Numerical analysis of Volterra integro-differential equations

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by Jason Anthony Roberts.

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Declaration

No part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other institution of learning. However some parts of the material contained herein have been previously published:

- Chapter 3 is to be presented at the 16th IMACS World Congress on Scientific Computation, Applied Mathematics and Simulation, Lausanne, August 2000and will appear in the proceedings volume (co-authored with J.T. Edwards & N.J. Ford).

- Parts of chapter 4 have appeared in presentations given at the 17th Biennial Conference on Numerical Analysis, Dundee, June 1997 and at the IMACS World Congress on Scientific Computation, Applied Mathematics and Simulation, Berlin, July 1997. The work appears in the IMACS proceedings and the Manchester Centre for Computational Mathematics Numerical Analysis Technical Report (number 311), October 1997. It has also appeared in the Journal of Integral Equations and Applications, Winter 1998, Vol 10(4) (all are co-authored with N.J. Ford & C.T.H. Baker).

- The work described in chapter 5 has been published by the Manchester Centre for Computational Mathematics as a Numerical Analysis Technical Report (number 312), October 1997 (co-authored with N.J. Ford, J.T. Edwards & L.E. Shaikhet). It has also been submitted to the Journal of Stability and Control: Theory and Applications.

- Results from Chapter 6 were presented at the IMACS Advances in Scientific Computing and Mathematical Modelling Conference, Alicante, June 1998. They have also appeared in the International Journal of Applied Science and Computations, Vol 6 (1), 1999 (co-authored with J.T. Edwards).

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Abstract

We consider numerical methods applied to equations of the general form

$$y'(t) = G\left(t, y(t), \int_{-\infty}^{t} K(t, s, y(s))ds\right).$$

The case where $K(t, s, y(s)) = 0$ for $s < 0$ is considered first and in greatest detail.

We

- give a numerical analysis of a linear integro-differential equation that is directly relevant to the nonlinear work contained herein.

- give general results which guarantee that asymptotic stability of the zero solution to equations of the form $y'(t) = -\int_0^t k(t-s)f(y(s))ds$ is preserved (under conditions) in the numerical approximation. We develop further these results so that they are applicable to a more general class of numerical scheme, θ-methods.

- consider a special case for further analysis; we give necessary and sufficient conditions for the presence of asymptotic stability in numerical approximations to the zero solution. This approach involves imposing a condition on the parameter values. We show that such parameter values exist and that our earlier theory applies to a non-empty set of equations.

- examine the case $K(t, s, y(s)) \neq 0$ for $s < 0$. We show that our result, which guarantees that asymptotic stability of the zero solution is preserved in the numerical approximation, is sufficient but not necessary.

Chapter 1

Volterra integral and functional equations

We present in this chapter a brief description of some of the Volterra equations which have been analysed by previous authors (Brunner & Van der Houwen [13], Linz [53], for example). We use some of these results later to investigate related equations. Although we discuss integral equations and functional equations separately, integral equations can be thought of as a special case of functional equations. The material found in this chapter may be found in more abstract form in [30], or in [13], [15] and [53]. We collect these results together for convenience. We discuss linear equations first because, in general, results for nonlinear problems are related to and weaker than results for the linear problem. There is also a strong link between our choice of nonlinear problems and linear equations in that the linear part of our problems is the dominant part.

1.1 Volterra integral equations

An integral equation is one in which the unknown quantity occurs under an integral sign. We are concerned, for the moment, with the special case where the unknown is a function of one real variable. Volterra integral equations have variable regions of integration (dependent on the independent variable). Equations where the region of integration is fixed are called Fredholm equations and are not under discussion here. We can categorise Volterra integral equations as follows : Volterra integral equations of the first kind and Volterra integral equations of the second kind. We concern ourselves only with second kind equations and note that first kind equations may be considered as a special case of second kind equations. These categories can be further subdivided into two sub classes - linear and nonlinear equations.

1.1.1 Linear Volterra integral equations of the second kind

The general form of a linear Volterra Integral Equations (VIE) of the second kind is

$$y(t) = f(t) + \int_0^t k(t,s)y(s)ds \qquad (1.1.1)$$

We now present some results regarding the existence and uniqueness of solutions to (1.1.1) and the qualitative properties of such solutions.

These results can be found in many standard integral equation texts.

Theorem 1.1.1 ([53], page 30) *If $k(t,s)$ is continuous in $0 \le s \le t \le T$ and $f(t)$ is continuous in $0 \le t \le T$, then the integral equation (1.1.1) possess a unique continuous solution for $0 \le t \le T$.*

Theorem 1.1.1 can be proved using Picard iterations.† A series solution to equation (1.1.1) may also sometimes be obtained using such a technique but in many practical situations this is not possible due to the nature of the integrals. Theorem 1.1.1 is based upon a contraction mapping argument. An alternative existence-uniqueness theorem, derived using the method of continuation, is given below.

Theorem 1.1.2 ([53], page 32) *Assume that in (1.1.1)*

(i) $f(t)$ is continuous in $0 \le t \le T$,

(ii) for every continuous function h and all $0 \le \tau_1 \le \tau_2 \le t$ the integrals

$$\int_{\tau_1}^{\tau_2} k(t,s)h(s)ds$$

and

$$\int_0^t k(t,s)h(s)ds$$

are continuous functions of t,

(iii) $k(t,s)$ is absolutely integrable with respect to s for all $0 \le t \le T$,

† see page 3.

(iv) there exist points $0 = T_0 < T_1 < T_2 < \ldots < T_N = T$ *such that, for all* i *and all* $T_i \leq t \leq T_{i+1}$,

$$\int_{Ti}^{min(t,T_{i+1})} | k(t,s) | \, ds \leq \alpha < 1,$$

where α *is independent of* t *and* i,

(v) for every t *in* $[0,T]$

$$\lim_{\delta \to 0+} \int_t^{(t+\delta)} | k(t+\delta, s) | \, ds = 0$$

then (1.1.1) has a unique continuous solution for $0 \leq t \leq T$.

The benefit of having these results available is that it is a standard procedure to adapt results for nonlinear problems from the linear results. One can use Picard iterates (known as the method of successive approximations or the Picard method) to derive other theorems, in the following way. Consider equation (1.1.1). The Picard method consists of the simple iteration

$$y_n(t) = f(t) + \int_0^t k(t,s) y_{n-1}(s) ds, \; n = 1, 2, \ldots \tag{1.1.2}$$

with

$$y_0(t) = f(t). \tag{1.1.3}$$

For convenience, we introduce

$$\phi_n(t) = y_n(t) - y_{n-1}(t), \; n = 1, 2, \ldots \tag{1.1.4}$$

$$\phi_0(t) = f(t). \tag{1.1.5}$$

Subtracting the same equation with n replaced by n-1 from (1.1.2) yields

$$\phi_n(t) = \int_0^t k(t,s) \phi_{n-1}(s) ds, \; n = 1, 2, \ldots. \tag{1.1.6}$$

From (1.1.3) we have

$$y_n(t) = \sum_{i=0}^{n} \phi_i(t) \tag{1.1.7}$$

Examining the computation of the ϕ_n's, we see that

$$\phi_1(t) = \int_0^t k(t,s)f(s)ds \tag{1.1.8}$$

$$\phi_2(t) = \int_0^t k(t,s)\phi_1(s)ds \tag{1.1.10}$$

$$= \int_0^t k(t,s)\int_0^s k(s,\tau)\,f(\tau)\,d\tau ds. \tag{1.1.11}$$

If $k(t,s)$ and $f(t)$ are continuous then we have

$$\phi_2(t) = \int_0^t \int_\tau^t k(t,s)k(s,\tau)\,dsf(\tau)\,d\tau = \int_0^t k_2(t,\tau)\,f(\tau)\,d\tau \tag{1.1.12}$$

where

$$k_2(t,\tau) = \int_\tau^t k(t,s)k(s,\tau)\,ds.$$

A similar result holds for the other $\phi_n(t)$ and it follows by induction that

$$\phi_n(t) = \int_0^t k_n(t,s)f(s)ds \tag{1.1.13}$$

where $k_n(t,s) = \int_s^t k(t,\tau)\,k_{n-1}(\tau,s)\,d\tau$, with $k_1(t,s) = k(t,s)$. The k_n are called the *iterated kernels*. From (1.1.7) we have

$$y_n = f(t) + \int_0^t \Gamma(t,s)f(s)ds, \tag{1.1.14}$$

where

$$\Gamma_n(t,s) = \sum_{i=1}^{n} k_i(t,s), \tag{1.1.15}$$

If $k(t,s)$ is continuous and

$$|\, k(t,s) \,| \leq k, \;\; 0 \leq s \leq t \leq T,$$

then

$$|\, k_n(t,s) \,| \leq \frac{k^n(t-s)^{n-1}}{(n-1)!}.$$

Therefore

$$\Gamma(t,s) = \sum_{i=1}^{\infty} k_i(t,s)$$

is uniformly convergent for $0 \leq s \leq t \leq T$. The function $\Gamma(t,s)$ is the *resolvent kernel* for $k(t,s)$. The resolvent kernel allows us to write down the form of the unique solution to (1.1.1) under certain conditions.

Theorem 1.1.3 ([53], page 36) *If $k(t,s)$ and $f(t)$ are continuous, then the unique continuous solution of (1.1.1) is given by*

$$y(t) = f(t) + \int_0^t \Gamma(t,s) f(s) ds. \tag{1.1.16}$$

The following theorem gives an alternative way of deriving the resolvent kernel.

Theorem 1.1.4 ([53], 37) *Under the assumption of theorem 1.1.3 the resolvent*
kernel $\Gamma(t,s)$ satisfies the equation

$$\Gamma(t,s) = k(t,s) + \int_s^t k\left(t,\tau\right)\Gamma\left(\tau,s\right) d\tau,\ 0 \leq s \leq t \leq T. \tag{1.1.17}$$

Equation (1.1.17) involves an unknown function of two variables and is not generally useful for computing the resolvent kernel. There is a special case, however, where a simplification applies.

Definition 1.1.5 (Convolution kernels) *If the kernel of (1.1.1) is a function of $(t-s)$ only, that is, $k(t,s) = k(t-s)$, then k is said to be a difference (or convolution) kernel.*

Theorem 1.1.6 ([53], page 38) *If k is a difference kernel, and $k(t)$ and $g(t)$ are continuous, then the unique continuous solution of (1.1.1) is given by*

$$y(t) = f(t) + \int_0^t R(t-s) f(s) ds, \tag{1.1.18}$$

where the resolvent kernel $R(t)$ is the solution of

$$R(t) = k(t) + \int_0^t k(t-s) R(s) ds. \tag{1.1.19}$$

Note that when k is a convolution, the solution is a convolution of R and f. When an explicit solution to an equation cannot be found, it is sometimes possible to obtain information on qualitative behaviour of such a solution. For example, one can seek results on smoothness of the solution, bounds and asymptotic behaviour. Below are some established results.

Theorem 1.1.7 ([53], page 39) *If*

(i) $f(t)$ *is* p *times continuously differentiable in* $[0,T]$,

(ii) $\left(\frac{\partial^j}{\partial t^j}\right) k(t,s)$ *is continuous in* $0 \le s \le t \le T$ *for all* $j = 0, 1, \ldots, p$,

(iii) $\left(\frac{d^q}{dt^q}\right) k_r(t)$ *is continuous in* $[0,T]$ *for all* $q \ge 0$ *and* $r \ge 0$ *such that* $r + q \le p - 1$,

then the solution of (1.1.1) is p *times continuously differentiable in* $[0,T]$.

Theorem 1.1.8 ([53], page 40) *Assume that the kernel* $k(t,s)$ *in (1.1.1) is absolutely integrable with respect to* s *for all* $0 \le t \le T$ *and that the equation has a continuous solution. Assume also that there exist functions* $F(t)$ *and* $K(t,s)$ *satisfying*

$$\begin{aligned} | f(t) | &< F(t), & 0 \le t \le T, \\ | k(t,s) | &< K(t,s), & 0 \le s \le t \le T, \end{aligned}$$

and such that the integral equation

$$Y(t) = F(t) + \int_0^t K(t,s)Y(s)ds \tag{1.1.20}$$

has a continuous solution $Y(t)$ *for* $0 \le t \le T$. *Then*

$$| y(t) | < Y(t), \quad 0 \le t \le T.$$

The next two theorems regarding qualitative behaviour require constraints on the kernel $k(t,s)$.

Theorem 1.1.9 ([53], page 43) *If in*

$$y(t) = 1 + \int_0^t k(t,s)y(s)ds, \tag{1.1.21}$$

we have

(i) $k(t,s) \le 0$,

(ii) $\frac{\partial}{\partial t}k(t,s) \ge 0$,

for all $0 \le s \le t \le T$, *then the solution of (1.1.21) satisfies* $0 \le y(t) \le 1$.

Theorem 1.1.10 ([53], page 43) *If the conditions of Theorem (1.1.9) hold, and if, in addition*

(i) $\lim_{t\to\infty} \int_0^t k(t,s)ds = -\infty$,

(ii) $\lim_{t\to\infty} y(t)$ *exists,*

then

$$\lim_{t\to\infty} y(t) = 0.$$

Theorems which use particular properties of the kernel are quite common. In Chapter 4, the results of Levin and Nohel for nonlinear equations also require a number of constraints.

Theorem 1.1.11 ([53], page 45) *Let* $f(t)$, $k(t,s)$, $\Delta f(t)$, $\Delta k(t,s)$ *be continuous and bounded by* $|k(t,s)| \le K$, $|\Delta k(t,s)| \le \Delta K$, $|f(t)| \le F$, $|\Delta f(t)| \le \Delta F$.

Let $\hat{y}(t)$ *be the solution of*

$$\hat{y}(t) = f(t) + \Delta f(t) + \int_0^t \{k(t,s) + \Delta k(t,s)\}\hat{y}(s)ds. \tag{1.1.22}$$

Then $\hat{y}(t)$ *satisfies*

$$\begin{aligned} |\hat{y}(t) - y(t)| &\le \{\Delta F + \Delta K t (F + \Delta F) e^{(K+\Delta K)t}\} e^{Kt} \quad (1.1.23) \\ &= O(\Delta F) + O(\Delta K), \end{aligned}$$

where $y(t)$ *is the solution of (1.1.1).*

Theorem 1.1.11 is known as a Gronwall-type theorem.

When the kernel of (1.1.1) is unbounded, it is often convenient to rewrite this equation as

$$y(t) = f(t) + \int_0^t p(t,s)k(t,s)y(s)ds, \tag{1.1.24}$$

where $p(t,s)$ represents the part with the non-smooth behaviour. We now have the following theorem.

Theorem 1.1.12 ([53], page 48) *Assume that in (1.1.1)*

(i) $f(t)$ *is continuous in* $0 \leq t \leq T$,

(ii) $k(t,s)$ *is continuous in* $0 \leq s \leq t \leq T$,

(iii) for each continuous function h *and all* $0 \leq \tau_1 \leq \tau_2 \leq t$ *the integrals*

$$\int_{\tau_1}^{\tau_2} p(t,s)k(t,s)h(s)ds$$

and

$$\int_0^t p(t,s)k(t,s)h(s)ds$$

are continuous functions of t,

(iv) $p(t,s)$ *is absolutely integrable with respect to* s *for all* $0 \leq t \leq T$,

(v) there exist points $0 = T_0 < T_1 < T_2 < \cdots < T_N = T$ *such that with* $t \geq T_i$

$$\int_{T_i}^{\min(t,T_{i+1})} | p(t,s) | ds \leq \alpha < 1, \tag{1.1.25}$$

where

$$K = \max_{0 \leq s \leq t \leq T} | k(t,s) |,$$

(vi) for every $t \geq 0$

$$\lim_{\delta \to 0^+} \int_t^{t+\delta} | p(t+\delta, s) | ds = 0 \tag{1.1.26}$$

Then (1.1.24) has a unique continuous solution in $0 \leq t \leq T$.

Many results of the kind presented here can be generalised to systems of second kind VIE's. The system

$$\begin{aligned} y_1(t) &= f_1(t) + \int_0^t \sum_{i=1}^n k_{1i}(t,s) y_i(s) ds \\ y_2(t) &= f_2(t) + \int_0^t \sum_{i=1}^n k_{2i}(t,s) y_i(s) ds \\ &\vdots \\ y_n(t) &= f_n(t) + \int_0^t \sum_{i=1}^n k_{ni}(t,s) y_i(s) ds \end{aligned}$$

can be rewritten as

$$\mathbf{y}(t) = \mathbf{f}(t) + \int_0^t \mathbf{k}(t,s)\mathbf{y}(s) ds \tag{1.1.27}$$

Using the vector norm

$$\| \mathbf{y}(t) \| = \max_{1 \leq i \leq n} | y_i(t) |, \tag{1.1.28}$$

the induced matrix norm is

$$\| \mathbf{k}(t,s) \| = \max_{1 \leq i \leq n} \sum_{j=1}^n | k_{ij}(t,s) | . \tag{1.1.29}$$

The following theorems for system (1.1.27) are generalisations of theorems for a single equation. The norms defined above make the proofs formally the same.

Theorem 1.1.13 ([53], page 46) *If* $\mathbf{f}(t)$ *and* $\mathbf{k}(t,s)$ *are continuous in* $0 \leq s \leq t \leq T$ *(meaning that all components are continuous), then the system (1.1.27) has a unique continuous solution for* $0 \leq t \leq T$.

Theorem 1.1.14 ([53], page 47) *If the system (1.1.27) possesses a unique continuous solution* $\mathbf{y}(t)$ *in* $0 \leq t \leq T$, *such that* $\mathbf{k}(t,s)\mathbf{y}(s)$ *is absolutely integrable and if*

$$\| \mathbf{f}(t) \| < F(t),$$
$$\| \mathbf{k}(t,s) \| < K(t,s)$$

with continuous K and F, then

$$\| \mathbf{y}(t) \| < Y(t),$$

where $Y(t)$ is the continuous solution of

$$Y(t) = F(t) + \int_0^t K(t,s)Y(s)ds. \tag{1.1.30}$$

1.1.2 Nonlinear Volterra integral equations of the second kind

We consider the nonlinear VIE

$$y(t) = f(t) + \int_0^t k(t,s,y(s))ds, \quad 0 \le t \le T \tag{1.1.31}$$

Many of the results for linear equations are extended by putting conditions on the kernel k of (1.1.31). The method of successive approximations can be generalised for nonlinear equations by ensuring that k is Lipschitz continuous in its third argument; i.e.

$$| k(t,s,w) - k(t,s,z) | \le L \mid w - z \mid \tag{1.1.32}$$

where L is independent of t, s, w and z. The successive iterates are defined by

$$y_n(t) = f(t) + \int_0^t k\left(t, s, y_{n-1}(s)\right) ds \tag{1.1.33}$$

with $y_0(t) = f(t)$.

The method can be used to prove the following theorem.

Theorem 1.1.15 ([53], page 52) *Assume that in (1.1.31) the functions $f(t)$ and $k(t,s,u)$ are continuous in $0 \le s \le t \le T$ and $-\infty < u < \infty$, and that furthermore the kernel satisfies a Lipschitz condition of the form (1.1.32). Then (1.1.31) has a unique continuous solution for all finite T.*

Theorems have been developed for equations with more general kernels, as below. We can use a number of these results to show that, for the equations we consider in Chapters 4, 5 and 6, unique solutions exist under particular conditions.

Theorem 1.1.16 ([53], page 55) *Consider (1.1.31) with $f(t)$ continuous in $0 \le t \le T$. Assume that there exist constants α, β, L such that* *(i)* $\alpha < f(t) < \beta, 0 \leq t \leq T,$

(ii) for all $0 \leq s \leq t \leq T$ *and* $\alpha < u < \beta$*, the kernel* $K(t,s,u)$ *is continuous in all variables,*

(iii) for all $0 \leq s \leq t \leq T$ *and* $\alpha < w < \beta$, $\alpha < z < \beta$ *the kernel satisfies the Lipschitz condition*

$$| K(t,s,w) - K(t,s,z) | \leq L | w - z | .$$

Then there exists a $\delta > 0$ *such that (1.1.31) has a unique continuous solution in* $0 \leq t \leq \delta$.

Some qualitative results for linear equations have been extended, for the nonlinear case, as follows:

Theorem 1.1.17 ([53], page 58) *If* $f(t)$ *is* p *times continuously differentiable in* $[0,T]$ *and*
$K(t,s,u)$ *is* p *times continuously differentiable with respect to all three arguments in* $0 \leq s \leq t \leq T$, $-\infty < u < \infty$,
then the solution of (1.1.31) is p *times continuously differentiable.*

Theorem 1.1.18 ([53], page 58) *Assume that the conditions of Theorem (1.1.15) hold. Let* $y(t)$ *be the continuous solution of (1.1.31) and let* $Y(t)$ *be another continuous function satisfying* $Y(t) = F(t) + \int_0^t H(t,s,Y(s))\,ds$, *where* $F(t)$ *and* $H(t,s,u)$ *are continuous in all arguments, and the following conditions hold:*

(i) $| f(t) | < F(t), 0 \leq t \leq T,$

(ii) for all functions $z_1(t)$, $z_2(t)$ *such that* $| z_1(t) | \leq z_2(t)$, *the inequality*

$$| K(t,s,z_1(t)) | \leq H(t,s,z_2(t))$$

holds for all $0 \leq s \leq t \leq T$.

Then

$$| y(t) | < Y(t), \quad 0 \leq t \leq T.$$

Theorem 1.1.19 ([53], page 60) *Assume that the conditions of Theorem (1.1.15) are satisfied. Let $\hat{y}(t)$ be an approximate solution to equation (1.1.31) such that*

$$r(t) = f(t) + \int_o^t K\left(t, s, \hat{y}(s)\right) ds - \hat{y}(t). \qquad (1.1.34)$$

Then

$$| \, y(t) - \hat{y}(t) \, | \leq Re^{Lt},$$

where $y(t)$ is the solution of (1.1.31) and $R = \max | \, r(t) \, |$.

Theorem 1.1.20 ([53], page 60) *Let the conditions of Theorem (1.1.15) hold, and let $y(t)$ be the continuous solution of the equation*

$$y(t) = 1 + \int_0^t K(t, s, y(s))ds. \qquad (1.1.35)$$

Assume that the kernel satisfies the following conditions:

(i) for all u and $0 \leq s \leq t \leq T$,

$$K(t, s, u) \geq 0 \quad if \quad u \leq 0,$$
$$K(t, s, u) \leq 0 \quad if \quad u \geq 0;$$

(ii) for all $u \geq 0$ and $0 \leq s \leq t \leq T$, the function $K(t, s, u)$ is a nondecreasing function of t.

Then $0 \leq y(t) \leq 1$.

Theorem 1.1.21 ([53], page 61) *Assume that the conditions of Theorem (1.1.20) are satisfied. Furthermore, assume that for every $a > 0$ we have*

$$\lim_{t \to \infty} \int_0^t K\left(t, s, Y(s)\right) ds = -\infty$$

for every $Y(t)$ satisfying $a \leq Y(t) \leq 1$.
Then

$$\lim_{t \to \infty} y(t) = 0,$$

if the limit exists.

The following theorem is a result for equations with unbounded kernels.

Theorem 1.1.22 ([53], page 62) *Consider the equation*

$$y(t) = f(t) + \int_0^t K(t, s, y(s))\, ds \tag{1.1.36}$$

where

(i) $f(t)$ *is continuous in* $0 \le t \le T$,

(ii) $K(t, s, u)$ *is a continuous function in* $0 \le s \le t \le T$, $-\infty < u < \infty$,

(iii) the Lipschitz condition

$$| K(t, s, w) - K(t, s, z) | \le L | w - z |$$

is satisfied for $0 \le s \le t \le T$ *and all* w *and* z,

(iv) $p(t, s)$ *satisfies conditions (iii)-(vi) of Theorem (1.1.12) with* K *replaced by* L *and* $K(t, s, h(s))$ *instead of* $k(t, s)h(s)$.

Then (1.1.36) has a unique continuous solution in $0 \le t \le T$.

As with linear equations, many results for a single nonlinear equation can be extended to apply to a system of nonlinear equations.

Theorem 1.1.23 ([53], page 62) *Consider the system of equations*

$$\mathbf{y}(t) = \mathbf{f}(t) + \int_0^t \mathbf{K}(t, s, \mathbf{y}(s))\, ds. \tag{1.1.37}$$

Assume that

(i) $\mathbf{f}(t)$ *is continuous (i.e. every component is continuous),*

(ii) $\mathbf{K}(t, s, \mathbf{u})$ *is a continuous function for* $0 \le s \le t \le T$, $-\infty < \| \mathbf{u} \| < \infty$,

(iii) the kernel satisfies the Lipschitz condition

$$\| \mathbf{K}(t, s, \mathbf{w}) - \mathbf{K}(t, s, \mathbf{z}) \| \leq L \| \mathbf{w} - \mathbf{z} \|$$

where the norm is as defined in (1.1.6).

Then (1.1.37) has a unique continuous solution in $0 \leq t \leq T$.

For Theorem (1.1.3) there is no analogous result for nonlinear equations. However, a result has been established which establishes a relationship between certain nonlinear equations and related linear forms.

Theorem 1.1.24 ([53], page 64) *Consider the nonlinear equation*

$$y(t) = f(t) + \int_s^t k(t, s)\{y(s) + H(s, y(s))\}ds. \tag{1.1.38}$$

Let $\Gamma(t, s)$ *be the resolvent kernel for* $k(t, s)$ *as given in (1.1.15). Let* $Y(t)$ *be defined by*

$$Y(t) = f(t) + \int_0^t \Gamma(t, s)f(s)ds. \tag{1.1.39}$$

Assume that all functions involved are continuous and such that (1.1.38) has a unique continuous solution on $[0, T]$ *and* $\Gamma(t, s)$ *is a continuous function. Then* $y(t)$ *satisfies*

$$y(t) = Y(t) + \int_0^t \Gamma(t, s)H(s, y(s))ds. \tag{1.1.40}$$

1.1.3 Convolution equations

The convolution equations discussed here are a special case of linear VIEs of the second kind. They are studied in great depth due to their practical use in mathematical modelling. The results presented in this section are directly relevant to Chapter 3, in which our linear test equation is a convolution equation.

Definition 1.1.25 *The linear convolution equation of the second kind is defined as*

$$y(t) = f(t) + \int_0^t k(t - s)y(s)ds \tag{1.1.41}$$

Hence a convolution equation is one with a convolution kernel. Consider first, a kernel of the form

$$k(t-s) = \sum_{i=0}^{n} a_i (t-s)^i \tag{1.1.42}$$

so that the equation (1.1.41) reduces to

$$y(t) = f(t) + \sum_{i=0}^{n} a_i \int_0^t (t-s)^i y(s) ds. \tag{1.1.43}$$

Theorem 1.1.26 ([53], page 79) *If $f(t)$ is continuous on $[0,T]$, then the solution to (1.1.43) is given by*

$$y(t) = f(t) + \sum_{i=0}^{n} a_i w_i(t),$$

where the w_i are the solution of the system

$$w_0'(t) = f(t) + \sum_{i=0}^{n} a_i w_i(t),$$

$$\begin{aligned} w_i'(t) &= i w_{i-1}(t), \quad i = 1, 2, \ldots, n, \\ w_0(0) &= w_1(0) = \ldots = w_n(0) = 0. \end{aligned}$$

Theorem 1.1.27 ([53], page 83) *If $k(t)$ is of polynomial form (1.1.42), then its resolvent kernel $R(t)$ is the solution of the homogeneous linear differential equation with constant coefficients*

$$R^{n+1}(t) = a_0 R^{(n)}(t) + a_1 R^{(n-1)}(t) + \cdots + n! a_n R(t),$$

with initial conditions

$$\begin{aligned} R(0) &= a_0, \\ R'(0) &= a_1 + a_0^2, \\ R''(0) &= 2a_2 + a_0\left(a_1 + a_0^2\right) + a_1 a_0, \\ &\vdots \end{aligned}$$

Laplace transforms play a large part in studying linear convolution equations of the second kind. First, we define a shorthand notation for writing down the convolution of two functions, y_1 and y_2:

$$(y_1 * y_2)(t) = \int_0^t y_1(t-s)y_2(s)ds \tag{1.1.44}$$

The Laplace transform of a function $y(t)$ is denoted here by y^* or $\mathcal{L}(y)$. It is defined as

$$y^*(w) = \mathcal{L}(y)(w) = \int_0^\infty e^{-wt}y(t)dt \tag{1.1.45}$$

The Laplace transform operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1}$ given by

$$\mathcal{L}^{-1}(u)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{wt}u(w)dw \tag{1.1.46}$$

such that

$$\mathcal{L}^{-1}(y^*)(t) = y(t) \tag{1.1.47}$$

The useful application of Laplace transforms arises primarily from the following theorem.

Theorem 1.1.28 ([53], page 84) *Let y_1 and y_2 be two functions which are absolutely integrable over some interval $[0,T]$ and which are bounded in every finite subinterval not including the origin. If, furthermore $\mathcal{L}(y_1)$ and $\mathcal{L}(y_2)$ are absolutely convergent for $w \geq w_0$, then $\mathcal{L}(y_1 * y_2) = \mathcal{L}(y_1) \cdot \mathcal{L}(y_2)$, $\Re w \geq w_0$. In other words, the Laplace transform of a convolution is the product of the individual transforms.*

Consider equation (1.1.41). This can be rewritten in the following form

$$y = f + k * y. \tag{1.1.48}$$

Applying the Laplace transform to this, noting that $\mathcal{L}$ is a linear operator, gives

$$y^* = f^* + k^*y^*. \tag{1.1.49}$$

Solving this for y^*,

$$y^* = \frac{f^*}{1-k^*}, \tag{1.1.50}$$

and applying the inverse Laplace transform, gives the solution

$$y = \mathcal{L}^{-1}\left(\frac{f^*}{1-k^*}\right). \tag{1.1.51}$$

In practice, it is difficult to solve equations analytically using the Laplace transform method because the inverse Laplace transform is only known for a relatively small number of functions. The main advantage of the Laplace transform method is that it allows us to obtain some qualitative information regarding the behaviour of the solution in a simple way.

Results obtained for linear VIEs of the second kind can be applied immediately to linear second kind convolution equations. Since the latter are a special case of the former, the results sometimes simplify. For example, theorem 1.1.9 becomes

Theorem 1.1.29 *If the kernel $k(t)$ in the equation*

$$y(t) = 1 + \int_0^t k(t-s)y(s)ds \tag{1.1.52}$$

satisfies $k(t) < 0$ and $k'(t) > 0$ then the solution of (1.1.52) satisfies $0 \leq y(t) \leq 1$.

Important results regarding the asymptotic behaviour of solutions to (1.1.41) have been developed by Paley and Wiener (see [62]).

If we know something about the asymptotic behaviour of the f, what restrictions must we put on the kernel k so that the solution y displays the same asymptotic behaviour? According to [30],† if $f \in L^1(\mathbb{R}^+;\mathbb{C}^n)$, or $f \in L^\infty(\mathbb{R}^+;\mathbb{C}^n)$, then a necessary and sufficient condition exists for y to display the same asymptotic behaviour as f. This condition is $r \in L^1(\mathbb{R}^+;\mathbb{C}^{n\times n})$, where r is the resolvent of the kernel k. A proof of this is given in [30]. For the condition itself to hold, other conditions must be satisfied and these are presented in the Paley-Wiener theorems. We present the theorems without proof. Their results are discussed in more detail and proofs are presented in [30].

