Computational Aspects of Singularity Theory

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Abstract

In this thesis we develop computational methods suitable for performing the symbolic calculations common to local singularity theory. For classification theory we employ the unipotent determinacy techniques of Bruce, du Plessis, Wall and complete transversal theorems of Bruce, du Plessis. The latter results are, as yet, unpublished and we spend some time reviewing these results, extending them to filtrations of the module $m_n.\mathcal{E}(n,p)$ other than the standard filtration by degree. Weighted filtrations and filtrations induced by the action of a nilpotent Lie algebra are considered. A computer package called `Transversal` is developed. This is written in the mathematical language `Maple` and performs calculations such as those mentioned above and those central to unfolding theory. We discuss the package in detail and give examples of calculations performed in this thesis.

Several classifications are obtained. The first is an extensive classification of map-germs $(\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^4,0)$ under $\mathcal{A}$-equivalence. We also consider the classification of function-germs $(\mathbf{C}^p,0) \longrightarrow (\mathbf{C},0)$ under $\mathcal{R}(\mathcal{D})$-equivalence: the restriction of $\mathcal{R}$-equivalence to source coordinate changes which preserve a discriminant variety, $\mathcal{D}$. We consider the cases where $\mathcal{D}$ is the discriminant of the A_2 and A_3 singularities, extending the results of Arnol'd. Several other simple singularities are discussed briefly; in particular, we consider the cases where $\mathcal{D}$ is the discriminant of the A_4, D_4, D_5, D_6, and E_k singularities.

The geometry of the singularities $(\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^4,0)$ is investigated by calculating the adjacencies and several geometrical invariants. For the given source and target dimensions, the invariants associated to the double point schemes and $\mathcal{L}$-codimension of the germs are particularly significant.

Finally we give an application of computer graphics to singularity theory. A program is written (in C) which calculates and draws the family of profiles of a surface rotating about a fixed axis in $\mathbf{R}^3$, the resulting envelope of profiles, and several other geometrical features. The program was used in recent research by Rycroft. We review some of the results and conclude with computer produced images which demonstrate certain transitions of the singularities on the envelope.

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To my parents.

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Chapter 1

Introduction and Preliminary Material

One of the central themes of local singularity theory is the classification of germs of smooth maps under various types of equivalence. This includes the study of the possible deformations of a map-germ using versal unfoldings. The main objective of this thesis is to develop algorithms suitable for implementation on a computer which perform such tasks.

1.1 Introduction and Historical Background

The foundational work of Mather in the 1960's defined the now standard equivalence relations central to local singularity theory and gave algebraic criteria for determinacy and stability. (We refer to Section 1.3 for an explanation of the terminology and a review of basic singularity theory.) The work of Gaffney in the 1970's, [Ga2, Ga3] led to workable determinacy estimates which were used to perform classifications in several areas, e.g., [duP]. This deals mainly with the classification of map-germs under $\mathcal{A}$-equivalence. However, at around the same time extensive lists of function-germs under $\mathcal{R}$-equivalence were obtained. Many people can be attributed to this work but probably the most relevant are the Russian school of singularity theorists under Arnol'd; a general reference is [AGV]. The subject of determinacy was essentially wrapped up by the definitive results of [BduPW] which discusses determinacy in terms of unipotent group actions and provides excellent determinacy estimates. The other major technical tool used to perform classifications is the 'complete transversal' result of Bruce and du Plessis, [BduP]. This work is, as yet, unpublished and we spend sometime reviewing it in Chapter 2. Another central topic is the calculation of versal unfoldings (a notion due to Thom), used to analyse the deformation of map-germs. Again, there are algebraic criteria for calculating versal unfoldings. A beautiful proof of the 'fundamental theorem of versal unfoldings' may be found in [Mart2, Chapter XIV], [Mart1, Section I]. For a review of all these results, at least up until the early 1980's, we refer to the survey article of Wall, [Wal].

This is where our current research begins. The key point to all the aforementioned calculations is that the algebraic criteria may be reduced to finite symbolic problems (albeit very large ones occasionally) which may be performed by a computer. This is especially the case now we have the powerful determinacy and complete transversal results of Bruce, du Plessis and Wall at our disposal — the resulting classification method is very efficient when implemented on a computer. All the aforementioned algebraic criteria reduce to very similar symbolic calculations on a computer and we have developed a 'classification package' which performs all these calculations and related ones such as checking the hypotheses of the Mather Lemma, [MathIV, Lemma 3.1], (a useful tool in any classification). Indeed, most calculations in local singularity which may be reduced to the finite dimensional analogue of jet-groups acting on jet-spaces may be implemented.

Our classification package (called `Transversal`) is written in the mathematical language `Maple` and consists of a number of routines (or 'Maple functions'). It supports the equivalence relations defined by the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$ (plus several others, see Chapter 6). Its real utilisation comes from calculations involving the $\mathcal{A}$-group. Here the tangent space to an $\mathcal{A}$-orbit is very hard to work with, involving two separate module structures. Many of these calculations are just plain tedious and it is helpful to have a computer 'churn out' the answers, while others are bordering on impossibility without computer aid. One must question the potential use of the computer package until substantial classifications have been obtained. To achieve such a benchmark we perform a number of classifications in this thesis (such classifications are, of course, of immense interest within the field of singularity theory and its applications). We consider the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ under $\mathcal{A}$-equivalence. This is probably the most extensive and computationally involved $\mathcal{A}$-classification carried out to date. The classification of functions germs under coordinate changes in the source which preserve a given discriminant variety, is a variant on $\mathcal{R}$-classification which is computationally more demanding. We perform such classifications using the computer as well.

Several important $\mathcal{A}$-classifications have been performed in the past; for example the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$, see [Rie] (and [duP] for some of the earlier results); also the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^3, 0)$, see [Mo2, Mo1]. Such classifications can now be performed in a relatively short amount of time using the computer (c.f., the original calculations needed, though one would expect things to be easier when the answers are already known!). We must stress that the existence of such classifications helped enormously in the development of our computer programs. The classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ has been considered by West at Liverpool, at least in its earlier stages, [We]. Similarly, these results provided an important independent check to the results found by the computer. Several other colleagues at Liverpool have put the classification package through its paces (or are presently doing so), in particular, Hawes is considering the classification of map-germs $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ and its applications to robotics, [Haw]. We take this opportunity to acknowledge all of the above work as helping toward the development of our classification package. Similar work by Ratcliffe dealt with the use of computational methods in the $\mathcal{A}$-classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^3, 0)$, significantly extending Mond's results, [Rat1]. The original program was written in Pascal and only dealt with this particular classification. However, we have learnt recently that the program has been re-written in Maple and deals with a lot more cases. The packages have not been directly compared, only discussed in personal communications, so we cannot really say much. We will be sufficiently vague and simply add that the underlying algorithms of the packages are similar but the overall goals seem to be quite different!

There are many applications of singularity theory where tedious calculations are routine. Hopefully, our classification package will be of some aid in such areas. For example, in differential geometry, the study of the contact of a *family* of submanifolds with a curve or surface can be reduced to a problem in $\mathcal{A}$-classification, [B6, Section 7]. The classification of the singularities which arise in rigid motions (a branch of robotics) can also be reduced, via transversality results, to an $\mathcal{A}$-classification of map-germs up to a given codimension, [Hob].

1.2 Thesis Overview

In Chapter 2 we review the technical tools such as complete transversals, refining them slightly. For example, we formulate complete transversal theorems which work with unipotent group actions, this ties in with the determinacy theorems of [BduPW] — to obtain an efficient classification method it is convenient to use the *same* subgroup of $\mathcal{K}$ in the determinacy *and* complete transversal calculations. This also allows the use of a finer filtration of the module of map-germs than the standard filtration by degree. We make some attempt at generalising the complete transversal theorems to work with non-standard filtrations. Filtrations defined by a system of weights are considered.

In Chapter 3 we consider the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ under $\mathcal{A}$-equivalence. This includes a list of all the simple singularities and a classification of the corank 1 singularities up to codimension 11. We give several examples of the calculations which give rise to series and include a comprehensive summary of the computer results which provide the resulting stratification of the jet-spaces (for those who feel the urge to read it!).

In Chapter 4 we study the geometry of the singularities discovered in Chapter 3. We give the adjacency diagrams of the simple singularities (verifying that these are indeed simple). Several geometric invariants (such as the those obtained from the multiple points schemes discussed in [MarMo]) are defined. We calculate these invariants for the simple singularities and remark on some relations. The calculation of these invariants can be extremely tedious, as can recognition of map-germs (needed when calculating the adjacencies). Our classification package `Transversal` was of help in both of these aspects.

In Chapter 5 we consider the classification of function germs under diffeomorphisms in the source which preserve a given discriminant variety. We review the corresponding determinacy results and formulate a classification method which uses weighted filtrations. We classify function-germs on the discriminants of the simple singularities: A_k, D_k and E_k; extending the lists found in [A2]. The classification is performed by computer, as is the calculation of the Saito vector fields which generate the module of vector fields tangent to the given discriminant (this module is needed to so that we can perform the classification).

The final two chapters describe our computer work. In Chapter 6 we describe the classification package, `Transversal`. We give a rather technical (though comprehensive) description akin to a reference manual. To make this more 'user friendly' a number of examples of the calculations performed in this thesis are given. These show the precise syntax, given response and appropriate interpretation. The chapter finishes with a description of the programming strategy and underlying algorithms.

We change the theme from computer algebra to computer graphics in the final chapter, Chapter 7, using computers to investigate projections of surfaces. A program is developed for calculating and drawing the family of profiles of a surface which rotates about a fixed axis in 3-space. Is has been used to investigate the geometry and, in particular, the singularities of the envelope of profiles, in recent research by Rycroft, [Ryc]. We review some of the work and conclude with computer produced images which demonstrate the findings.

The classification package and profile/envelope drawing program were demonstrated to several people at the European Singularity Project (ESP) Workshop on Applications of Singularity Theory held at Liverpool University, 29th March – 2nd April 1993 and advertised in [GKMT]. The classification package was also demonstrated at the NSF Regional Geometry Institute held at Amherst, Massachusetts, July 1992. The author would like to thank the Department of Pure Mathematics, University of Liverpool; and the SERC for financial support toward attending this conference.

1.3 Basic Singularity Theory

We will review the techniques and notation of basic singularity theory. We only consider the local case and work with germs of smooth maps $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ ('smooth' will always mean C^∞) or germs of analytic maps $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$, depending on the context. Most of what follows deals with the real case, the definitions in the complex case are similar. As a standard reference we cite the survey article of Wall, [Wal]. In addition we refer to [MathIII], [BduPW], for determinacy results and [Mart2], [Mart1], for unfolding theory. Our notation will be based on these references.

