{36}{127}]\!]$.

CHAPTER FIVE

IRREDUCIBILITY OF $\mathcal{V}_n$

We establish the irreducibility of a combinatorial model for the varieties V_n for $n \leq 7$, which is conjectured to be equivalent to irreducibility of V_n and thus the connectedness of W_n. We shall again be constructing equivalences between matings of degree two laminations, but in contrast to chapter 4 we will be dealing exclusively with degree two branched coverings which are equivalent to rational maps. It is hoped that the results of this chapter can be generalised and taken further, with the aim of proving the irreducibility of V_n for all n.

§5.1 Combinatorial model for V_n.

The following result and corollary suggest a method for proving the connectedness of W_n:

Proposition 5.1.1. *Each connected component of W_n intersects non-trivially with the hyperbolic component of a polynomial in RM_2.*

Proof. See [R2] □

Corollary 5.1.2. *W_n is connected if the hyperbolic components of all quadratic polynomials in RM_2 with 0 of period n lie in the same connected component of W_n. Then, the irreducibility of V_n follows by 1.3.2.*

Note that in the above corollary "polynomial" means a degree two rational map of the form $z \mapsto \dfrac{z^2 + az + b}{z^2}$, with $c_2 = -2b/a$ fixed. This is equivalent (as a branched covering) to a polynomial.

Satisfying the condition of 5.1.2 in not something that we are able to tackle directly, at least not in general (see Chapter 6 for some examples). Instead, we resort again to laminations and their matings.

Let $\mathcal{V}_n' \subset QML$ be the set of period n minor leaves. For $\mu \in \mathcal{V}_n'$ define D_μ to be the component of $\{z \in \hat{\mathbb{C}} \,|\, |z| \leq 1\} \setminus \overline{\nu}$ which contains 0, where ν is the minimal leaf in QML such that $\nu < \mu$. Let $\mathcal{V}_n$ denote the following *combinatorial model* for V_n: $\mathcal{V}_n = \bigcup_{\mu \in \mathcal{V}_n'} \{\mu\} \times D_\mu / \sim$, where $\sim$ is defined by $(\mu_1, x_1) \sim (\mu_2, x_2)$ if and only if $x_1 \in \nu_1$ and $x_2 \in \nu_2$ for minimal minor leaves ν_1 and ν_2, both of the same period $\leq n$ and $s_{\mu_1} \sqcup s_{\nu_1} \simeq_\phi s_{\mu_2} \sqcup s_{\nu_2}$, where $\phi(X(s_{\mu_1})) = X(s_{\mu_2})$ and $\phi(X(\tau \circ s_{\nu_1} \circ \tau)) = X(\tau \circ s_{\nu_2} \circ \tau)$ and $\tau(z) = 1/z$.

Note that all such matings are rational by 2.5.1.

The reason we consider this combinatorial model is the following conjecture, the proof toward which substantial progress has been made.

Conjecture: (Rees) $\mathcal{V}_n$ connected implies W_n connected (and thus V_n irreducible).

This enables us to tackle the problem of irreducibility with the combinatorial methods at our disposal.

By the above conjecture we have an equivalent statement to 5.1.2:

$\mathcal{V}_n$ is connected if all the polynomial matings $s_a \sqcup s_0$ can be connected, where $\mathcal{L}_0$ is the empty lamination. Our strategy, therefore, is to investigate possible connections between such matings, for which our main tool is the following statement, which follows directly from the definition of $\mathcal{V}_n$.

5.1.3. Let μ_x and μ_y be periodic minor leaves and suppose that $s_x \sqcup s_y$ is equivalent to a rational map. Then if there exist μ_a and μ_b such that $s_a \sqcup s_b \simeq s_x \sqcup s_y$, then the polynomials associated to μ_a and μ_x lie in the same connected component, $\mathcal{V}'$ of $\mathcal{V}_n$, and the polynomials associated to μ_b and μ_y lie in the same connected component, $\mathcal{V}''$ of $\mathcal{V}_n$.

From now on, instead of saying that the polynomial-equivalent maps $s_a \sqcup s_0$

and $s_x \sqcup s_0$ lie in the same connected component of W_n, we shall abuse our notation somewhat and say that μ_a and μ_x are "connected".

§5.2 Connections for $n = k$.

We consider the simplest case: that is, where $k = n$. We consider equivalences of the form $s_x \sqcup s_y \simeq s_a \sqcup s_b$, where each minor leaf is of period n.

Let us start with the example $n = k = 3$. We want to show that $\mathcal{V}_3$ is connected by showing that the polynomials associated to the lamination maps with period 3 minor leaves lie in a connected set — We "connect up" the minor leaves $[\![\frac{1}{7}, \frac{2}{7}]\!]$, $[\![\frac{5}{7}, \frac{6}{7}]\!]$ and $[\![\frac{3}{7}, \frac{4}{7}]\!]$. Note that this result occurs in Wittner's thesis [W].

Theorem 5.2.1. *$s_{\frac{1}{7}} \sqcup s_{\frac{3}{7}} \simeq s_{\frac{3}{7}} \sqcup s_{\frac{1}{7}}$ and $s_{\frac{6}{7}} \sqcup s_{\frac{3}{7}} \simeq s_{\frac{3}{7}} \sqcup s_{\frac{6}{7}}$*

Proof. We use a similar construction to that used in Theorem 4.1.2. In $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ there is a triangle made up of the leaves $[\![\frac{1}{7}, \frac{2}{7}]\!]$, $[\![\frac{2}{7}, \frac{4}{7}]\!]$ and $[\![\frac{4}{7}, \frac{1}{7}]\!]$, which form the boundary of an invariant region T under s_q, and the periodic leaves in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ share a common endpoint with two of these triangular leaves. (In $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ there is the invariant arc $[\![\frac{1}{3}, \frac{2}{3}]\!]^{-1}$.) There exists an invariant circle C under $s_{\frac{1}{7}} \sqcup s_{\frac{3}{7}}$ up to isotopy: C has arcs in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\} \setminus T$ close to the periodic leaves that make up the triangle, such that these arcs are cyclically permuted in the same way as these leaves; these arcs extend into $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ so that they enclose the leaves $[\![\frac{3}{7}, \frac{4}{7}]\!]^{-1}$, $[\![\frac{1}{7}, \frac{6}{7}]\!]^{-1}$ and $[\![\frac{2}{7}, \frac{5}{7}]\!]^{-1}$ and so that $X(s_{\frac{3}{7}})$ is contained in $In(C)$.

Then, let ϕ be an orientation preserving homeomorphism $C \to S^1$, $\phi(X(s_{\frac{1}{7}}))$ is in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ and $\phi(X(s_{\frac{3}{7}}))$ is in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$. Furthermore $\phi(T)$ is contained in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$. The mating equivalent via ϕ to $s_{\frac{1}{7}} \sqcup s_{\frac{3}{7}}$ must contain the leaves $[\![\frac{1}{3}, \frac{2}{3}]\!]$, $[\![\frac{1}{7}, \frac{2}{7}]\!]^{-1}$, $[\![\frac{2}{7}, \frac{4}{7}]\!]^{-1}$ and $[\![\frac{4}{7}, \frac{1}{7}]\!]^{-1}$ (see figure 5.2a). So $\phi(T)$ is the region bounded by $[\![\frac{1}{7}, \frac{2}{7}]\!]^{-1}$, $[\![\frac{2}{7}, \frac{4}{7}]\!]^{-1}$ and $[\![\frac{4}{7}, \frac{1}{7}]\!]^{-1}$ up to isotopy (and $\phi([\![\frac{1}{3}, \frac{2}{3}]\!]^{-1}) = [\![\frac{1}{3}, \frac{2}{3}]\!]$). Since both critical points are of period three, this forces the mating to be $s_{\frac{3}{7}} \sqcup s_{\frac{1}{7}}$ because the image minor leaf μ must be such that $\mu > [\![\frac{1}{3}, \frac{2}{3}]\!]$. Thus $s_{\frac{1}{7}} \sqcup s_{\frac{3}{7}} \simeq_\phi s_{\frac{3}{7}} \sqcup s_{\frac{1}{7}}$.

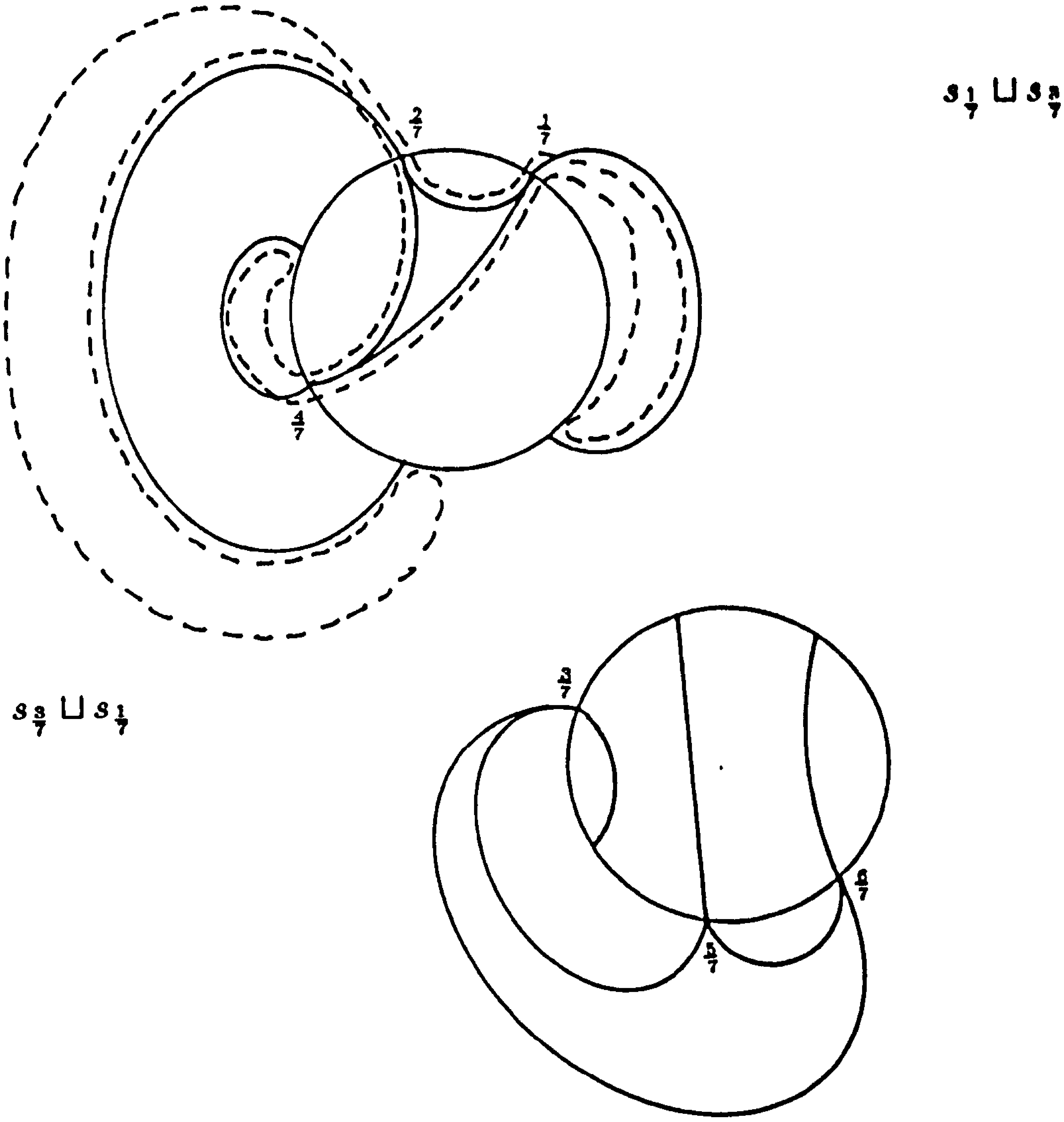


Figure 5.2a

The argument for $s_{\frac{6}{7}} \sqcup s_{\frac{3}{7}} \simeq s_{\frac{3}{7}} \sqcup s_{\frac{6}{7}}$ is similar. □

We introduce some notation:

If $\mu_a \times D_{\mu_a}$ and $\mu_b \times D_{\mu_b}$ lie in the same connected component of $\mathcal{V}_n$, we write $\mu_a \bowtie \mu_b$.

Corollary 5.2.2. *$\mathcal{V}_3$ is connected.*

Proof. By 5.2.1 and 5.1.3 $[\frac{1}{7}, \frac{2}{7}] \bowtie [\frac{3}{7}, \frac{4}{7}]$ and $[\frac{3}{7}, \frac{4}{7}] \bowtie [\frac{5}{7}, \frac{6}{7}]$, so $[\frac{1}{7}, \frac{2}{7}] \bowtie [\frac{5}{7}, \frac{6}{7}]$.
□

Notice that the proof of 5.2.1 relied on the fact the endpoints of periodic leaves in $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$ coincided with endpoints of periodic leaves in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$. This is what enabled the particular circle C to be constructed. We generalise this construction to minor leaves of any period n, for which there is an n-sided invariant region.

Theorem 5.2.3. *Let* $\exp(2\pi iq)$ *be an endpoint of* μ_q*, a minimal (in QML) period* n *minor leaf. Then, for all period* n *minor leaves* μ_p*, such that* μ_p *has an endpoint in the set* $\{s_q^j(\exp(2\pi iq)) \mid 1 \leq j \leq n\} \cup \{s_{1-q}^j(\exp(2\pi i(1-q))) \mid 1 \leq j \leq n\}$, $\mu_q \bowtie \mu_p$.

Proof. Firstly let us establish which leaves are of the type of μ_p above. Since μ_q is minimal, its n forward images form an n-gon: two of its vertices are those of the minor leaf; the other $n-2$ vertices are at the endpoint of a unique minor leaf in *QML*. These minor leaves are not minimal: if one was, both endpoints would be vertices, i.e. it would be a side of the polygon; however, a side of this polygon cannot be a minor leaf because the image of the associated major leaf is μ_q. Let us call these minor leaves (where they exist) $\mu_j(q)$ for $1 \leq j < n$, where $\mu_j(q)$ has an endpoint at the angle jq (mod1) on S^1.

We aim to connect up all the minor leaves with the above endpoints via a unique (to each conjugate pair of polygons) symmetric minor leaf.

Lemma 5.2.4. *The minor leaf with an endpoint at the left-most (angle nearest to 1/2) point of* $\{s_q^j(\exp(2\pi iq)) \mid j \leq n\}$ *is symmetric about the real axis. This means that this minor leaf is identical to the one with an endpoint at the left-most endpoint of* $\{s_{1-q}^j(\exp(2\pi i(1-q))) \mid j \leq n\}$.

Proof. Let x be the point in question. Let $\ell_x = \ell_x(\mu_q)$ be the leaf joining x to $1-x$. Then ℓ_x is of period n because its endpoints are, and the periodic orbit of ℓ_x has no self-intersections because each periodic leaf has complex-conjugate endpoints (exactly one of which is in $\{s_q^j(\exp(2\pi iq)) \mid j \leq n\}$). We need to show that ℓ_x is the image of the longest of these n leaves: now clearly $\ell_x > [\![\frac{1}{3}, \frac{2}{3}]\!]$ and is the shortest period n leaf in the region $\mathcal{R}([\![\frac{1}{3}, \frac{2}{3}]\!])$; hence its pre-image is longer than the pre-image of any of the other leaves in $\mathcal{R}([\![\frac{1}{3}, \frac{2}{3}]\!])$; but one of these leaves in $\mathcal{R}([\![\frac{1}{3}, \frac{2}{3}]\!])$ is the minor leaf—it must therefore be ℓ_x. □

Examples: For $n = 5$ we have the polygon $[\![\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}]\!]$ and its conjugate — in this case $\ell_x = [\![\frac{15}{31}, \frac{16}{31}]\!]$. However, we also have the polygon $[\![\frac{9}{31}, \frac{10}{31}, \frac{18}{31}, \frac{20}{31}, \frac{5}{31}]\!]$ and its conjugate, for which $\ell_x = [\![\frac{13}{31}, \frac{18}{31}]\!]$ — see figure 5.2b.