Theorem 1.1.30 ([30], page 45) *Let $k \in L^1(\mathbb{R}^+;\mathbb{C}^{n\times n})$. Then the resolvent r of k satisfies*

$$r \in L^1\left(\mathbb{R}^+;\mathbb{C}^{n\times n}\right)$$

if and only if

$$\det[I + \hat{k}(z)] \neq 0,\ \Re z \geq 0.^{‡}$$

† for the vector-valued equation of the form (1.1.41)

‡ $\hat{k}(z)$ is Laplace transform of k.

Theorem 1.1.31 ([30], page 46) *Let* $k \in L^1(\mathbb{R}; \mathbb{C}^{n\times n})$. *Then there is a function*

$$r \in L^1\left(\mathbb{R}; \mathbb{C}^{n\times n}\right)$$

satisfying the two equations $r + k * r = r + r * k = k$, *if and only if*

$$\det[I + \hat{k}(z)] \neq 0, \ \Re z = 0.$$

This function r, *known here as the whole line resolvent of* k, *is unique.*

Theorem 1.1.30 is known as the half-line Paley-Wiener theorem, and Theorem 1.1.31 is known as the whole-line Paley-Wiener theorem. These theorems are stated for systems of equations. They can be simplified to special cases for scalar equations. The discrete versions of Paley & Wiener play an important role in numerical methods.

1.2 Volterra integro-differential equations

Volterra integro-differential equations (VIDEs) involve derivatives of the unknown function as well as integral terms. The presence of both derivatives and integrals allows for many different forms of the equation; each form having its own advantages and disadvantages when deciding on a method of (approximate) solution. This variety makes classification of VIDEs rather difficult. A few examples of VIDEs are given below.

Examples

$$y'(t) - \int_0^t k(t,s)y(s)ds = f(t) \tag{1.2.1}$$

$$y''(t) + a(t)y'(t) + b(t)y(t) + \int_0^t k(t,s)y(s)ds = f(t). \tag{1.2.2}$$

It is sometimes convenient to reduce VIDEs to a single second kind VIE or a system of VIEs of the second kind, and apply the generalised forms of results obtained for single VIEs to the system when considering solutions and qualitative behaviour. For example, equation (1.2.1) can be reduced to a system of two VIEs by direct integration. Results on the properties of VIDEs are usually derived from results for VIEs and ODEs.

1.2.1 Linear Volterra integro-differential equations of convolution type

In this section (which is directly relevant to our test equation in Chapter 3), we consider the integro-differential equation

$$y' + (\mu * y) = f,\ y(0) = y_0, \tag{1.2.3}$$

where y is the unknown function and f is the forcing term. This equation is discussed in some detail in [30] and this text is the source for much of the material here. We begin first, by differentiating the convolution equation of the second kind

$$y(t) + \int_0^t k(t-s)y(s)ds = f(t),\ t \in \mathbb{R}^+ \tag{1.2.4}$$

to give

$$y'(t) + k(0)y(t) + \int_0^t k'(t-s)y(s)ds = f'(t),\ t \in \mathbb{R}^+;\ y(0) = f(0). \tag{1.2.5}$$

This is an example of the more general equation

$$y'(t) + \int_0^t \mu y(t-s)ds = f(t),\ t \in \mathbb{R}^+;\ y(0) = y_0, \tag{1.2.6}$$

where $\mu(s)$ is a measure.

Grippenberg *et al* [30] shows that the solution of equation (1.2.6) can be given by a variation of constants formula, similar to the corresponding formula for ordinary differential equations (ODEs) (see [37], for example). Laplace Transforms can be used to obtain the formula. Taking transforms of both sides of (1.2.6):

$$wy^*(w) - y_0 + \mu^*(w)y^*(w) = f^*(w), \tag{1.2.7}$$

and hence

$$y^*(w) = [wI + \mu^*(w)]^{-1}(y_0 + f^*(w)). \tag{1.2.8}$$

Note that the Laplace transform of the Borel measure μ is defined as

$$\mu^*(w) = \int_0^t \mu(s) e^{-ws} ds, \tag{1.2.9}$$

and the convolution of a measure μ and a function a (both defined on $\mathbb{R}^+$ is defined as the function

$$(\mu * a)(t) = \int_0^t \mu(s) a(t-s) ds. \tag{1.2.10}$$

Suppose that there exists a locally integrable function r whose Laplace transform is given by

$$r^*(w) = [wI + \mu^*(w)]^{-1}. \qquad (1.2.11)$$

Then y is given by the variation of constants formula

$$y(t) = r(t)y_0 + (r * f)(t), \ t \in \mathbb{R}^+. \qquad (1.2.12)$$

Equation (1.2.6) can be written as

$$y'(t) + (\mu * y)(t) = f(t), \ t \in \mathbb{R}^+; \ y(0) = y_0. \qquad (1.2.13)$$

1.2.2 Nonlinear Volterra integro-differential equations of convolution type

Nonlinear convolution VIDEs are the main type equation we study in chapters 4, 5 and 6. An example of a nonlinear VIDE with a convolution kernel is

$$y'(t) + \int_0^t k(t-s)g(y(s))ds = f(t), \ t \in \mathbb{R}^+. \qquad (1.2.14)$$

Special cases of this equation have been studied (for example, see [50], [51], [13]).

We consider the homogeneous equation

$$y'(t) = -\int_0^t k(t-s)g(y(s))ds, \ t \in \mathbb{R}^+, \qquad (1.2.15)$$

under the following conditions: (1) $k(t)$ is completely monotone.† (2) $g(y) \in C(-\infty,\infty), yg(y) > 0 (y \neq 0)$ (and hence $g(y)$ and y always have the same sign, $g(0) = 0$). (3) $G(y) := \int_0^y g(\xi)d\xi \to \infty$ as $|y| \to \infty$. Levin and Nohel in [50] give the following theorem for equation (1.2.15) subject to the above conditions.

Theorem 1.2.1 (Levin and Nohel) *Any solution $y(t)$ of (1.2.15) subject to (1), (2), (3) is asymptotically stable‡ providing $k(t)$ is not the constant function $k(0)$.*

† See chapter 5 of [30] for definition and discussion of 'completely monotone'.

‡ See definition 1.5.2 for definition of 'asymptotically stable'.

1.3 Delay Volterra integro-differential equations

A Delay Volterra integro-differential equation (DVIDE) is an equation involving the derivative of the unknown function, an integral operator of Volterra type, and a dependency on a previous state of the system. An example of a DVIDE is the equation

$$y'(t) = \xi y(t) + \nu \int_{t-\tau}^{t} k(t-s)y(s)ds + g(t), \ t > 0, \tag{1.3.1}$$

where ξ, ν are constant. This is a linear DVIDE with a convolution and a fixed finite delay τ. This is an example of the far more general kernel,

$$y'(t) = G\left(t, y(t), \int_{t-\tau}^{t} K(t,s,y(s))ds\right), \ t \geq 0. \tag{1.3.2}$$

We consider a specific delay equation in Chapter 7.

1.4 An informal definition of stability

Different texts have different definitions of stability. It is important,to avoid ambiguity, that we give a clear definition of what we mean by stability. Saaty and Bram [69] give a good informal account of stability in relation to ODEs (which is relevant, since VIDEs and IEs can be expressed as ODEs). The informal definition follows:

"One is usually interested in the solution of a differential equation if the initial conditions or the right side of the equation is changed. This change corresponds to a real-life situation in which, for example, the differential equation maybe an idealisation which assumes a calm atmosphere, whereas the true state is described by a turbulent atmosphere which buffets a missile and hence comprises a disturbance of its trajectory. The simplest case is that in which the initial conditions are changed (disturbed). The existence theorem guarantees a unique solution of the differential equation for each choice of initial conditions. Thus for two initial conditions, i.e. the original and the disturbed, we obtain two solution, and the following question arises: Will the difference between the trajectories remain bounded, tend to zero, oscillate, or grow without bound as time goes on? If the trajectories remain bounded or oscillate (without any growth or decay) then the system is stable. If the trajectories tend to zero then the system is asymptotically stable. Otherwise the system is unstable."

There exist a variety of methods for investigating the stability of a system. For example: linearisation (see [54] for example), Lyapunov's Direct Method (see [57] for example), frequency domain methods(see [61] for example). This list is not exhaustive. We are interested in the method attributed to Lyapunov because it is widely used in the stability of analysis of nonlinear Volterra equations (see [16], [51], [50], for example).

1.5 Stability of Volterra integral and functional equations: Lyapunov approach

Lyapunov's method is described in [57]. However, here we choose Saaty and Bram's description of the method (see [69]) because of its clarity. They describe the method informally as it is applied to ODEs:

"Lyapunov's second method examines the stability of a differential equation without the use of explicit functions. It generally applies to a free system, i.e. an unforced system having the origin as a point of equilibrium. Stability itself is concerned with deviations about an equilibrium point. Thus stability means that if the initial conditions of the trajectory of an undisturbed motion are disturbed slightly from equilibrium at the origin, then subsequent motions remain in a small neighbourhood of the origin."

1.5.1 Stability of solutions to ordinary differential equations

Here we re-present precise definitions of stability, instability and asymptotic stability of equilibrium points of planar autonomous systems, i.e. systems of the form

$$\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}). \tag{1.5.1}$$

These definitions can be found in [35].

Definition 1.5.1 *An equilibrium point $\bar{\mathbf{y}}$ of an autonomous system* $\mathbf{y}' =$ y) is said to be stable if, for any given $\epsilon > 0$, there is a $\delta > 0$ (depending only on ϵ) such that, for every $\mathbf{y}_0$ for which $\|\mathbf{y}_0 - \bar{\mathbf{y}}\| < \delta$, the solution $\phi(t, \mathbf{y}_0)$ of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ through $\mathbf{y}_0$ at $t = 0$ satisfies the inequality $\|\phi(t, \mathbf{y}_0) - \bar{\mathbf{y}}\| < \epsilon$ for all $t \geq 0$. The equilibrium $\bar{\mathbf{y}}$ is said to be unstable if it is not stable, that is, there is an $\nu > 0$ such that, for any $\delta > 0$, there is an $\mathbf{y}_0$ with $\|\mathbf{y}_0 - \bar{\mathbf{y}}\| < \delta$ and $t_{\mathbf{y}_0} > 0$ such that $\|\phi(t_{\mathbf{y}_0}, \mathbf{y}_0) - \bar{\mathbf{y}}\| = \nu$.*

* [35] uses 'planar' autonomous system.

Definition 1.5.2 *An equilibrium point* $\bar{\mathbf{y}}$ *is said to be asymptotically stable if it is stable and, in addition, there is an* $r > 0$ *such that* $\|\phi(t, \mathbf{y}_0) - \bar{\mathbf{y}}\| \to 0$ *as* $t \to +\infty\, \forall\, \mathbf{y}_0$ *satisfying* $\|\mathbf{y}_0 - \bar{\mathbf{y}}\| < r$.

In [35] the following theorem is presented.

Theorem 1.5.3 *If all the eigenvalues of the coefficient matrix* $A \in \mathbb{R}^{n \times n}$ *in the linear system* $\mathbf{y}' = A\mathbf{y}$ *have negative real parts, then its equilibrium point* $\bar{\mathbf{y}} = \mathbf{0}$ *is asymptotically stable. Moreover, there are positive constants* K *and* α *such that*

$$\|e^{At}\mathbf{y}_0\| \leq Ke^{-\alpha t}\|\mathbf{y}_0\| \; \forall t \geq 0, \; \mathbf{y}_0 \in \mathbb{R}^n.$$

If one of the eigenvalues of the coefficient matrix A *has positive real part, then the equilibrium point* $\bar{\mathbf{y}} = \mathbf{0}$ *is unstable.*

We now introduce Lyapunov functions and briefly state some results of Lyapunov [57].

Theorem 1.5.4 (Lyapunov) *Let* $\bar{\mathbf{y}} = \mathbf{0}$ *be an equilibrium point of* $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ *and* V *be a positive definite differentiable function with continuous first derivatives, on a neighbourhood* U *of* $\mathbf{0}$.†

(i) If $V'(\mathbf{y}) \leq 0$ *for* $\mathbf{y} \in U \setminus \{0\}$, *then* $\mathbf{0}$ *is stable.*

(ii) If $V'(\mathbf{y}) < 0$ *for* $\mathbf{y} \in U \setminus \{0\}$, *then* $\mathbf{0}$ *is asymptotically stable.*

(iii) If $V'(\mathbf{y}) > 0$ *for* $\mathbf{y} \in U \setminus \{0\}$, *then* $\mathbf{0}$ *is unstable.*

Definition 1.5.5 *A positive definite function* V *on an open neighbourhood* U *of the origin is said to be a Lyapunov function for* $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ *if* $V'(\mathbf{y}) \leq 0$ *for all* $\mathbf{y} \in U \setminus \{0\}$. *When* $V'(\mathbf{y}) < 0$ *for all* $\mathbf{y} \in U \setminus \{0\}$, *the function* V *is called a strict Lyapunov function.*

Using a Lyapunov function in this way to determine the stability of an equilibrium point is known as Lyapunov's direct method. A recognised difficulty in applying this method is that, for some systems, it may not be easy to find an appropriate positive definite function (see, for example, [44], p. 107-108).

1.5.2 Direct method of Lyapunov for difference equations

For the analysis of discrete equations we use discrete Lyapunov functions. The conventional discrete Lyapunov theory is for difference equations of fixed

† $V'(y)$ means: $\frac{\partial V}{\partial y^i} \cdot \frac{dy^i}{dt} = \frac{\partial V}{\partial y^i} f^i(y) = f(V)$, $f : \mathbb{R}^n \to \mathbb{R}^n$.

$V(0) = 0$.

finite order. We present this first, followed by a variation which applies to difference equations of unbounded order. The latter method is directly relevant to our discretisations in later chapters. This is because the integral term in the integro-differential equations leads to difference equations of unbounded order when discretised.

Direct method of Lyapunov for difference equations of bounded order

This material is a re-presentation from [22] of an adaption of Lyapunov's method for difference equations of bounded order.

Consider the difference equation

$$y(n+1) = f(y(n)) \tag{1.5.2}$$

where $f : G \to \mathbb{R}^k$, $G \subset \mathbb{R}^k$, is continuous. We assume that y^* is an equilibrium point of (1.5.2), that is $f(y^*) = y^*$.

Let $V : \mathbb{R}^k \to \mathbb{R}$ be defined as a real valued function. The variation of V relative to (1.5.2) would then be defined as

$$\Delta V(y) = V(f(y)) - V(y) \tag{1.5.3}$$

and

$$\Delta V(y(n)) = V(f(y(n))) - V(y(n)) = V(y(n+1)) - V(y(n)). \tag{1.5.4}$$

Notice that if $\Delta V(y) \leq 0$, then V is nonincreasing along solutions of (1.5.2). The function V is said to be a *Lyapunov function* on a subset H of $\mathbb{R}^k$ if

(i) V is continuous on H and

(ii) $\Delta V(y) \leq 0$ whenever y and $f(y) \in H$.

Let $B(y, \gamma)$ denote the open ball in $\mathbb{R}^k$ of radius γ and centre y defined by $B(y, \gamma) = \{z \in \mathbb{R}^k : \| z - y \| < \gamma\}$. For the sake of brevity, $B(0, \gamma)$ will henceforth be denoted by $B(\gamma)$. We say that the real-valued function V is *positive definite* at y^* if

(i) $V(y^*) = 0$ and

(ii) $\Delta V(y) > 0$ for all $y \in B(y^*, \gamma)$, for some $\gamma > 0$.

We now present Lyapunov's first Stability Theorem for difference equations of bounded order.

Theorem 1.5.6 (Elaydi [22]) *If V is a Lyapunov function for equation (1.5.2) on a neighbourhood H of the equilibrium point y^*, and V is positive definite with respect to y^*, then y^* is stable. If, in addition, $\Delta V(y) < 0$, whenever y, $f(y) \in H$ and $y \neq y^*$, then y^* is asymptotically stable. Moreover, if $G = H = \mathbb{R}^k$ and $V(y) \to \infty$ as $\| y \| \to \infty$ then y^* is globally asymptotically stable.*

Direct method of Lyapunov for difference equations of unbounded order

This method is presented in [16] in some detail. Earlier references to discrete Lyapunov functions are contained in [48], for example. We present the required results below.

Definition 1.5.7 (Crisci *et al* [16]) *For the difference equation*

$$y_{n+1} = f(n, y_n, \dots, y_0) \tag{1.5.5}$$

a discrete Lyapunov function $V(n, y_n, \dots, y_0)$ is a function that satisfies the hypotheses of Theorem 1.5.8.

Theorem 1.5.8 (Crisci *et al* [16]) *Let $V_i(y_0, y_1, \cdots, y_i)$ be, for each natural number i, a scalar function continuous with respect to all its arguments, which satisfies:*

1. $V_0(0) = 0$

2. $V_i(y_0, y_1, \cdots, y_i) \geq \omega_1(||y_i||)$

3. $\Delta V_i = V_{i+1}(y_0, y_1, \cdots, y_i, y_{i+1}) - V_i(y_0, y_1, \cdots, y_i) \leq 0$ then the solution $y_n = 0$ of (1.5.5) is stable.

4. *If, in addition, $\Delta V_i \leq -\omega_2(||y_i||)$ then the solution of (1.5.5) is asymptotically stable.*

Here, the functions ω_i are assumed to be scalar increasing functions that satisfy $\omega_i(0) = 0$.

1.5.3 Direct method of Lyapunov for Volterra integro-differential equations

This section describes a generalisation of Lyapunov's direct method. This generalisation is described in more detail by Crisci *et al* [16]. Consider the nonlinear system of VIDE's

$$\mathbf{y}'(t) = \mathbf{G}(t, \mathbf{y}(t)) + \int_0^t \mathbf{k}(t, s, \mathbf{y}(s))ds, t \geq 0, \tag{1.5.6}$$

subject to the initial conditions

$$\mathbf{y}(0) = \mathbf{y}_0, \tag{1.5.7}$$

where $\mathbf{y}, \mathbf{G}$ and $\mathbf{k}$ are real continuous functions satisfying local Lipschitz conditions with respect to $\mathbf{y}$, and $\mathbf{G}(t, \mathbf{0}) \equiv \mathbf{0}; \mathbf{k}(t, s, \mathbf{0}) \equiv \mathbf{0}$ for all $t, s \geq 0$.

Definition 1.5.9 (Crisci *et al* **[16])** *The trivial solution of equation (1.5.6) defined by the zero initial condition* $\mathbf{y}(0) = \mathbf{0}$ *will be called*

(i) stable if for any $\epsilon > 0$ *there exists* δ_ϵ *such that* $||\mathbf{y}(t)|| \leq \epsilon, t \geq 0$, *for any initial vector* $\mathbf{y}_0$ *with* $||\mathbf{y}_0|| \leq \delta$;

(ii) asymptotically stable if it is stable and $\lim_{t\to\infty} \mathbf{y}(t) = 0$, *for all vectors* $\mathbf{y}_0 \in D$ *where* D *is some neighbourhood of the origin. Sometimes* D *is called the domain of attraction of the trivial solution.*

Definition 1.5.10 (Crisci *et al* **[16])** *A functional* $\mathbf{V}(t, \mathbf{y}_t)$ *is called positive definite (decrescent) if there exists a function* $\omega_1(r)$ *(function* $\omega_2(r)$*) such that* $\mathbf{V}(t, \mathbf{y}_t) \geq \omega_1(||\mathbf{y}(t)||)$ *(such that* $\mathbf{V}(t, \mathbf{y}_t) \leq \omega_2(\sup_{-t \leq \theta \leq 0} ||\underline{y}(t+\theta)||)$.

Lemma 1.5.11 (Crisci *et al* **[16])** *The trivial solution of (1.5.6) is asymptotically stable if there exists a scalar continuous functional* $\mathbf{V}(t, \mathbf{y}_t)$ *which, for any solution* $\mathbf{y}(t)$ *of the problem (1.5.6), (1.5.7), belonging to* D *for all* t, *is positive definite, decrescent and its right upper total derivative denoted as* $\frac{d^+\mathbf{V}(t,\mathbf{y}_t)}{dt}$ *is negative definite.*

Theorem 1.5.12 (Crisci *et al* **[16])** *Assume that*

$$a > \sup_{t>0} \int_0^\infty \kappa(t+s, t)ds \tag{1.5.8}$$

where

$$a = -\sup_{t\geq 0, \mathbf{y}\in D} \gamma(\mathbf{g}(t,\mathbf{y})) \tag{1.5.9}$$

and k is of the form

$$\|k(t,s,\mathbf{y})\| \leq \kappa(t,s)\|\mathbf{y}\|. \tag{1.5.10}$$

Then the trivial solution of (1.5.6) is asymptotically stable.

1.6 Some applications giving rise to Volterra integral and functional equations

In this section we discuss the application of Volterra equations to the modelling of real-life situations. We present some example models which have been studied previously and are indicative of the sort of situations where Volterra equations occur.

"Volterra equations arise most naturally in certain types of time-dependent problems whose behaviour at time t depends not only on the state at that time, but also on the states at previous times." (Linz [53], page 13). Models of such problems are called systems with memory. The integral term in the model (and, in delay equations, terms involving a time lag) provide a natural way for modelling memory in a system. A clear example of where Volterra equations are naturally suited is population dynamics. The birth rate of a population is, to a large extent, dependent on a previous state of the population; the gestation period of the species, for example, must be taken into account. Similarly, a large natural disaster in a population's past which greatly reduces its size is also going to a have a bearing on the population's current state. Equations which model population dynamics are often termed evolutionary equations. Such areas of study can be grouped under the more general heading of mathematical biology.

Biology and ecology are not the only fields to benefit from Volterra equations. They also occur in mechanical systems and renewal theory. Integral equations are also useful in models where differential equations may first seem to be the more natural choice. This is because differential equations may be represented by integral equations. Conversely, differential equations may the preferred choice for some models when integral equations seem more appropriate.

1.6.1 An application to population dynamics

The study of Volterra equations began with research, by Volterra, into population dynamics (see [72]). Consider the Malthus model for population growth

$$\frac{dN(t)}{dt} = \alpha N(t), \; t \geq 0, \tag{1.6.1}$$

with initial condition

$$N(0) = N_0, \tag{1.6.2}$$

where $N(t)$ is the number of individuals alive in the population at time t and α is a constant representing the difference between the birth and death rates. The Malthus model is very simplistic and, as a result, flawed in that the rate of change of the population $\frac{dN(t)}{dt}$ depends only on the number of individuals alive at time t. The population will either grow exponentially ($\alpha > 0$), decay exponentially ($\alpha < 0$) or remain constant ($\alpha = 0$).

The assumptions made by the Malthus model are rarely realistic and models which are to be of any use must be far more complicated. For example, suppose that the environment in which the population lives changes over time, possibly due to a reduction in the food supply or pollution. Now, instead of α being a constant, it will change with time t. However, not only will it be determined by the current state of the population but also the past states of the population will have some effect (since they have had an effect on the current state of the environment). Thus α now becomes a function of t incorporating a history-dependent (or memory) term. For example,

$$\alpha(t) = \alpha_0 - \int_0^t k(t-s)N(s)ds. \tag{1.6.3}$$

Equation (1.6.1) now becomes the Volterra integro-differential equation

$$\frac{dN(t)}{dt} = N(t)\left(\alpha_0 - \int_0^t k(t-s)N(s)ds\right), \tag{1.6.4}$$

with a convolution kernel. The equation used by Volterra in his research was actually

$$\frac{dN(t)}{dt} = N(t)\left(\alpha_0 - \alpha_1 N^2(t) - \int_0^t k(t-s)N(s)ds\right), \tag{1.6.5}$$

with the $-\alpha_1 N^2(t)$ term introduced to account for competition for resources between members of the population. This term tends to inhibit the growth of the population.

1.6.2 An application to renewal problems in industry

Applying Volterra equations to renewal problems was first discussed by Feller (see [24]). Suppose a machine in a factory requires a particular component which is subject to failure over time. In general, the failure time is a random variable with a probability density function $\rho(t)$ such that the probability of failure in a small interval $(t, t+\Delta t)$, of a component which was new at time t', is $\rho(t-t')\Delta t$. If every component eventually fails then

$$\int_{t'}^{\infty} \rho(t-t')dt = 1, \tag{1.6.6}$$

for every t', so that $\rho(t)$ must satisfy

$$\int_{0}^{\infty} \rho(t)dt = 1. \tag{1.6.7}$$

If the component is replaced when it fails (and the next component is replaced on failure, and so on) then of practical interest to the maintainers of the machine is the renewal density, $h(t)$ say, which measures the probability for the need of a replacement. We define $h(t)$ so that the probability that a replacement must be made in the interval $(t, t+\Delta t)$ is given by $h(t)\Delta t$. The probability that a replacement is needed is the sum of the probability that the first failure occurs in $(t, t+\Delta t)$, and the probability that a renewal was made at time t', followed by another failure after $t-t'$ units of time. Adding all contributions together and taking the limit as $\Delta t \to 0$ yields the renewal equation

$$h(t) = \rho(t) + \int_{0}^{t} h(t')\rho(t-t')dt'. \tag{1.6.8}$$

1.6.3 An application to nuclear reactor dynamics

Functional differential equations (FDEs) are often used to model the dynamics and stability of nuclear reactors. We briefly state three models which are discussed in [44] (page 16). The model involves delays which actually occur in the reactor for a number of reasons; i.e. the time it takes heat to diffuse to another part of the reactor, the snapping time of the control system, etc. The first two models are each a pair of coupled delay differential equations and the third model is a system of four such equations.

$$\left.\begin{aligned} \dot{x}(t) &= (ax(t) + by(t-h))(1+x(t)), \\ \dot{y}(t) &= x(t) - y(t); \end{aligned}\right\} \tag{1.6.9}$$

$$\left.\begin{array}{l}\dot{x}(t) = (\phi(x(t-h_1)) + \psi(y(t-h_2)))(1+x(t)),\\ \dot{y}(t) = x(t) - y(t);\end{array}\right\} \tag{1.6.10}$$

These first two models above do not take into account the delayed neutrons. The third model below does.

$$\left.\begin{array}{l}\dot{x}(t) = (a_1\theta_1(t) + a_2\theta_2(t))(1+x(t)) - a_3(x(t)-y(t)),\\ \dot{y}(t) = a_4(x(t)-y(t)),\\ \gamma\dot{\theta}_1(t) = (1-a)x(t) - b(\theta_1(t)-\theta_2(t)),\\ \dot{\theta}_2(t) = \alpha\theta_2(t-h) - \theta_2(t) + ax(t) + b(\theta_1(t)-\theta_2(t)).\end{array}\right\} \tag{1.6.11}$$

In each model, $x(t)$ is the relative change of neutron density, $y(t), \theta_1(t), \theta_2(t)$ are proportional to the relative change in temperature of the reactor, fuel and de-acceleration device, respectively. In the third model (1.6.11), the delay h is the time of liquid fuel transportation along a circular contour.

These models may be studied in their current form or integrated first to produce systems of VDIEs.

1.6.4 Another application to population dynamics

This model discussed here appears in [30] (page 5). Consider a population x which has an age distribution $x(t,a), t \geq 0, a \geq 0$. Therefore $\int_A x(t,a)da$ is the number of individuals with age in the set A at time t. It is assumed that the process of aging and dying is modelled by the balance law

$$\frac{\partial y(t,a)}{\partial t} + \frac{\partial y(t,a)}{\partial a} = -m(a)y(t,a), \tag{1.6.12}$$

where the non-negative function m denotes the age-dependent death rate. Suppose now that the birth process satisfies the integral equation

$$y(t,0) = \int_{\mathbb{R}+} y(t,a)b(a)da, \tag{1.6.13}$$

where b is the age-dependent fertility. The initial age distribution $y(0,a) = \phi(a)$ is known. Equation (1.6.12) can be solved using the method of characteristics (a known analytical method for certain types of partial differential equations (PDEs)). Using the initial condition, the following may be obtained

$$y(t,a) = \begin{cases} \phi(a-t)e^{\left(-\int_0^t m(s+a-t)ds\right)}, & 0 \leq t < a,\\ y(t-a,0)e^{\left(-\int_0^a m(s)ds\right)}, & t \geq a. \end{cases} \tag{1.6.14}$$

Substituting (1.6.14) into (1.6.13) yields the linear Volterra equation

$$y(t,0) + \int_0^t k(t-s)y(s,0)ds = f(t), \; t \geq 0, \tag{1.6.15}$$

where

$$k(t) = -b(t)e^{\left(-\int_0^t m(s)ds\right)}, \tag{1.6.16}$$

and

$$f(t) = \int_{\mathbb{R}+} \phi(s)e^{\left(-\int_0^t m(s+\sigma)d\sigma\right)}b(t+s)ds. \tag{1.6.17}$$

Equation (1.6.15) is the classical renewal equation again.

Chapter 2

Numerical methods

The purpose of this chapter is to introduce the numerical methods that we use in later chapters for solving integro-differential equations. We introduce the idea of combining a numerical method for solving ODEs (a simple Euler rule) with a quadrature rule (a θ method) to produce a hybrid numerical method for solving VIDEs. This is the technique we use in later chapters. The reason we need numerical methods is that relatively few equations encountered in practice can be solved analytically. Techniques for finding approximate solutions must be used. One such technique is to use numerical methods.

Numerical methods are used to provide a discrete equation which then has to be solved. Therefore, to approximate a Volterra equation we must discretise the problem and solve this new problem. This is a complicated process. Even for simple numerical schemes, the analysis is not straightforward. In [39], Iserles quotes Nick Trefethen, saying that "any mathematical problem, upon discretisation, becomes a more challenging mathematical problem". As well as discussing the methods we use, we also give references to alternative numerical methods.

We concentrate on discussing linear multi-step methods and Runge-Kutta methods because our choice of numerical scheme for solving equations in later chapters falls into both categories. Although the method is simple it is important in that results derived for this method provide a basis from which results can be developed for complicated methods (either linear multistep or Runge-Kutta). This approach has proved useful in the fields of both ODEs (see [38]) and Volterra integral equations (see [26]). As such, we expect it to be equally useful in the field of Volterra integro-differential equations. The problems we are concerned with can be thought of as initial value problems of differential systems. It is therefore appropriate to discuss numerical methods from this standpoint.

2.1 Convergence and consistency of numerical methods

When discussing numerical methods, we need to consider convergence, consistency and stability. We use [49] as a source of reference for this section. Consider the initial value problem

$$y' = f(t, y), \; y(a) = \nu, \; f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, \; \nu = [\nu_1, \nu_2, \ldots, \nu_m]^T. \quad (2.1.1)$$

Suppose that this problem satisfies the conditions for the following existence-uniqueness theorem:

Theorem 2.1.1 ([49], page 5) *Let $f(t, y)$, where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, be defined and continuous for all (t, y) in the region D defined by $a \leq t \leq b$, $-\infty < y_{(i)} < \infty$ (where $y_{(i)}$ is the ith component of y), $i = 1, 2, \ldots, m$, where a and b are finite, and let there exist a constant L such that*

$$\| f(t, y) - f(t, y^*) \| \leq L \| y - y^* \| \quad (2.1.2)$$

holds for every $(t, y), (t, y^*) \in D$. Then for any $\Upsilon \in \mathbb{R}^m$, there exists a unique solution $y(t)$ of the problem (2.1.1), where $y(t)$ is continuous and differentiable for all $(t, y) \in D$.