The $\mathbf{R}$-algebra of smooth function-germs $(\mathbf{R}^n, 0) \longrightarrow \mathbf{R}$ will be denoted $\mathcal{E}_n$, the $\mathbf{C}$-algebra of analytic function-germs $(\mathbf{C}^n, 0) \longrightarrow \mathbf{C}$ will be denoted $\mathcal{O}_n$. Both are local rings with maximal ideal, m_n, the germs with zero target. (See [BL, Chapters 1 and 4], for example.) The set of map-germs $(\mathbf{R}^n, 0) \longrightarrow \mathbf{R}^p$ (respectively, $(\mathbf{C}^n, 0) \longrightarrow \mathbf{C}^p$) is an $\mathcal{E}_n$-module (respectively, $\mathcal{O}_n$-module) and will be denoted $\mathcal{E}(n,p)$ (respectively, $\mathcal{O}(n,p)$). The manifolds $\mathbf{R}^n$ and $\mathbf{R}^p$ will be referred to as the *source* and *target* of a map-germ $(\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^p,0)$ (analogously for the complex case).

1.3.1 The Standard Mather Groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$

$\mathcal{R}$ is defined to be the group of germs of diffeomorphisms $(\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$, $\mathcal{L}$ the group of germs of diffeomorphisms $(\mathbf{R}^p,0) \longrightarrow (\mathbf{R}^p,0)$, and $\mathcal{A}$ the direct product $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. We define actions '$\cdot$' of $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ on $m_n.\mathcal{E}(n,p)$ by

$$\begin{aligned} h \cdot f &= f \circ h^{-1}, \quad h \in \mathcal{R}, \\ h' \cdot f &= h' \circ f, \quad h' \in \mathcal{L}, \\ (h,h') \cdot f &= h' \circ f \circ h^{-1}, \quad (h,h') \in \mathcal{A}, \end{aligned}$$

where $f \in m_n.\mathcal{E}(n,p)$. $\mathcal{R}$ (respectively, $\mathcal{L}$) is often called the group of smooth coordinate changes in the source (respectively, target).

$\mathcal{C}$ is defined to be the group of germs of diffeomorphisms $(\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p, 0)$ which project to the identity on $\mathbf{R}^n$ and leave fixed the subspace $\mathbf{R}^n \times \{0\}$. Thus $H \in \mathcal{C}$ can be written in the form

$$H(x,y) = (x, \tilde{H}(x,y))$$

where $\tilde{H} : (\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p,0)$ and $\tilde{H}(x,0) = 0$ for $x \in \mathbf{R}^n$ near zero. We define an action '$\cdot$' of $\mathcal{C}$ on $m_n.\mathcal{E}(n,p)$ by

$$(x, H \cdot f(x)) = H(x, f(x)), \quad H \in \mathcal{C}, \quad f \in m_n.\mathcal{E}(n,p).$$

$\mathcal{C}$ can be thought of as the group of germs of diffeomorphisms $(\mathbf{R}^p,0) \longrightarrow (\mathbf{R}^p,0)$ parametrized by $x \in \mathbf{R}^n$. Define $h_x(y) = \tilde{H}(x,y)$, for x near zero; h_x is (necessarily) the germ of a diffeomorphism. The previous formula can be written as

$$H \cdot f(x) = h_x(f(x)).$$

$\mathcal{K}$ is defined to be the group of germs of diffeomorphisms $(\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p, 0)$ which can be written in the form

$$H(x,y) = (h(x), \tilde{H}(x,y))$$

where h is a map-germ $(\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$ (necessarily a diffeomorphism), $\tilde{H}$ a map-germ $(\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p,0)$ and $\tilde{H}(x,0) = 0$ for $x \in \mathbf{R}^n$ near zero. We define an action '$\cdot$' of $\mathcal{K}$ on $m_n.\mathcal{E}(n,p)$ by

$$(x, H \cdot f(x)) = H(h^{-1}(x), f(h^{-1}(x))), \quad H \in \mathcal{K}, \quad f \in m_n.\mathcal{E}(n,p),$$

that is

$$H \cdot f(x) = h_x(f(h^{-1}(x))).$$

$\mathcal{K}$ is often called the *contact group*; we refer to [MathIII, Section 2] for properties and geometrical interpretations of $\mathcal{K}$ and $\mathcal{C}$.

$\mathcal{C}$ is a normal subgroup of $\mathcal{K}$ and $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ can be identified with subgroups of $\mathcal{K}$ by identifying $h \in \mathcal{R}$, $h' \in \mathcal{L}$ with the map-germs

$$(x,y) \mapsto (h(x), y), \qquad (x,y) \mapsto (x, h'(y)),$$

of $\mathcal{K}$. With these identifications, $\mathcal{K}$ becomes the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$ in the sense that $\mathcal{C}$ is a normal subgroup of $\mathcal{K}$ and each element of $\mathcal{K}$ can be written uniquely in the form $h \circ c$ where $h \in \mathcal{R}$ and $c \in \mathcal{C}$.

The Mather groups are used to define the standard equivalence relations on $m_n.\mathcal{E}(n,p)$, they are all subgroups of $\mathcal{K}$. By a *group of equivalences* we will mean a subgroup $\mathcal{G}$ of $\mathcal{K}$. $\mathcal{G}$ is often one of $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$ but may be any subgroup of $\mathcal{K}$.

1.3.2 Tangent Spaces and Lie Algebras

We define the 'tangent space', θ_f, to $\mathcal{E}(n,p)$ at f to be the $\mathcal{E}_n$-module of germs of smooth vector fields along f. So $\xi \in \theta_f$ if $\xi : (\mathbf{R}^n, 0) \longrightarrow T(\mathbf{R}^p)$ and $\pi_p \circ \xi = f$ where $\pi_p : T(\mathbf{R}^p) \longrightarrow \mathbf{R}^p$ is the natural projection from the tangent bundle $T(\mathbf{R}^p)$ of $\mathbf{R}^p$, to $\mathbf{R}^p$. We define $\theta_n = \theta_{1_{R^n}}$, $\theta_p = \theta_{1_{R^p}}$ where 1_{R^n} and 1_{R^p} denote the (germs at 0 of the) identity maps on $\mathbf{R}^n$, $\mathbf{R}^p$, respectively. The Lie algebra of a group of equivalences $\mathcal{G} \subset \mathcal{K}$ is defined in 2.7. We will just note that $L\mathcal{R} = m_n.\theta_n$ and $L\mathcal{L} = m_p.\theta_p$, for now.

We define the $\mathcal{E}_n$-homomorphism

$$\begin{aligned} tf : \theta_n &\longrightarrow \theta_f \\ \phi &\mapsto df \circ \phi \end{aligned}$$

and the $\mathcal{E}_p$-homomorphism (via $f^* : \mathcal{E}_p \longrightarrow \mathcal{E}_n$, $\alpha \mapsto \alpha \circ f$ for $\alpha \in \mathcal{E}_p$)

$$\begin{aligned} wf : \theta_p &\longrightarrow \theta_f \\ \psi &\mapsto \psi \circ f. \end{aligned}$$

The tangent spaces to the orbits of the standard Mather groups are then given by

$$L\mathcal{R}\cdot f = tf(m_n.\theta_n), \qquad L\mathcal{L}\cdot f = wf(m_p.\theta_p), \qquad L\mathcal{C}\cdot f = f^*(m_p).\theta_f,$$

$$L\mathcal{A}\cdot f = L\mathcal{R}\cdot f + L\mathcal{L}\cdot f, \qquad L\mathcal{K}\cdot f = L\mathcal{R}\cdot f + L\mathcal{C}\cdot f.$$

(Here we follow [BduPW] notationally — this provides natural generalisations in contrast to the old notation $T(\mathcal{G}\cdot f)$.)

The above notation is convenient for theoretical purposes but, in practice, we apply the following observations. θ_f is a free $\mathcal{E}_n$-module of rank p, for if $(y_1,\ldots,y_p)$ is a system of local coordinates on $(\mathbf{R}^p,0)$ then the vector fields

$$(\partial/\partial y_1)\circ f,\ldots,(\partial/\partial y_p)\circ f$$

along f, form a free basis for θ_f. We can therefore identify θ_f with $\mathcal{E}(n,p)$ and the above tangent spaces can be written as

$$\begin{aligned} L\mathcal{R}\cdot f &= m_n.\{\partial f/\partial x_1,\ldots,\partial f/\partial x_n\} \\ L\mathcal{L}\cdot f &= f^*(m_p).\{e_1,\ldots,e_p\} \\ L\mathcal{C}\cdot f &= f^*(m_p).\mathcal{E}_n.\{e_1,\ldots,e_p\} \end{aligned}$$

where $e_1,\ldots,e_p$ are the standard basis vectors of $\mathbf{R}^p$ (considered as elements of $\mathcal{E}(n,p)$).

Note that $L\mathcal{A}\cdot f$ has a mixed module structure. We can, of course, think of it as an $\mathcal{E}_p$-module (via f^*) but lose a lot of the structure in doing so.

1.3.3 Finite Determinacy of Map-Germs

One of the central ideas in singularity theory is to replace the space of germs $m_n.\mathcal{E}(n,p)$ with the space of k-jets $J^k(n,p) = m_n.\mathcal{E}(n,p)/m_n^{k+1}.\mathcal{E}(n,p)$, for some k. This is a finite dimensional vector space which can be identified with the space of Taylor polynomials of such germs. For a given subgroup $\mathcal{G}$ of $\mathcal{K}$ we define $\mathcal{G}_k$ to be the subgroup of $\mathcal{G}$ consisting of all elements of $\mathcal{G}$ whose k-jet is equal to the identity. These are normal subgroups and we define the jet-groups $J^k\mathcal{G} = \mathcal{G}/\mathcal{G}_k$. The action of $\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$ induces an action of $J^k\mathcal{G}$ on $J^k(n,p)$ which is a Lie group action. The idea is to study the action of $\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$ using the action of $J^k\mathcal{G}$ on $J^k(n,p)$. This is discussed in detail in Chapter 2. Denote the projection of $f \in m_n.\mathcal{E}(n,p)$ into the jet-space $m_n.\mathcal{E}(n,p)/m_n^{k+1}.\mathcal{E}(n,p)$ by j^kf.

We say f is *k-$\mathcal{G}$-determined* if any map-germ g with $j^k g = j^k f$ is $\mathcal{G}$-equivalent to f. Once we know a map-germ is k-determined for some k, it is sufficient to work in the k-jet-space to classify the $\mathcal{G}$-orbits.

Mather gave a characterisation theorem for determinacy, [MathIII, Theorem 3.5], [Wal, Theorem 1.2]. The theorems in [BduPW] can give excellent estimates for the determinacy degree (the least k for which a map-germ is k-determined) and we use these in practice. We refer to [BduPW] and Chapter 2 for a detailed discussion on such determinacy theorems.

In order for a computer to perform determinacy checks we need to reduce the criteria to finite problems in linear algebra. When $L\mathcal{G} \cdot f$ is an $\mathcal{E}_n$-module (for example when $\mathcal{G} = \mathcal{R}$ or $\mathcal{K}$) we can use the Nakayama Lemma.

