Local and Global Aspects of the Module Theory of Singular Curves

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ABSTRACT

Let X be an integral projective curve with a singular point x. We study $\mathcal{M} = \mathcal{M}_T(n,d)$, the moduli space of torsion free sheaves of rank n and degree d on X, via a study of the module theory of $\mathcal{O}_x$. For example, we derive a formula for the dimension of the tangent space to $\mathcal{M}$ at a point corresponding to a stable sheaf $\mathcal{F}$ in terms of local invariants of $\mathcal{F}$ and the genus of X.

Following the work of Bhosle on generalised parabolic bundles (applied to the case of x a node or a cusp) we prove that there exists a projective scheme $\mathcal{M}_P(n,d)$, of bundles on the normalisation, $\tilde{X}$, of X with suitable 'extra' structure, and a surjective morphism $\Psi : \mathcal{M}_P(n,d) \longrightarrow \mathcal{M}_T(n,d)$ which restricts to an isomorphism on the preimage of the space of stable locally free sheaves. Applying this we give an upper bound on the dimension of $\mathcal{M}_T(n,d)$ and prove that $\mathcal{M}_T(n,d)$ is connected if $g(\tilde{X}) \geq 2$.

In the particular case $n = 1$, where $\mathcal{M}_T(1,d) = \overline{J(X)}$ is the compactified Jacobian, the space $\mathcal{M}_P(1,d)$ fibres over $J(\tilde{X})$ and Ψ is finite. $\mathcal{M}_P(1,d)$ is generally singular, but in some cases gives a normalisation of $\overline{J(X)}$. We apply our methods to study $\overline{J(X)}$ for some particular singularities, describing the stratification of $\overline{J(X)}$ according to local type for curves with simple singularities. In a final chapter we extend these ideas to look at the compactified Jacobian of a curve with unibranched singularities.

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CONTENTS

INTRODUCTION

Let X be an integral projective curve with a singular point x. We are interested in studying $\mathcal{M} = \mathcal{M}_T(n,d)$, the moduli space of torsion free sheaves of rank n and degree d on X. Many natural questions arise: how many components does $\mathcal{M}$ have and what is its dimension? Is $\mathcal{M}$ reduced; what are its singularities? At a point corresponding to a (stable) sheaf $\mathcal{F}$ what is the tangent space? Is there any natural way of constructing a normalisation or a desingularisation of $\mathcal{M}$? Which torsion free sheaves can arise as limits of locally free sheaves? Does there exist a natural stratification of $\mathcal{M}$?

Relatively little seems to be known about $\mathcal{M}$ in general, but what is clear is that many of the answers to these questions depend on the type of singularities that X has; for instance, a fundamental theorem of Rego says that $\mathcal{M}$ is irreducible if and only if all the singular points of X have embedding dimension 2. Rather more is known about $\mathcal{M}$ in the special cases when $n = 1$ or when all the singularities of X are simple nodes.

Let us briefly review the history of the subject. Consider first the case where X is smooth.

The Jacobian of a smooth curve is the oldest and best understood example of a moduli space of vector bundles. Of course, the Jacobian variety is much older than the notion of a vector bundle, but rank 1 bundles have many other interpretations; it was essentially with the work of Weil that the modern viewpoint began to emerge.

The investigations into vector bundles of higher ranks began with the special cases of X rational (Grothendieck) and X elliptic (Atiyah); in both cases a fairly direct approach led to a classification theorem. It became clear, however, that for a general curve some sort of restriction would be necessary before one could obtain a classification. Mumford's idea of *stability* (in Geometric Invariant Theory) proved to be the key: restricting to stable bundles (of given rank and degree) allows one to construct classifying spaces, moreover, in any family the stable bundles form an open subset. By adding semistable bundles under a suitable equivalence relation the moduli spaces can be completed to projective varieties. These spaces have now been much studied and they seem to arise naturally in a wide variety of problems. One technique used in their study is to consider what happens when X degenerates into a 'nice' singular curve (see, e.g. [Sundaram] and [Teixidor]).

When X has singularities we can still construct moduli spaces for vector bundles (or equivalently locally free sheaves), however, these spaces are not compact; one way to complete them to projective schemes is to allow degenerations of locally free sheaves into non-locally free sheaves. The existence proof for moduli spaces for these (stable) torsion free sheaves is then very similar to that for bundles, although technically more involved. The first results, due to [D'Souza] concerned $\mathcal{M}_T(1,d) = \overline{J(X)}$—the compactified Jacobian of X. These were later studied by Rego, Altman and Kleiman: an application of Iarrobino's work on punctual Hilbert schemes ([Altman, Iarrobino and Kleiman]) implies that the compactified Jacobian of an integral projective curve lying in a smooth surface is an integral projective scheme, whilst [Rego 1] gives a proof of the converse. The relative case is considered in [Altman and Kleiman 2]. In another direction [Oda and Seshadri] studied reducible curves with just nodes as singularities. The main factor influencing the complexity of the moduli spaces is the complexity of the graph of the curve (in terms of the number of generating trees).

For sheaves of higher ranks the paper [Rego 2] settled the reducibility question, whilst other works have mostly concentrated on the case where X is a (possibly reducible) curve with only nodes as singularities (e.g. [Seshadri], [Sundaram], [Bhosle 1,2,3] and [Teixidor]).

Seshadri had introduced the idea of a parabolic vector bundle and had used these to construct a desingularisation of a moduli space of bundles on a nonsingular curve. From some further work of Seshadri it seemed that, for a nodal curve, moduli of bundles with this extra structure living on the normalisation of the curve might give some similar desingularisation. The key steps towards this were taken by Bhosle; allowing parabolic structures over divisors she generalised the idea of a parabolic bundle. For a nodal curve such a generalised parabolic bundle (GPB) can be regarded as consisting of a vector bundle on the normalisation, $\tilde{X}$, together with a glueing of the fibres over the singular point. This gives rise to a correspondence between GPBs and torsion free sheaves on the singular curve which preserves degree, and extends to a morphism of moduli spaces. In the case when rank and degree are coprime this does indeed give a desingularisation of the space of torsion free sheaves; e.g., in the rank 1 case one obtains a $\mathbf{P}^1$-bundle over $J(\tilde{X})$.

One of the objects of this thesis to describe how this approach can be extended to deal with an arbitrary singularity x on an irreducible curve. Two problems are encountered: firstly, the module theory of the local ring at a singularity can be very complicated; secondly, one needs to choose a divisor on $\tilde{X}$ lying over x, and it is not obvious how one should make this choice.

The following construction lies at the heart of Bhosle's work. Suppose (for notational convenience) that $x \in X$ is the *only* singular point; let $\pi_* : \tilde{X} \longrightarrow X$ be the normalisation map and let D be an effective divisor on $\tilde{X}$ with $Supp(D) \subset \pi^{-1}(x)$. If E is a rank n vector bundle on $\tilde{X}$ and if $F_1(E)$ is an $\mathcal{O}_x$-submodule of $E \otimes \mathcal{O}_D$ then one can associate a torsion free sheaf $\mathcal{E}$ to the pair $(E, F_1(E))$ using the short exact sequence

$$(*) \quad 0 \longrightarrow \mathcal{E} \longrightarrow \pi_* E \longrightarrow (E \otimes \mathcal{O}_D)/F_1(E) \longrightarrow 0.$$

Note that for a fixed bundle E the set of all such pairs with $\dim(F_1(E))$ fixed is parametrised by a certain subscheme of a Grassmannian (these subschemes have already been studied by [Rego 1]; he used them to study $\overline{J(X)}$ in the case X rational). We then choose $\dim(F_1(E))$ so that $\deg \mathcal{E} = \deg E$. Let $I \subset \tilde{\mathcal{O}}_x$ be the ideal defining D, and let C be the conductor of $\mathcal{O}_x$ in $\tilde{\mathcal{O}}_x$. Our main results are summarised in the theorem below.

Theorem (see (3.4.4), (3.4.9) and (4.3.4)).

1. If $I \subset C$ then every locally free sheaf $\mathcal{E}$ of rank n and degree d on X arises in this way for a unique $(E, F_1(E))$.

2. If $I \subset C^2$ then every torsion free sheaf $\mathcal{E}$ of rank n and degree d on X arises in this way for some $(E, F_1(E))$.

Moreover, if $I \subset C$, one can construct a moduli space $\mathcal{P}_I$ for these objects and one obtains a corresponding morphism $\Psi : \mathcal{P}_I(n,d) \longrightarrow \mathcal{M}_T(n,d)$. Then, letting $\mathcal{M}_V^s(n,d) \subset \mathcal{M}_T(n,d)$ denote the moduli space of stable locally free sheaves on X we have:

1′. The restriction $\Psi : \Psi^{-1}\mathcal{M}_V^s(n,d) \longrightarrow \mathcal{M}_V^s(n,d)$ is an isomorphism.

2′. If $I \subset C^2$ then $\Psi : \mathcal{P}_I(n,d) \longrightarrow \mathcal{M}_T(n,d)$ is surjective.

In certain cases (in fact, when $C = m_x$ and $(n,d) = 1$) the space $\mathcal{P}_C(n,d)$ gives a non-singular compactification of the space of locally free sheaves on X, although the map Ψ is not generally surjective. In contrast, when $\mathcal{O}_x$ is Gorenstein and $I = C$ the map $\Psi : \mathcal{P}_I(n,d) \longrightarrow \mathcal{M}_T(n,d)$ is already surjective, even though $\mathcal{M}_T(n,d)$ may have many components.

The main virtue of this construction is that the space $P_I(n,d)$ is very much easier to study than $\mathcal{M}_T(n,d)$. In particular, when $n = 1$ the map Ψ is finite and $\mathcal{P}_I(1,d)$ fibres over $J(\tilde{X})$; thus many questions about $\overline{J(X)}$ reduce to local problems. In some cases ($\mathcal{O}_x$ of type A_n, $n \le 4$ or D_4) we prove that $\mathcal{P}_I(1,d)$ is a normalisation of $\overline{J(X)}$. As another application we obtain a new upper bound on the dimension of $\mathcal{M}_T(n,d)$.

Our underlying philosophy in this thesis is to try to understand $\mathcal{M}_T(n,d)$ via an understanding of $TFmod(\mathcal{O}_x)$—the category of finite torsion free modules over $\mathcal{O}_x$. In particular we find a formula for the dimension of the tangent space to $\mathcal{M}_T(n,d)$ at a point corresponding to a stable torsion free sheaf $\mathcal{F}$ depending just on the genus $g(X)$ and the isomorphism type of $\mathcal{F}_x \in TFmod(\mathcal{O}_x)$. After developing some local algebra it proves possible to calculate this efficiently in a large number of cases.

In the case where $\mathcal{O}_x$ is of finite representation type (i.e., has only finitely many isomorphism classes of indecomposable torsion free $\mathcal{O}_x$-modules) one then obtains a stratification of $\mathcal{M}_T(n,d)$ (at least when $(n,d) = 1$) according to the local type of a sheaf at x. [A remark on terminology: In this thesis, by a *stratification* of a scheme Y we will always mean a finite collection of pairwise disjoint, irreducible, locally closed subschemes U_i, covering Y, such that, for all i, the dimension of the Zariski tangent space to Y is constant on U_i.] We describe the resulting stratification of $\overline{J(X)}$ when X is a curve with a single simple singularity and draw the diagrams of the associated partial orderings.

We now make some remarks about questions which we do not consider in this thesis. All of our detailed applications of the theorem above concern the rank 1 case. It was felt that compactified Jacobians were already complicated enough to be interesting to study, and also that a large amount of further work would have been necessary in order to develop sufficient techniques to say much in higher ranks; indeed, a good understanding of the compactified Jacobian would be a prerequisite. We do not succeed in identifying the type of singularities that the compactified Jacobian has; our methods yield a certain amount of information concerning, e.g., multiplicity, but additional methods would seem to be required to give a complete answer.

Another omission is that we do not consider the relative case, working at all times over a fixed curve. It seems that there would be problems in doing this: one cannot expect the schemes $P_I(n,d)$ (above) to fit together for a 'general' family (see [Rego 1]).

The major outstanding problem with this work at present concerns infinitesimal deformations: in particular to understand which deformations (of various objects) are *obstructed*. This is essential in trying to identify non-reduced structures. A major point would be to try to get an infinitesimal version of the map Ψ; we know that this will be neither injective nor surjective in general.

It does seem that there is considerable scope for further work on these questions.

The organisation of material in this thesis is as follows.

In chapter 1 we review some aspects of the theory of curves and their singularities, including a look at the (generalised) Jacobian of a singular curve; all of this is well known.

Fundamental in our approach will be a good understanding of the local theory of sheaves at a singularity, so in chapter 2 we discuss the module theory of 1-dimensional Cohen-Macaulay local rings. The most important fact is that torsion free modules are 'bounded' in an appropriate sense, enabling us to construct parameter spaces representing all torsion free modules of a given rank. We review the module classification of [Greuel and Knörrer] when x is a simple singularity. We also spend some time explaining how to calculate certain homology groups; the results being used in later chapters.

In chapter 3, using results from chapter 2, we study the category of torsion free sheaves on the singular curve and the relationship with sheaves on a (partial) normalisation. Next, for each suitable choice of a divisor over the singular point, we introduce the category of parabolic Modules and a functor into the category of torsion free sheaves (as above). The main results give conditions under which this is onto. Finally, stability is considered. The definition of stability for parabolic Modules depends on the choice of a weight and we must prove that for a suitable choice the correspondence Ψ 'preserves' stability. Unfortunately, when $(n, d) \neq 1$ this is not true, but there does exist a 'sufficiently good' approximation.

Chapter 4 sees the extension of these ideas to the level of families of sheaves and moduli spaces. We prove that the existence of a moduli space for parabolic Modules follows from the existence of a moduli space for GPBs; but we do not prove the latter, referring instead to [Bhosle 1,2].

The remaining 2 chapters give applications of the theorems of chapter 4 to the study of compactified Jacobians of curves with particular singularities. In chapter 5 we describe in some detail the situation for the simple plane curve singularities: describing the stratification of $\overline{J(X)}$ and calculating the dimension of the tangent space at each point. In the case of x of type A_3, A_4 or D_4 we identify the normalisation of $\overline{J(X)}$.

In the final chapter we consider a combinatorial approach which leads to a partial description of the stratification of the compactified Jacobian when the curve has an analytically irreducible singularity. This is given in terms of 'semigroup modules' determined by the semigroup of the singularity. We describe how this works in some particular cases. The methods and examples of this chapter provide a good testing ground for conjectures about compactified Jacobians.

CHAPTER 1

PRELIMINARIES ON CURVES AND SINGULARITIES

In this chapter, taking the opportunity to fix our notation and conventions, we review some basic facts concerning curves and coherent sheaves (§1.1), and look at singularities and their fundamental invariants (§1.2). In §1.3 we review the theory of Cartier divisors and the Picard group, which leads on to the generalised Jacobian—our first example of a moduli space.

§1.1 Remarks on Curves

We will work over an algebraically closed field k which can generally be taken as arbitrary, although we might wish to make some restrictions on its characteristic when considering examples. Throughout this thesis we consider X an irreducible and reduced (i.e., integral) projective curve, and in later chapters we will assume that X has at least 1 singular point. It should be remarked that it would also be possible to develop the theory in the case of X being reducible, although there would then be extra sources of complication.

Write $\mathcal{O}_X$ for the structure sheaf of X and K for the field of rational functions on X. All sheaves of $\mathcal{O}_X$-modules will be assumed to be coherent. If $\mathcal{F}$ is such a sheaf then we can define cohomology groups $\mathrm{H}^i(X,\mathcal{F})$ for $i \geq 0$: these are finite dimensional vector spaces and, since X has dimension 1, $\mathrm{H}^i(X,\mathcal{F}) = 0$ for $i \geq 2$; write $\mathrm{h}^i(X,\mathcal{F}) = \dim_k \mathrm{H}^i(X,\mathcal{F})$. We define the *Euler Characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \mathrm{h}^0(X,\mathcal{F}) - \mathrm{h}^1(X,\mathcal{F}).$$

By the *genus*, g, of X we will always mean the arithmetic genus given by

$$g = g(X) = 1 - \chi(\mathcal{O}_X).$$

For coherent sheaves $\mathcal{E},\mathcal{F}$ we denote by $\mathbf{Hom}(\mathcal{E},\mathcal{F})$ the sheaf of $\mathcal{O}_X$-homomorphisms from $\mathcal{E}$ to $\mathcal{F}$; $\mathrm{Hom}(\mathcal{E},\mathcal{F}) = \mathrm{H}^0(X,\mathbf{Hom}(\mathcal{E},\mathcal{F}))$ denotes the group of globally defined homomorphisms. We can define right derived functors (in the second variable) $\mathbf{Ext}^i(\mathcal{E},\mathcal{F})$ and $\mathrm{Ext}^i(\mathcal{E},\mathcal{F})$ respectively; see [Hartshorne III 6] for their basic properties.

There is a duality theorem, which, since our assumptions on X imply that X is Cohen-Macaulay, takes the following form.

1.1.1 Duality theorem (see [Hartshorne III 7.6]).

On X, there exists a canonical dualising sheaf ω_X—which is a coherent torsion free sheaf of rank one—such that for any coherent sheaf $\mathcal{F}$ we have canonical isomorphisms

$$\mathrm{Ext}^i(\mathcal{F},\omega_X) \simeq \mathrm{H}^{1-i}(X,\mathcal{F})^\vee; \quad i = 0,1.$$

Recall that the *rank* of a sheaf $\mathcal{F}$ is the dimension of $\mathcal{F}_x \otimes K$ over K at one (and hence all) point(s) x of X ($\mathcal{F}_x$ denoting the stalk of $\mathcal{F}$ at x, K being equal to the field of fractions of the local ring $\mathcal{O}_x$). If $\mathcal{F}$ is a coherent torsion free sheaf we write $\mathcal{F}^* = \mathbf{Hom}(\mathcal{F},\omega_X)$ for the sheaf dual to $\mathcal{F}$; later on we may also refer to the sheaf $\mathcal{F}^\vee = \mathbf{Hom}(\mathcal{F},\mathcal{O}_X)$.

The other essential fact concerning the sheaf $\mathcal{F}$ is the Riemann-Roch Theorem, which we can take as giving the definition of the *degree* (denoted $\deg \mathcal{F}$) of a coherent sheaf.

1.1.2 Riemann-Roch. If $\mathcal{F}$ is a coherent sheaf then

$$\chi(\mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rank}(\mathcal{F})(1-g).$$

An immediate consequence is that degree is additive on short exact sequences: i.e., if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves on X then

$$\deg(\mathcal{F}) = \deg(\mathcal{F}') + \deg(\mathcal{F}'').$$

§1.2 Singularities

Proofs of most of the following facts can be found in [Serre 1].

X has a normalisation $\pi : \tilde{X} \longrightarrow X$, with $\tilde{X}$ non-singular, π birational (for more details on this see §3.1). For $x \in X$ we write $\tilde{\mathcal{O}}_x$ for the normalisation of the local ring $\mathcal{O}_x$ of X at x, i.e., $\tilde{\mathcal{O}}_x$ is the integral closure of $\mathcal{O}_x$

in K. The ring $\tilde{\mathcal{O}}_x$ is semilocal, the number of maximal ideals of $\tilde{\mathcal{O}}_x$ equals the number of branches of X at x. Write $m = m_x$ for the maximal ideal of $\mathcal{O}_x$. Its main invariants are as follows: $\mathcal{O}_x$ has *multiplicity* $e = e(x) = \dim_k(\tilde{\mathcal{O}}_x/m\tilde{\mathcal{O}}_x)$ and *embedding dimension* $\text{emb.dim}(x) = \dim_k(m/m^2)$. We define $\delta(x) = \dim_k(\tilde{\mathcal{O}}_x/\mathcal{O}_x)$, $\delta(X) = \sum_{x\in X} \delta(x)$.

The (arithmetic) genus of X satisfies $g(X) = g(\tilde{X}) + \delta(X)$ (cf. (3.2.1/2)).

We define the *sheaf of conductors* $\mathcal{C}$ on X to be $\text{Ann}(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$, locally $\mathcal{C}_x$, the conductor of $\mathcal{O}_x$ in $\tilde{\mathcal{O}}_x$, is the largest ideal of $\mathcal{O}_x$ which is an ideal in $\tilde{\mathcal{O}}_x$. Define $\tilde{\delta}(x) = \dim_k(\tilde{\mathcal{O}}_x/\mathcal{C}_x)$.

Proposition 1.2.1 see [Serre 1].

For $x \in X$ a singular point $\delta(x) + 1 \leq \tilde{\delta}(x) \leq 2\delta(x)$.

Both cases of equality will turn out to be important. If $\delta(x) + 1 = \tilde{\delta}(x)$ then [Serre 1] says x is 'defined by a modulus', here I will call such $\mathcal{O}_x$ *cubical* (cf. Ch.6). These singularities are characterised as follows:

Proposition 1.2.2 (again, see [Serre 1]).

$\mathcal{O}_x$ is cubical $\Leftrightarrow \mathcal{C}_x = m_x \Leftrightarrow e(x) = \delta(x) + 1 \Rightarrow \text{emb.dim}(x) = e(x)$.

At the other extreme $\tilde{\delta}(x) = 2\delta(x) \Leftrightarrow \omega_x \simeq \mathcal{O}_x$, when we say $\mathcal{O}_x$ is *Gorenstein*. Then, call X *Gorenstein* if for each singular point $x \in X$, $\mathcal{O}_x$ is Gorenstein; so X is Gorenstein $\Leftrightarrow \omega_X$ is locally free. For example a plane curve (or, more generally, any local complete intersection—see (2.1.2)) is Gorenstein.

It is necessary to have some idea of when 2 singularities, x and x', should be considered as being isomorphic. Requiring that the corresponding local rings be isomorphic does not give a good definition because the algebraic local ring carries too much global information; consequently a better definition is that of analytic equivalence—x and x' are *analytically equivalent* if the corresponding local rings have isomorphic completions. So all non-singular points are analytically equivalent, for example. Note that all the integral invariants defined above are preserved under taking completions (see, e.g., [Matsumura]). In this thesis we are basically interested in singularities up to analytic equivalence—this is enough to distinguish all the so called 'simple singularities', for instance.

In fact, this notion of isomorphism is still too fine; for example, two singularities consisting of 4 lines through a point in a plane are not analytically equivalent unless the cross ratios are the same. It seems, therefore, that we should consider the cruder notion of 'equisingularity'. However, we do not develop the tools to handle this here.

§1.3 The Generalised Jacobian

In this section we review the theory of divisors on X, the group $\mathrm{Pic}(X)$ and the associated variety—the generalised Jacobian. Much of this is taken from [Hartshorne].

Let $\mathcal{K} = K \times X$ be the constant sheaf of the field of rational functions on X: $\mathcal{O}_X$ is a subsheaf of $\mathcal{K}$. Let $\mathcal{K}^*$ (resp. $\mathcal{O}_X^*$) denote the sheaf of invertible elements in $\mathcal{K}$ (resp. $\mathcal{O}_X$). [It should always be clear from the context whether an upper asterisk refers to 'duality' or to 'the group (or sheaf) of invertible elements'.]

Definitions. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{K}^*/\mathcal{O}_X^*$; i.e., an element of $\mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}_X^*)$.
Consider the exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}^* \longrightarrow \mathcal{K}^*/\mathcal{O}_X^* \longrightarrow 0.$$

Taking global sections gives an exact sequence

$$k^* = \mathrm{H}^0(X, \mathcal{O}_X^*) \longrightarrow \mathrm{H}^0(X, \mathcal{K}^*) \xrightarrow{\phi} \mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}_X^*) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X^*) \longrightarrow 0.$$

Note that $\mathcal{K}^*$ is flasque and so $H^1(X, \mathcal{K}^*) = 0$. A Cartier divisor in the image of ϕ is said to be *principal*, and 2 Cartier divisors are called *linearly equivalent* if they differ by a principal divisor. Cartier divisors modulo linear equivalence form a group, $\mathrm{CaCl}\, X$, and from the above exact sequence we see that $\mathrm{CaCl}\, X$ is canonically isomorphic to $\mathrm{H}^1(X, \mathcal{O}_X^*)$.

If D is a Cartier divisor on X then it can be represented by giving an open cover $\{U_i\}$ of X, and, for each i, an element $f_i \in \mathrm{H}^0(U_i, \mathcal{K}^*)$ such that for each pair i, j the difference f_i/f_j is an element of $\mathrm{H}^0(U_i \cap U_j, \mathcal{O}_X^*)$. Thus $1/f_i$ generates a submodule of $\mathcal{K}^*$ over U_i and, because f_i/f_j is invertible on $U_i \cap U_j$; these submodules glue together to give a subsheaf, $\mathcal{L}(D)$ of $\mathcal{K}$. Indeed $\mathcal{L}(D)$ is an invertible sheaf. Invertible sheaves under tensor product form a group $\mathrm{Pic}(X)$ —the *Picard* group of X.

Proposition 1.3.1 (see [Hartshorne] II 6.13).

1. The above gives a bijection between the set of Cartier divisors on X and the set of invertible subsheaves of $\mathcal{K}$.
2. $\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.
3. D_1 and D_2 are linearly equivalent if and only if $\mathcal{L}(D_1) \simeq \mathcal{L}(D_2)$.
4. Consequently, we obtain an isomorphism $\mathrm{CaCl}\, X \longrightarrow \mathrm{Pic}(X)$.

Proposition 1.3.2. If X is singular with normalisation $\tilde{X}$ then $\mathrm{Pic}(X)$ and $\mathrm{Pic}(\tilde{X})$ are related by a short exact sequence of groups:

$$(\dagger) \quad 0 \longrightarrow \bigoplus_{x \in X} \tilde{\mathcal{O}}_x^*/\mathcal{O}_x^* \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow 0.$$

Proof. Write $A = \oplus_{x \in X} \tilde{\mathcal{O}}_x^*/\mathcal{O}_x^*$ and consider this as a sheaf of groups on X with finite support. On X there is an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \pi_* \mathcal{O}_{\tilde{X}}^* \longrightarrow A \longrightarrow 0.$$

Taking global sections gives an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X^*) &\longrightarrow \mathrm{H}^0(X, \pi_* \mathcal{O}_{\tilde{X}}^*) \longrightarrow \mathrm{H}^0(X, A) \\ &\longrightarrow \mathrm{H}^1(X, \mathcal{O}_X^*) \longrightarrow \mathrm{H}^1(X, \pi_* \mathcal{O}_{\tilde{X}}^*) \longrightarrow 0. \end{aligned}$$

Now $\mathrm{H}^0(X, \mathcal{O}_X^*) = k^* = \mathrm{H}^0(X, \pi_* \mathcal{O}_{\tilde{X}}^*)$ and $\mathrm{H}^0(X, A) = A$. Since π has finite fibres the higher direct image sheaves of π_* vanish and so there is an isomorphism $\mathrm{H}^1(X, \pi_* \mathcal{O}_{\tilde{X}}^*) \simeq \mathrm{H}^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*)$. Hence the result follows on making the canonical identifications $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*)$ and $\mathrm{Pic}(\tilde{X}) = \mathrm{H}^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*)$.

A Cartier divisor D represented by $\{U_i, f_i\}$ is said to be *effective* if, for each i, $f_i \in H^0(U_i, \mathcal{O}_X)$. Any such divisor defines a closed subscheme Y of X and it is easy to see that $\mathcal{L}(-D) = \mathcal{I}_Y$ is then the ideal sheaf of Y. Moreover, the degree of $\mathcal{L}(D)$ is the dimension of $\mathcal{O}_X/\mathcal{I}_Y$, i.e., the number of zeros of the f_i counted with appropriate multiplicity. Conversely, any closed subscheme supported on non-singular points, being locally defined by the vanishing of a single function on X, gives rise to an effective divisor. This, together with the group property of $\mathrm{Pic}(X)$, shows that invertible sheaves of all degrees exist. Thus degree gives a surjective homomorphism $\mathrm{Pic}(X) \longrightarrow \mathbf{Z}$: let $\mathrm{Pic}^0(X)$ be the kernel of this homomorphism.

$\mathrm{Pic}^0(X)$ has a natural algebraic structure; we state the result from the modern viewpoint. A *family of invertible sheaves of degree d* on X parametrised by a variety S is, by definition, an invertible sheaf $\mathcal{F}$ on $S \times X$ such that for each $s \in S$, $\mathcal{F}_s$, the restriction of $\mathcal{F}$ to $\{s\} \times X \simeq X$, has degree d.

Theorem 1.3.3 [Grothendieck]. For each integral projective curve X there exists a pair $(J,\mathcal{U}) = (J,\mathcal{U})(X,d)$ consisting of a g dimensional quasi-projective variety J and a family, $\mathcal{U}$, of degree d invertible sheaves on X parametrised by J with the following universal property. For any family $\mathcal{F}$ of invertible sheaves of degree d parametrised by S there exists a unique morphism $\phi : S \longrightarrow J$ such that $\mathcal{F}_s \simeq ((\phi \times 1)^*\mathcal{U})_s$ for all points s of S. In other words $(J,\mathcal{U})(X,d)$ is a *fine moduli space* for degree d invertible sheaves on X.

J is called the *generalised Jacobian* of X; strictly speaking there is one, J^d, for each d, but choosing any degree d invertible sheaf $\mathcal{M}$ the correspondence $\mathcal{L} \mapsto \mathcal{L}\otimes\mathcal{M}$ gives an isomorphism $J^0 \longrightarrow J^d$. It follows easily from the universal property that J^0 is an algebraic group, and that, as a group, $J^0 \simeq \mathrm{Pic}^0(X)$: we often write simply $J(X)$ for $J^0(X)$. In the case when X is non-singular $J(X)$ is an abelian variety, however, if X has singularities then $J(X)$ is only quasi-projective; in fact, there is an exact sequence of algebraic groups analogous to (†) (see 1.3.2) and so $J(X)$ is the extension of the abelian variety $J(\tilde{X})$ by a linear group (see [Oort]).

CHAPTER 2

1-DIMENSIONAL LOCAL RINGS AND TORSION FREE MODULES

§2.1 Introduction

In this chapter we concentrate entirely on the local situation. The basic question we ask is 'how can we describe the set of all isomorphism classes of torsion free modules over a 1-dimensional Cohen-Macaulay local ring?' All the subsequent chapters will depend on this description. A basic fact is that any finite torsion free module is isomorphic to both a submodule of a free module and to a module containing a free module in such a way that the quotients both have finite dimension. Moreover, minimising these dimensions, we obtain important integer invariants for the module, and from the resulting short exact sequences it is often possible to compute homology groups. Extending this idea we find that we can represent every torsion free module of given rank as a point of a certain Grassmannian: this geometric construction is fundamental in the later chapters. These parameter spaces have also been studied in [Rego 1] and [Greuel and Pfister].

All modules will be assumed to be finite. If x is non singular then $\mathcal{O}_x$ is a PID and so every torsion free $\mathcal{O}_x$-module is free. If x is singular, however, then there always exist torsion free modules which are *not* free, e.g. the maximal ideal m_x, which requires at least two generators. We will describe the main invariants of these modules (§2.2) and give a 'geometric' approach to classification (§2.3), illustrating the module classification for various examples in §2.4. Finally in §2.5 we give some homology calculations (e.g., for the space of self extensions of a module) that will be used later. First, however, we need to recall some standard facts about the ring $\mathcal{O}_x$ itself.

We will drop the x in the notation throughout the rest of the chapter. Thus $\mathcal{O}$ will be a 1-dimensional Cohen-Macaulay local ring with maximal ideal m and field of fractions K. $\tilde{\mathcal{O}}$ is its integral closure in K and its conductor in $\tilde{\mathcal{O}}$ is C; $\delta = \dim_k \tilde{\mathcal{O}}/\mathcal{O}$, $\tilde{\delta} = \dim_k \tilde{\mathcal{O}}/C$. The ring $\tilde{\mathcal{O}}$ is finitely generated over $\mathcal{O}$.

The Cohen-Macaulay property of $\mathcal{O}$ implies the following: There exists a *canonical ideal* (also called a *dualising module*), ω, of $\mathcal{O}$: i.e., if M is any finite torsion free module, and if $M^* = \mathrm{Hom}_{\mathcal{O}}(M,\omega)$, then $M^{**} \simeq M$. (The case when $\omega \simeq \mathcal{O}$ is the Gorenstein case.) This is the local version of the Serre duality theorem, although it is not quite trivial to show this;—a proof of this fact is given in (3.1.6). We make some more remarks on ω at the end of this section.

Theorem 2.1.1. '$\mathcal{O}$ is Gorenstein' is equivalent to each of the following statements:

1. $\mathcal{O}$ has injective dimension 1;

2. $\mathrm{Ext}^i_{\mathcal{O}}(k,\mathcal{O}) = 0$ for all $i \geq 2$;

3. $\mathrm{Ext}^i_{\mathcal{O}}(k,\mathcal{O}) = 0$ for some $i \geq 2$;

4. $\dim_k(m^{-1}/\mathcal{O}) = 1$.
   For a proof see [Bass]. With regard to **4.** recall that if M is any fractional ideal of $\mathcal{O}$ then $M^{-1} = \{x \in K | x.M \subset \mathcal{O}\}$. For $\mathcal{O}$ Gorenstein $m^{-1} = \mathrm{End}(m)$ is the unique minimal overring of $\mathcal{O}$.

The property of being a local complete intersection is related to the Gorenstein property.

Theorem 2.1.2.

1. If $\mathcal{O}$ is a complete intersection ring (e.g., if emb.dim$\mathcal{O} = 2$) then $\mathcal{O}$ is Gorenstein.

2. If emb.dim$\mathcal{O} = 3$ then it is also true that $\mathcal{O}$ Gorenstein $\Rightarrow$ $\mathcal{O}$ is a complete intersection ring. (This is not true if emb.dim$\mathcal{O} \geq 4$.)

   For a proof of **1.** see [Matsumura], **2.** is due to Serre—see [Serre 2].

When $\mathcal{O}$ is not regular it has infinite homological dimension; in fact projective modules are very sparse:

Lemma 2.1.3 [Altman and Kleiman 1]. Let M be a finite $\mathcal{O}$-module; then the following are equivalent:

1. M is free;

2. M is projective;

3. M is flat;

4. $\mathrm{Tor}_1^{\mathcal{O}}(M,k) = 0$.
   In fact, all other finite $\mathcal{O}$-modules require infinite projective resolutions.

For modules of finite injective dimension the picture is similar (remember that if an $\mathcal{O}$-module has finite injective dimension then this dimension equals the dimension of the ring [Matsumura]):

Lemma 2.1.4 [Matlis]. For an ideal I of $\mathcal{O}$ the following are equivalent:

1. I is a canonical ideal;

2. I has injective dimension 1;

3. $\mathrm{Ext}^1_{\mathcal{O}}(k,I) \simeq k$;

4. K and K/I are injective $\mathcal{O}$-modules.

If any of these hold then also $\mathrm{Ext}^1(I,I) = 0$ (this follows easily from **4**).

Note that the canonical ideal is determined only up to isomorphism, consequently M^*, the dual of a module M, is defined only as an abstract module unless a specific canonical ideal is given: Then, regarding M as a submodule of K^n ($n = \mathrm{rank} M$), M^* is defined as a submodule of K^n and $M^{**} = M$. When more precision is required we will define *the* canonical ideal ω to be the maximal canonical ideal of $\mathcal{O}$—so, if $\mathcal{O}$ is Gorenstein $\omega = \mathcal{O}$—and define duality with respect to this choice of ω.

Suppose now that $\mathcal{O}$ is not Gorenstein: we want to calculate $\dim(\mathcal{O}/\omega)$.

Lemma 2.1.5. If $\mathcal{O}$ is not Gorenstein then $\dim(\mathcal{O}/\omega) = 2(\tilde{\delta} - \delta)$.
Also, if $\omega' \simeq \omega$ with $\mathcal{O} \subset \omega' \subset \tilde{\mathcal{O}}$ then $\dim(\omega'/\mathcal{O}) = 2\delta - \tilde{\delta}$.

Proof. For any $\mathcal{O}$-submodule, I, of $\tilde{\mathcal{O}}$ define the *conductor* of I to be $C(I) = \mathrm{Ann}(\tilde{\mathcal{O}}/I)$. Note that $C(I) = \mathrm{Hom}(\tilde{\mathcal{O}}, I)$, so

$$C(\omega) = \mathrm{Hom}(\tilde{\mathcal{O}}, \omega) = \tilde{\mathcal{O}}^*.$$

More generally, if $0 \neq z \in K$ then $\mathrm{Hom}(z\tilde{\mathcal{O}}, I) = z^{-1}C(I)$. On writing $C = z\tilde{\mathcal{O}}$ it follows that $C^* = z^{-1}C(\omega)$, and so $C \cdot C^* = C(\omega)$.

It is possible to find $\omega' \simeq \omega$ with

$$\mathcal{O} \subset \omega' \subset \tilde{\mathcal{O}}$$

so that the quotient $\omega'/\mathcal{O}$ has the minimum dimension possible (see (2.2.1) for a proof of this). Clearly $\omega' \cdot \tilde{\mathcal{O}} = \mathcal{O} \cdot \tilde{\mathcal{O}} = \tilde{\mathcal{O}}$ and, since $1 \in \omega'$ we claim that

$$C(\omega') = C(\mathrm{End}(\omega')) = C(\mathcal{O}) = C.$$

To prove the first equality, note that one inclusion is obvious, so suppose that $x \in C(\omega') - C(\mathrm{End}(\omega'))$; then, $\exists\ y \in \tilde{\mathcal{O}}$ such that $xy \notin \mathrm{End}(\omega')$, i.e. $\exists\ z \in \omega'$ such that $xyz \notin \omega'$. But this is not possible since $xy \in C(\omega') \Rightarrow xyz \in C(\omega') \subset \omega'$. Note also that $\mathrm{End}(\omega') = \mathcal{O}$ which gives the second equality.

Dualising the above, $(\omega')^* = \mathcal{O}'$ is a proper ideal of $\mathcal{O}$, maximal amongst those isomorphic to $\mathcal{O}$, $C(\omega) = C(\mathcal{O}')$ and $\omega \cdot \tilde{\mathcal{O}} = \mathcal{O}' \cdot \tilde{\mathcal{O}}$. Since we assume $\mathcal{O}$ is not Gorenstein $\mathcal{O}' \neq \mathcal{O}$. Now, since $\tilde{\mathcal{O}} \simeq \mathcal{O}' \cdot \tilde{\mathcal{O}} \subset \mathcal{O}$, we must have $\mathcal{O}' \subset C$, hence $\omega \subset C$ and $C^* \subset \mathcal{O}$ by duality. Maximising $\mathcal{O}'$ is equivalent to maximising $\mathcal{O}' \cdot \tilde{\mathcal{O}}$, thus $\mathcal{O}' \cdot \tilde{\mathcal{O}} = C$ from which it follows that $C = C^*$ and $C(\omega) = C^2$.