NB The above theorem is equivalent to Theorem 1.1.23 for systems of integral equations.

Consider the general numerical method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\phi_f\left(y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n; h\right), \quad (2.1.3)$$

with appropriate starting values

$$y_\mu = \nu_\mu(h), \mu = 0, 1, \ldots, k-1, \quad (2.1.4)$$

where the subscript f on the right-hand side indicates that the dependence of ϕ on $y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n$ is through the function $f(t, y)$. Lambert [49], p. 24 imposes the following two conditions on (2.1.3):

1. $\phi_{f\equiv 0}\left(y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n; h\right) \equiv 0$,

2. $$\| \phi_f\left(y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n; h\right) - \phi_f\left(y^*_{n+k}, y^*_{n+k-1}, \ldots, y^*_n, t^*_n; h\right) \| \leq M \sum_{j=0}^{k} \| y_{n+j} - y^*_{n+j} \|,$$

   where M is a constant.

A formal definition of convergence of the method follows:

Definition 2.1.2 ([49], page 27) *The method defined by (2.1.3) and (2.1.4) is said to be convergent if, for all initial value problems satisfying the hypotheses of Theorem 2.1.1, we have that*

$$\lim_{h \to 0} y_n = y(t)$$

holds for all $t \in [a, b]$ and for all solutions $\{y_n\}$ of the difference equation in (2.1.3) satisfying the method's starting conditions.A method which is not convergent is said to be divergent.

Note that if convergence is to be a property of a method, then convergence must take place for all initial value problems.

We now consider what is meant by consistency. The numerical method would be an infinitely accurate representation of the differential system if the difference equation (2.1.3) were satisfied exactly when we replaced the numerical solution y_{n+j} at t_{n+j}, by the exact solutions $y(t_{n+j})$, for $j = 0, 1, 2, \ldots, k$. We therefore take as a measure of accuracy the value of the residual R_{n+k} which results on making the substitution. We thus define R_{n+k} by

$$R_{n+k} = \sum_{j=0}^{k} \alpha_j y(t_{n+j}) - h\phi_f(y(t_{n+k}), y(t_{n+k-1}), \ldots, y(t_n), t_n; h). \quad (2.1.5)$$

R_{n+k} is essentially the local truncation error (the error that is introduced at each step). As the stepsize of the numerical method gets smaller, we would hope that the residual would get smaller, so that for h infinitely close to zero, the residual is negligible. We define consistency as follows:

Definition 2.1.3 ([49], page 28) *The method defined by (2.1.3) and (2.1.4) is said to be consistent if, for all initial value problems satisfying the hypotheses of Theorem 2.1.1, the residual R_{n+k} defined by (2.1.5) satisfies*

$$\lim_{h \to 0} \frac{1}{h} R_{n+k} = 0.$$

NB The word 'consistent' is shorthand for the phrase 'consistent with the differential system'.

It is documented in [49] that convergence implies consistency but the converse is not true. It can happen that the difference system produced by applying a numerical method to a given initial value problem suffers an in-built instability which persists even in the limit as $h \to 0$ and prevents convergence. The form of stability we are considering here is called zero-stability

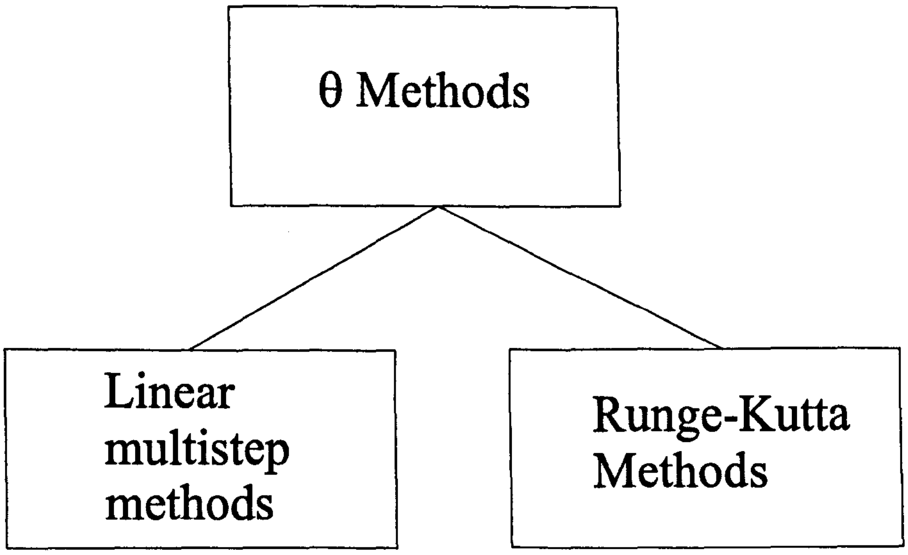

Figure 2.2.1: Two principal types of numerical methods

because it is concerned with stability of the difference system in the limit as h tends to zero. Any numerical method applied to (2.1.1) will introduce errors due to discretisation and round-off, and these could be interpreted as being equivalent to perturbing the problem; if the original continuous problem is not stable then no numerical method has any hope of producing an acceptable solution. The same will be true if the difference equation produced by the method is itself over-sensitive to perturbations.

2.2 Two principal methods

We will spend most time considering the linear θ methods.† This choice of numerical method falls into two categories of methods simultaneously - linear multistep methods and Runge-Kutta methods. It is therefore appropriate that we should briefly discuss the characteristics of both these types of methods. We discuss these methods with respect to the classical ODE initial value problem (because this is what they were originally designed for). We note that the Volterra equations under investigation in this thesis may be rewritten as initial value problems. Later, we discuss how the methods are applied to Volterra integral and integro-differential equations without having to rewrite them as a system of ODEs. The θ method is linear in y_n and f_n and is a one-step method. It is a simple method but has very low accuracy. Linear multistep methods achieve higher accuracy by retaining linearity with

† See page 35A overleaf for a description of θ methods.

Consider the integral $\int_a^b f(x)dx$ approximated by the quadrature rule $h\sum_{j=0}^{n} w_j^{(n)} f(a+jh)$, where $h = \frac{b-a}{n}$. A θ-method is a quadrature method where the weights $w_j^{(n)}$ are of the form $\{w_0^{(n)}, w_1^{(n)}, \ldots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{\theta, 1, \ldots, 1, 1-\theta\}$. **NB** The explicit Euler rules corresponds to $\theta = 1$, the implicit Euler rule corresponds to $\theta = 0$ and the Trapezium rule corresponds to $\theta = \frac{1}{2}$.

respect to y_{n+j} and f_{n+j}, $j = 0, 1, \ldots, k$, but sacrificing the one-step format. Runge-Kutta methods develop differently from Euler's method; higher order is achieved by retaining the one-step form but sacrificing the linearity. "With linear multistep methods it is easy to tell when we ought to change the steplength but hard to change it, while with Runge-Kutta methods it is hard to tell when to change the steplength but easy to change it." [49], page 149.

2.2.1 Linear multistep methods

Linear multistep methods are a subclass of the general method described by (2.1.3) and (2.1.4) in which the function $\phi_f(y_{n+k}, y_{n+k-1}, \ldots, y_n, t_n; h)$ defined by (2.1.3) takes the form of a linear combination of the values of the function f evaluated at (t_{n+j}, y_{n+j}), $j = 0, 1, \ldots, k$. Using the notation

$$f_{n+j} \equiv f(t_{n+j}, y_{n+j}), \; j = 0, 1, \ldots, k$$

we define a linear multistep method in standard form by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.2.1}$$

where α_j and β_j are constants subject to the conditions

$$\alpha_k = 1, \; |\alpha_0| + |\beta_0| \neq 0.$$

The first of these conditions removes the arbitrariness that arises from the fact that we could multiply both sides of (2.2.1) without altering the method. The method (2.2.1) is clearly explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$. Previously in this chapter, we introduced the residual of a numerical method as a measure of the method's accuracy. By forming a Taylor expansion about some suitable value of t, we could express the residual as a power series in h. The power of h in the first non-vanishing term is then an indication of accuracy. We do this for Euler's method and the trapezoidal method as examples (taking t_n as the origin of the expansions). Using the fact that $y' = f(t, y)$, we obtain

$$R_{n+1} = y(t_{n+1}) - y(t_n) - hy'(t_n) = \frac{h^2}{2} y^{(2)}(t_n) + O(h^3), \tag{2.2.2}$$

and

$$R_{n+1} = y(t_{n+1}) - y(t_n) - \frac{h}{2}(y'(t_{n+1}) + y'(t_n)) = -\frac{h^3}{12} y^{(3)}(t_n) + O(h^4), \tag{2.2.3}$$

respectively, from which we conclude that the trapezoidal method is the more accurate by one power of h.

2.2.2 Runge-Kutta methods

The general s-stage Runge-Kutta method for the initial value problem (2.1.1) is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i \tag{2.2.4}$$

where

$$k_i = f\left(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j\right), \; i = 1, 2, \ldots, s. \tag{2.2.5}$$

It is assumed that the following condition holds:

$$c_i = \sum_{j=1}^{s} a_{ij}, \; i = 1, 2, \ldots, s. \tag{2.2.6}$$

It is convenient to display the coefficients occuring in (2.2.4), (2.2.5) in the following form, known as a Butcher array:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & & \vdots \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

We define the s-dimensional vectors c and b and the $s \times s$ matrix A by

$$c = [c_1, c_2, \ldots, c_s]^T, \; b = [b_1, b_2, \ldots, b_s]^T, \; A = [a_{ij}]. \tag{2.2.7}$$

If in (2.2.4) and (2.2.5) we have that $a_{ij} = 0$ for $j \geq i$, $i = 1, 2, \ldots, s$, then the method is an explicit (sometimes called 'classic') Runge-Kutta method (NB this corresponds to A being strictly lower triangular). Otherwise the method is implicit. Note that implicit Runge-Kutta methods pose an even more daunting computational problem than implicit linear multistep methods.

2.3 Application to Volterra integral equations

Consider the Volterra integral equation of the second kind

$$y(t) = g(t) + \int_0^t k(t, s, y(s))\, ds, \; y(0) = y_0, \; t \in I = [0, T]. \tag{2.3.1}$$

Numerical methods for solving the (2.3.1) are reviewed in great detail in [13]. The methods covered include adaptions of linear multistep methods and

Runge-Kutta methods to cover this type of equation. The simplest technique (which is common to both subclasses) is the method of using a quadrature rule to approximate the integral on a finite uniform mesh. Assume that the given interval I is replaced by the mesh points t_n, $n = 0, \ldots, N$ (with $t_N = T$), and let y_n denote a numerical approximation to the exact solution $y(t_n)$ at t_n. Consider the quadrature formula

$$y_{n+1} = g(t_{n+1}) + h \sum_{j=0}^{n+1} \omega_j^{(n+1)} k((n+1)h, jh, y_j), \; y(0) = y_0, \tag{2.3.2}$$

where $\omega_j^{(n)}$ are the quadrature weights. The use of a quadrature rule with a uniform mesh is aptly suited to convolution equations (which is our main topic in this thesis); convolution integral equations have a number of features that can be preserved if a uniform mesh is used (see [2]). Our previous discussion of convergence, consistency and numerical stability is equally applicable here (according to Baker [1], "it is not surprising to find that there is a close connection between methods for ODEs and for classical Volterra integral equations of the second kind."). The numerical solution of Volterra integral equations by this method has been studied extensively during the 1990's and earlier. See, for example, [28], [26], [27].

2.4 Application to Volterra integro-differential equations

Consider the Volterra integro-differential equation

$$y'(t) = g(t) + \int_0^t k(t, s, y(s))\, ds, \; y(0) = y_0. \tag{2.4.1}$$

A logical approach to solving this problem numerically would be to use an ODE method to approximate the derivative on the left and a quadrature rule to approximate the integral on the right hand side of the equation. Since we are choosing methods which are common to both of the subclasses of methods (linear multistep and Runge-Kutta) we choose to approximate the derivative on the right hand side with the simple Euler rule so that our numerical method looks like:

$$\frac{y_{n+1} - y_n}{h} = g(t_{n+1}) + h \sum_{j=0}^{n+1} \omega_j^{(n+1)} k((n+1)h, jh, y_j), \; y(0) = y_0. \tag{2.4.2}$$

The work which follows in later chapters uses the above approach and builds on work previously done. See, for example, [12], [8], [9], [16], [21], [13]. More complicated (but less general) methods have been studied. See, for example, [41], [56], [4], [75].

We are specifically interested in using θ-methods to approximate the integral. This is a simple form of quadrature rule where the weights are defined as follows:

$$\left\{\omega_0^{(n+1)}, \omega_1^{(n+1)}, \ldots, \omega_n^{(n+1)}, \omega_{n+1}^{(n+1)}\right\} = \{\theta, 1, \ldots, 1, 1-\theta\} \tag{2.4.3}$$

where $0 \leq \theta \leq 1$. This class of methods includes (amongst others) the forward Euler rule, the backward Euler rule and the trapezoidal rule.

2.5 Other numerical methods

Here we briefly mention other numerical methods which may be employed for solving Volterra integro-differential equations. These other methods are not the focus of this thesis. Nevertheless, they are important, well-established methods which are widely used. We give a list of references for further reading in these areas.

For more detail on linear multistep methods see [13], [49], [41], [70], [58] and [56]. Similarly, there are a multitude of Runge-Kutta methods which are not discussed here. See, for example [13], [49], [5], [10] and [55].

We have not discussed here quadrature methods based on, say Newton-Cotes formulas, Gauss-Legendre formulas, Radau formulas or Lobatto formulas, to name but a few. See [13] for a discussion of these methods.

Collocation methods (some of which resemble Runge-Kutta-Nyström methods - see [13]) are based on the principle of approximating the exact solution to the integro-differential equation in a suitably chosen finite-dimensional function space, which is usually a subspace of the space containing the solution. This approximation will not, in general, satisfy the equation at a point not belonging to this finite subset. Collocation methods are studied in depth in [13], [6] and [11].

Predictor-Corrector methods employ two different numerical methods to approximate an equation at a particular point. An explicit method (known as the predictor) is first used to give an 'initial guess' at the exact solution. This 'guess' is then passed as a starting value to a second method (of a higher order than the first and usually implicit), known as the 'corrector' which provides a better approximation to the value. Subsequent approximations are continuously passed to the second method providing a succession of better approximations of the solution at that point until a desired level of accuracy is reached. Once this has been reached the predictor-corrector method moves onto the next discrete point on the mesh and the process is repeated. For a detailed analysis of this approach see [49].

Chapter 3

A linear integro-differential equation close to bifurcation points

In this chapter we investigate the qualitative behaviour of numerical approximations to a Volterra integro-differential equation, with a view to highlighting the problems introduced by adopting numerical approaches. We choose a linear, well-understood problem of the form

$$y'(t) = -\int_0^t e^{-\lambda(t-s)} y(s)ds, \qquad y(0) = 1$$

and write down a general numerical scheme which is often employed for such equations. We aim to show that although the stability theory for the continuous problem is straightforward, this is not necessarily the case for an analogous discrete problem, and that care needs to be taken in such an approach. We briefly re-present (for convenience) the details of the bifurcation points for the continous problem and then we derive the bifurcation points for the discrete problem. We show that as the stepsize of the numerical scheme decreases, the bifurcation points tend towards those of the continuous scheme. We illustrate our results with some numerical experiments.

3.1 Introduction

The qualitative behaviour of numerical approximations of solutions of functional differential equations is an important area for analysis. The aim is to ensure that, even over long time intervals, the behaviour of the numerical solution reflects accurately that of the true solution.

There is a well-established stability theory for equations of the form

$$y'(t) = g(t) + \xi y(t) + \nu \int_0^t y(s)ds, \ \nu \neq 0 \tag{3.1.1}$$

and the performance of numerical methods applied to (3.1.1) has been investigated. (See for example [4], [12], [13]). This is a natural starting point for the analysis of nonlinear problems that can be linearised in the form (3.1.1) but this analysis does not extend to all classes of problem. Many real-world model equations have convolution kernels with fading memory, and it is our concern in this chapter to consider the qualitative behaviour of numerical solutions to this class of problem. The previous analysis does not apply in this case.

In this chapter, we consider in detail the solution by numerical techniques of the integro-differential equation

$$y'(t) = -\int_0^t e^{-\lambda(t-s)} y(s)ds, \ y(0) = 1 \tag{3.1.2}$$

The equation depends on the value of the single parameter λ and is chosen for ease of analysis. For λ real and negative, the kernel is of growing memory type. For λ real and positive, the kernel has a fading memory effect. This is a linear equation whose analytical solution displays surprisingly rich dynamical behaviour even for real values of the parameter λ and it is this behaviour that we want to consider for the numerical scheme. We view this as a prototype problem that will provide insight into the behaviour of more complicated equations. In fact there are four real intervals of λ values in each of which the solutions of equation (3.1.2) behave qualitatively differently. It turns out that the numerical approximation of this behaviour of the original system is not altogether straightforward.

We consider the following questions:

1. does the numerical scheme display the same four qualitatively different types of long term behaviour as are found in the true solution

2. are the interval ranges for the parameter giving rise to the changes in behaviour of the solution λ the same as in the original problem?

Our discussion is informed by existing results on stability ranges for the parameter values of the integro-differential equation. We will also compare our results with those from some similar investigations (of Hopf bifurcation points) relating to delay differential equations that give us additional insight into the behaviour of the solution close to bifurcation points under discretisation.

3.2 Analytical background

We consider the equation (3.1.2). One can easily establish (by considering for example an equivalent ordinary differential equation) that for real values of λ the solution to (3.1.2) bifurcates at $\lambda = 0, \pm 2$. We have the following qualitative behaviour:

A1. When $\lambda \geq 2$, $y \to 0$ as $t \to \infty$, with no oscillations

A2. When $0 < \lambda < 2$, $y \to 0$ as $t \to \infty$, with infinitely many oscillations

A3. When $\lambda = 0$, $y(t) = \cos(t)$; (persistent oscillations)

A4. When $-2 < \lambda < 0$, the solutions contains infinitely many oscillations of increasing magnitude

A5. When $\lambda \leq -2$, the solution grows without any oscillations.

While the continous theory for (3.1.2) is well established, the analysis of numerical techniques is not so straightforward. To illustrate this, figure 3.2.1 shows a numerical solution to (3.1.2) with $\lambda = 0$ and $\alpha = 1$. In other words, figure 3.2.1 is an approximation to the cosine function; the numerical solution is a decaying oscillation which does not represent the true qualitative behaviour.

3.3 Numerical analysis

We solve (3.1.2) numerically using a low-order scheme based on combining a linear multistep method for solving ODE's with a θ-method quadrature rule for performing the integration. A stability analysis of this and other low-order methods (such as those based on a forward Euler rule, and a trapezoidal rule) has been performed for other examples of linear integro-differential equations and a wide range of integral equations, see [8], [12], [13], [9] and [58] for example. In particular, [8] illustrates clearly that, although a stability analysis of the continuous problem may be straightforward, this is not necessarily the case with the discrete form. We would anticipate that our numerical method will give us four intervals of λ where the solutions

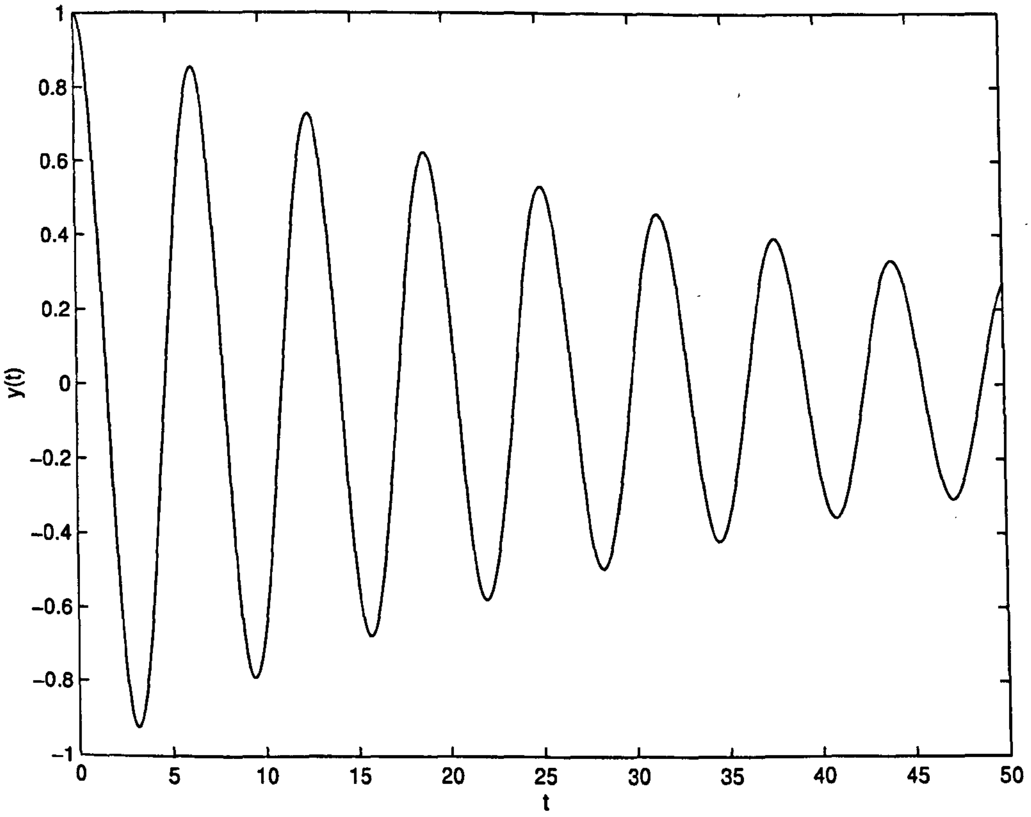

Figure 3.2.1: Solving (3.1.2), $\lambda = 0$, using a backward Euler scheme with stepsize 0.05.

to the discrete scheme behave qualitatively differently (as with the continuous problem), however we know from investigation of bifurcation points for numerical solution of delay differential equations (see [76]) and indeed from stability analysis of integro-differential equations that the bifurcation points may arise at the *wrong* values of the parameter. Based on previous experience we would expect this difference to be dependent upon the stepsize h of the numerical method and on the choice of method itself. Furthermore, based on previous work ([76], [29]), one might expect the bifurcation points of the discrete scheme to approach the bifurcation points of the continuous problem as $h \to 0$. We will show that the approximation of the bifurcation points is to the order of the method.

The numerical method is of the form

$$y_{n+1} - y_n = -h^2 \left(\theta e^{-\lambda(n+1)h} y_0 + \sum_{j=1}^{n} e^{-\lambda h(n+1-j)} y_j + (1-\theta) y_{n+1} \right), \ y_0 = 1 \tag{3.3.1}$$

with $0 \leq \theta \leq 1$ and we set $0 < h < 1$. We know from [1] that when a θ method is applied to the test equation

$$y(t) - \lambda \int_{t_0}^{t} y(s) ds = g(t), \ y(t_0) = y_0 \tag{3.3.2}$$

it is convergent and consistent. We also know from [1] that every ODE method generates a corresponding integral equation method. In fact ODE methods provide approximations to indefinite integrals.

The equation (3.3.1) is equivalent to

$$\left(1 + h^2(1-\theta)\right) y_{n+2} + \left(h^2 \theta e^{-\lambda h} - 1 - e^{-\lambda h}\right) y_{n+1} + e^{-\lambda h} y_n = 0 \tag{3.3.3}$$

For the corresponding characteristic equation

$$\left(1 + h^2(1-\theta)\right) k^2 + \left(h^2 \theta e^{-\lambda h} - 1 - e^{-\lambda h}\right) k + e^{-\lambda h} = 0 \tag{3.3.4}$$

we can derive conditions on the nature of its roots:

B1. We have real and distinct roots when

$$\lambda > -\frac{1}{h} \ln \left(\frac{1 + 2h^2 - h^2\theta - 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1} \right)$$

B2. We have real and equal roots when

$$\lambda = -\frac{1}{h} \ln \left(\frac{1 + 2h^2 - h^2\theta - 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1} \right)$$

B3. We have complex roots when

$$-\frac{1}{h} \ln \left(\frac{1 + 2h^2 - h^2\theta - 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1} \right) > \lambda$$

$$> -\frac{1}{h} \ln \left(\frac{1 + 2h^2 - h^2\theta + 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1} \right)$$

B4. We have real and equal roots when

$$\lambda = -\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta+2\sqrt{-h^2\left(h^2\theta-1-h^2\right)}}{h^4\theta^2-2h^2\theta+1}\right)$$

B5. We have real and distinct roots when

$$\lambda < -\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta+2\sqrt{-h^2\left(h^2\theta-1-h^2\right)}}{h^4\theta^2-2h^2\theta+1}\right).$$

Consider case B1. The dominant root of the characteristic equation is

$$k_1 = \frac{1+e^{-\lambda h}-h^2\theta e^{-\lambda h}-\sqrt{\left(h^2\theta e^{-\lambda h}-1-e^{-\lambda h}\right)^2-4e^{-\lambda h}\left(1+h^2\left(1-\theta\right)\right)}}{2\left(1+h^2\left(1-\theta\right)\right)}. \tag{3.3.5}$$

Since $k_1 \in \mathbb{R}$, it is easy to see that k_1 is monotone, increasing as (as a function of λ) for $\lambda > -\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta-2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right)$. As $\lambda \to \infty$, $k_1 \to \frac{1}{1+h^2(1-\theta)}$. Therefore $|k_1| < 1$ and under condition B1, all solutions to (3.3.1) are asymptotically stable.

Now consider case B2. The single root of the characteristic equation becomes

$$k_1 = \frac{1+e^{-\lambda h}-h^2\theta e^{-\lambda h}}{2\left(1+h^2\left(1-\theta\right)\right)}. \tag{3.3.6}$$

It can be seen that

$$|k_1| = \frac{1}{2\left(1+h^2-h^2\theta\right)}+\frac{\left(1-h^2\theta\right)}{2e^{\lambda h}\left(1+h^2-h^2\theta\right)} \tag{3.3.7}$$

It is clear that

$$\frac{1}{2\left(1+h^2-h^2\theta\right)} \leq \frac{1}{2} \tag{3.3.8}$$

Since λ, h are positive, we know that

$$\frac{\left(1-h^2\theta\right)}{2e^{\lambda h}\left(1+h^2-h^2\theta\right)} < \frac{1}{2} \tag{3.3.9}$$

NB We know this is a strict inequality because $\lambda \neq 0$ and $h \neq 0$. Therefore $|k_1| < 1$ and it follows that (3.3.1) has asymptotically stable solutions which do not oscillate.

For case B3, we can write the roots of the characteristic equation as

$$k_1 = \frac{1 + e^{-\lambda h} - h^2\theta e^{-\lambda h} \pm i\sqrt{4e^{-\lambda h}\left(1 + h^2\left(1-\theta\right)\right) - \left(h^2\theta e^{-\lambda h} - 1 - e^{-\lambda h}\right)^2}}{2\left(1 + h^2\left(1-\theta\right)\right)}. \tag{3.3.10}$$

Firstly, we note that since the roots have a non-zero imaginary part, $\arg k \neq 0$ and so we have solutions which oscillate. Both roots have modulus:

$$|k| = \frac{1}{e^{\frac{\lambda h}{2}}\sqrt{1 + h^2\left(1-\theta\right)}} \tag{3.3.11}$$

dependent on λ. If $|k| < 1$ then the solution is asymptotically stable, with infinitely many oscillations of decreasing magnitude. If $|k| = 1$ then the solution is stable but not asymptotically stable and the solution exhibits persistent oscillations. Finally, if $|k| > 1$ then the solution is unstable (i.e. the solution diverges) with infinitely many oscillations of increasing magnitude. By setting $|k| = 1$ and solving for λ we find that

$$\lambda = \frac{1}{h}\ln\left(\frac{1}{1 + h^2\left(1-\theta\right)}\right).$$

Thus, when

$$\frac{1}{h}\ln\left(\frac{1}{1 + h^2\left(1-\theta\right)}\right) < \lambda < -\frac{1}{h}\ln\left(\frac{1 + 2h^2 - h^2\theta - 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1}\right)$$

we have asymptotically stable solutions which oscillate, when

$$\lambda = \frac{1}{h}\ln\left(\frac{1}{1 + h^2\left(1-\theta\right)}\right)$$

we have a solution which exhibits persistent oscillations, and when

$$-\frac{1}{h}\ln\left(\frac{1 + 2h^2 - h^2\theta + 2\sqrt{-h^2\left(h^2\theta - 1 - h^2\right)}}{h^4\theta^2 - 2h^2\theta + 1}\right) < \lambda < \frac{1}{h}\ln\left(\frac{1}{1 + h^2\left(1-\theta\right)}\right)$$

we have an oscillating solution which diverges.

The analysis for case B4 is identical to that for B2 with the exception that λ is negative.The modulus of the root of the characteristic equation is still given by

$$|k_1| = \frac{1}{2\left(1+h^2-h^2\theta\right)} + \frac{\left(1-h^2\theta\right)}{2e^{\lambda h}\left(1+h^2-h^2\theta\right)} \tag{3.3.12}$$

but now we are dealing with a negative λ. Therefore we have the two inequalities

$$\frac{1}{2\left(1+h^2-h^2\theta\right)} \leq \frac{1}{2} \tag{3.3.13}$$

and

$$\frac{\left(1-h^2\theta\right)}{2e^{\lambda h}\left(1+h^2-h^2\theta\right)} > \frac{1}{2} \tag{3.3.14}$$

It can be seen that when (3.3.13) becomes an equality, (3.3.14) is large enough to make the sum of the two greater than 1. And as the left hand side of (3.3.13) decays, the left hand side of (3.3.14) grows at a greater rate (exponential growth). Therefore $|k_1| > 1$ and we have asymptotically stable solutions (remembering that $\arg k = 0$ and there are therefore no oscillations in the solution.

Finally we consider case B5. The dominant root of the characteristic equation is

$$k_1 = \frac{1+e^{-\lambda h}-h^2\theta e^{-\lambda h}-\sqrt{\left(h^2\theta e^{-\lambda h}-1-e^{-\lambda h}\right)^2-4e^{-\lambda h}\left(1+h^2\left(1-\theta\right)\right)}}{2\left(1+h^2\left(1-\theta\right)\right)}. \tag{3.3.15}$$

as it was under case B1. Since $k_1 \in \mathbb{R}$, it is easy to see that k_1 is monotone, increasing in magnitude as λ decreases beyond

$$-\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta+2\sqrt{-h^2\left(h^2\theta-1-h^2\right)}}{h^4\theta^2-2h^2\theta+1}\right).$$

Therefore $|k_1| > 1$ and we can say that, under condition B2, we have a divergent solution. Also, since $\arg k = 0$, the solution does not oscillate.

We summarise our results in the following theorem.