Lemma 1.1 (Nakayama) *Let R be a commutative ring, M an ideal such that for $x \in M$, $1 + x$ is a unit. Let C be an R-module, A and B R-submodules of C with A finitely generated. If $A \subset B + M.A$ then $A \subset B$.*

Proof. See [BL, Lemma 4.15], [Wal, Lemma 1.4]. □

Example. $R = \mathcal{E}_n$, $M = m_n$, $C = m_n$, $A = m_n^{k+1}$, $B = L\mathcal{R}_1 \cdot f$, where f is a function germ $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$. To check the determinacy criterion

$$m_n^{k+1} \subset L\mathcal{R}_1 \cdot f$$

we need only verify

$$m_n^{k+1} \subset L\mathcal{R}_1 \cdot f + m_n^{k+2}.$$

However, in the case $\mathcal{G} = \mathcal{A}$, $L\mathcal{A} \cdot f$ is not an $\mathcal{E}_n$-module. We now use the following result due to du Plessis.

Lemma 1.2 *Let C be a finitely generated $\mathcal{E}_n$-module, $B \subset C$ a finitely generated $\mathcal{E}_n$-submodule, $A \subset f^*(m_p).C$ a finitely generated $\mathcal{E}_p$-submodule (via f^*) and M a proper, finitely generated ideal in $\mathcal{E}_n$ such that for $x \in M$, $1 + x$ is a unit. If*

$$M.C \subset A + B + M.(f^*(m_p) + M).C$$

then

$$M.C \subset A + B.$$

Proof. [BduPW, Lemma 2.6]. □

Example. $C = \mathcal{E}(n,p)$, $A = L\mathcal{L}_1 \cdot f$, where $f \in m_n.\mathcal{E}(n,p)$, $B = L\mathcal{R}_1 \cdot f$, $M = m_n^{k+1}$. To check the determinacy criterion

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{A}_1 \cdot f$$

we need only verify

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{A}_1 \cdot f + m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p) + m_n^{2k+2}.\mathcal{E}(n,p).$$

1.3.4 Versal Unfoldings

An s-parameter *unfolding* of a map-germ $f_0 \in m_n.\mathcal{E}(n,p)$ is a map-germ

$$\begin{array}{rcl} F : (\mathbf{R}^n \times \mathbf{R}^s, 0) & \longrightarrow & (\mathbf{R}^p \times \mathbf{R}^s, 0) \\ (x,u) & \mapsto & (f(x,u),u) \end{array}$$

such that $f_0(x) = f(x,0)$. The notation $f_u(x) = f(x,u)$ is often employed; f_u can be thought of as a deformation of f_0, parametrized smoothly by $u \in \mathbf{R}^s$.

We will consider the case $\mathcal{G} = \mathcal{A}$ in what follows. The definitions and results for other subgroups of $\mathcal{K}$ are analogous. Two unfoldings $F, G : (\mathbf{R}^n \times \mathbf{R}^s, 0) \longrightarrow (\mathbf{R}^p \times \mathbf{R}^s, 0)$ of f_0 are *isomorphic* if there exists germs of diffeomorphisms

$$\begin{array}{rcl} \phi : (\mathbf{R}^n \times \mathbf{R}^s, 0) & \longrightarrow & (\mathbf{R}^n \times \mathbf{R}^s, 0) \\ \psi : (\mathbf{R}^p \times \mathbf{R}^s, 0) & \longrightarrow & (\mathbf{R}^p \times \mathbf{R}^s, 0) \end{array}$$

which are s-parameter unfoldings of the identity maps on $\mathbf{R}^n$ and $\mathbf{R}^p$, respectively, and $G = \psi \circ F \circ \phi^{-1}$. So $\phi_0 = 1_{\mathbf{R}^n}$, $\psi_0 = 1_{R^p}$ and ϕ_u, ψ_u are (necessarily) germs of diffeomorphisms of $\mathbf{R}^n$, $\mathbf{R}^p$, respectively, for small u. Thus $g_u = \psi_u \circ f_u \circ \phi_u^{-1}$ and g_u is $\mathcal{A}$-equivalent to f_u via diffeomorphisms in the source and target which are parametrized smoothly by $u \in \mathbf{R}^s$ (for small u).

Remark. The germs f_u, g_u, ϕ_u, ψ_u cannot be considered as germs at 0 with target 0 (for $u \neq 0$). This situation, where the origin is not fixed, is often called $\mathcal{A}_e$*-equivalence.* If we need to keep the origin fixed then the map-germs ϕ, ψ must satisfy $\phi(0,u) = 0$ and $\psi(0,u) = 0$, in addition, for all (small) u. The terms $\mathcal{A}_e$*-unfolding* and $\mathcal{A}$*-unfolding* are sometimes used to clarify the context.

Given $h : (\mathbf{R}^t, 0) \longrightarrow (\mathbf{R}^s, 0)$ we define the *pull-back* of F by h, denoted h^*F, to be the t-parameter unfolding

$$(h^*F)(x,v) = (f(x,h(v)),v).$$

F and G are said to be *equivalent* if there exists a diffeomorphism $h : (\mathbf{R}^s,0) \longrightarrow (\mathbf{R}^s,0)$ such that G is isomorphic to h^*F (this is an equivalence relation). If G is now some t-parameter unfolding of f_0 (so t does not necessarily equal s), we say G is *induced* from F if there exists a smooth map-germ $h : (\mathbf{R}^t,0) \longrightarrow (\mathbf{R}^s,0)$ such that G is isomorphic to h^*F. The fundamental definitions of unfolding theory follow.

Definitions.

1. F is *versal* if every unfolding of f_0 is induced from F.
2. F is *trivial* if it is isomorphic to the constant unfolding (in s parameters), $(x,u) \mapsto (f_0(x),u)$.
3. f_0 is *stable* if all unfoldings of f_0 are trivial.

To say F is versal means all other unfoldings of f_0 are described by F, up to isomorphism and a reparametrization of $\mathbf{R}^s$.

We now come to the fundamental existence theorem on versal unfoldings. The results here use the identifications of the 'tangent spaces' discussed in Section 1.3.2. Given an unfolding $F(x,u) = (f(x,u),u)$, the *initial speeds*, $\dot{F}_i \in \mathcal{E}(n,p)$, of F are defined by

$$\dot{F}_i(x) = \partial f/\partial u_i(x,0), \quad \text{for} \quad i = 1,\ldots,s.$$

The $\mathcal{A}_e$*-tangent space* of $f_0 \in \mathcal{E}(n,p)$ is defined by

$$L\mathcal{A}_e \cdot f_0 = \mathcal{E}_n.\langle \partial f_0/\partial x_1, \ldots, \partial f_0/\partial x_n \rangle + \mathcal{E}_p.\{e_1,\ldots,e_p\},$$

and the $\mathcal{A}_e$*-codimension* by

$$\mathcal{A}_e\text{-Codim}(f_0) = \dim_{\mathbf{R}}(\mathcal{E}(n,p)/L\mathcal{A}_e \cdot f_0).$$

Theorem 1.3 *F is versal if and only if*

$$L\mathcal{A}_e \cdot f_0 + \mathbf{R}.\{\dot{F}_1,\ldots,\dot{F}_s\} = \mathcal{E}(n,p).$$

Suppose $c = \mathcal{A}_e$-Codim$(f_0) < \infty$ and $g_1, \ldots, g_c \in \mathcal{E}(n,p)$ form an **R**-spanning set for the complementary space to $L\mathcal{A}_e \cdot f_0$ in $\mathcal{E}(n,p)$. Defining the unfolding

$$F(x,u) = \left(f(x) + \sum_{i=1}^{c} u_i g_i(x), u \right)$$

we find that $\dot{F}_i = g_i$.

Corollary 1.4 *$F(x,u) = (f(x) + \sum_i u_i g_i(x), u)$ is a versal unfolding of f_0.*

Corollary 1.5 *f_0 has a versal unfolding if and only if $\mathcal{A}_e$-Codim$(f_0) < \infty$.*

We also recall the following.

Theorem 1.6 *f_0 is stable if and only if $\mathcal{A}_e$-Codim$(f_0) = 0$.*

Define $c = \mathcal{A}_e$-Codim(f_0). The least number of parameters for a versal unfolding of f_0 is c. We call a c-parameter unfolding of f_0 a *miniversal* unfolding.

Theorem 1.7 *All miniversal unfoldings of f_0 are equivalent.*

For a proof of the above theorems and further discussion we refer to [Mart2, Chapters XIII and XIV], [Mart1, Section I], [Wal, Section 3]. The results in the $\mathcal{A}$-case are the same, only we use the *$\mathcal{A}$-codimension*

$$\mathcal{A}\text{-Codim}(f_0) = \dim_{\mathbf{R}} \left(m_n.\mathcal{E}(n,p) / L\mathcal{A} \cdot f_0 \right)$$

and require an **R**-spanning set $g_1, \ldots, g_c \in m_n.\mathcal{E}(n,p)$ for the complementary space to $L\mathcal{A} \cdot f_0$ in $m_n.\mathcal{E}(n,p)$.

1.3.5 Discriminants

We will require one further concept related to unfoldings, that of a discriminant. For simplicity we will restrict to the specific setting used in this thesis, rather than embark on a general discussion. In what follows $f_0 : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ will be the germ of an analytic function with an isolated singularity at 0. It follows that f is finitely $\mathcal{R}$-determined and therefore has a versal $\mathcal{R}_e$-unfolding

$$\begin{array}{rcl} F:(\mathbf{C}^n\times\mathbf{C}^s,0) & \longrightarrow & (\mathbf{C}\times\mathbf{C}^s,0) \\ (x,u) & \mapsto & (f(x,u),u) \end{array}$$

(with $\dot{F}_i$ spanning the $\mathbf{C}$-vector space $\mathcal{O}_n/L\mathcal{R}_e\cdot f_0=\mathcal{O}_n/\langle\partial f_0/\partial x_1,\ldots,\partial f_0/\partial x_n\rangle$). The (germ of the) set

$$\Sigma F=\{\,(x,u)\;:\;\partial f/\partial x_1=\cdots=\partial f/\partial x_n=0\quad\text{at}\quad(x,u)\,\}$$

is called the *critical set* of F. The *discriminant* of F, DF, is defined to be the (germ of the) set

$$\{\,u\in\mathbf{C}^s\;:\;\exists x\in\mathbf{C}^n\quad\text{with}\quad f(x,u)=0\quad\text{and}\quad(x,u)\in\Sigma F\,\}.$$

Note that *any* unfolding of f_0 has a well-defined discriminant, F need not be versal.

Proposition 1.8 *Any two versal unfoldings with the same number of parameters have diffeomorphic discriminants*

By the discriminant of a singularity f_0 we will mean the discriminant of a miniversal unfolding of f_0. This is well-defined up to diffeomorphism. We will consider the discriminants of the simple singularities in Chapter 5. For example, the discriminant of the A_2 singularity $f_0:(\mathbf{C},0)\longrightarrow(\mathbf{C},0)$, $f_0(x)=x^3$, is the standard cusp $4u_1^3+27u_2^2=0$ and the discriminant of the A_3 singularity $f_0(x)=x^4$, is the swallowtail surface.