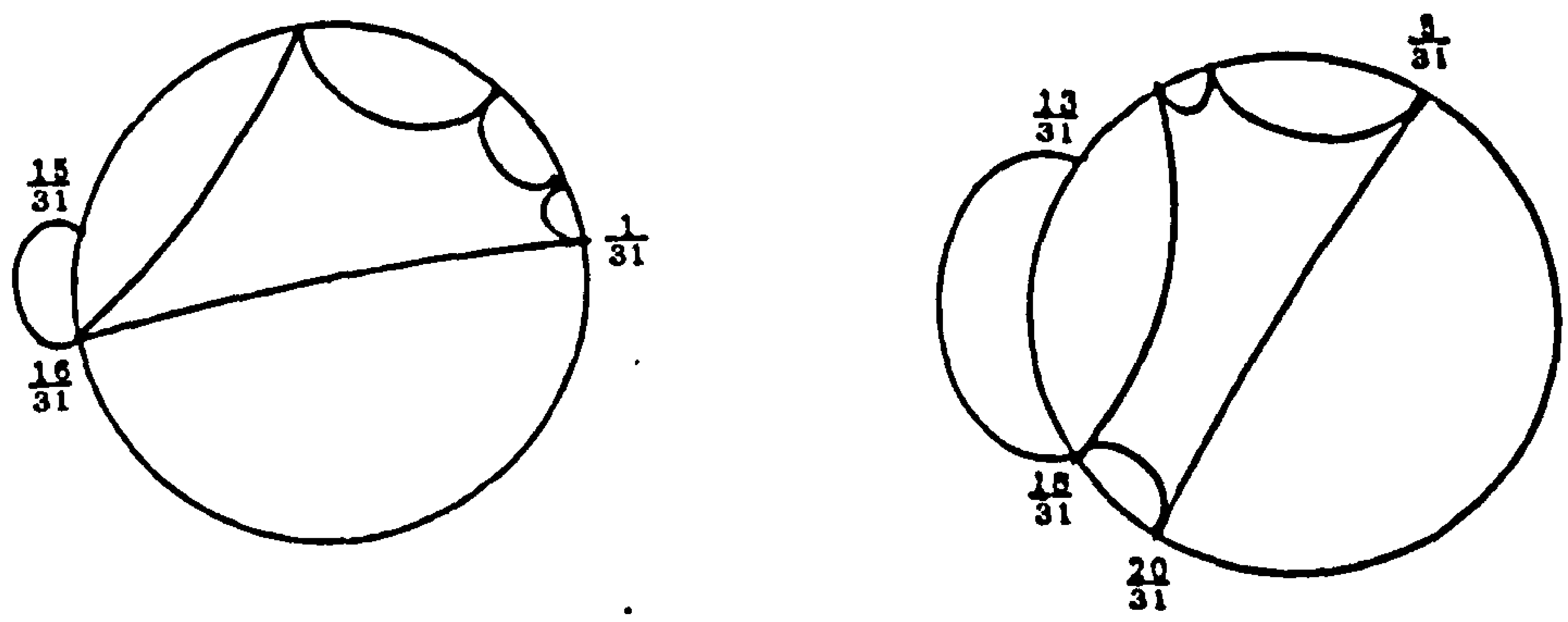


Figure 5.2b

We now construct an invariant circle $C = C(s_q \sqcup s_{1-p})$ (cf. 4.1.2). There are again non-isotopic ways of constructing such a circle, but we will make a standard choice, as follows. The forward orbit of the leaf μ_q forms the polygonal boundary of a region P in $\{z \in \hat{\mathbb{C}} \mid |z| \leq 1\}$ which is invariant up to isotopy.

Let G denote the components of $\{z \in \hat{\mathbb{C}} \mid |z| \geq 1\} \setminus \bigcup \mathcal{L}_p^{-1}$ which intersect $X(s_{1-p})$ and are adjacent to vertices of P. Let us rotate P by a small amount in an anticlockwise direction to get a region P'. Now perturb each component of G, so that a component G_0 becomes G_0', and so that $\overline{G_0'} \cap S^1$ lies strictly inside a component of $S^1 \setminus \{\text{vertices of } P\}$. These intervals for the different components are then disjoint. Then connect each vertex of P' to the nearby component of G', which lies in a clockwise direction from the vertex, by a simple arc, thickened into a small tubular neighbourhood. The resulting construction is a connected set intersecting S^1 in a small arc for each vertex of P'. Let C be the boundary of this set. C is then clearly invariant up to isotopy relative to $s_q \sqcup s_{1-p}$. Note that C separates $X(s_{1-p})$ from $X(s_q)$.

Let ϕ be an orientation-preserving homeomorphism of $\hat{\mathbb{C}}$, which maps C to the unit circle, such that $0 \mapsto 0$ and $\infty \mapsto \infty$. Assuming that ϕ induces an equivalence of branched coverings, let g be the map which we will to show is equivalent to $s_q \sqcup s_{1-p}$ via ϕ. So $g \simeq \phi \circ (s_q \sqcup s_{1-p}) \circ \phi^{-1}$. $\phi(C) = S^1$ must be an invariant circle. Thus $X(s_q)$ is mapped into the unit disc, and this set of image points must be periodic of period n. Similarly $X(s_{1-p})$ is mapped outside of the unit disc to a period n orbit. Now, g is the mating of two polynomial functions, because of the invariant circle $\phi(C)$ (see §2.5). We establish exactly what these are:

For the purposes of the following lemma we choose q on S^1 to be the endpoint of μ_q which maps to the other endpoint under s_q, where this happens.

Lemma 5.2.5. *Let* $x = 2^j(1-q)$ (mod1) *and* $y = 2^{n+1-j}(1-q)$ (mod1). *Then* $s_q \sqcup s_x \simeq s_y \sqcup s_q$, *where* $1 \leq j < n$, *except for the case where* $2^j q$ *is an endpoint of* μ_q.

Proof. P is the invariant region bounded by the n-gon $\{\mu_j(q) \mid 1 \leq j < n\}$ and P' and C are defined in 5.2.4 above, as well as the invariant set of arcs γ_i for $1 \leq i \leq n$ with endpoints on C. These arcs are subsets of S^1 close to the forward orbit of q. Let U_∞ be the component of $\hat{\mathbb{C}} \setminus C$ which contains ∞. Then the set of arcs $\cup_{1\leq i \leq n}\gamma_j$ bound a polygon Π in U_∞ and each component of $U_\infty \setminus \Pi$ contains exactly one point of $X(s_x)$ because μ_x is a minimal minor leaf. Let γ_j also separate $\tau \circ s_x^j \circ \tau(\infty)$ from Π, where $\tau(z) = 1/z$.

Let us write α_i for the arc of C which is close to that side of P where α_i is close to $s_q^{i-1}\mu_q$. Then γ_1 has one endpoint in α_j, and in general γ_i has one endpoint in α_{i+j-1}, where $\alpha_{n+k} = \alpha_n$. then γ_{n+2-j} has one endpoint in α_1. Let β_i be an arc from $s_q^i(0)$ to $\gamma_{n+1+i-j}$ where β_i runs from $s_q^i(0)$ to a neighbourhood of α_i without crossing S^1 or C, and then runs close to α_i to the endpoint $y_i = y'_{n+1+i-j}$ of $\gamma_{n+1+i-j}$ of α_i. Then by the above construction we have an isotopy χ which preserves S^1 and $X(f)$, and $\chi(\beta_{j-1})$ is a component of $f^{-1}(\beta_j)$ for $j > 1$ (thus determined by x_{j-1}) and $\chi(\beta_n)$ is the subarc of $f^{-1}(\beta_1)$ which joins 0 to one of the (two) points $f^{-1}(x_1)$.

There is a homeomorphism ψ_1 of $\hat{\mathbb{C}}$ which maps ∞ to 0, S^1 to itself with orientation reversed, and maps y'_k to $\exp(2\pi i(2^{k-1}q))$. Thus ψ_1 maps y_k to $\exp(2\pi i(2^{n+k-j}q))$ and in particular y_1 to $\exp(2\pi i(2^{n+1-j}q))$. If we write $\psi(z) = (\psi_1(z))^{-1}$, then ψ maps y_1 to $\exp(2\pi i(2^{n+1-j}(1-q)))$. Then by 2.4.3 we see that $s_q \sqcup s_x \simeq s_y \sqcup s_q$. □

We are now in a position to finish the proof of theorem 5.2.3. By 5.2.5, we have connections between pairs of minor leaves μ_a and μ_b, where $a = 2^j(1-q)\bmod 1$ and $b = 2^{n+1-j}(1-q)\bmod 1$. But this connects all of these leaves $\{s_q^j(\exp(2\pi i(1-q)))\mid 1 \leq j < n\}$ together, in particular to the unique minor leaf given by lemma 5.2.4. By symmetry the set of minor leaves $\{s_q^j(\exp(2\pi i q))\mid 1 \leq j < n\}$ is connected in the same way and to the same leaf given by lemma 5.2.4. Thus $\mu_q \bowtie \mu_y$ and $\mu_x \bowtie \mu_q$ by 5.1.3. □

Example (see figure 5.2c): Let $\mu_p = [\![\frac{7}{15}, \frac{8}{15}]\!]$.

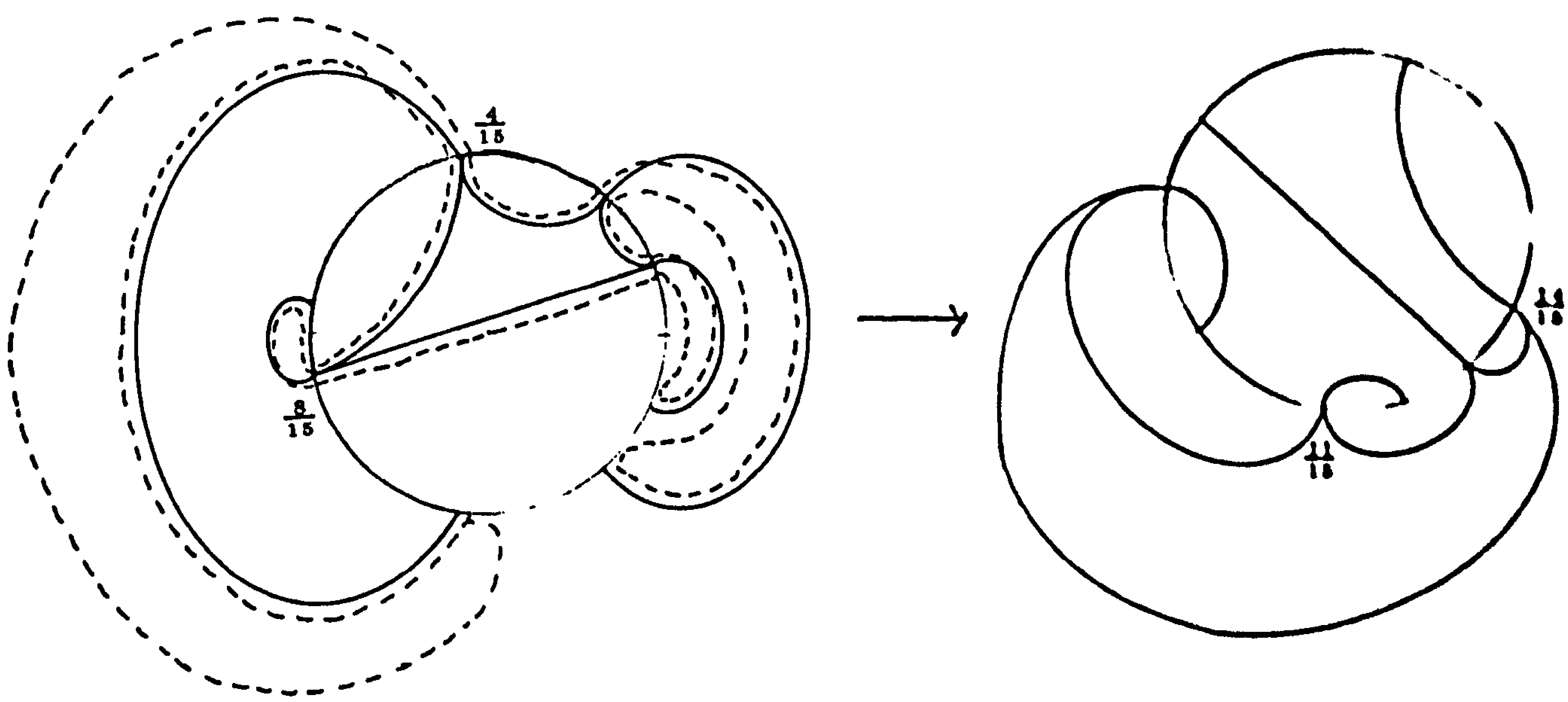


Figure 5.2c

By 5.2.5, when $n = 4$ and $j = 3$ we get $s_{\frac{1}{15}} \sqcup s_{\frac{8}{15}} \simeq s_{\frac{11}{15}} \sqcup s_{\frac{1}{15}}$, and $s_{\frac{14}{15}} \sqcup s_{\frac{7}{15}} \simeq s_{\frac{4}{15}} \sqcup s_{\frac{14}{15}}$. (Taking $\mu_q = [\![\frac{3}{15}, \frac{4}{15}]\!]$ gives us the same equivalences, but inverted.) Thus $[\![\frac{1}{15}, \frac{2}{15}]\!] \bowtie [\![\frac{11}{15}, \frac{12}{15}]\!] \bowtie [\![\frac{7}{15}, \frac{8}{15}]\!] \bowtie [\![\frac{13}{15}, \frac{14}{15}]\!] \bowtie [\![\frac{3}{15}, \frac{4}{15}]\!]$, connecting five of the six period four minor leaves.

However, we do not have sufficient information to prove that $\mathcal{V}_4$ is connected because the leaf $[\![\frac{2}{5}, \frac{3}{5}]\!]$ does not have endpoints on an invariant quadrilateral.

Because of lemma 5.2.4 and 5.2.5 each conjugate pair of period n minimal minor leaves allows us to connect a total of $2n-3$ period n minor leaves. Any two distinct such conjugate pairs do not give rise to minor leaves common to both, except for the symmetric leaf given by lemma 5.2.4. Why not? let ℓ be such a leaf, which has endpoints on conjugate n-gons. Then, $\ell \subset \mathcal{R}([\frac{1}{3}, \frac{2}{3}])$, otherwise it would intersect other leaves in QML. Also, it must be symmetric so that it does not intersect another leaf in its periodic orbit. Then it must be the symmetric leaf of 5.2.4. by the proof of that lemma.

§5.3 Connections for $n \neq k$.

We extend the possibilities for connecting up polynomials by considering minor leaves which are not minimal. There is a strong analogy with Chapter 4, Section 1.

Theorem 5.3.1. *Suppose that $s_q \sqcup s_x \simeq s_y \sqcup s_q$, as determined by 5.2.5 such that μ_q, μ_x and μ_y are all of some period $k < n$. Then if $\mu_{q'} > \mu_q$ and is of period n, $s_{q'} \sqcup s_x \simeq s_{y'} \sqcup s_q$, where $\mu_{y'}$ has the minimal leaf μ_y in QML and $\mu_{q'}$ and $\mu_{y'}$ are of the same period, n.*

Proof. Let the circle C and the arc set $\{\gamma_i \,|\, 1 \leq i \leq n\}$ be defined as in 5.2.5. Lemma 5.2.5 selects the minimal leaf μ_y given μ_q and μ_x (by comparison, in 4.1.2 there is only one possible choice): C and $\bigcup_i \gamma_i$ are again invariant under $s_{q'} \sqcup s_x$ and we have ϕ constructed in the same manner with $s_{q'} \sqcup s_x \simeq_\phi s_{y'} \sqcup s_q$ for some y', and where $\phi(C) = S^1$ and $\phi(\infty) = \infty$. Then there is an arc set $\bigcup_i \phi(\gamma_i)$, invariant under $s_{y'} \sqcup s_q$, which satisfies the conditions of 2.4.2. Thus $\mu_{y'} > \mu_y$. □

Ideally, we would like to sharpen this result, so that we can specify exactly what $\mu_{y'}$ is in general, in an analogous way to §4.4. However, this is in general more involved than for the results we obtained in Chapter 4. Instead, we will show examples of the conclusion of 5.3.1, which will be enough to show the irreducibility of $\mathcal{V}_n$ for particular examples.