Theorem 3.3.1 *Consider solving the Volterra integro-differential equation (3.1.2) subject to the initial condition* $y(0) = 1$, *with the numerical scheme (3.3.1)* ($0 \leq \theta \leq 1$, $0 < h < 1$), *under the equivalent discrete initial conditions* $y_0 = 1$. *Then:*

C1. *When* $-\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta-2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right) \le \lambda$, *the solution to (3.3.1) is asymptotically stable with no oscillations*

C2. *When* $\frac{1}{h}\ln\left(\frac{1}{1+h^2(1-\theta)}\right) < \lambda < -\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta-2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right)$ *the solution to (3.3.1) is asymptotically stable with infinitely many oscillations of decreasing magnitude*

C3. *When* $\lambda = \frac{1}{h}\ln\left(\frac{1}{1+h^2(1-\theta)}\right)$, *the solution to (3.3.1) exhibits persistent oscillation about 0 with equal magnitude*

C4. *When* $-\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta+2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right) < \lambda < \frac{1}{h}\ln\left(\frac{1}{1+h^2(1-\theta)}\right)$, *the solution to (3.3.1) diverges to* ∞ *with infinitely many oscillations of increasing magnitude*

C5. *When* $\lambda \le -\frac{1}{h}\ln\left(\frac{1+2h^2-h^2\theta+2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right)$, *the solution diverges to* ∞ *without any oscillations.*

The limits of the bifurcation points in theorem 3.3.1 as h tends to zero correspond to the bifurcation points of the original continuous problem. It is easy to show this, using L'Hospital's rule.

We can also prove that, for $-2 \le \lambda \le 2$, the numerical scheme is of order h (the same order as the Θ method):

Consider the bifurcation point

$$f(h,\theta) = -\frac{1}{h}\ln\left(\frac{1+2h-h^2\theta-2\sqrt{-h^2\left(h^2\theta-1-h^2\right)}}{h^4\theta^2-2h^2\theta+1}\right).$$

This may be rewritten as

$$\begin{aligned} f(h,\theta) &= -\frac{1}{h}\ln\left(1+\left(2h^2-h^2\theta-2h\sqrt{1+\left(h^2-h^2\theta\right)}\right)\right) \\ &\quad +\frac{1}{h}\ln\left(1-2\left(h^2\theta-h^4\theta^2\right)\right). \end{aligned}$$

We factorise the first term and expand the square root term using the binomial expansion as follows:

$$
\begin{aligned}
f(h,\theta) &= -\frac{1}{h}\ln(1-2h(1+(h^2-h^2\theta))^{\frac{1}{2}}+2h^2-h^2\theta) \\
&\quad +\frac{1}{h}\ln(1-2(h^2\theta-h^4\theta^2)) \\
&= \frac{-1}{h}\ln(1-2h(1+\frac{1}{2}(h^2-h^2\theta) \\
&\quad \frac{1}{8}(h^2-h^2\theta)^2+\cdots)+2h^2-h^2\theta) \\
&\quad +\frac{1}{h}\ln(1-2(h^2\theta-h^4\theta^2)) \\
&= -\frac{1}{h}((-2h(1+\frac{1}{2}(h^2-h^2\theta) \\
&\quad -\frac{1}{8}(h^2-h^2\theta)^2+\cdots-2h^2+h^2\theta))+\cdots) \\
&\quad -\frac{1}{h}(-(2h^2\theta-h^4\theta^2)-\frac{(2h^2\theta-h^4\theta^2)^2}{2}-\cdots) \\
&= 2+\frac{O(h^2)}{h} \\
&= 2+O(h)
\end{aligned}
$$

The same analysis can be applied to the bifurcation point

$$
-\frac{1}{h}\ln\left(\frac{1+2h-h^2\theta+2\sqrt{-h^2(h^2\theta-1-h^2)}}{h^4\theta^2-2h^2\theta+1}\right).
$$

3.4 Numerical experiments

To corroborate our analytical work, we illustrate our results graphically. The numerical scheme reduces to three well-known methods for $\theta=1$ (the forward Euler rule), $\theta=0$ (the backward Euler rule) and $\theta=\frac{1}{2}$ (the trapezium rule) being used for the integration. Figures 3.4.1, 3.4.2 and 3.4.3 illustrate the variation in the magnitudes and arguments of the characteristic roots of the equation (3.3.4) as λ varies, for the different methods. The dominant root is clearly visible. Figures 3.4.4, 3.4.5 and 3.4.6 show how the bifurcation points change as h varies.

The results presented in this chapter show that the well-established stability theory based on the analysis of equation (3.1.1) gives only a very limited insight into the qualitative behaviour of solutions of the class of convolution equations with fading memory kernel that we have considered here. In later chapters we observe that the qualitative behaviour of numerical solutiosns

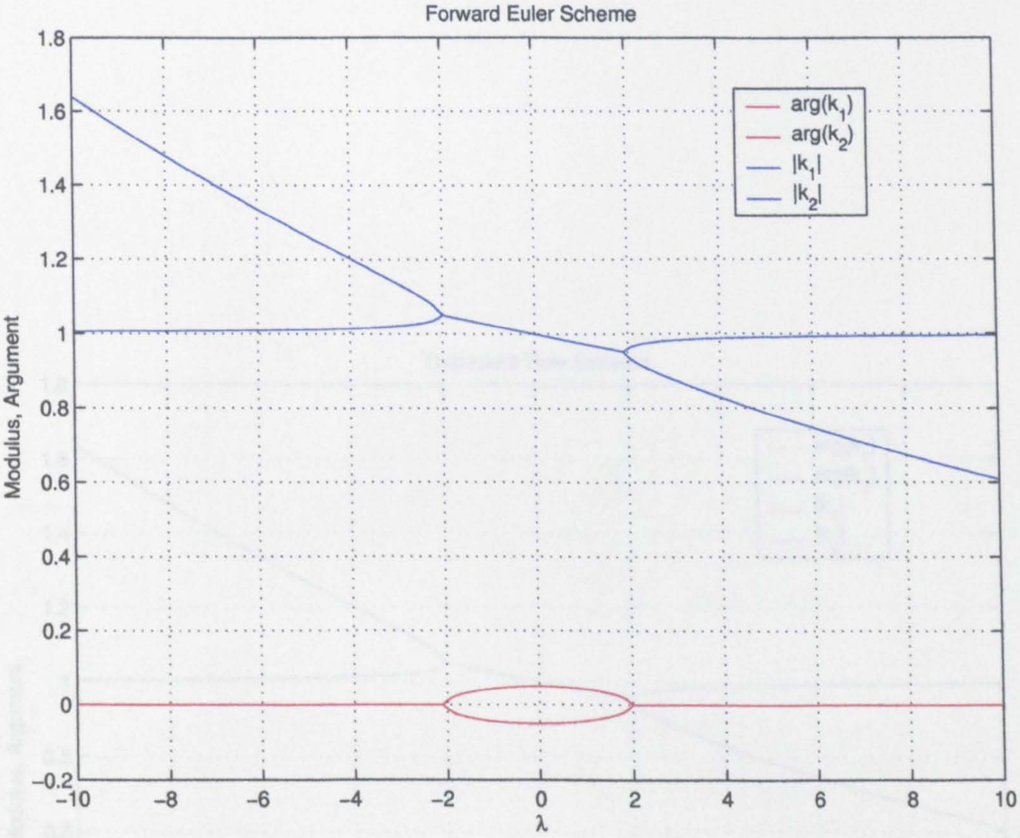

Figure 3.4.1: Plotting λ versus $|k|$ and $\arg k$ for $h = 0.05$, $\theta = 1$

to equations of this type may have surprising features and our consideration here of the prototype problem (3.1.2) illustrates how this unexpected behaviour may arise. The results we have presented here show that, for these simple methods at least, the bifurcation parameters are approximated in the numerical scheme to the order of the method.

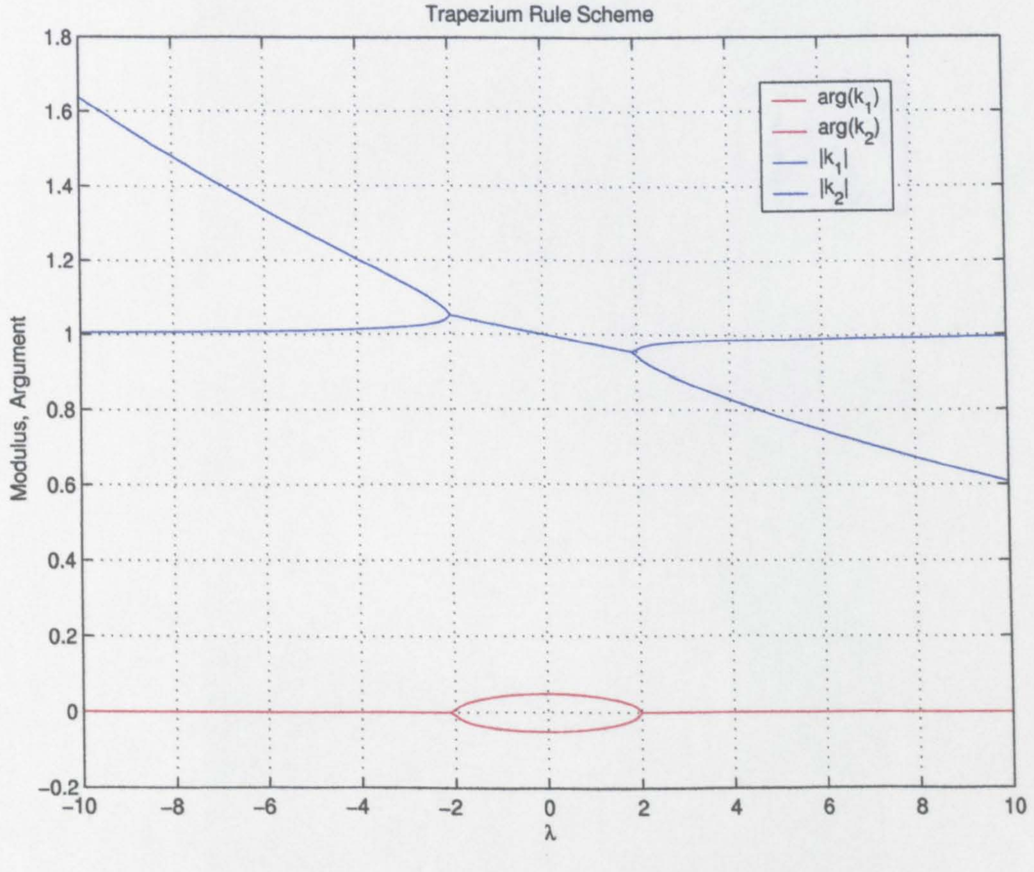

Figure 3.4.2: Plotting λ versus $|k|$ and $\arg k$ for $h = 0.05$, $\theta = \frac{1}{2}$

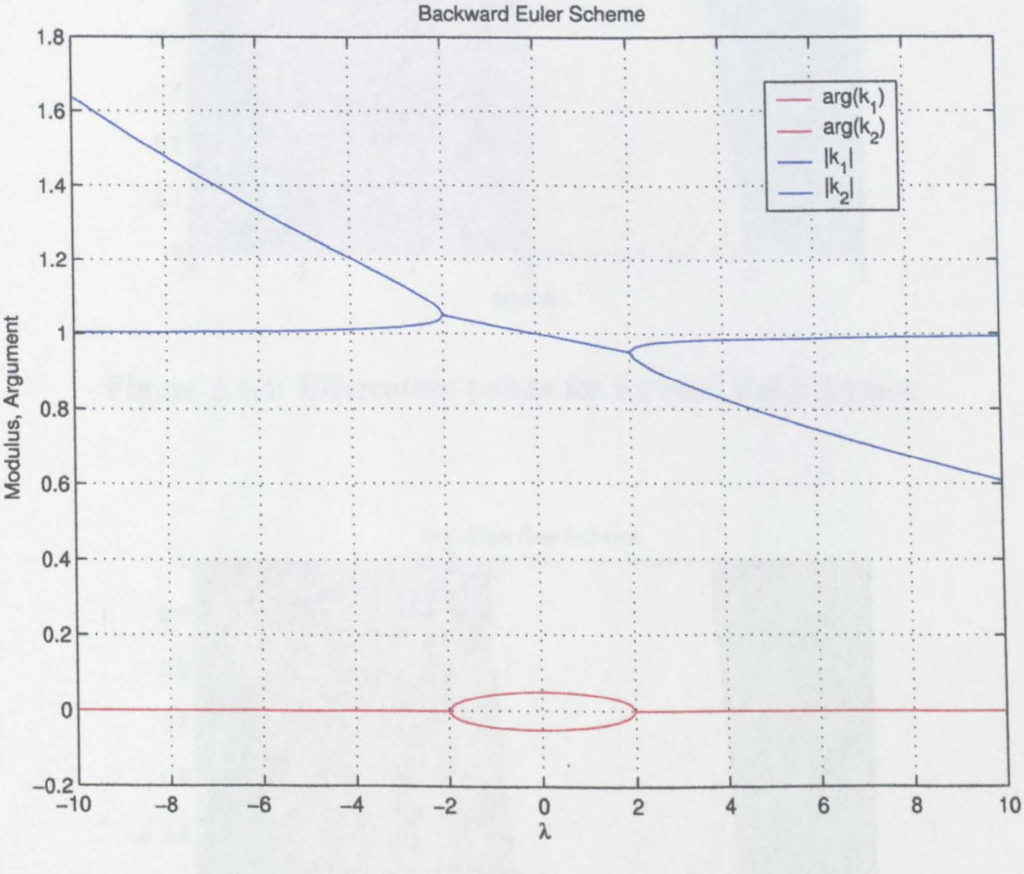

Figure 3.4.3: Plotting λ versus $|k|$ and $\arg k$ for $h = 0.05$, $\theta = 0$

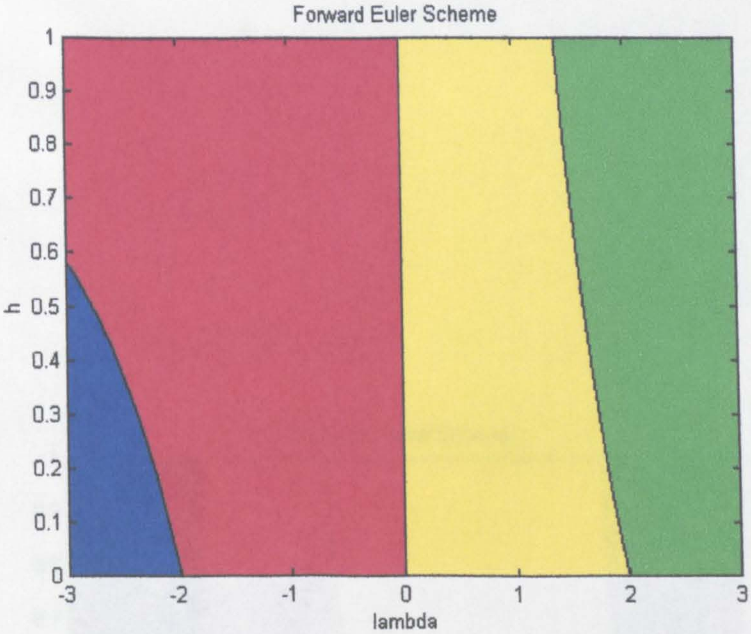

Figure 3.4.4: Bifurcation points for forward Euler scheme

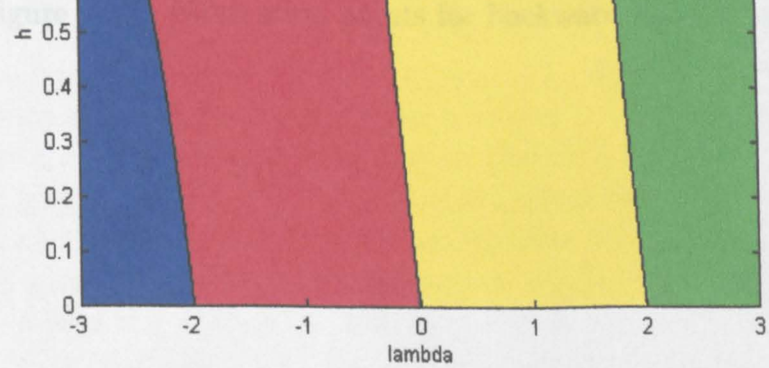

Figure 3.4.5: Bifurcation points for trapezium rule scheme

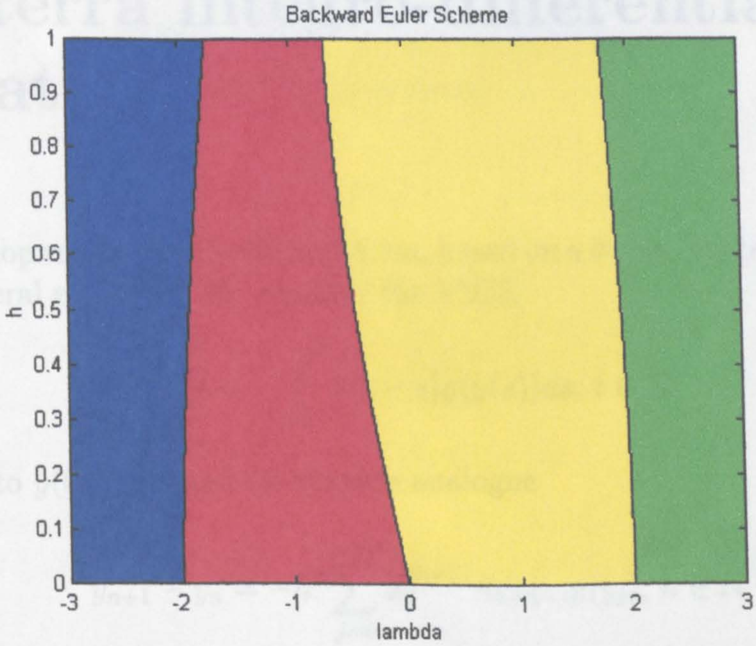

Figure 3.4.6: Bifurcation points for backward Euler scheme

Chapter 4

A homogeneous nonlinear Volterra integro-differential equation

We develop results for the discrete form, based on a θ-rule quadrature method, of a general equation. We consider the VIDE

$$y'(t) = -\int_0^t k(t-s)g(y(s))ds, \; t \in \mathbb{R}^+ \tag{4.0.1}$$

subject to $y(0) = y_0$, and its discrete analogue

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} \omega_j^{(n+1)} A_{n+1-j} g(y_j), \; n \in \mathbb{N}. \tag{4.0.2}$$

Levin and Nohel [50] give an analysis of the qualitative behaviour of solutions to (4.0.1) by means of methods based on deriving a Lyapunov function for the solution. In this chapter we restate Levin and Nohels' results for convenience and rework some of their analysis to aid us later. We analyse the qualitative behaviour of solutions to (4.0.2) basing our analysis on the material given by Levin and Nohel in their discussion of (4.0.1), thereby developing an analysis for the discrete case which is analogous to the continuous case. We give a theorem on the qualitative behaviour of solutions to (4.0.2) and we extend the analysis of both the continuous and discrete equations to wider classes of equations. We consider what conditions it would be natural to impose on the numerical method to guarantee that the qualitative behaviour of solutions to (4.0.1) will be preserved in the solutions of the discrete scheme and illustrate our results with some numerical examples.

4.1 The continuous case

Equation (4.0.1) has been analysed by Levin and Nohel [50]. They investigate the solutions of (4.0.1)as $t \to \infty$, where $k(t)$ is completely monotonic on $0 \le t \le \infty$ and where $g(y)$ is a (nonlinear) spring. Under this hypothesis, Levin and Nohel show (4.0.1) to be relevant to certain physical applications, such as nuclear reactor dynamics. Note that when $g(y) \equiv y$, (4.0.1) becomes linear. The linear case is well documented (see [13], for example). Also note that if $k(t) \equiv k(0)$ then (4.0.1) reduces to the nonlinear oscillator equation

$$y''(t) = -k(0)g(y(t)) \tag{4.1.1}$$

Consider (4.0.1) subject to the following conditions:

H1 $k \in C[0, \infty)$ is completely monotone,

H2 $g(x) \in C(\infty, \infty)$, $xg(x) > 0 (x \neq 0)$ (and hence x and $g(x)$ always have the same sign and $g(0) = 0$),

H3 $G(x) := \int_0^x g(\xi) d\xi \to \infty$ as $|x| \to \infty$.

Levin and Nohel [50] give the following theorem:

Theorem 4.1.1 (Levin and Nohel) *Any solution $u(t)$ of (4.0.1) subject to H1, H2, H3 satisfies $u(t) \to 0$ as $t \to \infty$ providing the L^1 function $k(t)$ is non null.*

Theorem 4.1.1 applies only when k is not constant. If $k(t) \equiv 0$ then $y(t)$ is constant and therefore stable. If $k(t) \equiv k(0) \neq 0$ then the solution $u(t)$ may be stable but not asymptotically stable. For example, the equation $y'(t) = -\int_0^t y(s) ds$ is equivalent to the equation of simple harmonic motion $y''(t) = -y(t)$, whose solution is stable but not asymptotically stable. Since the L^1 kernel k has fading memory (because it is completely monotone), $y'(t) = 0$ can only be satisfied as $t \to \infty$ if $\lim_{s \to \infty} g(y(s)) = 0$. This is only possible if $y(s) \to 0$ as $s \to \infty$. Thus it is clear, without sophisticated analysis, that 0 is the only possible finite limit for the solution $u(t)$. We may weaken H1 without compromising our result as follows: we need only assume that $(-1)^j k^{(j)}(t) \ge 0$ for $j = 0, 1, 2$ and k is not constant.

The full proof of Theorem 4.1.1 requires analysis beyond the use of a Lyapunov function. However, [50] also contains the following result which forms the basis for a discrete Lyapunov investigation.

Theorem 4.1.2 (Levin & Nohel) *The zero solution of (4.0.1) subject to H1, H2, H3 is asymptotically stable providing that $k(t) \in L^1$ is non null.*

Proof: We reproduce the main stages of the proof, given by [50], for a comparison with the discrete case. The proof is based on the construction of a Lyapunov function of the form

$$V(t) = G(u(t)) + \frac{1}{2}\int_0^t \int_0^t k(\tau + s)g(u(t-\tau))g(u(t-s))d\tau ds. \qquad (4.1.2)$$

We may show that the function $V(t)$ defined here is positive as follows: $G(u(t))$ is positive because 0 is the only possible limit for the solution function $u(t)$. Also $K(\tau, s) = k(\tau+s)$ is a kernel of positive type since k is completely monotone and completely monotone kernels are of positive type. The second term in the the expression for V is therefore non-negative. We may express $V'(t)$ as follows:

$$\begin{aligned} V(t) &= G(u(t)) + \frac{1}{2}\int_0^t \int_0^t k(\tau + s)g(u(t-\tau))g(u(t-s))d\tau ds \\ &= G(u(t)) + \frac{1}{2}\int_0^t \int_0^t k(2t - \tau - s)g(u(\tau))g(u(s))d\tau ds \end{aligned}$$

Therefore,

$$\begin{aligned} V'(t) &= G'(u(t))u'(t) + \int_0^t \int_0^t k'(2t-\tau-s)g(u(\tau))g(u(s))d\tau ds \\ &\quad + \int_0^t k(t-s)g(u(t))g(u(s))ds. \end{aligned}$$

Now

$$G'(u(t))u'(t) = -g(u(t))\int_0^t k(t-s)g(u(s))ds \qquad (4.1.3)$$

by (4.0.1). Hence by substituting,

$$V'(t) = \int_0^t \int_0^t k'(\tau+s)g(u(t-\tau))g(u(t-s))d\tau ds. \qquad (4.1.4)$$

Since $k(t)$ is completely monotone, $-k'(\tau + s)$ is a kernel of positive type.† Hence $V'(t) \leq 0$ with equality only for $t = 0$.

We note that inhomogeneous equations of the form

$$y'(t) + \int_0^t k(t-s)g(y(s))ds = f(t, y(t)),\ t \in \mathbb{R} \qquad (4.1.5)$$

† See [30], p. 492 for definition.

can also be analysed. Levin and Nohel give an analysis of equations of this form in [51] but this analysis does not appear to be amenable to discrete analogues. We note however that the analysis of Theorem 4.1.2 can be extended simply to give a corresponding result for (4.1.5).

Corollary 4.1.3 (Ford) *The conclusions of Theorem 4.1.1 are also valid for the equation (4.1.5) subject to the additional condition that $f(t,0) = 0$ and $\xi f(t,\xi) \leq 0$ whenever $\xi \neq 0$.*

Proof: We adapt the expression in (4.1.3):

$$G'(u(t))u'(t) = g(u(t))f(t,u(t)) - \int_0^t k(t-s)g(u(s))ds. \qquad (4.1.6)$$

It follows that the expression for $V'(t)$ includes the additional term

$$g(u(t))f(t,u(t)),$$

Since $\xi f(t,\xi) \leq 0$ and x and $g(x)$ always have the same sign with $g(0) = 0$, we can conclude that $g(u(t))f(t,u(t)) \leq 0$. The final conclusion then follows as before.

4.2 The discrete case

A natural approach to the numerical solution of equations of the form (4.0.1) would be the combination of a differential equation method with a quadrature rule for the integral. We consider a simple approach of this type.

With a θ-rule as a quadrature method, we analyze

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} \omega_j^{(n+1)} k_{n+1-j} g(y_j), \; \omega_j^{(n+1)} \geq 0, \qquad (4.2.1)$$

where $\{\omega_0^{(n+1)}, \omega_1^{(n+1)}, \dots, \omega_n^{(n+1)}, \omega_{n+1}^{(n+1)}\} = \{\theta, 1, \dots, 1, 1-\theta\}$, $y_0 = y(0)$.

Following on from Corollary 4.1.3, we remark here that our analysis can also be applied to equations of the form

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} \omega_j^{(n+1)} k_{n+1-j} g(y_j) + f(n, y_n). \qquad (4.2.2)$$

The construction of a Lyapunov function for the discrete equation is less straightforward than one might hope. In particular, we need to be very careful that the function $V(n, y_j)$ that we define does not depend on future values of the solution sequence y_j. In other words, we require that $V(n, y_j)$ does not depend on any y_i with $i > n$. As we shall see, it proves to be impossible to give a complete discrete analogue of the continuous theory.

To begin with, we give a proof of asymptotic stability of the zero solution of equation (4.2.1) when only positive perturbations are permitted. This may seem rather restrictive, but nevertheless can be a useful result when the function y has a particular physical or biological meaning which implies that only nonnegative values of y_n are possible. (This situation covers, for example, models of population size or of concentration of a drug in the bloodstream.) Furthermore, it is our conjecture (supported by experimental evidence) that, for a wide class of kernels k and for suitable choice of initial value y_0 and $h > 0$, $y_n \geq 0$ for every n.

We now give our first theorem on qualitative behaviour of solutions to (4.2.1). The proof of the theorem is based on Lyapunov's method applied to difference equations.

Theorem 4.2.1 *For the equation (4.2.1), we make the following assumptions:*

H4. For each natural number n, the matrix $A(n) := (A_{ij}) = (k_{2n-i-j})$ *is a positive definite matrix and that the matrix* $A^{\ddagger}(n) := (A^{\ddagger}_{ij}) = (k_{2n+2-i-j} - k_{2n-i-j})$ *is a negative semi-definite matrix,*

H5. The function $g(u)$ *satisfies conditions H2 and H3 of Theorem 4.1.1 and is also nondecreasing.*

H6. The solution values satisfy $y_j \geq 0$ *for each* $j \geq 0$.

H7. The weights $\omega_j^{(n)}$ *are given by a* θ*-method with* $0 \leq \theta \leq \frac{1}{2}$. *Thus we insist that the* θ*-method is A-stable.*†

Then, for every $\epsilon > 0$ *there is a corresponding* $\delta_\epsilon > 0$ *and a natural number* N_ϵ *for which* $|y_0| < \delta_\epsilon$ *implies* $|y_n| < \epsilon$ *for each* $n > N_\epsilon$. *If, in addition to H4-H7 above,* $A^{\ddagger}(n)$ *is, for each* n*, a negative definite matrix, then* $y_n \to 0$ *as* $n \to \infty$.

In other words, the stability, respectively asymptotic stability, of the zero solution (subject to H6) is preserved under discretisation by a θ*-method with* $0 \leq \theta \leq \frac{1}{2}$ *provided the admissable perturbations produce a solution satisfying H6.*

† See [49], p. 224 for definition.

Proof: Following a similar approach to the one used in the proof of Theorem 4.1.2, we shall exhibit a Lyapunov function, this time for the sequence $\{y_n\}$ which satisfies the equation (4.2.1). The conclusions given in the statement of Theorem 4.2.1 then follow by Theorem 1.5.8.

Define

$$V(n,\{y_j\}_0^n) := \frac{h^2}{2}\sum_{j=0}^{n}\sum_{i=0}^{n}\omega_i^{(n)}\omega_j^{(n)}k_{2n-i-j}g(y_i)g(y_j) + G(n,\{y_j\}) \quad (4.2.3)$$

where $G(n,\{y_j\}_0^n)(\geq 0)$, $G(n,0)=0$ will be defined later. As in the previous proof, we shall show that $V(n,\{y_j\})$ defined in this way has the properties required of a Lyapunov function for $\{y_j\}$.

Clearly, $V(n,0)=0$ and $V(n,\{y_j\})\geq 0$ because, by hypothesis, $A(n)$ is a positive definite matrix for each n and $G(n,\{y_j\})\geq 0$, $G(n,0)=0$.

Next we demonstrate that $V(n+1,\{y_j\}) - V(n,\{y_j\}) \leq 0$.

$$\begin{aligned} V(n+1,\{y_j\}) - V(n,\{y_j\}) &= \frac{h^2}{2}\sum_{j=0}^{n}\sum_{i=0}^{n}(\omega_i^{(n+1)}\omega_j^{(n+1)}k_{2n+2-i-j}g(y_i)g(y_j) \\ &\quad -\omega_i^{(n)}\omega_j^{(n)}k_{2n-i-j}g(y_i)g(y_j)) \\ &\quad +h^2(1-\theta)g(y_{n+1})\sum_{j=0}^{n}\omega_j^{(n+1)}k_{n+1-j}g(y_j) \\ &\quad +\frac{h^2}{2}(1-\theta)^2k_0g(y_{n+1})^2 \\ &\quad +G(n+1,\{y_j\}) - G(n,\{y_j\}). \end{aligned}$$

Now define

$$G(n,\{y_j\}) := \sum_{j=1}^{n}\omega_j^{(n)}g(y_j)(y_j - y_{j-1}) + y_0M, \quad (4.2.4)$$

$$G(0,\{y_j\}) = y_0M \quad (4.2.5)$$

where M is some positive constant chosen to make $G(n,\{y_j\}) \geq 0$. For example, with our hypotheses, we can choose $M = \max_{t\in[0,y_0]} g(t) = g(y_0)$.