1.3.6 Miscellaneous Results

Further concepts and results from singularity theory and its neighbouring fields will be recalled throughout this thesis as they are required (most of the time, at least!). We finish by giving a couple of results which are used extensively throughout the classifications in this thesis.

In any classification of map-germs the simple singularities are extremely important. We adopt the definition of Arnol'd, [AGV, p.184]. Let X be a manifold and G a Lie group which acts on X. The *modality* of a point $x\in X$ under the action of G on X is the least number m such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be *simple*, if its modality is 0, that is a sufficiently small neighbourhood intersects only a finite number of orbits. The *modality* of a finitely determined map-germ is defined to be the modality of a sufficient jet in the jet-space under the action of the jet-group. Let $\mathcal{G}$ be one of $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$. The following unpublished result due to J.W. Bruce allows us to count moduli.

Theorem 1.9 *Let W be a smooth constructible subset of the jet-space $J^k(n,p)$ and for $w \in W$ define*

$$d(w) = \dim \left\{ (T_w(J^k\mathcal{G} \cdot w) + T_w W)/T_w(J^k\mathcal{G} \cdot w) \right\}.$$

Then, given an integer $r \geq 1$, if the set $\{ w \in W : d(w) \leq r-1 \}$ is a constructible subset of W of smaller dimension, then every germ f with $j^k f \in W$ is of $\mathcal{G}$-modality r or greater.

In particular, to identify non-simple germs we have the following Corollary; this was noted in [B5].

Corollary 1.10 *Let W be a smooth constructible subset of the jet-space $J^k(n,p)$. Suppose that the set*

$$\{ w \in W : T_w(J^k\mathcal{G} \cdot w) \supset T_w W \}$$

is a constructible subset of W of smaller dimension. Then no germ f with $j^k f \in W$ is $\mathcal{G}$-simple.

The above theorem and its corollary hold for the groups $\mathcal{R}(\mathcal{D})$ and weighted filtrations defined in Chapter 5 as well.

The following proposition (more of a remark really) concerns 'scaling' coordinate changes in $\mathcal{A}$-equivalence. Let $(x_1, \ldots, x_n)$ be a system of local coordinates on $(\mathbf{R}^n, 0)$ and $(y_1, \ldots, y_p)$ a system of local coordinates on $(\mathbf{R}^p, 0)$. For a given k-jet $j^k f$, say, the complete transversal theorems in Chapter 2 provide a family of $J^{k+1}(n,p)$ orbits over $j^k f$. We attempt to reduce the parameters appearing in these families to a unit (usually ± 1 in the real case and 1 in the complex case) by applying scaling coordinate changes:

$$\begin{aligned} (x_1, \ldots, x_n) &\mapsto (a_1 x_1, \ldots, a_n x_n) \\ (y_1, \ldots, y_n) &\mapsto (b_1 y_1, \ldots, b_n y_n) \end{aligned}$$

for $a_i, b_i \in \mathbf{R}$, $a_i, b_i \neq 0$. The problem of simplifying the orbits using scaling is made easy by the following proposition.

Proposition 1.11 *Replacing a_i by e^{λ_i} and b_i by e^{μ_i}, where $\lambda_i, \mu_i \in \mathbf{R}$ reduces the problem to one in linear equations.*

This technique is best described by examples, we refer to Chapter 3 where it is used extensively.

Chapter 2

Complete Transversals in Generalised Jet-Spaces

2.1 Introduction and Preliminary Material

Complete transversals provide an efficient method for obtaining the orbits when a Lie group acts on an affine space. We apply this to the action of jet-groups on jet-spaces, obtaining a classification technique which proceeds inductively at the jet-level. The method is due to Bruce and du Plessis and is a direct generalisation of the work of Dimca and Gibson in the $\mathcal{K}$-case, [DG]. It can also be likened with the spectral sequence methods of Arnol'd for the classification of functions under $\mathcal{R}$-equivalence, [A4], [AGV, Chapter 14]. Our main reference for the preliminary material is [BduP], but is as yet unpublished so we review the relevant material below. We omit the original proofs, though they may be found in [Wi]. The aim of this chapter is to generalise these results to jet-spaces other than those defined by the standard filtration by degree. In particular, this allows the use of 'unipotent' groups and the associated 'nilpotent' filtration — this is a very powerful technique when used with the 'unipotent' determinacy theorems of [BduPW], especially in $\mathcal{A}$ classification. We also discuss the use of complete transversals in weighted filtrations. Such filtrations are related to nilpotent filtrations (in a certain sense) but are more natural to use in specific applications.

Complete transversals have been used in several classifications in the past, see for example [BG4], [Tar], [Hob], [GH], [Wi]. The theorems developed in this chapter have been used in the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ (Chapter 3 and [We]) and in the classification of function germs on discriminant varieties (Chapter 5). Similar methods have been developed by Bruce and independently by Ratcliffe using 'triviality theorems' (Thom-Levine) as the main technical tool, [BG5], [Rat1, Rat2]. The approach below, due to Bruce and du Plessis, is more general and uses Lie group actions and the Mather Lemma as its main technical tool.

We start with a result due to Mather specifically concerned with Lie group actions and the calculation of the orbits.

Lemma 2.1 (Mather Lemma) *Let G be a Lie group acting smoothly on a finite dimensional manifold V. Let X be a connected submanifold of V. Then X is contained in a single orbit of G if and only if*

1. *for each $x \in X$, $T_xX \subset T_x(G \cdot x) = LG \cdot x$;*
2. *dim $T_x(G \cdot x)$ is constant for all $x \in X$.*

See [MathIV, Lemma 3.1]. The basic complete transversal theorem is a corollary to the Mather Lemma.

Theorem 2.2 *Let G be a Lie group acting smoothly on an affine space A, and let W be a subspace of V_A with*

$$LG \cdot (x + w) = LG \cdot x \tag{2.1}$$

for all $x \in A$ and all $w \in W$. Then

1. *for all $x \in A$ we have*

$$x + \{LG \cdot x \cap W\} \subset G \cdot x \cap \{x + W\};$$

2. *if $x_0 \in A$ and T is a vector subspace of W satisfying*

$$W \subset T + LG \cdot x_0$$

then for any $w \in W$ there exists $g \in G$ and $t \in T$ such that

$$g \cdot (x_0 + w) = x_0 + t.$$

Proof. See [BduP]. Part *1* follows from the hypothesis 2.1 and the Mather Lemma; Part *2* is then a consequence of Part *1*. Note that we are identifying the tangent space at $x \in A$ with the underlying vector space V_A. □

Remarks 2.3

(1). Theorem 2.2 Part *2* says that the transversal T to the orbit of x_0 contains a representative for the G-orbits of *every* element in the affine subspace $x_0 + W$ of A. T will therefore be referred to as a *complete transversal.*

(2). The hypothesis 2.1 says that for all $x \in A$ the tangent space to the orbit of a point in $x + W$ is the same for all points, being equal to the tangent space to the orbit $G \cdot x$ at x. This is the crucial condition for the theory to work. However, in practice we usually replace condition 2.1 by the sharper condition

$$l \cdot (x + w) = l \cdot x \tag{2.2}$$

for all $x \in A$, $w \in W$ and $l \in LG$. (This is the infinitesimal version of $g \cdot (x + w) = g \cdot x + w$ for $g \in G$, which is therefore a sufficient alternative.) In fact, a sharper result can now be obtained, namely that there exists a closed connected Lie subgroup H of G such that for any $x \in A$ we have

$$x + \{LG \cdot x \cap W\} = H \cdot x.$$

This follows from [BduPW, Lemma 4.3], details can be found in [BduP].

(3). For a general definition and discussion of affine spaces we refer to [Por, Chapter 4]. In all our applications A will just be a vector space V say, with $W \subset V_A = V$. The affine subspace $x_0 + W$ is therefore an affine subspace of a *vector space* obtained by translating the vector subspace W by x_0.

The classification result used in the aforementioned references now follows. We firstly develop the notation. The standard k-jet-space $m_n.\mathcal{E}(n,p)/m_n^{k+1}.\mathcal{E}(n,p)$ is denoted $J^k(n,p)$. Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $\mathcal{G}_k$ be the normal subgroup of $\mathcal{G}$ consisting of those germs whose k-jet is equal to that of the identity. The standard k-jet-group is defined to be the quotient group $\mathcal{G}/\mathcal{G}_k$ and is denoted $J^k\mathcal{G}$. This is a Lie group and acts on the affine space $J^k(n,p)$; see [MathIII, Section 7] and Sections 2.2 and 2.3 below. Let H^k denote the image of $m_n^k.\mathcal{E}(n,p)$ in $J^k(n,p)$, the vector subspace of $J^k(n,p)$ consisting of the homogeneous jets of degree k.

Corollary 2.4 *Let $\mathcal{G}$ be one of the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$, and consider $\mathcal{G}_1$. Then given $f \in m_n.\mathcal{E}(n,p)$ and $T \subset H^{k+1}$ a vector subspace of H^{k+1} such that*

$$H^{k+1} \subset L(J^{k+1}\mathcal{G}_1) \cdot j^{k+1}f + T,$$

we have for every $(k+1)$-jet $j^{k+1}g$ ($g \in m_n.\mathcal{E}(n,p)$) with $j^k g = j^k f$ that $j^{k+1}g$ is in the same $J^{k+1}\mathcal{G}_1$ orbit as $j^{k+1}f + t$ for some $t \in T$.

Proof. The proof is given in [BduP] in slightly more generality. □

Remark. We will not have cause to use the above corollary and have included it just to give a flavour of the results. The basic line of the proof is to set $A = J^{k+1}(n,p)$, $W = H^{k+1}$ and $G = J^{k+1}\mathcal{G}_1$ in Theorem 2.2. The condition 2.2 follows from the standard approximation lemmas (a suitable version is [duP, Sublemma 2.2] with restriction to $N \times \{0\}$, and setting $l = k$, $k = 2$, $R = m_n^2$ (the symbols on the left-hand side of these equations refer to those which appear in [duP, Sublemma 2.2]); c.f., Lemma 3.4 of the same paper). This lemma gives $l \cdot (f+h) - l \cdot f \in m_n^{k+2}.\mathcal{E}(n,p)$ for $h \in m_n^{k+1}.\mathcal{E}(n,p)$, but only applies if $l \in L\mathcal{G}_1$ hence the need to use $\mathcal{G}_1$ above instead of $\mathcal{G}$. Thus, the corollary can be used to obtain representatives for all $(k+1)$-jets with k-jet f up to $\mathcal{G}_1$ equivalence, and in practice this provides a powerful tool for $\mathcal{G}$ equivalence too. We obtain a list of representatives for the $\mathcal{G}$ orbits with possible (though usually few) redundancies. One can make this more efficient by using larger subgroups of $\mathcal{G}$ than $\mathcal{G}_1$ — we discuss this in Section 2.3. The complete transversal theorems do not work for the whole group $\mathcal{G}$ as the following counter-examples indicate.