Using duality,

$$\dim \omega/\tilde{\mathcal{O}}^* = \dim \tilde{\mathcal{O}}/\mathcal{O} = \delta \text{ and}$$
$$\dim C^*/\tilde{\mathcal{O}}^* = \dim \tilde{\mathcal{O}}/C = \tilde{\delta}.$$

Thus

$$\dim \mathcal{O}/\tilde{\mathcal{O}}^* = 2\tilde{\delta} - \delta \text{ and } \dim \mathcal{O}/\omega = 2(\tilde{\delta} - \delta).$$

Note also that

$$\dim \omega'/\mathcal{O} = \dim \omega'/\omega - \dim \mathcal{O}/\omega = \tilde{\delta} - 2(\tilde{\delta} - \delta) = 2\delta - \tilde{\delta}.$$

§2.2 Module Invariants

Recall that if M is an $\mathcal{O}$-module the *rank* of M is defined to be $\dim_K(M \otimes_{\mathcal{O}} K)$. (If $\mathcal{O}$ has zero divisors, i.e., if the curve X is reducible, then the rank of a module (or sheaf) needs to be defined separately along each branch.)

In general there may exist uncountably many isomorphism classes of indecomposable $\mathcal{O}$-modules of given rank. However we do have the following proposition which shows that the modules are, in some sense, 'bounded'.

Proposition 2.2.1. If M is a torsion free $\mathcal{O}$-module of rank n, then

1. M is isomorphic to an $\mathcal{O}$-module M' with
$$\mathcal{O}^n \subset M' \subset \tilde{\mathcal{O}}^n.$$

2. Dually, M is isomorphic to an $\mathcal{O}$-module M'' with
$$(\tilde{\mathcal{O}}^*)^n \subset M'' \subset \omega^n \subset \mathcal{O}^n.$$
In particular, if $\mathcal{O}$ is Gorenstein then
$$C^n \subset M'' \subset \mathcal{O}^n.$$

Proof.

1. Rank of $M = n$ implies $M\tilde{\mathcal{O}} \simeq \tilde{\mathcal{O}}^n$. Embedding M in K^n we can choose n elements in M giving such an isomorphism; let F be the rank n free module spanned by these elements. So we have inclusions $F \subset M \subset F.\tilde{\mathcal{O}} = M.\tilde{\mathcal{O}}$. Choosing an isomorphism $F \simeq \mathcal{O}^n$ now gives the result.

2. Using **1.** it is possible to write $\mathcal{O}^n \subset M^* \subset \tilde{\mathcal{O}}^n$. Now dualising implies
$$(\tilde{\mathcal{O}}^n)^* \subset (M^*)^* \subset \mathcal{O}^n$$
but $(M^*)^* \simeq M$.

Lemma 2.2.2. If M and N are torsion free $\mathcal{O}$-modules of rank n with $\mathcal{O}^n \subset M, N \subset \tilde{\mathcal{O}}^n$, then any isomorphism $\phi : M \longrightarrow N$ is the restriction of an automorphism of $\tilde{\mathcal{O}}^n$. i.e., it is induced by an element of $GL_n(\tilde{\mathcal{O}})$.

Proof. Viewing M and N as submodules of K^n
$$\mathrm{Hom}_{\mathcal{O}}(M, N) = \{\phi \in M_n(K) | \phi(M) \subset N\}.$$

So any isomorphism ϕ can be represented as a non-singular matrix (ϕ_{ij}) with entries in K. We need to prove that $\phi_{ij} \in \tilde{\mathcal{O}}$. For any $(x_1, \ldots, x_n) \in M$ there exists a non-zero element $c \in \mathcal{O}$ such that $c(x_1, \ldots, x_n) \in C^n$. Then $c\phi_{ij}(x_j) = \phi_{ij}(cx_j) = cx_j\phi_{ij}(1)$. So $\phi_{ij}(x_j) = x_j\phi_{ij}(1)$, and $\phi_{ij}(1) \in \tilde{\mathcal{O}}$, since $\phi(\mathcal{O}^n) \subset \tilde{\mathcal{O}}^n$.

Definitions 2.2.3. When M is an $\mathcal{O}$-module of rank n, define

$$\ell(M) = \delta n - \dim_k(M\tilde{\mathcal{O}}/M).$$

Alternatively, using the above,

$$\ell(M) = \min\{\dim_k(M/F) \mid F \subset M,\ F \simeq \mathcal{O}^n\}.$$

Thus $\ell(\mathcal{O}^n) = 0$, $\ell(\tilde{\mathcal{O}}^n) = \delta n$, and if $M \not\simeq \mathcal{O}^n, \tilde{\mathcal{O}}^n$ then $0 < \ell(M) < \delta n$. As another example, it follows from (2.1.5) that $\ell(\omega) = 2\delta - \tilde{\delta}$.

So, for each M, there is a short exact sequence

$$\sigma_1 : \qquad 0 \longrightarrow \mathcal{O}^n \longrightarrow M \longrightarrow T_M \longrightarrow 0$$

where $\dim_k T_M = \ell(M)$. T_M is well defined up to isomorphism (using 2.2.2). Viewing M as a submodule of $\mathcal{O}^n$ via $M \subset \tilde{\mathcal{O}}^n \simeq C^n \subset \mathcal{O}^n$ gives a short exact sequence

$$\sigma_2 : \qquad 0 \longrightarrow M \longrightarrow \mathcal{O}^n \longrightarrow T'_M \longrightarrow 0$$

where $\dim_k T'_M = n\tilde{\delta} - \ell(M)$. Again, one sees that T'_M is well defined.

Remarks. The sequence σ_2 is not necessarily minimal: that is, we might be able to write M as a submodule of $\mathcal{O}^n$ such that the quotient has smaller dimension. The general form above is sufficient for our purposes.

In [Greuel and Pfister] an invariant, $\delta(M)$, was defined to be $\dim_k(M\tilde{\mathcal{O}}/M)$ (see also chapter 6). For us it seems to be more convenient to define this invariant from the other direction: $\ell(M)$ can be thought of as measuring how far M is from being projective.

Definition. Define the *index*, $i(M)$ of a module M to be

$$i(M) = \min\{\ell(M), \ell(M^*)\}.$$

In many of our examples the index of M measures the codimension of modules isomorphic to M in a space of all modules of given rank (cf. §2.3 and chapter 5).

To end this section we present a lemma which gives an upper bound for the number of generators required by an $\mathcal{O}$-module of given rank. If M is a rank n torsion free $\mathcal{O}$-module write $r = r(M)$ for the minimal number of elements in a generating set of M. Then the equality $r = \dim_k(M/mM)$ follows from Nakayama's lemma (see [Matsumura Theorem 2.3]). From 2.2.1.(1) we clearly have $n \leq r \leq n(1+\delta)$. In fact:

Lemma 2.2.4. $r(M) \leq ne$ (where e=multiplicity of $\mathcal{O}$).

Proof: (closely following that for ideals in [Sally].)

We use the fact that there exists an element $t \in m$ which is *superficial* for $\mathcal{O}$, i.e., $tm^j = m^{j+1}$ for all sufficiently large j. We may assume $M \subset \mathcal{O}^n$. Then $\dim(\mathcal{O}^n/t\mathcal{O}^n) = \dim(\mathcal{O}^n/t\mathcal{O}^n) + \dim(t\mathcal{O}^n/tM) - \dim(\mathcal{O}^n/M) = \dim(\mathcal{O}^n/tM) - \dim(\mathcal{O}^n/M) = \dim(M/tM)$. Also, applying this to $M = m^j$, with j large enough, we have $\dim(\mathcal{O}^n/t\mathcal{O}^n) = ne$.
The exact sequence

$$0 \longrightarrow mM/tM \longrightarrow M/tM \longrightarrow M/mM \longrightarrow 0$$

gives

$$\begin{aligned}\dim(M/mM) &= \dim(M/tM) - \dim(mM/tM)\\ &= ne(x) - \dim(mM/tM),\end{aligned}$$

so $r(M) \leq ne$.

§2.3 Parameter Spaces for Modules

We make some general remarks on the classification problem for finite torsion free $\mathcal{O}$-modules. There are 2 different approaches; firstly that of trying to list canonical forms, and, secondly, constructing parameter spaces containing representatives of every isomorphism class. The attempt to list canonical forms proceeds by fixing all discrete invariants, then writing down a system of generators satisfying some minimality properties; these generating sets will contain some parameters, different choices of which might give isomorphic modules. It is then necessary to calculate the orbits under the action of the group of automorphisms of $\tilde{\mathcal{O}}^n$ (see, e.g., [Schappert]). For some examples of lists see §2.4.

The purpose of this section is to make a few comments on the second approach. The parameter spaces discussed here will occur frequently in what follows and we develop their properties as these are needed. More details on these spaces can be found in [Rego 1] and [Greuel and Pfister].

Let $I \subset \mathcal{O}$ be any non-zero ideal of $\mathcal{O}$. Then $V = \tilde{\mathcal{O}}/I$ is an $\mathcal{O}$-module with finite dimension as a vector space over k. If M is an $\mathcal{O}$-module with $I^n \subset M \subset \tilde{\mathcal{O}}^n$ then the projection $p : \tilde{\mathcal{O}}^n \longrightarrow V^n$ takes M to a subspace of V^n which is also a submodule. Conversely if W is any $\mathcal{O}$-submodule of V^n then $p^{-1}(W) = N$ is an $\mathcal{O}$-module with $I^n \subset N \subset \tilde{\mathcal{O}}^n$. We say that N is *represented* as a submodule of V^n.

Note that a subspace W of V^n is a submodule if and only if $\mathcal{O}W \subset W$ which happens if and only if $u.W = W$ for all units, $u \in (\mathcal{O})^*$, or, equivalently, $\bar{u}.W = W$ for all $\bar{u} \in ((\mathcal{O}/I)^*)/k^*$, since I annihilates and k^* acts trivially.

The set of all subspaces W of V^n of a given dimension, $a \leq n.\dim_k(\tilde{\mathcal{O}}/I)$, is parametrised by the Grassmannian $\mathrm{Gr}(a, V^n)$. The finite dimensional unipotent group $((\mathcal{O}/I)^*)/k^*$ acts on this space, and so, by the above, submodules of V^n of dimension a are parametrised by the set of fixed points for this action. This set has a natural structure as a connected closed subscheme of $\mathrm{Gr}(a, V^n)$ ('the fixed point subscheme'—see [Fogarty]) and we denote it by $\mathrm{Gr}^{\mathcal{O}}(a, V^n)$. The *geometric* properties of this scheme will be investigated later; what we want to demonstrate now is that it is possible to choose the pair (I, a) so that every isomorphism class of rank n torsion free modules is represented in $\mathrm{Gr}^{\mathcal{O}}(a, V^n)$.

Theorem 2.3.1. Let ω denote the (maximal) canonical ideal. Write $I = \tilde{\mathcal{O}}^*$, $V = \tilde{\mathcal{O}}/I$ and $\ell = \dim \mathcal{O}/I$. Then every isomorphism class of rank n torsion free modules is represented in $\mathrm{Gr}^{\mathcal{O}}(n\ell, V^n)$.
In particular if $\mathcal{O}$ is Gorenstein we may take $I = C$.

Proof. The proof follows that of [Rego 1] (for the rank 1, Gorenstein case). Observe that, certainly, the free module of rank n is represented in

$\mathrm{Gr}^{\mathcal{O}}(n\ell, (\tilde{\mathcal{O}}/I)^n)$. Recall that every rank n module is isomorphic to a module M with $I^n \subset M \subset \omega^n \subset \mathcal{O}^n$ (2.2.1).

Let $z_1, \ldots, z_r$ be the generators of the maximal ideals in $\tilde{\mathcal{O}}$. $M\tilde{\mathcal{O}} \simeq \tilde{\mathcal{O}}^n$, working in K^n, there exists $\underline{z} \in M$ such that $M \cdot \tilde{\mathcal{O}} = \underline{z}\tilde{\mathcal{O}}^n$. On choosing a suitable basis we can write

$$\underline{z} = (u_1 \prod z_j^{s_{1j}}, u_2 \prod z_j^{s_{2j}}, \ldots, u_n \prod z_j^{s_{nj}})$$

where the u_i are units in $\tilde{\mathcal{O}}$ and the s_{ij} are non-negative integers (since $M \subset \tilde{\mathcal{O}}^n$); let $s = \sum_{i,j} s_{ij}$. Now $\dim \mathcal{O}^n/M \leq s$ since $\underline{z}\mathcal{O}^n \subset M$ implies

$$\dim \mathcal{O}^n/M = \dim \mathcal{O}^n/\underline{z}\mathcal{O}^n \leq$$

$$\dim \underline{z}\tilde{\mathcal{O}}^n/\underline{z}\mathcal{O}^n + \dim \tilde{\mathcal{O}}^n/\underline{z}\tilde{\mathcal{O}}^n - \dim(\tilde{\mathcal{O}}^n/\mathcal{O}^n) = n\delta + s - n\delta = s.$$

For $1 \leq i \leq n$ and $1 \leq j \leq r$ choose non-negative integers t_{ij} such that each $t_{ij} \leq s_{ij}$ and $\sum t_{ij} = \dim(\mathcal{O}^n/M)$ Define

$$\underline{y} = \left(\prod z_j^{-t_{1j}}, \prod z_j^{-t_{2j}}, \ldots, \prod z_j^{-t_{nj}}\right) \text{ and } M' = \underline{y}M.$$

Now certainly $I^n \subset M'$ and

$$M' = \underline{y}M \subset \underline{z}^{-1}M \subset \tilde{\mathcal{O}}^n$$

(where $\underline{z}^{-1}$ is defined by the requirement that $\underline{z}^{-1}.\underline{z} = (1, 1, \ldots, 1)$). Moreover, by the condition on Σt_{ij} it is clear that $\dim M'/I = \dim \mathcal{O}^n/I = n\ell$ and thus M' is represented by a point of $\mathrm{Gr}^{\mathcal{O}}(n\ell, V^n)$.

Remarks.

1. Some of these schemes are considered in detail in chapter 5.

2. The result says that a suitable choice of ideal is $\tilde{\mathcal{O}}^*$, which is either C (if $\mathcal{O}$ is Gorenstein) or C^2 (otherwise). Of course $\dim(\mathcal{O}/C^2)^n = 2\tilde{\delta}n$ is very 'large', and one might expect that it would be possible to represent every module as a subspace of a space of smaller dimension—for example, in [Greuel and Pfister] (also (6.3.1)) it is proved that, for rank 1 modules in the case when $\tilde{\mathcal{O}}$ is a local ring, a space of dimension 2δ will do. This may well generalise; however, if $\tilde{\mathcal{O}}$ is only semilocal, there is no canonical choice of ideal achieving this dimension. On the other hand, $\tilde{\mathcal{O}}^*$ provides a natural choice for all cases.

3. In the case of rank 1 modules when $\mathcal{O}$ is analytically irreducible (i.e., $\tilde{\mathcal{O}}$ is a local ring) two different subspaces correspond to isomorphic modules if and only if they lie in the same orbit under the action of the group $(\tilde{\mathcal{O}}/I)^*$ on $\mathrm{Gr}^{\mathcal{O}}(a, V^n)$. Greuel and Pfister prove that there is a stratification of $\mathrm{Gr}^{\mathcal{O}}(a, V^n)$ (given by fixing certain invariants—cf. chapter 6) such that the quotient can be defined on each strata; this leads to the construction of moduli spaces for rank 1 torsion free modules (with suitable fixed invariants).

4. Note that these schemes are analytic invariants of the singularity, in other words if we first take the completion $\hat{\mathcal{O}}$ of $\mathcal{O}$, replacing I by $\hat{I} = I \otimes \hat{\mathcal{O}}$ etc. the same spaces result. As a consequence of this we see that the classifying finite torsion free modules over $\mathcal{O}$ is equivalent to classifying finite torsion free modules having the same rank along each branch over $\hat{\mathcal{O}}$:—since $\hat{\mathcal{O}}$ may have zero divisors an $\hat{\mathcal{O}}$-module M is defined to be ***torsion free*** if it is torsion free with respect to ***regular*** elements of $\hat{\mathcal{O}}$.

The group $(\mathcal{O}/I)^*$ is a subgroup of the group $G = (\tilde{\mathcal{O}}/I)^*$ which also acts on $\mathrm{Gr}(n\ell, V^n)$, and fixed points for G will correspond to $\tilde{\mathcal{O}}$-submodules of V^n. Write $\mathrm{Gr}^{\tilde{\mathcal{O}}}(n\ell, V^n)$ for the associated fixed point subscheme.

Proposition 2.3.2. Let W_M denote the set of points of $\mathrm{Gr}^{\mathcal{O}}(n\ell, V^n)$ which represent M. Assume this set is not empty. Then

$$\overline{W_M} \cap \mathrm{Gr}^{\tilde{\mathcal{O}}}(n\ell, V^n) \neq \emptyset.$$

Proof. Take $y \in W_M$ and write $G(y)$ for the G-orbit of y. Since the action of G commutes with that of $(\mathcal{O}/I)^*$ all points of $G(y)$ will also represent M. Now G also acts on $\overline{G(y)} \subset \overline{W_M}$, and since G is connected and abelian by [Borel 10.4] we can conclude that G has a fixed point in $\overline{G(y)}$.

§2.4 Examples

We want to describe the module classification explicitly for some examples. In common with many such problems in Algebra the classification falls into one of three types: *finite, tame* or *wild* (see [Drozd and Greuel]). The distinction between the 2 infinite types will not be of much importance here, and the finite type singularities will form the basis of most of our examples. The rings $\mathcal{O}_x$ with only finitely many isomorphism classes of indecomposable modules are characterised by a theorem of Greuel and Knörrer, they also gave the module classification in these cases. The main purpose of this section is to review this.

We assume that k has characteristc 0.

Theorem 2.4.1 Singularities of Finite Type [Greuel and Knörrer].

1. Any complete local Gorenstein ring of dimension 1 which has only finitely many isomorphism classes of indecomposable torsion free modules is either regular or isomorphic to the complete local ring of a simple singularity. That is, one of the following types:

$$A_{n;n\geq 1}, D_{n;n\geq 4}, E_6, E_7, \text{ or } E_8.$$

2. Further, any other 1-dimensional complete local ring of finite type is the unique minimal overring of one of the above. We denote the corresponding singularities by

$$D^-_{n;n\geq 4}, E_6^-, E_7^-, \text{ and } E_8^-.$$

Remarks. 'A_n^-' does not occur in this last list since the overring in this case is Gorenstein—this fact goes back to [Bass]. As an example of what is happening geometrically, D_4 is the (simple) planar triple point, whilst D_4^- is the non-planar triple point.

Theorem 2.4.2 Rank 1 Module Classification for Simple Singularities.

The complete classification is found in [Greuel and Knörrer], we are interested in the rank 1 (torsion free) modules (i.e., the modules having rank 1 along each branch).

Let $\mathcal{O}$ be a 1-dimensional complete local ring of finite type. Then every rank 1 torsion free module is isomorphic to either **1.** an overring of $\mathcal{O}$, or **2.** the dual of an overring of $\mathcal{O}$.

Note that the overrings occurring are also necessarily of finite type. We list below the module classification for each of the above singularities. For each module we give a generating set and indicate if it is isomorphic to an overring, in which case we write $M \sim B$, if B is the type of the singularity to which this overring corresponds. In each case M_0 is the free module (not always listed) and M_δ is the normalisation. The dual of a module M is denoted by M^*, if no M^* appears in the list then $M \simeq M^*$ (of course, $(M^*)^* \simeq M$). We will use the notation given below for these modules consistently in future calculations (notably in §2.5 and chapter 5).

Case i $A_{2\delta-1} : \mathcal{O} = k[[(t,t),(t^\delta,0)]]$.
$M_i = \langle 1,(t^{\delta-i},0)\rangle, \quad 0 \le i \le \delta, \quad M_i \sim A_{2\delta-1-2i}$.
(The notation means that

$$M_i = \mathcal{O} + (t^{\delta-i},0)\cdot\mathcal{O} \subset \tilde{\mathcal{O}} = k[[t]] \oplus k[[t]].)$$

Case ii $A_{2\delta} : k[[t^2,t^{2\delta+1}]]$.
$M_i = \langle 1,t^{2\delta+1-2i}\rangle, \quad 0 \le i \le \delta, \quad M_i \sim A_{2\delta-2i}$.

Case iii $D_{2\delta-2} : k[[(t,t,0),(t^{\delta-2},0,t)]]$;
$D^-_{2\delta-2} : k[[(t,t,0),(t^{\delta-2},0,0),(0,0,t)]], \quad \delta \ge 3$.
$M_i = \langle 1,(t^{\delta-1-i},0,0)\rangle, \quad 1 \le i \le \delta-2, \quad M_i \sim D^-_{2(\delta-i)}$.
$M_i^* = \langle (1,1,0),(t^{\delta-i-2},0,1)\rangle, \quad 1 \le i \le \delta-2$.
$M^1_{\delta-1} = \langle 1,(1,0,0)\rangle, \quad M^2_{\delta-1} = \langle 1,(0,1,0)\rangle, \quad M^{1,2}_{\delta-1} \sim A_1$.
$M_\delta = \langle (1,0,0),(0,1,0),(0,0,1)\rangle$.
$N_i = \langle 1,(0,0,1),(t^{\delta-i},0,0)\rangle, \quad 2 \le i \le \delta-1, \quad N_i \sim A_{2\delta-2i-1}$.

Case iv $D_{2\delta-1} : k[[(t^2,0),(t^{2\delta-3},t)]]$; $D^-_{2\delta-1} : k[[(t^2,0),(t^{2\delta-3},0),(0,t)]], \ \delta \ge 3$
$M_i = \langle 1,(t^{2\delta-2i-1},0)\rangle, \quad 1 \le i \le \delta-1, \quad M_i \sim D^-_{2(\delta-i-1)}, \ M_{\delta-1} \sim A_1$.
$M_i^* = \langle (1,0),(t^{2\delta-2i-3},1)\rangle, \quad 1 \le i \le \delta-2$,
$M_\delta = \langle 1,(1,0),(t,0)\rangle$.
$N_i = \langle 1,(1,0),(t^{2\delta-2i+1},0)\rangle, \quad 2 \le i \le \delta-2, \quad N_i \sim A_{2(\delta-1)}$.

Case v $E_6 : k[[t^3, t^4]]$; $E_6^- : k[[t^3, t^4, t^5]]$.
$M_1 = \langle 1, t^5 \rangle$, $M_1^* = \langle 1, t \rangle$, $M_2 = \langle 1, t^2 \rangle$, $M_3 = \langle 1, t, t^2 \rangle$.
$M_1 \sim E_6^-$, $M_2 \sim A_2$.

Case vi $E_7 : k[[(t^2, t), (t^3, 0)]]$; $E_7^- : k[[(t^2, t), (t^3, 0), (t^4, 0)]]$.
$M_1 = \langle 1, (t^4, 0) \rangle$, $M_1^* = \langle 1, (t, 0) \rangle$, $M_2 = \langle 1, (t^2, 0) \rangle$,
$M_2^* = \langle (t, 1), (1, 0) \rangle$, $M_3 = \langle 1, (t, 0), (t^2, 0) \rangle$,
$N_3 = \langle 1, (1, 0) \rangle$, $M_4 = \langle 1, (1, 0), (t, 0) \rangle$,
$M_1 \sim E_7^-$, $M_2 \sim D_5^-$, $N_3 \sim A_2$, $M_3 \sim A_1$.

Case vii $E_8 : k[[t^3, t^5]]$; $E_8^- : k[[t^3, t^5, t^7]]$.
$M_1 = \langle 1, t^7 \rangle$, $M_1^* = \langle 1, t^2 \rangle$, $M_2 = \langle 1, t^4 \rangle$,
$M_2^* = \langle 1, t \rangle$, $M_3 = \langle 1, t^2, t^4 \rangle$, $M_4 = \langle 1, t, t^2 \rangle$.
$M_1 \sim E_8^-$, $M_2 \sim E_6^-$, $M_3 \sim A_2$.

Remarks.

1. The special feature of the module classification in these cases, that every rank 1 module is isomorphic to either (i) an overring; or, (ii) the dual of an overring, is certainly not true for many, more complicated singularities. However, it is also true that this feature does not characterise the simple singularities (e.g. $k[[t^3, t^7]]$ also has this property—see chapter 6).

2. For the simple plane curve singularities note that the number i appearing as the subscript in the classification is the index (defined in 2.2.3) of the module. Thus, $\text{index}(M) = \ell(M)$ if M is isomorphic to an overring, and $\text{index}(M) = \ell(M) - 1$ otherwise.

3. In general the more maximal ideals that $\tilde{\mathcal{O}}$ has the more modules $\mathcal{O}$ has. The extra module for E_7 as compared to E_8 is a reflection of this; both singularities have multiplicity 3 and $\delta = 4$ but an E_7 singularity is made up of 2 branches, whilst an E_8 singularity is unibranched. Similarly D_{even} compared with D_{odd}.

4. For the A type singularities the above classification is the complete list of indecomposable torsion free modules. For the other examples there exist

(finitely many) indecomposable modules of higher ranks; these are listed in [Greuel and Knörrer].

5. There are various ways of representing the classification diagrammatically. It is, for example, possible to draw the Auslander-Reiten quivers for these singularities, see e.g., [Yoshino].

§2.5 Some Homology

We gather together here several results and calculations which will be needed at various times later on. We want to understand firstly tensoring by $\tilde{\mathcal{O}}$, and secondly the space of self extensions of a module.

In general the tensor product of two torsion free $\mathcal{O}$-modules is not torsion free: it can sometimes be useful to have a bound on the dimension of the torsion submodule of such a product. In view of our applications we consider only the case of tensoring by $\tilde{\mathcal{O}}$. We make some basic observations in the two lemmas that follow.

Lemma 2.5.1. If M is a torsion free $\mathcal{O}$-module of rank n then

1. $\mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, M) \simeq \mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T_M)$, and

2. There is a short exact sequence

$$0 \longrightarrow \tilde{\mathcal{O}}^n \longrightarrow M \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \longrightarrow T_M \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \longrightarrow 0,$$

with $\ell(M) \le \dim_k(T_M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}) \le e\ell(M)$.

Proof. Consider the exact sequence

$$\sigma_1 : \qquad 0 \longrightarrow \mathcal{O}^n \longrightarrow M \longrightarrow T_M \longrightarrow 0$$

tensored with $\tilde{\mathcal{O}}$ over $\mathcal{O}$. We get:

$$0 \to \mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, M) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T_M) \xrightarrow{\phi} \tilde{\mathcal{O}}^n \longrightarrow M \otimes \tilde{\mathcal{O}} \longrightarrow T_M \otimes \tilde{\mathcal{O}} \longrightarrow 0.$$

Since $\tilde{\mathcal{O}}^n$ has no torsion the map ϕ must be zero, giving the isomorphism of **1.** and the short exact sequence of **2.** Now let $V = \{v_1, ..., v_{\ell(M)}\}$ be a basis for T_M over k. If $\{f_1, ..., f_e\}$ generates $\tilde{\mathcal{O}}$ over $\mathcal{O}$ then $B = \{v_i \otimes f_j | 1 \le i \le \ell(M), 1 \le j \le e\}$ generates $T_M \otimes \tilde{\mathcal{O}}$ over $\mathcal{O}$. But if $g \in \mathcal{O}$ then $g(v_i \otimes f_j) = gv_i \otimes f_j$

and gv_i can be written as a k-linear combination of elements of V, so B also generates $T_M \otimes \tilde{\mathcal{O}}$ over k. Hence $\dim_k(T_M \otimes \tilde{\mathcal{O}}_x) \leq e\ell(M)$. Equality is not obtained in general, but the canonical injection $T_M \longrightarrow T_M \otimes \tilde{\mathcal{O}}, v \longmapsto v \otimes 1$ implies $\dim_k(T_M \otimes \tilde{\mathcal{O}}_x) \geq \ell(M)$.

Lemma 2.5.2. The torsion submodule of $M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ is isomorphic to $\mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T'_M)$. In particular, using **2** above, this has dimension $\leq e\ell(M)$.

Proof. Consider the exact sequence

$$\sigma_2 : \qquad 0 \longrightarrow M \longrightarrow \mathcal{O}^n \longrightarrow T'_M \longrightarrow 0$$

tensored with $\tilde{\mathcal{O}}$ over $\mathcal{O}$. We get:

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T'_M) \longrightarrow M \otimes \tilde{\mathcal{O}} \xrightarrow{\phi} \tilde{\mathcal{O}}^n \longrightarrow T'_M \otimes \tilde{\mathcal{O}} \longrightarrow 0.$$

So torsion$(M \otimes \tilde{\mathcal{O}}_x) \subset \ker\phi$, since $\tilde{\mathcal{O}}^n$ is torsion free. But $\mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T'_M)$ is a finite dimensional submodule of $M \otimes \tilde{\mathcal{O}}$, so $\mathrm{Tor}_1^{\mathcal{O}}(\tilde{\mathcal{O}}, T'_M) \subset$ torsion$(M \otimes \tilde{\mathcal{O}}_x)$. The result follows by exactness.

We now turn our attention to some extension problems. A basic fact that we shall use without further comment is that one can calculate Ext groups after first taking completions: i.e., if M and N are $\mathcal{O}$-modules then $\hat{M} = M \otimes \hat{\mathcal{O}}$ and there is a canonical identification of $\mathcal{O}$-modules

$$\mathrm{Ext}^i_{\mathcal{O}}(M, N) \cong \mathrm{Ext}^i_{\hat{\mathcal{O}}}(\hat{M}, \hat{N}).$$

This follows from the fact that $\hat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$ and that Ext^i is finite dimensional over k (cf. [Matsumura]).

Our philosophy in what follows is that calculation of Ext is made a lot easier by judicious use of the dualising module.

Lemma 2.5.3. If M is torsion free then

$$\mathrm{Ext}^i_{\mathcal{O}}(M, \omega) = 0, \ \forall i \geq 1.$$

Proof. Note that for any module, N, $\mathrm{Ext}^i_{\mathcal{O}}(M, \omega) = 0$, $\forall i \geq 2$, since ω has injective dimension 1.

If M has rank n then since M is finite and torsion free there is an exact sequence

$$(*) \qquad 0 \longrightarrow M \longrightarrow \omega^n \longrightarrow T \longrightarrow 0$$

where $\dim_k T < \infty$. $\mathrm{Hom}(\,\cdot\,,\omega)$ applied to $(*)$ gives an exact sequence

$$\ldots \longrightarrow 0 = \mathrm{Ext}^1(\omega^n,\omega) \longrightarrow \mathrm{Ext}^1(M,\omega) \longrightarrow \mathrm{Ext}^2(T,\omega) = 0,$$

so $\mathrm{Ext}^1(M,\omega)$ is also zero.

Remark. The modules M for which $\mathrm{Ext}^1(M,\omega) = 0$ are the *maximal Cohen-Macaulay* modules. It is also not hard to show that every non-zero maximal Cohen-Macaulay module is torsion free (see e.g., [Yoshino]).

Lemma 2.5.4. If M is a finite torsion free module then

$$\mathrm{Ext}^i_{\mathcal{O}}(M,M) \simeq \mathrm{Ext}^i_{\mathcal{O}}(M^*,M^*), \ \forall i \geq 0.$$

Proof. Let

$$F^\bullet : \qquad \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

be a projective (i.e., free) resolution of M.

Then, dualising

$$F^{\bullet^*} : \qquad 0 \longrightarrow M^* \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow \ldots$$

is also exact, since $\mathrm{Ext}^i(M,\omega) = 0, \ \forall i \geq 1$. Let $G = \mathrm{Hom}(\,\cdot\,,M)$ and $H = \mathrm{Hom}(M^*,\,\cdot\,)$: then $F^{\bullet^*}$ is acyclic for H—i.e., $\mathrm{Ext}^i(M^*,F_j^*) = 0, \ \forall i,j \geq 1$ since each $F_j^* \simeq \omega^{N_j}$ for some N_j. Hence we can calculate $\mathrm{Ext}^\bullet(M,M)$ as the homology of $G(F^\bullet)$ and $\mathrm{Ext}^\bullet(M^*,M^*)$ as the homology of $H(F^{\bullet^*})$.

The result follows on noting that

$$H(F^{\bullet^*}) = G(F^\bullet)^{**} = G(F^\bullet).$$

In fact, we are mainly interested in $\mathrm{End}(M)$ and $\mathrm{Ext}^1(M,M)$. If M has rank 1 then the natural inclusion $\mathcal{O} \hookrightarrow \mathrm{End}(M)$ has cokernel of finite dimension:

Lemma 2.5.5. If M is a rank one torsion free module then

$$\dim_k(\mathrm{End}(M)/\mathcal{O}) = \dim_k \mathrm{Ext}^1(T'_M,M) - \tilde{\delta} + \ell(M).$$

Proof. Recall that T'_M is a module of k-dimension $\tilde{\delta} - \ell(M)$ defined by an exact sequence

$$\sigma_2 \qquad 0 \longrightarrow M \longrightarrow \mathcal{O} \longrightarrow T'_M \longrightarrow 0.$$

Hom($\cdot$, M) gives a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{End}(M) \longrightarrow \mathrm{Ext}^1(T'_M, M) \longrightarrow 0.$$

These can be combined, using the inclusion $\mathcal{O} \hookrightarrow \mathrm{End}(M)$ induced by homothety, into a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathcal{O} & \longrightarrow & T'_M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathrm{End}(M) & \longrightarrow & \mathrm{Ext}^1(T'_M, M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \dfrac{\mathrm{End}(M)}{\mathcal{O}} & \longrightarrow & \dfrac{\mathrm{Ext}^1(T'_M, M)}{T'_M} & \longrightarrow & 0
\end{array}$$

and, using the isomorphism on the bottom row, and calculating dimensions from the last column the result follows.

Remarks.

1. If M is an overring of $\mathcal{O}$ then $\mathrm{Hom}_{\mathcal{O}}(M, M) = \mathrm{End}_M(M) = M$.

2. For rank 1 modules, which is the only case where we actually carry out the calculations, $\mathrm{Ext}^1(T'_M, M)$ is most easily computed using this lemma; for higher ranks finding a sequence analagous to σ_2 and computing the cokernel after applying Hom($\cdot$, M) gives a reasonably efficient method. (Alternatively one can generalise the lemma.)

2.5.6 Procedure for calculating $\mathrm{Ext}^1_{\mathcal{O}}(M, M)$.

Of course, this could be done by finding a projective resolution of M: however, the following method is much quicker in practice. Suppose that M is a torsion free module of rank n. We can find an embedding $M \subset \omega^n$ giving an exact sequence

$$(*) \qquad 0 \longrightarrow M \longrightarrow \omega^n \longrightarrow T \longrightarrow 0$$

where $\dim_k T < \infty$. Applying $\mathrm{Hom}(M, \cdot\,)$ gives an exact sequence

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Hom}(M,M) \xrightarrow{\phi} \mathrm{Hom}(M,\omega^n) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}(M,T) \longrightarrow \mathrm{Ext}^1(M,M) \longrightarrow \mathrm{Ext}^1(M,\omega) = 0.$$

Thus

$$\dim_k \mathrm{Ext}^1(M,M) = \dim_k \mathrm{Hom}(M,T) - \dim_k(\mathrm{coker}(\phi)).$$

In practice we can often choose $(*)$ so that ϕ is an isomorphism.

Corollary 2.5.7. Suppose $\mathcal{O}$ is Gorenstein and M is an overring: $\mathcal{O} \subset M \subset \tilde{\mathcal{O}}$. Then $\mathrm{Ext}^1(M,M) = \mathrm{Hom}(M^*, \mathcal{O}/M^{-1})$.

Proof. Recall (2.5.4) that $\mathrm{Ext}^1(M,M) = \mathrm{Ext}^1(M^*,M^*)$ and that $M^* \simeq M^{-1}$ if $\mathcal{O}$ is Gorenstein. For M^{-1} there is an exact sequence

$$0 \longrightarrow M^{-1} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/M^{-1} \longrightarrow 0.$$

On applying $\mathrm{Hom}(M^{-1}, \cdot\,)$ the first map (ϕ, above in the sequence $(\dagger)$) is the isomorphism

$$\mathrm{Hom}(M^{-1}, M^{-1}) \simeq (M^{-1})^{-1}.$$

Remark. So, in the above situation calculating $\mathrm{Ext}^1(M,M)$ reduces to calculating $\mathrm{Hom}(N,T)$ for N a torsion free module and T a torsion module. Clearly, a homomorphism $N \longrightarrow T$ is determined by specifying the images of a generating set for N. So $\dim .\mathrm{Hom}(N,T) \le r(N) \cdot \dim T$. On the other hand, relations between the generators of N might restrict this dimension. In fact, it is sufficient to look at relations modulo the annihilator of T. More precisely, if $A = \mathrm{Ann}(T)$ then, by $\mathcal{O}$-linearity, $A.N$ is in the kernel of any homomorphism $N \longrightarrow T$. Thus

$$\mathrm{Hom}_{\mathcal{O}}(N,T) = \mathrm{Hom}_{\mathcal{O}}(N/A.N, T) = \mathrm{Hom}_{\mathcal{O}/A}(N/A.N, T).$$

This reduces the calculation of $\dim_k \mathrm{Hom}(N,T)$ to a finite dimensional problem: list (up to the action of $\mathcal{O}$) all relations between generators of N giving non-trivial relations between the corresponding generators of $N/A.N$, and, for each one, count the contribution to $\dim_k \mathrm{Hom}(N,T)$. In practice, at least for our examples, this is straightforward.

We will now carry out some of these calculations:—for the rank 1 modules over the local rings of the simple plane curve singularities of §2.4.

Definition. When M is a rank one torsion free module define

$$\chi_1(M) = \dim_k \operatorname{Ext}^1_{\mathcal{O}}(M, M) - \dim_k(\operatorname{End}(M)/\mathcal{O}).$$

Theorem 2.5.8. If M is a rank 1 module over the local ring of a simple plane curve singularity then

$$\chi_1(M) = i = \operatorname{index}(M).$$

Proof. This, necessarily, proceeds case by case, although the above lemmas significantly reduce the amount of calculation required. Firstly we recall that every rank 1 module over a simple curve singularity is either isomorphic to an overring, or, it is dual to such a module: In particular in all cases, using the notation of (2.4.2), we have

$$\operatorname{End}(M_i) = M_i = \operatorname{End}(M_i^*), \ \operatorname{End}(N_i) = N_i,$$

$$\text{so } \dim_k(\operatorname{End}(M)/\mathcal{O}) = \text{ index of } M.$$

It remains to show that $\dim_k \operatorname{Ext}^1(M, M) = 2i(M)$ for each M. Since $\operatorname{Ext}^1(M, M) = \operatorname{Ext}^1(M^*, M^*)$ (2.5.4) we need only check this for one of each pair (M, M^*). Note that for the case M_0 (the free module) the result is trivial.

Case i: A_n

Lemma 2.5.9. For the modules M_k, M_j over a ring of type A_n

$$\dim_k \operatorname{Ext}^1(M_k, M_j) = 2\min\{k, j\}.$$

Proof. We consider the cases n odd/n even in parallel.