Calculation method.

We number arcs of circles C_j in the same way as we did in Theorem 4.1.2 This time, however, the process is complicated by the lack of symmetry inherent in Theorem 5.3.1: Theorem 4.1.2, by comparison, deals with conjugate limbs of M.

All the results that follow rely upon the construction of circles C_j , for some appropriate j, with labellings analogous to those we dealt with in Chapter 4. We do not describe a general algorithm for producing these labellings however, although such an algorithm surely exists. Instead we construct the explicit examples necessary to deal with all minor leaves of period less than or equal to seven.

Let $\Pi_j(f)$ be defined, as in chapter 4, as the set $f^{-j}(\pi_0)$. Also define C_j and $\mathcal{G}_j$ in a similar fashion. Then ϕ_j satisfies the following important properties:

1. $\phi_j(\Pi_j) \subset \{z \in \hat{\mathbb{C}} \mid |z| \geq 1\}$.
2. $\phi_j \circ f(\pi) = g \circ \phi_{j+1}(\pi)$ for all $\pi \in \Pi_{j+1}$.

Note on presentation of results.

We construct equivalences between rational matings. As with non-rational matings, there is the following symmetry: if $s_x \sqcup s_y \simeq_\phi s_a \sqcup s_b$, then $s_{1-x} \sqcup s_{1-y} \simeq_{\overline{\phi}} s_{1-a} \sqcup s_{1-b}$, where $\overline{\phi}(\overline{z}) = \overline{\phi(z)}$. By lemma 5.1.3, we have that $\mu_x \bowtie \mu_a$ and $\mu_y \bowtie \mu_b$ imply that $\mu_{1-x} \bowtie \mu_{1-a}$ and $\mu_{1-y} \bowtie \mu_{1-b}$. Therefore, in the following sections, we will quote results about connections between leaves, often leaving unwritten the connections which follow easily from the above symmetry.

§5.4 The case $k = 3$.

Recall that a standard polygon, for $\mu_q > [\![\frac{1}{7}, \frac{2}{7}]\!]$, is one in $\mathcal{L}_q \cap \mathcal{L}_{[\frac{1}{7}, \frac{2}{7}]}$ (c.f 4.2.). We now establish the labelling on standard polygons beyond the leaf $[\frac{1}{7}, \frac{2}{7}]$. The labelling is common to all standard polygons in the laminations $\mathcal{L}_q$, such that $\mu_q > [\![\frac{1}{7}, \frac{2}{7}]\!]$.

Let $\Pi'_j = \Pi'_j(\mu_q)$ be defined as $\{\pi \in \Pi_j \mid \pi > \mu_p\}$, where μ_p is the minimal leaf $\mu < \mu_q$. We label polygons in Π'_j below, adopting the notation $[x_1, x_2, x_2] \to [y_1, y_2, y_3]$, where the y_i are the labels for the x_i, so that $\phi_j(x_i) = \overline{y_i}$. Note that the order in which the y_i are presented is important.

Theorem 5.4.1. *Let $\mu_q > [\![\frac{1}{7}, \frac{2}{7}]\!]$. For any equivalence of the form $s_q \sqcup s_{\frac{2}{7}} \simeq s_{q'} \sqcup s_{\frac{3}{7}}$, the labellings associated to the triangles in Π'_m, for $3 \leq m \leq 6$, and where these polygons are standard, are as follows:*

Π_3 :

$$[\![\tfrac{9}{56}, \tfrac{11}{56}, \tfrac{15}{56}]\!] \to [\tfrac{29}{56}, \tfrac{23}{56}, \tfrac{25}{56}]$$

Π_4 :

$$[\![\tfrac{23}{112}, \tfrac{25}{112}, \tfrac{29}{112}]\!] \to [\tfrac{71}{112}, \tfrac{65}{112}, \tfrac{67}{112}]$$

Π_5 :

$$[\![\tfrac{37}{224}, \tfrac{39}{224}, \tfrac{43}{224}]\!] \to [\tfrac{113}{224}, \tfrac{107}{224}, \tfrac{109}{224}]$$

$$[\![\tfrac{51}{224}, \tfrac{53}{224}, \tfrac{57}{224}]\!] \to [\tfrac{99}{224}, \tfrac{93}{224}, \tfrac{95}{224}]$$

Π_6 :

$$[\![\tfrac{65}{448}, \tfrac{67}{448}, \tfrac{71}{448}]\!] \to [\tfrac{253}{448}, \tfrac{247}{448}, \tfrac{249}{448}]$$

$$[\![\tfrac{79}{448}, \tfrac{81}{448}, \tfrac{85}{448}]\!] \to [\tfrac{239}{448}, \tfrac{233}{448}, \tfrac{235}{448}]$$

$$[\![\tfrac{93}{448}, \tfrac{95}{448}, \tfrac{99}{448}]\!] \to [\tfrac{281}{448}, \tfrac{275}{448}, \tfrac{277}{448}]$$

$$[\![\tfrac{107}{448}, \tfrac{109}{448}, \tfrac{113}{448}]\!] \to [\tfrac{267}{448}, \tfrac{261}{448}, \tfrac{263}{448}]$$

$$[\![\tfrac{121}{448}, \tfrac{123}{448}, \tfrac{127}{448}]\!] \to [\tfrac{197}{448}, \tfrac{191}{448}, \tfrac{193}{448}]$$

Proof. Let $C_0 = C$, where C is as described in 5.2.4.

Let us consider $\mu_q > [\![\frac{1}{7}, \frac{2}{7}]\!]$. Then $\pi_0 = [\![\frac{1}{7}, \frac{2}{7}, \frac{4}{7}]\!]$ and the sides of π_0 are labelled by $\frac{4}{7}, \frac{1}{7}$ and $\frac{2}{7}$ respectively: i.e., $\phi_0(\exp(\frac{2\pi i}{7})) = \exp(\frac{6\pi i}{7})$ etc. This is a direct consequence of 5.2.1.

Then this forces (by property 2 of ϕ_1) that the leaves of $\pi_1 = [\![\frac{1}{14}, \frac{9}{14}, \frac{11}{14}]\!]$ are labelled by $\frac{11}{14}, \frac{1}{14}$ and $\frac{9}{14}$ respectively. i.e., $\phi_1(\exp(\frac{2\pi i}{14})) = \exp(\frac{6\pi i}{14})$ etc.

As we can see from figure 5.4a, there is an element of $\mathcal{G}_1$ joining π_0 to π_1, the boundaries of which define part of C_1. Now, the two triangles of Π_2 are connected by elements of $\mathcal{G}_2$, which must not intersect the element of $\mathcal{G}_1$. This means that the triangle $[\![\frac{9}{28}, \frac{11}{28}, \frac{15}{28}]\!]$ is labelled (by ϕ_2) $(\frac{1}{28}, \frac{23}{28}, \frac{25}{28})$ and vice-versa. e.g., $\phi_2(\exp(\frac{2\pi i}{28})) = \exp(\frac{38\pi i}{28})$.

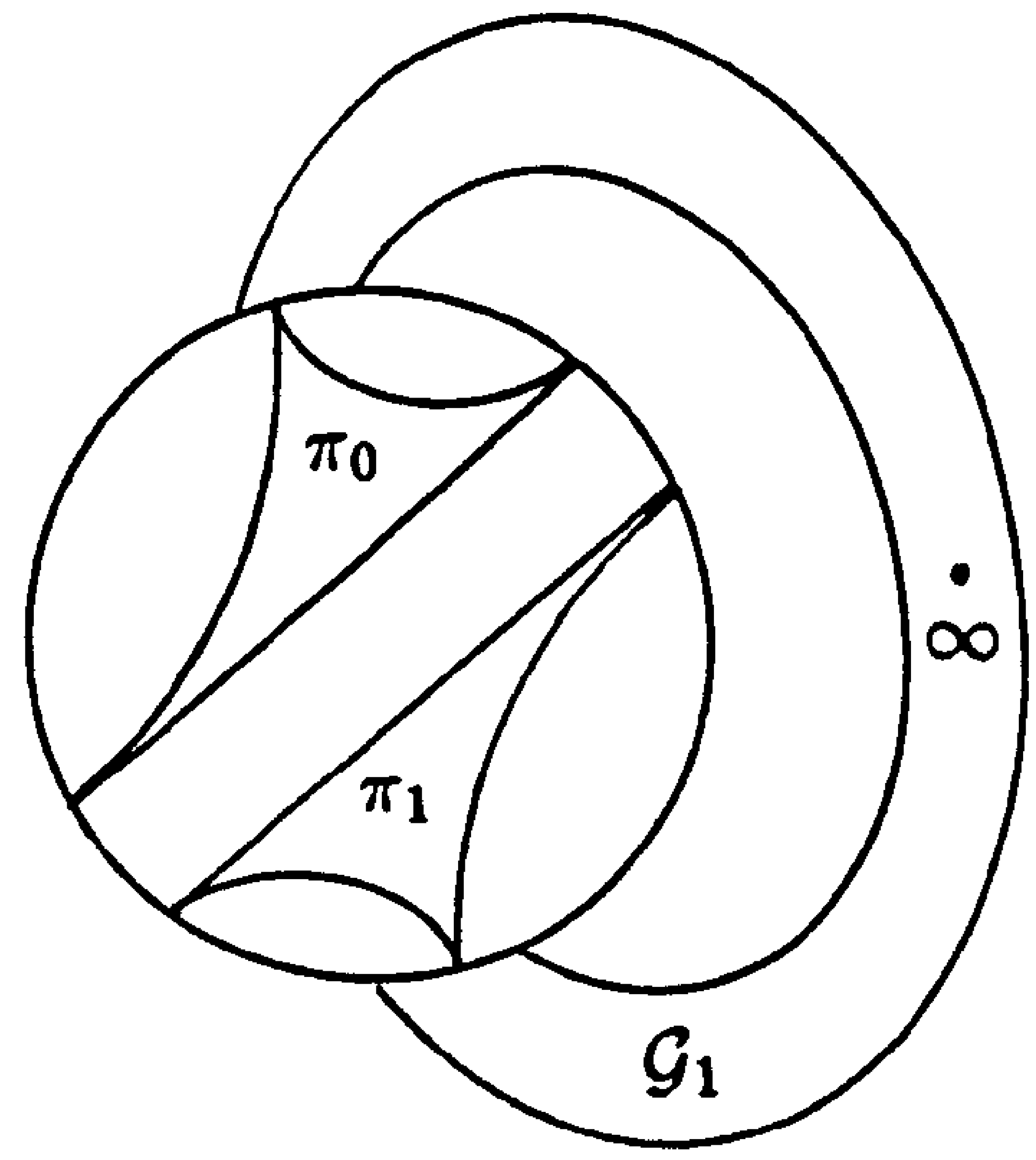


Figure 5.4a

We continue by observing the relevant connections via elements of $\mathcal{G}_j$. (See figure 5.4b, where elements of $\mathcal{G}_j$ are represented by the lines across the middle third of 5.4b, and the upper and lower thirds are parts of the unit disc in $\hat{\mathbb{C}}$.) The triangle (in Π_3) $\llbracket \frac{9}{56}, \frac{11}{56}, \frac{15}{56} \rrbracket$ is labelled $(\frac{29}{56}, \frac{23}{56}, \frac{25}{56})$: It is connected to $\llbracket \frac{23}{28}, \frac{25}{28}, \frac{1}{28} \rrbracket$ by an element of $\mathcal{G}_3$. Note that it must be this triangle connected to $\llbracket \frac{23}{28}, \frac{25}{28}, \frac{1}{28} \rrbracket$ because all other triangles in Π_3 are separated by $\mathcal{G}_1$ or the element of $\mathcal{G}_2$ illustrated (of course, no two elements of $\mathcal{G}$ can intersect). Each further connection illustrated is forced in exactly the same way:

$\llbracket \frac{23}{112}, \frac{25}{112}, \frac{29}{112} \rrbracket$ is connected between $\llbracket \frac{1}{7}, \frac{2}{7}, \frac{4}{7} \rrbracket$ and $\llbracket \frac{9}{14}, \frac{11}{14}, \frac{1}{14} \rrbracket$ by some $g \in \mathcal{G}_4$. $\llbracket \frac{37}{224}, \frac{39}{224}, \frac{43}{224} \rrbracket$ is connected to $\llbracket \frac{93}{112}, \frac{95}{112}, \frac{99}{112} \rrbracket$ by some $g \in \mathcal{G}_5$, which in turn is connected to $\llbracket \frac{9}{56}, \frac{11}{56}, \frac{15}{56} \rrbracket$ by some $g \in \mathcal{G}_4$. $\llbracket \frac{51}{224}, \frac{53}{224}, \frac{57}{224} \rrbracket$ is connected to $\llbracket \frac{9}{56}, \frac{11}{56}, \frac{15}{56} \rrbracket$ by some $g \in \mathcal{G}_5$. $\llbracket \frac{65}{448}, \frac{67}{448}, \frac{71}{448} \rrbracket$ is connected between $\llbracket \frac{1}{7}, \frac{2}{7}, \frac{4}{7} \rrbracket$ and $\llbracket \frac{23}{28}, \frac{25}{28}, \frac{1}{28} \rrbracket$ by elements of $\mathcal{G}_6$ and $\mathcal{G}_5$. $\llbracket \frac{79}{448}, \frac{81}{448}, \frac{85}{448} \rrbracket$ is connected between $\llbracket \frac{23}{28}, \frac{25}{28}, \frac{1}{28} \rrbracket$ $\llbracket \frac{9}{56}, \frac{11}{56}, \frac{15}{56} \rrbracket$ by some $g \in \mathcal{G}_6$. $\llbracket \frac{93}{448}, \frac{95}{448}, \frac{99}{448} \rrbracket$ is connected to $\llbracket \frac{23}{112}, \frac{25}{112}, \frac{29}{112} \rrbracket$ by elements of $\mathcal{G}_6$ and $\mathcal{G}_5$. $\llbracket \frac{107}{448}, \frac{109}{448}, \frac{113}{448} \rrbracket$ is connected to $\llbracket \frac{23}{112}, \frac{25}{112}, \frac{29}{112} \rrbracket$ by some $g \in \mathcal{G}_6$. $\llbracket \frac{121}{448}, \frac{123}{448}, \frac{127}{448} \rrbracket$ is connected to $\llbracket \frac{51}{112}, \frac{53}{112}, \frac{57}{112} \rrbracket$ by some $g \in \mathcal{G}_6$.