Now

$$
\begin{aligned}
G(n+1,\{y_j\}) - G(n,\{y_j\}) &= (1-\theta)g(y_{n+1})(y_{n+1}-y_n) + \theta g(y_n)(y_n - y_{n-1}) \\
&= -h^2(1-\theta)g(y_{n+1})\sum_{j=0}^{n+1}\omega_j^{(n+1)}k_{n+1-j}g(y_j) \\
&\quad -h^2\theta g(y_n)\sum_{j=0}^{n}\omega_j^{(n)}k_{n-j}g(y_j).
\end{aligned}
$$

It follows, taking into account the change in the weight index from $\omega_i^{(n+1)}$ to $\omega_i^{(n)}$, that

$$
\begin{aligned}
& V(n+1,\{y_j\}) - V(n,\{y_j\}) \\
&= \frac{h^2}{2}\left(\sum_{j=0}^{n}\sum_{i=0}^{n}\omega_i^{(n)}\omega_j^{(n)}(k_{2n+2-i-j} - k_{2n-i-j})g(y_i)g(y_j)\right) \\
&\quad + h^2\theta g(y_n)\sum_{j=0}^{n}\omega_{j=0}^{(n)}\omega_j^{(n)}(k_{n+2-j} - k_{n-j})g(y_j) \\
&\quad + \frac{h^2}{2}\theta^2 k_2 g(y_n)^2 - \frac{h^2}{2}(1-\theta)^2 k_0 g(y_{n+1})^2.
\end{aligned}
$$

By hypothesis, the matrix of order $n+1$ with (i,j) entry $(k_{2n+2-i-j} - k_{2n-i-j})$ is negative semi-definite, and so the first two terms in the righthand side of this expression are less than or equal to 0. The condition $0 \le \theta \le \frac{1}{2}$ combines with the observation that $k_2 - k_0 \le 0$ and the fact that g is nondecreasing to yield the result that $V(n+1,\{y_j\}) - V(n,\{y_j\}) \le 0$, as required. Moreover, if $(k_{2n+2-i-j} - k_{2n-i-j})$ is negative definite, then it follows that

$$
V(n+1,\{y_j\}) - V(n,\{y_j\}) < \frac{h^2}{2}(k_2\theta^2 g(y_n)^2 - k_0(1-\theta)^2 g(y_{n+1})^2) < 0
$$

. The conclusions of the theorem follow from Theorem 1.5.8 by choosing the function $\omega(s)$ to be an increasing function on the interval $[0, y_0]$ bounded above by $\frac{h^2}{2}(k_0 - k_2)\theta^2 g(s)^2$.

It is worth making a number of remarks about our analysis of the discrete case.

1. It is possible to undertake a similar analysis and to reach a similar conclusion to that given in Theorem 4.2.1 by a direct argument and without recourse to a discrete Lyapunov function. The direct argument is based

on showing that the sequence $\{y_n\}$ is a decreasing sequence of nonnegative values. We proceed in this chapter with the Lyapunov approach, however the following chapter discusses the alternative approach.

2. The analysis also applies to perturbations of the zero solution that are restricted to taking negative values. This follows since $xg(x) > 0$ for all nonzero x.

3. The sequence $\{k_j\}$ we have considered has fading memory, since it is in l^1 and is completely monotone. It follows from (4.2.1) that

   (a) the only possible limit, λ, of the sequence $\{y_j\}$ must satisfy $g(\lambda) = 0$ (and so $\lambda = 0$),

   (b) if $y_n \geq 0$ for $n \geq N$, then there is a $J \geq N$ for which $y_{j+1} - y_j \leq 0$ for every $j \geq J$,

   (c) if $y_n \leq 0$ for $n \leq N$, then there is a $J \geq N$ for which $y_{j+1} - y_j \geq 0$ for every $j \geq J$.

4. In either of the last two cases, we can easily construct a Lyapunov function for $\{y_n\}$ as we did in our proof of Theorem 4.2.1. The only addition to the analysis we gave is that the constant M must be changed to ensure that $G > 0$ is sufficient (but not necessary) for the conclusions of Theorem 4.2.1.

We summarise these remarks in the following theorem, whose proof is identical to the proof of Theorem 4.2.1 apart from the choice of constant M.

Theorem 4.2.2 *For the equation (4.2.1), assume H4, H5, H7 are satisfied as in Theorem 4.2.1. Let y_0 be given. Then either*

(i) the sequence y_n exhibits infinitely many changes of sign, or

(ii) for every $\epsilon > 0$ there is a corresponding $\delta_\epsilon > 0$ and a natural number N_ϵ for which $|y_0| < \delta_\epsilon$ implies $|y_n| < \epsilon$ for each $n > N_\epsilon$.

If, in addition to H4, H5 and H7 above, $A^\ddagger(n)$ is a negative definite matrix, then either y_n changes sign infinitely often or $y_n \to 0$ as $n \to \infty$.

Note. All of the above analysis can be repeated with hardly any additional work for equations of the form (4.2.2), under the conditions $f(t,0) = 0$, $\xi f(t,\xi) < 0$. We obtain the following corollary.

Corollary 4.2.3 *Under the additional hypothesis that the function $f(t,\xi)$ satisfies $f(t,0) = 0$ and $\xi f(t,\xi) \leq 0$ for $\xi \neq 0$, the conclusions of Theorem 4.2.2 also apply to the equation (4.2.2).*

4.3 An example equation

In this section we consider the particular integro-differential equation

$$y'(t) = -\int_0^t e^{-\lambda(t-s)} y^3(s) ds. \qquad (4.3.1)$$

NB We return to this equation in chapter 5 to illustrate the complexity of the problem of choosing a suitable discrete analogue.

For λ real and positive, this equation satisfies the conditions of Theorem 4.1.1 and Theorem 4.1.2. We can therefore conclude that the zero solution of (4.3.1) is asymptotically stable and that every solution satisfies $y(t) \to 0$ as $t \to \infty$ whatever initial value $y(0)$ we choose. Further, from our analysis, we can predict that the numerical solution will either oscillate infinitely many times or will satisfy $y_n \to 0$. Our conjecture in the previous section suggests that, for sufficiently small starting value y_0, the solution will satisfy $y_n \geq 0$.

However, before we proceed with our numerical experiments, we present an alternative analysis of (4.3.1) by converting it to a second order ordinary differential equation, in order to give us some further insights into the problem.

Various attempts at finding an analytical solution have been unsuccessful; including applying Laplace transforms to (4.3.1) and also by first converting the equation to an ODE and attempting to use elementary ODE techniques. However, it is useful to note that if we consider the particular case where $\lambda = 0$ and we are given the initial conditions $y(0) = 1$, $y'(0) = 0$ then the equation can be solved as follows: Differentiating both sides of (4.3.1) with respect to t yields

$$y''(t) = -\lambda y'(t) - y^3(t). \qquad (4.3.2)$$

Setting $\lambda = 0$, we have

$$y''(t) = -y^3(t). \qquad (4.3.3)$$

Multiplying both sides by $y'(t) \Rightarrow$

$$y'(t)y''(t) = -y^3(t)y'(t). \tag{4.3.4}$$

Integrating both sides with respect to $t \Rightarrow$

$$\frac{(y'(t))^2}{2} = -\frac{y^4(t)}{4} + C. \tag{4.3.5}$$

The initial conditions can be used to show that $C = \frac{1}{4}$. Therefore,

$$\sqrt{2}y'(t) = \sqrt{1 - y^4(t)} \tag{4.3.6}$$

i.e.

$$\sqrt{2}\int_1^y \frac{1}{\sqrt{1-\psi^4}}d\psi = \int_0^t dt. \tag{4.3.7}$$

Therefore,

$$\sqrt{2}\int_1^y \frac{1}{\sqrt{1-\psi^4}}d\psi = t. \tag{4.3.8}$$

Although the integral on the left hand side cannot be evaluated explicitly using elementary techniques, we can use the original differential equation to obtain a geometric proof that the solution is periodic (see Figure 4.3.1) and we can use results from the study of elliptic integrals to obtain the period of oscillation. The equation (4.3.5) represents an ovoid closed curve in the $(\frac{dy}{dt}, y)$ plane (see figure 4.3.1).

We can proceed further with equation (4.3.8) by using a suitable change of variable to obtain an elliptic integral. We give two suitable substitutions here and show that both lead to the same elliptic integral. We know, from the original second order equation, that $dy = pdt$, where $p = \frac{dy}{dt}$ and $dp = -y^3dt$. Therefore, $t = \int \frac{dy}{p} = -\int \frac{dp}{y^3}$.

Method 1: Taking the positive square root for equation (4.3.8), let $y = +\sqrt{\cos(\phi)}$. Therefore $y^2 = \cos(\phi)$ and $2ydy = -\sin(\phi)d\phi$.

$$t = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\cos(\phi)}}d\phi. \tag{4.3.9}$$

Method 2: From letting $p = \frac{dy}{dt}$ we know that

$$t = -\int_0^p \frac{dp}{(1-2p^2)^{\frac{3}{4}}} \tag{4.3.10}$$

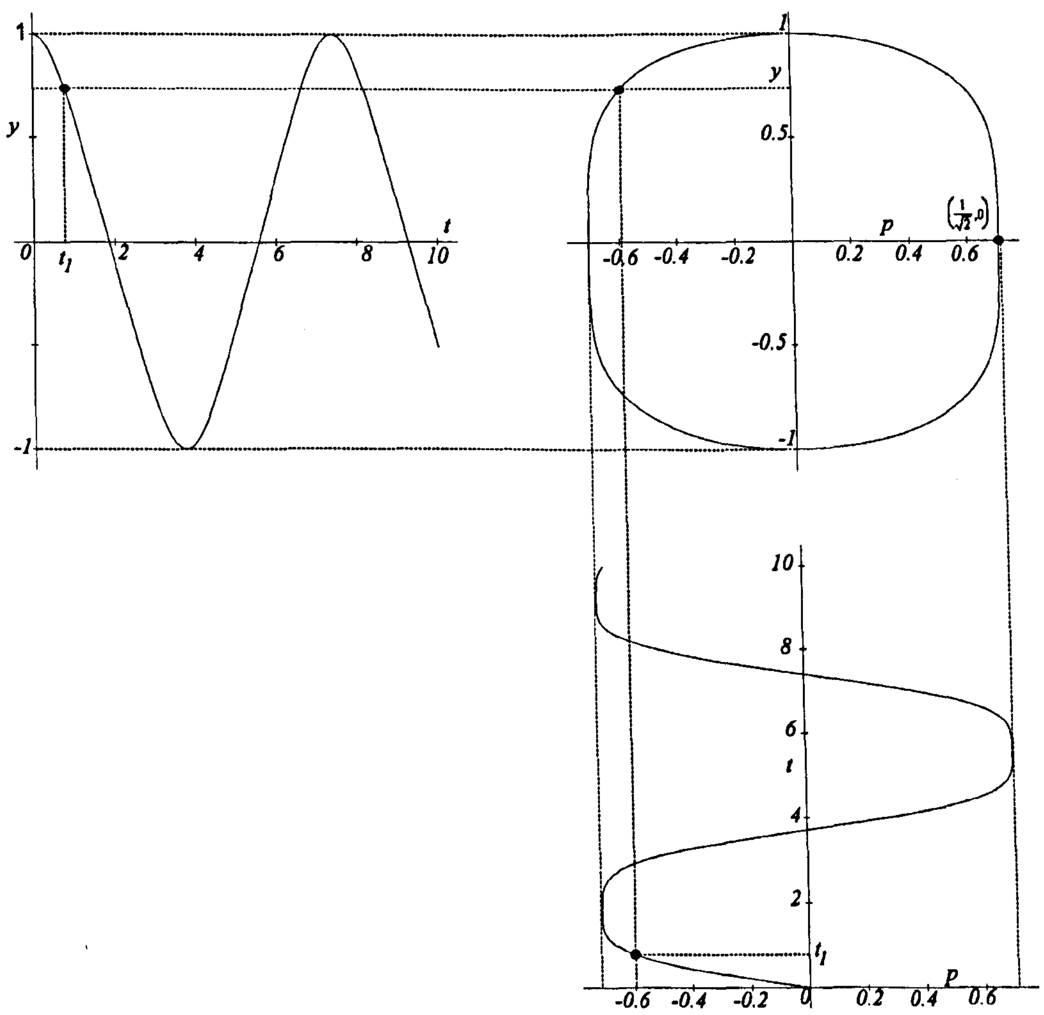

Figure 4.3.1: Geometric proof of periodicity

Let $\sqrt{2}p = \sin(\theta)$ so that $\sqrt{2}\frac{dp}{d\theta} = \cos(\theta)$. Therefore,

$$t = -\frac{1}{\sqrt{2}}\int \frac{\cos(\theta)d\theta}{(\cos^2(\theta))^{\frac{3}{4}}} = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\cos(\theta)}}d\theta \tag{4.3.11}$$

which is identical to equation (4.3.9). Because of periodicity,

$$\begin{aligned}
\text{Period} &= 4\left(\frac{1}{\sqrt{2}}\int_{\frac{\pi}{2}}^{0} \frac{1}{\sqrt{\cos(\phi)}}d\phi\right) \\
&= 2\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos(\phi)}}d\phi \\
&= \sqrt{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \\
&= 7.4163
\end{aligned}$$

which is correct to 4 decimal places. So the solution to the original problem is periodic with a period of approximately 7.4. Although we know that (4.3.1) satisfies Nohel and Levins' conditions for asymptotic stability (for positive λ), it is interesting to note that when we write the problem as a pair of coupled first order ODE's we can find a different Lyapunov function to show that the system possesses an asymptotically stable solution as expected. Writing (4.3.1) as two coupled equations:

$$p = \frac{dy}{dt} \tag{4.3.12}$$

$$\frac{dp}{dt} = -\lambda y_2 - y_1^3 \tag{4.3.13}$$

Let

$$V = 2y_2^2 + y_1^4 (\geq 0) \tag{4.3.14}$$

Therefore

$$\begin{aligned}
V' &= 4p\frac{dp}{dt} + 4y^3p \\
&= 4p(-\lambda p - y^3) + 4y^3p
\end{aligned}$$

i.e.

$$V' = -4\lambda p^2 < 0 \tag{4.3.15}$$

Thus, $V(y,p)$ is a Lyapunov function for the system (4.3.12), (4.3.13) and the zero solution is asymptotically stable.

We now conduct some numerical experiments. The VIDE (4.3.1) satisfies conditions H1, H2 and H3, using θ-methods with $\theta = 0, \frac{1}{2}, 1$. For $\theta = 0, \frac{1}{2}$ the discrete equation satisfies H4, H5 and H7, and we will see that, for small enough $y_0 > 0$, H6 is also satisfied. For $\theta = 1$, the hypothesis H7 is violated and we are able to make some interesting observations about the numerical solution in this case.

We consider three discrete equations ($\theta = 0, \frac{1}{2}, 1$) and compare the long term solutions obtained for different initial values y_0.

First we consider the implicit Euler rule ($\theta = 0$) which provides an implicit scheme:

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} e^{-\lambda(n+1-j)h} y_j^3, \; n \geq 1. \tag{4.3.16}$$

Previous experience with other types of problem (i.e. integral equations, see [28] for example) indicates that we could expect a highly stable scheme to result. Indeed, one can derive a definite result on qualitative behaviour of the solution in this case by the method of stability by first approximation, see, for example, [48]. Figures 4.3.2 and 4.3.3 show values of the solution y_n of (4.3.16) for fixed $h = 0.1$ when the initial value of y_0 takes different values. After some oscillations (according to the initial value y_0) $y_n \to 0$ as $n \to \infty$. The diagrams indicate that the zero solution is asymptotically stable for $h = 0.1$. In practice, when one has a priori knowledge that $y(t) \geq 0$, one would discard oscillatory solutions $\{y_n\}$ as unrealistic.

Second, we consider the use of the trapezium rule ($\theta = 0.5$) to provide an alternative implicit scheme:

$$y_{n+1} - y_n = -h^2 \left(\sum_{j=1}^{n} e^{-\lambda(n+1-j)h} y_j^3 + \frac{y_{n+1}^3 + y_0^3}{2} \right), \; n \geq 1. \tag{4.3.17}$$

Again, we have reason to expect good stability behaviour. Figures 4.3.4 and 4.3.5 exhibit similar results to (4.3.16).

For our third scheme we have used the explicit Euler rule ($\theta = 1$) for comparison:

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n} e^{-\lambda(n-j)h} y_j^3, \; n \geq 1. \tag{4.3.18}$$

Here we are using an explicit scheme for evaluating the sequence $\{y_n\}$. Previous experience leads us to suspect that the scheme may exhibit poor stability

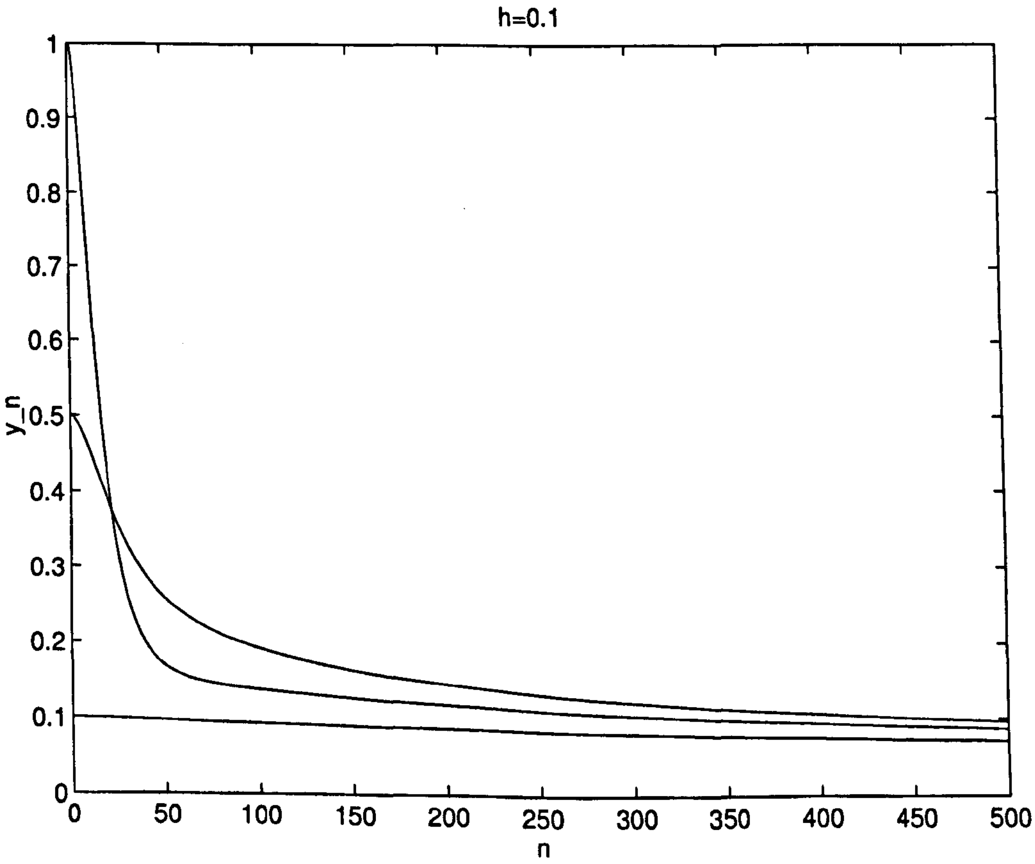

Figure 4.3.2: $\theta = 0$. With small initial value, the solution tends slowly to 0.

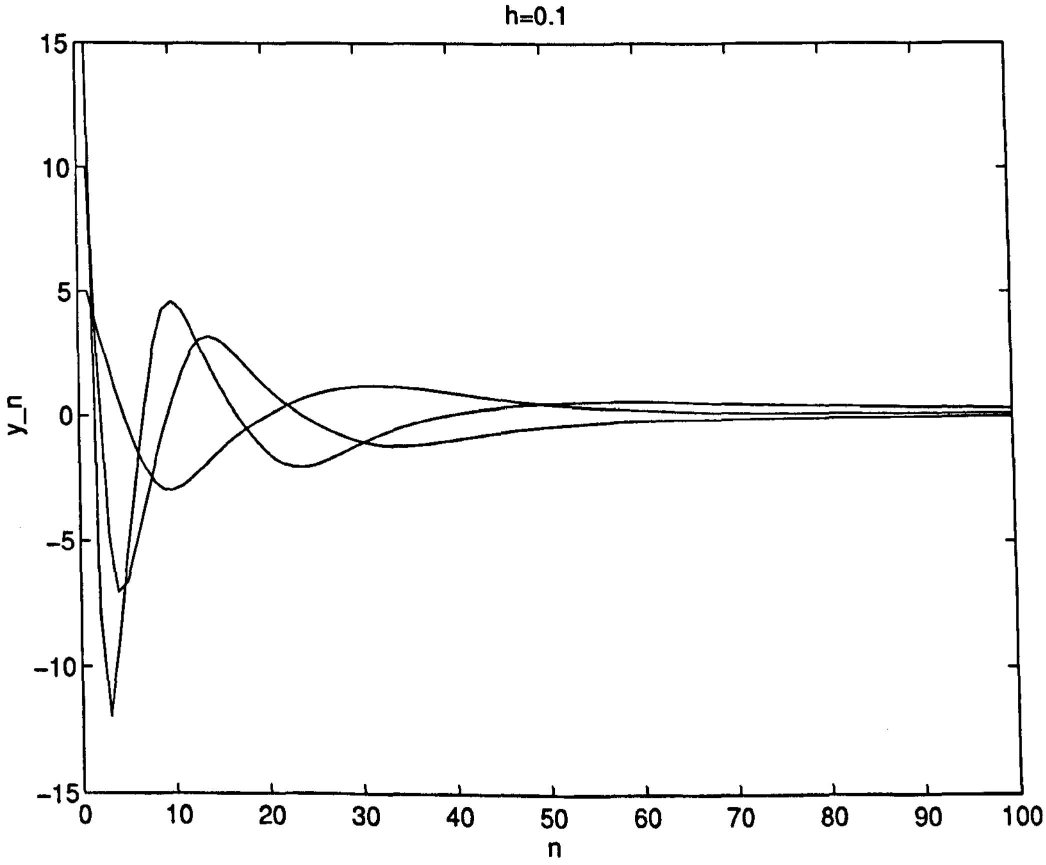

Figure 4.3.3: $\theta = 0$. With larger initial value, the solution tends to 0 after several oscillations.

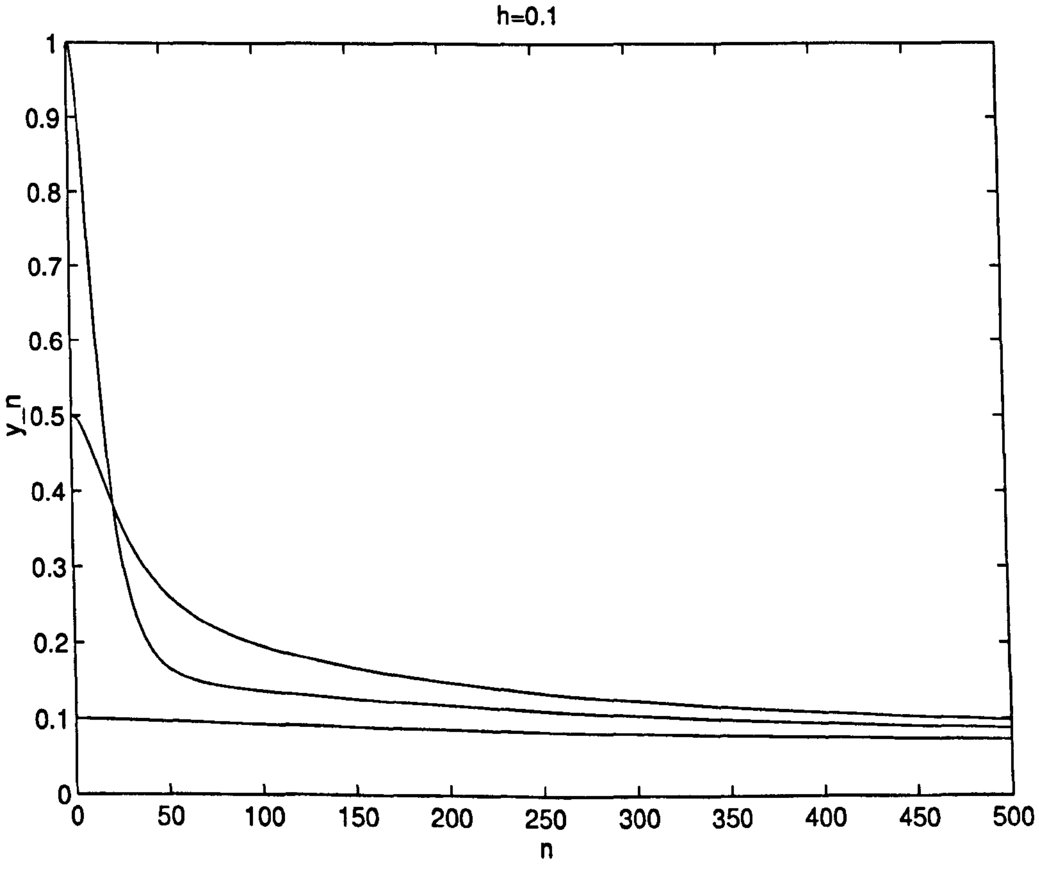

Figure 4.3.4: $\theta = 0.5$. With small initial value, the solution tends slowly to 0.

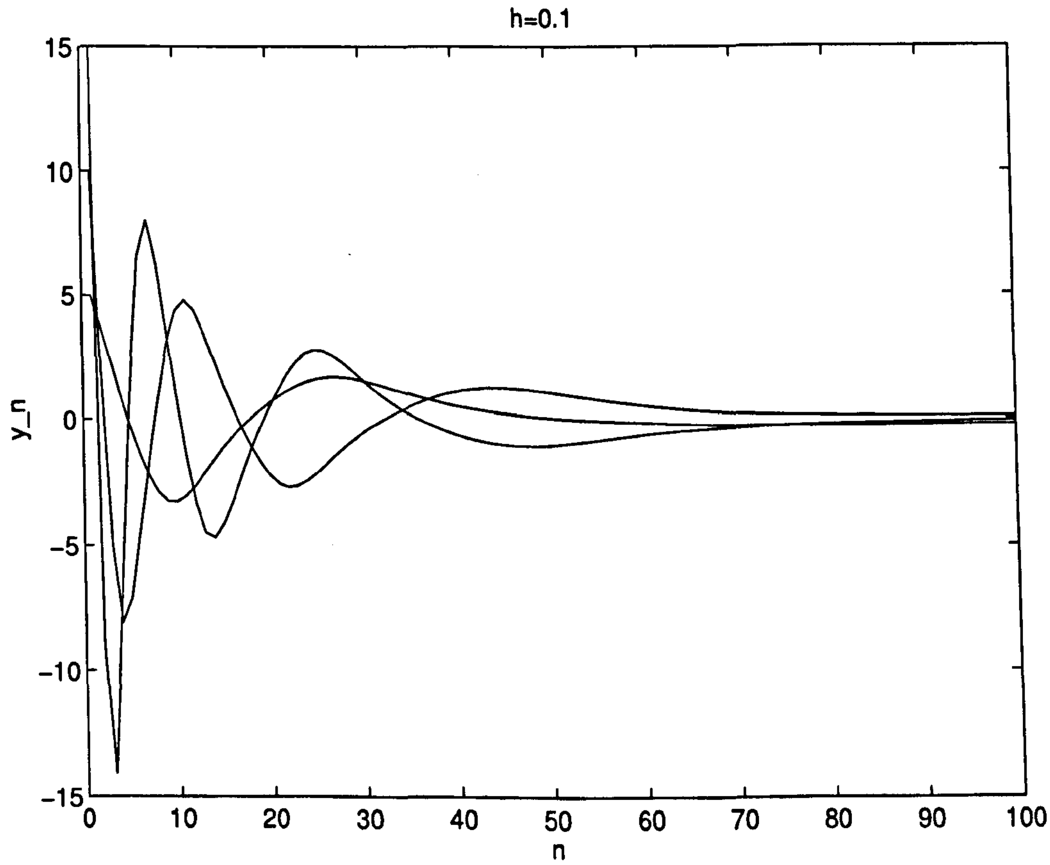

Figure 4.3.5: $\theta = 0.5$. With larger initial value, the solution tends to 0 after several oscillations.

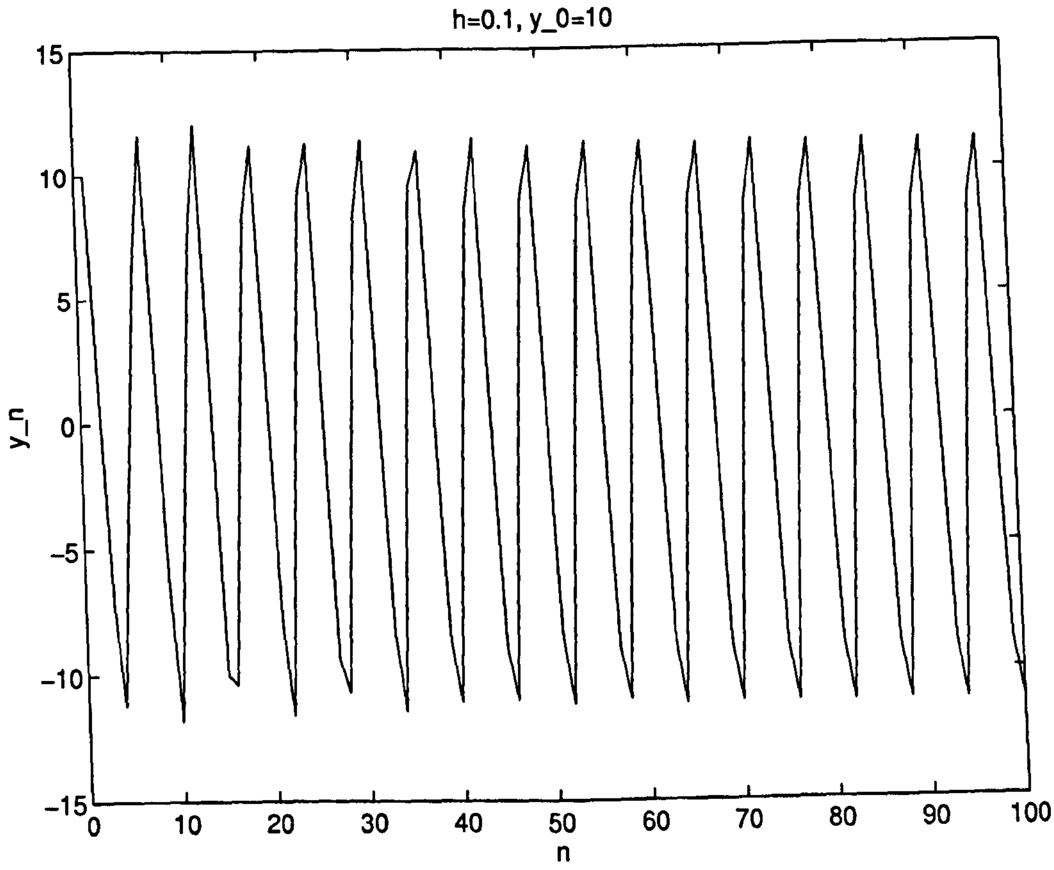

Figure 4.3.6: $\theta = 1$. A particular choice of y_0 yields persistent oscillations.