Write M_j as an ideal:

$$M_j \simeq I_j = \begin{cases} \langle (t^j, t^j), (t^\delta, 0) \rangle \\ \langle t^{2j}, t^{2\delta+1} \rangle \end{cases}.$$

Write T_j for the quotient:

$$(\dagger) \qquad 0 \longrightarrow I_j \longrightarrow \mathcal{O} \longrightarrow T_j \longrightarrow 0.$$

T_j has dimension j—a basis is $\{b_0, \ldots, b_{j-1}\}$ where

$$b_i = \begin{cases} (t^i, t^i) \\ t^{2i} \end{cases}, 0 \leq i \leq \delta.$$

Now $\mathrm{Hom}(I_k, \cdot\,)$ on (†) gives:

$$0 \longrightarrow \mathrm{Hom}(I_k, I_j) \longrightarrow \mathrm{Hom}(I_k, \mathcal{O}) \longrightarrow \mathrm{Hom}(I_k, T_j) \longrightarrow \mathrm{Ext}^1(I_k, I_j) \longrightarrow 0$$

Note that $\mathrm{Hom}(I_k, I_j) \simeq M_i$, where $i = \min\{j, k\}$; $\mathrm{Hom}(I_k, I_j)$ contains 1 if $k \geq j$, otherwise the element of smallest degree is (t^{j-k}, t^{j-k}) (resp. $t^{2(j-k)}$). Hence the first map is an isomorphism if $k \geq j$, otherwise the cokernel has dimension $j - k$.

Now to calculate $\mathrm{Hom}(I_k, T_j)$ we follow the advice above: $\mathrm{Ann}T_j = I_j$, if $k \geq j$ then $I_k \subset I_j$, and since there are no relations between the generators of I_k mod I_k^2 there will be none mod $I_k.I_j$. It follows that in this case $\dim_k \mathrm{Hom}(I_k, T_j) = 2j$. However, if $k < j$ then we have $(t^k, t^k)\cdot(t^\delta, 0) = 0$ (resp. $t^{2k}\cdot t^{2\delta+1} = 0$) in $I_k/I_k.I_j$. It follows that under any homomorphism the image of the second generator of I_k on the subspace spanned by $\{b_0, \ldots, b_{j-k-1}\}$ must be 0, and $\dim \mathrm{Hom}(I_k, T_j) = j + k$.

Thus for $k \geq j$, $\dim \mathrm{Ext}^1(I_k, I_j) = 2j$, and for $k < j$, $\dim \mathrm{Ext}^1(I_k, I_j) = j + k - (j - k) = 2k$.

Case ii: D_n.

We present the details for the case 'n even', the case 'n odd' being essentially identical.

For $D_{2\delta-2}$ we have $\mathcal{O} = k[[(t,t,0), (t^{\delta-2}, 0, t)]]$, and the conductor C is the ideal $\langle (t^{\delta-1}, 0, 0),\ (0, t^{\delta-1}, 0),\ (0, 0, t^2)\rangle$

Taking the module M_i^* we find that this is isomorphic to the ideal

$$I_i = \langle (t^i, t^i, 0), (t^{\delta-2}, 0, t)\rangle = M_i^{-1}.$$

Thus $S_i = \mathcal{O}/I_i$ has a basis

$$\{(1,1,1),\ (t,t,0), \ldots, (t^{i-1}, t^{i-1}, 0)\} = \{b_0, \ldots, b_{i-1}\}.$$

Note that $\mathrm{Ann}(S_i) = I_i$ and there are no relations between the generators of I_i

$\bmod(I_i^2)$. Hence $\mathrm{Hom}(M_i^*, S_i) = \mathrm{Ext}^1(M_i, M_i)$ (2.5.7) has dimension $2i$, which deals with the modules of types M_i, M_i^* for $1 \le i \le \delta - 2$.

$N_i \simeq J_j = \langle (0,0,t^2),\ (t^{\delta-1},0,0),\ (t^{i-1},t^{i-1},0)\rangle$; $M_\delta \simeq C = J_\delta$. $T_i = \mathcal{O}/J_i$ has a basis

$$\{(1,1,1),\ (t,t,0),\ldots,(t^{i-2},t^{i-2},0),\ (t^{\delta-2},0,t)\} = \{b_0,\ldots,b_{i-2},b_{i-1}\}.$$

In $\mathrm{Hom}(N_i, T_i)$ we have the relation

$$(t,t,0)\cdot(0,0,t^2) = 0 \Rightarrow (0,0,t^2) \mapsto 0 \text{ on the subspace } < b_0, b_1, \ldots, b_{i-3} > .$$

Also

$$(t^{\delta-2},0,t)\cdot(t^{i-1},t^{i-1},0) = (t^i,t^i,0)\cdot(t^{\delta-1},0,0)$$

$$\Rightarrow (t^{i-1},t^{i-1},0),(t^{\delta-1},0,0) \mapsto 0 \text{ on } < b_0 > .$$

So $\dim_k \mathrm{Hom}(N_i, T_i) = 3i - (i-2) - 2 = 2i$ as required.

The remaining cases are $M_{\delta-1}^j$, $(j = 1,2)$. Take $j = 2$: $M_{\delta-1}^2 \simeq J = \langle (t^{\delta-2},0,t),(0,t^{\delta-1},0)\rangle$. The annihilator of the quotient $\mathcal{O}/J$ is J, and there are no relations mod J^2. Thus, since $\mathcal{O}/J$ has dimension $\delta - 1$, $\dim_k \mathrm{Ext}^1(M_{\delta-1}^2, M_{\delta-1}^2) = 2(\delta - 1)$.

Case iii: E_6. $C = \langle t^6,\ t^7,\ t^8\rangle$. To deal with the cases M_1, M_1^* notice that $M_1^{-1} = m$ so

$$\mathrm{Hom}(M_1^*, \mathcal{O}/M_1^{-1}) = \mathrm{Hom}(M_1^*, k)$$

which clearly has dimension 2 since M_1^* is generated by 2 elements.

For M_2 we find that

$$M_2^{-1} = I = \langle t^4,\ t^6\rangle.$$

$T = \mathcal{O}/I$ has a basis $\{t^4, t^6\}$, and it is easily seen that $\mathrm{Hom}(I,T)$ has dimension 4.

Finally, M_3: $M_3 \simeq C$ and $\mathcal{O}/C$ has a basis

$$\{1,\ t^3,\ t^4\} = \{b_0, b_1, b_2\}$$

The relations

$$t^4 \cdot t^6 = t^3 \cdot t^7 \text{ and}$$

$$t^4 \cdot t^7 = t^3 \cdot t^8$$

imply that under any map $M_3 \longrightarrow \mathcal{O}/C$ the images of all 3 generators on $< b_0 >$ must be zero. Thus $\dim_k \mathrm{Hom}(M_3, \mathcal{O}/C) = 6$.

Case iv: E_7 **and** E_8. We present details for E_7. Here $C = \langle (t^5,0),\ (t^6,0),\ (0,t^3)\rangle$. As for E_6 the cases M_1, M_1^* follow easily from the fact that $M_1^{-1} = m$.

$M_2^* \simeq I_2 = \langle (t^4,t),(t^3,0)\rangle$. $T_2 = \mathcal{O}/I$ has a basis $\{(1,1),\ (t^2,t)\}$ and $\mathrm{Hom}(M_2^*, T_2)$ has dimension 4.

$M_3 \simeq I_3 = \langle (t^4,t^2),\ (t^5,0),\ (t^6,0)\rangle$. $T_3 = \mathcal{O}/I_3$ has a basis

$$\{(1,1),\ (t^2,t),\ (t^3,0)\} = \{b_0, b_1, b_2\}.$$

From $(t^3,0)\cdot(t^4,t^2) = (t^2,t)\cdot(t^3,0)$ and $(t^3,0)\cdot(t^5,0) = (t^2,t)\cdot(t^6,0)$ we see that all 3 generators map to zero on $< b_0 >$.

$N_3 \simeq J_3 = \langle (t^3,0),\ (0,t^3)\rangle$. $S_3 = \mathcal{O}/J_3$ has a basis $\{(1,1),\ (t^2,t),\ (t^4,t^2)\}$. $\mathrm{Hom}(N_3, S_3)$ has dimension 6.

$M_4 \simeq C$, $T_4 = \mathcal{O}/C$ has a basis

$$\{(1,1),\ (t^2,t),\ (t^3,0),\ (t^4,t^2)\} = \{b_0, b_1, b_2, b_3\}$$

There are relations

$$(t^3,0)\cdot(t^5,0) = (t^2,1)\cdot(t^6,0) \text{ and}$$
$$(t^3,0)\cdot(0,t^3) = 0.$$

So $(t^6,0)$ must have zero image on $< b_0, b_1 >$ and the other 2 generators have zero image on $< b_0 >$; whence $\dim \mathrm{Hom}(M_4, T_4) = 4.3 - 4 = 8$, thus completing the proof of (2.5.8).

CHAPTER 3

CATEGORIES OF SHEAVES AND FUNCTORS

The basic question underlying the discussion of this chapter is 'what is the relationship between sheaves on X and sheaves on its normalisation, $\tilde{X}$?' Because $\tilde{X}$ has lower genus than X there are far fewer sheaves on $\tilde{X}$, and, so, to get something like a bijection one needs to consider sheaves on $\tilde{X}$ with some extra structure. Thus, after making some remarks on torsion free sheaves on X (§3.1) and the functors π_* and π^* (§3.2), we introduce *parabolic Modules* in §3.3. To each parabolic Module one can associate a torsion free sheaf on X, and in §3.4 we look at how to set things up so that every locally (resp. torsion) free sheaf on X is represented by some parabolic Module. §3.5 considers the question of stability, which is essential in constructing the moduli spaces of chapter 4. A key part of this chapter concerns the behaviour of 'degree' under various functors.

§3.1 Torsion Free Sheaves

This section is about how it is possible to 'globalise' certain statements about torsion free modules over $\mathcal{O}_x$—often one can carry out constructions locally and then glue together the results. The following propositions illustrate this. With the exception of (3.1.8/9) there is nothing essentially new in this section; however, for lack of convenient references we provide fairly complete details.

Proposition 3.1.1. Let x be a singular point of X, and let $\mathcal{O}_x{}' \subset \tilde{\mathcal{O}}_x$ be an overring of $\mathcal{O}_x$. Then:

1. There exists a subsheaf $\mathcal{O}'$ of $\mathcal{K} = K \times X$ such that $\mathcal{O}'|_{X-x} = \mathcal{O}_X|_{X-x}$ and the stalk of $\mathcal{O}'$ at x is $\mathcal{O}_x{}'$;

2. There exists an integral projective curve X' and a natural finite surjective morphism $\pi : X' \longrightarrow X$ such that the restriction of π to $X - \pi^{-1}(x)$ is an isomorphism and, if $\mathcal{O}_{X'}$ is the structure sheaf of X', then $\pi_*\mathcal{O}_{X'}$ is the sheaf $\mathcal{O}'$ of 1.
   Further, the genus of X' is given by $g(X') = g(X) - \ell(\mathcal{O}_x{}')$.

Proof.

1. For any affine open set, U of X, define $\mathcal{O}'(U)$ as a subring of $\mathcal{K}(U) = \mathrm{H}^0(U, \mathcal{K})$ by

$$\mathcal{O}'(U) = \{f \in \mathcal{K}(U) \mid f|_{U-x} \in \mathcal{O}(U-x),\ \varinjlim f \in \mathcal{O}_x{}'\}.$$

   By the direct limit of f we mean the image of f under the canonical map $\mathcal{K}(U) \longrightarrow K$ arising from taking the direct limit over affine open subsets of U containing x. Glue together the $\mathcal{O}'(U)$ for various U to form a subsheaf, $\mathcal{O}'$, of $\mathcal{K}$. Then $\mathcal{O}'$ equals $\mathcal{O}_X$ away from x and $(\mathcal{O}')_x = \mathcal{O}'_x$ since $\mathcal{K}$ is flasque. Note that $\mathcal{O}'$ is certainly coherent since $\mathcal{O}'_x \subset \tilde{\mathcal{O}}_x \Rightarrow \mathcal{O}'_x$ is finitely generated over $\mathcal{O}_x$, so $\mathcal{O}'$ is a sheaf with the required properties.

2. (Compare [Hartshorne II Ex.5.17].) For an affine open U write $A = H^0(U, \mathcal{O}_X)$, $A' = H^0(U, \mathcal{O}')$ and U' for the affine curve associated to A'. The inclusion $A \subset A'$ gives a finite surjective morphism $\pi_U : U' \longrightarrow U$. Glueing together pieces we obtain a curve X' with structure sheaf $\mathcal{O}'_X$ such that if π is the resulting map $X' \longrightarrow X$ then $\mathcal{O}' = \pi_* \mathcal{O}_{X'}$. X' is clearly integral and proper over k, and so is projective (indeed, any one dimensional scheme which is proper over an algebraically closed field is projective—see [Hartshorne II.2 Ex.5.8]). The formula for the genus follows from computing the cohomology associated to the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}' = \pi_* \mathcal{O}_{X'}$ (cf. §3.2.1/2).

It is clearly possible to generalise the above in order to deal with several singularities at once. Suppose $\{x_1, \dots x_s\}$ is the set of all singular points of X. Then, given a set of overrings, $\mathcal{O}'_{x_i}$, of the $\mathcal{O}_{x_i}$, there exists a corresponding sheaf, $\mathcal{O}'$, and a corresponding curve X' together with a canonical map $\pi : X' \longrightarrow X$. We call such a curve a ***partial normalisation*** of X. In particular, if $\mathcal{O}'_{x_i} = \tilde{\mathcal{O}}_{x_i}$ for each i then X' is the normalisation, $\tilde{X}$, of X.

In view of (2.2.1) we can also generalise this to construct torsion free sheaves locally isomorphic to any given set of torsion free modules having the same rank; more precisely:

Proposition 3.1.2. Suppose, that for each singular point x_i of X, we have a rank n torsion free module, M_i, over $\mathcal{O}_{x_i}$; then there exists a torsion free sheaf $\mathcal{F}$ on X with

$$\mathcal{O}_X^n \subset \mathcal{F} \subset \pi_* \mathcal{O}_{\tilde{X}}^n$$

and

$$\mathcal{F}_{x_i} \simeq M_i \qquad \forall i = 1, \ldots, s.$$

Further, $\dim_k(\mathcal{F}/\mathcal{O}_X{}^n) = \sum_i \ell(M_i)$.

Proof. By (2.2.1) we can assume that $\mathcal{O}_{x_i}^n \subset M_i \subset \tilde{\mathcal{O}}_{x_i}^n$ for each i. Then, as in the proof of (3.1.1(**1**)), one can construct a subsheaf of $\mathcal{K}^n$ with the required properties.

So every torsion free sheaf of rank n is *locally* isomorphic to one between $\mathcal{O}_X^n$ and $\mathcal{O}_{\tilde{X}}^n$. In rank 1, since isomorphisms of modules are achieved by multiplying by elements of K^*, to say that $\mathcal{F}$ and $\mathcal{F}'$ are locally isomorphic is precisely to say that there is an invertible sheaf $\mathcal{L}$ such that $\mathcal{F} = \mathcal{F}' \otimes \mathcal{L}$. Hence:

Corollary 3.1.3. Up to tensoring with an invertible sheaf every rank 1 torsion free sheaf is isomorphic to a unique sheaf $\mathcal{F}$ with

$$\mathcal{O}_X \subset \mathcal{F} \subset \pi_* \mathcal{O}_{\tilde{X}}.$$

In higher ranks, whilst this is no longer true, we can still find global analogues of the sequences σ_1, σ_2 of §2.2. For a sheaf $\mathcal{F}$ define torsion sheaves $T_{\mathcal{F}} = \oplus_i T_{\mathcal{F}_{x_i}}$ and $T'_{\mathcal{F}} = \oplus_i T'_{\mathcal{F}_{x_i}}$.

Proposition 3.1.4. If $\mathcal{F}$ is a torsion free sheaf on X then there are locally free sheaves $\mathcal{E}, \mathcal{E}'$ on X and short exact sequences:

$$\Sigma 1. \qquad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow T_{\mathcal{F}} \longrightarrow 0;$$

$$\Sigma 2. \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}' \longrightarrow T'_{\mathcal{F}} \longrightarrow 0.$$

Proof. By (2.2.1/2) we can find, for each i, modules $M_i = \phi_i \mathcal{F}_{x_i}$ with $\mathcal{O}_{x_i}^n \subset M_i \subset \tilde{\mathcal{O}}_{x_i}^n$, where the ϕ_i are automorphisms of K^n. Thus

$$\phi_i^{-1} \mathcal{O}_{x_i}^n \subset \mathcal{F}_{x_i} \subset \phi_i^{-1} \tilde{\mathcal{O}}_{x_i}^n.$$

We can extend this to some open sets, U_i, around each point x_i:

$$\phi_i^{-1}\mathcal{O}_X^n(U_i) \subset \mathcal{F}(U_i) \subset \phi_i^{-1}\pi_*\mathcal{O}_{\tilde{X}}^n(U_i).$$

We can assume that the U_i cover X (if not: repeat the construction at the 'missing' points). $\phi_i \cdot \phi_j^{-1}$ gives us glueing functions on each $U_i \cap U_j$ and so allows us to construct the locally free sheaf $\mathcal{E}$ required for $\Sigma 1$. The other part is similar.

The remainder of this section concerns the Hom functors; we generalise the results of chapter 2. The following lemma is useful.

Lemma 3.1.5. If $\mathcal{E}$ and $\mathcal{F}$ are torsion free sheaves on X then there is a natural isomorphism of vector spaces

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathrm{H}^1(X, \mathbf{Hom}(\mathcal{E},\mathcal{F})) \oplus \bigoplus_{x\in X} \mathrm{Ext}^1_{\mathcal{O}_x}(\mathcal{E}_x, \mathcal{F}_x).$$

In particular, if $\mathcal{E}$ is locally free then

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathrm{H}^1(X, \mathbf{Hom}(\mathcal{E},\mathcal{F})).$$

Proof. This is a consequence of the degeneration of the local-global spectral sequence for Ext when $\dim X = 1$ (see [Godement II,7.3.3]).

Duality. Recall that the dual sheaf of $\mathcal{F}$ is denoted $\mathcal{F}^* = \mathbf{Hom}(\mathcal{F}, \omega_X)$, so, e.g., $\omega_X = (\mathcal{O}_X)^*$. For any non-zero torsion free modules N and M over a domain A there is a canonical injection $\eta : M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,N),N)$ given by $\eta : m \mapsto \phi_m$ where ϕ_m is defined by $\phi_m(f) = f(m)$. Similarly, we get a canonical injection of sheaves $\eta : \mathcal{E} \longrightarrow \mathbf{Hom}(\mathbf{Hom}(\mathcal{E},\mathcal{F}),\mathcal{F})$ for each pair of torsion free sheaves on X. The following proposition, that in the case $\mathcal{F} = \omega_X$ this map is an isomorphism, involves showing that the localisation of the dualising sheaf at a point really is a dualising module.

Proposition 3.1.6. For any torsion free sheaf $\mathcal{F}$ of rank n on X

1. The canonical map $\eta : \mathcal{F} \hookrightarrow \mathcal{F}^{**}$ is an isomorphism;

2. $\deg \mathcal{F}^* = n(2g-2) - \deg \mathcal{F}$.

Proof. For a general singular curve, when ω_X need not be locally free, this is slightly more involved than one might expect.

1. From the duality theorem (1.1.1) we have

$$\mathrm{H}^0(\mathcal{F}) \simeq \mathrm{Ext}^1(\mathcal{F},\omega_X)^\vee.$$

Applying this to $\mathcal{F} = \mathcal{O}_X$ and using (3.1.5) gives $\mathrm{h}^1(\omega_X) = 1$.
It is enough to prove that η_x is an isomorphism for each $x \in X$, i.e., that ω_x is a canonical ideal for $\mathcal{O}_x$, or, by (2.1.4), that $V = \mathrm{Ext}^1_{\mathcal{O}_x}(k,\omega_x) = k$ at each point x. Note that $V = \mathbf{Ext}^1_{\mathcal{O}_X}(k_x,\omega_X)$, where k_x denotes the 1-dimensional torsion sheaf supported at x. Let $\mathcal{I}_x$ be the ideal sheaf of x; consider the exact sequence

$$0 \longrightarrow \mathcal{I}_x \longrightarrow \mathcal{O}_X \longrightarrow k_x \longrightarrow 0.$$

Dualising gives

$$0 \longrightarrow \omega_X \longrightarrow \mathcal{I}_x^* \longrightarrow V \longrightarrow 0.$$

and, taking cohomology:

$$0 \longrightarrow \mathrm{H}^0(X,\omega_X) \longrightarrow \mathrm{H}^0(X,\mathcal{I}_x^*) \longrightarrow$$

$$\mathrm{H}^0(X,V) \longrightarrow \mathrm{H}^1(X,\omega_X) \longrightarrow \mathrm{H}^1(X,\mathcal{I}_x^*) \longrightarrow 0.$$

Now $\mathrm{H}^0(X,\mathcal{I}_x) = 0$ and so, by Riemann-Roch, $\mathrm{h}^1(X,\mathcal{I}_x) = g$. Hence, $\mathrm{h}^0(X,\mathcal{I}_x^*) = g$ by duality and since, also, $\mathrm{h}^0(X,\omega) = g$ the first map in this long exact sequence must be an isomorphism. So, using the fact that $\mathrm{h}^1(X,\omega) = 1$ we must have $\dim V = \mathrm{h}^0(X,V) \le 1$. The result follows on noticing that $\dim V = 0$ is not possible—for example consider an exact sequence

$$0 \longrightarrow M \longrightarrow \omega_x \longrightarrow k \longrightarrow 0,$$

for M a torsion free module. Applying $\mathrm{Hom}(k,\ \cdot\ )$ to this sequence shows that k is a subspace of V.

2. Using Riemann-Roch and duality we have the two equations

$$h^0(\mathcal{F}) - h^0(\mathcal{F}^*) = \deg\mathcal{F} - n(g-1),$$

$$h^0(\mathcal{F}^*) - h^0(\mathcal{F}^{**}) = \deg\mathcal{F}^* - n(g-1).$$

From **1.** we know that $\mathcal{F} = \mathcal{F}^{**}$, so, adding these equations:

$$2n(g-1) = \deg \mathcal{F} + \deg \mathcal{F}^{*} \quad \text{as required.}$$

The following lemma is standard.

Lemma 3.1.7. If $\mathcal{F}$ is a rank n torsion free sheaf and if $\mathcal{L}$ is an invertible sheaf on X then

$$\textbf{1.} \qquad \deg(\mathcal{F} \otimes \mathcal{L}) = \deg \mathcal{F} + n \deg \mathcal{L}.$$

If $\mathcal{E}$ is a locally free sheaf then $\mathbf{Hom}(\mathcal{E}, \mathcal{F}) \simeq \mathcal{E}^{\vee} \otimes \mathcal{F}$ where $\mathcal{E}^{\vee} = \mathbf{Hom}(\mathcal{E}, \mathcal{O}_X)$.

$$\textbf{2.} \qquad \deg \mathcal{E}^{\vee} = -\deg \mathcal{E} \text{ and } \deg \mathcal{E}^{\vee} \otimes \mathcal{E} = 0.$$

Proof. For **1** cf. [Newstead]; in **2** to prove the first statement take determinants, whilst the second follows from **1** using induction on the rank.

Our basic method in dealing with torsion free sheaves is to try and reduce a given problem to a purely local problem. The following, together with (3.1.5), shows that, in the case when $\mathcal{F}$ is *simple*—i.e., when $\mathrm{H}^0(X, \mathbf{End}(\mathcal{F})) = k$— we can reduce the calculation of $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ to a local problem that was dealt with in §2.5.

Proposition 3.1.8. If $\mathcal{F}$ is simple of rank n then

$$\mathrm{h}^1(X, \mathbf{End}(\mathcal{F})) = n^2(g-1) + 1 + \sum_{x \in X} \left(n(n\tilde{\delta}(x) - \ell(\mathcal{F}_x)) - e_{1_x} \right),$$

where $e_{1_x} = \dim \mathrm{Ext}^1_{\mathcal{O}_x}(T'_{\mathcal{F}_x}, \mathcal{F}_x)$.

Proof. $\mathcal{F}$ has an embedding

$$(*) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow T'_{\mathcal{F}} \longrightarrow 0$$

where $\mathcal{E}$ is locally free (3.1.4). Note that

$$\chi(\mathcal{E}) - \chi(\mathcal{F}) = \chi(T'_{\mathcal{F}}) = \sum_x \left(n\tilde{\delta}(x) - \ell(\mathcal{F}_x) \right).$$

Applying $\mathbf{Hom}(\,\cdot\,, \mathcal{F})$ to $(*)$ gives a short exact sequence of sheaves

$$0 \longrightarrow \mathbf{Hom}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathbf{Hom}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathbf{Ext}^1(T'_{\mathcal{F}}, \mathcal{F}) \longrightarrow 0.$$

Now taking cohomology gives a long exact sequence

$$\mathrm{H}^0(X,\mathbf{Hom}(\mathcal{E},\mathcal{F})) \hookrightarrow \mathrm{H}^0(X,\mathbf{Hom}(\mathcal{F},\mathcal{F})) \longrightarrow \mathrm{H}^0(X,\mathbf{Ext}^1(T'_{\mathcal{F}},\mathcal{F}_x))$$
$$\longrightarrow \mathrm{H}^1(X,\mathbf{Hom}(\mathcal{E},\mathcal{F})) \longrightarrow \mathrm{H}^1(X,\mathbf{Hom}(\mathcal{F},\mathcal{F})) \longrightarrow 0.$$

Since $\mathcal{F}$ is simple the second term has dimension 1, whilst the first is zero (—a non-zero map $\mathcal{E} \longrightarrow \mathcal{F}$ would split the exact sequence $(*)$, since $\mathcal{F}$ is simple).

The Riemann-Roch formula applied to $\mathbf{Hom}(\mathcal{E},\mathcal{F}) \simeq \mathcal{E}^\vee \otimes \mathcal{F}$ gives

$$\begin{aligned}\mathrm{h}^1(\mathcal{E}^\vee \otimes \mathcal{F}) &= n^2(g-1) - \deg(\mathcal{E}^\vee \otimes \mathcal{F})\\ &= n^2(g-1) + \sum_x \left(n^2\tilde{\delta}(x) - n\ell(\mathcal{F}_x)\right)\end{aligned}$$

(working out $\deg(\mathcal{E}^\vee \otimes \mathcal{F})$ from $(*)$ tensored with $\mathcal{E}^\vee$, using (3.1.7)). So the result follows on taking the alternating sum of dimensions in the long exact sequence above.

Corollary 3.1.9. If $\mathcal{F}$ has rank 1 then it is simple (indeed, it is stable—see §3.5) and we can simplify the result using (2.5.5); combining with (3.1.5) we find that

$$\dim \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = g + \sum_{x\in X} \chi_1(\mathcal{F}_x)$$
$$= g + \sum_{x\in X} \left(\dim \mathrm{Ext}^1_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{F}_x) - \dim(\mathrm{End}_{\mathcal{O}_x}\mathcal{F}_x)/\mathcal{O}_x\right).$$

Note that, by (2.5.8) we have thus calculated the dimension of this vector space for any rank 1 torsion free sheaf over a curve having only simple plane curve singularities: in this case we have

$$\dim \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = g + \sum_{x\in X} i(\mathcal{F}_x).$$

§3.2 Functors Induced From $\pi : \tilde{X} \longrightarrow X$

Write $Coh(X)$ for the category of coherent sheaves on X and $TFS(X)$ for the category of torsion free (coherent) sheaves on X. We write $Vect(X)$ for the category of vector bundles on X, or, equivalently the category of locally free sheaves. The terms "locally free sheaf" and "vector bundle" will often be used interchangeably. We have $Vect(X) \subset TFS(X)$ and $Vect(\tilde{X}) = TFS(\tilde{X})$. Note that if $\mathcal{F}$ is a torsion free sheaf then the restriction, $\mathcal{F}|_{X_{\text{reg}}}$, of $\mathcal{F}$ to the set of regular points of X is locally free. By a *subbundle* of a torsion free sheaf we will always mean a subsheaf such that the quotient is torsion free. A remark on notation: we tend to use capital roman letters (E, F, etc.) for locally free sheaves on $\tilde{X}$, and script letters ($\mathcal{E}$, $\mathcal{F}$, etc.) for torsion free sheaves on X.

π_* gives a functor $Coh(\tilde{X}) \longrightarrow Coh(X)$ which is exact—as the higher direct image sheaves of π_* vanish because the fibres of π are zero dimensional. A consequence of this is that, for any $\mathcal{F} \in Coh(\tilde{X})$, there are natural isomorphisms

$$\mathrm{H}^i(\tilde{X},\mathcal{F}) \simeq \mathrm{H}^i(X,\pi_*\mathcal{F}) \ \ \forall i \geq 0$$

—see [Hartshorne III Ex.8.1]. This enables us to calculate the effect of π_* on degree:

Lemma 3.2.1. If $E \in Coh(\tilde{X})$ has rank n then

$$\deg \pi_*E = \deg E + n\delta(X).$$

Proof. From the above $\chi(\pi_*E) = \chi(E)$ and so, by Riemann-Roch

$$\begin{aligned}\deg \pi_*E &= \deg E + n\left(g(X) - g(\tilde{X})\right)\\ &= \deg E + n\delta(X).\end{aligned}$$

If $\mathcal{F}$ is a torsion free sheaf of rank $n \geq 1$ and degree d we define the *slope*, μ of $\mathcal{F}$ to be $\mu = d/n$. So the above lemma says that π_* increases slope by $\delta(X)$.

In exactly the same way one proves the following.

Corollory 3.2.2. If $\pi' : X' \longrightarrow X$ is any partial normalisation of X then

$$\pi'_* : Coh(X') \longrightarrow Coh(X) \text{ is exact}$$

and increases slope by

$$\delta'(X) = g(X) - g(X').$$

Remark. If T is a torsion sheaf on $\tilde{X}$ (or, more generally, on any partial normalisation of X) then $\deg \pi_*T = \deg T = \dim T$. Suppose $Supp(T) \subset \pi^{-1}(x)$ for some $x \in X$; then as an $\mathcal{O}_x$- module π_*T is simply T with the

$\tilde{\mathcal{O}}_x$-action restricted to an $\mathcal{O}_x$-action. Hence, we tend to abuse notation and write T for $\pi_* T$.

If $\mathcal{F}$ is a torsion free sheaf on X then $\pi^*\mathcal{F}$ is not torsion free unless $\mathcal{F}$ is locally free. However, there is an induced functor $\tilde{\pi}^* = \pi^*/\text{torsion} : TFS(X) \longrightarrow Vect(\tilde{X})$.

Proposition 3.2.3. Relating π_* and π^* we have two standard facts:

1. The adjoint property: A natural isomorphism of groups
$$\mathrm{Hom}_{\mathcal{O}_{\tilde{X}}}(\pi^*\mathcal{F}, E) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \pi_* E),$$
for a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, and a sheaf of $\mathcal{O}_{\tilde{X}}$-modules E. This gives us canonical maps $\mathcal{F} \longrightarrow \pi_*\pi^*\mathcal{F}$, and $\pi^*\pi_* E \longrightarrow E$. These will be isomorphisms when restricted to X_{reg}, and $\pi^{-1}(X_{\text{reg}})$ respectively.

2. A natural isomorphism of sheaves
$$\pi_*\pi^*\mathcal{F} \simeq \pi_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Proof. The second statement follows from the definitions of these functors, noting that π is an open map. The first can be found in [Hartshorne II 5].

To proceed further we need to consider the effect of π^* and $\tilde{\pi}^*$ on degree.

Proposition 3.2.4. Let $\mathcal{F}$ be a rank n torsion free sheaf on X. Then

1. $$\deg \mathcal{F} \le \deg \pi^*\mathcal{F} \le \deg \mathcal{F} + n \sum_{x \in X} ((e(x) - 1)\delta(x)), \quad \text{and}$$
2. $$\deg \tilde{\pi}^*\mathcal{F} = \deg \mathcal{F} - \ell(\mathcal{F}) = \deg \mathcal{F} - \sum_{x \in X} \ell(\mathcal{F}_x).$$

Proof. These statements are global versions of the lemmas 2.5.1/2. Let $\mathcal{F}$ be a torsion free sheaf of rank n. Consider π^* of the sequence $\Sigma 1$ of (3.1.4). Using the lemmas mentioned above we obtain a short exact sequence:

$$\Sigma 1' \qquad 0 \longrightarrow \pi^*\mathcal{E} \longrightarrow \pi^*\mathcal{F} \longrightarrow \bigoplus_{x \in X} T_{\mathcal{F}_x} \otimes_{\mathcal{O}_x} \tilde{\mathcal{O}}_x \longrightarrow 0$$

Note, that if $\mathcal{E}$ is locally free on X then $\deg \pi^*\mathcal{E} = \deg \mathcal{E}$—this follows from the fact that $\pi^*\mathcal{O}_X = \mathcal{O}_{\tilde{X}}$.

So from this exact sequence we see that:

$$\deg \pi^* \mathcal{F} - \deg \mathcal{F} = \sum \dim(T_{\mathcal{F}_x} \otimes \tilde{\mathcal{O}}_x) - \sum \dim T_{\mathcal{F}_x}$$

Together with the inequalities in (2.5.1) this gives the first part of the proposition. For the other part note that the injection $\mathcal{F} \longrightarrow \pi_* \pi^* \mathcal{F}/\text{torsion}$ is locally

$$\mathcal{F}_x \longrightarrow \mathcal{F}_x \otimes \tilde{\mathcal{O}}_x/\text{torsion} = \mathcal{F}_x \cdot \tilde{\mathcal{O}}_x.$$

Hence the dimension of the cokernel is $\sum_x (n\delta(x) - \ell(\mathcal{F}_x))$, so, since π_* increases slope by $\delta(X)$ (3.2.1)

$$\deg(\pi^* \mathcal{F}/\text{torsion}) - \deg(\mathcal{F}) = -\sum_x \ell(\mathcal{F}_x).$$

Corollory 3.2.5. When E is a vector bundle on $\tilde{X}$,

$$E \simeq \tilde{\pi}^* \pi_* E.$$

Proof. Applying the second part of the proposition to $\pi_* E$, we see that $\deg(\pi^* \pi_* E/\text{torsion}) - \deg(E) = 0$. Hence the generic isomorphism $\tilde{\pi}^* \pi_* E \longrightarrow E$ of (3.2.3) is an isomorphism.

Corollory 3.2.6. The functor π_* gives a bijection from the set of isomorphism classes of vector bundles on $\tilde{X}$ to the set of isomorphism classes of sheaves locally isomorphic to $\tilde{\mathcal{O}}_x^n$ (for some n) on X.

Proof. The inverse is given by $\mathcal{F} \longmapsto \tilde{\pi}^* \mathcal{F}$. We know, (3.2.5), that $\tilde{\pi}^* \pi_* \cong 1_{Vect(\tilde{X})}$; on the other hand, $\mathcal{F} \longrightarrow \pi_* \tilde{\pi}^* \mathcal{F}$ is a generic isomorphism (3.2.3) and, if $\mathcal{F}$ is locally isomorphic to $\tilde{\mathcal{O}}_x^n$ then we have shown the degrees are equal and so this must be an isomorphism.

Remark 3.2.7. Of course, if $\pi' : X' \longrightarrow X$ is a partial normalisation then we also get functors $(\pi')^*$ and $(\tilde{\pi'})^*$. Since not every sheaf on X' is locally free the picture is more complicated than for $\tilde{X}$. The important observation is that one can generalise (3.2.5/6) to the case of torsion free sheaves of $\mathcal{O}_{X'}$-modules on X', i.e., every torsion free sheaf of $\pi'_* \mathcal{O}_{X'}$-modules on X is the direct image of a unique torsion free sheaf on X'. (Cf. [Hartshorne II Ex.5.17]—π'_* induces an equivalence between the appropriate categories since π' is affine.)

§3.3 Parabolic Structures

In this section we introduce the category of generalised parabolic bundles and various subcategories. Our definitions will be restricted versions of those found in [Bhosle 1, 2].

Definitions. Let Y be a curve, D an effective divisor supported on Y_{reg} and let E be a torsion free sheaf on Y. Write $\mathcal{O}_D = \mathcal{O}_Y/\mathcal{O}(-D)$ for the Artinian ring supported on D, and write $F_0(E) = E \otimes_{\mathcal{O}_Y} \mathcal{O}_D$—this is a vector space of dimension $\mathrm{rk}E.\deg D$. A *parabolic structure* on E over D is, by definition, a subspace $F_1(E)$ of $F_0(E)$. The pair, $\tau = (a, D)$ where $a = \dim F_1(E)$, is called the *type* of the parabolic structure.

More generally, one could allow a flag, $F_0(E) \supset F_1(E) \supset \ldots \supset F_r(E)$, of subspaces (see [Bhosle 1]), but we shall not use this.

In our context the curve Y will be $\tilde{X}$ and the divisor D will have support on the preimage of a singular point of X. Let $\{x_1, \ldots, x_s\}$ be the set of all singular points of X.

Definition. A *generalised parabolic bundle* —GPB for short—$(E, \underline{F}_1(E))$ on $\tilde{X}$ is a vector bundle E together with a set of parabolic structures, $\underline{F}_1(E) = \{F_1^i(E), \ldots, F_1^i(E)\}$, $F_1^i(E)$ being a parabolic structure over a divisor D_i with support of D_i contained in the set $\pi^{-1}(x_i)$. Note in particular that the divisors D_i are pairwise disjoint. Define a divisor $D = \sum_i D_i$, so that $\mathcal{O}_D = \oplus_i \mathcal{O}_{D_i}$; write $F_0(E) = E \otimes \mathcal{O}_D = \oplus_i F_0^i(E)$ and $F_1(E) = \oplus_i F_1^i(E) \subset F_0(E)$.

We will only define non-zero morphisms between two GPBs, $(E, \underline{F}_1(E))$ and $(E', \underline{F}_1(E'))$, if their parabolic structures are defined over the same divisors $\underline{D} = \{D_1, \ldots, D_s\}$. A *morphism* of GPBs, $\phi : (E, \underline{F}_1(E)) \longrightarrow (E', \underline{F}_1(E'))$, is a morphism of bundles, $\phi : E \longrightarrow E'$, such that, if $\bar{\phi}_i : E \otimes \mathcal{O}_{D_i} \longrightarrow E' \otimes \mathcal{O}_{D_i}$ is the induced map, we have, for each i

$$\bar{\phi}(F_1^i(E)) \subset F_1^i(E')$$

The morphism, ϕ, above, is an *isomorphism* if ϕ is an isomorphism of vector bundles and $\bar{\phi}_i(F_1^i(E)) = F_1^i(E')$ for each i.

We write $GPB(\tilde{X})$ for the category of generalised parabolic bundles on $\tilde{X}$, and $GPB(\tilde{X}, \underline{D})$ for the subcategory corresponding to bundles with parabolic structures over a fixed set of divisors $\underline{D}$.

If $(E, \underline{F}_1(E))$ is a GPB and $E' \subset E$ is a subbundle then the *canonical parabolic structure* on E' is given by setting $F_1^i(E') = F_0^i(E') \cap F_1^i(E)$. Then a *parabolic subbundle* of $(E, \underline{F}_1(E))$ is defined to be a subbundle E' of E with the canonical parabolic structure.