All the labellings on this set of triangles are then forced by the labellings on the triangles they are connected to or between, as quoted above. $\square$

This is sufficient for the following:

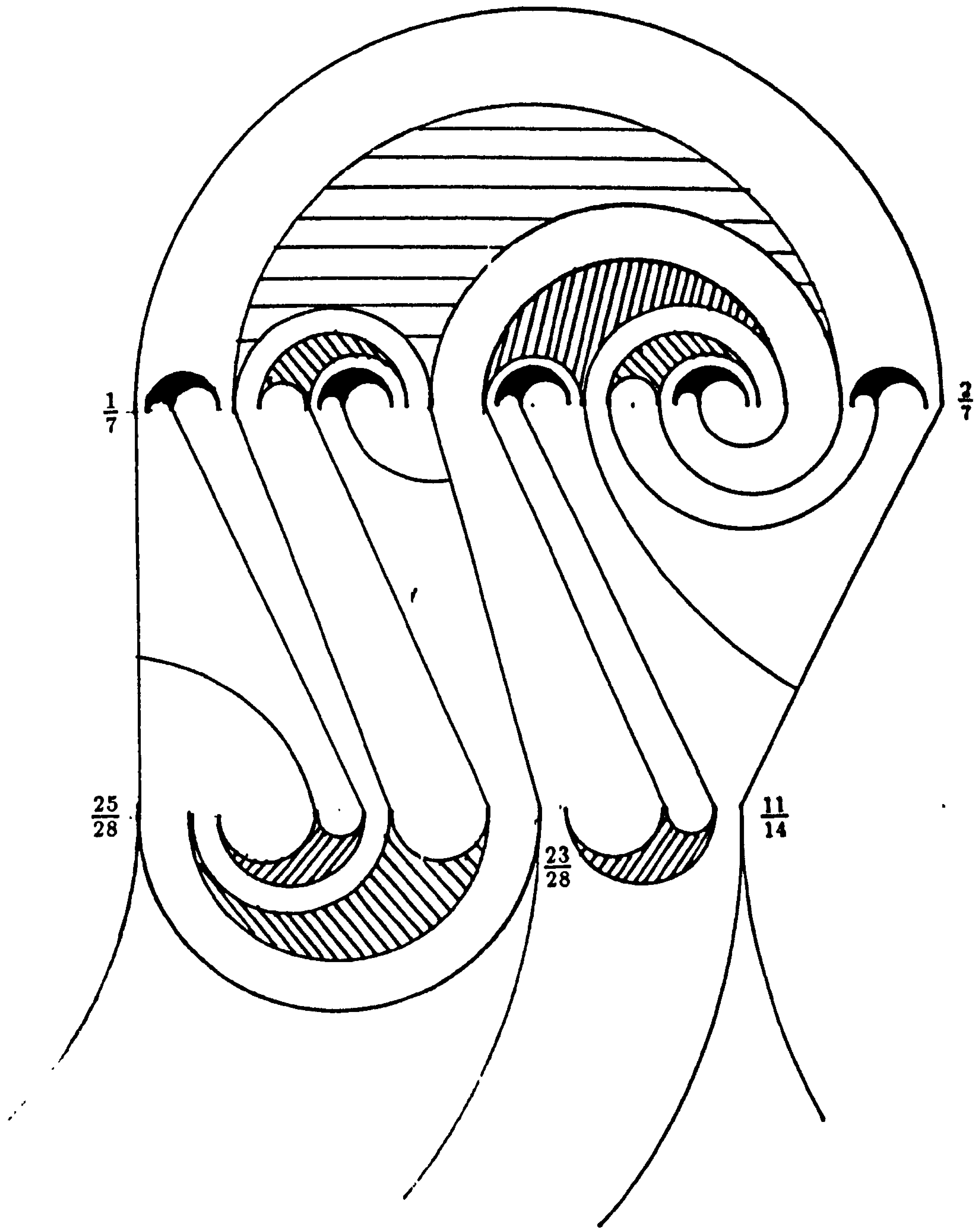


Figure 5.4b

Corollary 5.4.2. *The following connections occur:*

$$[\![\tfrac{3}{15},\tfrac{4}{15}]\!] \bowtie [\![\tfrac{2}{5},\tfrac{3}{5}]\!]$$

$$[\![\tfrac{5}{31},\tfrac{6}{31}]\!] \bowtie [\![\tfrac{15}{31},\tfrac{16}{31}]\!]$$

$$[\![\tfrac{7}{31},\tfrac{8}{31}]\!] \bowtie [\![\tfrac{13}{31},\tfrac{18}{31}]\!]$$

$$[\![\tfrac{10}{63},\tfrac{17}{63}]\!] \bowtie [\![\tfrac{28}{63},\tfrac{35}{63}]\!]$$

$$[\![\tfrac{11}{63},\tfrac{12}{63}]\!] \bowtie [\![\tfrac{30}{63},\tfrac{33}{63}]\!]$$

$$[\![\tfrac{13}{63},\tfrac{14}{63}]\!] \bowtie [\![\tfrac{23}{63},\tfrac{24}{63}]\!]$$

$$[\![\tfrac{15}{63},\tfrac{16}{63}]\!] \bowtie [\![\tfrac{26}{63},\tfrac{37}{63}]\!]$$

$$[\![\tfrac{19}{127},\tfrac{20}{127}]\!] \bowtie [\![\tfrac{57}{127},\tfrac{70}{127}]\!]$$

$$[\![\tfrac{21}{127},\tfrac{22}{127}]\!] \bowtie [\![\tfrac{63}{127},\tfrac{64}{127}]\!]$$

$$[\![\tfrac{23}{127},\tfrac{24}{127}]\!] \bowtie [\![\tfrac{61}{127},\tfrac{66}{127}]\!]$$

$$[\![\tfrac{25}{127},\tfrac{34}{127}]\!] \bowtie [\![\tfrac{75}{127},\tfrac{76}{127}]\!]$$

$$[\![\tfrac{26}{127},\tfrac{33}{127}]\!] \bowtie [\![\tfrac{51}{127},\tfrac{52}{127}]\!]$$

$$[\![\tfrac{27}{127},\tfrac{28}{127}]\!] \bowtie [\![\tfrac{46}{127},\tfrac{49}{127}]\!]$$

$$[\![\tfrac{29}{127},\tfrac{30}{127}]\!] \bowtie [\![\tfrac{71}{127},\tfrac{72}{127}]\!]$$

$$[\![\tfrac{31}{127},\tfrac{32}{127}]\!] \bowtie [\![\tfrac{53}{127},\tfrac{74}{127}]\!]$$

$$[\![\tfrac{35}{127},\tfrac{36}{127}]\!] \bowtie [\![\tfrac{54}{127},\tfrac{73}{127}]\!]$$

Note: We have omitted the results which follow easily by symmetry. e.g., $[\![\tfrac{59}{63},\tfrac{60}{63}]\!] \bowtie [\![\tfrac{5}{63},\tfrac{6}{63}]\!]$. We shall also do this in many of the results which follow.

Proof. The homeomorphisms ϕ_j: What we do is to establish, for a given period n minor leaf, what value of j is sufficiently large to establish the leaf $\mu_{q'}$ uniquely, where $s_q \sqcup s_{\frac{6}{7}} \simeq_{\phi_j} s_{q'} \sqcup s_{\frac{3}{7}}$ and q is an endpoint of one of the leaves on the left above.

Let q be an odd-denominator rational. Then let $A(q)$ denote the arc of S^1 from $\exp(2\pi i q)$ to $\exp(2\pi i q')$, where q' is the next point in the periodic orbit of q following the circle anti-clockwise. Then, we have a set of n arcs of the form $(2^j q \pmod 1, 2^j q' \pmod 1)$. Let an arc (a_1, a_2) be called critical if the angle measured from a_1 to a_2 anti-clockwise is more than half of the circumference of the unit circle. An odd-denominator rational is determined uniquely by the order of its forward iterates under $z \mapsto 2z \pmod 1$ on $\{z \mid z \in [0,1]/0 \sim 1\}$ together with the placing of critical arcs between consecutive iterates. e.g., the unique period 4 number under $z \mapsto 2z \pmod 1$ for which there is a critical arc between the third and fourth iterates is $\frac{1}{15}$ — critical arc is $(\frac{8}{15}, \frac{1}{15})$.

Theorem 5.4.1 gives us $\phi_j(x)$ for x on appropriate polygons in Π'_j. These are pre-periodic even-denominator rationals — we aim to use these to approximate periodic odd-denominator rationals, using the principle that an even-denominator rational with the same order of n iterates on S^1 together with the same critical arc arrangement determines that unique odd-denominator rational. We attach the leaf μ_q to some pre-periodic leaf in $\pi \subset \Pi'_j$, which maps forward to the fixed triangle. We justify the above principle: let x be the pre-periodic approximation to q periodic, which satisfies the above order and critical gap condition. Then $s_x \sqcup s_{\frac{1}{7}} \simeq s_q \sqcup s_{\frac{1}{7}}$, by the continuous deformation of x to q in a way which does not affect the equivalence class: we can define arcs β_j as the straight line arcs which join $s_x^j(0)$ to $2^{j-1}q$, and apply 2.4.3 to obtain the equivalence.

This gives us an equivalence of the form $s_q \sqcup s_{\frac{1}{7}} \simeq s_{q'} \sqcup s_{\frac{3}{7}}$, which by 5.1.3 means that $\mu_q \bowtie \mu_{q'}$. This is the approach we take generally.

We show that $s_{\frac{1}{5}} \sqcup s_{\frac{3}{7}} \simeq s_{\frac{2}{5}} \sqcup s_{\frac{1}{7}}$. The leaf $[\![\frac{3}{15}, \frac{4}{15}]\!]$ is approximated by the pre-periodic leaf $[\![\frac{11}{56}, \frac{15}{56}]\!]$, which by 5.4.1 is labelled by $\frac{23}{56}$. Thus $\phi_3(\exp(2\pi i \frac{11}{56})) = \exp(2\pi i \frac{33}{56})$. Consider the first four iterates of $\frac{33}{56}$ under $z \mapsto 2z \pmod 1$:

$$\frac{33}{56} \mapsto \frac{5}{28} \mapsto \frac{5}{14} \mapsto \frac{5}{7}.$$

Compare to the periodic orbit of $\frac{3}{5}$:

$$\frac{3}{5} \mapsto \frac{1}{5} \mapsto \frac{2}{5} \mapsto \frac{4}{5}.$$

In both cases the order of points in S^1 is 1423, where the digit j refers to the j-th number in the above iterated sequence. The arc $A(\frac{33}{56})$ is $(\frac{33}{56}, \frac{5}{7})$, corresponding to $A(\frac{3}{5}) = (\frac{3}{5}, \frac{4}{5})$. The critical arcs are $A(\frac{5}{14})$ and $A(\frac{5}{7})$, which correspond to $A(\frac{2}{5})$ and $A(\frac{4}{5})$ respectively. Thus our condition is satisfied and $[\frac{3}{15}, \frac{4}{15}] \bowtie [\frac{2}{5}, \frac{3}{5}]$.

We pick out some of the results. All the others follow straightforwardly as above. Consider the period 6 tuning $[\frac{10}{63}, \frac{17}{63}]$: this is approximated by the leaf $[\frac{1}{7}, \frac{2}{7}]$, labelled $\frac{4}{7}$, whose orbit shadows that of $\frac{35}{63}$.

$$\frac{4}{7} \mapsto \frac{1}{7} \mapsto \frac{2}{7} \mapsto \frac{4}{7} \mapsto \frac{1}{7} \mapsto \frac{2}{7}$$

However this does not help us, because of the repetition of points in the period six orbit, so we use a more direct method, using the fact that $[\frac{10}{63}, \frac{17}{63}]$ is a tuning: $s_{\frac{10}{63}} = s_{\frac{1}{3}} \vdash s_{\frac{1}{7}}$. Then $(s_{\frac{1}{3}} \vdash s_{\frac{1}{7}}) \sqcup s_{\frac{3}{7}} \simeq (s_{\frac{3}{7}} \vdash s_{\frac{1}{3}}) \sqcup s_{\frac{1}{7}}$. And $s_{\frac{3}{7}} \vdash s_{\frac{1}{3}} = s_{\frac{28}{63}}$.

Next, we consider the case where the standard polygons (as labelled by theorem 5.4.1) do not suffice. We need to find the labellings on some thin non-standard (see §4.4 for terminology) polygons for the maps where μ_q is $[\frac{25}{127}, \frac{34}{127}]$ or $[\frac{26}{127}, \frac{33}{127}]$ — these are type (2) minor leaves in the terminology of §4.4 and the only examples of this type we consider. For these two minor leaves the pre-periodic leaf $[\frac{11}{56}, \frac{15}{56}]$ is not sufficient to determine $\mu_{q'}$ (the third and sixth iterate are identical).

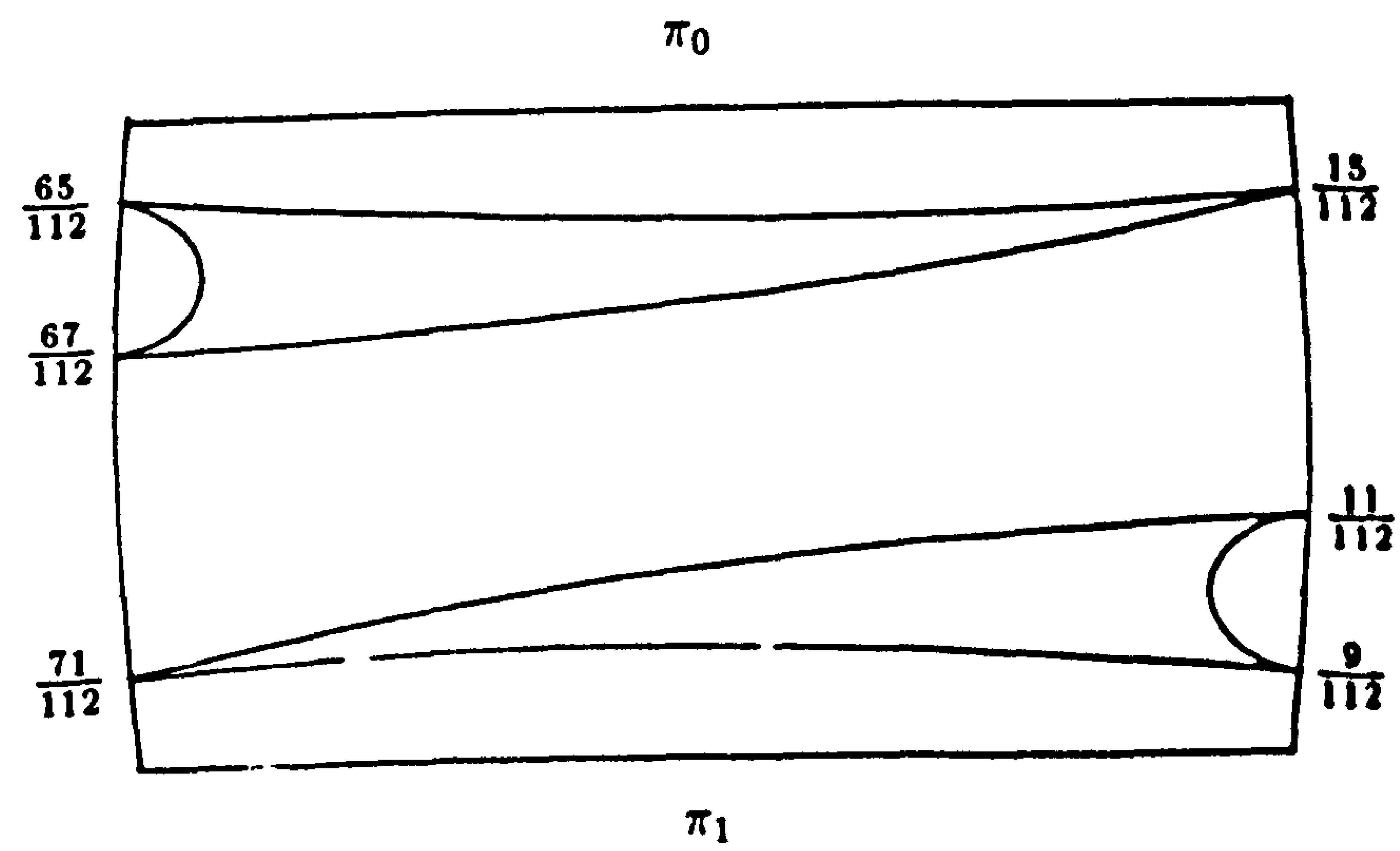


Figure 5.4c

The thin polygons we have to use are pre-images of $[\frac{9}{112}, \frac{11}{112}, \frac{71}{112}]$ and $[\frac{15}{112}, \frac{65}{112}, \frac{67}{112}]$, which are labelled $[\frac{85}{112}, \frac{79}{112}, \frac{81}{112}]$ and $[\frac{25}{112}, \frac{29}{112}, \frac{23}{112}]$ respectively — see figure 5.4c. Pulling these back three times under $s_q \sqcup s_{\frac{3}{7}}$ we can select unambiguously the correct labelling relative to the standard polygons already dealt with to get:

Π_7 : $$[\tfrac{177}{896}, \tfrac{179}{896}, \tfrac{239}{896}] \to [\tfrac{533}{896}, \tfrac{527}{896}, \tfrac{529}{896}]$$

$$\left[\tfrac{233}{896}, \tfrac{235}{896}, \tfrac{183}{896}\right] \to \left[\tfrac{361}{896}, \tfrac{365}{896}, \tfrac{359}{896}\right].$$

Then for $[\frac{25}{127}, \frac{34}{127}]$ we get $\frac{535}{896} \mapsto \frac{87}{448} \mapsto \frac{87}{224} \mapsto \frac{87}{112} \mapsto \frac{31}{56} \mapsto \frac{3}{28} \mapsto \frac{3}{14}$, which corresponds to the orbit $\frac{76}{127} \mapsto \frac{25}{127} \mapsto \frac{50}{127} \mapsto \frac{100}{127} \mapsto \frac{73}{127} \mapsto \frac{19}{127} \mapsto \frac{38}{127}$.