Examples 2.5 (Counter Examples for the $\mathcal{G}$ Case)

Let $\mathcal{G}$ be one of $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$. We shall assume the complete transversal theorem holds for $\mathcal{G}$ and obtain an absurd determinacy result as a contradiction. Suppose the complete transversal theorem holds for $\mathcal{G}$: then given $f \in m_n.\mathcal{E}(n,p)$ and $T \subset H^{k+1}$ such that

$$H^{k+1} \subset L(J^{k+1}\mathcal{G}) \cdot j^{k+1}f + T,$$

we have for every $(k+1)$-jet $j^{k+1}g$ with $j^k g = j^k f$ that $j^{k+1}g$ is in the same $J^{k+1}\mathcal{G}$ orbit as $j^{k+1}f + t$ for some $t \in T$. We pursue similar arguments to [BduPW] Theorems 1.9 and 2.1 to show a map-germ $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ which satisfies $m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{G} \cdot f$ is k-$\mathcal{G}$-determined. Indeed, by [MathIII] if this holds then f is finitely $\mathcal{G}$-determined, and by the (assumed) complete transversal result for any $g \in m_n.\mathcal{E}(n,p)$ such that $j^l g = j^l f$ (any $l \geq k$) we have $j^{l+1}g$ is in the same $J^{l+1}\mathcal{G}$ orbit as $j^{l+1}f$. Thus $j^k g = j^k f$ implies g is in the same $\mathcal{G}$ orbit as f, so f is k-determined. However, there are well-known counter-examples to this result. An obvious consideration is function-germs $(\mathbf{R},0) \longrightarrow (\mathbf{R},0)$ under $\mathcal{R}$ equivalence — the statement 'if $m_1^{k+1} \subset L\mathcal{R}\cdot f$ then f is k-$\mathcal{R}$-determined' implies x^{k+1} is k-determined which is absurd.

2.2 Non-Standard Filtrations of $m_n.\mathcal{E}(n,p)$ and the Associated Jet-Spaces

In this section we develop the idea of generalised jet-spaces. We allow finer filtrations of $m_n.\mathcal{E}(n,p)$ and $\mathcal{G}$ than the standard one by degree, the idea being to replace the action of $\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$ by the action of a Lie group on a smooth manifold by forming the quotient groups and modules. We impose further conditions so that a workable complete transversal theorem holds. The use of non-standard filtrations can give substantial improvements to the efficiency of the calculations; the jet-space can be partitioned into smaller spaces and the jet-group is generally larger than the $\mathcal{G}_1$ group required in the previous section. This generality incorporates the 'nilpotent filtrations' introduced by Bruce and du Plessis, and the 'weighted filtrations' introduced by Arnol'd, and allows us to give compact proofs for the corresponding complete transversal theorems — see Section 2.3 and Section 2.4. (In the standard case we can extend the definition to give jet-spaces for maps $f : X \longrightarrow Y$ between manifolds X and Y. These jet-spaces are invariant under coordinate changes and this allows the construction of the jet bundles, as in [GG, Chapter II]. Generally, such a construction does not work for non-standard filtrations (for example, weighted filtrations). However, we are concerned with local theory and need only work with map-germs $(\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^p,0)$ for suitable n and p (or equally well over $\mathbf{C}$, depending on the context). Our definitions therefore only apply to the local case, using filtrations of the module $m_n.\mathcal{E}(n,p)$, and there is no attempt at generalisation.)

As a preliminary we define a filtration of a module M to be a strictly decreasing chain of submodules $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ and a filtration of a group G to be a strictly decreasing chain of subgroups $G = G_0 \supset G_1 \supset G_2 \supset \cdots$.

Definition 2.6 Given a subgroup $\mathcal{G}$ of $\mathcal{K}$ acting on $m_n.\mathcal{E}(n,p)$, by a *jet-filtration* we mean a filtration $\{M_k\}$ of the module $m_n.\mathcal{E}(n,p)$ together with a filtration

$\{G_k\}$ of the group $\mathcal{G}$ by *normal* subgroups G_k such that the following hold.

1. Each M_k has finite codimension in $m_n.\mathcal{E}(n,p)$ (as a real or complex vector space, as appropriate). Thus, $m_n.\mathcal{E}(n,p)/M_k$ is a finite dimensional vector space for all k.

2. Each quotient group $\mathcal{G}/G_k$ is Lie group and there is a Lie group action

$$\mathcal{G}/G_k \times m_n.\mathcal{E}(n,p)/M_k \longrightarrow m_n.\mathcal{E}(n,p)/M_k$$

induced from the action of $\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$. So that in the notation described next we have $j^k\phi \cdot j^k f = j^k(\phi \cdot f)$.

Notation. Given a jet filtration, which we will denote by $F = (\{M_k\}, \{G_k\})$, we define the k-jet-space to be the finite dimensional vector space $m_n.\mathcal{E}(n,p)/M_k$ and denote this $J^k_F(n,p)$. For $k \geq 1$ we define H^k_F to be the image of M_{k-1} in $J^k_F(n,p)$, this is the subspace of 'homogeneous jets of degree k'. The k-jet-group is defined to be the Lie group $\mathcal{G}/G_k$, which is denoted $J^k_F\mathcal{G}$. We denote the image of $f \in m_n.\mathcal{E}(n,p)$ in $J^k_F(n,p)$ by $j^k_F f$ and the image of $\phi \in \mathcal{G}$ in $J^k_F\mathcal{G}$ by $j^k_F\phi$. In most applications the context is clear and we drop the subscript 'F'. The symbol $\sim$ will denote the equivalence of germs (jets) under the action of $\mathcal{G}$ (respectively $J^k_F\mathcal{G}$).

If $\mathcal{G}$ is a subgroup of $\mathcal{K}$ we define $L\mathcal{G}$ by differentiating curves which lie in $\mathcal{G}$; this corresponds to the finite dimensional situation where we obtain the Lie algebra of a Lie group (equivalently the tangent space to the Lie group at the identity). A precise definition follows.

Definition 2.7 Let M^m be some m-dimensional manifold. A map-germ $\phi : (M^m, x) \longrightarrow (\mathcal{K}, 1)$ is called C^τ if the induced map

$$\Phi : (M^m, x) \times (\mathbf{R}^{n+p}, 0) \longrightarrow (\mathbf{R}^{n+p}, 0)$$

is of class C^τ in the usual sense. We define a curve in $\mathcal{K}$ by taking $M = \mathbf{R}$ with coordinate t and obtain a C^τ vector field germ on $\mathbf{R}^{n+p}$ at 0,

$$z \mapsto \partial\Phi(t,z)/\partial t|_{t=0}.$$

The set of all such vector fields is denoted $L\mathcal{K}$. The set of all vector fields arising from C^τ curves ϕ with image in $\mathcal{G}$ is denoted $L\mathcal{G}$. We define $J^k(L\mathcal{K})$ (and $J^k(L\mathcal{G})$) by taking such curves ϕ and projecting into the jet-group before differentiating to give the vector field.

For the cases $L\mathcal{R}$, $L\mathcal{L}$, $L\mathcal{A}$, $L\mathcal{C}$ and $L\mathcal{K}$ and the standard filtration by degree these are Lie algebras (though this is not known for general $\mathcal{G}$) and there are standard coordinate versions. Also, the induced action of the Lie algebra $L\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$ carries over to an action of $J^k(L\mathcal{G})$ on $J^k(n,p)$ $(= LJ^k(n,p))$ where we have $j^k l \cdot j^k f = j^k(l \cdot f)$ for $l \in L\mathcal{G}$ and $f \in m_n.\mathcal{E}(n,p)$. All these follow from [MathIII, Section 7.3, 7.4] (the methods extend to jet-filtrations).

Example 2.8 (The Standard Filtration by Degree)
See [MathIII, Section 7]. We filter $m_n.\mathcal{E}(n,p)$ by the chain of submodules $M_k = m_n^{k+1}.\mathcal{E}(n,p)$ and filter $\mathcal{K}$ by the normal subgroups $\mathcal{K}_k$ consisting of all $H \in \mathcal{K}$ whose (standard) k-jet at $0 \in \mathbf{R}^n \times \mathbf{R}^p$ is equal to the k-jet at 0 of the identity; that is $\mathcal{K}_k = (1_{n+p} + m_{n+p}^{k+1}.\mathcal{E}(n+p, n+p)) \cap \mathcal{K}$. For subgroups $\mathcal{G}$ of $\mathcal{K}$ this generalises by setting $\mathcal{G}_k = \mathcal{K}_k \cap \mathcal{G}$. Consider the case where $\mathcal{G}$ is one of $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$. Now $J^k\mathcal{G} = \mathcal{G}/\mathcal{G}_k$ is a Lie group and acts smoothly on $J^k(n,p) = m_n.\mathcal{E}(n,p)/m_n^{k+1}.\mathcal{E}(n,p)$. (This also follows as a corollary to the results in Section 2.4 on weighted filtrations). We remark that since $\mathcal{G} \subset \mathcal{K} \subset m_{n+p}.\mathcal{E}(n+p, n+p)$ one could form the jet-group by defining it as the image of $\mathcal{G}$ in the quotient *module* $J^k(n+p, n+p)$ — if fact this is often the case in the literature. In this example we can show that the two are equivalent. Although this definition is easier to formulate, the quotient group definition ensures that the resulting object is indeed a group and provides an easy criterion for the existence of a well-defined action on $J^k(n,p)$ (see Section 2.4, for example).

A classification using a jet-filtration can therefore proceed inductively. If at some k-jet-level $j^k f \sim j^k g$ then at the $(k+1)$-jet-level $j^{k+1}f \sim j^{k+1}g + j^{k+1}h$ for some $j^{k+1}h \in H^{k+1}$, so we need only work with a representative for each orbit at the k-level and determine the corresponding representatives at the $(k+1)$-level. The process stops for determined germs. Given a jet-filtration $F = (\{M_k\}, \{G_k\})$, we say a map-germ $f \in m_n.\mathcal{E}(n,p)$ is k-determined if any map-germ g with $j_F^k f = j_F^k g$ is $\mathcal{G}$-equivalent to f. That is, $f + M_k$ is contained in the $\mathcal{G}$-orbit of f. (See [duP, Section 4], we are just asking if f is M_k-$\mathcal{G}$-determined.) Now for a k-determined germ it is easy to see that $j_F^k g \sim j_F^k f \Leftrightarrow f \sim g$. So the classification of a certain class of finitely determined germs (up to a given codimension, say) is equivalent to the classification of the associated class of orbits in the jet-spaces, and the problem is therefore reduced from the realm of Frechet manifolds to the action of Lie groups on affine spaces — a finite dimensional situation.

Before generalising the complete transversal theorem we recall some useful technical conditions which many of the standard groups used in singularity theory satisfy.