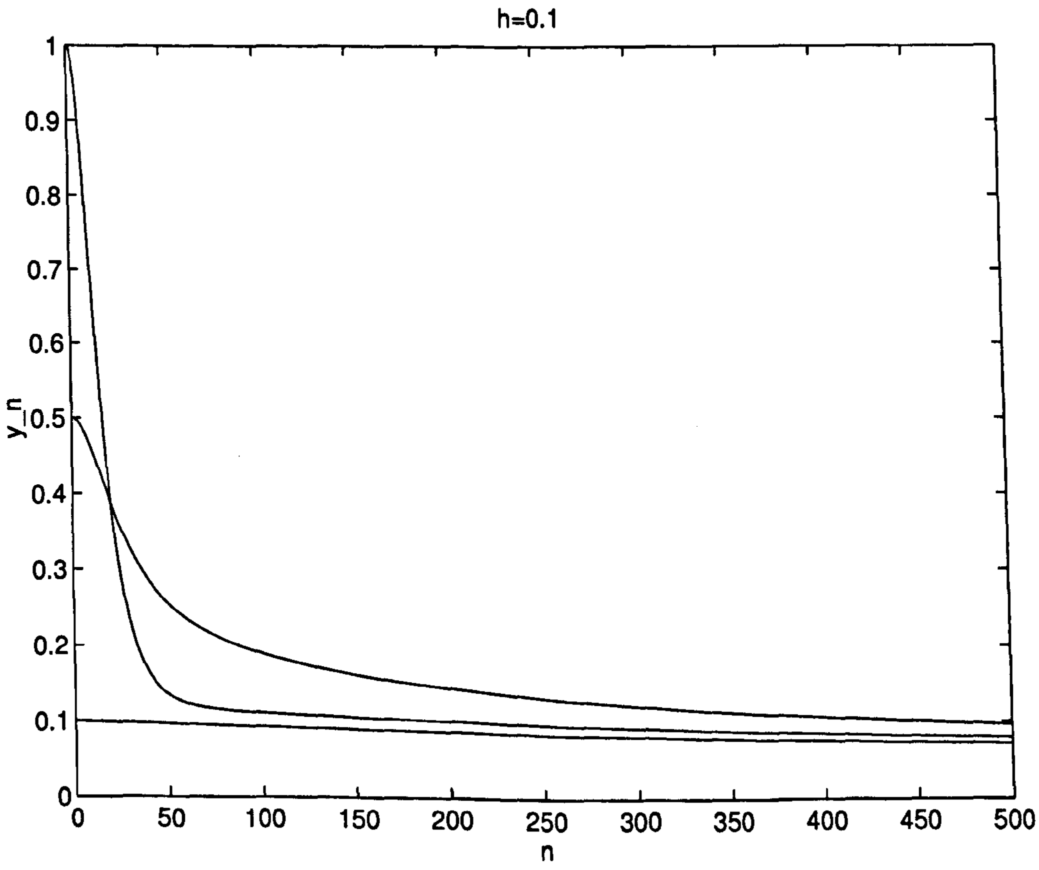

Figure 4.3.7: $\theta = 1$. Smaller choices of y_0 lead to solutions that tend to 0.

properties. Indeed, this scheme does not satisfy H7. Figures 4.3.6 and 4.3.7 give examples demonstrating where the use of an explicit rule leads to problems. For sufficiently small values of y_0, the solution of (4.3.18) satisfies $y_n \to 0$ as $n \to \infty$. However, we can observe that, for a particular choice of y_0, the numerical solution exhibits (spurious or unrealistic) persistent oscillations.

In our example, for each of the discrete schemes we have considered, when we choose our step length h sufficiently small, it appears from our calculations that, for sufficiently small perturbations of the initial value from zero, the solution tends to zero. For the two implicit schemes, we have seen no evidence that persistent oscillations do, in fact, arise. For the explicit scheme we considered, there is likely to be a relationship between the choice of h and the size of perturbation of the initial value from zero if persistent oscillations are to be avoided. However, more analysis is needed to predict a precise relationship between the numerical method, the step length and the stability of the zero solution for different choices of perturbation y_0.

We give this insight. We can seek (directly) solutions of equations (4.3.16), (4.3.17) and (4.3.18) that exhibit stable oscillations of period two. It is easy to show that no solutions of this type arise for equations (4.3.16) and (4.3.17) whatever choice of initial value y_0 we make. However, for equation (4.3.18), solutions of this type do exist and the amplitude of the oscillations is $\frac{1}{h}\left(2+2e^{-\lambda h}\right)^{\frac{1}{2}}$.

For fixed h, λ, there is one possible amplitude of oscillatory solution of period 2. Persistent oscillations of period 2 can arise when the initial perturbation of the zero solution is precisely $\frac{1}{h}\left(2+2e^{-\lambda h}\right)^{\frac{1}{2}}$. Further, as $h \to 0$, the size of the necessary perturbation approaches ∞. One can adopt a similar approach to calculate the amplitude of persistent oscillations of period greater than 2. The behaviour we have observed is consistent with our expectation that, for suffciently small perturbations of the zero solution (the size of perturbation depending on h) the solution will not exhibit persistent oscillations of this type even for the explicit scheme (4.3.18).

Chapter 5

Stability of a difference analogue for a nonlinear integro-differential equation of convolution type

In this chapter we focus more on our example equation in the previous chapter, in order to give us greater insight into the problem of choosing a suitable discrete analogue. Although our results are dependent on some strict conditions we show, in chapter 6, that they are not too restrictive and so the results are useful. Consider the nonlinear integro-differential equation

$$y'(t) = -k \int_0^t e^{-\lambda(t-s)} y^3(s)ds, \; k, \lambda > 0. \tag{5.0.1}$$

It is easy to prove that the zero solution of this equation is stable. For example, put $y_1(t) = y(t)$, $y_2(t) = y'(t)$. Differentiating the equation (5.0.1), we obtain

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= -ky_1^3(t) - \lambda y_2(t). \end{aligned}$$

The function $V = ky_1^4(t) + 2y_2^2(t)$ is a Lyapunov function for this system since $V' = -4\lambda y_2^2(t) \leq 0$. We note also hat $V' < 0$ unless $y_2 = 0$.

It is obvious that one continuous system can have several discrete analogues according to the choice of numerical scheme, however not all of these analogues need be (asymptotically) stable. The problem we are interested in is to determine how one may construct a difference analogue of continuous asymptotically stable systems which will be asymptotically stable.

In the next section, we propose one possible difference analogue of the equation (5.0.1) and asymptotic stability of the zero solution of this discrete scheme will be proved. The proof we give is based on the general method of Lyapunov functional construction (developed for systems with aftereffect).

5.1 Construction of difference analogues

We consider here two possible choices of difference analogue:

For the first, we shall divide the interval $[0,t]$ into $n+1$ intervals of length $h>0$. In this way $t=(n+1)h$, $s=jh$, $j=0,1,\dots,n$, $y_j=y(jh)$, and the right-hand side of equation (5.0.1) takes the form

$$-kh\sum_{j=0}^{n} e^{-h\lambda(n+1-j)}y_j^3.$$

Using $\frac{y_{n+1}-y_n}{h}$ to approximate $y'(t)$ in the point $t=(n+1)h$ as a result we obtain the explicit difference scheme:

$$y_{n+1}=y_n-kh^2\sum_{j=0}^{n} e^{-h\lambda(n+1-j)}y_j^3,\ n=0,1,\dots. \tag{5.1.1}$$

Denoting $a=e^{-h\lambda}$, we transform the right-hand side of the equation (5.1.1) in the following way

$$y_{n+1}=y_n-akh^2y_n^3+a\left(-kh^2\sum_{j=0}^{n-1} e^{-h\lambda(n-j)}y_j^3\right). \tag{5.1.2}$$

Differencing successive expressions of the form (5.1.1) leads to

$$y_1=y_0-akh^2y_0^3, \tag{5.1.3}$$

$$y_{n+1}=y_n-akh^2y_n^3+a\left(y_n-y_{n-1}\right),\ n=1,2,\dots. \tag{5.1.4}$$

The equation (5.1.4 has four parameters: k,λ,h,y_0. We denote

$$x_n=\frac{y_n}{y_0},\ \tau=h\lambda,\ a=e^{-\tau},\ \gamma=k\frac{y_0^2}{\lambda^2}, \tag{5.1.5}$$

and finally obtain the equation

$$x_0=1,\ x_1=1-\gamma a\tau^2,$$

$$x_{n+1} = x_n - \gamma a \tau^2 x_n^3 + a(x_n - x_{n-1}),\ n = 1, 2, \dots, \tag{5.1.6}$$

which has only two parameters: γ and τ.

We can follow a simliar approach, based on the use of analternative approximation for the integral term. We divide the interval $[0, t]$ into n intervals of fixed length $h > 0$. We put $t = nh$, $s = jh$, $j = 0, 1, 2, \dots, n$; $y_j = y(jh)$ and we use the approximation

$$-kh \sum_{j=1}^{n} e^{-h\lambda(n-j)} y_j^3$$

for the right hand side of equation (5.0.1). Using $\frac{y_{n+1} - y_n}{h}$ for approximation of the derivative $y'(t_n)$ we obtain:

$$y_1 = y_0,\ y_{n+1} = y_n - kh^2 \sum_{j=1}^{n} e^{-h\lambda(n-j)} y_j^3,\ n = 1, 2, 3, \dots. \tag{5.1.7}$$

Following the same approach as above, this difference scheme may be expressed in the two-paramter form

$$x + 1 = x_0 = 1,\ x_{n+1} = x_n - \gamma \tau^2 x_n^3 + a(x_n - x_{n-1}),\ n = 1, 2, 3, \dots. \tag{5.1.8}$$

5.2 Lyapunov functional construction

The general method of Lyapunov functional construction for difference equations [46] consists of four steps. We describe the approach in detail here as it applies to (5.1.6) and then we state the corresponding results for equation (5.1.8).

Step 1. Represent the equation (5.1.6) in the form

$$x_{n+1} = F(x_n) + \Delta G_n(x_{n-1}),$$

where

$$F(x) = x - \gamma a \tau^2 x^3,\ G(x) = ax,\ \Delta G_n(x_{n-1}) = a(x_n - x_{n-1}).$$

Step 2. Consider the auxiliary difference equation without delay

$$z_0 = 1,\ z_{n+1} = F(z_n) = z_n - \gamma a \tau^2 z_n^3,\ n = 01, 2, \dots. \tag{5.2.1}$$

The function $v(z) = z^2$ is a Lyapunov function for this equation. In fact, using (5.2.1) we get

$$\begin{aligned}\Delta v_n &= v(z_{n+1}) - v(z_n) \\ &= z_{n+1}^2 - z_n^2 \\ &= (z_n - \gamma a\tau^2 z_n^3)^2 - z_n^2 \\ &= \left(\gamma a\tau^2\right)^2 z_n^4 \left(z_n^2 - \frac{2}{\gamma a\tau^2}\right).\end{aligned}$$

It is obvious that, if we impose the condition

$$\gamma a\tau^2 < 2$$

we have $\Delta v_n < 0$ for all $n = 0, 1, \ldots$ providing $z_n \neq 0$.

Step 3. We will construct the Lyapunov functional V_n for the system (5.1.6) in the form $V_n = V_{1n} + V_{2n}$, where

$$V_{10} = v(x_0) = x_0^2,$$

$$\begin{aligned}V_{1n} &= v(x_n - G(x_{n-1})) \\ &= (x_n - ax_{n-1})^2, \; n = 1, 2, \ldots.\end{aligned}$$

Calculating ΔV_{1n}, $n = 1, 2, \ldots$, we obtain

$$\begin{aligned}\Delta V_{1n} &= (x_{n+1} - ax_n)^2 - (x_n - ax_{n-1})^2 \\ &= (x_n - \gamma a\tau^2 x_n^3 - ax_{n-1})^2 - (x_n - ax_{n-1})^2 \\ &= (\gamma a\tau^2)^2 x_n^6 - 2\gamma a\tau^2 x_n^4 + 2\gamma(a\tau)^2 x_n^3 x_{n-1}.\end{aligned}$$

Since

$$\begin{aligned}2|x_n^3 x_{n-1}| &\leq x_n^2(x_n^2 + x_{n-1}^2) \\ &= x_n^4 + x_n^2 x_{n-1}^2 \\ &\leq x_n^4 + \frac{1}{2}\left(x_n^4 + x_{n-1}^4\right) \\ &= \frac{3}{2}x_n^4 + \frac{1}{2}x_{n-1}^4,\end{aligned}$$

then

$$\Delta V_{1n} \leq (\gamma a\tau^2)^2 x_n^6 - \gamma(a\tau)^2\left(\frac{2}{a} - \frac{3}{2}\right)x_n^4 + \frac{1}{2}\gamma(a\tau)^2 x_{n-1}^4.$$

Analogously for $n = 0$:

$$\begin{aligned}\Delta V_{10} &= V_{11} - V_{10} \\ &= (x_1 - ax_0)^2 - x_0^2 \\ &= (1 - \gamma a\tau^2 - a)^2 - 1 \\ &= a^2(1+\gamma\tau^2)^2\left(1 - \frac{2}{a(1+\gamma\tau^2)}\right).\end{aligned}$$

It is easy to see that, for small enough τ (and, indeed, for large enough τ) $\Delta V_{10} < 0$.

Step 4. Choosing the functional V_{2n} in the form $V_{20} = 0$, $V_{2n} = \frac{1}{2}\gamma(a\tau)^2 x_{n-1}^4$, $n = 1, 2, \dots$, for the functional $V_n = V_{1n} + V_{2n}$ we get

$$\Delta V_n \le -(\gamma a\tau^2)^2 x_n^4 (f(\tau) - x_n^2),\ n = 0, 1, \dots, \tag{5.2.2}$$

where

$$f(\tau) = \frac{2(e^\tau - 1)}{\gamma\tau^2}.$$

It is obvious that

$$\lim_{\tau\to 0} f(\tau) = \infty,\ \lim_{\tau\to\infty} f(\tau) = \infty.$$

Simple investigations of $f(\tau)$ show that $f(\tau)$ has a minimum value of approximately $\frac{3.088}{\gamma}$ for τ close to 1.6.

Let us suppose that the sequence $\{x_n\}$ is bounded and that there exists a τ for which

$$x_n^2 \le k < f(\tau),\ n = 1, 2, \dots, \tag{5.2.3}$$

In this way, $\Delta V_n < 0$ for all $n = 0, 1, \dots$ as long as $x_n \ne 0$. We note, for example, that if each x_n^2 is bounded by $\frac{3}{\gamma}$ then (5.2.3) will be satisfied for all $\tau > 0$. On the other hand, for larger bounds on x_n^2, (5.2.3) will be satisfied for a restricted range of τ. It can be shown that, under suitable additional conditions, all solutions $\{x_n\}$ are bounded and therefore this assumption may be justified. Further investigation of this feature of solutions will be the subject of the next chapter.

Remark The corresponding analysis for the alternative discrete form (5.1.8) proceeds as follows:

1. We choose $F(x) = x - a\gamma\tau^2 x^3$ and $G(x) = ax$.

2. We proceed as above and show that, whenever $\gamma\tau^2 < 2$, $\Delta v_n < 0$ for $n = 1, 2, \dots$ so long as $z_n \ne 0$.

3. We can show that

$$\Delta V_{1n} = \gamma^2\tau^4 x_n^6 - 2\gamma\tau^2 x_n^4 + 2\gamma\tau^2 a x_n^3 x_{n-1},\ n = 1, 2, \ldots$$

so

$$\Delta V_{1n} \le \gamma^2\tau^4 x_n^6 - \gamma\tau^2 x_n^4\left(2 - \frac{3a}{2}\right) + \frac{1}{2}\gamma a\tau^2 x_{n-1}^4$$

and

$$\Delta V_{10} = (x_1 - a x_0)^2 - x_0^2 = a^2 - 2a = a(a-2) < 0.$$

4. We put

$$V_{20} = 0,\ V_{2n} = \frac{1}{2}\gamma a\tau^2 x_{n-1}^4,\ n = 1, 2, 3, \ldots$$

and we can then show that

$$\Delta V_n \le -\gamma^2\tau^4 x_n^4(f(\tau) - x_n^2),$$

where

$$f(\tau) = \frac{2(1 - e^{-\tau})}{\gamma\tau^2}.$$

It is easy to show that f is a strictly decreasing function for positive τ with $f(0) = \infty$ and $f(\infty) = 0$. Thus, for any constant $M > 0$ there exists $\tau_0 > 0$ such that $f(\tau) \ge M$ for every positive $\tau \le \tau_0$. We conclude that, if $\{x_n^2\}$ is assumed bounded, then for sufficiently small $\tau > 0$, $f(\tau) \ge x_n^2$, $n = 0, 1, 2, \ldots$.

5.3 Proof of asymptotic stability

From (5.2.2), (5.2.3) it follows

$$\sum_{j=0}^{n} \Delta V_j = V_{n+1} - V_0 \le -(\gamma a\tau^2)^2 \sum_{j=0}^{n} x_j^4(f(\tau) - x_j^2) < 0. \tag{5.3.1}$$

Since

$$V_{n+1} = (x_{n+1} - a x_n)^2 + \frac{1}{2}\gamma(a\tau)^2 x_n^4 \ge \frac{1}{2}\gamma(a\tau)^2 x_n^4$$

and

$$V_{n+1} \le V_0 = x_0^2 = 1,$$

from here it follows that

$$x_n^4 \leq \frac{2}{\gamma(a\tau)^2}.$$

Using (5.1.5) we obtain

$$y_n^2 = \left(\frac{2}{k}\right)^{\frac{1}{2}} \frac{|y_0|}{ah}.$$

Therefore for any $\epsilon > 0$ there exists $\delta = \left(\frac{k}{2}\right)^{\frac{1}{2}} ah\epsilon^2$ such that $|y_n| < \epsilon$, $n > 0$, if $|y_0| < \delta$. In other words, we have shown that the zero solution of the equation (5.1.1) is stable.

We can show further that the zero solution is asymptotically stable. From (5.3.1) and the fact that $V_{n+1} \geq 0$ it follows that

$$\sum_{j=1}^{\infty} x_j^4(f(\tau) - x_j^2) \leq \frac{V_0}{(\gamma a\tau^2)^2}. \tag{5.3.2}$$

The convergence of the series in the left-hand part of (5.3.2) implies that

$$\lim_{n\to\infty} x_n^4(f(\tau) - x_n^2) = 0$$

Now the function

$$x^4\left(f(\tau) - x^2\right)$$

has zeros at $x = 0, \pm\sqrt{f(\tau)}$ so for large n the values of x_n must lie in arbitrary small neighbourhoods of these three points. But by hypothesis,

$$x_n^2 \leq k < f(\tau)$$

so the two outer points are excluded. Hence

$$\lim_{n\to\infty} x_n = 0.$$

$$\lim_{n\to\infty} y_n = 0.$$

Remark A similar argument applies to solutions of the equation (5.1.8).

We summarise our conclusions in the following:

Theorem 5.3.1 *We assume that k, λ, y_0 are given and we solve equations (5.1.6) and (5.1.8) for solutions $\{y_n\}$ for a fixed value of $h > 0$.*

1. *For equation (5.1.6) there is a constant $M > 0$ for which all bounded solutions with $y_n^2 \leq M$ satisfy $y_n \to 0$ regardless of the step size $h > 0$.*

2. *For equation (5.1.6), we consider bounded solutions with $y_n^2 \leq N$ with $N > M$. There exist constants $\tau_{N1}, \tau_{N2} > 0$ such that, if h is chosen so that $0 < h\lambda < \tau_{N1}$ or $h\lambda > \tau_{N2}$ then $y_n \to 0$.*

3. *For equation (5.1.8), consider bounded solutions with $y_n^2 \leq Q$. There exists a constant $\tau_Q > 0$ such that, if h is chosen so that $0 < h\lambda < \tau_Q$ then $y_n \to 0$.*

Remark

In our statement of this Theorem, we have considered the behaviour of bounded solutions of the discrete equations. We can observe from our work in the previous chapter that unbounded solutions may arise with particular combinations of y_0, $h\lambda$. We can further take note that we proved in the previous chapter that, for a different discretisation of (5.0.1) and for sufficiently small $h > 0$ all solutions of the discrete equation are bounded by $|y_0|$. For our explicit methods, we have conducted numerical experiments which indicate a similar result. Our calculations indicate that, if $f(\tau) > 1$, then the solution of the equation (5.1.6) satisfies the condition $|x_n| \leq 1$, $n = 1, 2, \ldots$. (Correspondingly, $|y_n| \leq |y_0|$.) Figures 5.3.1, 5.3.2 and 5.3.3 show the behaviour of the solution of the equation (5.1.6) corresponding to different values of the parameters τ, γ and function $f(\tau)$.

5.4 One generalisation

In this section we generalise our results of the previous section for a wider class of equations. Consider the nonlinear integro-differential equation

$$y'(t) = -k \int_0^t e^{-\lambda(t-s)} y^m(s) ds, \ k, \lambda, m > 0. \tag{5.4.1}$$

The analogues of the equations (5.1.4) and (5.1.6) for the equation (5.4.1) are

$$y_1 = y_0 - akh^2 y_0^m,$$

$$y_{n+1} = y_n - akh^2 y_n^m + a(y_n - y_{n-1}), \ n = 1, 2, \ldots . \tag{5.4.2}$$

and

$$x_0 = 1, \ x_1 = 1 - \gamma a \tau^2,$$

$$x_{n+1} = x_n - \gamma a \tau^2 x_n^m + a(x_n - x_{n-1}), \ n = 1, 2, \ldots , \tag{5.4.3}$$

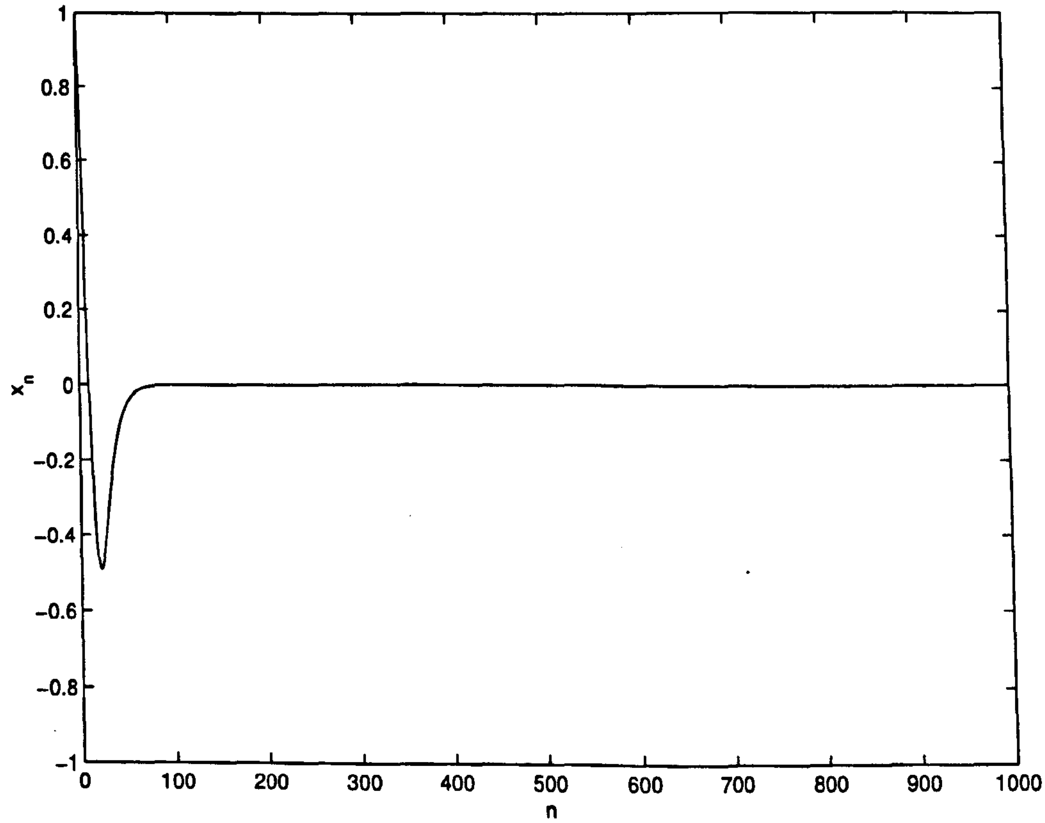

Figure 5.3.1: Numerical scheme (5.1.6) with $\tau = 0.1$, $\gamma = 6$

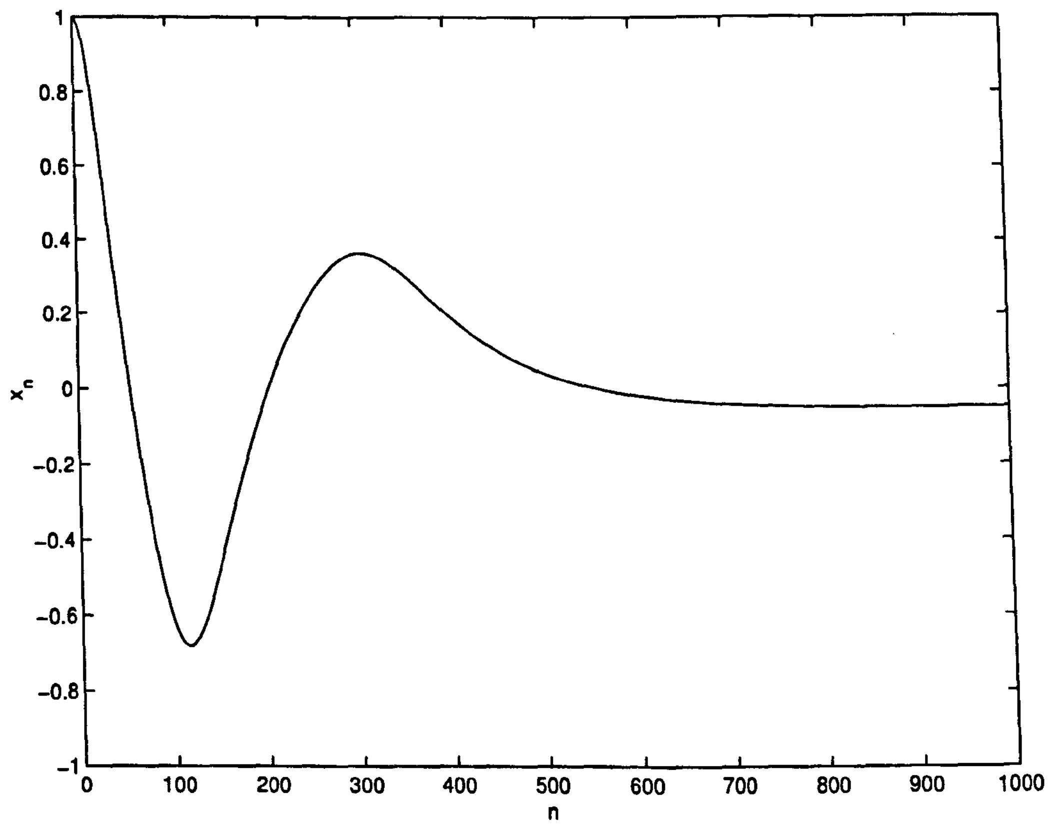

Figure 5.3.2: Numerical scheme (5.1.6) with $\tau = 0.01$, $\gamma = 15$

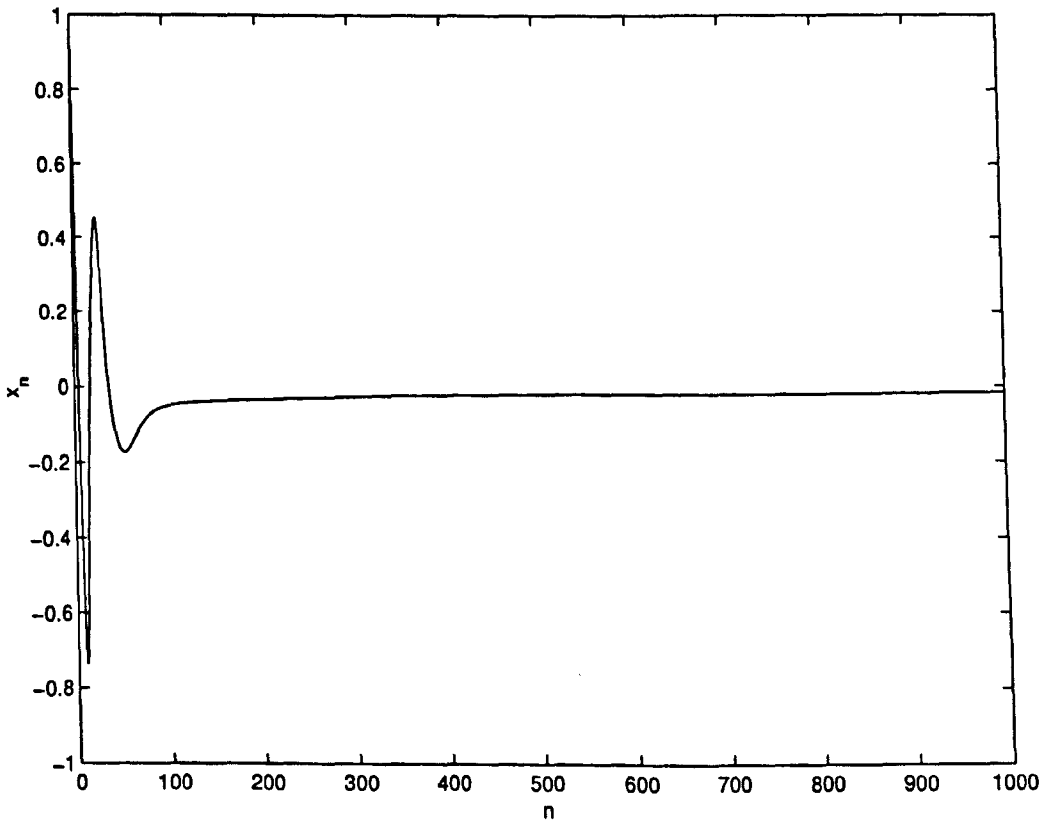

Figure 5.3.3: Numerical scheme (5.1.6) with $\tau = 0.1$, $\gamma = 20$

where

$$x_n = \frac{y_n}{y_0}, \ \tau = h\lambda, \ a = e^{-\tau}, \ \gamma = k\frac{y_0^{m-1}}{\lambda^2}.$$

Let us assume that parameter m has the form

$$m = 2^{l+1} - 1, \ l = 0, 1, 2, \dots .$$

In this case the permitted values of m are: $1, 3, 7, 15, 31, \dots$. In particular, if $l = 0$ (or $m = 1$) the equation (5.4.1) is the equation (5.0.1) which we have already analysed.