Also for $[\frac{26}{127}, \frac{33}{127}]$ we get $\frac{367}{896} \mapsto \frac{367}{448} \mapsto \frac{143}{224} \mapsto \frac{31}{112} \mapsto \frac{31}{56} \mapsto \frac{3}{28} \mapsto \frac{3}{14}$, which corresponds to the orbit $\frac{51}{127} \mapsto \frac{102}{127} \mapsto \frac{77}{127} \mapsto \frac{27}{127} \mapsto \frac{54}{127} \mapsto \frac{108}{127} \mapsto \frac{89}{127}$;
or $\frac{45}{112} \mapsto \frac{45}{56} \mapsto \frac{17}{28} \mapsto \frac{3}{14} \mapsto \frac{3}{7} \mapsto \frac{6}{7} \mapsto \frac{5}{7}$, which corresponds to the orbit $\frac{52}{127} \mapsto \frac{104}{127} \mapsto \frac{81}{127} \mapsto \frac{35}{127} \mapsto \frac{70}{127} \mapsto \frac{13}{127} \mapsto \frac{26}{127}$. □

Corollary 5.4.3. *$\mathcal{V}_4$ is connected.*

Proof. By the example after 5.2.5, it is sufficient to show that $[\frac{3}{15}, \frac{4}{15}] \bowtie [\frac{2}{5}, \frac{3}{5}]$, which follows by 5.4.2. □

§5.5 The case $k = 4$.

We have the following chains of connection for $\mathcal{V}_5$ by 5.2.5:

$$[\tfrac{1}{31}, \tfrac{2}{31}] \bowtie [\tfrac{3}{31}, \tfrac{4}{31}] \bowtie [\tfrac{7}{31}, \tfrac{8}{31}] \bowtie [\tfrac{15}{31}, \tfrac{16}{31}] \bowtie [\tfrac{23}{31}, \tfrac{24}{31}] \bowtie [\tfrac{27}{31}, \tfrac{28}{31}] \bowtie [\tfrac{29}{31}, \tfrac{30}{31}]$$

$$[\tfrac{9}{31}, \tfrac{10}{31}] \bowtie [\tfrac{5}{31}, \tfrac{6}{31}] \bowtie [\tfrac{19}{31}, \tfrac{20}{31}] \bowtie [\tfrac{13}{31}, \tfrac{18}{31}] \bowtie [\tfrac{25}{31}, \tfrac{26}{31}] \bowtie [\tfrac{11}{31}, \tfrac{12}{31}] \bowtie [\tfrac{21}{31}, \tfrac{22}{31}]$$

$[\frac{14}{31}, \frac{17}{31}]$ is still isolated. This leaves us with $\mathcal{V}_5$ in at most three connected pieces.

Theorem 5.5.1. *Let $\mu_q > [\frac{1}{15}, \frac{2}{15}]$. For the equivalences $s_q \sqcup s_{\frac{7}{15}} \simeq s_{q'} \sqcup s_{\frac{1}{15}}$ and $s_q \sqcup s_{\frac{11}{15}} \simeq s_{q''} \sqcup s_{\frac{1}{15}}$, the labellings associated to the quadrilaterals in Π'_m, for $4 \le m \le 6$, are as follows:*

$s_q \sqcup s_{\frac{7}{15}}$:

Π_4 : $\left[\frac{17}{240}, \frac{19}{240}, \frac{23}{240}, \frac{31}{240}\right] \to \left[\frac{53}{240}, \frac{61}{240}, \frac{47}{240}, \frac{49}{240}\right]$

Π_5 : $\left[\frac{47}{480}, \frac{49}{480}, \frac{53}{480}, \frac{61}{480}\right] \to \left[\frac{83}{480}, \frac{91}{480}, \frac{77}{480}, \frac{79}{480}\right]$

Π_6 : $\left[\frac{77}{960}, \frac{79}{960}, \frac{83}{960}, \frac{91}{960}\right] \to \left[\frac{233}{960}, \frac{241}{960}, \frac{227}{960}, \frac{229}{960}\right]$

$\left[\frac{107}{960}, \frac{109}{960}, \frac{113}{960}, \frac{121}{960}\right] \to \left[\frac{263}{960}, \frac{271}{960}, \frac{257}{960}, \frac{259}{960}\right]$

$s_q \sqcup s_{\frac{11}{16}}$:

$$\Pi_4: \qquad [\![\tfrac{17}{240}, \tfrac{19}{240}, \tfrac{23}{240}, \tfrac{31}{240}]\!] \to [\tfrac{121}{240}, \tfrac{107}{240}, \tfrac{109}{240}, \tfrac{113}{240}]$$

$$\Pi_5: \qquad [\![\tfrac{47}{480}, \tfrac{49}{480}, \tfrac{53}{480}, \tfrac{61}{480}]\!] \to [\tfrac{271}{480}, \tfrac{257}{480}, \tfrac{259}{480}, \tfrac{263}{480}]$$

$$\Pi_6: \qquad [\![\tfrac{77}{960}, \tfrac{79}{960}, \tfrac{83}{960}, \tfrac{91}{960}]\!] \to [\tfrac{541}{960}, \tfrac{527}{960}, \tfrac{529}{960}, \tfrac{533}{960}]$$

$$[\![\tfrac{107}{960}, \tfrac{109}{960}, \tfrac{113}{960}, \tfrac{121}{960}]\!] \to [\tfrac{451}{960}, \tfrac{437}{960}, \tfrac{439}{960}, \tfrac{443}{960}]$$

Proof. Refer to figures 5.5a and 5.5b. All the connections are forced. □

Corollary 5.5.2. *The following connections occur (as well the symmetric ones):*

$$[\tfrac{3}{31}, \tfrac{4}{31}] \bowtie [\tfrac{25}{31}, \tfrac{26}{31}]$$
$$[\tfrac{5}{63}, \tfrac{6}{63}] \bowtie [\tfrac{47}{63}, \tfrac{48}{63}]$$
$$[\tfrac{7}{63}, \tfrac{8}{63}] \bowtie [\tfrac{46}{63}, \tfrac{53}{63}]$$
$$[\tfrac{9}{127}, \tfrac{10}{127}] \bowtie [\tfrac{99}{127}, \tfrac{100}{127}]$$
$$[\tfrac{11}{127}, \tfrac{12}{127}] \bowtie [\tfrac{97}{127}, \tfrac{98}{127}]$$
$$[\tfrac{13}{127}, \tfrac{14}{127}] \bowtie [\tfrac{103}{127}, \tfrac{104}{127}],$$
$$[\tfrac{15}{127}, \tfrac{16}{127}] \bowtie [\tfrac{93}{127}, \tfrac{102}{127}]$$
$$[\tfrac{3}{31}, \tfrac{4}{31}] \bowtie [\tfrac{14}{31}, \tfrac{17}{31}]$$
$$[\tfrac{5}{63}, \tfrac{6}{63}] \bowtie [\tfrac{28}{63}, \tfrac{35}{63}]$$
$$[\tfrac{7}{63}, \tfrac{8}{63}] \bowtie [\tfrac{29}{63}, \tfrac{34}{63}]$$
$$[\tfrac{9}{127}, \tfrac{10}{127}] \bowtie [\tfrac{63}{127}, \tfrac{64}{127}]$$
$$[\tfrac{11}{127}, \tfrac{12}{127}] \bowtie [\tfrac{57}{127}, \tfrac{70}{127}]$$
$$[\tfrac{13}{127}, \tfrac{14}{127}] \bowtie [\tfrac{59}{127}, \tfrac{68}{127}]$$
$$[\tfrac{15}{127}, \tfrac{16}{127}] \bowtie [\tfrac{58}{127}, \tfrac{69}{127}]$$

Proof. The results follow as for 5.4.2. — the standard polygon pre-periodic leaves are sufficient to determine all the above connections, so there are no exceptional cases as in 5.4.2. □

Corollary 5.5.3. *V_5 is connected.*

Figure 5.5a

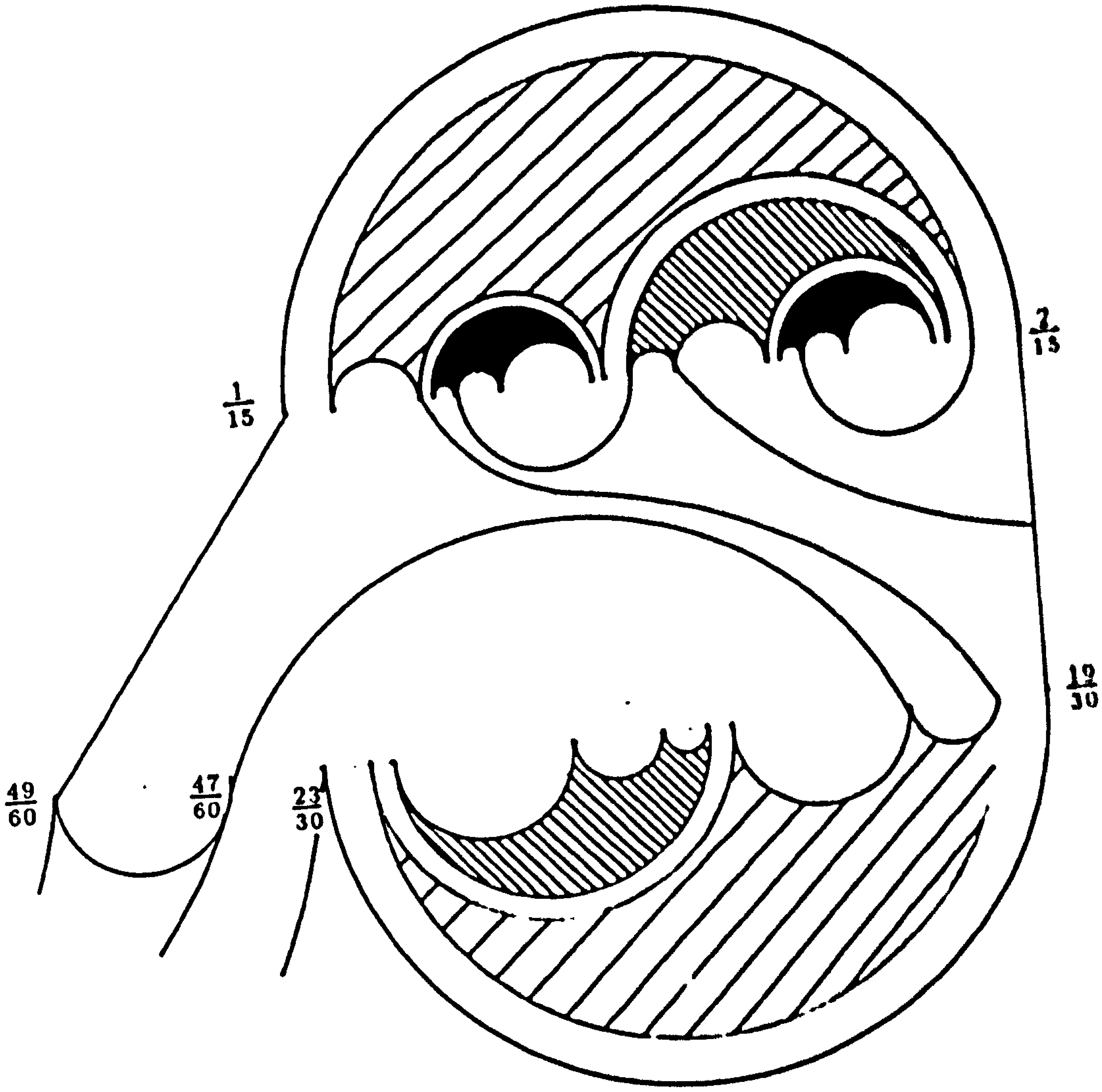
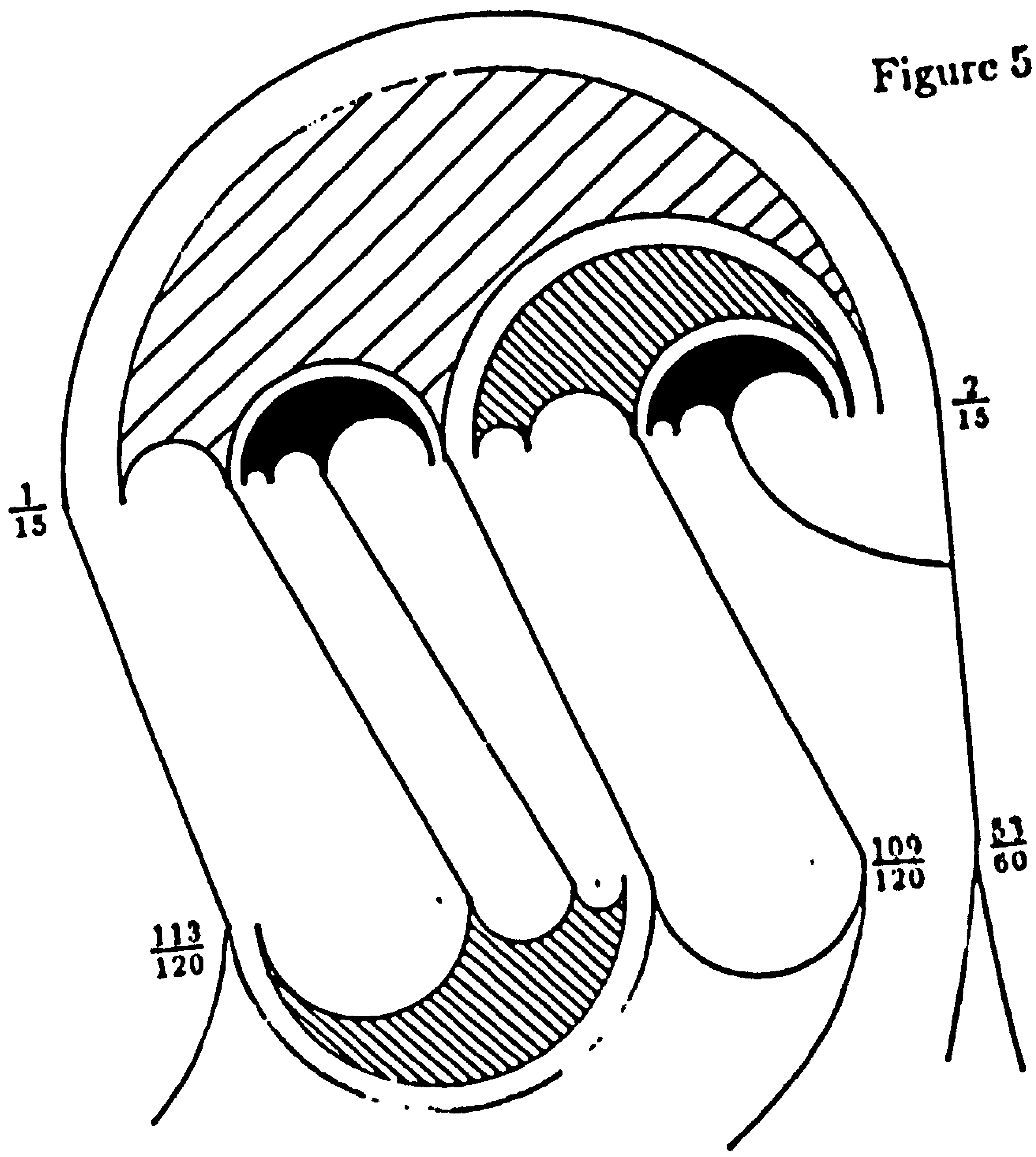


Figure 5.5b



Proof. By 5.5.2, $[\![\frac{3}{31},\frac{4}{31}]\!] \bowtie [\![\frac{25}{31},\frac{26}{31}]\!]$, connecting all the leaves except for one (see above example at start of this section). Also by 5.5.2, $[\![\frac{3}{31},\frac{4}{31}]\!] \bowtie [\![\frac{14}{31},\frac{17}{31}]\!]$. □

§5.6 The case $k = 5$.