Definition 2.9 Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$. We call $\mathcal{G}$ *jet-closed* if for each $r \geq 1$, $J^r\mathcal{G}$ is a closed subgroup of $J^r\mathcal{K}$

A jet-closed subgroup satisfies the following property which is extremely useful in practical applications of the complete transversal theorems:

$$J^s(L\mathcal{G}) \subset L(J^s\mathcal{G}) \quad \text{for all} \quad s.$$

In many cases (e.g., the standard Mather groups) we have equality, this is useful for determinacy results. If a jet-closed group $\mathcal{G}$ satisfies $J^s(L\mathcal{G}) = L(J^s\mathcal{G})$ for all s then we call it *fibrant*. Using the standard filtration we find that $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$ are all jet-closed and fibrant. Further examples for this filtration are given via the following concept. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$, then $\mathcal{H}$ is said to be *strongly closed* in $\mathcal{G}$ if $\mathcal{H}_s = \mathcal{G}_s$ (the subgroups with s-jet the identity) for some s (equivalently $\mathcal{G}_s \subset \mathcal{H}$), and $J^s\mathcal{H}$ is closed in $J^s\mathcal{G}$. (This definition applies to the standard filtration — we shall not have cause to generalise it.) Now, a strongly closed subgroup $\mathcal{H}$ of a jet-closed group $\mathcal{G}$ is itself jet-closed. If, in addition, $\mathcal{G}$ is fibrant then so is $\mathcal{H}$. See [BduPW, Section 4] for more details.

We now come to a generalisation of the complete transversal theorem.

Theorem 2.10 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$, $L \subset L\mathcal{G}$, and $(\{M_i\}, \{G_i\})$ a jet filtration such that for all $s \geq 0$*

1. *J^sL (a subset of $J^s(L\mathcal{G})$) is a Lie subalgebra of $L(J^s\mathcal{G})$;*
2. *for all $f \in m_n.\mathcal{E}(n,p)$, $h \in M_s$ and $l \in L$ we have $l \cdot (f+h) - l \cdot f \in M_{s+1}$.*

Then for $f \in m_n.\mathcal{E}(n,p)$, $k \geq 1$ and T a subspace of $H^{k+1} \subset J^{k+1}(n,p)$ such that

$$J^{k+1}L \cdot j^{k+1}f + T \supset H^{k+1},$$

we have any k-jet j^kg with $j^kg \sim_{J^k\mathcal{G}} j^kf$ has $(k+1)$-jet $j^{k+1}g \sim_{J^{k+1}\mathcal{G}} j^{k+1}f + t$ for some $t \in T$. Such a space T will be referred to as a complete transversal.

Proof. $J^{k+1}\mathcal{G}$ is a Lie group and acts on the vector space $J^{k+1}(n,p)$. Now, by hypothesis, $J^{k+1}L$ is a Lie subalgebra of $L(J^{k+1}\mathcal{G})$ so there exists a (unique, connected) Lie subgroup G of $J^{k+1}\mathcal{G}$ with Lie algebra $LG = J^{k+1}L$; see [War, Theorem 3.19], for example. The assumption $j^k g \sim j^k f$ is equivalent to $g \sim f + h$ for some $h \in M_k$, that is $j^{k+1}g \sim j^{k+1}f + j^{k+1}h$ where $j^{k+1}h \in H^{k+1}$. In the complete transversal theorem 2.2 take G as above, $A = J^{k+1}(n,p)$ and $W = H^{k+1}$. Then the condition 2.1 (in fact the sharper condition 2.2) follows since any element of $LG = J^{k+1}L$ may be written as $j^{k+1}l$ for some $l \in L\mathcal{G}$ so

$$j^{k+1}l \cdot (j^{k+1}f + j^{k+1}h) - j^{k+1}l \cdot (j^{k+1}f) = j^{k+1}(l \cdot (f+h) - l \cdot f)$$

which is equal to zero by hypothesis 2 above. Thus by Theorem 2.2 we have $j^{k+1}f + j^{k+1}h \sim_G j^{k+1}f + t$ for some $t \in T$, but G is a subgroup of $J^{k+1}\mathcal{G}$ so the result follows. □

The condition 'J^sL is a Lie subalgebra of $L(J^s\mathcal{G})$' is presented in terms of jet-groups; it is more desirable to work directly with $\mathcal{G}$ and $L\mathcal{G}$. However, for many of the cases which arise in applications it is already known that $L\mathcal{G}$ is a Lie algebra and that $J^s(L\mathcal{G}) \subset L(J^s\mathcal{G})$ for all s. Then, for some predetermined subalgebra L of $L\mathcal{G}$ it follows that J^sL is a Lie subalgebra of $L(J^s\mathcal{G})$ for all s. (One often takes $L = L\mathcal{G}$, for example, in the $\mathcal{A}_1$ complete transversal theorem.) In particular we have the following.

Corollary 2.11 *In the complete transversal Theorem 2.10, condition 1 may be replaced by the requirements that $\mathcal{G}$ is jet-closed and $L\mathcal{G}$ is a Lie algebra with L a Lie subalgebra.*

Finally, we also note that condition 2 may be replaced by the weaker condition $L \cdot (f+h) - L \cdot f \subset M_{s+1}$. However, the stated (sharper) condition holds in practice. See also Remarks 2.3 part 2.

2.3 Nilpotent Lie Algebras and the Associated Filtration

In this section we show how the $\mathcal{G}_1$ complete transversal theorems described in Section 2.1 can be extended to, for example, unipotent subgroups of $\mathcal{K}$. The jet-filtration is induced by the corresponding nilpotent Lie algebra and is generally a lot finer than the standard filtration by degree.

The ideas here are due to Bruce and du Plessis and will appear in the future alongside the important foundational work reviewed in Section 2.1. As yet the only reference for the results given below is the preprint [B5], although a similar version of these results appears in [BduPW, Section 2], essentially the proof of Theorem 2.1. We take this opportunity to introduce the ideas; review the basic results needed on nilpotent representations of Lie algebras; and provide proofs in the setting of Lie group actions using the ideas of Section 2.2. This also serves to introduce the natural faithful representation of the Lie algebra $L(J^1\mathcal{K})$. Not only is this an informative exercise (especially for persons unfamiliar with the results) but in our proofs we put the emphasis on 'nilpotent' subspaces L of $L\mathcal{K}$, providing a workable classification technique in practice. Usually we cannot use the whole Lie algebra $L\mathcal{G}$ of a given subgroup $\mathcal{G}$ of $\mathcal{K}$ (see Example 2.5) but need to restrict to some 'nilpotent' subspace L. However, we do not need to concern ourselves with the existence of a (unipotent) subgroup $\tilde{\mathcal{G}}$ of $\mathcal{G}$ such that $L\tilde{\mathcal{G}} = L$.

Throughout this section we will assume that $m_n.\mathcal{E}(n,p)$ and $\mathcal{K}$ are filtered using the standard filtration by degree. The contrast with the induced 'nilpotent' filtration will be clear from the notation.

Firstly we discuss the singularity theory. There is a natural faithful representation of the Lie algebra $L(J^1\mathcal{K})$ on $\mathbf{R}^{n+p}$. Recall that $\mathcal{K}$ is the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$ and an element $H \in \mathcal{K}$ can be written (via a representative of the germ) in the form

$$H(x,y) = (h(x), \tilde{H}(x,y))$$

where $h : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is (necessarily) a diffeomorphism and $\tilde{H}(x,0) = 0$ for $x \in \mathbf{R}^n$ near to zero. Necessarily $h_x : (\mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$ defined by $h_x(y) = \tilde{H}(x,y)$, for x near zero, is a germ of a diffeomorphism. ($\mathcal{C}$ can be thought of as the group of diffeomorphisms of $(\mathbf{R}^p, 0)$ parametrized by $x \in \mathbf{R}^n$.) Now since $\tilde{H}(x,0) = 0$, writing $\tilde{H}$ as $(\tilde{H}_1, \ldots, \tilde{H}_p)$, we have $\tilde{H}_j \in m_p.\mathcal{E}_{n+p}$ for $j = 1, \ldots, p$ (m_p being the subring of $\mathcal{E}_{n+p}$ of germs which vanish on $\mathbf{R}^n \times \{0\}$). So we can write

$$\tilde{H}_j = \sum_{k=1}^{p} y_k \tilde{H}_{jk}, \quad \text{where} \quad \tilde{H}_{jk} \in \mathcal{E}_{n+p},$$

and the 1-jet of $\tilde{H}_j$ is just $\sum_{k=1}^{p} \tilde{H}_{jk}(0) y_k$. Thus, the 1-jet of $\tilde{H}$ is naturally an element of $GL(p,\mathbf{R})$. Now $J^1\mathcal{R} \cong GL(n,\mathbf{R})$ and it is not hard to show $J^1\mathcal{C} \cong GL(p,\mathbf{R})$ and $J^1\mathcal{K} \cong GL(n,\mathbf{R}) \oplus GL(p,\mathbf{R})$. We summarise this and describe the natural representation at the Lie algebra level.

Lemma 2.12 *$J^1\mathcal{K} \cong GL(n,\mathbf{R}) \oplus GL(p,\mathbf{R})$. $L(J^1\mathcal{K}) \cong gl(n,\mathbf{R}) \oplus gl(p,\mathbf{R})$, where*

$gl(n,\mathbf{R})$ denotes the Lie algebra of $GL(n,\mathbf{R})$ consisting of n by n matrices with Lie bracket $[N_1,N_2]=N_1N_2-N_2N_1$, and similarly for $gl(p,\mathbf{R})$. The map

$$\begin{array}{rcl} gl(n,\mathbf{R})\oplus gl(p,\mathbf{R}) & \longrightarrow & gl(n+p,\mathbf{R}) \\ (M,N) & \mapsto & \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \end{array}$$

is a faithful representation of the Lie algebra $L(J^1\mathcal{K})$ on $\mathbf{R}^{n+p}$.

Proof. The first part was discussed above; then taking Lie algebras and using the standard identifications gives the second part. It is easy to check the given map is a Lie algebra homomorphism; it is clearly faithful. □

Remark 2.13 There is another natural representation of $L(J^1\mathcal{K})$ induced by the action of the Lie group $J^1\mathcal{K}$ on $J^1(n,p)$. Generally, if a Lie group G acts linearly on a vector space V over $\mathbf{R}$, then for each $l \in LG$ the map $v \mapsto l\cdot v$ is a linear map $V \longrightarrow V$. In fact we have a representation of the Lie algebra LG on V, $LG \longrightarrow End(V)$, since, by assumption, we have a Lie group homomorphism $\phi : G \longrightarrow Aut(V)$ and the assertion is then nothing more than the fact that the differential $d\phi : LG \longrightarrow End(V)$ is a Lie algebra homomorphism (see, for example, [War, Chapter 3]). In our situation we have an action of $J^1\mathcal{K} \cong GL(n,\mathbf{R}) \oplus GL(p,\mathbf{R})$ on $J^1(n,p) \cong M(n,p)$ (the space on n by p matrices). This is just $(G,H,X) \mapsto GXH^{-1}$ for $G \in GL(n,\mathbf{R})$, $H \in GL(p,\mathbf{R})$ and $X \in M(n,p)$. The corresponding action of $L(J^1\mathcal{K})$ on $M(n,p)$ is (it turns out) given by $(M,N,X) \mapsto MX - XN$ for $M \in gl(n,\mathbf{R})$, $N \in gl(p,\mathbf{R})$ and $X \in M(n,p)$. This is in many ways a more natural representation of $L(J^1\mathcal{K})$; however, it is not faithful. The crucial point for constructing the finer filtrations, as will be seen below, is that we have a nilpotent representation. The nilpotency of this representation only works at the 1-jet level. However, the nilpotency of the faithful representation described earlier carries over to the higher jet-levels and ensures we can construct the filtration.