We construct a Lyapunov functional V_n for the equation (5.4.3) in the form $V_n = V_{1n} + V_{2n}$, where as before

$$V_{1n} = (x_n - ax_{n-1})^2, \ n = 1, 2, \dots .$$

Calculating ΔV_{1n}, $n = 1, 2, \dots$, we obtain

$$\Delta V_{1n} = (x_{n+1} - ax_n)^2 - (x_n - ax_{n-1})^2$$

$$= (\gamma a\tau^2)^2 x_n^{2m} - 2\gamma a\tau^2 x_n^{m+1} + 2\gamma(a\tau)^2 x_n^m x_{n-1}.$$

It is easy to see that

$$2|x_n^m x_{n-1}| \le x_n^{m-1}(x_n^2 + x_{n-1}^2) = x_n^{m+1} + x_n^{m-1}x_{n-1}^2$$

$$\le x_n^{m+1} + \frac{1}{2}x_n^{m-3}(x_n^4 + x_{n-1}^4) = \left(1 + \frac{1}{2}\right) x_n^{m+1} + \frac{1}{2}x_n^{m-3}x_{n-1}^4$$

$$\le \left(1 + \frac{1}{2}\right) x_n^{m+1} + \frac{1}{2^2}x_n^{m-7}(x_n^8 + x_{n-1}^8) = \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) x_n^{m+1} + \frac{1}{2^2}x_n^{m-7}x_{n-1}^8$$

$$\le \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right) x_n^{m+1} + \frac{1}{2^3}x_n^{m-15}x^1 6_{n-1} \le \dots$$

$$\le \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^l}\right) x_n^{m+1} + \frac{1}{2^l}x_n^{m+1-2^{l+1}}x_{n-1}^{m+1}$$

$$= \left(2 - \frac{1}{2^l}\right) x_n^{m+1} + \frac{1}{2^l}x_{n-1}^{m+1}.$$

In this way

$$\Delta V_{1n} \le (\gamma a\tau^2)^2 x_n^{2m} - \gamma(a\tau)^2 \left(\frac{2}{a} - 2 + \frac{1}{2^l}\right) x_n^{m+1} + \frac{1}{2^l}\gamma(a\tau)^2 x_{n-1}^{m+1}.$$

Choosing the functional V_{2n} in the form $V_{2n} = \frac{1}{2l}\gamma(a\tau)^2 x_{n-1}^{m+1}$, $n = 1, 2, \ldots$, for the functional $V_n = V_{1n} + V_{2n}$ we get

$$\Delta V_n \leq -(\gamma a \tau^2)^2 x_n^{m+1}(f(\tau) - x_n^{m-1}), \ n = 1, 2, \ldots,$$

where

$$f(\tau) = \frac{2(e^\tau - 1)}{\gamma \tau^2}.$$

We suppose that there exists a τ for which

$$x_n^{m-1} < f(\tau), \ n = 0, 1, \ldots. \tag{5.4.4}$$

In this way, $\Delta V_n < 0$ for all $n = 0, 1, \ldots$ whenever $x_n \neq 0$.

By analogy with Theorem 5.3.1 we get

Theorem 5.4.1 *We assume that k, λ, y_0 are given and we solve equations (5.4.3) for solutions $\{y_n\}$ for a fixed value of $h > 0$.*

1. *There is a constant $M > 0$ for which all bounded solutions with $y_n^2 \leq M$ satisfy $y_n \to 0$ regardless of the step size $h > 0$.*

2. *Consider bounded solutions with $y_n^2 \leq N$. There exist constants $\tau_{N1}, \tau_{N2} > 0$ such that, if h is chosen so that $0 < h\lambda < \tau_{N1}$ or $h\lambda > \tau_{N2}$ then $y_n \to 0$.*

Remark

In the linear case ($m = 1$) the condition (5.4.4) has the trivial form $f(\tau) > 1$. From here we get the following statement. Let $h > 0$ such that

$$\frac{2(e^{\lambda h} - 1)}{kh^2} > 1.$$

Then the zero solution of the equation (5.4.2) with $m = 1$ is asymptotically stable. NB Leonid Shaikhet of Donetsk can now extend the results to any odd power; i.e. $m = 1, 3, 5, 7, 9, \ldots$.

Chapter 6

The existence of bounded solutions of discrete analogues for a nonlinear integro-differential equation

Here we prove that for a certain choice of discretisation of

$$y'(t) = -\kappa \int_0^t e^{-\lambda(t-s)} y^m(s)ds, \; y(0) = y_0, \qquad (6.0.1)$$

$$\kappa, \lambda, m > 0, t \in \mathbb{R}^+, m = 2^{l+1} - 1, l = 0, 1, 2, \dots,$$

all solutions are bounded provided the parameters κ, λ fall within certain ranges. This allows us to apply the principal theorem from the previous chapter to prove that for these parameter values all solutions are asymptotically stable. Preliminary numerical calculations lead us to believe that there also exist ranges of parameter values for which no asymptotically stable solutions occur. In addition, our numerical experiments provide strong evidence for the existence of ranges of parameter values within which a wealth of different solution behaviour occurs, providing strong evidence for the onset of chaotic behaviour.

6.1 Existence of bounded solutions to two particular discrete analogues

In this section we prove that for certain pairs of values of γ and τ, with definitions given in (5.1.5) the equation

$$x_0 = 1, \; x_1 = 1 - \gamma a \tau^2,$$

$$x_{n+1} = x_n - \gamma a \tau^2 x_n^m + a(x_n - x_{n-1}) \tag{6.1.1}$$

has a bounded solution.

Theorem 6.1.1 *Consider the discretisation (6.1.1) of the Volterra integro-differential equation (6.0.1). Then* $|x_n| < 1$ *for all* $n = 1, 2, \ldots$ *provided that*

$$\gamma a \tau^2 < 2, \tag{6.1.2}$$

$$a < \frac{1}{2}, \tag{6.1.3}$$

and

$$\begin{cases} \frac{2}{\gamma\tau^2} < 1 & \text{when } 1 \le \left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}} \quad (i) \\ \frac{m-1}{m(m\gamma a\tau^2)^{\frac{1}{m-1}}} < 1 - 2a & \text{when } \left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}} < 1 \le \tilde{x} \quad (ii) \\ 1 < \frac{2(1-a)}{\gamma a\tau^2} & \text{when } \left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}} < \tilde{x} < 1 \quad (iii) \end{cases} \tag{6.1.4}$$

where $\tilde{x}$ *is the unique nonzero real solution to the nonlinear equation*

$$x(\gamma a \tau^2 x^{m-1} - 1) = \frac{m-1}{m(m\gamma a\tau^2)^{\frac{1}{m-1}}} \tag{6.1.5}$$

for $x \in \left(\left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}}, \infty\right)$.

Proof: Assume $|x_k| < 1$ for $k = 1, 2, \ldots, n$. Then either $x_n = 0$ or $x_n \neq 0$. If $x_n = 0$ then

$$|x_{n+1}| = |a||x_{n-1}| < 1 \tag{6.1.6}$$

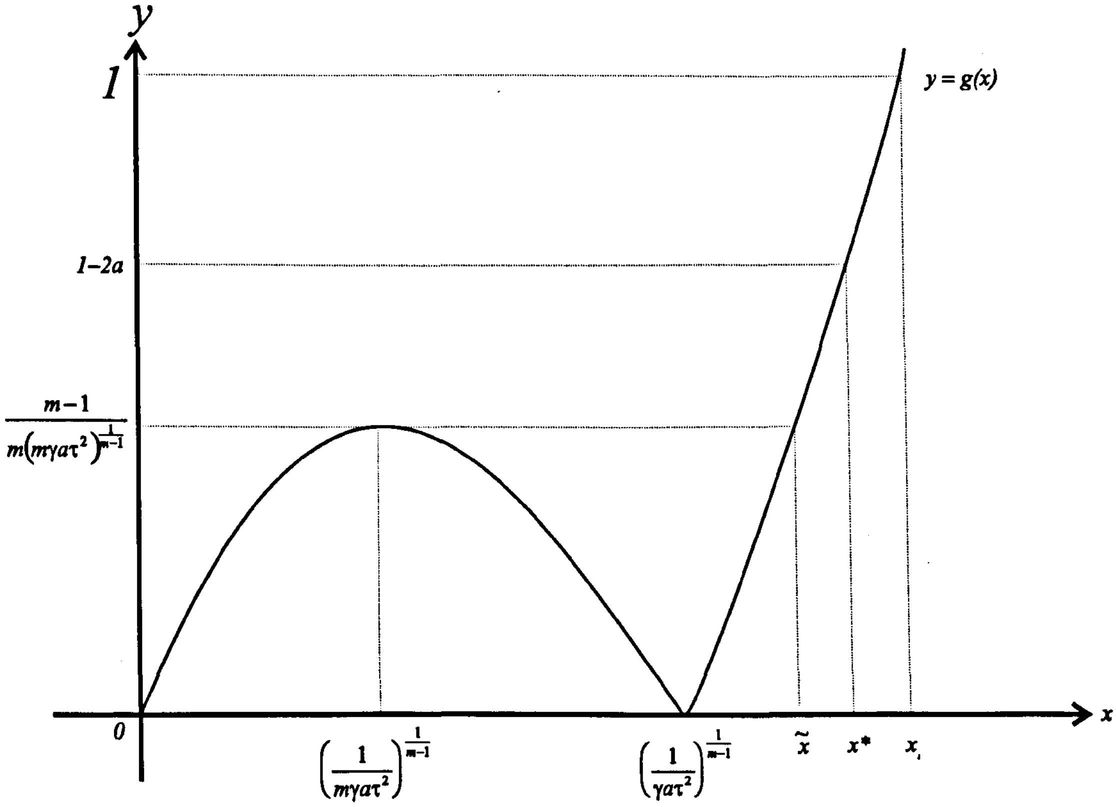

Figure 6.1.1: Graph of $y = g(x)$ for $m = 3$ with domain extended to $(0, \infty)$

because $a = e^{-\tau} < 1$ and $|x_{n-1}| \leq 1$ by hypothesis. If $x_n \neq 0$ then

$$\begin{aligned} |x_{n+1}| &= |x_n - \gamma a \tau^2 x_n^m + a(x_n - x_{n-1})| & (6.1.7) \\ &\leq |x_n||1 - \gamma a \tau^2 x_n^{m-1}| + 2a & (6.1.8) \end{aligned}$$

The right hand side of the inequality (6.1.8) is strictly bounded above by unity provided

$$|x_n||1 - \gamma a \tau^2 x_n^{m-1}| < 1 - 2a. \tag{6.1.9}$$

Consider the function $g : (0, 1) \rightarrow \mathbb{R}^+$

$$g(x) = x|1 - \gamma a \tau^2 x^{m-1}|, \ 0 < x < 1. \tag{6.1.10}$$

We show that, under the hypotheses of the theorem, $g(x) < 1 - 2a$ for every $x \in (0, 1)$. We consider separately three intervals along the x-axis where $x = 1$ could lie (see figure 6.1.1).

I1: $0 < x < 1 \le \left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}}$. Note that $x = \left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}}$ is the point at which g assumes its maximum value $\frac{m-1}{m(m\gamma a\tau^2)^{\frac{1}{m-1}}}$. Clearly

$$\max_{x\in(0,1)} g(x) < \sup_{x\in(0,1)} g(x) = 1 - \gamma a\tau^2$$

since g is an increasing function over this region. Therefore we can guarantee

$$\max_{x\in(0,1)} g(x) < 1 - 2a$$

by requiring that $1 - \gamma a\tau^2 < 1 - 2a$, or, equivalently, by ensuring that $\frac{2}{\gamma\tau^2} < 1$ in condition (6.1.4*(i)*).

I2: $0 < x < 1 \le \tilde{x}$. The horizontal tangent through the local maximum possessed by $g(x)$ at $\left(\left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}}, \frac{m-1}{m(m\gamma a\tau^2)^{\frac{1}{m-1}}}\right)$ intersects the curve $y = g(x)$ again at $x = \tilde{x}$ where $\tilde{x}$ is the positive real solution to the nonlinear equation

$$x\left(\gamma a\tau^2 x^{m-1} - 1\right) = \frac{m-1}{m\left(m\gamma a\tau^2\right)^{\frac{1}{m-1}}} \qquad (6.1.11)$$

such that $x > \left(\frac{1}{\gamma a\tau^2}\right)^{\frac{1}{m-1}}$ since $g(x)$ is monotonically increasing in this region. Clearly

$$\max_{x\in(0,1)} g(x) \le \max_{x\in(0,\tilde{x})} g(x) = g\left(\left(\frac{1}{m\gamma a\tau^2}\right)^{\frac{1}{m-1}}\right) = g(\tilde{x}) = \frac{m-1}{m\left(m\gamma a\tau^2\right)^{\frac{1}{m-1}}}.$$

And so again we can see that

$$\max_{x\in(0,1)} g(x) < 1 - 2a$$

by imposing condition (6.1.4*(ii)*).

I3: $0 < x < 1$ such that $\tilde{x} < 1 < x^* < x_1$ where x^*, x_1 are defined by the relations $g(x^*) = 1 - 2a$, $g(x_1) = 1$. Clearly, in this case,

$$\max_{x\in(0,1)} g(x) < \sup_{x\in(0,1)} g(x) = \gamma a\tau^2 - 1$$

and

$$\max_{x\in(0,1)} g(x) < 1 - 2a$$

is guaranteed by ensuring that $\gamma a\tau^2 - 1 < 1 - 2a$ which it is by condition (6.1.4*(iii)*).

(N.B. A completely analogous argument applies for the region $-1 < x < 0$.)

To complete the induction argument for $x_n \neq 0$ we show that $|x_1| \leq 1$, or in other words,

$$-1 < 1 - \gamma a\tau^2 < 1. \tag{6.1.12}$$

This follows since $\gamma a\tau^2 > 0$ $(\gamma, a, \tau > 0)$, and by (6.1.2).

The following is the corresponding theorem for the discrete analogue to (6.0.1)

$$x_0 = x_1 = 1,$$

$$x_{n+1} = x_n - \gamma\tau^2 x_n^m + a(x_n - x_{n-1}),\ n = 1, 2, \dots, \tag{6.1.13}$$

The proof is similar.

Theorem 6.1.2 *Consider the discretisation (6.1.13) of the Volterra integro-differential equation (6.0.1). Then $|x_n| < 1$ for all $n = 1, 2, \dots$ providing that*

$$\gamma\tau^2 < 2, \tag{6.1.14}$$

$$a < \frac{1}{2}, \tag{6.1.15}$$

and

$$\begin{cases} \frac{2a}{\gamma\tau^2} < 1 & \text{when } 1 \leq \left(\frac{1}{m\gamma\tau^2}\right)^{\frac{1}{m-1}} \quad (i) \\ \frac{m-1}{m(m\gamma\tau^2)^{\frac{1}{m-1}}} < 1 - 2a & \text{when } \left(\frac{1}{m\gamma\tau^2}\right)^{\frac{1}{m-1}} < 1 \leq \tilde{x} \quad (ii) \\ 1 < \frac{2(1-a)}{\gamma\tau^2} & \text{when } \left(\frac{1}{m\gamma\tau^2}\right)^{\frac{1}{m-1}} < \tilde{x} < 1 \quad (iii) \end{cases} \tag{6.1.16}$$

where $\tilde{x}$ is the unique nonzero real solution to the nonlinear equation

$$x(\gamma\tau^2 x^{m-1} - 1) = \frac{m-1}{m(m\gamma\tau^2)^{\frac{1}{m-1}}} \tag{6.1.17}$$

for $x \in \left(\left(\frac{1}{m\gamma\tau^2}\right)^{\frac{1}{m-1}}, \infty\right)$.

Remarks: Theorems 6.1.1 and 6.1.2 give *sufficient* conditions on γ and τ for the existence of bounded solutions to the discrete schemes (6.1.1) and (6.1.13) respectively. The use of the rather coarse triangle inequality bounds leads us to speculate that the conditions are not also necessary. We explore this further in the next section.

6.2 Some numerical experiments

With $m = 3$, scheme (6.1.1) becomes

$$x_{n+1} = x_n - \gamma a\tau^2 x_n^3 + a(x_n - x_{n-1}), n = 1, 2, \ldots, x_0 = 1, x_1 = 1 - \gamma a\tau^2. \tag{6.2.1}$$

Similarly, scheme (6.1.13) becomes

$$x_{n+1} = x_n - \gamma\tau^2 x_n^3 + a(x_n - x_{n-1}), n = 1, 2, \ldots, x_0 = 1, x_1 = 1. \tag{6.2.2}$$

First we consider the scheme (6.2.1). Figure 6.2.1 shows the values of γ, τ such that the conditions 6.1.4*(i),(ii),(iii)* respectively are satisfied. The dark region indicated in figure 6.2.2 represents a numerically generated plot of γ and τ values for which the inequality $|g(x)| < 1 - 2a$ is satisfied, where x was allowed to vary from -1 to 1 in steps of 0.001. This gives the entire range of admissible pairings under the restriction of the triangle inequality in (6.1.8). The union of the three sets shaded in figure 6.2.1 can be seen to be identical to the set of pairings shaded in figure 6.2.2. Next we show that the region in figure 6.2.2 is a proper subset of the class of (γ, τ) pairings for the scheme (6.2.1) for which all solutions satisfy $|x_n| < 1$. By iterating the scheme (6.1.7) we obtained figure 6.2.3 which gives the complete region for which $|x_n| < 1$.

By way of illustration of the poorer behaviour of the scheme (6.2.2) we present figures 6.2.4 and 6.2.5.

Remarks For both iterative schemes (6.2.1) and (6.2.2) we observe from the graphs that the way we conduct our analysis in section 2 restricts our choice of parameters (γ, τ) compared with our numerical experiments. It can also be observed that, although analytically we have the desired results (i.e. that the iterative schemes do indeed possess only bounded solutions under certain choices of parameters), from a numerical analysis standpoint the admissible (γ, τ) pairings found by the analysis do not admit small values of h since the values of τ are bounded below away from zero. Figure 6.2.3 suggests that in practice there is no such restriction.

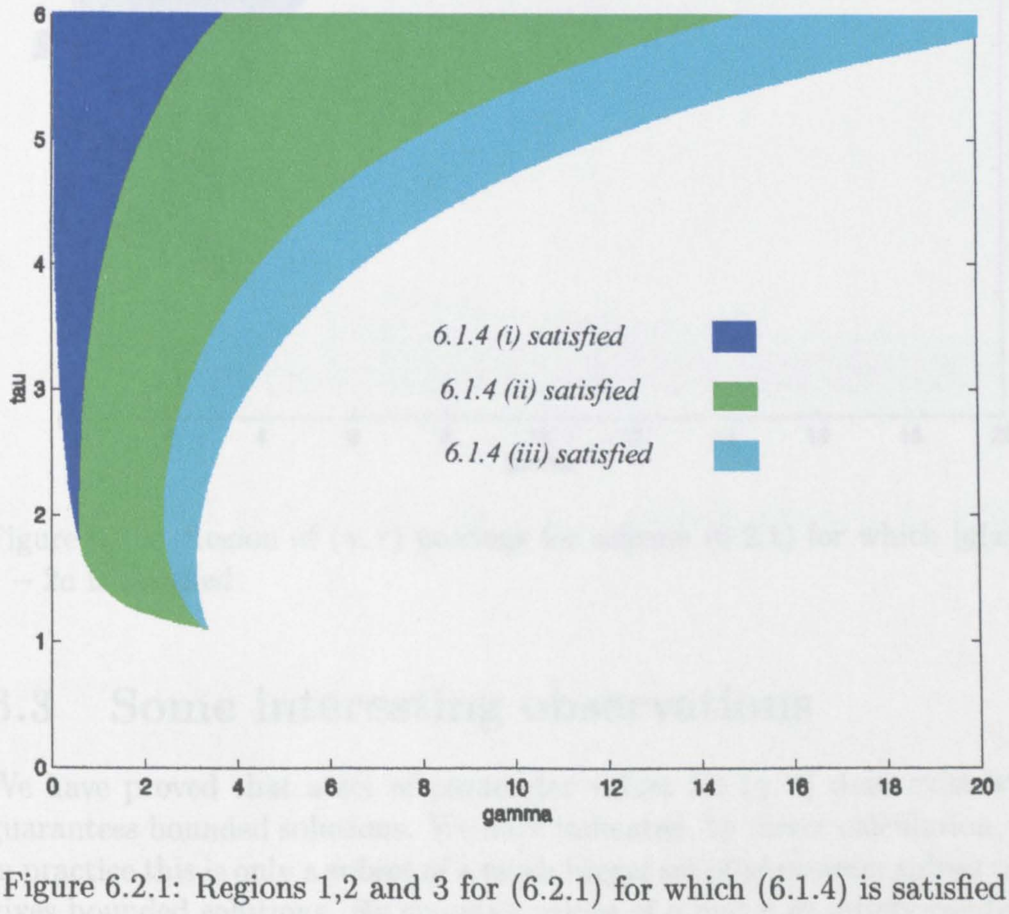

Figure 6.2.1: Regions 1,2 and 3 for (6.2.1) for which (6.1.4) is satisfied

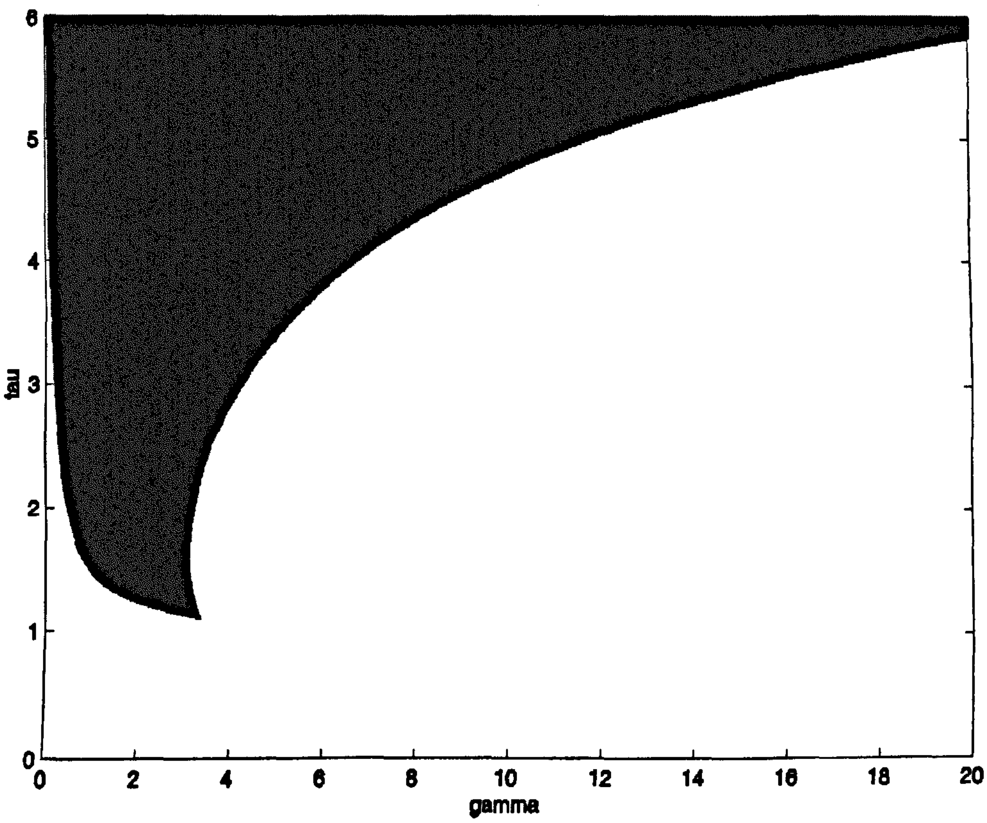

Figure 6.2.2: Region of (γ, τ) pairings for scheme (6.2.1) for which $|g(x)| < 1 - 2a$ is satisfied

6.3 Some interesting observations

We have proved that a set of parameter values for (γ, τ) does exist which guarantees bounded solutions. We have indicated, by direct calculation, that in practice this is only a subset of a much bigger set of parameter values which gives bounded solutions. By choosing values of γ and τ to satisfy conditions (6.1.2), (6.1.3) and (6.1.4*(iii)*) or (6.1.14), (6.1.15) and (6.1.16*(iii)*) we can show that the assumptions of Chapter 4 (section 2) are satisfied. In the former case we have the following theorem.

Theorem 6.3.1 *For the equation*

$$y_{n+1} - y_n = -\kappa h^2 \sum_{j=0}^{n} e^{-h\lambda(n+1-j)} y_j^3, \; y_0 \, given, \qquad (6.3.1)$$

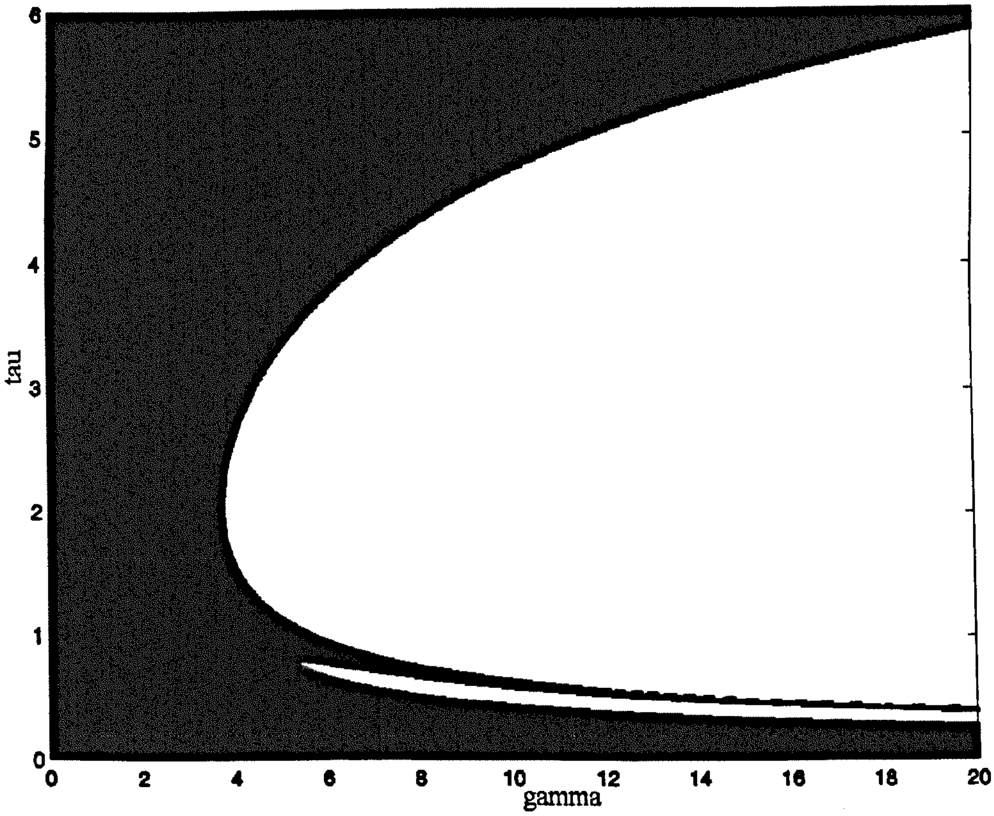

Figure 6.2.3: Region of admissible (γ, τ) pairings obtained by iterating (6.2.1).

assume κ, λ, h, y_0 satisfy

$$\kappa h^2 y_0^2 e^{-\lambda h} < 2, \tag{6.3.2}$$

$$e^{-\lambda h} < \frac{1}{2}, \tag{6.3.3}$$

and

$$1 < \frac{2\left(e^{\lambda h} - 1\right)}{\kappa h^2 y_0^2} \ \textit{when} \ \frac{1}{3\kappa h^2 e^{-\lambda h}} < \tilde{y}^2 < y_0^2, \tag{6.3.4}$$

where $\tilde{y}$ is the unique nonzero real solution to the nonlinear equation

$$\kappa h^2 e^{-\lambda h} y^3 - y = \frac{2y_0}{3\sqrt{3\kappa h^2 y_0^2 e^{-\lambda h}}} \tag{6.3.5}$$

for $y > \frac{1}{\sqrt{3\kappa h^2 e^{-\lambda h}}}$; then every solution to the equation (6.3.1) satisfies $y_n \to 0$ as $n \to \infty$; hence they are all asymptotically stable.

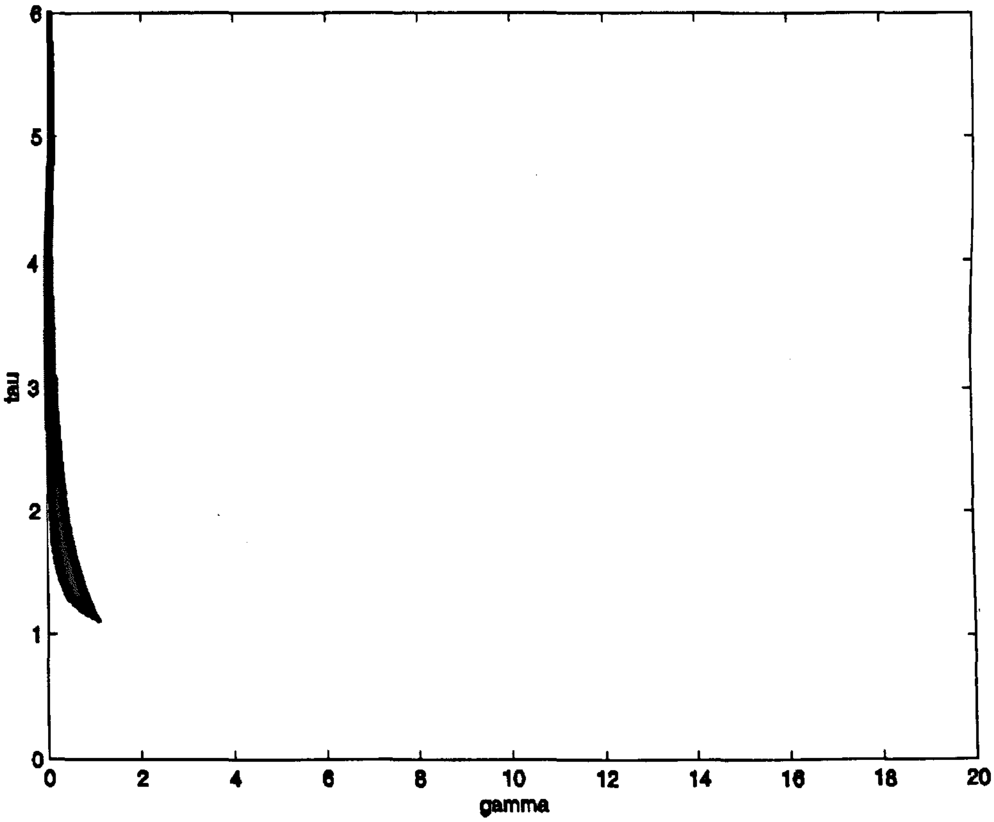

Figure 6.2.4: Region of (γ, τ) pairings for scheme (6.2.2) for which (6.1.16) is satisfied

In conducting our numerical experiments, we discovered a region in the (γ, τ) plane (see figure 6.2.3) where there is a finger-shaped region of non-convergence. We can zoom in on this region to see that it actually contains further regions of convergence and non-convergence. This is illustrated in figures 6.3.1, 6.3.2 and 6.3.3 where the images appear to be indicative of chaotic behaviour. We have established analytically, for certain parameter values, the existence of periodic solutions. Furthermore we have found numerically solutions that undergo many oscillations and then either diverge or converge to zero. Solutions decreasing monotonically to zero have also been found.