Similarly if E'' is a quotient bundle of E, with projection map $p : E \longrightarrow E''$, the *canonical parabolic structure* on E'' is given by $F_1^i(E'') = \bar{p}(F_1^i(E))$. A *parabolic quotient bundle* of E then means a quotient bundle with the canonical parabolic structure.

Our main interest here will be in generalised parabolic bundles which also have an $\mathcal{O}_X$-module structure. More precisely we say a GPB, $(E, \underline{F}_1(E))$, is a *parabolic Module* (over X) if, for each i, the subspaces $F_1^i(E) \subset E \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{D_i}$ are also sub $\mathcal{O}_{x_i}$-modules (where $E \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{D_i}$ is considered as a $\mathcal{O}_{x_i}$-module via π_*), or, equivalently, if $F_1(E)$ under π_* is a sheaf of $\mathcal{O}_X$-modules. Then a morphism of parabolic Modules is a morphism of GPBs which restricts to a morphism of $\mathcal{O}_{x_i}$-modules over D_i. Write $PMod(\tilde{X})$ for the category of parabolic Modules (—or $PMod(\tilde{X}, \underline{D})$ if we wish to fix extra structure).

Lemma 3.3.1.

1. If $(E, \underline{F}_1(E))$ is a parabolic Module, and E' is a subbundle of E (resp. E'' is a quotient bundle of E) then E' (resp. E'') with the canonical parabolic structure is a parabolic Module.

2. Any morphism of GPBs, $(E, \underline{F}_1(E))$ and $(E', \underline{F}_1(E'))$, which are parabolic Modules is a morphism of parabolic Modules. I.e., $PMod(\tilde{X}, \underline{D})$ is a *full subcategory* of $GPB(\tilde{X}, \underline{D})$.

Proof.

1. This is clear—e.g., $F_1^i(E) = E' \otimes \mathcal{O}_{D_i} \cap F_1^i(E) \subset E \otimes \mathcal{O}_{D_i}$ is the intersection of two submodules of an $\mathcal{O}_{x_i}$-module.

2. The map from $F_0(E)$ to $F_0(E')$ preserves $\tilde{\mathcal{O}}_x$-structure at each x, so re-

stricts to a morphism of $\mathcal{O}_x$-modules if both $F_1(E)$ and $F_1(E')$ have $\mathcal{O}_X$-structure.

Given a GPB, $\underline{E} = (E, \underline{F}_1(E))$, set $a_i = \dim F_1^i(E)$ and call the set of all data $\underline{\tau} = \{a_i, D_i\}$ the *type* of $\underline{E}$. Note that choosing a divisor D_i supported on $\pi^{-1}(x_i)$ for x_i a singular point is equivalent to choosing an ideal I_{D_i} in $\tilde{\mathcal{O}}_{x_i}$ (i.e., I_{D_i} is the ideal defining D_i).

Lemma 3.3.2. Fix a bundle E of rank n on $\tilde{X}$; choose a set of parabolic structures $\underline{\tau} = \{a_i, D_i\}$ and write $V_i = \tilde{\mathcal{O}}_{x_i}/I_{D_i}$—so, V_i is the vector space underlying $\mathcal{O}_{D_i}$. Then:

1. The set of all GPBs $(E, \underline{F}_1(E))$ of type $\underline{\tau}$ over E is parametrised by a product of Grassmannians $\prod_i \mathrm{Gr}(a_i, V_i^n)$;
2. The set of all parabolic Modules of type $\underline{\tau}$ over E is parametrised by a product of fixed point subschemes $\prod_i \mathrm{Gr}^{\mathcal{O}_{x_i}}(a_i, V_i^n)$.

Proof. This is immediate from the considerations of §2.3.

Remark. In the above lemma it is possible, in either case, that 2 different points correspond to isomorphic objects. In fact

$$(E, \underline{F}_1(E)) \simeq (E, \underline{F}_1'(E)) \Leftrightarrow \exists\ \phi \in \mathrm{Aut}(E) \text{ such that } \bar{\phi}(F_1(E)) = F_1'(E).$$

So it follows that the correspondence is one-to-one if and only if E is simple.

§3.4 The Functors Ψ_* and Ψ^*

For each category of parabolic Modules $PMod(\tilde{X}, \underline{D},)$ as above, we will define a functor $\Psi_* : PMod(\tilde{X}, \underline{D}) \longrightarrow TFS(X)$; these can be thought of as generalisations of π_*.

We fix a category $PMod(\tilde{X}, \underline{D})$ of parabolic Modules, writing $D = \sum D_i$. If $(E, \underline{F}_1(E))$ is an object in this category define $\mathcal{E} = \Psi_*(E, \underline{F}_1(E))$ as follows:

Using the surjection of sheaves $\mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0$ on $\tilde{X}$ we have a surjection of sheaves $\pi_* E \longrightarrow E \otimes \mathcal{O}_D \longrightarrow 0$ on X. Define $\mathcal{E}$ to be the kernel of the composition of this map with the projection $E \otimes \mathcal{O}_D \longrightarrow (E \otimes \mathcal{O}_D)/F_1(E) \longrightarrow 0$, i.e., $\mathcal{E}$ is defined by the short exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_* E \longrightarrow \frac{E \otimes \mathcal{O}_D}{F_1(E)} \longrightarrow 0.$$

To see that this gives a functor suppose that $\phi : (E, \underline{F}_1(E)) \longrightarrow (E', \underline{F}_1(E'))$ is a morphism of parabolic Modules.

The resulting diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{E} & \longrightarrow & \pi_* E & \longrightarrow & E \otimes \mathcal{O}_D & \longrightarrow & \dfrac{E \otimes \mathcal{O}_D}{F_1(E)} & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}' & \longrightarrow & \pi_* E' & \longrightarrow & E' \otimes \mathcal{O}_D & \longrightarrow & \dfrac{E' \otimes \mathcal{O}_D}{F_1(E')} & \longrightarrow 0
\end{array}$$

is commutative, since $\phi(F_1(E)) \subset F_1(E')$, so we have an induced map of sheaves $\phi : \mathcal{E} \longrightarrow \mathcal{E}'$ which is uniquely determined by the requirement that the resulting diagram is commutative. The functorial properties of Ψ_* are then easy to check.

Note that Ψ_* preserves rank, i.e., $\mathrm{rk}\mathcal{E} = \mathrm{rk}E$.

Proposition 3.4.1 Basic properties of Ψ_*.

1. Ψ_* is an exact functor.

2. $\deg \mathcal{E} - \deg E = \mathrm{rk}E(\delta - \deg D) + \dim(F_1(E))$, in particular, Ψ_* preserves degree for $(E, \underline{F}_1(E))$ with $\dim(F_1(E))/\mathrm{rk}E = \deg D - \delta$.

3. The isomorphism class of $\mathcal{E}_{x_i}$ as an $\mathcal{O}_{x_i}$-module depends on the choice of subspace $F_1^i(E) \subset E \otimes \mathcal{O}_{D_i}$. If $F_1^i(E)$ is an $\tilde{\mathcal{O}}_{x_i}$-module for each i then $\mathcal{E} \simeq \pi_* E'$ for some bundle E' on $\tilde{X}$.

Proof.

1. This is easy to check, using the 3x3 lemma in the abelian category $Coh(X)$.

2. From the defining exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_* E \longrightarrow \frac{E \otimes \mathcal{O}_D}{F_1(E)} \longrightarrow 0$$

we get : $\deg \mathcal{E} = \deg \pi_* E - \deg((E \otimes \mathcal{O}_D)/F_1(E))$

$$= \deg E + \mathrm{rk}E\delta - \mathrm{rk}E \deg D + \dim(F_1(E)).$$

Thus $\deg \mathcal{E} - \deg E = \mathrm{rk}E(\delta - \deg D) + \dim(F_1(E))$ as required.

3. If $F_1(E) \in Coh(\tilde{X})$, and if E' is the vector bundle on $\tilde{X}$ given by

$$0 \longrightarrow E' \longrightarrow E \longrightarrow (E \otimes \mathcal{O}_D)/F_1(E) \longrightarrow 0$$

then clearly $\pi_* E' = \mathcal{E}$.

In general, if $\mathrm{rk} E = n$, the local version of the exact sequence defining $\mathcal{E}$ at a singular point x_i is:

$$0 \longrightarrow \mathcal{E}_{x_i} \longrightarrow \tilde{\mathcal{O}}^n_{x_i} \longrightarrow \frac{\mathcal{O}^n_{D_i}}{F^i_1(E)} \longrightarrow 0,$$

i.e.,

$$\mathcal{E}_{x_i} \simeq \ker\left(\tilde{\mathcal{O}}^n_{x_i} \longrightarrow \frac{\mathcal{O}^n_{D_i}}{F^i_1(E)}\right).$$

So $\mathcal{E}_{x_i} \simeq p^{-1}(F^i_1(E))$ where p is the projection $p : \tilde{\mathcal{O}}^n_{x_i} \longrightarrow \mathcal{O}^n_{D_i}$. Hence, from the considerations of chapter 2 on the module theory we have:

Corollory 3.4.2. If each divisor D_i is sufficiently large then every isomorphism class of torsion free $\mathcal{O}_{x_i}$-modules is represented as $\mathcal{E}_{x_i}$ for some $\mathcal{E} = \Psi_*(E, \underline{F}_1(E))$ with $(E, \underline{F}_1(E))$ in $PMod(\tilde{X}, \underline{D})$. The eventual aim in this section is to prove a global version of this statement.

Given a category $PMod(\tilde{X}, \underline{D})$ as above, there is no consistent way of defining an 'inverse' functor $\Psi^* : TFS(X) \longrightarrow PMod(\tilde{X}, \underline{D})$ in general. However, in certain cases it is possible to define a functor Ψ^* on $Vect(X)$.

Definitions. Let x be a point of X; we will call an ideal $I \subset \mathcal{O}_x$ *normal* if $I \simeq \tilde{\mathcal{O}}_x$ as an $\mathcal{O}_x$-module. Similarly, call a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ *normal* if each stalk, I_x, is normal. So, for example, the sheaf $\mathcal{C}$ is normal and every other normal sheaf of ideals is a subsheaf of this. If D is a divisor on $\tilde{X}$ then call D *normal* (with respect to X) if $\pi_*\mathcal{O}(-D)$ is a normal sheaf of ideals. Note that D is normal if and only if $\pi_*\mathcal{O}(-D) \subset \mathcal{C}$.

Now, fix a category of parabolic Modules, $PMod(\tilde{X}, \underline{D})$, such that $D = \sum_i D_i$ is normal. Suppose that D_i is defined by an ideal $I_i \subset \tilde{\mathcal{O}}_{x_i}$; set $\ell_i = \dim \mathcal{O}_{x_i}/I_i$, so $\dim \tilde{\mathcal{O}}_{x_i}/I_i = \ell_i + \delta(x_i)$. In particular notice that, since I_i is normal, $\mathcal{O}_{x_i}/I_i \otimes \tilde{\mathcal{O}}_{x_i} = \tilde{\mathcal{O}}_{x_i}/I_i$. Then, if $\mathcal{E}$ is a locally free sheaf on X, set

$\Psi^*(\mathcal{E}) = (E, \underline{F}_1(E))$, where $E = \pi^*\mathcal{E}$ and $F_1(E)$ is the subspace

$$\mathcal{E} \otimes_{\mathcal{O}_X} \left(\frac{\mathcal{O}_X}{\pi_*\mathcal{O}(-D)}\right) \subset \left(\mathcal{E} \otimes \left(\frac{\mathcal{O}_X}{\pi_*\mathcal{O}(-D)}\right)\right) \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_{\tilde{X}} = \pi^*\mathcal{E} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_D.$$

For example, in the case where D_i is the divisor defined by $\mathcal{C}_{x_i}$ we have

$$\dim F_0^i(E) = \mathrm{rk}E(\dim(\tilde{\mathcal{O}}_x/\mathcal{C}_{x_i})) = \mathrm{rk}E(\tilde{\delta}(x_i)) \quad \text{and}$$
$$\dim F_1^i(E) = \mathrm{rk}E(\dim(\mathcal{O}_x/\mathcal{C}_{x_i})) = \mathrm{rk}E(\tilde{\delta}(x_i) - \delta(x_i)).$$

Lemma 3.4.3. We suppose D is normal and use the above notation. Then, if $\mathcal{E}$ is a locally free sheaf on X we have $\mathcal{E} \simeq \Psi_*\Psi^*\mathcal{E}$.

Proof. Write $n = \mathrm{rk}E$ and $\mathcal{E}' = \Psi_*\Psi^*\mathcal{E}$. The short exact sequence defining $\mathcal{E}'$ is

$$0 \longrightarrow \mathcal{E}' \longrightarrow \pi_*\pi^*\mathcal{E} \longrightarrow \frac{\pi^*\mathcal{E} \otimes (\mathcal{O}_{\tilde{X}}/\mathcal{O}(-D))}{\mathcal{E} \otimes (\mathcal{O}_X/\pi_*\mathcal{O}(-D))} = \mathcal{E} \otimes \pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X \longrightarrow 0$$

So

$$\deg\mathcal{E}' = \deg\mathcal{E} + n\delta(X) - n\sum_i(\delta(x_i) + \ell_i - \ell_i) = \deg\mathcal{E}.$$

On the other hand there is a canonical injection $\mathcal{E} \longrightarrow \pi_*\pi^*\mathcal{E}$, which is an isomorphism away from singular points: the cokernel is $\mathcal{E} \otimes (\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$, so there is a map $\mathcal{E} \longrightarrow \mathcal{E}'$ and this is an isomorphism since $\deg\mathcal{E} = \deg\mathcal{E}'$.

Corollory 3.4.4. Under these conditions Ψ_* determines a one to one correspondence between parabolic Modules $(E, \underline{F}_1(E))$ with $\dim F_1^i(E) = \mathrm{rk}E\ell_i$ for each i, $\Psi_*(E, \underline{F}_1(E))$ locally free and locally free sheaves on X.

Proof. By the lemma it is enough to check that if $(E, \underline{F}_1(E))$ is such a parabolic Module then $\Psi^*\Psi_*(E, \underline{F}_1(E)) = (E, \underline{F}_1(E))$. If $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$ then, since $\deg\pi^*\mathcal{E} = \deg E$, we must have $\pi^*\mathcal{E} \simeq E$. To see that $F_1^i(E) = \mathcal{E} \otimes \mathcal{O}_{x_i}/I_i$ note that the defining sequence for $\mathcal{E}$ is

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_*\pi^*\mathcal{E} \longrightarrow \frac{\pi^*\mathcal{E} \otimes (\oplus_i\tilde{\mathcal{O}}_{x_i}/I_i)}{\oplus_i F_1^i(E)} \longrightarrow 0;$$

but $\pi_*\pi^*\mathcal{E}/\mathcal{E} = \mathcal{E} \otimes \pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$, so $F_1^i(E) = \mathcal{E} \otimes \mathcal{O}_{x_i}/I_i$.

We want to generalise the above and describe the fibre of Ψ_* in $PMod(\tilde{X}, \underline{\tau})$ over a torsion free sheaf $\mathcal{E}$ which is not locally free. We assume throughout that D is normal and use notation as above.

Suppose $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$. So there is an exact sequence

$$(*) \quad 0 \longrightarrow \mathcal{E} \longrightarrow \pi_* E \longrightarrow \frac{E \otimes \mathcal{O}_D}{F_1(E)} \longrightarrow 0.$$

Firstly, in (3.4.5–7), we will find necessary conditions that $(E, \underline{F}_1(E))$ must satisfy.

Lemma 3.4.5. Write $E_0 = \tilde{\pi}^* \mathcal{E}$, then E arises as an extension

$$(\dagger) \quad 0 \longrightarrow E_0 \longrightarrow E \longrightarrow \oplus T_i \longrightarrow 0,$$

where T_i is a torsion sheaf supported on $\pi^{-1}(x_i)$ with $\deg T_i = \ell(\mathcal{E}_{x_i})$.

Proof. Applying $\tilde{\pi}^*$ to the sequence $(*)$ gives us an injection of sheaves $E_0 \longrightarrow E$; the dimension of the cokernel can then be calculated using (3.2.1).

Now, to look at $F_1(E)$ concentrate at some $x_i = x$, and to simplify notation drop the i ($\ell = \dim \mathcal{O}_x / I \ldots$ etc.).

Lemma 3.4.6.

1. $\dim \mathcal{E}_x / I \cdot \mathcal{E}_x = n\ell + \ell(\mathcal{E}_x)$.

2. Consider the natural map

$$\phi : \mathcal{E}_x / I \cdot \mathcal{E}_x \longrightarrow E \otimes \mathcal{O}_D$$

arising from tensoring the local version of $(*)$ with $\mathcal{O}_x / I$ over $\mathcal{O}_x$. $F_1(E)$ is the image of $\mathcal{E}_x / I \cdot \mathcal{E}_x$ under ϕ. The map ϕ is not in general injective; in fact $\dim \ \ker(\phi) \leq \dim T / I \cdot T$.

3. If $\dim F_1^i(E) = \ell(\mathcal{E}_{x_i})$ for each i (so that $\deg \mathcal{E} = \deg E$) then $T / I \cdot T = T$ and $\ker(\phi') \simeq T$ where ϕ' is the map $E_0 \otimes \mathcal{O}_D \longrightarrow E \otimes \mathcal{O}_D$.

Proof.

1. Consider the short exact sequence $0 \longrightarrow \mathcal{O}_x^n \longrightarrow \mathcal{E}_x \longrightarrow T_{\mathcal{E}_x} \longrightarrow 0$ (2.2.3); since $I \subset \mathcal{C}_x \subset \mathrm{Ann} T_{\mathcal{E}_x}$ we have $I\mathcal{E}_x = I\mathcal{O}_x^n$. So $\dim_k(\mathcal{E}_x / I\mathcal{E}_x) = n\ell + \dim(T_{\mathcal{E}_x}) = n\ell + \ell(\mathcal{E}_x)$.

2. $F_1(E) = \mathrm{im}(\phi)$ is clear on examining the local version of $(*)$ (3.4.1). We can factorise ϕ as

$$\mathcal{E}_x / I\mathcal{E}_x \longrightarrow E_0 \otimes \mathcal{O}_D \longrightarrow E \otimes \mathcal{O}_D.$$

The first map is an injection—since it corresponds to the minimal embedding $\mathcal{E}_x \longrightarrow \tilde{\mathcal{O}}_x^n$, and to calculate the kernel of the second consider the sequence $\pi_*(\dagger)$ (from (3.4.5)) tensored with $\mathcal{O}_x/I$ over $\mathcal{O}_x$. This gives rise to an exact sequence

$$0 \longrightarrow \ker(\phi') \longrightarrow E_0 \otimes \mathcal{O}_D \xrightarrow{\phi'} E \otimes \mathcal{O}_D \longrightarrow T/I \cdot T \longrightarrow 0.$$

Since, by **1.**, the middle two terms both have dimension $(\delta + \ell)n$ we see that $\dim \ker(\phi') = \dim(T/I \cdot T)$.

3. $\dim F_1(E) = \ell(\mathcal{E}_x) = \dim T$ implies that we must have $T = T/I \cdot T$. The torsion $\tilde{\mathcal{O}}_x$-module, T, splits as a direct sum

$$T = \oplus_j \tilde{\mathcal{O}}_x/m_j^{v_{1j}} \oplus \ldots \oplus \oplus_j \tilde{\mathcal{O}}_x/m_j^{v_{nj}}.$$

where $\{m_j\}$ is the set of all maximal ideals of $\tilde{\mathcal{O}}_x$. For a suitable choice of basis in $E_0 \otimes \mathcal{O}_D$ the torsion submodule $\ker(\phi')$ has the same decomposition. This enables us to construct the required isomorphism $T \simeq \ker(\phi')$. In particular we can regard T as a submodule of $\mathcal{E}_x/I \cdot \mathcal{E}_x$.

Lemma 3.4.7. Write V for the vector space $\mathcal{O}_D$. Then, under the condition of **3** in the above lemma we must have that $\mathcal{E}_x$ is represented in $\mathrm{Gr}^{\mathcal{O}_x}(n\ell, V^n)$ (see §2.3).

Proof. Using (3.4.6) we have $T \subset \mathcal{E}_x/I \cdot \mathcal{E}_x \subset V^n$, $\dim T = \ell(\mathcal{E}_x)$ and $\dim(\mathcal{E}_x/I \cdot \mathcal{E}_x)/T = n\ell$. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{E}_x & \hookrightarrow & (E_0)_x & \xrightarrow{\Phi} & E_x & & \\ & & & & \downarrow & & \downarrow p & & \\ 0 & \longrightarrow & T & \longrightarrow & E_0 \otimes \mathcal{O}_D & \xrightarrow{\phi'} & E \otimes \mathcal{O}_D & \longrightarrow & T \longrightarrow 0 \end{array}$$

where the bottom row is exact. Now the image, F_1, of $\mathcal{E}_x$ in $E \otimes \mathcal{O}_D$ has dimension $n\ell$, hence, by commutativity, we must have $I^n \subset \Phi(\mathcal{E}_x)$ and so $p^{-1}(F_1) = \Phi(\mathcal{E}_x) \simeq \mathcal{E}_x$. So this gives a submodule of V^n of the correct dimension representing $\mathcal{E}_x$, and therefore the required point in $\mathrm{Gr}^{\mathcal{O}_x}(n\ell, V^n)$.

Conversely, we will now show that if $\mathcal{E}$ is a torsion free sheaf on X and D is a normal divisor such that $\mathcal{E}$ is locally represented then there exists $(E, \underline{F}_1(E))$ with $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$. More precisely:

Proposition 3.4.8. Suppose that, for each i, $\mathcal{E}_{x_i}$ is represented in $\mathrm{Gr}^{\mathcal{O}_{x_i}}(n\ell_i, V_i^n)$ by a submodule F_1^i then:

1. There exists a parabolic Module $(E, \underline{F}_1(E))$ with $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$. This is uniquely determined by the given F_1^i.

2. On $\mathrm{Gr}^{\mathcal{O}_{x_i}}(n\ell_i, V_i^n)$ there is a natural action of the group $((\tilde{\mathcal{O}}_{x_i}/I_i)^*)^n$. One obtains the same parabolic Module $(E, \underline{F}_1(E))$ given another set of representatives $F_1^{i\prime}$ which lie in the same orbit of $((\tilde{\mathcal{O}}_{x_i}/I_i)^*)^n$ on each $\mathrm{Gr}^{\mathcal{O}_{x_i}}(n\ell_i, V_i^n)$.

Proof. We need to reverse the above argument. Well, $p^{-1}(F_1^i) \simeq \mathcal{E}_{x_i} \subset p^{-1}(V_i^n) = \tilde{\mathcal{O}}_{x_i}^n$. On the other hand we have $\mathcal{E}_x \cdot \tilde{\mathcal{O}}_{x_i} = (E_0)_{x_i} \subset \tilde{\mathcal{O}}_{x_i}^n$. This gives us $T_i = \tilde{\mathcal{O}}_{x_i}^n / E_{0_{x_i}}$ of dimension $\ell(\mathcal{E}_x)$ (cf. 3.4.6). Write $E_{0_{x_i}} = \Phi_i \tilde{\mathcal{O}}_{x_i}^n$ for suitable $\Phi_i \in \mathrm{Aut}(K^n)$, so that the image of $\mathcal{E}_{x_i}$ under $p \cdot \Phi_i$ equals F_1^i. Now we want to construct the bundle E by an extension which locally agrees with Φ_i. We have specified the local version of the extension (†) at each point and this determines the locally free sheaf E. Hence we obtain $(E, \underline{F}_1(E))$. The locally free sheaf E will be unchanged if we replace some Φ_i by $u \cdot \Phi_i$ where u is an element of $GL_n(\tilde{\mathcal{O}}_{x_i})$ which preserves the given choice of bases: i.e. if $u \in (\tilde{\mathcal{O}}_{x_i}^*)^n$. This deals with **2**. Finally, to see that $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$, notice that from the construction of $(E, \underline{F}_1(E))$

$$\mathcal{E} \subset \mathcal{E}' = \ker\left(\pi_* E \longrightarrow \frac{E \otimes \mathcal{O}_D}{F_1(E)}\right),$$

but $\deg \mathcal{E} = \deg \mathcal{E}'$ and so $\mathcal{E} = \mathcal{E}'$.

Corollory 3.4.9. Suppose further that, for each x_i,

$$(*) \qquad I_i \subset \begin{cases} \mathcal{C}_{x_i} & \text{if } \mathcal{O}_{x_i} \text{ is Gorenstein} \\ \mathcal{C}_{x_i}^2 & \text{otherwise} \end{cases}$$

then for each torsion free sheaf $\mathcal{E}$ on X there exists $(E, \underline{F}_1(E)) \in PMod(\tilde{X}, \underline{D})$ with $\deg E = \deg \mathcal{E}$ and $\Psi_*(E, \underline{F}_1(E)) = \mathcal{E}$.

Proof. By (2.3.1) we know that under these conditions $\mathcal{E}$ is locally represented. Thus the result follows immediately from the above proposition.

Corollary 3.4.10. In the case rank $\mathcal{E} = 1$ the fibre of Ψ_* over $\mathcal{E}$ is finite. If, further, the curve X has just 1 branch at each singular point then there is at most 1 preimage of each $\mathcal{E}$.

Proof. By (3.4.8(**2**)) it is enough to show that in the rank 1 case every module is represented only finitely many times in $\mathrm{Gr}^{\mathcal{O}_{x_i}}(\ell_i, V_i)$ up to the action of $\tilde{\mathcal{O}}^*_{x_i}$. This is proved in [Rego 1; 2.3]. For further details and examples of how this works in practice see (5.3.3).

Remark. If rank > 1 then the fibre is not generally finite; see [Bhosle 1] for an example.

§3.5 Stability

The notion of stability is essential in constructing moduli spaces for 'vector bundle type' objects. Most of the material in this section is standard, and can be found in one of [Newstead], [Seshadri] or [Bhosle 1]; we will define stability and develop its basic properties, simultaneously, for objects in each of the categories $Vect(\tilde{X})$, $TFS(X)$, $GPB(\tilde{X},\underline{D})$, $PMod(\tilde{X},\underline{D})$. One of the points of our method of presentation is to emphasize that a good notion of stability behaves in exactly the same way for bundles with parabolic structures as it does for vector bundles: indeed, many of the proofs are the same.

In the discussion that follows the word 'bundle' will be used to mean an object in one of these categories, and 'subbundle' will mean 'subbundle with the canonical parabolic structure', etc.

For bundles with parabolic structures the flag terms should be allowed to affect stability to some extent. We fix a *weight* $0 \leq \alpha < 1$ and define

$$\text{par.}\deg(E, \underline{F}_1(E)) = \deg E + \alpha.\dim F_1(E).$$

We call $\alpha.\dim F_1(E) = \mathrm{wt}E$ the weight of $(E, \underline{F}_1(E))$. Ideally we would like to take $\alpha = 1$ (cf. 3.5.9.) but in this case the resulting definition of stability lacks many desirable properties. At the end of the section we explain how to choose α so that Ψ_* preserves stability.

Definitions. For a bundle E define the *(parabolic) slope* of E to be $\mu(E) = (\text{par})\deg E/\operatorname{rk} E$.

We say E is *stable (resp. semistable)* if for all proper subbundles E' of E, $\mu(E') < \mu(E)$ (resp. $\mu(E') \leq \mu(E)$). By a *proper subbundle* we mean a non-zero subbundle of corank ≥ 1. Notice that this definition depends on the choice of α ($\alpha = 0$ is 'ordinary' stability).

Remark. Equivalently, E is stable (resp. semistable) if for all proper quotients E'', $\mu(E'') > \mu(E)$ (resp. $\geq$).

Lemma 3.5.1.

1. If E is a bundle of rank 1 then E is stable.

2. If E is a parabolic Module then E is stable (resp. semistable) as a parabolic Module if and only if E is stable (resp. semistable) as a generalised parabolic bundle.

Proof. 1. is immediate from the definition, whilst 2. follows from (3.2.1).

Lemma 3.5.2. Suppose $f : E \to F$ is a morphism of bundles in one of the above categories, which is a generic isomorphism of the underlying bundles. Then $(\mathrm{par})\deg E \leq (\mathrm{par})\deg F$, with equality if and only if f is an isomorphism of bundles.

Proof. f gives us a short exact sequence of sheaves

$$0 \longrightarrow E \xrightarrow{f} F \longrightarrow T \longrightarrow 0$$

where T is a torsion sheaf with $\dim T = \deg F - \deg E$. Moreover we have $\dim F_1(E) - \dim F_1(F) \leq \dim T$ i.e. $\mathrm{wt}E - \mathrm{wt}F \leq \alpha \dim T$.
So

$$\begin{aligned} \mathrm{par.}\deg E - \mathrm{par.}\deg F &= \deg E + \mathrm{wt}E - (\deg F + \mathrm{wt}F) \\ &\leq (\alpha - 1)\dim T < 0 \end{aligned}$$

if $\dim T \neq 0$, as $\alpha < 1$.

Corollory 3.5.3. Let $f : E \longrightarrow F$ be a morphism of *semistable* bundles with $\mu = \mu(E) = \mu(F)$. Then

1. f has constant rank.

2. If E and F are stable then either $f = 0$ or f is an isomorphism.

Proof. We can factor f as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & & & \downarrow f & & \downarrow h & & \\ 0 & \longleftarrow & F'' & \longleftarrow & F & \longleftarrow & F' & \longleftarrow & 0 \end{array}$$

where $h : \mathrm{coim}(f) = E'' \longrightarrow F' = \mathrm{im}(f)$ is a generic isomorphism. Since E and F are semistable

$$(*) \qquad \mu(F') \leq \mu \leq \mu(E'')$$

and now by the lemma we must have $\mu(F') = \mu(E'')$ and so h is an isomorphism, hence f is of constant rank.

For 2, by $(*)$ we must have, if $f \neq 0$, $\quad E = \mathrm{im}(f) = F$ by stability of E and F.

Corollory 3.5.4 Stable$\Rightarrow$Simple.

If E is a stable bundle then any non-zero morphism $f : E \longrightarrow E$ is multiplication by a scalar.

Proof. If λ is an eigenvalue of f over some point y, then $(f - \lambda Id) : E \longrightarrow E$ is not an isomorphism, so by (3.5.3) $f - \lambda Id = 0$, i.e. f is multiplication by λ.

Corollory 3.5.5. From the above we can conclude that, if c_μ is a subcategory of one of the above categories consisting of all semistable E with $\mu(E) = \mu$, then c_μ is an abelian category; the simple objects in c_μ are the stable bundles by (3.5.4). We can apply the Jordan-Hölder Theorem—If E is an object in c_μ then there exists a filtration in c_μ

$$E = E_r \supset E_{r-1} \supset \dots \supset E_0 = 0$$

such that E_i/E_{i-1} is stable with $\mu(E_i/E_{i-1}) = \mu$ for each $i = 1, \dots, r$.
Define $\mathrm{gr}.E = \bigoplus_i (E_i/E_{i-1})$. This does not depend on the choice of filtration. We say objects E, F of c_μ are *s-equivalent* if $\mathrm{gr}.E \simeq \mathrm{gr}.F$.

Lemma 3.5.6 Semistable Torsion Free Sheaves are Bounded.

Write e' for the maximum of the multiplicities of the points of X. Let $\mathcal{F}$ be a semistable torsion free sheaf on X of rank n and degree d with

$$d > n(2g - 2 + e').$$

Then

(A) $\mathcal{F}$ is generated by its sections;

(B) $\mathrm{H}^1(X, \mathcal{F}) = 0$

Proof. see [Newstead 5.2']; the bound given here is somewhat lower in general, but the same proof works by virtue of the bound on the maximal number of generators of a torsion free module in (2.2.4).

As an immediate consequence, applying this to $E(-D)$, we obtain boundedness for GPBs (on a non-singular curve).

Corollary 3.5.7. Let $(E, \underline{F}_1(E))$ be a semistable GPB; suppose

$$\mu(E, \underline{F}_1(E)) > n\left(2g - 1 + \sum_i \deg D_i\right).$$

Then:

(A) E is generated by sections and for each i the canonical map $\mathrm{H}^0(\tilde{X}, E) \longrightarrow \mathrm{H}^0(\tilde{X}, E \otimes \mathcal{O}_{D_i})$ is onto;

(B) $\mathrm{H}^1(\tilde{X}, E) = 0$.

Next we explain how stability behaves under π_* and Ψ_*. Recall (2.2.1), that if E is a vector bundle on $\tilde{X}$ then $\mu(\pi_* E) = \mu(E) + \delta$ ($\delta = \delta(X)$).

Lemma 3.5.8. If E is a vector bundle on $\tilde{X}$ then E is stable (resp. semistable) if and only if $\pi_* E$ is stable (resp. semistable) on X.

Proof. First suppose that $\pi_* E$ is stable but E is not stable. Then there exists a subbundle F of E with $\mu(F) \geq \mu(E)$, but then $\pi_* F$ is a subbundle of $\pi_* E$ and

$$\mu(\pi_* F) = \mu(F) + \delta \geq \mu(E) + \delta = \mu(\pi_* E)$$

contradicting stability of $\pi_* E$.

Now suppose that E is stable but that $\pi_* E$ is not stable, i.e. there exists a torsion free subsheaf $\mathcal{F}$ of $\pi_* E$ with $\mu(\mathcal{F}) \geq \mu(E) + \delta$. Consider the subbundle F of E generated by $\tilde{\pi}^* \mathcal{F}$. Now, using (3.2.4),

$$\mu(F) \geq \mu(\tilde{\pi}^* \mathcal{F}) \geq \mu(\mathcal{F}) - \delta \geq \mu(E)$$

contradicting the stability of E.

Similarly for the semistable case.

Remark. The same proof works if $\pi : X' \longrightarrow X$ is any partial normalisation, replacing δ by $\delta' = \delta(X) - \delta(X')$.

We seek to generalise this result to Ψ_*. Note that if $(E, \underline{F}_1(E))$ is a parabolic Module (or a GPB) then stability of $(E, \underline{F}_1(E))$ will depend on the choice of α, although, of course, $\Psi_*(E, \underline{F}_1(E))$ does not depend on α.

We specialise to $\alpha = 1$, and say $(E, \underline{F}_1(E))$ is *1-stable* (resp. *1-semistable*) if for all proper parabolic subbundles E' of E

$$\frac{\deg E' + \dim F_1(E')}{\mathrm{rank} E'} < \frac{\deg E + \dim F_1(E)}{\mathrm{rank} E} \quad (\text{resp. } \leq).$$

Remark. This notion of 1-stability does not share the properties of stability as outlined above. In particular the analogue of (3.5.2) fails—e.g. If $D = y$ then the short exact sequence

$$0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \longrightarrow k_y \longrightarrow 0$$

with $\dim F_1(\mathcal{O}(-y)) = 1$, $F_1(\mathcal{O}) = \{0\}$ gives a map between 1-stable parabolic bundles of the same slope which is not an isomorphism. It is also essentially this which means that the construction of the moduli space of GPBs does not immediately extend to the 1-stable case. However the usefulness of 1-stable parabolic modules is shown by the following.

Lemma 3.5.9. Let $PMod(\tilde{X}, \underline{D})$ be a category of parabolic modules with the divisor $D = \sum_i D_i$ normal; D_i defined by an ideal $I_i \subset \mathcal{C}_{x_i}$ and $\deg D_i = \delta(x_i) + \ell_i$, where $\ell_i = \dim(\mathcal{O}_{x_i}/I_i)$; write $\ell = \sum_i \ell_i$. Suppose $(E, \underline{F}_1(E))$ is a parabolic module in this category with $\dim F_1^i(E) = \ell_i \operatorname{rk} E$ for each i.

Then $(E, \underline{F}_1(E))$ is 1-stable (resp. 1-semistable) if and only if $\mathcal{E} = \Psi_*(E, \underline{F}_1(E))$ is stable (resp. semistable) .

Proof. First suppose $\mathcal{E}$ is stable but $(E, \underline{F}_1(E))$ is not 1-stable, so there exists a subbundle $(E', F_1(E'))$ of $(E, \underline{F}_1(E))$ with

$$\mu(E', \underline{F}_1(E')) = \frac{\deg E' + \dim F_1(E')}{\operatorname{rk} E'} \geq \frac{\deg E + \dim F_1(E)}{\operatorname{rk} .E} = \mu(E, \underline{F}_1(E)).$$

By functoriality, $\Psi_*(E', F_1(E'))$ gives a subsheaf $\mathcal{E}'$ of $\mathcal{E}$. Using (3.4.1)

$$\deg \mathcal{E} = \deg E \Rightarrow \mu(\mathcal{E}) = \mu(E, \underline{F}_1(E)) - \ell, \quad \text{so}$$

$$\deg \mathcal{E}' = \deg E' - \operatorname{rk} .E'\ell + \dim F_1(E') \Rightarrow \mu(\mathcal{E}') = \mu(E, \underline{F}_1(E)) - \ell.$$

So $\mathcal{E}'$ contradicts the stability of $\mathcal{E}$.

Conversely, suppose $(E, \underline{F}_1(E))$ is 1-stable but there exists a subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$. Let $F_0 = \tilde{\pi}^*\mathcal{F}$ and let F be the subbundle of E generated by F_0. We claim that the parabolic subbundle $(F, \underline{F}_1(F))$ contradicts stability of $(E, \underline{F}_1(E))$: for this, using the first part it is enough to prove that $\mathcal{F}$ is a subsheaf of $\Psi_*(F, \underline{F}_1(F))$, or equivalently that there is a subspace $F_1'(F) \subset F_1(F)$ such that $\mathcal{F} = \Psi_*(F, \underline{F}_1'(F))$.

Consider the composite map

$$\mathcal{F} \longrightarrow \pi_* F_0 \longrightarrow \pi_* F.$$

The quotient has the form $F \otimes \mathcal{O}_D / F_1'(F)$ for some $F_1'(F) \subset F \otimes \mathcal{O}_D$ (using the fact that D is normal, cf. (3.4.5)). Now $\mathcal{F} \subset \mathcal{E}$ and $\pi_* F \subset \pi_* E$ implies that $F_1'(F) \subset F_1(E)$, and hence $F_1'(F) \subset F_1(F)$ as required.

The proof in the semistable case is similar.

For GPBs $(E, \underline{F}_1(E))$ with weight α close to 1, the conditions for stability and 1-stability are similar. More precisely:

Lemma 3.5.10. Let $(E, \underline{F}_1(E))$ be a GPB with rank $E = n$ and parabolic structures over D_i. Suppose α satisfies

$$1 - 1/(\deg D . n(n-1)) < \alpha < 1.$$

Then

1. If $(E, \underline{F}_1(E))$ is semistable it is 1-semistable.
2. If $(E, \underline{F}_1(E))$ is 1-stable it is stable.
3. Suppose further that rank E and degree E are coprime and $\dim(F_1^i(E)) = a_i . \operatorname{rk} E$ for a_i an integer, then all four conditions are equivalent.

Proof. If $(F, \underline{F}_1(F))$ is a proper parabolic subbundle of $(E, \underline{F}_1(E))$ write

$$B_F = n . \deg F - \operatorname{rk} F . \deg E$$

$$A_F = \operatorname{rk} F . \dim F_1(E) - n . \dim F_1(F)$$

Now E is stable (resp. semistable) if and only if $B_F < \alpha A_F$ for all subbundles F (resp. $\leq$). To get the condition for 1-(semi)stability we just replace the α by 1.