We will describe the exact correspondence between the vector field notation used for $L(J^1\mathcal{K})$ in practice, and the matrix notation used in the identification with $gl(n,\mathbf{R}) \oplus gl(p,\mathbf{R})$. A 1-jet

$$j = (a_{11}x_1 + \cdots + a_{n1}x_n, \ldots, a_{1p}x_1 + \cdots + a_{np}x_n) \in J^1(n,p)$$

is identified with the n by p matrix $A = (a_{ij}) \in M(n,p)$. An element of $J^1\mathcal{K}$ is identified with an element of $GL(n,\mathbf{R}) \oplus GL(p,\mathbf{R})$ and acts thus

$$(G,H,A) \mapsto GAH^{-1}$$

for $G \in GL(n,\mathbf{R})$, $H \in GL(p,\mathbf{R})$ and $A \in M(n,p)$. To begin with consider the action of the $GL(n,\mathbf{R})$ part of $J^1\mathcal{K}$ on $M(n,p)$. $G = (g_{ij})$ maps $A = (a_{ij})$ to $(\sum_{k=1}^{n} g_{ik}a_{kj})$; in terms of jets this map is

$$
\begin{aligned}
j \quad \mapsto \quad & \left(\left(\sum_{k=1}^{n} g_{1k}a_{k1}\right)x_1 + \cdots + \left(\sum_{k=1}^{n} g_{nk}a_{k1}\right)x_n,\right. \\
& \left.\ldots, \left(\sum_{k=1}^{n} g_{1k}a_{kp}\right)x_1 + \cdots + \left(\sum_{k=1}^{n} g_{nk}a_{kp}\right)x_n\right).
\end{aligned}
$$

The 1-jet of the diffeomorphism $(\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$ corresponding to $G = (g_{ij})$ such that the actions are compatible is therefore

$$
(x_1,\ldots,x_n) \quad \mapsto \quad \left(\sum_{k=1}^{n} g_{k1}x_k,\ldots,\sum_{k=1}^{n} g_{kn}x_k\right).
$$

Now, at the Lie algebra level, given $M = (m_{ij}) \in gl(n,\mathbf{R})$, $\gamma(t) = 1_n + tM$ is a path through 1_n in $GL(n,\mathbf{R})$ (for small t) such that $\gamma'(0) = M$. Using the above, the 1-parameter path F of diffeomorphisms corresponding to this is therefore given by

$$
F(x_1,\ldots,x_n,t) = \left(x_1 + t\sum_{k=1}^{n} m_{k1}x_k,\ldots,x_n + t\sum_{k=1}^{n} m_{kn}x_k\right),
$$

and 'differentiating with respect to t' corresponds to forming the vector field $dF(\partial/\partial t)$ (c.f., [duP, Section 1]). Putting $t = 0$ then gives the required vector field, an element of $J^1(L\mathcal{R}) \subset J^1\theta_n$:

$$
\begin{aligned}
dF\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0} &= \sum_{i=1}^{n} \frac{\partial(x_i \circ F)}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial x_i} \\
&= \left(\sum_{k=1}^{n} m_{k1}x_k\right)\frac{\partial}{\partial x_1} + \cdots + \left(\sum_{k=1}^{n} m_{kn}x_k\right)\frac{\partial}{\partial x_n}.
\end{aligned}
$$

In particular, this vector field corresponds to the matrix $(m_{ij}) \in gl(n,\mathbf{R})$ *whose* r*th column consists of the coefficients of the* $\partial/\partial x_r$ *term.* A similar result holds for the correspondence between the vector fields in $J^1(L\mathcal{C})$ and matrices (m_{ij}) in $gl(p,\mathbf{R})$. *The vector field*

$$
\left(\sum_{k=1}^{p} m_{k1}y_k\right)\frac{\partial}{\partial y_1} + \cdots + \left(\sum_{k=1}^{p} m_{kp}y_k\right)\frac{\partial}{\partial y_p}
$$

corresponds to the matrix (m_{ij}) *whose* r*th column consists of the coefficients of the* $\partial/\partial y_r$ *term.*

Now we recall some of the basic definitions regarding nilpotent endomorphisms and representations of Lie algebras.

Let V be a finite dimensional vector space over a field $\mathbf{F}$ (in our case $\mathbf{R}$). An endomorphism $\alpha \in End(V)$ is called *nilpotent* if $\alpha^n = 0$ for some n. If L is a Lie algebra, with V an L-module, and S a subset of L, then we say S is *nilpotent on* V if $S^n V = 0$ for some n. That is we have a representation $\rho : L \longrightarrow End(V)$ (i.e., a Lie algebra homomorphism into $End(V)$) of L on V and require $\rho(S)^n = 0$ for some n. The two notions are related (in the case $S = L$) by the following version of Engel's theorem which says that L is nilpotent on V if (and only if) it is a Lie algebra of nilpotent endomorphisms; the latter being easier to check in practice.

Theorem 2.14 *Take L and V as above, together with a representation of L on V. Then L is nilpotent on V ($L^n V = 0$ for some n) if (the image under the representation of) every element of L is a nilpotent endomorphism.*

Proof. [Hoch, Section VII, 1.5]. □

Note the difference between L being nilpotent on some V and L being a nilpotent Lie algebra. Recall that L is *nilpotent* if its *lower central series* $\{L^i\}$; a descending chain of ideals of L defined by $L^0 = L$, $L^{i+1} = [L, L^i]$; ends at 0, that is $L^n = 0$ for some n. We shall be interested in the case when the representation of $L(J^1\mathcal{K})$ on $\mathbf{R}^{n+p}$, described above, is nilpotent for some subalgebra L of $L(J^1\mathcal{K})$.

Remark. Since this representation is faithful it follows that L is a nilpotent Lie algebra. From [Hoch, Section XVI, 4.2], over a field $\mathbf{F}$ of characteristic 0, an affine algebraic group G is unipotent if and only if its Lie algebra LG is a nilpotent Lie algebra. More precisely, the category of unipotent affine algebraic F-groups and the category of nilpotent F-Lie algebras are naturally equivalent. The results can therefore be interpreted in terms of unipotent groups. This proved to be the key idea in determinacy, [BduPW]. It would be interesting to apply the theory of unipotent groups to the complete transversal theorems. In practice, however, it is sufficient to work with nilpotent Lie algebras, and we will restrict to this setting.

The following will also be of use later.

Proposition 2.15 *Let L be a subalgebra of $gl(V)$, V a finite dimensional vector space. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists a flag $\{V_i\}$ in V stable under L, with $L(V_i) \subset V_{i-1}$ for all i. In particular, there exists a basis of V relative to which the matrices of L are all strictly upper triangular.*

Proof. [Hum, Chapter I, Section 3.3]. □

Returning to the singularity theory, we are in a position to define the 'nilpotent filtration' once we have the following.

Proposition 2.16 *Let $L \subset L\mathcal{K}$ be such that J^1L is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbf{R}^{n+p}$. Then given an integer $r \geq 1$ there is an integer $k_r \geq 1$ such that*

$$L^{k_r} \cdot (m_n^r.\mathcal{E}(n,p)) \subset m_n^{r+1}.\mathcal{E}(n,p).$$

Proof. We shall work with standard coordinates; then any $l \in L\mathcal{K}$ can be written in the form

$$l = \sum g_i \partial/\partial x_i \oplus \sum h_i \tilde{h}_i \partial/\partial y_i \in L\mathcal{R} \oplus L\mathcal{C}$$

where $g_i \in m_n$, $h_i \in \mathcal{E}_n$ and $\tilde{h}_i \in m_p$ (multiplication via f^* in the latter case). Take $f \in m_n^r.\mathcal{E}(n,p)$. If $g_i \in m_n^2$ then

$$g_i \partial/\partial x_i(f) \in m_n^{r+1}.\mathcal{E}(n,p),$$

and if $h_i \in m_n$ or $\tilde{h}_i \in m_p^2$ then

$$h_i \tilde{h}_i \partial/\partial y_i(f) = h_i \tilde{h}_i(f) e_i \in m_n^{r+1}.\mathcal{E}(n,p),$$

so we need only consider $g_i \partial/\partial x_i$ with $g_i \in m_n \setminus m_n^2$ and $\tilde{h}_i \partial/\partial y_i$ with $\tilde{h}_i \in m_p \setminus m_p^2$, that is the terms in J^1L. By assumption J^1L is nilpotent on $\mathbf{R}^{n+p}$, so by Proposition 2.15 the matrices in the representation (Lemma 2.12) are conjugate to strictly upper triangular matrices. We can assume these matrices are indeed strictly upper triangular in what follows. Writing l in vector field notation (see the earlier discussion) we have

$$\begin{aligned} g_i \partial/\partial x_i &= (m_{1i} x_1 + m_{2i} x_2 + \cdots + m_{i-1,i} x_{i-1}) \partial/\partial x_i \\ \tilde{h}_i \partial/\partial y_i &= (\tilde{m}_{1i} y_1 + \tilde{m}_{2i} y_2 + \cdots + \tilde{m}_{i-1,i} y_{i-1}) \partial/\partial y_i \end{aligned} \qquad (2.3)$$

for $m_{ij}, \tilde{m}_{ij} \in \mathbf{R}$. Now order the monomial vectors $x_1^{r_1} \dots x_n^{r_n} e_i \in m_n^r.\mathcal{E}(n,p)$ to produce a sequence $\{v_i\}$ as follows. Start with those of degree r: order the vectors $x_1^{r_1} \dots x_n^{r_n} e_1$ using lexicographic order (starting with $v_1 = x_n^r e_1$); next order the vectors $x_1^{r_1} \dots x_n^{r_n} e_2$ in the same way using lexicographic order; and so on, continuing with those of degree $r+1$ and higher. Then by 2.3 we have $l \cdot v_i = v_{i+i'}$ for some $i' \geq 1$. $J^1 L$ is a finite dimensional vector space so we need only consider a finite number of (basis) vectors l. Thus, for large enough t, any product $l_1 \dots l_t$ of elements of L must satisfy $l_1 \dots l_t \cdot v \in m_n^{r+1}.\mathcal{E}(n,p)$ for $v \in m_n^r.\mathcal{E}(n,p)$ and the result follows. □

Definition 2.17 (Nilpotent Filtration) Let $L \subset L\mathcal{K}$ be such that $J^1 L$ is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbf{R}^{n+p}$. We define for integers $r \geq 1$ and $s \geq 0$ the *nilpotent filtration*

$$M_{r,s}(L) = \sum_{i \geq s} L^i \cdot (m_n^r.\mathcal{E}(n,p)) + m_n^{r+1}.\mathcal{E}(n,p).$$

For $r = 0$ we just define $M_{0,0}$ to be $m_n.\mathcal{E}(n,p)$ for consistency. The associated *jet-space* $J^{r,s}(n,p)$ is then defined to be $m_n.\mathcal{E}(n,p)/M_{r,s}(L)$, and by the homogeneous terms of degree (r,s) we mean the image of the space $M_{r,s-1}(L)$ in this quotient, and denote this $H^{r,s}$.