By 5.2.5:

$$[\![\tfrac{1}{63},\tfrac{2}{63}]\!] \bowtie [\![\tfrac{3}{63},\tfrac{4}{63}]\!] \bowtie [\![\tfrac{7}{63},\tfrac{8}{63}]\!] \bowtie [\![\tfrac{15}{63},\tfrac{16}{63}]\!] \bowtie [\![\tfrac{31}{63},\tfrac{32}{63}]\!]$$
$$\bowtie [\![\tfrac{47}{63},\tfrac{48}{63}]\!] \bowtie [\![\tfrac{55}{63},\tfrac{56}{63}]\!] \bowtie [\![\tfrac{59}{63},\tfrac{60}{63}]\!] \bowtie [\![\tfrac{61}{63},\tfrac{62}{63}]\!]$$

Theorem 5.6.1. *Let $\mu_q > [\![\frac{1}{31},\frac{2}{31}]\!]$ For the equivalences $s_q \sqcup s_{\frac{27}{31}} \simeq s_{q'} \sqcup s_{\frac{1}{31}}$, $s_q \sqcup s_{\frac{23}{31}} \simeq s_{q'} \sqcup s_{\frac{1}{31}}$, and $s_q \sqcup s_{\frac{15}{31}} \simeq s_{q'} \sqcup s_{\frac{1}{31}}$, the labellings associated to the pentagons in Π'_m, for $5 \le m \le 6$, are respectively as follows:*

$s_q \sqcup s_{\frac{15}{31}}$:

Π_5 : $\quad [\![\frac{33}{992},\frac{35}{992},\frac{39}{992},\frac{47}{992},\frac{63}{992}]\!] \to [\frac{101}{992},\frac{109}{992},\frac{125}{992},\frac{95}{992},\frac{97}{992}]$

Π_6 : $\quad [\![\frac{95}{1984},\frac{97}{1984},\frac{101}{1984},\frac{109}{1984},\frac{125}{1984}]\!] \to [\frac{163}{1984},\frac{171}{1984},\frac{187}{1984},\frac{157}{1984},\frac{159}{1984}]$

$s_q \sqcup s_{\frac{23}{31}}$:

Π_5 : $\quad [\![\frac{33}{992},\frac{35}{992},\frac{39}{992},\frac{47}{992},\frac{63}{992}]\!] \to [\frac{233}{992},\frac{249}{992},\frac{219}{992},\frac{221}{992},\frac{225}{992}]$

Π_6 : $\quad [\![\frac{95}{1984},\frac{97}{1984},\frac{101}{1984},\frac{109}{1984},\frac{125}{1984}]\!] \to [\frac{419}{1984},\frac{435}{1984},\frac{405}{1984},\frac{407}{1984},\frac{411}{1984}]$

$s_q \sqcup s_{\frac{27}{31}}$:

Π_5 $\quad [\![\frac{33}{992},\frac{35}{992},\frac{39}{992},\frac{47}{992},\frac{63}{992}]\!] \to [\frac{497}{992},\frac{467}{992},\frac{469}{992},\frac{473}{992},\frac{481}{992}]$

Π_6 $\quad [\![\frac{95}{1984},\frac{97}{1984},\frac{101}{1984},\frac{109}{1984},\frac{125}{1984}]\!] \to [\frac{1055}{1984},\frac{1025}{1984},\frac{1027}{1984},\frac{1031}{1984},\frac{1039}{1984}]$

Proof. This follows in a similar way to 5.4.1. □

Corollary 5.6.2. *The following connections occur:*

$$[\tfrac{3}{63}, \tfrac{4}{63}] \bowtie [\tfrac{57}{63}, \tfrac{58}{63}]$$

$$[\tfrac{3}{63}, \tfrac{4}{63}] \bowtie [\tfrac{49}{63}, \tfrac{50}{63}]$$

$$[\tfrac{3}{63}, \tfrac{4}{63}] \bowtie [\tfrac{30}{63}, \tfrac{33}{63}]$$

$$[\tfrac{5}{127}, \tfrac{6}{127}] \bowtie [\tfrac{111}{127}, \tfrac{112}{127}]$$

$$[\tfrac{5}{127}, \tfrac{6}{127}] \bowtie [\tfrac{99}{127}, \tfrac{100}{127}]$$

$$[\tfrac{5}{127}, \tfrac{6}{127}] \bowtie [\tfrac{60}{127}, \tfrac{67}{127}]$$

$$[\tfrac{7}{127}, \tfrac{8}{127}] \bowtie [\tfrac{117}{127}, \tfrac{118}{127}]$$

$$[\tfrac{7}{127}, \tfrac{8}{127}] \bowtie [\tfrac{94}{127}, \tfrac{101}{127}]$$

$$[\tfrac{7}{127}, \tfrac{8}{127}] \bowtie [\tfrac{61}{127}, \tfrac{66}{127}]$$

Proof. As before. □

Theorem 5.6.3. *Let $\mu_q > [\frac{9}{31}, \frac{10}{31}]$. For the equivalences $s_q \sqcup s_{\frac{26}{31}} \simeq s_{q'} \sqcup s_{\frac{9}{31}}$, $s_q \sqcup s_{\frac{11}{31}} \simeq s_{q'} \sqcup s_{\frac{9}{31}}$, and $s_q \sqcup s_{\frac{13}{31}} \simeq s_{q'} \sqcup s_{\frac{9}{31}}$, the labellings associated to the pentagons in Π_m, for $5 \leq m \leq 6$, are as follows:*

$s_q \sqcup s_{\frac{13}{31}}$:

Π_5 $\quad [\frac{289}{992}, \frac{297}{992}, \frac{299}{992}, \frac{315}{992}, \frac{319}{992}] \to [\frac{195}{992}, \frac{165}{992}, \frac{173}{992}, \frac{175}{992}, \frac{191}{992}]$

Π_6 : $\quad [\frac{599}{1984}, \frac{607}{1984}, \frac{609}{1984}, \frac{625}{1984}, \frac{629}{1984}] \to [\frac{381}{1984}, \frac{351}{1984}, \frac{359}{1984}, \frac{361}{1984}, \frac{377}{1984}]$

$s_q \sqcup s_{\frac{11}{31}}$:

Π_5 : $\quad [\frac{289}{992}, \frac{297}{992}, \frac{299}{992}, \frac{315}{992}, \frac{319}{992}] \to [\frac{625}{992}, \frac{629}{992}, \frac{599}{992}, \frac{607}{992}, \frac{609}{992}]$

Π_6 : $\quad [\frac{599}{1984}, \frac{607}{1984}, \frac{609}{1984}, \frac{625}{1984}, \frac{629}{1984}] \to [\frac{1121}{1984}, \frac{1125}{1984}, \frac{1095}{1984}, \frac{1101}{1984}, \frac{1105}{1984}]$

$s_q \sqcup s_{\frac{26}{31}}$:

Π_5 : $\quad [\frac{289}{992}, \frac{297}{992}, \frac{299}{992}, \frac{315}{992}, \frac{319}{992}] \to [\frac{423}{992}, \frac{419}{992}, \frac{443}{992}, \frac{413}{992}, \frac{421}{992}]$

Π_6 : $\quad [\frac{599}{1984}, \frac{607}{1984}, \frac{609}{1984}, \frac{625}{1984}, \frac{629}{1984}] \to [\frac{1291}{1984}, \frac{1307}{1984}, \frac{1311}{1984}, \frac{1281}{1984}, \frac{1289}{1984}]$

Proof. See figure 5.6a for illustration of $\mathcal{R}([\frac{9}{31}, \frac{10}{31}])$.

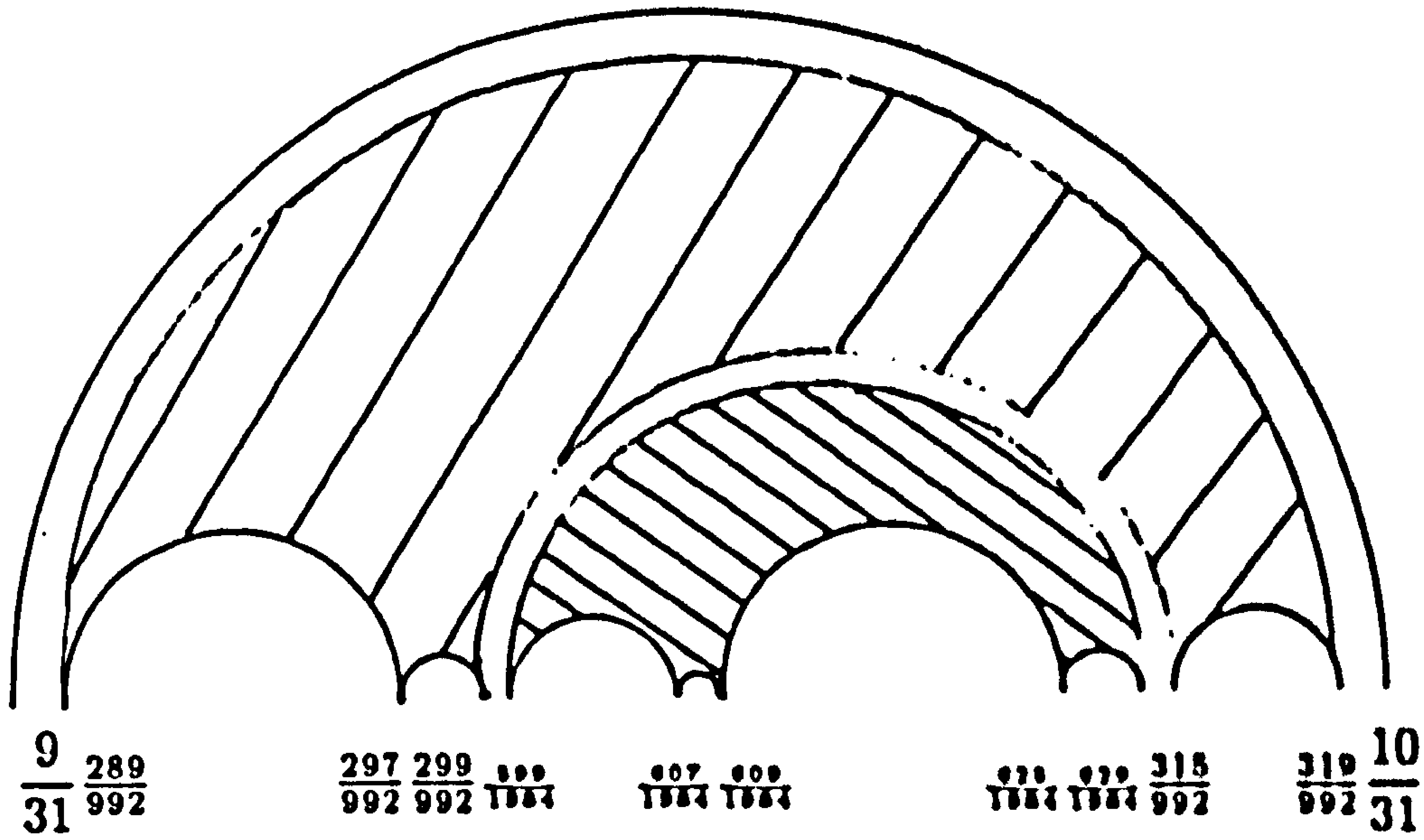


Figure 5.6a

We leave out the details. Note that the leaf $[\frac{19}{63}, \frac{20}{63}]$ is in the gap between the two pentagons illustrated. □

Corollary 5.6.4. *The following connections occur:*

$$[\tfrac{19}{63}, \tfrac{20}{63}] \bowtie [\tfrac{51}{63}, \tfrac{52}{63}]$$

$$[\tfrac{19}{63}, \tfrac{20}{63}] \bowtie [\tfrac{22}{63}, \tfrac{25}{63}]$$

$$[\tfrac{19}{63}, \tfrac{20}{63}] \bowtie [\tfrac{28}{63}, \tfrac{35}{63}]$$

$$[\tfrac{37}{127}, \tfrac{38}{127}] \bowtie [\tfrac{93}{127}, \tfrac{102}{127}]$$

$$[\tfrac{37}{127}, \tfrac{38}{127}] \bowtie [\tfrac{47}{127}, \tfrac{48}{127}]$$

$$[\tfrac{37}{127}, \tfrac{38}{127}] \bowtie [\tfrac{54}{127}, \tfrac{73}{127}]$$

$$[\tfrac{39}{127}, \tfrac{40}{127}] \bowtie [\tfrac{103}{127}, \tfrac{104}{127}]$$

$$[\tfrac{39}{127}, \tfrac{40}{127}] \bowtie [\tfrac{45}{127}, \tfrac{50}{127}]$$

$$[\tfrac{39}{127}, \tfrac{40}{127}] \bowtie [\tfrac{55}{127}, \tfrac{56}{127}]$$

Proof. As before. □

Corollary 5.6.5. *$\mathcal{V}_6$ is connected.*

Proof. We combine the results of 5.2.3, 5.4.2, 5.5.2, 5.6.2 and 5.6.4 in Table 5.6.6.

Notation: the table shows all the period six minor leaves, indicated by the numerators of the angles of their endpoints and where the denominator is 63 in all cases. A symbol in the table demonstrates that all the minor leaves which have that symbol as an entry are connected, as shown in one of the above theorems. e.g., the leaves which have Pa as an entry are all connected together by corollary 5.6.2. The symbols T_j refer to leaves which are identified using pre-images of the triangle $[\frac{1}{7}, \frac{2}{7}, \frac{4}{7}]$. Similarly Q_j refer to leaves for which we use pre-images of $[\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}]$; Pa and Pb to $[\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}]$ and $[\frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{18}{31}, \frac{20}{31}]$ respectively, and H to the hexagon $[\frac{1}{63}, \frac{2}{63}, \frac{4}{63}, \frac{8}{63}, \frac{16}{63}, \frac{32}{63}]$. The small parenthesised entries (e.g., (Pa)) are the analogous connections we observe hold by symmetry. Note that for the symmetric leaves in the lower part of the table the connections are those indicated as well as to the same letter but parenthesised. e.g., $[\frac{26}{63}, \frac{37}{63}]$ is connected to $[\frac{15}{63}, \frac{16}{63}]$ and $[\frac{47}{63}, \frac{48}{63}]$ by T_4 and (T_4) respectively.

Table 5.6.6.

V_6									
(1,2)	H							(H)	(61,62)
(3,4)	(H)	Pa					(Pa)	H	(59,60)
(5,6)		(Pa)	Q_1			(Q_1)	Pa		(57,58)
(7,8)	(H)		Q_2			(Q_2)		H	(55,56)
(10,17)			(Q_2)	T_1	(T_1)	Q_2			(46,53)
(11,12)		(Pb)		T_2	(T_2)		Pb		(51,52)
(13,14)		(Pa)		T_3	(T_3)		Pa		(49,50)
(15,16)	(H)		(Q_1)	T_4	(T_4)	Q_1		H	(47,48)
(19,20)		Pb					(Pb)		(43,44)
(22,25)		Pb					(Pb)		(38,41)
(23,24)				T_3	(T_3)				(39,40)
(26,37)				T_4					
(28,35)		Pb	Q_1	T_1					
(29,34)			Q_2						
(30,33)		Pa		T_2					
(31,32)	H								

We observe that there is a chain of connections from every leaf above to $\llbracket\frac{31}{63}, \frac{32}{63}\rrbracket$. e.g., $\llbracket\frac{23}{63}, \frac{24}{63}\rrbracket$ is connected to $\llbracket\frac{13}{63}, \frac{14}{63}\rrbracket$ by T_3, which is connected to $\llbracket\frac{59}{63}, \frac{60}{63}\rrbracket$ by (P_4), which in turn is connected to $\llbracket\frac{31}{63}, \frac{32}{63}\rrbracket$ by H. □

§5.7 The case $k = 6$.