1. Note: if $A_F \geq 0$ then $B_F \leq \alpha A_F \Rightarrow B_F \leq A_F$, as $0 < \alpha < 1$.
If $A_F < 0$, since $\operatorname{rk} F \geq 1$,

$$-A_F \leq n(n-1)\deg D \text{ i.e. } -A_F/(n(n-1)\deg D) \leq 1. \quad \text{So}$$

$$\alpha > 1 - 1/(\deg Dn(n-1)) \Rightarrow \alpha A_F < A_F - A_F/(n(n-1)\deg D) \leq A_F + 1.$$

Now $B_F \leq \alpha A_F \Rightarrow B_F < A_F + 1 \Rightarrow B_F \leq A_F$ since B_F, A_F are integers. Hence **1**.

2. This follows by an argument similar to that of **1**.

3. One can easily check that under the conditions stated the *equality* in the condition for 1-semistability can never occur. Hence 1-stability is equivalent to 1-semistability. The result then follows by using parts **1** and **2**.

Remark. Note that a choice of a good lower bound for α depends on the rank. Hence in the construction of moduli spaces for *GPBs* one fixes the rank before fixing α.

So far we have said nothing about the question of when stable bundles exist (for rank > 1). The most basic fact is that if $g(\tilde{X}) \geq 2$ then there exist stable vector bundles of all ranks and degrees on $\tilde{X}$ (see [Seshadri], for instance). Using this, and the results above, one can easily manufacture a large number of stable sheaves on X and stable parabolic Modules on $\tilde{X}$ (see (4.3.5)). There should exist stable locally free sheaves on the singular curve X as long as $g(X) \geq 2$. The most natural way to prove this statement would be via a deformation argument, considering X as a specialisation of a family of smooth curves. Such techniques lie beyond the scope of the present work, so in the next chapter we only claim that the moduli space of stable sheaves on a singular curve is non-empty if $g(\tilde{X}) \geq 2$.

CHAPTER 4

MODULI SPACES

§4.1 Existence Theorems

To each of the categories considered in the previous chapter it is possible to associate various moduli spaces of stable objects. In this section we will summarise the existence theorems for these spaces, describing some of their main properties and, later in the chapter, explaining how the functors of chapter 3 can be extended to give morphisms between them.

We begin by recalling some basic definitions; see [Newstead]. Let k-Sch denote the category of separated schemes of finite type over k. Let C be a subcategory of k-Sch; e.g. $C = TFS(X, n, d)$ is the category of semistable torsion free sheaves over a curve X with rank n and degree d. Suppose we have a suitable notion of a *family* of objects of C such as:

Definition. A family of objects of $TFS(X, n, d)$ parametrised by $S \in k$-Sch is a torsion free sheaf $\mathcal{F}$ over $S \times X$ which is flat over S and such that, for all points s of S, $\mathcal{F}_s \in TFS(X, n, d)$. Given $f : S' \longrightarrow S$ we get an induced family $(f \times 1)^* \mathcal{F}$ parametrised by S'.

Write

$$\underline{C} : k{-}Sch \longrightarrow Sets$$

for the functor which associates to $S \in k$-Sch the set of all families of objects of C parametrised by S. A pair $(\Phi, \mathcal{M}_C)$ is called a ***coarse moduli space*** for C if $\mathcal{M}_C$ is a k-scheme whose geometric points correspond bijectively to isomorphism classes of objects of C, and

$$\Phi : \underline{C} \longrightarrow Mor_{k-Sch}(\ \cdot\ , \mathcal{M}_C)$$

is a natural transformation having the universal property that any other such, $(\Phi', \mathcal{M}'_C)$, factors uniquely through Φ. A basic fact is that this is sufficient to determine uniquely the structure of $\mathcal{M}_C$ as a scheme. Further, $(\Phi, \mathcal{M}_C)$ is a ***fine moduli space*** if Φ is an isomorphism of functors; this implies the existence of a ***universal family*** (as for invertible sheaves (1.3.3), for example).

Below we state the theorems relating to the moduli problems for the categories of §3.5. For proofs and a fuller discussion [Newstead] and [Seshadri] treat the cases of vector bundles and torsion free sheaves, whilst [Seshadri] (see also [Mehta and Seshadri]) deal with (ordinary) parabolic bundles. For the generalised parabolic bundle case see [Bhosle]; the case of parabolic Modules follows from this, as explained in §4.2 where the correct notions of family are also discussed.

4.1.1 Vect($\tilde{X}$) see [Newstead, Theorem 5.8].

There exists a non-singular irreducible quasi-projective variety $\mathcal{M}_V(n,d)^s = \mathcal{M}^s_{Vect(\tilde{X})}(n,d)$ which is a coarse moduli space for vector bundles of rank n and degree d on $\tilde{X}$. $\mathcal{M}^s_V(n,d)$ has a natural compactification to a normal projective variety $\mathcal{M}_V(n,d) = \mathcal{M}_{Vect(\tilde{X})}(n,d)$ by adding in s-equivalence classes of semistable bundles. The tangent space at a point corresponding to a stable bundle E is $H^1(X, \mathbf{End}(E))$. $\dim \mathcal{M}_V(n,d) = n^2(g(\tilde{X})-1)+1$ when $\mathcal{M}_V(n,d)$ is not empty. If n and d are coprime then $\mathcal{M}_V(n,d) = \mathcal{M}^s_V(n,d)$ and $\mathcal{M}_V(n,d)$ is a fine moduli space.

4.1.2 TFS(X) see [Newstead, Theorem 5.8′].

There exists a quasi-projective scheme $\mathcal{M}^s_T(n,d) = \mathcal{M}^s_{TFS(X)}(n,d)$ which is a moduli space for stable torsion free sheaves of rank n and degree d on X. This has a natural compactification to a projective scheme $\mathcal{M}_T(n,d) = \mathcal{M}_{TFS(X)}(n,d)$ by adding in s-equivalence classes of semistable torsion free sheaves. The tangent space at a point corresponding to a stable sheaf $\mathcal{F}$ is $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ (a proof of this can be found in [Sorger]; see also remark 2 below). If n and d are coprime then $\mathcal{M}_T(n,d) = \mathcal{M}^s_T(n,d)$ and $\mathcal{M}_T(n,d)$ is a fine moduli space. In particular when $n=1$, $\mathcal{M}_T(1,d) = \overline{J(X)}$ is the *compactified Jacobian* of X.

Remarks.

1. $\mathcal{M}_T(n,d)$ is generally singular, even in the coprime case (see Chapters 5 and 6 for examples). In addition, $\mathcal{M}_T(n,d)$ may be reducible—in fact [Rego 2], $\mathcal{M}_T(n,d)$ is irreducible if and only if X is embeddable in a smooth

surface; also $\overline{J(X)}$ is reduced in this case [Altman, Iarrobino, Kleiman]. The dimension of $\mathcal{M}_T(n,d)$ depends on the type of singularities that X has (cf. Chapter 6); however, we do have $\mathcal{M}_{Vect(X)}(n,d) \subset \mathcal{M}_{TFS(X)}(n,d)$ as an open irreducible subset and $\dim \mathcal{M}_{Vect(X)}(n,d) = n^2(g(X)-1)+1$ (if not empty) and so when X is planar this is also the dimension of $\mathcal{M}_{TFS(X)}(n,d)$. More generally, we give an upper bound on the dimension in (4.3.7).

2. Note that at a stable point $[\mathcal{F}]$, using the moduli space property, the Zariski tangent spaces coincides with the space of families of sheaves $\mathcal{E}$ parametrised by $S = Spec(k[\epsilon]/\epsilon^2)$ such that $\mathcal{E} \otimes k \simeq \mathcal{F}$. This is the space of (first order) *infinitesimal deformations* of $\mathcal{F}$. Given such a deformation one obtains an exact sequence

$$0 \longrightarrow k\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

and flatness of $\mathcal{E}$ over S implies that $k\mathcal{E} \simeq \mathcal{F}$. In this way one can show that infinitesimal deformations correspond bijectively to elements of $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$.

4.1.3 GPB($\tilde{X}, \tau$) [Bhosle 1].

τ will denote the fixed parabolic structure, consisting of a fixed divisor D_i over each singular point $x_i \in X$. Fix, also, a rational weight $\alpha < 1$. There exists a non singular quasi-projective variety $\mathcal{M}_G^{\alpha-s}(n,d,\underline{a}) = \mathcal{M}^{\alpha-s}_{GPB(\tilde{X},\tau)}(n,d,\underline{a})$ which is a moduli space for α-stable generalised parabolic bundles $(E, \underline{F}_1(E))$ on $\tilde{X}$ of rank n and degree d with $\dim F_1^i(E) = a_i$, where $\underline{a} = (a_1, ..., a_N)$. Adding s-equivalence classes of α-semistable GPBs gives a normal projective variety $\mathcal{M}_G^{\alpha}(n,d,\underline{a}) = \mathcal{M}^{\alpha}_{GPB(\tilde{X},\tau)}(n,d,\underline{a})$. The dimension is given by

$$\begin{aligned}\dim \mathcal{M}_G^{\alpha}(n,d,\underline{a}) &= n^2(g(\tilde{X})-1)+1+\sum_i \dim \mathrm{Gr}(a_i, n \deg D_i) \\ &= n^2(g(\tilde{X})-1)+1+\sum_i a_i(n\deg D_i - a_i).\end{aligned}$$

If n and d are coprime then $\mathcal{M}_G^{s}(n,d,\underline{a})$ is a fine moduli space, and if, additionally,

$$(*): \quad n \text{ divides } a = \sum_i a_i \text{ and } 1 - 1/(a(n-1)) < \alpha < 1$$

then $\mathcal{M}_G^{\alpha-\bullet}(n,d,\underline{a}) = \mathcal{M}_G^{\alpha}(n,d,\underline{a})$. If α' is a rational weight also satisfying the inequality in $(*)$ then $\mathcal{M}_G^{\alpha}(n,d,\underline{a}) \simeq \mathcal{M}_G^{\alpha'}(n,d,\underline{a})$ and we will write simply $\mathcal{M}_G(n,d,\underline{a})$ and call it the moduli space of stable GPBs.

4.1.4 PMod$(\tilde{X},\tau)$.

Fix discrete invariants: τ, etc. as in (4.1.3). There exists a quasi-projective scheme $\mathcal{M}_P^{\alpha-\bullet}(n,d,\underline{a}) = \mathcal{M}^{\alpha-\bullet}_{PMod(\tilde{X},\tau)}(n,d,\underline{a})$ which is a moduli space for α-stable parabolic Modules of rank n, degree d with $\dim F_1^i(E) = a_i$ for each i. This has a natural compactification to a projective scheme $\mathcal{M}_P^{\alpha}(n,d,\underline{a}) = \mathcal{M}^{\alpha}_{PMod(\tilde{X},\tau)}(n,d,\underline{a})$. $\mathcal{M}_P^{\alpha-\bullet}(n,d,\underline{a})$ may be singular, indeed, it can be reducible and non-reduced. $\mathcal{M}_P^{\alpha}(n,d,\underline{a})$ is a closed subscheme of $\mathcal{M}_G^{\alpha}(n,d,\underline{a})$. If n and d are coprime then $\mathcal{M}_P^{\alpha-\bullet}(n,d,\underline{a})$ is a fine moduli space (the universal family on $\mathcal{M}_G^{\alpha-\bullet}(n,d,\underline{a})$ restricts to a universal family on $\mathcal{M}_P^{\alpha-\bullet}(n,d,\underline{a})$). If, additionally, $(*)$ of (4.1.3) is satisfied then $\mathcal{M}_P^{\alpha-\bullet}(n,d,\underline{a}) = \mathcal{M}_P^{\alpha}(n,d,\underline{a}) = \mathcal{M}_P(n,d,\underline{a})$.

§4.2 Remarks on Construction

The standard method of construction is via Mumford's Geometric Invariant Theory. One first finds a space, $\tilde{S}$, where all the required objects are represented, and where a reductive algebraic group G acts so that objects are isomorphic if and only if they lie in the same orbit. The moduli space desired is the quotient $\tilde{S}/G$ (if it exists). To prove that this quotient exists it is usually necessary to find a G-equivariant embedding of $\tilde{S}$ in a 'better understood' space Z where the quotient is known to exist (on a suitable open set). In practice Z is often a product of Grassmannians. It is not our intention to delve deeply into questions about taking quotients; rather, we will show how to get $\tilde{S}$ and appeal to the fact that $\tilde{S}$ is a closed subscheme of the appropriate scheme $\tilde{R}$ for the GPB case, where we know the quotient exists by the results of [Bhosle 1, 2].

First it is necessary to define the notion of a family of parabolic Modules. If $E \longrightarrow T \times \tilde{X}$ is a family of vector bundles of rank n and degree d parametrised by a k-scheme T, then write E_{D_i} for the restriction of E to $T \times D_i$. A family of GPBs should consist of a family of vector bundles E together with, for each i, a rank a_i subbundle, $F_1^i(E)$, of $(p_T)_* E_{D_i}$. This will be a family of parabolic Modules if for each $t \in T$ $(F_1^i(E))_t$ is an $\mathcal{O}_{x_i}$-submodule of $E_t \otimes \mathcal{O}_{D_j}$. Equivalently, we can make the definition below. Let $\mathrm{Gr}(a_i, E_{D_i})$ be the Grassmannian bundle of rank a_i subbundles of E_{D_i} over T, and let $\mathrm{Gr}^X(a_i, E_{D_i})$ be the fixed point scheme under the natural action of $(\mathcal{O}_{x_i}/I_{D_i})^*$. Let $\mathrm{Gr}(\underline{a}, E_{\underline{D}})$ (resp. $\mathrm{Gr}^X(\underline{a}, E_{\underline{D}})$) be the product over T of these spaces over the set of all the D_i. We view $\mathrm{Gr}^X(\underline{a}, E_{\underline{D}})$ as a closed subscheme of $\mathrm{Gr}(\underline{a}, E_{\underline{D}})$. Then *a family of generalised parabolic bundles* of rank n and degree d and type $\{a_1 \dots a_N\}$ parametrised by T is a pair (E, Φ), where $E \longrightarrow T \times \tilde{X}$ is a family of vector bundles of rank n and degree d over $\tilde{X}$ and Φ is a section of $\mathrm{Gr}(\underline{a}, E_{\underline{D}})$ over T. If Φ is a section of $\mathrm{Gr}^X(\underline{a}, E_{\underline{D}})$ then (E, Φ) is *a family of parabolic Modules.*

Given this, it is easy to construct moduli spaces in the rank 1 case, where stability is automatic:

Proposition 4.2.1. In the case of rank 1, the moduli space $\mathcal{M}_{GPB}(1, d, \underline{a})$ (resp. $\mathcal{M}_{PMod}(1, d, \underline{a})$) has a fibration over the Jacobian $J(\tilde{X})$ of $\tilde{X}$ with fibre $\mathrm{Gr}(\underline{a}, \mathcal{O}_{\underline{D}}) = \prod_i \mathrm{Gr}(a_i, \mathcal{O}_{D_i})$ (resp. $\mathrm{Gr}^X(\underline{a}, \mathcal{O}_{\underline{D}}) = \prod_i \mathrm{Gr}^X(a_i, \mathcal{O}_{D_i})$).

Proof. Let $\mathcal{U} \longrightarrow J(\tilde{X}) \times \tilde{X}$ be the Poincaré bundle, universal for families of line bundles of degree d on $\tilde{X}$. Write $\mathcal{U}_{\underline{D}}$ for the restriction of $\mathcal{U}$ to $J(\tilde{X}) \times \prod D_i$. As above we can form $\mathrm{Gr}(\underline{a}, \mathcal{U}_{\underline{D}})$ and $\mathrm{Gr}^X(\underline{a}, \mathcal{U}_{\underline{D}})$. These give the required moduli spaces, and the result is clear.

We now set up the problem in the general case so as to be able to apply Geometric Invariant Theory. Fix discrete invariants as in (4.1.3) (parabolic structures, rank and degree, α). Let $\mathcal{S}$ denote the corresponding set of semistable GPBs. Choose m_0 so that the conclusions of (3.5.7) are satisfied for any $E \in \mathcal{S}$. That is, we have that, for any $m \geq m_0$, $E(m)$ is generated by sections, $h^0(E(m)) = N$, $h^1(E(m)) = 0$ and $E(m) \longrightarrow E(m) \otimes \mathcal{O}_D$ is surjective. Let $Q = Quot(\mathcal{O}_{\tilde{X}}^N, P)$ denote the quot scheme parametrising quotients of $\mathcal{O}_{\tilde{X}}^N$ with Hilbert polynomial $P = P(E)$, denote the universal bundle on $Q \times \tilde{X}$ by $\mathcal{U}$. Define

$$R = \{q \in Q | \ \mathcal{U}_q \text{ is locally free and generated by sections,}$$

$$h^0(\mathcal{U}_q) = N, \text{ and } h^1(\mathcal{U}_q) = 0\}.$$

So $\mathcal{U}' = \mathcal{U}|_R$ is a family of vector bundles of rank n and degree d parametrised by R. Write $\tilde{R} = \mathrm{Gr}(\underline{a}, \mathcal{U}'_{\underline{D}})$ and $\tilde{S} = \mathrm{Gr}^X(\underline{a}, \mathcal{U}'_{\underline{D}})$. By the choice of m all elements of $\mathcal{S}$ are represented in $\tilde{R}$. Denote by $\tilde{R}^s$ (resp. $\tilde{R}^{ss}$) the open subscheme of $\tilde{R}$ corresponding to stable (resp. semistable) GPBs. Similarly define $\tilde{S}^s$, $\tilde{S}^{ss}$.

The group $SL(N)$ acts on ${\mathcal{O}_X}^N$ and hence on Q, $\tilde{R}$, $\tilde{R}^s$ and $\tilde{R}^{ss}$.

Lemma 4.2.2. The action of $SL(N)$ on $\tilde{R}$ restricts to an action of $SL(N)$ on each of $\tilde{S}$, $\tilde{S}^s$ and $\tilde{S}^{ss}$.

Proof. By (3.5.1) $\tilde{S}^s = \tilde{S} \cap \tilde{R}^s$ and $\tilde{S}^{ss} = \tilde{S} \cap \tilde{R}^{ss}$ so we just need to see that the action restricts to an action on $\tilde{S}$. But this follows from the fact (3.3.1) that $PMod$ is a full subcategory of GPB.

Now, to prove that the moduli spaces of (4.1.3) (resp. (4.1.4)) exist we must prove that there exists a good quotient of $\tilde{R}^{ss}$ (resp. $\tilde{S}^{ss}$) by $SL(N)$ and a geometric quotient of $\tilde{R}^s$ (resp. $\tilde{S}^s$) by $SL(N)$. A useful lemma is the following.

Lemma 4.2.3 Ramanathan's Lemma (see [Newstead Prop. 3.12.]).

Let G be a reductive group acting on varieties Y, Y_1. Suppose $f : Y_1 \longrightarrow Y$ is an affine G-morphism; then if there exists a good quotient of Y mod G there also exists a good quotient of Y_1.

Remark. Although only stated for varieties the result easily generalises to schemes, the key fact in either case being that the morphism is affine.

A consequence of this is that a good quotient of $\tilde{S}^{ss}$ exists if a good quotient of $\tilde{R}^{ss}$ exists.

The existence of a quotient for $\tilde{R}^{ss}$ has been proved by Bhosle in 2 different ways—using both the methods of Gieseker and of Simpson. We make some brief remarks. In either case, one wants to find a $SL(N)$-equivariant embedding $\phi : \tilde{R} \longrightarrow Z$ in a scheme Z, and a polarisation of Z such that, if Z^s (resp. Z^{ss}) denotes the corresponding set of stable (resp. semistable) points of Z, then $\phi(\tilde{R}^s) = Z^s$ (resp. $\phi(\tilde{R}^{ss}) = Z^{ss}$). It may be necessary to increase m some more to get such an embedding. It then follows from Mumford's theory that there exists a geometric (resp. good) quotient of Z^s (resp. Z^{ss}). Now an application of Ramanathan's Lemma gives the result for the moduli space of $GPBs$.

We should remark that for '$\alpha = 1$' the embedding $\tilde{R} \longrightarrow Z$ fails to have the required properties with respect to stability, so the proof breaks down. The statements of (4.1.3/4) regarding the choice of α essentially follow from (3.5.10) (again, see [Bhosle 1]).

§4.3 Morphisms

Given that it is possible to construct moduli spaces for the categories we have been considering it is natural to hope that the functors of Chapter 3 give rise to morphisms between these spaces. Of course, it is necessary to extend the definitions of these functors to families, and to prove flatness of the image families.

Lemma 4.3.1 Flatness.

1. If $\pi : X' \longrightarrow X$ is a partial normalisation of X' and if $\mathcal{F} \longrightarrow T \times X'$ is a family of torsion free sheaves, flat over T, then $(1 \times \pi)_* \mathcal{F}$ is a family of torsion free sheaves on X which is also flat over T.

2. If $\mathcal{F} \longrightarrow T \times X$ is a family of torsion free sheaves write $\mathcal{F}^* = \mathbf{Hom}(\mathcal{F}, p_X^* \omega_X)$ for the family of dual sheaves. Then, if $\mathcal{F}$ is flat over T the dual family $\mathcal{F}^*$ is also flat over T.

Proof.

1. Consider the more general set up of a morphism $f : Y \longrightarrow Z$ in k-Sch and a factorisation $f = h \circ g$, $Y \xrightarrow{g} Y_1 \xrightarrow{h} Z$. Suppose $\mathcal{F}$ is a coherent sheaf on Y flat over Z, then we claim that $g_* \mathcal{F}$ is also flat over Z. Look at $h_*(g_* \mathcal{F})$. Let $U \subset Z$ be an open set; by definition we have

$$h_*(g_*\mathcal{F})(U) = g_*\mathcal{F}(h^{-1}(U)) = \mathcal{F}(g^{-1}(h^{-1}(U))) = \mathcal{F}(f^{-1}(U)) = f_*\mathcal{F}(U).$$

So $f_*\mathcal{F}$ and $h_*(g_*\mathcal{F})$ are locally isomorphic and the flatness of $h_*(g_*\mathcal{F})$ over Z follows.

2. For a proof of this we refer to [Sorger, 3.1].

Proposition 4.3.2. Let $\pi : X' \longrightarrow X$ be a partial normalisation of X; write $\delta' = \delta(X) - \delta(X')$ Then for each pair (n, d), π_* gives a morphism

$$\pi_* : \mathcal{M}_{T(X')}(n, d) \longrightarrow \mathcal{M}_{T(X)}(n, d + n\delta').$$

Proof. We already know (3.5.8) that π_* maps stable (resp. semistable) sheaves to stable (resp. semistable) sheaves. Note that if $F \longrightarrow T \times \tilde{X}$ is a family of bundles on $\tilde{X}$ parametrised by a noetherian scheme T, then $(1 \times \pi)_*$ gives a family of torsion free sheaves on X which is flat over T by the above lemma. Thus, using the moduli space property, π_* induces the required morphism.

Remark. Note that, in the situation above $\pi^*/torsion$ does not give rise to a morphism in the other direction. This is because, for $\mathcal{F}$ a sheaf on X, $\deg \pi^*/torsion(\mathcal{F}) - \deg(\mathcal{F})$ depends on the local type of $\mathcal{F}$, and so flatness of a general family would not be preserved. This is basically telling us that the singularities of $\mathcal{M}_{T(X)}(n,d)$ are worse than those of $\mathcal{M}_{T(X')}(n,d)$.

Proposition 4.3.3. Duality on X induces an isomorphism

$$\underline{\omega} : \mathcal{M}_T(n,d) \longrightarrow \mathcal{M}_T(n, 2n(g-1)-d).$$

Moreover, $\underline{\omega} \circ \underline{\omega} : \mathcal{M}_T(n,d) \longrightarrow \mathcal{M}_T(n,d)$ is the identity.

Proof. If $\mathcal{F}$ is a torsion free sheaf on X then $\mu(\mathcal{F}^*) = 2(g-1) - \mu(\mathcal{F})$ (3.1.6). Since $\mathbf{Ext}^1(\mathcal{F}, \omega) = 0$ if, for example, $\mathcal{E}$ is a subsheaf of $\mathcal{F}$ which contradicts stability then $\mathcal{E}^*$ will be a quotient sheaf of $\mathcal{F}^*$ contradicting stability. Hence $\mathcal{F}$ is stable (resp. semistable) if and only if $\mathcal{F}^*$ is stable (resp. semistable) . Since (4.3.1(2)) tells us that the dual of a flat family is flat the result follows.

The main result of this section concerns Ψ_*. This result generalises [Bhosle, Theorem 4.2] where this theorem is proved for curves with nodes and cusps.

Theorem 4.3.4.

Fix parabolic type τ as follows: for each singular point $x_i \in X$, let $D_i \subset \tilde{X}$ be a divisor defined by an ideal, I_i, contained in the conductor $\mathcal{C}_x$ with $\deg D_i = \delta(x_i) + \ell_i$ ($\ell_i = \dim(\mathcal{O}_{x_i}/I_i)$). Let $\mathcal{M}_P(n,d,\underline{a})$ be the moduli space of parabolic Modules $(E, \underline{F}_1(E))$ on $\tilde{X}$ of rank n and degree d with parabolic structures over the D_i, $\dim F_1^i(E) = \ell_i.n$. [N.b. we choose weight α such that $1 - 1/(\sum(\delta(x_i) + \ell_i)n(n-1)) < \alpha < 1$.]

Then Ψ_* gives a proper morphism

$$\Psi_* : \mathcal{M}_P(n,d,\underline{a}) \longrightarrow \mathcal{M}_T(n,d)$$

with the properties:

1. The restriction of Ψ_* to the preimage of the (stable) locally free sheaves is an isomorphism:

$$\Psi_* : \Psi_*^{-1}\left(\mathcal{M}_V^s(n,d)\right) \xrightarrow{\sim} \mathcal{M}_V^s(n,d).$$

2. Suppose, also that $I_i \subset \tilde{O}_{x_i}^*$ for each i. Then, if either $(n,d) = 1$ or $g(\tilde{X}) \geq 2$ then Ψ_* is surjective.

Proof. In Chapter 3 we have established most of the required properties 'pointwise', in order to prove that Ψ_* gives a morphism it is necessary to show that we can extend the definition of Ψ_* to *families* of parabolic Modules. Let $(E, \underline{F}_1(E))$ be a family of semistable parabolic Modules parametrised by T, $E \longrightarrow T \times \tilde{X}$. Write $D = \sum_i D_i$ and $F_1(E)$ the subbundle of $E \otimes p_X^* \mathcal{O}_D$ given by $F_1(E) = \oplus F_1^i(E)$. Then define a family of torsion free sheaves $\mathcal{E} = \Psi_*(E, \underline{F}_1(E))$ on X parametrised by T by the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow (1 \times \pi)_* E \longrightarrow E \otimes p_X^* \mathcal{O}_D / F_1(E) \longrightarrow 0.$$

$\mathcal{E}$ is flat over T because the last two terms of this sequence are flat over T. Moreover, for each $t \in T$ this restricts to the definition of Ψ_* in §3.4. We know (3.5.9/10) that if $(E, \underline{F}_1(E))_t$ is a semistable parabolic Module it is 1-semistable and so $\mathcal{E}_t$ is also semistable. It follows that Ψ_* induces a morphism $\Psi_* : \mathcal{M}_P(n, d, \underline{a}) \longrightarrow \mathcal{M}_T(n,d)$, and this morphism is proper since these spaces are projective.

1. We have constructed the inverse pointwise on $\mathcal{M}_V(n,d)$ in (3.4.3/4). Note that if $\mathcal{E}$ is a stable locally free sheaf then the preimage $(E, F_1(E))$ is 1-stable, hence stable by (3.5.9). If $\mathcal{E} \longrightarrow T \times X$ is a family of stable torsion free sheaves then $\Psi^* \mathcal{E} = (E, F_1(E))$ is the family given by

$$E = (1 \times \pi)^* \mathcal{E},$$

$$F_1(E) = \mathcal{E} \otimes p_X^* \mathcal{O}_D \subset (1 \times \pi)^* \mathcal{E} \otimes p_X^* \mathcal{O}_D = E \otimes p_X^* \mathcal{O}_D.$$

2. By (3.4.8) we know that for any torsion free sheaf $\mathcal{F}$ on X there is a parabolic Module E on $\tilde{X}$ of the above form mapping to it under Ψ_*. Moreover we

know that if $\mathcal{F}$ is stable then E is 1-stable (3.5.9) and hence stable, by the choice of α (3.5.10). If $(n,d) = 1$ this is enough to prove surjectivity. If $(n,d) \neq 1$ then there would appear to be a problem involving semistables (which are not stable). We know that any semistable parabolic Module is 1-semistable, hence maps to a semistable torsion free sheaf. In contrast, we do not *a priori* know that any strictly semistable torsion free sheaf has a semistable parabolic Module mapping to it. However, since $\mathcal{M}_P(n,d)$ is projective the image of Ψ_* must be complete, and so to prove surjectivity it is enough to prove that stable sheaves are dense in any component of $\mathcal{M}_T(n,d)$. This we prove in (4.3.5) below under the assumption that $g(\tilde{X}) \geq 2$.

Remark. There is the possibility that some stable parabolic Modules map to torsion free sheaves which are only semistable. This situation is not ideal; one attempt to remedy this (at least in the case of a curve with nodes) is by enlarging the category of parabolic Modules, essentially by allowing bundles with torsion. This was done in the paper of [Narasimhan and Ramadas]; it seems that the extra complications introduced in this way are considerable.

For each singular point x_j of X let M_j be a rank n torsion free $\mathcal{O}_{x_j}$-module; write $\underline{M}$ for the set $\{M_j\}$. Let $U^s_{\underline{M}}$ be the locally closed subset of $\mathcal{M}_T(n,d)$ consisting of stable sheaves of type $\underline{M}$ (i.e those $\mathcal{F}$ with $\mathcal{F}_{x_j} \simeq M_j$ for each j). At a semistable point the local type will depend on the particular choice of a representative of the s-equivalence class, in general. However, we can still define $U_{\underline{M}} \subset \mathcal{M}_T(n,d)$ to be the set of points such that some representative of the s-equivalence class has type $\underline{M}$.

Theorem 4.3.5.

1. (i) $U_{\underline{M}}$ is irreducible and (ii) if $U^s_{\underline{M}} \neq \emptyset$ then $U_{\underline{M}} \subset \overline{U^s_{\underline{M}}}$.

2. Suppose $g(\tilde{X}) \geq 2$, then:

   (i) For all $\underline{M}$ there exist stable sheaves of type $\underline{M}$: i.e. $U^s_{\underline{M}} \neq \emptyset$;

   (ii) $\mathcal{M}_T(n,d) = \overline{\mathcal{M}^s_T(n,d)}$;

   (iii) $\mathcal{M}_T(n,d)$ is connected.

Proof.

1. Firstly we briefly recall how the proof works in the case of vector bundles on X—full details can be found in [Seshadri]. We can assume (after tensoring with a line bundle) that d is such that all semistable bundles of degree d are generated by sections (3.5.6). If E is a vector bundle generated by sections then there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{n-1} \longrightarrow E \longrightarrow \det E \longrightarrow 0.$$

As $\det E$ varies over $J(X)$ families of such extensions give an irreducible family containing all bundles of given degree which are generated by sections. In particular all semistable bundles of rank n and degree d can be parametrised by an irreducible family. The above proof does not work for sheaves of type $\underline{M}$, in general, since there is no analogue of the above sequence. However, we can deduce the result as follows. Choose inclusions $M_j \subset \mathcal{O}_{x_j}^n$ for each j. Given a locally free sheaf $\mathcal{E}$ and a basis for the stalk at each singular point we can use these to canonically define a subsheaf $\mathcal{F}$ of type $\underline{M}$ of $\mathcal{E}$. Conversely any such $\mathcal{F}$ can be written in this form. So given $\mathcal{F}$, a semistable sheaf of type $\underline{M}$, we can write $\mathcal{F} \subset \mathcal{E}$ and vary $\mathcal{E}$ as above. This will give an irreducible family containing all points of $U_{\underline{M}}$. Since stability is open, when stable sheaves of type $\underline{M}$ exist, it also shows that any semistable sheaf of type $\underline{M}$ has a deformation to a stable sheaf of type $\underline{M}$, which proves (ii).

2. When $M_j = \tilde{\mathcal{O}}_{x_j}^n$ for each j write $U_{\tilde{\mathcal{O}}}^s = U_{\underline{M}}^s$. If $g(\tilde{X}) \geq 2$ then there exist stable bundles on $\tilde{X}$. Moreover, by (3.2.6), (3.5.8) and (4.3.2) we have

$$U_{\tilde{\mathcal{O}}}^s = \pi_* \mathcal{M}_V^s(n, d - n\delta(X)) \neq \emptyset.$$

The non-emptyness of all $U_{\underline{M}}$ now follows: by (2.3.2) and (3.4.8) we can find $(E, \underline{E}_1(E))$ which is stable with $\Psi_*(E, \underline{E}_1(E)) \in U_{\tilde{\mathcal{O}}}^s$, and such that $\underline{E}_1(E) \in \overline{W_{\underline{M}}}$, where $W_{\underline{M}}$ denotes the set of parabolic Modules over E of type $\underline{M}$. Consider $\overline{W_{\underline{M}}}$ as a family of torsion free sheaves on X: since stability is open and $(E, \underline{E}_1(E))$ is stable we see that there must exist stable sheaves of type $\underline{M}$. Using this and part 1 we also see that any semistable sheaf has a stable deformation. Connectivity follows because for all $\underline{M}$ we have $\overline{U_{\underline{M}}} \cap U_{\tilde{\mathcal{O}}} \neq \emptyset$, and $U_{\tilde{\mathcal{O}}}$ is irreducible.

Each space $\mathcal{M}_P(n,d,\underline{a})$ as in (4.3.4) above can be regarded as a compactification of $\mathcal{M}^s_V(n,d)$—the moduli space of stable vector bundles of rank n and degree d on X. In many cases these will be more 'economical' compactifications than the moduli space of torsion free sheaves on X, e.g.:

Suppose that all the singular points of X are cubical. Recall from (1.2.2) that $x \in X$ is a cubical singularity if and only if the conductor C of $\mathcal{O}_x$ in $\tilde{\mathcal{O}}_x$ is m_x. Then, taking D to be the divisor on $\tilde{X}$ defined by C, the action of $\mathcal{O}_x/C$ on $\mathcal{O}_D$ is just the action of k. Hence every GPB with parabolic structure defined on D is automatically a parabolic Module. Consequently:

Theorem 4.3.6. Suppose all the singularities x_i of X are cubical; let D_i be the divisor defined by C_{x_i}; let $\mathcal{M}_{GPB}(n,d)$ be the moduli space of GPBs of rank n and degree d on $\tilde{X}$ with flag terms F_1 of dimension n on each D_i. Then, if $\mathcal{M}_V(n,d)$ is the moduli space of rank n, degree d vector bundles on X, $\mathcal{M}_{GPB}(n,d)$ is a compactification of $\mathcal{M}_V(n,d)$ with $\dim \mathcal{M}_{GPB}(n,d) = \dim \mathcal{M}_V(n,d)$. If n and d are coprime then $\mathcal{M}_{GPB}(n,d)$ is a non-singular compactification.

Proof. Most of this follows from the above remarks and from the properties of $\mathcal{M}_{GPB}(n,d)$ (4.1.3), and of Ψ_* (4.3.4). It remains to check the equality of dimensions:

$$\dim \mathcal{M}_{GPB}(n,d) = n^2(g(\tilde{X})-1)+1+\sum_i n(n\deg(D_i)-n)$$

$$= n^2(g(\tilde{X})-1)+1+n^2\delta = n^2(g(X)-1)+1 = \dim \mathcal{M}_V(n,d).$$

Remark. If all the x_i are also double points then the resulting map $\Psi_* : \mathcal{M}_{GPB}(n,d) \longrightarrow \mathcal{M}_T(n,d)$ is surjective, and, in fact, the converse is true (— since this is the only case where X is Gorenstein and if X is not Gorenstein $\mathcal{M}_T(n,d)$ has at least 2 components.)

We can apply (4.3.4) to give an upper bound on the dimension of the space of torsion free sheaves on a singular curve.

Theorem 4.3.7. Recall that for $x \in X$ $e(x)$ denotes the multiplicity of x. Suppose that $g(\tilde{X}) \geq 2$. We have

$$\dim \mathcal{M}_T(n,d) \leq n^2 \left(g(\tilde{X}) + \sum_{x \in X} \delta(x) e(x) - 1 \right) + 1.$$

Proof. By (4.3.4) for a suitable choice of parabolic structures we have a surjective map

$$\mathcal{M}_P(n,d) \longrightarrow \mathcal{M}_T(n,d), \quad \text{so}$$

$$\dim \mathcal{M}_T(n,d) \leq \dim \mathcal{M}_P(n,d).$$

On an open dense subset of $\mathcal{M}_P^s(n,d)$ the underlying bundle will be stable, thus

$$\dim \mathcal{M}_P(n,d) \leq \dim \mathcal{M}_{V(\tilde{X})}(n,d) + \sum_{x \in X} \dim \left(\mathrm{Gr}^X(a_i, \mathcal{O}^n_{D_i}) \right).$$

At a point of $\mathrm{Gr}^X(a_i, \mathcal{O}^n_{D_i})$ corresponding to a module M with $I_i^n \subset M \subset \tilde{\mathcal{O}}^n_{x_i}$ the tangent space can be identified with $T(M) = \mathrm{Hom}_{\mathcal{O}_x}(M/I_i^n, (\tilde{\mathcal{O}}_{x_i})^n/M)$—(see [Greuel and Pfister, 1.13]). Now $(\tilde{\mathcal{O}}_{x_i})^n/M$ has dimension $\delta(x).n$ and, by (2.2.4) M/I_i^n requires $\leq ne(x)$ generators. Thus $\dim T(M) \leq n^2 \delta(x) e(x)$. Hence, combining this with the fact that $\dim \mathcal{M}_{V(\tilde{X})} = n^2(g(\tilde{X}) - 1) + 1$ we find

$$\dim \mathcal{M}_T(n,d) \leq \dim \mathcal{M}_P(n,d) \leq n^2 \left(g(\tilde{X}) + \sum_{x \in X} \delta(x) e(x) - 1 \right) + 1.$$

Remark. This upper bound is, in most cases, far from being effective (consider a plane curve, for example). However, there are examples to indicate that one cannot hope to improve much on this result—see §6.4.

§4.4 The Rank One Case

We summarise our results for the rank 1 case, with which the remaining 2 chapters are concerned.