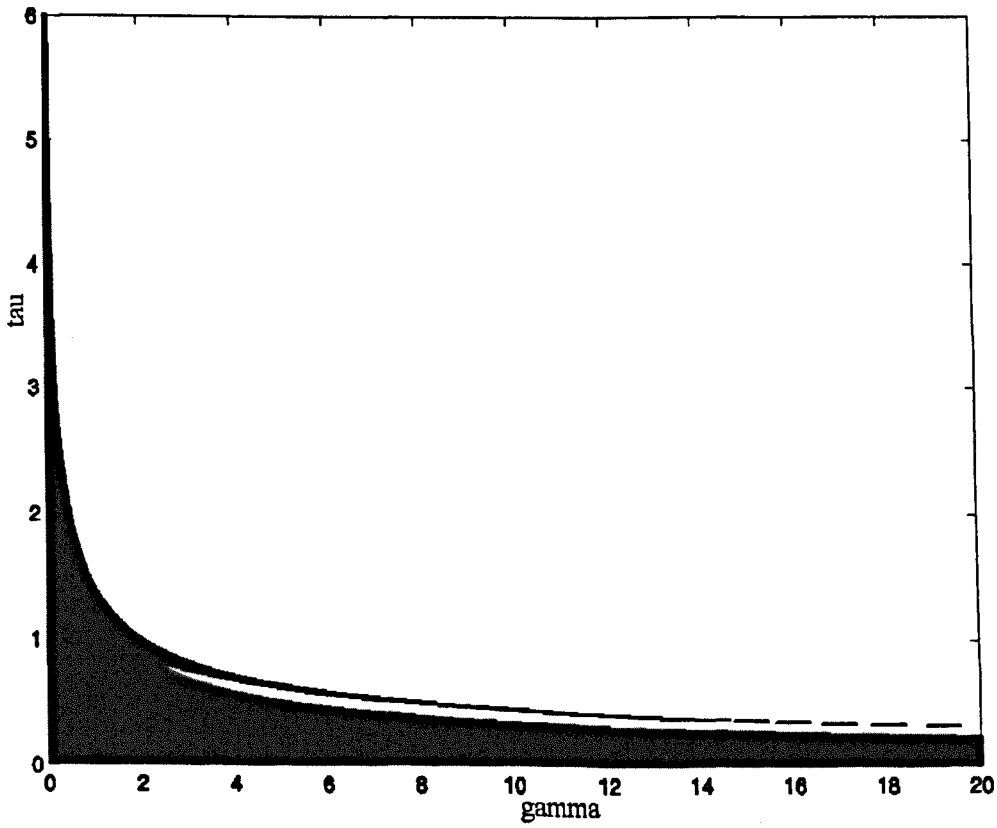

Figure 6.2.5: Region of admissible (γ, τ) pairings obtained by iterating (6.2.2).

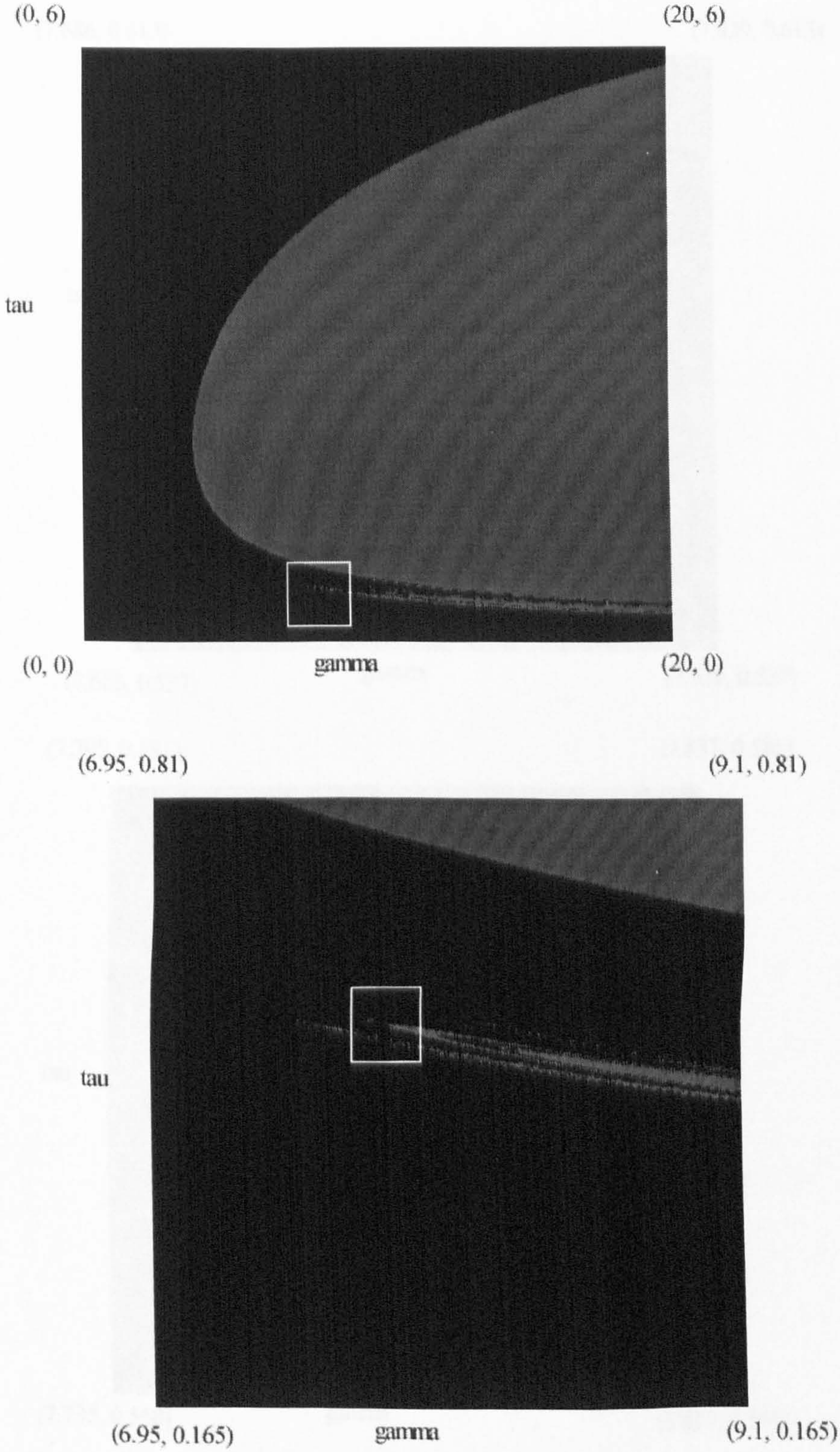

Figure 6.3.1: Successive close-ups of a region in the (γ, τ) plane. Darker region represents (γ, τ) pairings which lead to convergent solutions.

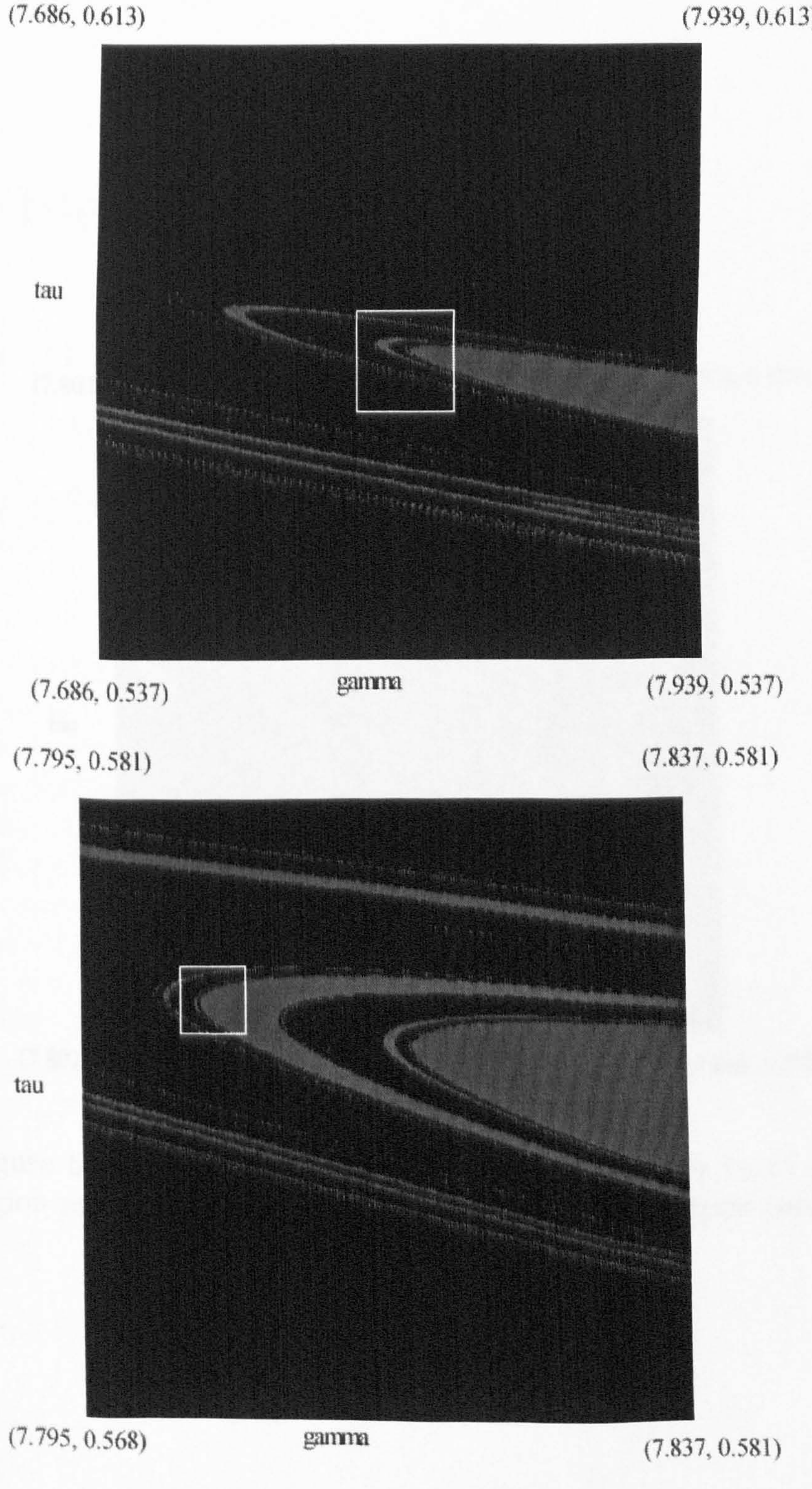

Figure 6.3.2: Successive close-ups of a region in the (γ, τ) plane. Darker region represents (γ, τ) pairings which lead to convergent solutions.

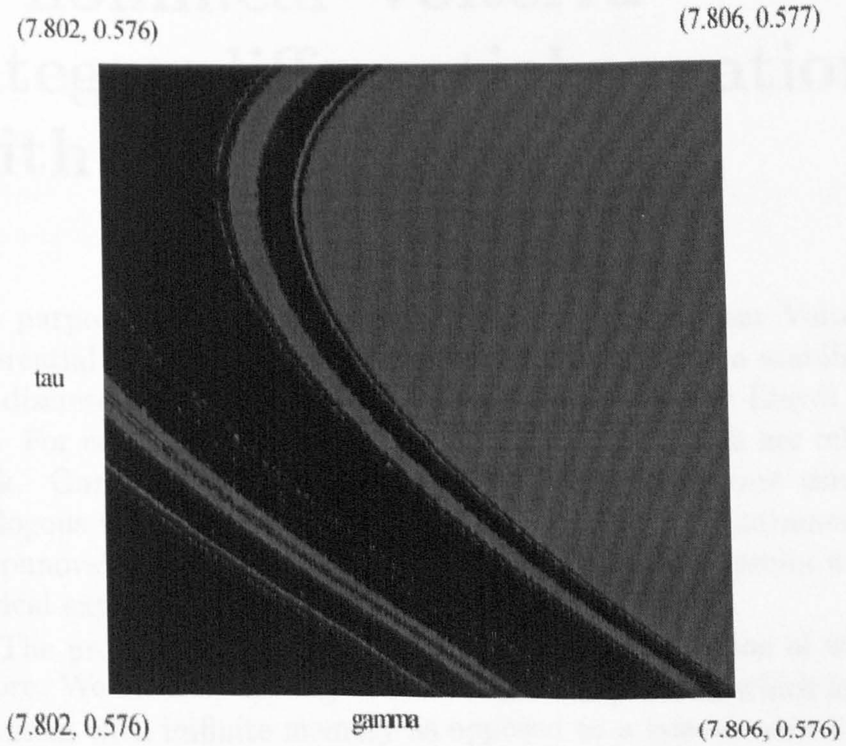

Figure 6.3.3: Successive close-ups of a region in the (γ, τ) plane. Darker region represents (γ, τ) pairings which lead to convergent solutions.

Chapter 7

A nonlinear Volterra integro-differential equation with infinite delay

The purpose of this chapter is to discretise a nonlinear Volterra integro-differential equation with infinite delay and to derive a stability result for the discrete case. The continuous case was studied by Elaydi & Cushman [20]. For convenience, we summarise their results which are relevant to our work. Our aim is to derive a result for the discrete case using a method analogous to that used for deriving the result for the continuous case, name Lyapunovs' Direct Method. Finally, we illustrate our results with some numerical experiments.

The problem considered here is a slight generalisation of what has gone before. We are now dealing with a continuous problem which is representing a system with inifinite memory as opposed to a system with finite memory. Such systems do play an important role in modelling real-life situations (see [44] for a wealth of examples).

7.1 Previous work

Consider the scalar problem:

$$y'(t) = f\left(t, y(t)\right) + \int_{-\infty}^{t} K\left(t, s, y(s)\right) ds \tag{7.1.1}$$

where $f(t,0) = K(t,s,0) = 0$, $f \in C[\mathbb{R}\times\mathbb{R},\mathbb{R}]$, $K \in C[\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R}]$, f_y, K_y are continuous. We say that $y(t,\phi)$ is a solution of (7.1.1) if it is continuous and satisfies (7.1.1) for $t \geq 0$ and it coincides with the initial function ϕ on $(-\infty, 0]$.

In [20], Elaydi & Cushman consider a generalisation of the above problem (i.e. a system of integro-differential equations with infinite delay). For convenience, we present here the scalar version of their result.

Lemma 7.1.1 (Elaydi & Cushman) *Let the function $g : (-\infty, 0] \to [1, \infty)$ be continuous with $g(0) = 1$, $g(\tau) \to \infty$ as $\tau \to \infty$. g is decreasing such that if $\phi \in Y$, $|\phi(s)| \leq \gamma g(s)$ for some $\gamma > 0$ and $-\infty < s \leq 0$, then $\int_{-\infty}^{0} K(t, s, \phi(s))\, ds$ is continuous for $t \geq 0$.*

For this g, denote by $(Y, |\cdot|_g)$ the Banach space of continuous functions $\phi : (-\infty, 0] \to \mathbb{R}$ such that $|\phi|_g = \sup_{-\infty < t \leq 0} \left| \frac{\phi(t)}{g(t)} \right|$ exists. All initial functions belong to $(Y, |\cdot|_g)$.†

Definition 7.1.2 (Elaydi & Cushman) *The zero solution of (7.1.1) is exponentially stable with positive constants L and α if, for $t \geq 0$,*

$$|y(t)| \leq Le^{-\alpha t} |\phi|_g.$$

Let

$$f(t, y) = f_y(t, 0)y + F(t, y) \tag{7.1.2}$$

and

$$K(t, s, y(s)) = K_y(t, s, 0)y + Q(t, s, y(s)). \tag{7.1.3}$$

Consider the linearised system

$$z'(t) = f_y(t, 0)z(t) + \int_0^t K_y(t, s, 0)z(s)ds. \tag{7.1.4}$$

Associated with (7.1.4) is the resolvent kernel $R(t, s)$ which is the unique solution of the equation

$$\frac{\partial}{\partial s} R(t, s) = -R(t, s)f_y(s, 0) - \int_s^t R(t, \sigma)K_y(\sigma, s, 0)d\sigma, \tag{7.1.5}$$

where $R(t, t) = I$ for $0 \leq s \leq t$. It has been shown in [31] that $R(t, s)$ exists for (7.1.5) and is continuous for $0 \leq s \leq t$. Let

$$H(t) = \int_{-\infty}^{0} K(t, s, \phi(s))ds \tag{7.1.6}$$

and

$$G(t) = \int_t^{\infty} R^2(s, t)ds. \tag{7.1.7}$$

† For each initial function ϕ, there will result a solution. For the zero initial function, we obtain the zero solution. Definition 7.1.2 considers the perturbation in the zero solution that arises from a non-zero initial function.

Theorem 7.1.3 (Elaydi & Cushman) *Assume that for (7.1.1) the following conditions hold:*

(i) $N \le G(t) \le L$,

(ii) $2G(t)\left(F(t,y(t)) + H(t) + \int_0^t K(t,s,y(s))ds\right)y(t) \le \frac{|y(t)|^2}{M}$, *where* $M > 1$,

(iii)
$$\left(-\int_t^\infty \int_t^s K_y(\sigma,t,0)R(s,\sigma)d\sigma \times R(s,t)ds - \int_t^\infty R(s,t)\int_t^s R(s,0)K_y(\sigma,t,0)d\sigma ds\right)y^2 \le 0.$$

Then the zero solution of (7.1.1) is exponentially stable.

7.2 The discrete case

Consider the discrete system

$$y_{n+1} - y_n = hf(n,y_n) + h^2\sum_{j=-\infty}^{n} K(n,j,y_j), \; h > 0. \tag{7.2.1}$$

We now linearise this system in a similar way to linearising (7.1.1). This gives:

$$\begin{aligned} y_{n+1} - y_n &= hf_y(n,0)y_n + hF(n,y_n) + hH_n \\ &\quad + h^2\sum_{j=0}^{n}\left(K_y(n,j,0)y_j + Q(n,j,y_j)\right), \end{aligned} \tag{7.2.2}$$

where $H_n = \sum_{j=-\infty}^{0}\left(K_y(n,j,0)y_j + Q(n,j,y_j)\right)$.

Definition 7.2.1 *Let*

$$G_n = h\sum_{j=n}^{\infty} R^2(j,n), \tag{7.2.3}$$

and

$$R(j,n+1) - R(j,n) = h\left(-R(j,n)f_y(n,0) - h\sum_{i=n}^{j} R(j,i)K_y(i,n,0)\right), \tag{7.2.4}$$

where $R(n,n) = I$.

Lemma 7.2.2

$$\begin{aligned} G_{n+1} - G_n &= -h^2 \sum_{j=n+1}^{\infty} R^2(j,n)f_y(n,0) \\ &- h^2 \sum_{j=n+1}^{\infty} R(j,n+1)R(j,n)f_y(n,0) \\ &- h^3 \sum_{j=n+1}^{\infty}\sum_{i=n}^{j} R(j,i)K_y(i,n,0)\left(R(j,n+1)+R(j,n)\right) - h. \end{aligned} \tag{7.2.5}$$

Proof:

$$\begin{aligned} G_{n+1} - G_n &= h\sum_{j=n+1}^{\infty} R^2(j,n+1) - h\sum_{j=n}^{\infty} R^2(j,n) \\ &= h\sum_{j=n+1}^{\infty} \left(R^2(j,n+1) - R^2(j,n)\right) - hR(n,n) \\ &= h\sum_{j=n+1}^{\infty} \left(R(j,n+1)+R(j,n)\right) \\ &\quad \cdot h\left(-R(j,n)f_y(n,0) - h\sum_{i=n}^{j} R(j,i)K_y(i,n,0)\right) - hI \\ &= -h^2 \sum_{j=n+1}^{\infty} R^2(j,n)f_y(n,0) \\ &\quad -h^2 \sum_{j=n+1}^{\infty} R(j,n+1)R(j,n)f_y(n,0) \\ &\quad -h^3 \sum_{j=n+1}^{\infty}\sum_{i=n}^{j} R(j,i)K_y(i,n,0)\left(R(j,n+1)+R(j,n)\right) - h. \end{aligned}$$

Theorem 7.2.3 *Assume that for the discrete problem (7.2.1), the following conditions hold:*

(H1)

$$\sum_{j=n}^{\infty} R^2(j,n)(y_{n+1}+y_n)\left(F(n,y_n)+H_n+h\sum_{j=0}^{n}(K_y(i,j,0)y_j+Q(n,j,y_j))\right)$$

$$\leq \frac{y_{n+1}^2}{M}, \tag{7.2.6}$$

for some $M > 1$.

(H2)

$$\sum_{j=n+1}^{\infty}\sum_{i=n}^{j} R(j,i)K_y(i,n,0)\left(R(j,n+1)+R(j,n)\right) \geq 0, \tag{7.2.7}$$

(H3) $|y_n|$, $f_y(n,0)$, $\sum_{j=n+1}^{\infty} R^2(j,n)$, $\sum_{j=n+1}^{\infty} R(j,n+1)R(j,n)$ *are all bounded.*

Then the zero solution of (7.2.1) is asymptotically stable.

Proof: We prove our result by means of Lyapunov's Direct Method. We define a Lyapunov function for our problem as follows. Let

$$V(t_n,y_n) = y_n^2 G_n \geq 0. \tag{7.2.8}$$

$$\begin{aligned}
\Delta V_n &= V_{n+1} - V_n \\
&= y_{n+1}^2 G_{n+1} - y_n^2 G_n \\
&= y_{n+1}^2 G_{n+1} - y_{n+1}^2 G_n + y_{n+1}^2 G_n - y_n^2 G_n \\
&= y_{n+1}^2 \left(G_{n+1} - G_n\right) + G_n \left(y_{n+1}^2 - y_n^2\right) \\
&= y_{n+1}^2 \left(G_{n+1} - G_n\right) + G_n \left(y_{n+1} + y_n\right)\left(y_{n+1} - y_n\right)
\end{aligned}$$

Using Lemma 7.2.2,

$$\begin{aligned}
\Delta V_n &= -hy_{n+1}^2 - h^2 \sum_{j=n+1}^{\infty} R^2(j,n) f_y(n,0) y_{n+1}^2 \\
&\quad - h^2 \sum_{j=n+1}^{\infty} R(j,n+1) R(j,n) f_y(n,0) y_{n+1}^2 \\
&\quad - h^3 \sum_{j=n+1}^{\infty} \sum_{i=n}^{j} R(j,i) K_y(i,n,0) \left(R(j,n+1) + R(j,n)\right) y_{n+1}^2 \\
&\quad + G_n(y_{n+1} + y_n)(y_{n+1} - y_n)
\end{aligned} \tag{7.2.9}$$

i.e.

$$\begin{aligned}
\Delta V_n &= -hy_{n+1}^2 - h^2 f_y(n,0) y_{n+1}^2 \sum_{j=n+1}^{\infty} R^2(j,n) \\
&\quad - h^2 f_y(n,0) y_{n+1}^2 \sum_{j=n+1}^{\infty} R(j,n+1) r(j,n) \\
&\quad - h^3 y_{n+1}^2 \sum_{j=n+1}^{\infty} \sum_{i=n}^{j} R(j,i) K_y(i,n,0) \left(R(j,n+1) + R(j,n)\right) \\
&\quad + G_n(y_{n+1} + y_n)(h f_y(n,0) + hF(n,y_n) + hH_n \\
&\quad + h^2 \sum_{j=0}^{n} (K_y(n,j,0) y_j + Q(n,j,y_j))).
\end{aligned} \tag{7.2.10}$$

Under conditions (H1) and (H2),

$$\begin{aligned}
\Delta V_n &\leq -hy_{n+1}^2 - h^2 f_y(n,0) y_{n+1}^2 \sum_{j=n+1}^{\infty} R^2(j,n) \\
&\quad - h^2 f_y(n,0) y_{n+1}^2 \sum_{j=n+1}^{\infty} R(j,n+1) R(j,n) + h \frac{y_{n+1}^2}{M}.
\end{aligned} \tag{7.2.11}$$

Therefore,

$$\Delta V_n < 0 \tag{7.2.12}$$

for small enough $h>0$ (under condition (H3)).

7.3 Numerical experiments

Consider the Volterra integro-differential equation

$$y'(t) = \alpha y(t) + \int_{-\infty}^{t} e^{-(t-s)} y^3(s) ds, \ t \geq 0 \tag{7.3.1}$$

subject to the initial function $\phi(t) = e^t$ for $t < 0$, with $\alpha < 0$ being constant.

Comparing this to our definition of the general equation, we note that:

$$f_y(t, 0) = \alpha,$$

$$F(t, y) = 0,$$

$$K_y(t, s, 0) = 0,$$

$$H(t) = \int_{-\infty}^{0} e^{-(t-s)} e^{3s} ds = \frac{1}{4e^t}.$$

We show that equation (7.3.1) satisfies the conditions of Theorem 7.1.3, as follows.

First, note that the resolvent kernel $R(t, s)$ of (7.3.1) is given by

$$\frac{\partial R}{\partial s} = -\alpha R(t, s).$$

$$\therefore R(t, s) = c(t) e^{-\alpha s}$$

$$R(t, t) = 1 \Rightarrow c(t) = e^{\alpha t}$$

Thus,

$$R(t, s) = e^{\alpha(t-s)}. \tag{7.3.2}$$

Therefore,

$$G(t) = \int_{t}^{\infty} \left(e^{\alpha(t-s)}\right)^2 ds = \frac{1}{2\alpha}. \tag{7.3.3}$$

So clearly G(t) is bounded and condition *(i)* of Theorem 7.1.3 is satisfied.

For our example equation, the left hand side of condition *(ii)* in Theorem 7.1.3 becomes

$$\frac{1}{\alpha}\left(\frac{1}{4e^t} + \int_{0}^{t} e^{-(t-s)} y^3(s) ds\right) y(t).$$

From our work in Chapter 3, and with the condition that α is negative, this expression is also negative and, as such, condition (ii) of Theorem 7.1.3 is

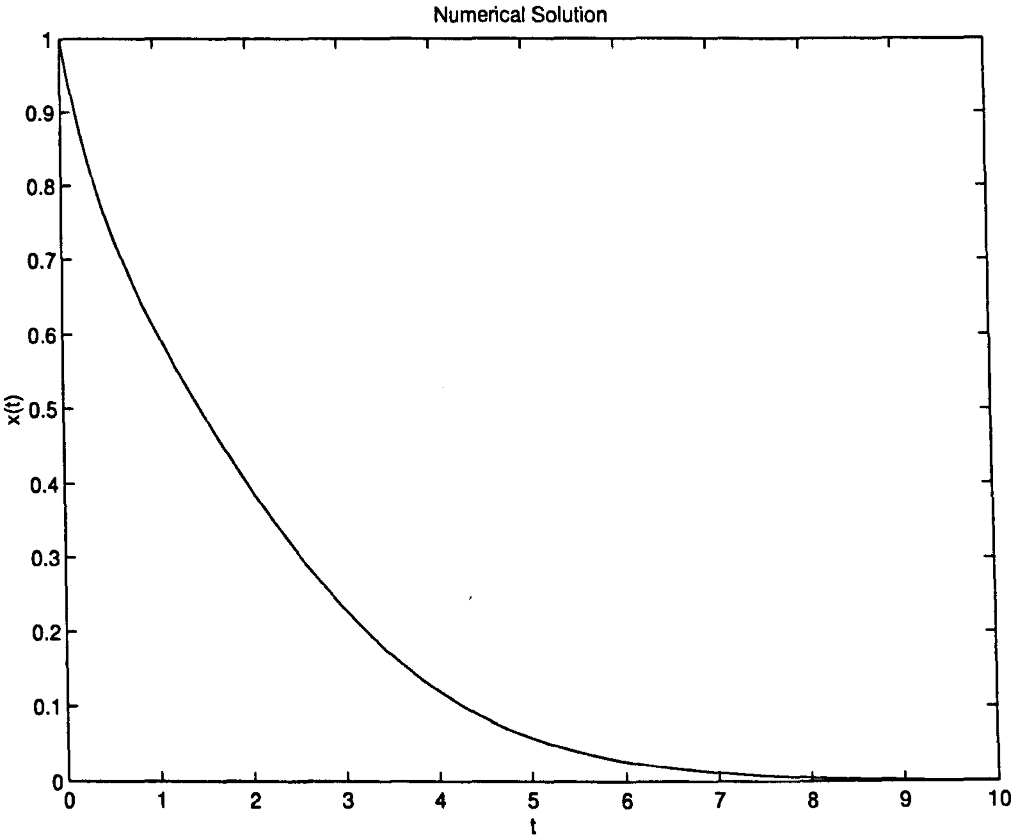

Figure 7.3.1: Numerical scheme (7.3.4) with $h = 0.01$, $\alpha = -1$

satisfied. With $K_y(t, s, 0) = 0$, condition (iii) is also clearly satisfied and therefore all of the conditions of Theorem 7.1.3 are satisfied.

Our discretisation of (7.3.1) is

$$y_{n+1} = y_n + h\alpha y_n + \frac{h}{4e^t} + h^2 \sum_{j=0}^{n} e^{-h(n-j)} y^3(s), \tag{7.3.4}$$

with $y_0 = 1$. Figure 7.3.1 demonstrates the good behaviour of the numerical solution. However, numerical experiments indicate that the discrete method actually fails for $\alpha > -0.68$ (approximately).

Chapter 8

Concluding Remarks

It is clear from the extensive references included herein that much work has been done in the past with regard to the analysis of the Volterra integro-differential equations. Similarly, much is known about the discretisation of such problems. However, it is accepted that gaps in the theory do exist. These gaps are hard to fill and are never filled completely. In a recent paper [1] Baker states that "the stability theory for numerical methods for Volterra equations is still incomplete though considerable advances have been made".

Baker [1] goes on to say that "with favourable assumptions (though not invariably), investigation of stability of a solution of a non-linear equation can be reduced to that of the solutions of a corresponding linear equation (stability in the first approximation). Alternative approaches invoke Lyapunov theory or *ad hoc* qualitative arguments.".

Chapter 3 illustrates that the approach of simplistic linearisation is not good enough for our chosen problems; our assumptions are not 'favourable enough' for such an approach to be productive.

In chapter 4, we see how hard it is to develop results for some of the simplest methods.

We have used Lyapunov theory to develop our results. It is generally considered to be a hard tool to use but extremely powerful (shown by the work in Chapter 5). Constructing suitable Lyapunov functionals for discrete systems is not a simple task and the use of discrete functionals that replicate continuous behaviour and a general construction method are both shown to be effective approaches. Many unsuitable functionals may be considered before a suitable one is found.

Chapter 6 confirms the applicability of our work in chapter 5. Some very strict conditions had to be imposed in chapter 5 in order to produce some results. This leads to the question: Are our restrictions so tight that we are developing results for a near-empty set of equations? Chapter 6 demonstrates that this is not the case.

Chapter 7 introduces work on systems with delay. A lot of work has been done on the continous theory (see [44]). However there are still gaps in theory for suitable numerical methods and these types of equations are becoming more frequent in attempts to model hereditary systems. Kolmanovskii and Myshkis [44] state that "numerous investigations have shown that temporal delays in an actual system have a considerable influence on the qualitative behaviour of the system."

To summarise, we have developed significant new results for some selected Volterra equations and we have highlighted an important tool (Lyapunov's direct method) for analysing the stability of discrete systems. The process of developing results for even the simplest numerical methods has been shown to be a difficult and long process, with much mathematical anaylsis being required. However, we have also demonstrated that it is an essential task because numerical methods do not always behave as they might first be expected to.

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