By 5.2.5 we have

$$\llbracket\tfrac{1}{127}, \tfrac{2}{127}\rrbracket \bowtie \llbracket\tfrac{3}{127}, \tfrac{4}{127}\rrbracket \bowtie \llbracket\tfrac{7}{127}, \tfrac{8}{127}\rrbracket \bowtie \llbracket\tfrac{15}{127}, \tfrac{16}{127}\rrbracket \bowtie \llbracket\tfrac{31}{127}, \tfrac{32}{127}\rrbracket \bowtie \llbracket\tfrac{63}{127}, \tfrac{64}{127}\rrbracket \bowtie$$
$$\bowtie \llbracket\tfrac{95}{127}, \tfrac{96}{127}\rrbracket \bowtie \llbracket\tfrac{111}{127}, \tfrac{112}{127}\rrbracket \bowtie \llbracket\tfrac{119}{127}, \tfrac{120}{127}\rrbracket \bowtie \llbracket\tfrac{123}{127}, \tfrac{124}{127}\rrbracket \bowtie \llbracket\tfrac{125}{127}, \tfrac{126}{127}\rrbracket$$

Theorem 5.7.1. *Let $\mu_q > \llbracket\frac{1}{63}, \frac{2}{63}\rrbracket$. For the equivalences $s_q \sqcup s_{\frac{x}{63}} \simeq s_{q'} \sqcup s_{\frac{1}{63}}$, where $x = 31, 47, 55, 59$ the labellings associated to the hexagons in Π_6', are respectively as follows:*

$$\begin{aligned}
\llbracket 65, 67, 71, 79, 95, 127\rrbracket &\to \llbracket 197, 205, 221, 253, 191, 193\rrbracket \\
&\to \llbracket 457, 473, 505, 443, 445, 449\rrbracket \\
&\to \llbracket 977, 1009, 947, 949, 953, 961\rrbracket \\
&\to \llbracket 2017, 1955, 1957, 1961, 1969, 1985\rrbracket
\end{aligned}$$

where the above numbers are numerators of fractions with denominator 3968.

Proof. Follows as for 5.4.1. We leave out the details.□

Corollary 5.7.2. *The following connections occur:*

$$\begin{aligned}
\llbracket\tfrac{1}{127}, \tfrac{2}{127}\rrbracket &\bowtie \llbracket\tfrac{121}{127}, \tfrac{122}{127}\rrbracket \\
\llbracket\tfrac{1}{127}, \tfrac{2}{127}\rrbracket &\bowtie \llbracket\tfrac{113}{127}, \tfrac{114}{127}\rrbracket \\
\llbracket\tfrac{1}{127}, \tfrac{2}{127}\rrbracket &\bowtie \llbracket\tfrac{97}{127}, \tfrac{98}{127}\rrbracket \\
\llbracket\tfrac{1}{127}, \tfrac{2}{127}\rrbracket &\bowtie \llbracket\tfrac{62}{127}, \tfrac{63}{127}\rrbracket
\end{aligned}$$

Proof. As before. □

Table 5.7.4.

V_7											
(1,2)	Ea									E_a	(125,126)
(3,4)	(Ea)	H							(H)	Ea	(123,124)
(5,6)		(H)	Pa_1					(Pa_1)	H		(121,122)
(7,8)	(Ea)		Pa_2					(Pa_2)		Ea	(119,120)
(9,10)	Eb		(Pa_2)	Q_1			(Q_1)	Pa_2		(Eb)	(117,118)
(11,12)				Q_2			(Q_2)				(115,116)
(13,14)		(H)		Q_3			(Q_3)		H		(113,114)
(15,16)	(Ea)		(Pa_1)	Q_4			(Q_4)	Pa_1		Ea	(111,112)
(17,18)	Eb									(Eb)	(109,110)
(19,20)					T_1	(T_1)					(107,108)
(21,22)	Ec				T_2	(T_2)				(Ec)	(105,106)
(23,24)			(Pb_2)	(Q_3)	T_3	(T_3)	Q_3	Pb_2			(103,104)
(25,34)	Eb		(Pb_1)	(Q_4)	T_4	(T_4)	Q_4	Pb_1			(93,102)
(26,33)			(Pa_2)		T_5	(T_5)		Pa_2			(94,101)
(27,28)			(Pa_1)	(Q_1)	T_6	(T_6)	Q_1	Pa_1			(99,100)
(29,30)		(H)		(Q_2)	T_7	(T_7)	Q_2		H		(97,98)
(31,32)	(Ea)				T_8	(T_8)				Ea	(95,96)
(35,36)	Eb				T_9	(T_9)				(Eb)	(91,92)
(37,38)	Ec		Pb_1					(Pb_1)		(Ec)	(89,90)
(39,40)			Pb_2					(Pb_2)			(87,88)
(41,42)	Ec									(Ec)	(85,86)
(43,44)	(Ec)									Ec	(83,84)
(45,50)	(Ec)		Pb_2					(Pb_2)		Ec	(77,82)
(46,49)					T_6	(T_6)					(78,81)
(47,48)			Pb_1					(Pb_1)			(79,80)
(51,52)					$T_5/(T_4)$	$T_4/(T_5)$					(75,76)
(55,56)	(Eb)		Pb_2		(T_7)	T_7		(Pb_2)		Eb	(71,72)
(53,74)	Ec				T_8						
(54,73)			Pb_1		T_9						
(57,70)				Q_2	T_1						
(58,69)				Q_4							
(59,68)	Eb			Q_3							
(60,67)			Pa_1								
(61,66)			Pa_2		T_3						
(62,65)		H									
(63,64)	Ea			Q_1	T_2						

Corollary 5.7.3. *V_7 is connected.*

Proof. We combine the results of 5.2.3, 5.4.2, 5.5.2, 5.6.2, 5.6.4 and 5.7.2. in table 5.7.4. The notation for Table 5.7.4 is as in Table 5.6.6. (The minor leaves are understood to have endpoints with denominator 127).

We observe that there is a chain of connections from every leaf to $[\frac{61}{127}, \frac{64}{127}]$. $\square$

CHAPTER SIX

GLOBAL PROPERTIES OF V_n

In this chapter we pull together some of our earlier results and try to sum up what we know about about W_n and V_n, and point out where and how our understanding can be taken further. We will be assuming throughout this chapter that every variety V_n is irreducible. We compute the genus of some of these varieties, making use of the calculations presented in §1.6.

§6.1 The genus of a plane curve.

By the genus of a variety in two variables, i.e., a plane curve, we mean its geometric genus as a Riemann surface, where we have removed the singular points. For the variety V_n therefore, this coincides with the geometric genus of W_n.

The genus of a variety, $g(V)$, which has no singular points, is given by the identity $g(V) = 1/2(n-1)(n-2)$, where n is the degree of the variety. So varieties of degree 1 and 2 have genus 0, and are isomorphic to the Riemann sphere. For example, V_3 and V_4 are isomorphic to the Riemann sphere. If the variety has singular points then $1/2(n-1)(n-2)$ gives an upper bound on the (geometric) genus.

We will calculate the genus of V_n ($n \leq 7$) by evaluating the effect of the singularities, for which we have a formula.

First we define *intersection number*. Recall that for the singularities we are considering, at each singularity we have V_n expressed (in local coordinates) as a product of the form $\prod_j (a - \alpha_j(b))$ where each $\alpha_j(b)$ is a power series in b, so that each term in the product defines one sheet or branch of V_n locally (see Chapters 1 and 3 — in fact we possibly have $\prod_j (b - \beta_j(a))$, though not in our examples in §1.6). Let two of these terms be given by $f_1 = a - \alpha_1 b$ and $f_2 = a - \alpha_2 b$, which determine branches B_1 and B_2 respectively. Then the intersection number of B_1 with B_2, denoted $B_1 \cdot B_2$ is given by $\dim_{\mathbf{C}} \frac{\mathbf{C}[a,b]}{< f_1, f_2 >}$, where $\mathbf{C}[a,b]$ is the ring of formal power series of a and b over $\mathbf{C}$ (assumed by default to be centred at 0) and $< f_1, f_2 >$ denotes the ideal generated by f_1 and f_2.

Proposition 6.1.1. *Let two branches of a variety intersect at $(a,b) = (0,0)$. Then, if there are local power series expansions for a in terms of b, namely*

$$a = \sum_{j=1}^{\infty} \alpha_j b^j \text{ and } a = \sum_{j=1}^{\infty} \alpha'_j b^j,$$

then the intersection number of these two branches is given by the least j such that $\alpha_j \neq \alpha'_j$.

Proof. Let k be the minimal j such that $\alpha_j \neq \alpha'_j$, so that $\alpha_j = \alpha'_j$ for all $1 \leq j < k$. The intersection number ν is given by

$$\nu = \dim_{\mathbf{C}} \frac{\mathbf{C}[a,b]}{< f, g >},$$

where $f = a - \sum_{j=1}^{\infty} \alpha_j b^j$ and $g = a - \sum_{j=1}^{\infty} \alpha'_j b^j$.

The ideal $< f, g >$ contains the elements $f - g = \alpha_k(1 + \sum_{j=1}^{\infty} \beta_j b^j) b^k$ and $(a - \alpha_1 b - \cdots - \alpha_{k-1} b^{k-1})$. So $\{< f, g > + b^i \mid 0 \leq i < k\}$ is a basis for the quotient ideal. □

We describe how we can use the information from 6.1.1 to calculate the genus of some V_n. Now, by [Bk], if we have a variety with known degree, and we have definite expressions for the branches of all the singularities of that variety, we are able to compute its genus. For this, we need *Milnor numbers*. Let f represent some variety at the origin, as above. Then the Milnor number of the singularity, μ, is given by

$$\mu = \dim_{\mathbf{C}} \frac{\mathbf{C}[a,b]}{< \frac{\partial f}{\partial a}, \frac{\partial f}{\partial b} >}.$$

Note that if $Df|_0$ is non-singular, then the Milnor number is 0. We can equally well define the Milnor number for a single branch of a singularity — this simply means that $f = 0$ is irreducible — so that for a smooth branch $\mu = 0$. In what follows, we write μ_j for the Milnor number of the j-th singularity of V_n, and μ_{ij} for the Milnor number of its i-th branch.

Theorem 6.1.2. *Let V be an irreducible variety of degree d, with k singularities. Then the genus of V is given by*

$$g(V) = \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{k} \frac{\mu_j + r_j - 1}{2},$$

where μ_j is the Milnor number of the j-th singularity, and r_j is the number of branches at the j-th singularity.

Proof. See [Bk] $\square$

Corollary 6.1.3. *The genus of an irreducible variety V_n is given by*

$$g(V_n) = \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{k} \sum_{i=1}^{r_j} \sum_{l=i+1}^{r_j} (B_i \cdot B_l),$$

where $B_i.B_l$ is the pairwise intersection number of branches at the j-th singularity.

Proof. By [Bk] the Milnor number of a singularity is expressible in terms of the Milnor numbers of its constituent branches by

$$\mu_j = \sum_{i=1}^{r_j} \mu_{ij} + \Big(\sum_{i=1}^{r_j} \sum_{l=i+1}^{r_j} 2(B_i \cdot B_l) \Big) + 1 - r_j,$$

where the μ_{ij} are the Milnor numbers of the branches of the j-th singularity.

By Theorem 3.0, all the branches at each singularity are smooth, so that $\mu_{ij} = 0$ for all i and j where defined. The result then follows by substitution into the equation in theorem 6.1.2. $\square$

§6.2 Irreducibility for V_5 and V_6.

It is already well established that the varieties V_n, for $n \leq 4$ are irreducible. We can also show that the varieties V_5 and V_6 are irreducible by fairly direct means. To do this we use Bezout's theorem and consider the real varieties which are subsets of V_5 and V_6. However, the discussion which follows is not of much use in general for reasons we shall explain.

Theorem 6.2.1. *Bezout's theorem. Let A_1 and A_2 be irreducible algebraic curves in $P^2\mathbb{C}$, with degrees m_1 and m_2 respectively. Then the sum of the intersection numbers of A_1 with A_2 is equal to $m_1 m_2$.*

Recall that V_5 and V_6 have singularities *at the point* $(a : b : c) = (-1 : 0 : 1)$. Let L be the straight line $a = -1$, a degree one curve. Consider the sets $V_5 \cap L$ and $V_6 \cap L$, which intersect at $(-1 : 0 : 1) = p$. Then (see §1.6) the intersection numbers of these two sets at p are 5 and 9 respectively by 6.1.1. But these are the degrees of V_5 and V_6 respectively, so this intersection point is of the maximum possible multiplicity because L is degree 1. Therefore, there can be no other intersection points between V_5 (or V_6) and L and hence all the components of V_5 and V_6 pass through this point. If we can show that all the sheets passing through this point lie in the same connected component of W_5 (or W_6), we have shown that V_5 and V_6 are irreducible (by 1.3.2).

Let $Re(V_j)$ be the subset of V_j for which b and $a(b)$ are real. (So $Re(V_j)$ is a real variety in its own right.) Then, if each branch of V_j in a neighbourhood of p contains a branch of $Re(V_j)$, as opposed to just the point p, it suffices to show that all the branches of $Re(V_j)$ at p are in the same connected component of V_j. This we do for $j = 5$ and $j = 6$.

Note on the real pictures. Because we know that all the singularities lie on the projective lines $b = 0$ and $c = 0$ we can change coordinates so that $Re(V_j)$ contains all the singularities in the finite plane: we can get them all on one two dimensional picture and see how the various branches connect up.

Figure 6.2a represents $Re(V_5)$. We can see that the two branches at p

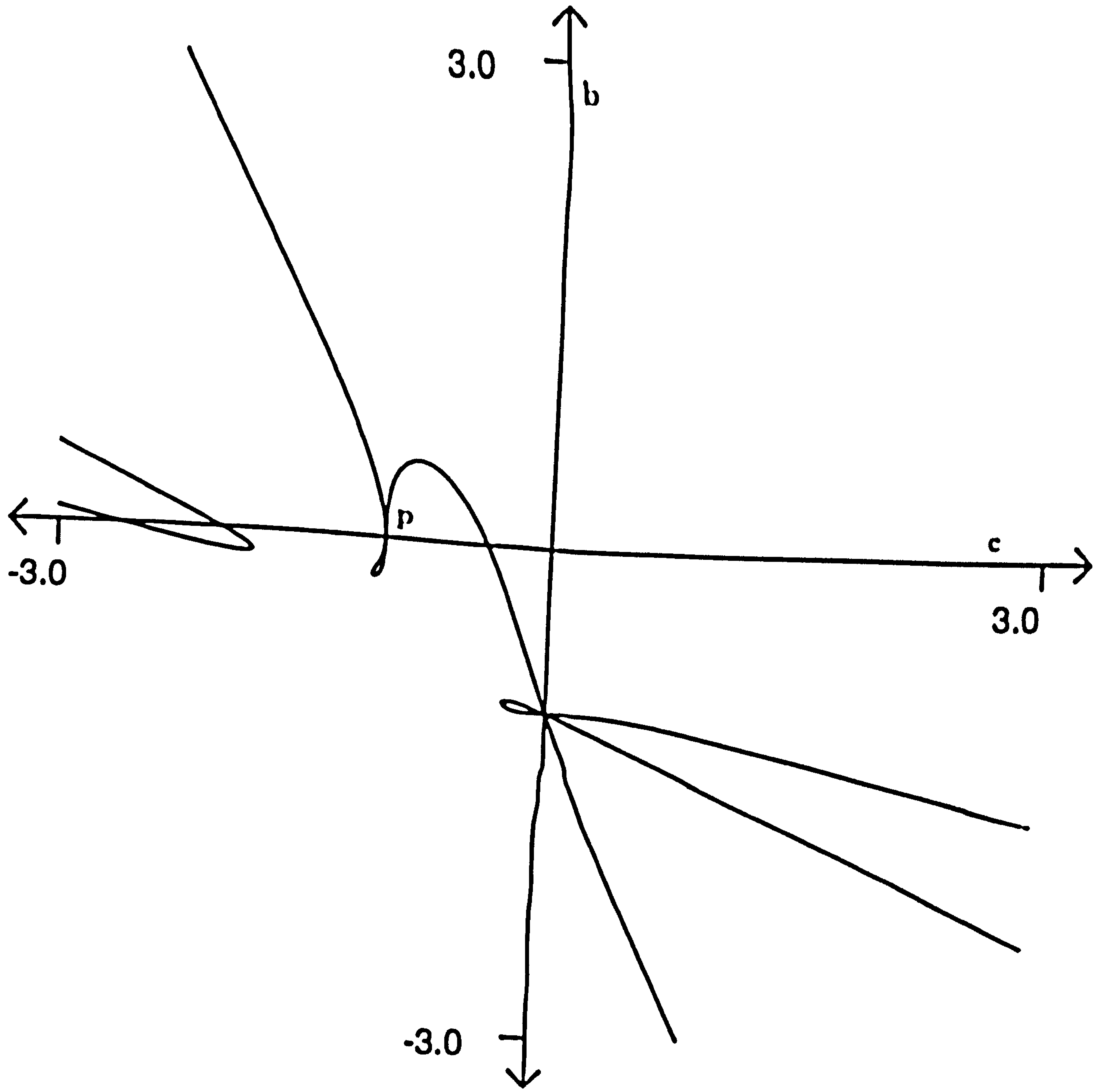


Figure 6.2a: $Re(V_3) \cap \{a = 1\}$.