Let X be a singular curve; by $\overline{J(X)}$ we will mean the compactified Jacobian of rank 1 degree 0 torsion free sheaves on X. Of course, $\overline{J(X)}$ contains the group variety $J(X)$ as an open subset, and we can consider the action of $J(X)$ on $\overline{J(X)}$. Let $\underline{M}$ be a collection of rank 1 torsion free $\mathcal{O}_{x_j}$-modules (one, M_j, for each singular point) and let $U_{\underline{M}}$ be the subset of $\overline{J(X)}$ of sheaves isomorphic to M_j at x_j. Recall, (3.1.2/3) that there exist sheaves locally isomorphic to a given set of modules and that any 2 locally isomorphic sheaves differ by an invertible sheaf. Thus $U_{\underline{M}}$ is a non-empty irreducible locally closed subvariety of $\overline{J(X)}$. We also see that $\dim U_{\underline{M}} \leq g = \dim J(X)$; in fact, if each M_j is isomorphic to an overring of $\mathcal{O}_{x_j}$ or to the dual of an overring then $U_{\underline{M}}$ has dimension $g(X) - \sum i(M_j)$(3.2.7). The tangent space to $\overline{J(X)}$ has constant dimension on $U_{\underline{M}}$: this dimension equals $g + \sum_i \chi_1(M_i)$ ((3.1.9), (4.1.2)). Duality implies that there are isomorphisms $U_{\underline{M}} \simeq U_{\underline{M}^*}$ (4.3.3).

The $U_{\underline{M}}$ cover $\overline{J(X)}$, so, in particular when each $\mathcal{O}_{x_j}$ has finite representation type we must have $\dim \overline{J(X)} = g$. In the next chapter we make a detailed study of the resulting stratification of $\overline{J(X)}$ in these cases. Of course, in general, to construct a sensible stratification, one would like to find a cruder equivalence relation on modules than isomorphism; some tentative steps in this direction are made in chapter 6.

A study of $\overline{J(X)}$ can often be made easier using properties of spaces of rank one parabolic Modules. Let $\underline{I}$ denote the choice of a normal ideal I_j of each $\mathcal{O}_{x_j}$. We will write $P_{\underline{I}}(X) = \mathcal{M}_{P(\underline{I})}(1, 0, \underline{a})$ for the associated space of rank 1 degree 0 parabolic Modules with $a_j = \dim(\mathcal{O}_{x_j}/I_j)$. This choice of the a_j ensures that Ψ_* preserves degree (3.4.1).

'Forgetting parabolic structure' gives a morphism

$$p : P_{\underline{I}}(X) \longrightarrow J(\tilde{X})$$

which is a fibration with fibre, $\prod \mathrm{Gr}^X(a_i, \tilde{\mathcal{O}}_{x_j}/I_j)$, a product of fixed point subschemes.

Theorem 4.4.1 see (3.4.10), (4.3.4).

Ψ_* gives a finite morphism

$$\Psi_* : P_{\underline{I}}(X) \longrightarrow \overline{J(X)}$$

such that

$$\Psi_* : \Psi_*^{-1}(J(X)) \longrightarrow J(X) \subset \overline{J(X)}$$

is an isomorphism. If $I_j \subset \tilde{\mathcal{O}}^*_{x_j}$ then Ψ_* is surjective.

We close this section with an example which shows the main reason why $P_{\underline{I}}(X)$ is easier to study than $\overline{J(X)}$. Recall (1.3.2) that $J(X)$ fibres over $J(\tilde{X})$; we show that this does not extend to a fibration of $\overline{J(X)}$ over $J(\tilde{X})$ in general.

Let X be a curve with a single node as its only singular point, and suppose $g(X) \geq 2$. Denote the node by x and its preimages in $\tilde{X}$ by y_1, y_2. Take I to be the maximal ideal of $\mathcal{O}_x$; then $P_I(X)$ is a $\mathbf{P}^1$ bundle over $J(\tilde{X})$ (see [Bhosle 1] or §5.2). For a line bundle L on $\tilde{X}$, $\Psi_*(L, F_1(L))$ is locally free on X unless $F_1(L) = L_{y_1}$ or $F_1(L) = L_{y_2}$. Thus we recover the sequence of algebraic groups

$$0 \longrightarrow k^* \longrightarrow J(X) \xrightarrow{\pi^*} J(\tilde{X}) \longrightarrow 0.$$

Let $V = ker \pi^* = \{\mathcal{L} \in J(X) |\ \pi^*\mathcal{L} = \mathcal{O}_{\tilde{X}}\}$ and let $\overline{V}$ be the closure of V in $\overline{J(X)}$. Then we must have

$$\overline{V} = \Psi_*(p^{-1}(\mathcal{O}_{\tilde{X}})),$$

since the latter is an irreducible closed set containing V as an open subset.

Consider the points of $p^{-1}(\mathcal{O}_{\tilde{X}}) - \Psi^* V$, namely $(L_i, F_1(L_i)) = (\mathcal{O}_{\tilde{X}}, k_{y_i})$, $i = 1, 2$. We find $\mathcal{L}_1 = \Psi_*(\mathcal{O}_{\tilde{X}}, k_{y_1})$ is given by the short exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \pi_*\mathcal{O}_{\tilde{X}} \longrightarrow k_{y_2} \longrightarrow 0,$$

i.e., $\mathcal{L}_1 = \pi_*\mathcal{O}(-y_2)$. Similarly, $\mathcal{L}_2 = \Psi_*(\mathcal{O}_{\tilde{X}}, k_{y_2}) = \pi_*\mathcal{O}(-y_1)$. Note that these are not isomorphic since $g(\tilde{X}) \geq 1$.

Now, $\mathcal{L}_1$ has 2 preimages in $P_I(X)$ (see (5.2.4)); the other one is

$$(L_1', F_1(L_1')) = (\mathcal{O}(y_1 - y_2), k_{y_2}).$$

Similarly, $\mathcal{L}_2$ has a preimage

$$(L_2', F_1(L_2')) = (\mathcal{O}(y_2 - y_1), k_{y_1}).$$

Thus $\Psi_{\bullet}^{-1}(\overline{V}) = p^{-1} \cup \{2 \text{ points}\}$. Moreover, we see that $\overline{J(X)}$ cannot fibre over $J(\tilde{X})$ because the fibres over $\mathcal{O}_{\tilde{X}}$ and $\mathcal{O}_{\tilde{X}}(y_1 - y_2)$ would intersect.

Remark. By the results of [D'Souza] the singular locus of $\overline{J(X)}$ consists of a normal crossing along the codimension 1 set of non-locally free sheaves. Using this and the above, we can also identify the map $\Psi_* : P_I(X) \longrightarrow \overline{J(X)}$. $P_I(X)$ has 2 distinguished sections over $J(\tilde{X})$:

$$\sigma_i : L \mapsto (L, k_{y_i}) \qquad i = 1, 2.$$

$\overline{J(X)}$ consists of $P_I(X)$ with the glueing

$$\sigma_1 \sim \sigma_2 \otimes \mathcal{O}_{\tilde{X}}(y_1 - y_2).$$

CHAPTER 5

EXAMPLES: SIMPLE SINGULARITIES

§5.1 Statement of Results

We apply the results of the previous chapter to some specific examples, all of these concern the rank one case. The main result of this chapter is the theorem below, which describes the stratification of the compactified Jacobian $\overline{J(X)}$ for X a curve with a single simple singularity, x. We should remark that the problem is purely local, so the extension to the case of several singularities is easy. Assume throughout that the ground field has characteristic zero. We split the detailed proof into cases; §5.2 deals with x of type A_n, §5.3 type D_n and §5.4 types $E_{6,7,8}$. In addition in §5.5 we explain what this tells us in the other finite representation cases: $D_n^-, E_{6,7,8}^-$. The main body of the chapter also contains a number of results about the schemes $P(X) = P_C(X)$ of §4.4; in some cases we show that these give normalisations of $\overline{J(X)}$.

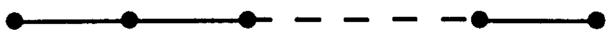
Theorem 5.1. Let X be a curve with a single simple singularity, x. Then $\overline{J(X)}$ is a projective variety. If M is a rank one torsion free module over O_x then let U_M be the subvariety corresponding to sheaves locally isomorphic to M at x.

1. U_M has codimension equal to $i(M) =$ index(M).
2. The tangent space to $\overline{J(X)}$ at a point of U_M has dimension $g + i(M)$.

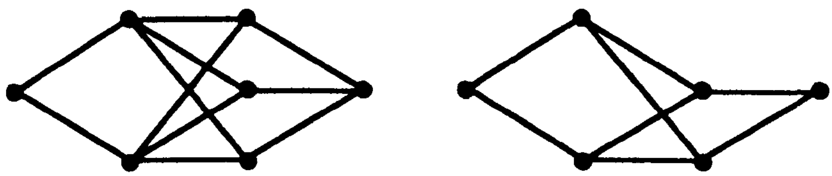
Further, the stratification of $\overline{J(X)}$ according to local type is given by the following diagrams. By this, we mean that each vertex corresponds to a locally closed subvariety, U_M for some torsion free module M, the codimension is equal to the length from the leftmost vertex ($U_{M_0} = J(X)$)—corresponding to locally free sheaves. Two vertices are joined if and only if the the closure of the component on the left contains the component on the right. In each case the total length of the diagram is equal to $\delta(x)$. Label the vertices so that the topmost chain from left to right corresponds to the sequence U_{M_0}, $U_{M_1}, \ldots, U_{M_\delta}$ (cf. §2.4). Then duality corresponds to reflection in the line joining the left and rightmost vertices, except in the penultimate column, where duality fixes each component. For clarity we have drawn the diagrams for D_4 (resp. D_5) separately from the general case of D_{even} (resp. D_{odd}).

Stratification Diagrams for $\overline{J(X)}$
for X with a Single Simple Singularity.

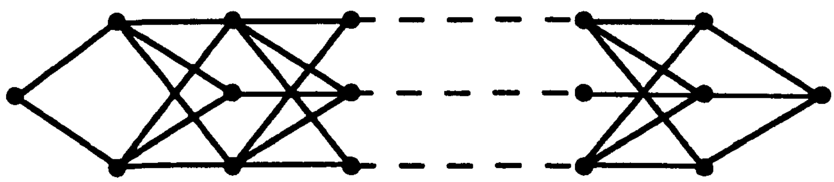
A_n



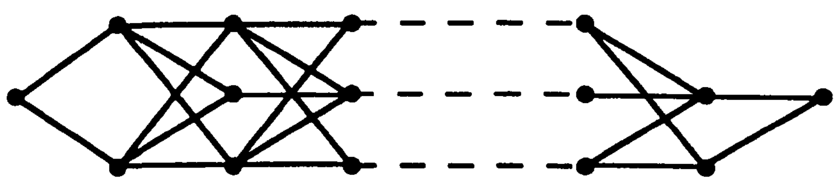
$D_{4/5}$



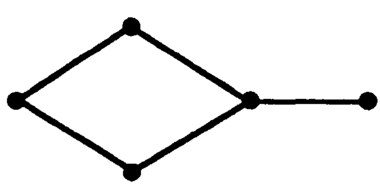
D_{even}



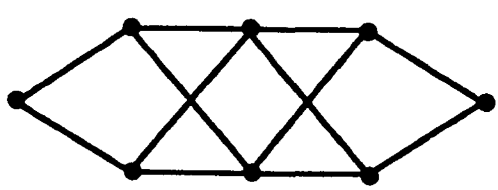
D_{odd}



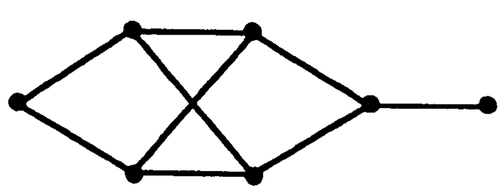
E_6



E_7



E_8



Proof. We leave the details to the later sections of the chapter and make here only a few general observations. The fact that $\overline{J(X)}$ is a projective variety follows from the fact that all these singularities are planar by the general result of [Altman, Iarrobino and Kleiman]. Alternatively, this follows from our other results since we directly verify that $\overline{J(X)}$ is irreducible, whilst the fact that $\overline{J(X)}$ is reduced follows from the fact that the dimension of the tangent space jumps by only 1 at each step, moving down the stratification. For the simple singularities the remark that every rank 1 module is either isomorphic to an overring or the dual of an overring (2.4.2) together with (3.2.5), which says that every sheaf locally isomorphic to an overring is a direct image of a unique sheaf on a partial normalisation, imply the conclusion of **1** (using also the properties of duality). The dimensions of the tangent spaces have already been calculated (see (2.5.8), (3.1.8/9) and §4.4).

As far as the diagrams go, we remark that we already know how many components there are of given codimension from §4.4 and the module classification. This is already enough to deal with the cases $A_n, D_{4,5}, E_6$. Also we know that if $\pi' : X' \longrightarrow X$ is a partial normalisation of X then $\pi'_*(\overline{J(X')})$ is a closed subvariety of $\overline{J(X')}$. Using this, and duality, in the remaining cases it is sufficient to prove that the closure of one of the codimension 1 strata really does contain all the codimension 2 strata: more specifically that the closure of U_{M_1} contains $U_{M_2^*}$. This we will verify in the section relevant to each particular case, using the method outlined below.

Let $P(X)$ denote the space of parabolic Modules with parabolic structures over the divisor D defined by $\mathcal{C}$ (flag terms of dimension δ). From §4.4 we have a finite surjective map $\Psi_* : P(X) \longrightarrow \overline{J(X)}$. Also there is a fibration $P(X) \longrightarrow J(\tilde{X})$ with fibre $\mathrm{Gr}^{\mathcal{O}_*}(\delta, \mathcal{O}_D)$. The stratification of $\overline{J(X)}$ according to local type can thus be deduced from that of $\mathrm{Gr}^{\mathcal{O}_*}(\delta, \mathcal{O}_D)$ according to module type. We proceed to prove that the above diagrams are correct by studying these schemes in detail—a similar method can also be used to prove statement **1**. In addition, we determine the number of preimages of each point in $\overline{J(X)}$ under Ψ_*. We will write $P(A_n)$ (etc.) for the reduced scheme $\mathrm{Gr}^{\mathcal{O}_*}(\delta, \mathcal{O}_D)_{\mathrm{red}}$ when x is of type A_n (etc.).

§5.2. A_n

We deal with the cases n odd/n even in parallel. The complete local ring at x is

$$A = \hat{\mathcal{O}}_x \simeq \begin{cases} k[[(t,t),(t^\delta,0)]] & \text{if } x \text{ is a node of type } A_{2\delta-1} \\ k[[t^2,t^{2\delta+1}]] & \text{if } x \text{ is a cusp of type } A_{2\delta} \end{cases}.$$

Note that $\delta(x) = \delta$: if $\bar{A} = \widetilde{(\hat{\mathcal{O}}_x)}$ denotes the normalisation then

$$\bar{A} = \begin{cases} k[[t_1]] \oplus k[[t_2]] \\ k[[t]] \end{cases}.$$

The conductor is

$$C = \begin{cases} \langle (t^\delta,0),(0,t^\delta) \rangle \\ \langle t^{2\delta}, t^{2\delta+1} \rangle \end{cases}.$$

Recall the classification of rank 1 torsion free modules over these rings as given in §2.4: for $0 \le j \le \delta$ there is a module M_j with

$$M_j \simeq_A \begin{cases} k[[(t,t),(t^{\delta-j},0)]] \\ k[[t^2,t^{2(\delta-j)+1}]] \end{cases}$$

which has generators $\begin{cases} \{(1,1),(t^{\delta-j},0)\} \\ \{1,t^{2(\delta-j)+1}\} \end{cases}$ over A and $i(M_j) = \ell(M_j) = j$.

Consider now the spaces $\mathrm{Gr}^{\mathcal{O}_x}(\delta,\mathcal{O}_D)$ and $P(A_n) = \mathrm{Gr}^{\mathcal{O}_x}(\delta,\mathcal{O}_D)_{\mathrm{red}}$.

Lemma 5.2.1. The cases $A_{2\delta-1}$ and $A_{2\delta}$ determine isomorphic subschemes of $\mathrm{Gr}(\delta,2\delta)$. In particular $P(A_{2\delta-1}) \simeq P(A_{2\delta})$.

Proof. Take the following bases of $\bar{A}/C$:

$$\{b_1,b_2,\ldots,b_{2\delta}\} = \begin{cases} \{(1,0),(0,1),(t,0),\ldots,(0,t^{\delta-1})\} \\ \{1, \quad t, \quad t^2,\ldots,t^{2\delta-1}\} \end{cases}.$$

$$s = \begin{cases} (t,t) \\ t^2 \end{cases}, \text{ the generator of } m/C \text{ acts by}$$

$$b_i \mapsto b_{i+2} \text{ (or } b_i \mapsto 0 \text{ if } i \ge 2\delta-1) \text{ in each case.}$$

It follows that the fixed point schemes determined by the action of $(A/C)^*$ are identical.

In order to describe these schemes we begin by working out the tacnode case (A_3; $\delta = 2$) explicitly. Note that if $\delta = 1$ then $\mathrm{Gr}^{\mathcal{O}_x}(\delta,\mathcal{O}_D) = \mathrm{Gr}(1,2) = \mathbf{P}^1$.

Proposition 5.2.2. $P(A_3)$ is a quadric cone in $\mathbf{P}^3$; there are 2 lines on the cone corresponding to points representing the maximal ideal, and the points representing the conductor are the cone point itself together with one other point on each of these lines.

Proof. The (complete) local ring is $A = k[[(t,t),(t^2,0)]]$; there are three modules M_0, M_1, M_2 isomorphic to $A, m^{-1} \simeq m, \bar{A} \simeq C$ respectively. Order a basis for $\bar{A}/C$ as $\{(1,0),(0,1),(t,0),(0,t)\}$.

We look for 2-dimensional subspaces F_1 of $\bar{A}/C$ which are A-modules. Note that, for F_1 to be a module, it is necessary and sufficient that

$$(a,b,c,d) \in F_1 \Rightarrow (t,t).(a,b,c,d) = (0,0,a,b) \in F_1.$$

Using this, if F_1 is a module then we can reduce a basis of F_1 to one of the following forms.

$$(i): \begin{Bmatrix} (1,\alpha,0,\beta) \\ (0,0,1,\alpha) \end{Bmatrix} \quad \alpha \neq 0.$$

$$(ii): \begin{Bmatrix} (1,0,0,\alpha) \\ (0,0,1,0) \end{Bmatrix} \quad (iii): \begin{Bmatrix} (0,1,\alpha,0) \\ (0,0,0,1) \end{Bmatrix} \quad \alpha \neq 0.$$

$$(iv): \begin{Bmatrix} (1,0,0,0) \\ (0,0,1,0) \end{Bmatrix} \quad (v): \begin{Bmatrix} (0,1,0,0) \\ (0,0,0,1) \end{Bmatrix} \quad (vi): \begin{Bmatrix} (0,0,1,0) \\ (0,0,0,1) \end{Bmatrix}.$$

Writing $p : \bar{A} \longrightarrow \bar{A}/C$ for the projection, $p^{-1}(F_1) \simeq A$ in case (i); $p^{-1}(F_1) \simeq m^{-1}$ in cases $(ii),(iii)$ and $p^{-1}(F_1) \simeq \bar{A}$ in cases $(iv),(v),(vi)$.

Taking the Plücker embedding of Gr$(2,4)$ in $\mathbf{P}^5$—

$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$$

—and ordering the coordinates on $\mathbf{P}^5$ as

$$(P_{12} : P_{13} : P_{14} : P_{23} : P_{24} : P_{34}),$$

the subspaces above correspond to the points:

$$\begin{array}{lll} (i) & (0:1:\alpha:\alpha:\alpha^2:-\beta) & \alpha \neq 0 \\ (ii) & (0:1:0:0:0:-\alpha) & \alpha \neq 0 \\ (iii) & (0:0:0:0:1:\alpha) & \alpha \neq 0 \\ (iv) & (0:1:0:0:0:0) & \\ (v) & (0:0:0:0:1:0) & \\ (vi) & (0:0:0:0:0:1). & \end{array}$$

$P(A_3)$ is the subvariety of $G(2,4)$ determined by these points. We see that $P_{12} = 0$ and $P_{14} = P_{23}$ on $P(A_3)$. Hence $P(A_3)$ can be regarded as a subvariety of $\mathbf{P}^3$ with equation $X_0^2 - X_1X_2 = 0$, i.e. $P(A_3)$ is a quadric cone. It is also clear that the points of type m^{-1} or of type $\bar{A}$ give two lines on $P(A_3)$. The cone point is of type $\bar{A}$ and there is one other such point on each line.

Alternatively, looking for fixed points directly,

$$((A/C)^*)/k^* = \{g_\lambda = (1,1) + \lambda(t,t)\}$$

acts on A/C by

$$g_\lambda : (y_1, y_2, y_3, y_4) \longmapsto (y_1, y_2, y_3 + \lambda y_1, y_4 + \lambda y_2).$$

In terms of the Plücker coordinates this gives

$$g_\lambda : (P_{12} : P_{13} : P_{14} : P_{23} : P_{24} : P_{34}) \longmapsto$$

$$(P_{12} : P_{13} : P_{14} + \lambda P_{12} : P_{23} : P_{24} + \lambda P_{12} : P_{34} + \lambda(P_{14} - P_{23}) + \lambda^2 P_{12}).$$

So, again, this gives us $P_{12} = (P_{14} - P_{23}) = 0$ for a fixed point. We remark that the subscheme $\mathrm{Gr}^{\mathcal{O}_\bullet}(2, \mathcal{O}_D)$ of $\mathrm{Gr}(2,4)$ is not reduced at the cone point: its tangent space has dimension 4 here (see (5.2.5) for a proof); this is similar to an example of [Fogarty]—the fixed point scheme for the action of $\mathbf{G}_a$ on $\mathbf{P}^1$ given by $(x : y) \mapsto (x + ty : y)$ consists of the point $(1 : 0)$ with non-reduced structure.

Proposition 5.2.3 Structure of $P(A_n)$.

1. $P(A_n)$ contains an open set, U_0, (of dimension δ) corresponding to subspaces F_1 with $p^{-1}(F_1) \simeq A$.

2. If n is even then $P(A_n) - U_0 \simeq P(A_{n-2})$.

3. If n is odd then $P(A_n) - U_0$ consists of 2 copies of $P(A_{n-2})$ intersecting in a $P(A_{n-4})$.

(So that this makes sense for all n define $P(A_0) = P(A_{-1}) = \{pt.\}$, $P(A_n) = \emptyset$ if $n < -1$.)

Proof. First fix some notation: use the basis of $\bar{A}/C$ and other notation given in (5.2.1.) and write π_j for the projection of $F_1 \subset \bar{A}/C$ onto the subspace of $\bar{A}/C$ spanned by b_j.

1. $p^{-1}(F_1) \simeq A$ implies that $\pi_1, \pi_2 \neq 0$ for n odd and $\pi_1 \neq 0$ for n even. So there exists u in F_1 with

$$u = (1, \alpha_1, \ldots)$$

with $\alpha_1 \neq 0$ if n is odd. Now $s^{\delta-1}(u) \neq 0$, so

$$\{u, s(u), \ldots s^{\delta-1}(u)\}$$

gives a basis for F_1. Subtracting suitable multiples of $s(u), s^2(u)$, etc. we can assume that

$$u = (1, \alpha_1, 0, \alpha_2, 0, \ldots, 0, \alpha_\delta).$$

In which case

$$p^{-1}(F_1) = \begin{cases} (1, \alpha_1 + \alpha_2 t + \ldots + \alpha_\delta t^{\delta-1})A & \text{if } n \text{ is odd} \\ (1 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_\delta t^{2\delta-1})A & \text{if } n \text{ is even} \end{cases}.$$

I.e., $p^{-1}(F_1) =$ (a unit of $\bar{A}$) $\times A$ in each case. It is clear that we cannot simplify u any further and thus this gives us the required open set U_0 of dimension δ.

2. If $F_1 \in P(A_n) - U_0$ (n even) then $\pi_1(F_1) = 0$. It follows, by a simple dimension count that F_1 must contain $(0, \ldots, 0, 1)$. Let $F_1' = \ker \pi_{2\delta}$, this is a $\delta - 1$ dimensional subspace of the space

$$\langle b_2, b_3, \ldots, b_{2\delta-1} \rangle$$

which is closed under the action of s, hence it corresponds to a point of a $P(A_{n-2})$.

3. This is very similar to the proof of **2.** If $F_1 \in P(A_n) - U_0$ then this time there are 2 possibilities, either $\pi_1 = 0$, or $\pi_2 = 0$—whence we get 2 copies of $P(A_{n-2})$, reasoning as above. Repeating the argument for these $P(A_{n-2})$s gives 4 copies of $P(A_{n-4})$, but 2 of these coincide (as in the A_3 example above), hence the result.

Corollary 5.2.4.

1. Up to $(\bar{A}/C)^*$ action each module M_j is represented $j+1$ (resp. 1) time(s) in $P(A_n)$ if n is odd (resp. even).

2. Choosing a normal ideal $I \subset C$ does not give any more preimages: i.e., $\mathrm{Gr}^A(\dim A/I, \bar{A}/I)_{\mathrm{red}} \simeq P(A_n)$.

Proof.

1. Note that in the proof above if F_1' represents M_j for A_{n-2} then F_1 represents M_{j+1} for A_n. So the result follows, using this as an inductive step and applying (5.2.3).

2. This is a consequence of the fact that there are precisely $j+1$ (resp. 1) ways of partitioning j into 2 (resp. 1) box(es): the full details of this are presented in (5.3.3) when we consider the case D_n.

Proposition 5.2.5. $P(A_n)$ is irreducible and nonsingular in codimension 1. In fact, for x a cusp of type A_n the tangent space to $\mathrm{Gr}^A(\delta, \mathcal{O}_D)$ at a point $[F_1]$, corresponding to a submodule F_1, is $\mathrm{Hom}_A(F_1, (\bar{A}/C)/F_1)$. This has dimension $\delta + 2\lfloor j/2 \rfloor$ when $p^{-1}(F_1) \simeq M_j$.

Proof. To prove irreducibility it is enough to look at the case n even—since by (5.2.1) $P(A_{2\delta}) \simeq P(A_{2\delta-1})$. And this is clear, either by looking at the proof of (5.2.3(2)) or by the fact that $\overline{J(X)}$ is irreducible and that the map $P(X) \longrightarrow \overline{J(X)}$ is one to one and onto.

The identification of the tangent space is due to [Greuel and Pfister 1.13]. We calculate the dimension of this space. Suppose $p^{-1}(F_1) \simeq M_j$. In the cusp case, up to multiplying a basis for F_1 by units of $\bar{A}$, we can write $F_1 = t^j M_j / C$. Then F_1 is generated over A by $\{t^j, t^{2\delta+1-j}\}$; the complement in $\bar{A}/C$ has basis $B = \{1, t, \ldots, t^{j-1}, t^{j+1}, t^{j+3}, \ldots, t^{2\delta-1-j}\}$.

$\mathrm{Hom}_A(F_1, (\bar{A}/C)/F_1)$ is spanned by elements of the form

$$\begin{cases} t^j \mapsto t^i \\ t^{2\delta+1-j} \mapsto 0 \end{cases} \text{ and } \begin{cases} t^j \mapsto 0 \\ t^{2\delta+1-j} \mapsto t^i \end{cases} \text{ where } i \in B.$$

Whilst all δ of the first type give rise to homomorphisms, a map of the latter type gives an A-module homomorphism if and only if

$$(t^2)^k t^{2\delta+1-j} = 0 \Rightarrow (t^2)^k t^i = 0.$$

It is clearly enough to check this for $k = \lfloor j/2 \rfloor$. For $i \equiv j (\mathrm{mod}\ 2)$ the condition is automatic: there are $\lfloor j/2 \rfloor$ such i in B. For $i \not\equiv j (\mathrm{mod}\ 2)$ the condition becomes $i + 2\lfloor j/2 \rfloor > 2\delta - 1 - j$.

$$\text{i.e.} \quad i > \begin{cases} 2\delta - 2j & \text{if } j \text{ is odd} \\ 2\delta - 1 - 2j & \text{if } j \text{ is even} \end{cases}.$$

Again there are $[j/2]$ such i in B, so the result on the dimensions follows. The singular set thus consists precisely of subspaces of type M_j for $j \geq 2$, and this set has codimension 2. Using the isomorphism $P(A_{2\delta}) \simeq P(A_{2\delta-1})$ of (5.2.1) the result for the nodal case also follows.

Now let us return to the global situation. Summing up, combining (4.4.1) with the results of this section ((5.2.2) and (5.2.5)) we have:

Theorem 5.2.6. Let X be a curve all of whose singularities are double points. Then $P(X)_{\mathrm{red}}$ is non-singular in codimension 1. If, further, $\delta(x) \leq 2$ at each singular point then $\Psi_* : P(X)_{\mathrm{red}} \longrightarrow \overline{J(X)}$ is the normalisation of $\overline{J(X)}$.

Remark. It would obviously be of interest to determine whether or not $P(A_n)$ is normal for all n.

§5.3 D_n

The schemes $P(D_n)$ are considerably more complicated than the schemes $P(A_n)$. We begin by analysing $P(D_4)$ in detail, where the presence of extra symmetry proves helpful.

Theorem 5.3.1. $P(D_4)$ is a cone on the del Pezzo surface S of degree 6 in $\mathbf{P}^6$; in particular $P(D_4)$ is normal. The complement of the set corresponding to the free module is a configuration of 6 planes, forming the hexagon of exceptional divisors on S.

Proof. Recall that for D_4, $\delta = 3$, $A = k[[(t,t,0),(t,0,t)]]$, $C = \langle (t^2,0,0),(0,t^2,0),(0,0,t^2)\rangle$. Take the following basis for $\bar{A}/C$:

$$\{b_1,\ldots,b_6\} = \{(1,0,0),(t,0,0),(0,1,0),(0,t,0),(0,0,1),(0,0,t)\}$$

and let $y_1 = (t,t,0)$ and $y_2 = (t,0,t)$ be the generators of the maximal ideal of A. There are modules

$$M_0, M_1, M_1^*, M_2^i \ (i = 1,2,3), M_3$$

—see §2.4 where the module M_1^3 was denoted N_2. We begin by determining which subspaces of $\bar{A}/C$ represent a given module. Suppose F_1 is a 3-dimensional submodule of $\bar{A}/C$; we specify F_1 by writing down a matrix whose rows give a basis for F_1. As before write π_i for the projection of F_1 onto the space spanned by b_i.

(*i*) Suppose that π_1, π_3 and π_5 are all non-zero. Then we can find (for dimensional reasons) a $u \in F_1$ such that $\pi_1(u), \pi_3(u), \pi_5(u) \neq 0$. Then $\{u, y_1(u), y_2(u)\}$ give a basis for F_1. We can reduce this to

$$\begin{pmatrix} 1 & 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 \\ 0 & 1 & 0 & \alpha_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \alpha_2 \end{pmatrix}$$

for $\alpha_1, \alpha_2 \neq 0$. Then $p^{-1}(F_1) \simeq A$; the set of all such points gives a 3-dimensional subvariety $U_0 \subset P(D_4)$.

(*ii*) Now suppose that exactly 2 of $\{\pi_1, \pi_3, \pi_5\}$ are non-zero, say just $\pi_5 = 0$, again we can find u with $\{u, y_1(u), y_2(u)\}$ giving a basis for F_1; this can be reduced to:

$$\begin{pmatrix} 1 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\alpha_1 \neq 0$. If $\alpha_2 \neq 0$ then $p^{-1}(F_1) \simeq M_1$; call the set of all such points U_5. If $\alpha_2 = 0$ then $p^{-1}(F_1) \simeq M_2^3$. Now, $U_5 \subset \overline{U}_0$; for, given $\alpha_1, \ \alpha_2 \neq 0$, consider the family of submodules, $F_{\lambda\mu}$ where

$$F_{\lambda\mu} \ni u_{\lambda\mu} = (1, 0, \alpha_1, 0, \lambda\alpha_2, \mu\alpha_2).$$

For $\lambda \neq 0$ this determines a point of U_0, but $\lambda = 0$ gives a point of U_5, and all points of U_5 arise in this way.

By symmetry we obtain also subvarieties $U_3, \ U_1 \subset \overline{U}_0$ (when π_3 or π_1 are zero, respectively).

(*iii*) Now suppose that only one of $\{\pi_1, \pi_3, \pi_5\}$ is non-zero,—say $\pi_1 \neq 0$. So F_1 contains $u = (1, \delta, 0, \alpha_2, 0, \alpha_1)$, $y_1(u) = y_2(u) = (0, 1, 0, 0, 0, 0)$. It is possible to write a third basis vector in the form $(0, 0, 0, \beta, 0, \gamma)$. Suppose first that $\beta \neq 0$, then this basis reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \gamma \end{pmatrix}.$$

If $\alpha_1, \ \gamma \neq 0$ then $p^{-1}(F_1) \simeq M_1^*$. Denote this set by $U_{3,5}$. Again it is not hard to see that $U_{3,5} \subset \overline{U}_0$. The boundary points of $U_{3,5}$ are as follows:

$$\begin{array}{ll} \alpha_1 = 0, \gamma \neq 0 & p^{-1}(F_1) \simeq M_2^1 \\ \alpha_1 \neq 0, \gamma = 0 & p^{-1}(F_1) \simeq M_2^2 \\ \alpha_1 = \gamma = 0 & p^{-1}(F_1) \simeq \bar{A}. \end{array}$$

On the other hand if $\beta = 0$, take $\gamma = 1$, then:

$$\begin{array}{ll} \alpha_2 \neq 0 & p^{-1}(F_1) \simeq M_2^3 \\ \alpha_2 = 0 & p^{-1}(F_1) \simeq \bar{A}. \end{array}$$

For the only other boundary point see (*iv*) below. As before we also have a similar picture for $U_{1,3}, U_{1,5}$— both are contained in $\overline{U_0}$ and the closure of each contains 3 lines, 1 representing each M_2^i.

(*iv*) The remaining possibility is $\pi_1 = \pi_3 = \pi_5 = 0$ whence F_1 has a basis

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

and $p^{-1}(F_1) \simeq \bar{A}$. Clearly this is contained in the closure of U_0.

Thus we have shown that $P(D_4)$ is irreducible. We now work out the equations defining it as a subvariety of $\mathrm{Gr}(3,6)$. Recall that U_0 is given as the set where we have a basis

$$\begin{pmatrix} \alpha_0 & 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 \\ 0 & \alpha_0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 & 0 & \alpha_2 \end{pmatrix}$$

for $\alpha_0, \alpha_1, \alpha_2 \neq 0$. This gives us the following non-zero Plücker coordinates.

$$\begin{aligned} X_0 = P_{146} = P_{245} = -P_{236} &= \alpha_0\alpha_1\alpha_2 \\ X_1 = -P_{124} &= \alpha_0^2\alpha_1 \\ X_2 = -P_{126} &= \alpha_0^2\alpha_2 \\ X_3 = -P_{234} &= \alpha_0\alpha_1^2 \\ X_4 = -P_{256} &= \alpha_0\alpha_2^2 \\ X_5 = -P_{346} &= \alpha_1^2\alpha_2 \\ X_6 = -P_{456} &= \alpha_1\alpha_2^2 \\ X_7 = -P_{246} &= \alpha_0\alpha_1\alpha_3. \end{aligned}$$

Regard $(X_0 : \ldots : X_7)$ as coordinates on a $\mathbf{P}^7$. The subvariety thus defined is a cone on a surface (cone point $(0 : \ldots 0 : 1)$): call this surface S. Looking at these expressions (omitting that for X_7) one can see that S is the image of the rational map $\mathbf{P}^2 - \to \mathbf{P}^6$ defined by the linear system of cubics passing through the points $(1:0:0), (0:1:0), (0:0:1) \in \mathbf{P}^2$. Thus S is the del Pezzo surface of degree 6 in $\mathbf{P}^6$, isomorphic to $\mathbf{P}^2$ blown up at 3 points (see [Manin] or [Hartshorne V. 4]). In particular S is smooth; we also prove this below.

There are relations between the coordinates:

$$(*) \qquad \begin{aligned} X_0^2 = X_1X_6 = X_2X_5 &= X_3X_4 \\ X_0X_1 &= X_2X_3 \\ X_0X_2 &= X_1X_4 \\ X_0X_3 &= X_1X_5 \\ X_0X_4 &= X_2X_6 \\ X_0X_5 &= X_3X_6 \\ X_0X_6 &= X_4X_5 \end{aligned}$$

We see below, on considering the Jacobian matrix, that these are sufficient to define $P(D_4) \subset \mathbf{P}^7$, and that all of these are necessary.

Let us look at the boundary points; notice that, from the first line of equations, for $1 \le i \le 6$, $X_i = 0 \Rightarrow X_0 = 0$. So suppose that $X_0 = 0$; now, for instance, if $X_1 \neq 0$ we obtain

$$X_4 = X_5 = X_6 = X_2X_3 = 0$$

giving us the 2 planes

$$\overline{U_{3,5}} \text{ with coordinates } (X_1 : X_2 : X_7) \qquad \text{and}$$

$$\overline{U_5} \text{ with coordinates } (X_1 : X_3 : X_7).$$

Analysing the other possibilities gives 4 more planes:

$$\begin{array}{cc} \overline{U_3} & (X_2 : X_4 : X_7); \\ \overline{U_{1,5}} & (X_3 : X_5 : X_7); \\ \overline{U_{1,3}} & (X_4 : X_6 : X_7); \\ \overline{U_1} & (X_5 : X_6 : X_7). \end{array}$$

The point $(0 : \ldots : 0 : 1)$ is common to all the planes and

$$\overline{U_i} \cap \overline{U_{j,k}} \text{ is a line } \Leftrightarrow i = j \text{ or } i = k.$$

gives the only other points of intersection of a pair of these planes. Each plane thus intersects 2 of the others in a line through the cone point, so the corresponding configuration in S is a hexagon. (But note that each plane also contains one other special line—the other coordinate axis representing M_2^i, for some i.)

We next want to show that the equations $(*)$ suffice to define S. So let S' be the subscheme defined by these equations; it is enough to show that S' is non-singular. To deal with points where $X_0 \neq 0$ consider the 4×4 submatrix of the Jacobian matrix of $(*)$ formed from the first 4 equations and the last 4 columns; this is

$$\begin{pmatrix} 0 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & 0 \\ X_4 & X_3 & 0 & 0 \\ X_2 & 0 & 0 & 0 \end{pmatrix}$$

which has non-zero determinant for $X_0 \neq 0$. Since S' has codimension 4 in this $\mathbf{P}^6$ we have that S' is non-singular at points where $X_0 \neq 0$. To deal with the boundary points is a simple matter of checking the possibilities; the key point to notice being that each X_i $(i = 1, \ldots, 6)$ occurs in precisely 4 of the equations of $(*)$, each time multiplied by a different X_j—in each case, for $X_0 = 0, X_i \neq 0$ this gives us a non-vanishing 4×4 minor. Hence S' is non-singular and $S = S'$.

Lemma 5.3.2. $P(D_4)$ is normal.

Proof. $P(D_4)$ is reduced and non-singular in codimension 1, the only singular point of $P(D_4)$ is the cone point. Using Serre's criterion for normality [Matsumura 23.8] it suffices to prove that the depth of the local ring at this point is at least 2. Let R denote the homogeneous coordinate ring of S; so $R = k[X_0, \dots, X_6]/I$ where I is the ideal generated by the equations $(*)$. Then R is also the coordinate ring of the affine cone on S. Let $m = \langle X_0, \dots, X_6 \rangle$ be the maximal ideal of the singular point. We show that m-depth$(R) \geq 2$. X_0 is not a zero-divisor in R, let $R_1 = R/X_0.R$. Then

$$R_1 \simeq k[X_1, \dots, X_6]/I_1 \qquad \text{where}$$

$$I_1 = \langle X_1X_4, X_1X_5, X_1X_6, X_2X_3, X_2X_5, X_2X_6, X_3X_4, X_3X_6, X_4X_5 \rangle.$$

Consider $F = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$; we claim this is not a zero divisor in R_1. Since I_1 is generated by monomials it is enough to check that $G \cdot F \neq 0$ for any monomial $0 \neq G \in R_1$,—but this is obvious: if X_i divides G then $G \cdot F$ will also have a non-zero term divisible by X_i.

Remark. In a similar way one can show that $P(D_5)$ is a cone on a surface S'' which is the image of $\mathbf{P}^2$ in $\mathbf{P}^6$ under the linear system of cubics passing through points p_1, p_2 having a given tangent at p_2.

Proposition 5.3.3. The number of preimages under Ψ_* of a sheaf locally isomorphic to a module M (which equals the number of components of $P(D_n)_M$— the subvariety of points representing M) is as follows.

1 $D_{2\delta-2}$:

M_0	$M_i (i < \delta - 1)$	M_i^*	N_i	$M_{\delta-1}^{1,2}$	M_δ
1	$2i + 1$	$2i + 1$	$3i - 2$	$2\delta - 2$	$3\delta - 2$

2 $D_{2\delta-1}$:

M_0	$M_i (i < \delta)$	M_i^*	N_i	M_δ
1	2	2	3	3

Proof. Before calculating these numbers we explain how one answers this question in general. Fix an ordering of the branches of the curve through the singular point. Suppose the conductor is

$$C = (t^{c_1}, t^{c_2}, \ldots, t^{c_r})\bar{A}.$$

For each module M, writing $M \subset \bar{A}$ in the usual way, we need to find the number, N, of

$$\underline{v} = (v_1, \ldots, v_r)$$

such that $|\underline{v}| = \sum_i v_i = \ell(M)$ and such that

$$C \subset t^{\underline{v}}(M)$$

where $t^{\underline{v}}$ denotes $(t^{v_1}, t^{v_2}, \ldots, t^{v_r})$. Note that

$$(*) \qquad C \subset t^{\underline{v}}(M) \Leftrightarrow C \subset t^{\underline{v}}C(M).$$

So on working out $C(M)$ for each module M the problem becomes very easy: if (as an $\bar{A}$ module)

$$C(M) = \langle (t^{a_1}, t^{a_2}, \ldots, t^{a_r}) \rangle,$$

then $(*)$ is satisfied if and only if $v_i \leq c_i - a_i$ for all i. Write $u_i = c_i - a_i$.

So we just need to work out $\underline{u} = (u_1, \ldots u_r)$ for M and to count all the $\underline{v}$ such that $\sum v_i = \ell(M)$ and $v_i \leq u_i$ for all i. This count is best done inductively on the number of branches: if $r = 1$ then the answer is 1 if $\ell(M) \leq u_1$ and zero otherwise; and for $r = 2$

$$N = \sharp\{j \mid j \leq u_2,\ 0 \leq \ell(M) - j \leq u_1\}$$

which, for example, equals $\ell(M) + 1$ if $\ell(M) \leq \min\{u_1, u_2\}$. Now apply this to A of type D_n.

1 $D_{2\delta-2}$.
$(c_1, c_2, c_3) = (\delta-1, \delta-1, 2)$, $i \leq \delta-2$. Consider first $M_i = \langle 1, (t^{\delta-1-i}, 0, 0) \rangle$; this gives us $\underline{u} = (i, i, 1)$. Now $\ell(M_i) = i$, $v_3 = 1$ gives i possibilities for (v_1, v_2) and $v_3 = 0$ gives $i + 1$, so, in total we have $N = i + i + 1 = 2i + 1$.

For M_i^* we again arrive at $\underline{u} = (i, i, 1)$, now $\ell(M_i^*) = i + 1$ and we get $N = i + 1 + i = 2i + 1$.

The above calculations are also valid for $M_{\delta-1}^1 = \langle 1, (1, 0, 0) \rangle$ and the case $M_{\delta-1}^2$ then follows by symmetry.

For N_i, $\underline{u} = (i-1, i-1, 2)$ and we have $N = i + i - 1 + i - 1 = 3i - 2$.

There remains $M_\delta = \bar{A}$: of course $C(\bar{A}) = \bar{A}$ and so $\underline{u} = (\delta - 1, \delta - 1, 2)$; we get $N = \delta - 1 + \delta + \delta - 1 = 3\delta - 2$.

2 $D_{2\delta-1}$. This is very similar, but simpler as there are only 2 branches. For both $M_i = \langle 1, (t^{2\delta-2i-1}, 0) \rangle$ $(i \le \delta - 1)$ and $M_i^* = \langle (1,0), (t^{2\delta-2i-3}, 1) \rangle$ we find $\underline{u} = (2i, 1)$ giving $N = 2$.

$M_\delta = \bar{A}$: $\underline{u} = (2\delta - 2, 2)$ and $N = 3$.

Similarly $N_i = \langle 1, (1,0), (t^{2\delta-2i+1}, 0) \rangle$ implies $\underline{u} = (2i - 2, 2)$ and $N = 3$.

Remark. In contrast with the case A_n we do not obtain the *maximum* possible number of representatives in $P(D_n)$. One can see this even in the case of D_4 where replacing 'C' by 'C^2' would give $N = 10$ representatives for $\bar{A}$: the additional possibilities for $\underline{v}$ being $(4, 4, 1)$ and its permutations.

Proposition 5.3.4. The diagrams given in (5.1) for the stratification of $\overline{J(X)}$ for x of type D_n are correct: i.e., (using the remarks in §5.1) for $1 \le i \le \delta - 2$,

$$U_{M_{i+1}^*} \subset \overline{U_{M_i}}.$$

Proof. We prove that in the case $i = 1$ there is a component in $P(D_n)$ representing M_1 whose closure contains a component representing M_2^*.

1 D_{even}. Consider the $(1,0,0)$ (shifted) component of $P(D_n)_{M_1}$ (5.3.3). A basis for a subspace corresponding to a point of this component is determined by the action of the generators of m on the image of the generating set for M_1; this basis can be reduced to the following form:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{\delta-2} & 0 & \alpha_{\delta-1} & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \alpha_1 & \dots & \alpha_{\delta-3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_1, \alpha_{\delta-1} \neq 0$. A subspace corresponding to a point in the $(2,1,0)$ component of $P(D_n)_{M_2^*}$ has a basis:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{\delta-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \alpha_1 & \dots & \alpha_{\delta-4} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_{\delta-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_1, \alpha_{\delta-2} \neq 0$. Notice that these matrices have $\delta - 1$ rows in common, the difference being row 1 of the top matrix compared with row $\delta - 1$ of the second matrix. Thus replace row 1 in the first matrix with

$$(0 \; \lambda \; 0 \; \dots \; 1 \; \lambda\alpha_1 \; \lambda\alpha_2 \; \dots \; \lambda\alpha_{\delta-2} \; 0 \; \alpha_{\delta-1} \; 0).$$

For $\lambda \neq 0$ this defines a point of the $(1,0,0)$ component representing M_1 (essentially we have just added the $(\delta-2)^{nd}$ row to a multiple of row 1) but for $\lambda = 0$ we obtain a matrix of the second form, hence a point of the $(2,1,0)$ component of points representing M_2^*. Moreover every such point arises from some such specialisation. Hence we have shown that $U_{M_2^*} \subset \overline{U_{M_1}}$ in $\overline{J(X)}$. This argument can be extended to prove that, for $i \leq \delta-2$, the closure of the $(i,0,0)$ component of $P(D_n)_{M_i}$ contains the $(i+1,1,0)$ component of $P(D_n)_{M_{i+1}^*}$.

2 D_{odd}. This is very similar, the closure of the $(1,0)$ component of $P(D_n)_{M_1}$ contains the $(3,0)$ component of $P(D_n)_{M_2^*}$. Again one finds that on simplifying bases the 2 matrices have $\delta - 1$ rows in common, and that a specialisation of a modified first row of the matrix for M_1 yields the matrix for M_2^*. For the sake of completeness we present the relevant 2 matrices below.

$$\begin{pmatrix} 0 & 1 & \alpha_2 & 0 & \alpha_3 & \dots & 0 & \alpha_{\delta-1} & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 1 & \alpha_2 & \dots & 0 & \alpha_{\delta-2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\alpha_1 \neq 0$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & \alpha_2 & 0 & \alpha_3 & \dots & 0 & \alpha_{\delta-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_2 & \dots & 0 & \alpha_{\delta-3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\alpha_1 \neq 0$.

§5.4 $E_{6,7,8}$

Our main goal in this section is to prove that the stratification diagrams for $\overline{J(X)}$ given in (5.1) are correct. As remarked in §5.1 we do not have to do anything further for E_6, so we proceed to E_7.

Proposition 5.4.1. Consider $P(E_7)$; the numbers of components representing a given module are as follows:

M_1, M_1^*	M_2, M_2^*	M_3	N_3	M_4
2	3	3	4	4

Proof. We recall the notation of (5.3.3). For E_7 the complete local ring is $A = k[[(t^2,t),(t^3,0)]]$, $C = (t^5,t^3)\cdot\bar{A}$.
$M_1 = \langle 1,(t^4,0)\rangle$, $M_1^* = \langle 1,(t,0)\rangle$; each has conductor $(t^3,t^2)\cdot\bar{A}$, so $\underline{u} = (2,1)$ and $N = 2$.
$M_2 = \langle 1,(t^2,0)\rangle$, $M_2^* = \langle (t,1),(1,0)\rangle$ give $\underline{u} = (3,2)$ and $N = 3$.
$M_3 = \langle 1,(t,0),(t^2,0)\rangle$, $\underline{u} = (4,2)$ and again $N = 3$.
$N_3 = \langle 1,(1,0)\rangle$ and so $\underline{u} = (3,3)$ and $N = 4$. Finally, looking at the conductor of A clearly $N = 4$ for $M_4 \simeq \bar{A}$.

Remark. Since the singularities E_6 and E_8 are unibranched there is only 1 component in $P(E_6)$ and $P(E_8)$ representing each module (3.4.10).

Proposition 5.4.2. The diagrams in §5.1 give the stratification of $\overline{J(X)}$ for X with a single singularity which is of type E_7 or E_8.

Proof. Recall the remarks in §5.1; it remains to prove that, in each case, $U_{M_2^*} \subset \overline{U_{M_1}}$. As for the D_n case we prove analogous statements about first $P(E_7)$ and then $P(E_8)$.
Take the $(1,0)$ component of $P(E_7)_{M_1}$. Consider the possible images of the generators of M_1 in a (4-dimensional) submodule F_1 of $\bar{A}/C$, the action of the generators of the maximal ideal on these gives a basis for F_1. This basis can be reduced to the form:

$$\begin{pmatrix} 0 & 1 & \alpha_1 & 0 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_2 \neq 0$. Now consider the $(2,1)$ component of $P(E_7)_{M_2^*}$. A typical point corresponds to a subspace with basis

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_2 \neq 0$. These matrices are the same except for the first row. Replace the first row in the first matrix by

$$(0\ \lambda\ 1\ 0\ 0\ \lambda\alpha_2\ \alpha_3\ 0).$$

Certainly for $\lambda \neq 0$ this gives a matrix of the first type, hence a point of U_{M_1}. Putting $\lambda = 0$ we obtain a matrix of the second type, and hence a point of $U_{M_2^*}$; moreover, we can obtain all such points this way. Thus $U_{M_2^*} \subset \overline{U_{M_1}}$ as required.

The E_8 case is very similar, we present the relevant matrices below.

M_1:

$$\begin{pmatrix} 0 & 1 & \alpha_1 & \alpha_2 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

M_2^*:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

As for E_7 it is clear all matrices of the second type arise as specialisations of those of the first type. (Actually, up to a permutation of columns, the matrices are essentially the same as for the E_7 case.)

§5.5 $D_n^-, E_{6,7,8}^-$

Our aims in this section are twofold: firstly to indicate how the stratifications in these cases compare with those for $D_n, E_{6,7,8}$; and secondly, by a detailed look at D_4^-, to explore the question of how different choices of divisor can give rise to quite different spaces of parabolic Modules.

Theorem 5.5.1. Suppose X is a curve with a single singularity which is of type D_n^- or $E_{6,7,8}^-$. Then $\overline{J(X)}$ has a stratification given by the appropriate diagram in §5.1 with the leftmost vertex removed. In particular:

1. $\overline{J(X)}$ is reducible with precisely two, g-dimensional components whoose intersection is $g-1$ dimensional.

2. U_M, the subvariety of sheaves locally isomorphic to M (at the singular point) has dimension $g-i(M)$. However, it is *not* true that the tangent space at a point of U_M has dimension $g+i(M)$, in general.

Proof. For the statements about the diagrams and the dimensions of the components there is nothing new to prove:—we know that in each case we have an injective map $\overline{J(X)} \longrightarrow \overline{J(X')}$ where X' is a curve with a singularity of type D_n or $E_{6,7,8}$. (Actually we should be more precise; we don't know of the existence of such an X' for any X subject to the above, but just for some X—those arising as partial normalisations—however, the properties we are trying to establish are insensitive to the birational character of the curve, so we assume that such an X' exists.) The image of this map is precisely the set of singular points of $\overline{J(X')}$ which we know about by (5.1). There remains the remark on the dimensions of the tangent spaces: I have not calculated all of these, but for E_6^- the tangent space at a point of $U_{\bar{A}}$ has dimension $g+4$ and $\operatorname{index}(\bar{A}) = \delta = 2$, this is a special case of a series of calculations carried out in §6.4.

We now look in detail at the non-planar triple point: D_4^-.
$A = k[[(t,0,0),(0,t,0),(0,0,t)]]$, the conductor C is just the maximal ideal m, so $\tilde{\delta} = \delta+1 = 3$. There are 6 isomorphism classes of ideals, for simplicity denote these

$$\begin{array}{ll} A & \\ \omega & = \langle (t,t,0),(t,0,t)\rangle \\ A_1^1 & \simeq \langle (t,0,0),(0,t,t)\rangle \\ A_1^2 & \simeq \langle (0,t,0),(t,0,t)\rangle \\ A_1^3 & \simeq \langle (0,0,t),(t,t,0)\rangle \\ \bar{A} & \qquad \simeq m. \end{array}$$

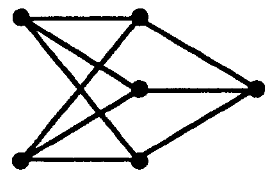
A_1^i is isomorphic to the overring of the partial normalisation given by 'pulling the i^{th} branch free'.

We will look at $P_{I_j}(D_4^-) = \operatorname{Gr}^A(\dim A/I_i, \bar{A}/I_j)_{\text{red}}$ for the following normal ideals I_j:

1. $I_1 = m = C$;

2. An ideal defining a divisor of degree $4 = 2\delta$, $I_2 = (t, t, t^2) \cdot \bar{A}$;
3. $I_3 = m^2 = C^2$.

Recall (5.1, 5.5.1) that $\overline{J(X)}$ has a stratification:



where the components correspond to the modules indicated below.

$$\begin{array}{ccc} A & A_1^1 & \\ & A_1^2 & \bar{A} \\ \omega & A_1^3 & \end{array}$$

Proposition 5.5.2. The spaces $P_{I_j}(D_4^-)$ each consist of a configuration of planes. The number of components representing a particular module is as follows.

1.

$$\begin{array}{ccc} 1 & 1 & \\ & 1 & 3 \\ 0 & 1 & \end{array}$$

2.

$$\begin{array}{ccc} 1 & 2 & \\ & 2 & 4 \\ 1 & 1 & \end{array}$$

3.

$$\begin{array}{ccc} 1 & 3 & \\ & 3 & 6 \\ 3 & 3 & \end{array}$$

Moreover, for any normal $I \subset C^2$ we do not obtain any more components than there are in case **3**.

Remark. A surprising feature is that in case **3** the dualising module is represented several times. A consequence of this is that the map

$$\Psi_* : P_{C^2}(X)_\omega \longrightarrow \overline{J(X)}_\omega$$

is not an isomorphism, and there can be no sensible duality theorem for parabolic Modules in this case.

Proof. First of all take $I = C$ and look at which modules are represented as 1-dimensional subspaces, F_1, of $\bar{A}/C$; this has a basis

$$\{(1,0,0),(0,1,0),(0,0,1)\}.$$

Let $0 \neq u \in F_1$, write $u = (\alpha_1, \alpha_2, \alpha_3)$; if all of these are non-zero then $p^{-1}(F_1) \simeq A$. On the other hand if just 1 of these $\alpha_i = 0$ then clearly $p^{-1}(F_1) \simeq A_1^i$. The only other possibilities, the 3 vertices of this triangle of reference in $\mathbf{P}^2$, give $p^{-1}(F_1) \simeq \bar{A}$. So we see that ω is not represented in $P_C(D_4^-) \simeq \mathbf{P}^2$.

Let us now examine $P_{C^2}(D_4^-)$ (the space $P_{I_2}(D_4^-)$ is a subscheme of this, as we shall indicate). $\bar{A}/C^2$ has dimension 6, so we want to find all 4-dimensional A-submodules of this. Order a basis for $\bar{A}/C^2$ as

$$\{(1,0,0),(t,0,0),(0,1,0),(0,t,0),(0,0,1),(0,0,t)\}$$

and adopt notation as in §5.2/3. Let y_1, y_2, y_3 be the 3 generators of m (i.e., $y_1 = (t,0,0)$ etc.). Let F_1 be a 4-dimensional submodule of $\bar{A}/C^2$

1. $\pi_1, \pi_3, \pi_5 \neq 0$. Then $y_i(F_1) \neq 0$ for $i = 1,2,3$ so a basis for F_1 reduces to

$$\begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The closure, $\overline{U_0}$ of the set of all such points in $P_{C^2}(D_4^-)$ is a copy of $P_C(D_4^-)$.

2. $\pi_1, \pi_3 \neq 0, \pi_5 = 0$. Now F_1 has a basis of the form

$$\begin{pmatrix} 1 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \end{pmatrix}$$

with one of $\alpha_1, \alpha_3 \neq 0$. If $\alpha_3 \neq 0$ then take $\alpha_3 = 1, \alpha_1 = 0$. For $\alpha_2, \alpha_4 \neq 0$ these points represent ω, and the closure, $\overline{U_5}$, of the set of these points gives us another $\mathbf{P}^2 \subset P_{C^2}(D_4^-)$. Moreover $\overline{U_0} \cap \overline{U_5}$ is the line of points

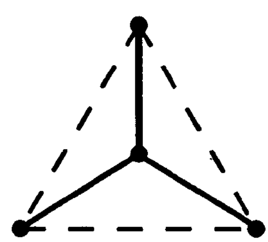
$$\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

representing A_1^3 (when $\beta_1, \beta_2 \neq 0$). We claim that the union, $\overline{U_0} \cup \overline{U_5}$, is a copy of $P_{I_2}(D_4^-)$:—to see this, strike out columns 2 and 4, and the rows (0 1 0 0 0 0) and (0 0 0 1 0 0) in all the above matrices.

Repeating the above in the other cases where just 1 of $\pi_1, \pi_3, \pi_5 = 0$ we obtain 2 more planes, $\overline{U_3}, \overline{U_1}$ of points representing ω, each intersecting $\overline{U_0}$ in a different line. Note also that, e.g, $\overline{U_3} \cap \overline{U_5}$ is the point with P_{1246} the only non-zero Plücker coordinate.

3. Two of $\{\pi_1, \pi_3, \pi_5\}$ are zero. This gives only 3 more points, i.e. the points representing $\bar{A}$ in $\overline{U_0}$.

To conclude, the dual graph of $P_{C^2}(D_4^-)$ is as shown below. (A vertex corresponds to a plane, 2 planes joined by a plain line meet in a line, 2 planes joined by a dotted line meet in a point.) The central vertex corresponds to the plane of points representing A.



Finally, note the number of partitions of 1 (resp. 2) into 3 boxes is 3 (resp. 6) so we really do have the maximum number of representatives of each module in $P_{C^2}(D_4^-)$.

CHAPTER 6

SINGULARITIES VIA SEMIGROUPS

In trying to understand the general structure of compactified Jacobians it is natural to look for a class of singularities which are somehow easier to handle, whilst at the same time being reasonably representative. Unibranched singularities form such a class. The semigroup associated to the complete local ring A of a unibranched curve singularity, arising from the fact that such a ring is a subring of a discrete valuation ring, is a useful invariant from which many properties of A can be determined (e.g. δ, $\tilde{\delta}$ and hence whether or not A is Gorenstein). Two unibranched plane curve singularities having the same semigroup are *equisingular*, in other words, they are members of the same family—this concept has been much studied by [Zariski]. For simplicity most of this chapter will concern monomial rings; one possible way to generalise our results would be to consider equisingular deformations of the curve.

The idea of this 'value set' invariant extends to rank 1 torsion free modules over A. Again, this gives the main discrete invariants for such a module: on the level of parameter spaces this corresponds to taking the Schubert cell decomposition. (These are also studied in [Greuel and Pfister], although with different applications in mind.) The resulting decompostion of the parameter space, despite being more complicated than that for a Grassmannian, is still useful, giving us a necessary condition for a module to be a limit of free modules, for instance.

There are other discrete invariants for modules; for example, the value set is not enough to determine the space of self extensions of a module. A consequence of this is that, in general, the value set decompostion is not quite enough to give a stratification of the compactified Jacobian of a curve with unibranched singularities. In this chapter we explore some of these questions through various examples: it is clear that there is scope for a lot of further work here.

The organisation of material in this chapter is as follows: In §6.1 we review the basic facts about these semigroups and the relationship with 1-dimensional local rings. §6.2 extends these ideas to rank 1 torsion free modules over such a ring, and shows how to classify the main discrete invariants of these modules. In §6.3, considering a fixed semigroup ring, we introduce a partial ordering on the set of isomorphism classes of semigroup modules and show how this leads to a partial answer of how to stratify the compactified Jacobian. We apply this analysis to the case of cubical singularities (§6.4) and to plane curve singularities with semigroup $< p, q >$ (§6.5). We also show that the compactified Jacobian of a curve with cubical singularities is generically reduced.

§6.1 Semigroups

Throughout this chapter we deal with A a complete local ring of an analytically irreducible curve singularity; i.e. A is a subring of $\bar{A} = k[[t]]$ such that $\bar{A}$ is finitely generated over A. The semigroup of A is defined by

$$\Gamma = \Gamma(A) = \{\gamma \in \mathbf{N} \mid \exists\, (t^\gamma + \text{ higher terms}) \in A\}.$$

In other words, if $v : \bar{A} \longrightarrow \mathbf{N}$ is the valuation on $\bar{A}$ then $\Gamma(A) = v(A)$ is the image of v restricted to A. Since A is a ring it is clear that Γ is a semigroup and $\Gamma \subset \Gamma(\bar{A}) = \mathbf{N}$. We adopt the convention here that $\mathbf{N}$ includes zero, as do all our semigroups. We will also adopt the convention that the word "semigroup" means a cofinite sub-semigroup of $\mathbf{N}$—i.e. the set $\mathbf{N} - \Gamma$ is finite. We refer to an element of $\mathbf{N} - \Gamma$ as a *gap* of Γ and denote the number of gaps of Γ by $\delta(\Gamma)$. $c(\Gamma)$, the *conductor* of Γ is defined to be the number c such that $c - 1$ is the highest gap of Γ. We also define the *multiplicity* of Γ to be $e(\Gamma) = \min\{0 \neq \gamma \in \Gamma\}$. Clearly $e(\Gamma(A)) = e(A)$. Note that the semigroup $\Gamma(A)$ has a unique minimal generating set (unlike the ring A) and so we refer to *the generators* of Γ. The number of generators of Γ will be called the *embedding dimension* of Γ. We write

$$\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$$

for the semigroup with generators $\{\gamma_1, \ldots, \gamma_r\}$. That Γ is cofinite is equivalent to the condition that $\operatorname{hcf}\{\gamma_1, \ldots, \gamma_r\} = 1$.

It is *not* true that the embedding dimension of $\Gamma(A)$ equals the embedding dimension of A—for example the ring $k[[t^4, t^6 + t^7]]$ has semigroup $\langle 4, 6, 13 \rangle$. However, to any Γ we can associate a ring

$$R(\Gamma) = k[[\{t^\gamma \mid \gamma \in \Gamma\}]]$$

which has the same embedding dimension. Such a ring will be called a *semigroup ring* (another name would be *monomial ring*).

The most basic facts about the semigroup of A are:

Lemma 6.1.1. $c(\Gamma(A)) = \tilde{\delta}(A)$ and $\delta(\Gamma(A)) = \delta(A)$.

Proof. Write $\Gamma = \Gamma(A)$. We know that $\bar{A}$ is finitely generated over A and $t^{\tilde{\delta}} \cdot \bar{A} \subset A$. It is clear that $c = c(\Gamma) \leq \tilde{\delta}$, so suppose that $c < \tilde{\delta}$. Let d be maximal such that there exists $f = t^d +$ (higher terms) in A but $t^d \not\in A$. But then if $d+1, d+2, \ldots$ are all in Γ then we can subtract these higher terms to get a contradiction. This shows that $d < c$; hence $t^c \bar{A} \subset A$ and $\tilde{\delta} = c$.

Using the first part it is then clear that the classes of t^g, as g runs through the gaps of Γ, give a basis for $\bar{A}/A$.

For all $c \geq \gamma \in \Gamma$, by the semigroup property, we must have that $c - 1 - \gamma$ is a gap, and so the inequality $\delta + 1 \leq c \leq 2\delta$ is easy to see. If Γ is such that $c = 2\delta$, i.e., $c - 1 - g$ is a gap if and only if $g \in \Gamma$, then Γ is called *symmetric*—so $\Gamma(A)$ is symmetric if and only if A is Gorenstein (cf. (6.2.4)). It is often useful to write down a given semigroup (or, more generally, any cofinite subset of N) as follows: e.g.

$$\Gamma = \langle 3, 7 \rangle : \ 0\ \underline{1}\ \underline{2}\ 3\ \underline{4}\ \underline{5}\ 6\ 7\ \underline{8}\ 9\ 10\ \underline{11}.$$

A number which appears underlined is a gap, and the others are, obviously, the elements of Γ; the last number we write down is the highest gap.

Examples

1. $\Gamma = \langle p, q \rangle$ where $p < q$ and $(p, q) = 1$. Then Γ is symmetric and $c(\Gamma) = (p-1)(q-1)$. To see this work mod p: q has order p mod p and the last gap must be $(p-1)q - p$, hence $c(\Gamma) = (p-1)q - p + 1 = (p-1)(q-1)$. It is also not hard to show that Γ is symmetric; $A = k[[t^p, t^q]]$ is a complete intersection, hence Gorenstein and so $\Gamma = \Gamma(A)$ is symmetric.

2. For semigroups with more than 2 generators there are many more possibilities and there is no formula giving c or δ. However, for Γ with 3 generators

the symmetric ones are characterised as follows (this is a consequence of the results of [Delorme]).

If Γ has 3 generators and is symmetric then

$$\Gamma = \langle n_1, pn_2, pn_3\rangle$$

where the n_i are distinct and $(n_2, n_3) = 1$, $(n_1, p) = 1$ and $n_1 \in \langle n_2, n_3\rangle$ (we do not need $n_1 < pn_2 < pn_3$). Moreover,

$$c(\Gamma) = (n_1 - 1)(p - 1) + p(n_2 - 1)(n_3 - 1).$$

So for example $\langle 4, 5, 6\rangle$ and $\langle 5, 6, 9\rangle$ are symmetric but $\langle 3, 5, 7\rangle$, or more generally $\langle p, q, r\rangle$ for p, q, r pairwise coprime are not symmetric.

Even if $n_1 \notin \langle n_2, n_3\rangle$ the formula above still gives an upper bound for c, because $p(c\langle n_2, n_3\rangle) + \langle p, n_1\rangle \subset \Gamma$.

Given semigroups Γ_1 and Γ_2 write $\Gamma_1 \vee \Gamma_2$ for the smallest semigroup containing Γ_1 and Γ_2. Thus Γ above has an expression

$$\Gamma = p\langle n_2, n_3\rangle \vee n_1\langle 1\rangle.$$

For semigroups, $\Gamma(A)$, with any number of generators the existence of a suitable 'decomposition' as above implies that A is a complete intersection (see [Delorme]). Of course, there are many more symmetric semigroups than these, e.g. $\langle 5, 6, 7, 8\rangle$.

The following result, although not used in what follows, illustrates some typical arguments concerning semigroups with this symmetry property.

Proposition 6.1.2. Suppose Γ is a symmetric semigroup of multiplicity $e > 2$. Then all the generators of Γ are less than $c(\Gamma)$.

Proof. Since Γ is symmetric there are $c/2$ elements of Γ less than $c(\Gamma)$. Let Γ' be the semigroup generated by these elements; we must show $\Gamma' = \Gamma$. Firstly, Γ' is cofinite, since the only way in which δ numbers less than 2δ can have a common factor is if the numbers are $0, 2, \ldots, 2(\delta - 1)$ and we have ruled this out by our assumption on the multiplicity. Secondly, we have an inequality $c(\Gamma') \leq 2\delta(\Gamma')$.

Since the numbers $c, c+1, \ldots, c+e-1$ are in Γ' and $e > 2$ we must therefore have that $c(\Gamma') = c(\Gamma)$ and so $\Gamma' = \Gamma$.

Corollary 6.1.3. If Γ is symmetric with multiplicity $e > 2$ then Γ requires at most $e-1$ generators.

Proof. Clearly no semigroup requires more than e generators, but, using the above, Γ needs no generator congruent to $c-1$ mod e.

§6.2 Semigroup Modules and Module Diagrams

We extend the definition of Γ to rank 1 torsion free modules over A.

If Γ is a semigroup (with the conventions of §6.1) by a Γ-*module* we mean a proper subset S of $\mathbf{Z}$ such that $\gamma + S \subset S$ for all $\gamma \in \Gamma$. Γ-modules S and S' are *isomorphic* if there is some $n \in \mathbf{Z}$ such that $S = S' + n$. Let $\Delta_0 = \Delta_{0,\Gamma}$ denote the set of all isomorphism classes of Γ-modules. For any Γ-module S with least element s we can write $\Gamma \subset -s + S \subset \mathbf{N}$. We refer to S as being in *normal form* if $s = 0$. Then define $\delta(S)$ to be the number of gaps of S, $\ell(S) = \delta(\Gamma) - \delta(S)$ and denote by $c(S) - 1$ the last gap of S. If S is in normal form then the set S is obtained from Γ by filling in a number $(\ell(S))$ of gaps, hence Δ_0 is a finite set: in fact $|\Delta_0| \leq 2^{\delta}$. We have a partial ordering on Δ_0 given by inclusion of Γ-modules in normal form. We define the *module diagram* $\Delta = \Delta_\Gamma$ to be the Hasse diagram of this partial ordering on Δ_0, and we think of Δ as being drawn so that ℓ is constant on each column, so that the leftmost vertex is Γ and the rightmost $\bar{\Gamma}$. Each edge can be labelled by a number according to which gap is filled in; for some examples see below (6.2.3).

Proposition 6.2.1 Structure of Δ. Fix a semigroup Γ and write $\delta = \delta(\Gamma)$.

1. Δ is connected: For each vertex S of Δ there is a chain of length $\ell(S)$ joining S to Γ and a chain of length $\delta(S)$ joining S to $\bar{\Gamma}$.

2. Δ is a subgraph of a cube of dimension δ. In fact Δ is a δ-cube if and only if all the generators of Γ are bigger than the conductor, i.e., $\Gamma = \langle \delta+1, \delta+2, \ldots, 2\delta+1 \rangle$. (Hence the justification for the name *cubical* for rings with this semigroup.)

3. Γ is symmetric iff there is just 1 edge leaving the vertex Γ in Δ.

4. There are at most $e-1$ edges leaving or entering any given vertex, and there are precisely $e-1$ edges to $\bar{\Gamma}$. The cube of maximum dimension contained in Γ is the $(e-1)$-cube between $S = \{0, e, e+1, \ldots, 2e-1, \ldots\}$ and $\bar{\Gamma}$.

Proof. Most of this is obvious. Edges leaving a vertex S are in 1 to 1 correspondence with numbers n such that $S' = S \cup \{n\}$ is a semigroup module. This happens if and only if $n + \Gamma \subset S'$. The possibilities are $n = g_r(S)$ where $g_r(S)$ is the highest gap of S congruent to r mod e (of course other elements of Γ may impose further restrictions). However, since S contains no gaps congruent to 0 mod e we see there are at most $e-1$ possibilities. On the other hand, $S \cup \{c(S)-1\}$ is always a Γ-module. Reversing the argument deals with edges entering a given vertex. To prove **3**, we know that Γ is symmetric if and only if all the gaps of Γ are of the form $c-1-\gamma$ for some $\gamma \in \Gamma$; so $c-1$ is the only edge leaving Γ. Conversely, if Γ is not symmetric then there is a number $n < e$ such that $c-1-n$ is a gap and another edge, $c-1-n$, leaving Γ.

Definition 6.2.2. For a Γ-module S (assumed to be in normal form) we define the dual

$$S^* = \{c(S) - 1 - s \mid s \in \mathbf{Z} - S\}.$$

Then S^* is also a Γ-module in normal form. We also have that $(S^*)^* = S$, and $\Gamma^* = \Gamma \Leftrightarrow \Gamma$ is symmetric. More generally we will denote Γ^* by Ω. Observe that $c(S) = c(S^*)$ and $\delta(S) + \delta(S^*) = c(S)$. We sometimes attach extra information to the graph Δ by linking a vertex to its dual.

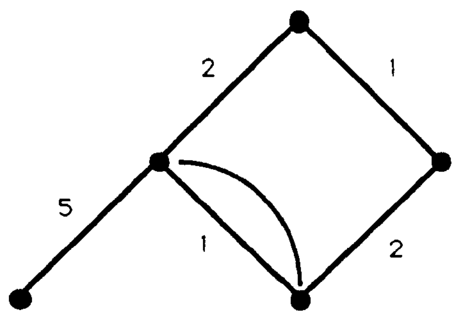
Examples 6.2.3.

1. A_{2r} : $\Gamma = \langle 2, 2r+1 \rangle$, Δ consists of a single chain joining Γ to $\bar{\Gamma}$. For all Γ-modules S we have $S^* = S$.

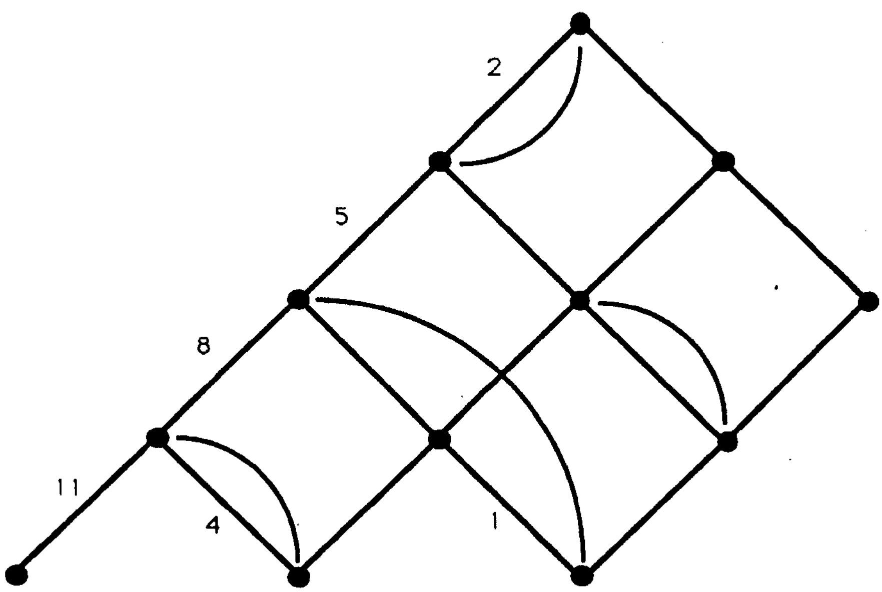
2. $\Gamma = \langle p, q \rangle$ for $2 < p < q$. Of course Δ can be quite complicated, but we can say something about its structure near to Γ. More precisely we can describe all the Γ-modules, $S \neq \Gamma$, with $c(S)$ maximal, i.e., $c(S) = c-1-p$. These all have the form $S_i = \Gamma \cup \{c-1, c-1-q, \ldots, c-1-iq\}$ for $0 \le i \le p-2$ (since $c-1-(p-2)q = q-p$). Moreover, knowing that $\delta(S) + \delta(S^*) = c(S)$,

we see that $S_i^* = S_{p-2-i}$.
If $q < 2p$ then there is a similar picture for modules S with $c(S) = c - 1 - q$.
We draw some pictures for $p = 3$—curves indicate dualities:

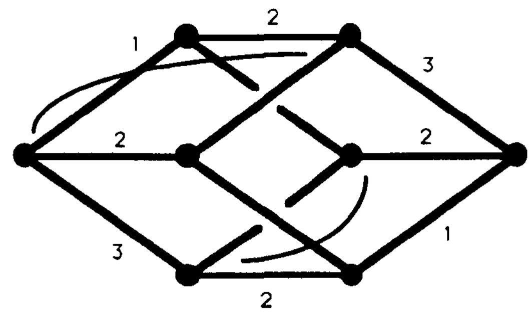
$\Gamma = \langle 3, 4 \rangle : \ 0\ \underline{\overline{1}}\ \underline{\overline{2}}\ 3\ 4\ \underline{\overline{5}}$



$\Gamma = \langle 3, 7 \rangle : \ 0\ \underline{\overline{1}}\ \underline{\overline{2}}\ 3\ \underline{\overline{4}}\ \underline{\overline{5}}\ 6\ 7\ \underline{\overline{8}}\ 9\ 10\ \underline{\overline{11}}$



3. Γ is cubical: $\Gamma_n = \langle n, n+1, \ldots \rangle$. Draw Δ so that the edges with lowest labels leaving a vertex appear at the top. As an example consider $\Gamma = \langle 4, 5, 6, 7 \rangle$.



In general, we can say that the top chain gives all the Γ-modules with maximal conductor, so duality reflects this chain—see §6.4 for more details. The bottom chain consists of cubical semigroups, S_n ($S_0 = \Gamma$), the dual of this chain is the penultimate column. The $(\delta - n)$ cube starting at S_n gives the semigroup modules for S_n, giving a nested sequence of diagrams Δ_{Γ_n}.

Module Theory

We now tie this up with the module theory of A. All modules considered are rank 1 torsion free. If M is such a module then writing M with $A \subset M \subset \bar{A}$ we form $\Gamma(M) = v(M)$ (the *value set* of M) in the obvious way—this will be a $\Gamma(A)$ module. $\Gamma(M)$ should be thought of as a discrete invariant of the module. Mimicking the proof for semigroups (6.1.1) one shows that $\ell(\Gamma(M)) = \ell(M)$.