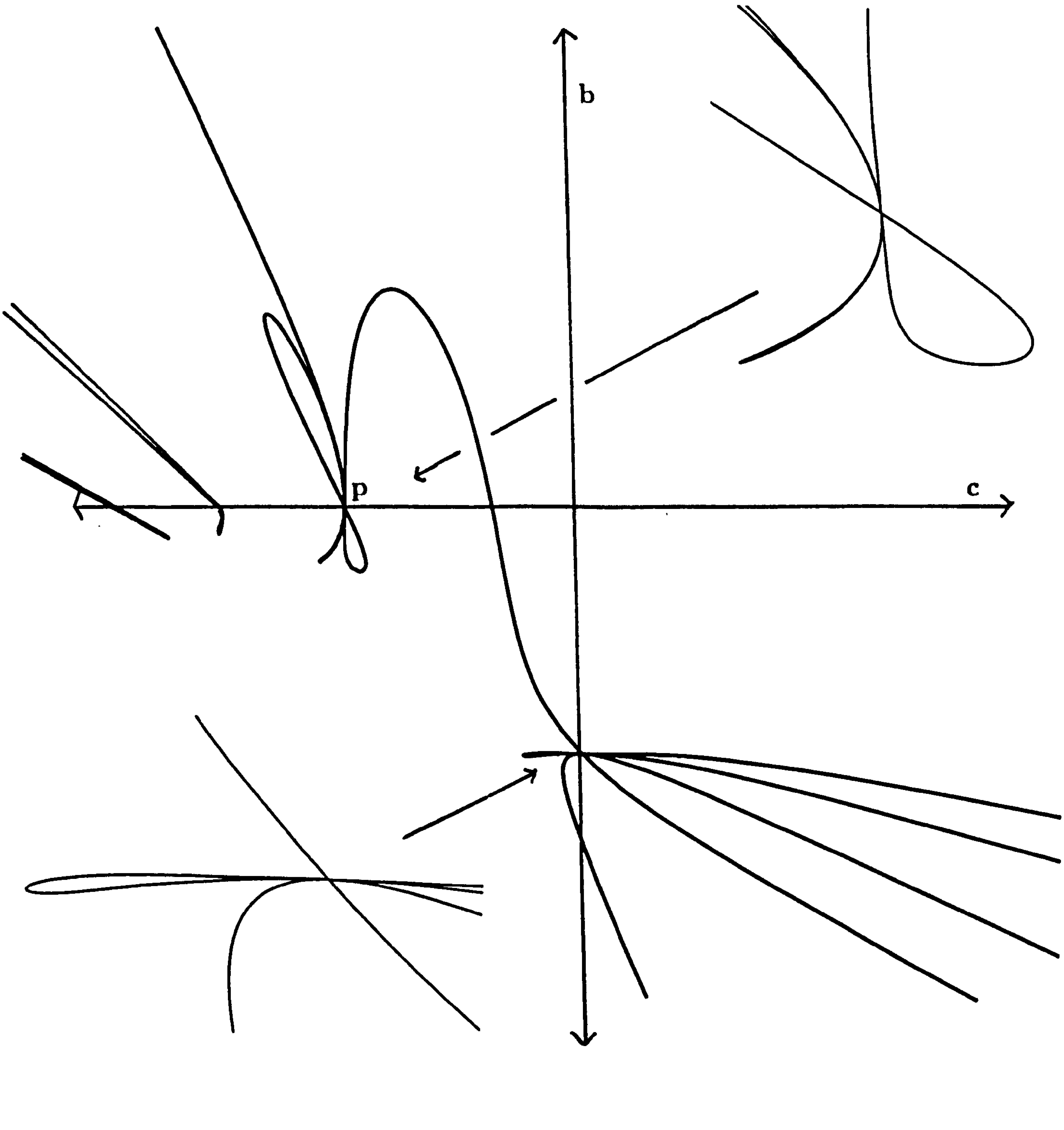


Figure 6.2b: $Re(V_6) \cap \{a = 1\}$.

are part of the same connected component of W_5, which is enough to show that V_5 is irreducible. However we can see see this directly by following the various branches through both singularities.

Figure 6.2b represents $Re(V_6)$. We cannot see directly that all the branches at p lie in the same connected component of W_6. However, we have local expansions here which tell us how the various arcs here connect up — this does suffice to show that all the branches lie in the same connected component of W_6. Again, we can see directly that W_6 is connected if we use the expansions to show how the arcs at $(-1:1:0)$ are connected.

Note: For $j \geq 7$, this line argument will not work. For example, V_7 has two branches at $p = (-1:0:1)$, for which the slopes are not real. Hence these two branches intersect locally with $Re(V_7)$ only at the point p, so any picture (as above) would not reveal anything directly useful.

However it does seem that in general the intersection $V_j \cap L$ is of maximum multiplicity (more about this later) so that the connectedness of local branches at p would be sufficient to show irreducibility. In particular, if the methods of the previous chapter could be used to prove that all minor leaves in the limb of $[\![\frac{1}{7}, \frac{2}{7}]\!]$ were connected together with those in the limb of $[\![\frac{5}{7}, \frac{6}{7}]\!]$, then this may suffice to show that all are connected. However, this seems likely to be just as difficult to prove as the full problem of Chapter 5.

§6.3 Examples of genus.

We have already shown that V_1, V_3 and V_4 are once, twice and four times punctured spheres respectively. We assume that V_5 and V_6 are irreducible.

Theorem 6.3.1. *The genus of V_5 is 1. Therefore, it is a one-holed torus with two points of self-intersection: one where two sheets meet tangentially; one where three sheets meet transversally.*

Proof. The degree of the variety V_5 is 5. Therefore, an upper bound on the genus is 6. In the following table we list the singularities of V_5, and the contribution they make to the genus (given by 6.1.2): on each row we list the pairwise intersection numbers of the sheets, in the order that they are presented in §1.6, followed by the total for the particular singularity.

V_5				
$(-1:0:1)$	1	1		2
$(-1:1:0)$	1	1	1	3

By 6.1.2 $g(V_5) = 6 - 2 - 3 = 1$. So W_5 is a Riemann surface of genus 1 with seven punctures. V_5 has two points of self-intersection: one is a simple double-point and the other is a simple triple-point, where simple means the branches meet transversally (see §1.6). □

Theorem 6.3.2. *The genus of V_6 is 6.*

Proof. The degree of the variety V_6 is 9. Therefore, an upper bound on the genus is 28. The following table is presented as for V_5.

V_6							
$(-1:0:1)$	1	1	1	4	2	2	11
$(-1/2:0:1)$	2						2
$(-1:1:0)$	1	1	1	2	2	2	9

By 6.1.2 $g(V_6) = 28 - 11 - 2 - 9 = 6$. So W_6 is a Riemann surface of genus 6 with eleven punctures. V_6 has three points of self-intersection. □

Example 6.3.3: V_7

In this case we merely obtain an upper bound for the genus.

In the following table each row has entries corresponding to the total intersection number of some branch a_j with branches given by a_i for $i > j$.

Since the degree of the variety V_7 is 21, an upper bound on the genus of V_7 is given by 190. From the table we have contributions which total $2 + 2 + 74 + 14 + 70 + 3 + 3 = 168$. So we have a value of $190 - 168 = 22$ for $g(V_7)$. So W_7

V_7											
$(\frac{3+\sqrt{5}}{2}:0:1)$	2										2
$(\frac{-3-\sqrt{5}}{2}:0:1)$	2										2
$(-1/2:0:1)$	2	2	2	2	2	4					14
$(-1:0:1)$	8	7	16	13	14	8	4	4			74
$(-1:1:0)$	10	9	8	7	7	5	9	6	6	3	70
$(\frac{-1-i}{2}:1:0)$	3										3
$(\frac{-1+i}{2}:1:0)$	3										3

is a Riemann surface of genus 22, with fifteen punctures (see §1.6). V_7 has seven points of self-intersection, provided that it is irreducible.

§6.4 Some remarks on further possible research.

It seems clear from the examples that there is some elementary relation between the position of leaves in the *QML* and the slopes of the corresponding branches at a singularity, just as there is between the minor leaves and the location of the singularities. The latter (see Chapter 4) can be summed up as:

Minor leaves have same or complex-conjugate minimal leaf.
$\Longleftrightarrow$ Associated polynomial hyperbolic components have the same singularity on their boundary.

We conjecture the following analogous relation (by next-to-minimal leaf we mean the least leaf greater than minimal one):

Minor leaves have same or complex-conjugate next-to-minimal leaf.
$\Longleftrightarrow$ Branches are tangent to each other.

Indeed, we can extend this conjecture further to: Let μ_q and $\mu_{q'}$ share n lesser pre-periodic leaves (i.e., there exist n leaves ℓ such that $\ell < \mu_q$ and $\ell < \mu_{q'}$). Then the corresponding branches are tangent to the n-th degree. This holds for the examples we have calculated in §1.6.

It is interesting to note that for most of the singularities that we have studied, the branches of the varieties have real slope. The cases where this fails to happen are the ones the ones that contain polynomials for which the minor leaf is non-regular. Then the two tangent slopes of the sheets corresponding to pairs of minor leaves that are partners (see §4.4 for definition) are complex-conjugate. e.g., The polynomials corresponding to $[\![\frac{19}{127}, \frac{20}{127}]\!]$ and $[\![\frac{91}{127}, \frac{92}{127}]\!]$ lie on one branch of V_7 at $(-1:0:1)$, and those corresponding to $[\![\frac{35}{127}, \frac{36}{127}]\!]$ and $[\![\frac{107}{127}, \frac{108}{127}]\!]$ on another, the tangents of which have slopes $\frac{1 \pm i}{2}$ by §1.6.

We return to considering the singularities of V_n at $(-1:0:1)$ (remarked on in §6.2): we conjecture that the intersection number of V_n with L is the same as the degree of its defining polynomial $p_n(a,b)$. Thus $V_n \cap L = p$ for all n.

The idea for a proof of the above: let μ_q be a minor leaf in the limb of $[\![\frac{1}{7}, \frac{2}{7}]\!]$. Then its position in the *QML* determines the pattern of its periodic critical orbit (and its symbolic dynamics). We know that the local branch of the variety which corresponds to μ_q (see Chapter 3) is expressed as a power series expansion $a = -1 + \alpha(b)$, the terms of which determine the degree of tangency and intersection number between branches. Now, the critical orbit pattern determines the expansion $a = -1 + \alpha(b)$, as the following example illustrates:

Example: $[\![\frac{7}{31}, \frac{8}{31}]\!]$, for which $\sigma = (1\ 2\ 0\ 0\ 0)$. We have seen that the third iterate of 0 is $b + \alpha(b)$. The next iterate is given by $\dfrac{(b+\alpha(b))^2 + (\alpha(b)-1)b + b}{(b+\alpha(b))^2}$, which we denote as I_4. In this case this quantity must be near 0. i.e., $\lim_{b\to 0} I_4 = 0$. We substitute $\alpha(b) = \alpha_1 b + \alpha_2 b^2 + O(b^3)$ and equate coefficients of b from the numerator and denominator of I_4. Note there are no constant terms. The b term in the numerator is $\alpha_1 b$; in denominator 0. Thus $\alpha_1 = 0$. The b^2 term in the numerator is $\alpha_1^2 b^2 + \alpha_2 b^2 = \alpha_2 b^2$; in the denominator $b^2 + \alpha_1^2 = b^2$. Thus $\alpha_2 = 1$. So $a = b^2 + O(b^3)$ on the branch corresponding to $[\![\frac{7}{31}, \frac{8}{31}]\!]$. We have more than this however. The above argument shows that *any* periodic leaf ($> [\![\frac{1}{7}, \frac{2}{7}]\!]$) for which the first four terms of σ are $(1\ 2\ 0\ 0)$ corresponds to a branch for which the local expansion is of the form $a = b^2 + O(b^3)$. Higher order terms can be calculated in the same way.

REFERENCES

[Af] Ahlfors, L. :" *Lectures on Quasi-conformal mappings* ," Von Nostrand (1966).

[AB] Ahlfors, L. and Bers, L. :" *Mapping theorem* ," Annals of Math. **72**, 385–404 (1960).

[Am] Ahmadi, D. :" *On the dynamics of certain rational maps of degree two* ," Thesis, Liverpool University, 1991.

[Bl] Blanchard, P. :" *Complex Analytic Dynamics on the Riemann Sphere* ," Bull. Amer. Soc. **11**, 85–141 (1984).

[Bk] Brieskorn :" *Plane algebraic curves* ," Birkhauser (1986).

[Br] Brolin, M. :" *Invariant sets under iteration of rational functions* ," Ark. Mat. **6**, 103–144 (1966).

[D] Douady, A. :" *Systèmes dynamiques holomorphes* ," Séminaire Bourbaki, Astérisque, **105-106** (1983), 39-63.

[DH1] Douady, A. and Hubbard, J. H. :" *Etudes dynamiques des polynômes complexes, Parts I and II* ," Math. d'Orsay (1985) .

[DH2] Douady, A. and Hubbard, J. H. :" *Itération des polynômes quadratiques complexes* ," C. R. Acad. Sci. Paris, Série I, t.**294**, 123-126 (1982).

[DH3] Douady, A. and Hubbard, J. H. :" *A proof of Thurston's topological classification of rational functions* ," Mittag-Leffler Preprint, 1985.

[Du] Duren, P. :" *Univalent Functions* ," New York Springer, 1983.

[F] Fatou, P. :" *Memoire sur les equations fonctionelles* ," Bull. Soc. Math. France **47** 161–271 (1919) **47** 33–96, 208–314(1920).

[G] Griffiths, P. A. :" *Introduction to algebraic curves* ," AMS Translations of mathematical monographs Vol. 76.

[J] Julia, G. :" *Iteration des applications fonctionelles* ," J. Math. Pure. Appl. 8 47–245 (1918).

[L] Levy, S. V. F. :" *Critically finite rational maps* ," Thesis, Princeton University, 1985.

[M1] Milnor, J. :" *Singular points of complex hypersurfaces* ," Princeton University press, 1968.

[M2] Milnor, J. :" *Dynamics in One Complex Variable: Introductory Lectures* ," SUNY Stony Brook, Preprint # 1990/5 (1990).

[M3] Milnor, J. :" *Hyperbolic components in spaces of polynomial maps* ," Preprint, 1988.

[M4] Milnor, J. :" *Remarks on quadratic rational maps* ," Preprint, 1991.

[R1] Rees, M. :" *Components of degree two hyperbolic rational maps* ," Invent. Math., **100** (1990), 357–382.

[R2] Rees, M. :" *A Partial Description of the Parameter Space of Rational Maps of degree two: Part 1* ," Acta Math., **168** (1992), 11–87.

[R3] Rees, M. :" *A Partial Description of the Parameter Space of Rational Maps of degree two: Part2* ," SUNY Stony Brook, Preprint # 1991/4.

[S] Sullivan, D. :" *Quasiconformal homeomorphisms and dynamics I, solution of the Fatou-Julia problem on wandering domains* ," Ann. Math. **122** 401–418 (1985).

[T] Thurston, W. P. :" *On the combinatorics of iterated rational maps* ," Preprint, Princeton Uni. and I. A. S. (1985).

[TL] Tan Lei :" *Accouplements des polynômes complexes* ," Thèse, Université Paris-sud, Orsay, 1987.

[W] Wittner, B. :" *On the bifurcation loci of rational maps of degree two* ," Thesis, Cornell University, 1988.