It is not true in general that for every $\Gamma(A)$-module S there is an A-module M with $\Gamma(M) = S$; however, this is true when A is a semigroup ring, e.g., we can take M to be the corresponding monomial module.

Notice that for A corresponding to a simple singularity (A_{even}, E_6 or E_8) the $\Gamma(A)$-module classification is exactly the same as the A-module classification. For more complicated rings one should think of a vertex in Δ as having certain parameters attached to it, so that any A-module giving that Γ-module is obtained by specifying values of these parameters. For example, for $\Gamma = \langle 3, 7\rangle$ we find that each vertex gives rise to only 1 module, except $S = 0\ \underline{1}\ \underline{2}\ 3\ 4\ \underline{5}$ where there is a 1-parameter family of modules $M_\lambda = A + (t^4 + \lambda t^5)A + t^8 A$ [Schappert]. Here $M_\lambda \not\cong M_\mu$ for $\lambda \neq \mu$. Also note that M_0 is special in that it requires 3 generators, wheras $t^8 \in A + (t^4 + \lambda t^5)A$ when $\lambda = 0$ (consider $t^3(t^4 + \lambda t^5) - t^7$). This example will reappear later on.

Lemma 6.2.4. For A modules M, N:

1. $\Gamma(\mathrm{Hom}_A(M, N)) \subset \{n \mid n + \Gamma(M) \subset \Gamma(N)\} = S$.
   Equality is not obtained in general, but $c\,(\Gamma(\mathrm{Hom}_A(M, N))) = c(S)$.

2. $\Gamma(\omega) = \Omega$, and for any module M, $\Gamma(M^*) = \Gamma(M)^*$.

Proof.

1. Any A-module homomorphism is multiplication by some element, u, of K. If we choose a representative of the isomorphism class of M so that the smallest valuation of any such u is 0 then we must have $\Gamma(M) \subset \Gamma(N)$ and, further $v(u') + \Gamma(M) \subset \Gamma(N)$ for any other $u' \in \mathrm{Hom}(M, N)$. As before (6.1.1) we can show that the conductors of the 2 modules are equal. For an example of inequality see below (6.2.5(2)).

2. Assume $A \subset \omega \subset \bar{A}$. From (2.1.5) we know that $c(\Gamma(\omega)) = c(\Gamma(A))$ and $\ell(\omega) = 2\delta - \tilde{\delta}$, i.e., $\delta(\Gamma(\omega)) + \delta = c$. We saw in the last section that Ω satisfies these two conditions. We claim that Ω is the only such Γ-module, since, for any Γ-module S, $c(S) - 1 - \gamma \in \mathbf{N} - S\ \forall \gamma \in \Gamma$. Now, given an A-module M, **1** tells us that $\Gamma(M^*) \subset \Gamma(M)^*$. Hence, applying this to M^*, $\Gamma(M) = \Gamma(M^{**}) \subset \Gamma(M^*)^*$. But, by duality $\Gamma(M^*)^* \supset \Gamma(M)$ and the two sets must be equal.

Applications 6.2.5.

1. If $u : M \longrightarrow N$ is an A-hom then the dimension of the cokernel is given by
$$\dim(\mathrm{coker}(u)) = \sharp\{\Gamma(N) - (\Gamma(M) + v(u))\}.$$

2. Consider $A = k[[t^6, t^7, t^8, \ldots]]$, and let M be the module generated by $\{1, t^2 + t^3, t^4\}$. Then $\Gamma(M) = 0\ \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}$, and $\{n \mid n + \Gamma(M) \subset \Gamma(M)\} = \Gamma(M)$.

However, it is easy to see that $\Gamma(\mathrm{End}(M)) = 0\,\underline{1}\,\underline{2}\,\underline{3}\,4\,\underline{5}$. Notice also that (at least for cubical rings) this is the counter example of smallest multiplicity. (To justify this last assertion, if $1+\Gamma(M) \subset \Gamma(M)$ then necessarily $M \simeq \bar{A}$.)

3. Calculation of $\mathrm{Ext}^1(M, M)$: this also depends on more than $\Gamma(M)$. As an example consider the family of modules M_λ over $A = k[[t^3, t^7]]$ as above: $M_\lambda = A + (t^4 + \lambda t^5)A + t^8 A$. We find that $M_\lambda^* \simeq M_\lambda$, in fact

$$M_\lambda \simeq M_\lambda^{-1} = \langle t^6 - \lambda t^7, t^{10}, t^{14} \rangle.$$

The quotient $T = A/M_\lambda^{-1}$ has basis

$$\{b_0, b_1, b_2\} = \{1, t^3, t^7\}.$$

By (2.5.7) there is an isomorphism $\mathrm{Ext}^1(M_\lambda, M_\lambda) \simeq \mathrm{Hom}(M_\lambda, T)$ and the latter can be calculated using the methods of §2.5. Note that $\mathrm{Ann}(T) = m^2$. Suppose first that $\lambda = 0$: then, amongst the generators, we have relations mod $m^2.M_0$

$$t^7.1 = t^3.t^4 \text{ and } t^7.t^4 = t^3.t^8.$$

These imply that there are no homs. $M_0 \longrightarrow T$ with non-zero image on $\langle b_0 \rangle$, and hence that $\dim \mathrm{Ext}^1(M_0, M_0) = 6$.

If $\lambda \neq 0$ then $t^8 = \frac{1}{\lambda} t^3(t^4 + \lambda t^5) - t^7$ is not needed as a generator so $\dim \mathrm{Ext}^1(M_\lambda, M_\lambda) \leq 6$. There is no analogue of the first relation above, however mod $m^2.M_\lambda$, $t^7.(t^4 + \lambda t^5) = t^3.t^8 = 0$ so the second generator must have zero image on $\langle b_0 \rangle$ under any homomorphism, and $\dim \mathrm{Ext}^1(M_\lambda, M_\lambda) = 5$.

For each M_λ note that $\mathrm{End}(M_\lambda) = M_\lambda$ so

$$\chi_1(M_\lambda) = \dim \mathrm{Ext}^1(M_\lambda, M_\lambda) - \dim(\mathrm{End}(M_\lambda)/A) = \begin{cases} 3 & \text{if } \lambda = 0 \\ 2 & \text{if } \lambda \neq 0 \end{cases}.$$

§6.3 Relation with the Compactified Jacobian

Henceforth we assume that A is a semigroup ring. Suppose that X is a curve with a single singularity which has a local ring whoose completion is A. Then we know that, as a set, points of $\overline{J(X)}$ are in 1-1 correspondence with points of $J(\tilde{X}) \times P_I(A)$ for a suitable ideal I of A. In fact we can take $I = t^{2\delta}\bar{A}$:

Lemma 6.3.1. Let S be a Γ-module in normal form, write $r(S) = c(S) - \delta(S)$ for the number of elements of S less than $c(S)$. Then

1. $r(S) \leq \delta$;
2. All rank 1 A-modules are represented in $\mathrm{Gr}^A(\delta, \bar{A}/I)$.

Proof.

1. $c(S) - 1$ is the top gap of S, and for all $c(S) > s \in S$ we must have $0 \leq c(S) - 1 - s \notin \Gamma$. Hence $r(S) \leq \delta$.
2. Let M be an A-module in normal form and let $\Gamma(M) = S$. By the first part
$$c(S) \leq \delta + \delta(S) = 2\delta - \ell(S).$$
As a consequence $I \subset t^{\ell(M)} \cdot M$ and $t^{\ell(M)}/I$ gives the required point in $\mathrm{Gr}^A(\delta, \bar{A}/I)$.

We will write simply $P(A)$ for the associated reduced fixed point subscheme. The first attempt to stratify $\overline{J(X)}$ would be into sets representing modules with a given value set S. Denote by U_S the corresponding subset of $P(A)$; this is what we are interested in studying in this section. We show U_S is locally closed and irreducible, and that there is another partial ordering on Δ_0 such that $\overline{U_S} \supset U_{S'}$ only if $S \leq S'$. This enables us to count the number of components of $P(A)$, for example. We can also calculate the dimension of U_S; note that if $\dim U_S = \rho(S) > \delta$ then there must be a $\rho(S) - \delta$ dimensional family of non-isomorphic A-modules having this value set S.

$P(A)$ is a subvariety of a Grassmannian $G = \mathrm{Gr}(\delta, 2\delta)$ and U_S is the intersection of an appropriate Schubert cell of G with $P(A)$. Firstly, we recall some facts about the Schubert cell decomposition of G.

Order a basis for $\bar{A}/I$ as

$$\{b_0, b_1 \ldots, b_{2\delta-1}\} = \{1, t, \ldots, t^{2\delta-1}\};$$

given an increasing sequence $\underline{a} = a_1, \ldots, a_\delta$ with $0 \leq a_1 < a_\delta < 2\delta - 1$ denote by $P_{\underline{a}}$ the corresponding Plücker coordinate. Define a partial ordering on the set of all these sequences by

$$\underline{a} \leq \underline{a}' \Leftrightarrow a_i \leq a_i' \;\; \forall\, i = 1, \ldots, \delta.$$

Then the *Schubert cell decomposition of* G is given by defining locally closed subschemes of G:

$$U_{\underline{a}} = \{P_{\underline{a}} \neq 0\} \cap \{\cap_{\underline{a}' < \underline{a}} P_{\underline{a}'} = 0\}.$$

These certainly cover G and one has the following standard theorem.

Theorem 6.3.2. Each $U_{\underline{a}}$ is an affine space and

1. $U_{\underline{a}'} \subset \overline{U_{\underline{a}}} \Leftrightarrow a' \leq a$.
2. $\dim_k U_{\underline{a}} = \sum_{i=1}^{\delta}(\delta - a_i + 1)$.

The proof becomes straightforward on writing down, in matrix form, a basis for a general point of $U_{\underline{a}}$.

We would like a version of this for $P(A)$. Assume throughout that $A = R(\Gamma)$ is a semigroup ring. Let S be a Γ-module S in normal form, and suppose $0 = s_1, s_2, \ldots, s_r$ are the elements of S less than $c(S)$ (allowing the empty set for $S = \bar{\Gamma}$). The number of terms in this sequence depends on S, but is always less than or equal to δ by (6.3.1). Now, extend this to a sequence of length δ by setting $s_{r+1} = c(S), s_{r+2} = c(S) + 1, \ldots, s_{\delta-1} = c(S) + \delta - r - 2$.

Definitions. For any Γ module S in normal form define $\tau(S)$ to be the sequence $\tau_0(S), \tau_1(S), \ldots, \tau_{\delta-1}(S)$ where $\tau_i(S) = s_i + \ell(S)$. As above, say $\tau(S) \leq \tau(S')$ iff $\tau_i(S) \leq \tau_i(S')$ for all i. This gives us another partial ordering on Δ_0. Now look at $U_S \subset P(A)$; a point belongs to U_S if and only if (i) $P_{\tau(S)} \neq 0$ and (ii) $P_{\tau(S')} = 0 \;\forall\; S' < S$. Of course, $U_S = P(A) \cap U_{\tau(S)}$.

Theorem 6.3.3. For each Γ-module S the component U_S of $P(A)$ representing modules M with $\Gamma(M) = S$ is an affine space. Also

1.

$$\overline{U_S} \supset U_{S'} \Rightarrow \tau(S) \leq \tau(S').$$

But the reverse implication does not hold in general.

2. $\rho(S) = \dim U_S$ can be calculated as follows: Let $0 = r_1 < \ldots < r_k$ be the generators of S less than $c(S)$; define S^i, $1 \leq i \leq k+1$, to be the Γ-module generated by $\{r_j | \; j < i\}$ ($S^0 = \emptyset$). For the i^{th} generator r_i set

$$N_i = \sharp\{g \in \mathbf{N} - S | \; g > r_i \text{ and } \gamma \in \Gamma, \; \gamma + r_i \in S^{i-1} \Rightarrow g + \gamma \in S\}.$$

So, e.g., $N_1 = \delta(S)$. Then

$$\rho(S) = \sum_{i=1}^{k} N_i.$$

Proof. We explain how to write down, in matrix form, a basis for any subspace giving a point of U_S, the assertions at the top, and the formula for $\rho(S)$ will follow from this. Beforehand, note that the implication in **1.** is immediate; a counter example to the reverse implication follows (6.3.4).

Suppose $0 = r_1 < r_2 < \ldots < r_k$ are the generators of S less than $c(S)$. We can assume that the first row of the matrix has the following form—for convenience refer to the first column as 'column zero' and write $\ell = \ell(S)$:

$$R_1 = 0 \ldots 0 \; 1 \; \alpha_1 \; \ldots \alpha_\delta \; 0 \ldots 0$$

i.e., R_1 has a 1 in column ℓ, an entry α_i in column j iff j is a gap in $S + \ell$ and zeros elsewhere. Now for all $\gamma \in \Gamma$ with $\gamma < 2\delta - \ell$ we get shifted rows $(R_1 + \gamma)$, consisting of γ zero entries followed by the first $2\delta - \gamma$ entries of R_1.

Now take r_2, write

$$R_2 = 0 \ldots 0,0,\ldots,0 \; 1 \; \alpha_1 \; \ldots \alpha_{N_2} \; 0 \ldots 0$$

where the first 1 occurs in column $r_2 + \ell$, and, to see what else to write proceed inductively: If $r_2 + \ell + 1$ is not a gap write 0; if $r_2 + \ell + 1$ is a gap then if

$r_2+\ell+1+\gamma$ is a gap for any $\gamma \in \Gamma$ the entry is determined as α_i for appropriate i, otherwise write $\alpha_{\delta+1}$. Continue in this way. Now acting on R_2 by elements of Γ gives us some more rows; by construction each of these will either coincide with shifts of R_1 (in which case discard it) or it will be distinct. Now proceed to R_3 with first non-zero entry a 1 in column $r_3+\ell \ldots$ and so on. The important point is that at each stage, under shifting by Γ the new rows will be consistent with those constructed before. Since we are working with a monomial ring all of the corresponding subspaces of $\bar{A}/I$ will be A-modules. It is clear that all M with $\Gamma(M) = S$ arise in this way for some choice of $\{\alpha_1, \ldots, \alpha_{\rho(S)}\}$. U_S is then an affine space with coordinates the α_i.

In certain cases the calculation of $\rho(S)$ simplifies:

Corollary 6.3.4.

Let $r_1 = 0, \ldots, r_k$ generate S mod $C(S)$. $\rho(S)$ satisfies:

1. $\rho(S) \geq \delta(S)$ with equality if $k = 1$.
2. $\rho(S) \leq \sum_i \sharp\{\text{gaps}, > r_i\}$ with equality if $\{r_i\}$ freely generate S mod $C(S)$, i.e., if for all pairs $\gamma_i \neq \gamma_j \in \Gamma$ with $r_i+\gamma_i = r_j+\gamma_j$ we have $r_i+\gamma_i \geq c(S)$.

Proof. This is clear from the proof of **2** in (6.3.3) above.

Example The following gives the promised counter example to the reverse implication in (6.3.6(**1**)). Take

$$\Gamma = \langle 5,6,7\rangle : 0\ \underline{\bar{1}}\ \underline{\bar{2}}\ \underline{\bar{3}}\ \underline{\bar{4}}\ 5\ 6\ 7\ \underline{\bar{8}}\ \underline{\bar{9}}$$

and

$$S = 0\ 1\ 2\ \underline{\bar{3}}\ \underline{\bar{4}}.$$

Then

$$\tau(\Gamma) = (0\ 5\ 6\ 7\ 10\ 11) \leq (4\ 5\ 6\ 9\ 10\ 11) = \tau(S).$$

But by (6.3.4(**2**)), since there are no relations $< 5 = c(S)$ between the **3** generators of S, $\rho(S) = 3.2 = 6 = \rho(\Gamma)$. Hence we cannot possibly have $U_S \subset \overline{U_\Gamma}$.

In order to find sufficient conditions for $U_{S'} \subset \overline{U_S}$ given $S \leq S'$ consider how one constructs a specialisation in the case of $\mathrm{Gr}(\delta, 2\delta)$. Given $\underline{a} \leq \underline{b}$ look at the matrix representing a general point of $U_{\underline{a}}$. Row i has first non-zero entry a 1 in column a_i, zeros in columns a_j, $j \neq i$, and other entries arbitrary. For each i with $a_i < b_i$ replace row i with the row having a 1 in column b_i, entries$< b_i$ multiplied by λ and other entries unchanged. For $\lambda \neq 0$ this altered matrix gives a point of $U_{\underline{a}}$, but $\lambda = 0$ gives the required specialisation to a general point of $U_{\underline{b}}$. The obstruction to this working for $U_S \subset P(A)$, is, of course, that we must preserve A-module structure at each stage. However, if $e(\Gamma) > c(S)$ then this problem disappears and we obtain:

Corollary 6.3.5.

1. Given S with $c(S) < e(\Gamma)$ we have

$$S' \geq S \Rightarrow U_{S'} \subset \overline{U_S}.$$

2. In particular, if Γ is cubical then, for all S, S'

$$U_{S'} \subset \overline{U_S} \Leftrightarrow S' \geq S.$$

Let us now look a little more closely at the partial ordering we have defined on Δ_0. We compare 2 semigroup modules which are joined in the diagram Δ.

Lemma 6.3.6. Suppose S and S' are semigroup modules such that $S' = S \cup \{s\}$ for some gap s of S. Then

$$S < S' \Leftrightarrow c(S) = s + 1,$$

i.e. s is the highest gap of S.

Proof. On passing from S to S' 'ℓ' increases by 1. Suppose $s_i = c(S)$, then, since $s < c(S)$, $s'_{i+1} = s_i$; but for $S \leq S'$ we need $s_i \leq s'_i + 1$ which happens iff $s'_i = c(S) - 1$.

For practical purposes we can simplify this partial ordering somewhat. For given i, $r_{\delta-i}(S) = 2\delta - 1 - i$ is the maximum possible. Define $\hat{r}(S)$ to be the sequence obtained by deleting all such terms from $r(S)$. Then the number of terms in $\hat{r}(S)$ is $r(S) = c(S) - \delta(S)$.

Lemma 6.3.7. $\tau(S) \leq \tau(S')$ if and only if (i) $r(S) \geq r(S')$ and (ii) $\tau_i(S) \leq \tau_i(S')$ $\forall i \leq r(S')$.

Proof. Obvious.

For some examples see §6.4/5, especially the end of §6.5.

As a closing remark we can say that using the ideas in this section many of the basic questions can be reduced to purely combinatorial problems, but that these combinatorics are quite complicated to handle.

§6.4 Examples: Cubical Singularities

Our intention here is to explore the situation in more detail for these relatively simple singularities. We describe the stratification of $\overline{J(X)}$ for the first few cases and also compute the dimensions of tangent spaces at various points. It turns out that $\overline{J(X)}$ is generically reduced, but it is possible that it could be highly non-reduced at the worst point, for example.

Let $\Gamma_n = \langle n, n+1, \ldots, 2n-1 \rangle$ and let $R_n = R(\Gamma_n)$ be the associated semigroup ring. We assume $n > 1$; then $\delta = n - 1$ and Δ_{Γ_n} is a δ-cube. The number of points in $\Delta_0^\ell = \{S \in \Delta_0 |\ \ell(S) = \ell\}$ is $\binom{n-1}{\ell}$. For $0 \leq i \leq n-2$ let

$$S_i = \{0, 1, \ldots, i, n, n+1, \ldots\} = \Gamma \cup \{1, 2, \ldots i\}.$$

Note that $c(S_i) = c(\Gamma)$ and $S_i^* = S_{n-2-i}$.

Proposition 6.4.1. (This can also be found in [Greuel and Pfister])

$P(R_n)$ has $n-1$ components, namely $\overline{U_{S_i}}$ for S_i as above. Also $\dim U_{S_i} = (i+1)(n-1-i)$. In particular

$$\dim P(R_n) = \begin{cases} (n^2-1)/4 \text{ if } n \text{ is odd} \\ n^2/4 \text{ if } n \text{ is even} \end{cases}$$

Proof. In each Δ_0^i S_i is the unique minimal element (with respect to '$\leq$'), and so by (6.3.5) there are at most $n-1$ components. We now check that $S_i \not\leq S_j$ for $i \neq j$. Otherwise we would have $i < j$, but then by duality we would also have $S_{n-2-i} \leq S_{n-2-j}$, forcing $n-2-i \leq n-2-j$ which is impossible. Hence there are precisely $n-1$ components.

To calculate $\rho(S_i)$ directly use the fact that we have no relations between the generators of any semigroup module modulo the conductor, and so by (6.3.6(2)),

$$\rho(S) = \sum_{s_i < c(S)} \sharp\{\text{gaps} > s_i\}$$

and for S_i this is just 'number of generators' times 'number of gaps', i.e $\rho(S_i) = (i+1)(n-1-i)$. Alternatively one can show that $\overline{U_{S_i}} \simeq \mathrm{Gr}(i+1, n-1)$.

For these semigroups it is also quite easy to calculate the dimensions of spaces of self-extensions. We consider R_{n-i} as a module over R_n and modules M_i with $\Gamma(M_i) = S_i$.

Proposition 6.4.2. Given any R_n module M the number $\chi_1(M)$ depends only on $\Gamma(M) \in \Delta_0$ and (if $n \geq 6$) $\Gamma(\mathrm{End}(M))$. For the particular modules R_{n-i} and for any module M_i with $\Gamma(M_i) = S_i$ we find:

1. $\chi_1(M_i) = i(n-2-i)\ \forall i$;

2. $\chi_1(R_{n-i}) = i(n-2),\ i < n-1$;

3. $\chi_1(\tilde{R}) = (n-1)^2$.

Proof. Recall our method of calculating χ_1 from (§2.5) and the comments of (6.2.5). Since the maximal ideal of R_n acts trivially on R_n/C the number of generators of an R_n-module will equal $1 + \ell(M)$, i.e., this depends only on the semigroup. Thus $\dim \mathrm{Ext}^1(M, M)$ depends only on $\Gamma(M)$ for any M. If $\Gamma(M) = S_i$ for some $i < n-1$ then $\Gamma(\mathrm{End}(M)) \subset \Gamma$ by (6.2.4), and so they must be equal; and further, $\chi_1(M)$ is then determined by S_i. So, for convenience, let M_i be the monomial module of S_i. Now calculate χ_1:

1. $\mathrm{End}(M_i) = R_n \Rightarrow \chi_1(M_i) = \dim \mathrm{Ext}^1(M_i, M_i)$. Now $M_i \subset \omega$ and the quotient is the finite dimensional module k^{n-2-i}. On applying $\mathrm{Hom}(M_i, \cdot)$ we get $R_n \subset M_{n-2-i}$ with cokernel k^{n-2}. Thus

$$\chi_1(M_i) = \dim \mathrm{Ext}^1(M_i, M_i) = \dim \mathrm{Hom}(M_i, k^{n-2-i}) - (n-2-i)$$

$$= (i+1)(n-2-i) - (n-2-i) = i(n-2-i).$$

2. $\mathrm{End}(R_{n-i}) = R_{n-i}$ and so $\chi_1(R_{n-i}) = \dim \mathrm{Ext}^1(R_{n-i}, R_{n-i}) - i$. Look at

$$0 \longrightarrow t^i \cdot R_n \longrightarrow \omega \longrightarrow k^{n-2} \longrightarrow 0$$

(valid for $i < n-1$). On applying $\mathrm{Hom}(R_{n_i}, \ \cdot \ )$ the first map becomes $R_{n-i} \subset \omega_{n-i}$ where $\omega_{n-i} = R_n + tR_n + \ldots t^{n-i-2}R_n + t^{n-i} + \ldots$ is the dualising module of R_{n-i}; the cokernel has dimension $(n-2-i)$.

$$\chi_1(R_{n-i}) = \dim \mathrm{Hom}(R_{n-i}, k^{n-2}) - (n-2-i) - i$$

$$= (i+1)(n-2) - (n-2-i) - i = i(n-2).$$

3. $\tilde{R}$: This is similar to the last case; now we must look at

$$0 \longrightarrow t^n \tilde{R} \longrightarrow \omega \longrightarrow k^{n-1} \longrightarrow 0.$$

We find

$$\chi_1(\tilde{R}) = n(n-1) - (n-1) = (n-1)^2.$$

In conclusion:

Theorem 6.4.3. If X is a curve with a single irreducible cubical singularity of multiplicity n then $\overline{J(X)}$ is generically smooth of dimension $g(\tilde{X}) + [n^2/4]$ and it has $n-1$ components. The tangent space at the worst points has dimension $g(\tilde{X}) + (n-1)^2$. Further, the closure of $J(X)$ consists of all sheaves of local type R_{n-i} for some i.

Proof. Most of this follows immediately from the above using the results established elsewhere in this thesis. To see generic smoothness, on any component points of local type S_i, for some i, are dense. We must check that $\dim(U_{S_i}) = \dim T_i$ is equal to the dimension of the tangent space at such a point. Using (6.4.1) the left hand side is

$$g(\tilde{X}) + \rho(S_i) = g(\tilde{X}) + (i+1)(n-1-i).$$

From above (6.4.2) the right hand side gives

$$\dim T_i = g(X) + \chi_1(M_i) = g(\tilde{X}) + (n-1) + i(n-2-i)$$

$$= g(\tilde{X}) + (i+1)(n-1-i)$$

as required.

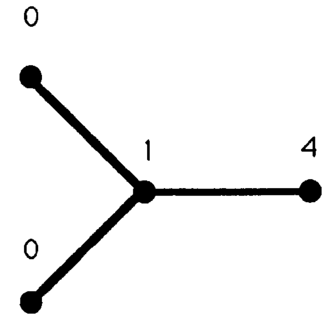
The final remark on the closure of $J(X)$ follows easily by induction using (4.3.5).

Remark. We would expect the same result to hold for any cubical singularity. [Rego 1], who also established this value of $\dim \overline{J(X)}$ in this case, claims that any curve singularity has a 'good' δ-constant deformation into a curve with a cubical singularity, and so $\dim \overline{J(X)} \le g + [(\delta+1)^2/4]$ by semicontinuity; unfortunately a proof of this claim does not seem to have been published.

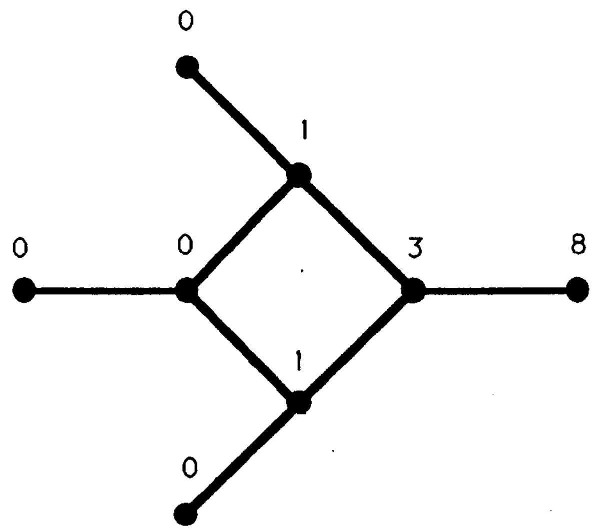
Examples 6.4.4. Finally, here are the stratification diagrams for $\overline{J(X)}$ in the cases $n = 3, 4$. As we have explained, the stratification in these cases is given by semigroup modules. The numbers attached to each vertex give the *embedding codimension* of that stratum. (The embedding codimension at a point is defined to be the dimension of the Zariski tangent space minus the dimension of the component on which the point lies, e.g., embedding codimension $= 0 \Leftrightarrow$ the point is smooth.) Note that the results above, together with duality, mean that we have already calculated the tangent spaces on all strata below except one; details for this will follow the second diagram.

The 'topmost' vertex in each diagram is that of locally free sheaves; duality corresponds to reflection in the horizontal axis.

$n = 3:$



$n = 4:$



In the diagram for $n = 4$ we had also to deal with the Γ-module

$$S = 0\,\underline{\overline{1}}\,2\,\underline{\overline{3}}.$$

There is a 1 parameter family of modules, $M_\lambda = R_4 + (t^2 + \lambda t^3)R_4$, having this invariant. For each M_λ note that $\mathrm{End}(M_\lambda) = M_\lambda$. $M_\lambda \subset \omega$ with quotient k, so one calculates $\chi_1(M_\lambda) = 2 - 1 = 1$. Since U_S is contained in the top dimensional component of $\overline{J(X)}$ these points are smooth.

Remark. An interesting observation concerning these modules is the following: We deduce that for each λ, X has a partial normalisation X_λ which has an A_4 point, but these partial normalisations are not isomorphic to each other. Another consequence is that on X there are an infinite number of rank 1 torsion free sheaves satisfying $\mathcal{F}^* = \mathcal{F}$ (since there is at least one such locally free sheaf on each X_λ). Such phenomena will occur on any curve with an analytically irreducible singularity of multiplicity ≥ 4.

§6.5 Examples: $\Gamma = \langle p, q \rangle$

Here we look briefly at singularities with semigroup $\langle p, q \rangle$ $(p < q)$. Since these are plane curve singularities we know that the compactified Jacobian is integral. It is known [Rego 1] that the boundary of $\overline{J(X)}$ has $p-1$ components. We identify what these are in terms of Γ-modules, and, as a consequence deduce that there exist curves in $\mathbf{P}^3$ with a single singularity such that the compactified Jacobian has an arbitrary number of components. Going back to plane curves, we prove that, in the case $q = p+1$, the tangent space at the worst points, U_Γ, has dimension $g + \delta$.

Let $\Gamma = \langle p, q \rangle$ for p, q coprime, $A = k[[t^p, t^q]]$. We refer back to (6.2.3(2)) for some facts about Γ-modules: Write

$$S_i = \Gamma \cup \{c-1, c-1-q, \ldots, c-1-iq\}$$

for $0 \leq i \leq p-2$. These are the Γ-modules ($\neq \Gamma$) with $c(S)$ maximal.

Proposition 6.5.1. $\rho(S_i) = \delta - 1$ $\forall$ i; hence U_{S_i} has codimension 1 in $P(A)$ and these are all such S.

Proof. Firstly, note that S_i is generated by $\{r_1, r_2\} = \{0, c-1-iq\}$ over Γ, and $c-1-iq = pq-p-(i+1)q = (p-i-1)q-p$, so there are relations between these generators $r_2 + p = r_1 + (p-i-1)q$. $\delta(S_i) = \delta - i - 1$; let $c-1-\gamma$ be a gap $> r_2$, for this gap to count towards ρ we must have $c-1-\gamma+\gamma' \in S_i$ $\forall \gamma' \in \Gamma$. In other words we must have $\gamma = jq+p$ for some $0 \leq j \leq i-1$. Whence $\rho(S_i) = \delta - i - 1 + i = \delta - 1$ as required. The other results follow because, by [Rego 1], we know that $P(A) - U_\Gamma$ has exactly $p-1$ components.

Corollary 6.5.2. Write $r = c(\Gamma) - 1 = pq - p - q$, let $\Gamma' = \langle p, q, r \rangle$, $A' = R(\Gamma')$. Then $P(A')$ is reducible with $p-1$ components. It follows that if X is the plane curve defined by $x^p z^{(q-p)} + y^q = 0$ then X has a partial normalisation X' corresponding to A' with $\overline{J(X')}$ having $p-1$ components.

Proof. Immediate from the above, since $P(A_1) = P(A) - U_\Gamma$.

Proposition 6.5.3. Let $A = k[[t^p, t^{p+1}]]$ then $\chi_1(\bar{A}) = \delta(A) = \frac{1}{2}p(p-1)$.

Proof. To begin, $\mathrm{End}(\bar{A}) = \bar{A}$ and $\ell(\bar{A}) = \delta$, so we have to show that $\dim \mathrm{Ext}^1(\bar{A}, \bar{A}) = 2\delta$. By (2.5.7) there is an isomorphism

$$\mathrm{Ext}^1(\bar{A}, \bar{A}) \simeq \mathrm{Hom}(\bar{A}, A/C).$$

So it must be shown that

$$\dim \mathrm{Hom}(\bar{A}, A/C) = p(p-1).$$

Since $\mathrm{socle}(A/C) = k^{p-1}$ and $\bar{A}$ requires p generators over A we obtain the inequality $\dim \mathrm{Hom}(\bar{A}, A/C) \geq p(p-1)$. But for each $0 < i \leq p-1$ we have the relation

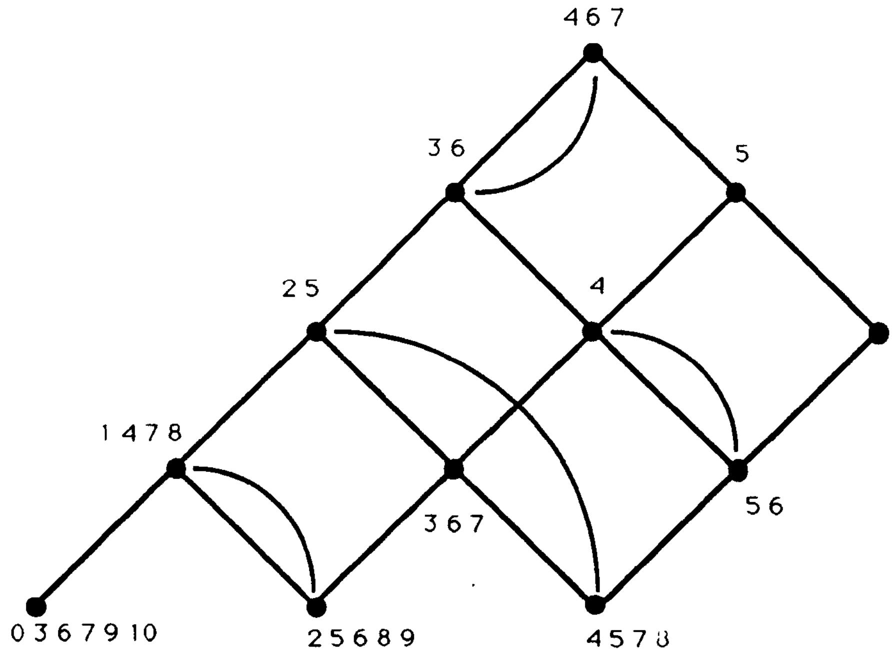
$$t^i \cdot t^p = t^{i-1} \cdot t^{p+1}.$$

Using this one sees that for any $\phi \in \mathrm{Hom}(\bar{A}, A/C)$ $\mathrm{image}(\phi) \subset \mathrm{socle}(A/C)$ and so there are no maps other than the above.

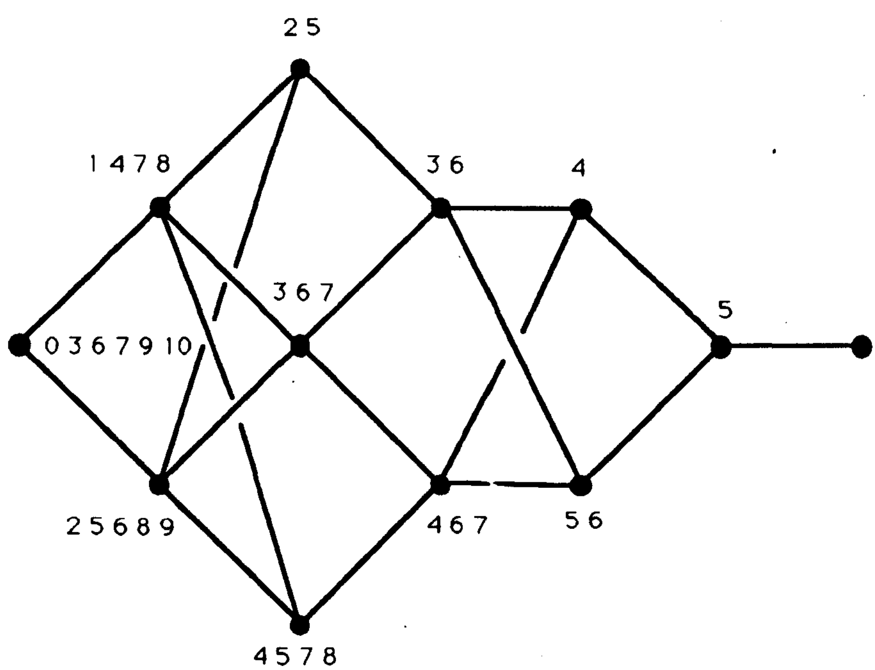
One should be able to deal with general (p, q) by a similar, but more complicated, argument. This seems to generalise the result we had for simple singularities, and one could conjecture $\chi_1(\bar{A}) = \delta(A)$ if and only if A has embedding dimension 2.

To close, we look at the example $\Gamma = \langle 3, 7 \rangle$ already considered earlier in this chapter. Recall that there is just one 1-parameter family of modules M_λ, with $\chi_1(M_0) = 3$ whilst $\chi_1(M_\lambda) = 2$ for $\lambda \neq 0$ (6.2.5(3)). This example illustrates the ideas of §6.3.

Here is the module diagram Δ with $\hat{\tau}$ (see (6.3.7)) written next to each vertex:



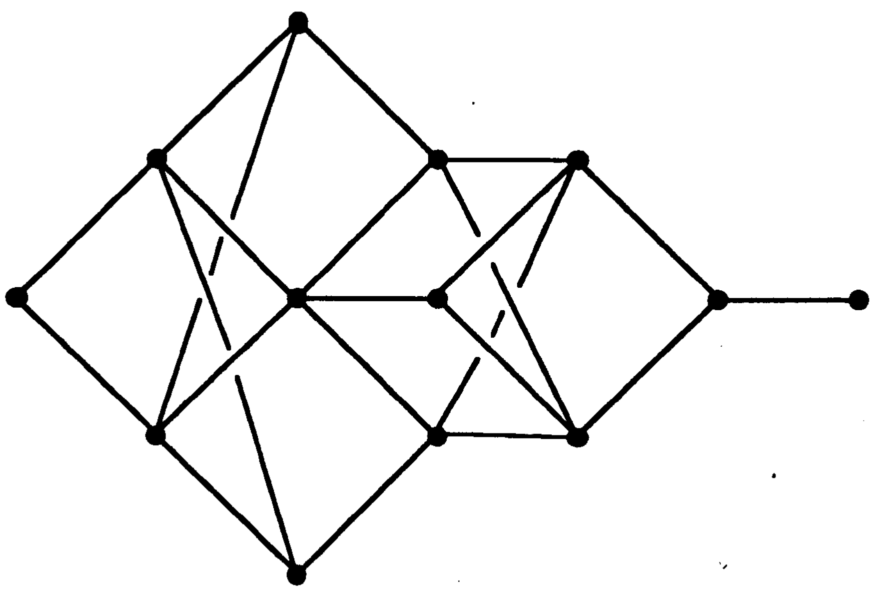
Draw the Hasse diagram for this partial ordering:



Now let X be a curve with a single singularity of this type, and look at $\overline{J(X)}$. In particular we want to draw a stratification diagram analogous to those of §5.1.

Since $\chi_1(M_\lambda)$ jumps by 1 at $\lambda = 0$, in the stratification diagram we require an extra vertex—for M_0, as compared to the previous diagram. Since (§6.3) the stratification does not precisely correspond to this partial ordering it is possible that some edges of the previous graph should be omitted. However, one can easily check (as in §5.3/4) that all of the indicated specialisations do in fact occur, although we omit a proof of this.

Stratification of $\overline{J(X)}$.



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