Observe that $M_{r,s}(L)$ is just the standard filtration by degree, only refined by the addition of the $\sum L^i \cdot (m_n^r.\mathcal{E}(n,p))$ terms.

Remarks. Firstly note that by the previous proposition the sum used to define $M_{r,s}(L)$ is finite. It also follows that for each r there is an integer k_r such that

$$M_{r,k_r} = M_{r,k_r+1} = M_{r,k_r+2} = \cdots = m_n^{r+1}.\mathcal{E}(n,p) = M_{r+1,0}.$$

So this filtration is *not* doubly indexed as might be thought at first, but can be thought of as a singly indexed sequence. This is therefore incorporated in the results of Section 2.2, only a more elaborate notation is required. The definition of $H^{r,s}$ as the image of $M_{r,s-1}(L)$ in $J^{r,s}(n,p)$ should make sense for $s = 0$ now as $M_{r,0} = M_{r-1,k_{(r-1)}}$.

It is worth qualifying the use of the faithful representation of Lemma 2.12 instead of the representation induced from the Lie group action as discussed in Remark 2.13. As a simple example consider the case $n = p = 2$ and note that the vector field $(x_1\partial/\partial x_1 + x_2\partial/\partial x_2) \oplus (-y_1\partial/\partial y_1 - y_2\partial/\partial y_2)$ acts nilpotently on 1-jets but not so on higher jets. Thus the crucial result of Proposition 2.16 need no longer hold. However, such a vector field is not nilpotent under the faithful representation on $\mathbf{R}^{n+p}$, being mapped to a diagonal matrix with 1's followed by (-1)'s along the diagonal — exceedingly 'un-nilpotent'!

Examples 2.18 We give examples by listing the generators for the spaces $H^{r,s}$ rather than describing the modules $M_{r,s}(L)$. Each (r,s)-jet-space is just a refinement of the standard r-jet-space (by degree), the generators for each successive $H^{r,s}$ give the extra monomials which arise when passing from the $(r, s-1)$-jet-space to the (r,s)-jet-space. We only consider the first few values of r, though there is an obvious pattern as one continues.

(1). Function germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$ with (x, y) as coordinates in the source and using the nilpotent Lie subalgebra of $L\mathcal{R}$

$$Sp\{x\partial/\partial y\} \oplus L\mathcal{R}_1.$$

In this example we achieve the finest filtration possible, obtaining a graded lexicographic order on the monomials.

(r,s)	Basis for $H^{r,s}$
(1,0)	$\{0\}$
(1,1)	$\{y\}$
(1,2) or (2,0)	$\{x\}$
(2,1)	$\{y^2\}$
(2,2)	$\{xy\}$
(2,3) or (3,0)	$\{x^2\}$
(3,1)	$\{y^3\}$
(3,2)	$\{xy^2\}$
(3,3)	$\{x^2y\}$
(3,4) or (4,0)	$\{x^3\}$

Generally such a fine filtration is not achieved, even for function germs, as the next example shows.

(2). Function germs $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}, 0)$ with (x, y, z) as coordinates in the source and using the nilpotent Lie subalgebra of $L\mathcal{R}$

$$Sp\{x\partial/\partial y, x\partial/\partial z, y\partial/\partial z\} \oplus L\mathcal{R}_1.$$

(r,s)	Basis for $H^{r,s}$
$(1,0)$	$\{0\}$
$(1,1)$	$\{z\}$
$(1,2)$	$\{y\}$
$(1,3)$ or $(2,0)$	$\{x\}$
$(2,1)$	$\{z^2\}$
$(2,2)$	$\{yz\}$
$(2,3)$	$\{y^2, xz\}$
$(2,4)$	$\{xy\}$
$(2,5)$ or $(3,0)$	$\{x^2\}$

In particular, note that there are two generators at the $(2,3)$-level.

(3). Map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$; we refer to the chapter dealing with this, in particular, Section 3.2.

We apply the results of Section 2.2 to obtain a complete transversal theorem for nilpotent filtrations. Firstly a couple of lemmas.

Lemma 2.19 *Let G be a connected Lie group acting smoothly on an affine space A; let B be a vector subspace of V_A. Then the action of G on A induces an action on A/B if and only if, for all $a \in A$, $b \in B$ and $l \in LG$*

$$l \cdot (a+b) - l \cdot a \in B.$$

Proof. [BduPW, 2.2]. □

Lemma 2.20 *Let $L \subset L\mathcal{K}$ be such that J^1L is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbf{R}^{n+p}$, so that the nilpotent filtration $M_{r,s}(L)$ is defined. Let $f \in m_n.\mathcal{E}(n,p)$, $h \in M_{r,s}(L)$ and $l \in L$. Then*

$$l \cdot (f+h) - l \cdot f \in M_{r,s+1}(L).$$

Proof. Follows in exactly the same way as [BduPW, 2.3]. □

If $L \subset L\mathcal{G}$ is such that J^1L is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbf{R}^{n+p}$ then the nilpotent filtration $M_{r,s}(L)$ defined in 2.17 exists. We filter the group $\mathcal{G}$ using the subgroups $\mathcal{G}_r$ used in the standard filtration by degree.

Explicitly, for all s we define $\mathcal{G}_{r,s}$ to the subgroup $(1_{n+p} + m_{n+p}^{r+1}.\mathcal{E}(n+p, n+p)) \cap \mathcal{G}$ of $\mathcal{G}$ consisting of those elements whose (standard) r-jet is equal to that of the identity. **Note:** we therefore have $J^{r,s}\mathcal{G} = J^r\mathcal{G}$ (the jet-group used in the standard filtration by degree) and $J^{r,s}L = J^rL$, for all s. The idea is to obtain a jet-filtration $(\{M_{r,s}(L)\}, \{G_{r,s}\})$, but this depends on the induced action of $J^{r,s}\mathcal{G}$ on $J^{r,s}(n,p)$ being well-defined. We do have the following.

Proposition 2.21 *If $\mathcal{G}$ is a fibrant subgroup of $\mathcal{K}$ and $J^1(L\mathcal{G})$ is nilpotent on $\mathbf{R}^{n+p}$ then $(\{M_{r,s}(L\mathcal{G})\}, \{G_{r,s}\})$ is a jet filtration.*

Following [BduPW] we will also denote $M_{r,s}(L\mathcal{G})$ by $M_{r,s}(\mathcal{G})$.

Proof. Since $M_{r,s}(\mathcal{G})$ contains $m_n^{r+1}.\mathcal{E}(n,p)$ it follows that it is of finite codimension in $m_n.\mathcal{E}(n,p)$ and $J^{r,s}(n,p)$ is therefore a finite dimensional vector space. From [MathIII, Section 7] the standard r-jet-group $J^r\mathcal{G}$ is a Lie group and acts on the standard r-jet-space $m_n.\mathcal{E}(n,p)/m_n^{r+1}.\mathcal{E}(n,p)$, which we will denote by A. Let B denote the image of $M_{r,s}(\mathcal{G})$ under the natural projection into A, and let j^rf denote the image of $f \in m_n.\mathcal{E}(n,p)$ in A and j^rl the image of $l \in L\mathcal{G}$ in $J^r(L\mathcal{G})$. Now, any element of $L(J^r\mathcal{G})$ may be written as j^rl for some $l \in L\mathcal{G}$ as $\mathcal{G}$ is fibrant, and any element of B may be written as j^rh for some $h \in M_{r,s}(\mathcal{G})$. Then

$$j^rl \cdot (j^rf + j^rh) - j^rl \cdot (j^rf) = j^r(l \cdot (f+h) - l \cdot f) \in B$$

since, by Lemma 2.20

$$l \cdot (f+h) - l \cdot f \in M_{r,s+1}(\mathcal{G}) \subset M_{r,s}(\mathcal{G}).$$

Thus, by Lemma 2.19, the action of $J^r\mathcal{G}$ on A induces an action on A/B. But $m_n^{r+1}.\mathcal{E}(n,p) \subset M_{r,s}(\mathcal{G}) \subset m_n.\mathcal{E}(n,p)$ and $B = M_{r,s}(\mathcal{G})/m_n^{r+1}.\mathcal{E}(n,p)$ so $A/B \cong m_n.\mathcal{E}(n,p)/M_{r,s}(\mathcal{G})$ by the (appropriate) isomorphism theorem for modules. Hence the action of $J^r\mathcal{G}$ on $m_n.\mathcal{E}(n,p)/m_n^{r+1}.\mathcal{E}(n,p)$ induces an action on $J^{r,s}(n,p)$. □

For a complete transversal theorem we can omit the requirement that $\mathcal{G}$ be fibrant. We can no longer be sure that $(\{M_{r,s}(\mathcal{G})\}, \{G_{r,s}\})$ is a jet-filtration but can work with subgroups of $J^{r,s}\mathcal{G}$ which act on $J^{r,s}(n,p)$ instead, and this suffices for applications of the theorem. We summarise the results below.

Theorem 2.22 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $L \subset L\mathcal{G}$ such that*

1. *J^rL (a subset of $J^r(L\mathcal{G})$) is a Lie subalgebra of $L(J^r\mathcal{G})$ for all $r \geq 0$;*

2. *the subalgebra J^1L of $L(J^1\mathcal{K})$ is nilpotent on $\mathbf{R}^{n+p}$.*

Then the following complete transversal result holds. For $f \in m_n.\mathcal{E}(n,p)$ and T a subspace of $H^{r,s+1} \subset J^{r,s+1}(n,p)$ such that

$$J^rL \cdot j^{r,s+1}f + T \supset H^{r,s+1},$$

we have any (r,s)-jet $j^{r,s}g \sim_{J^r\mathcal{G}} j^{r,s}f$ has $(r,s+1)$-jet $j^{r,s+1}g \sim_{J^{r+1}\mathcal{G}} j^{r,s+1}f + t$ for some $t \in T$. T will be referred to as an $(r,s+1)$ complete transversal.

Remark. We cannot guarantee the whole of $J^r\mathcal{G}$ acts on $J^{r,s}(n,p)$. However, there is some subgroup G of $J^r\mathcal{G}$ which acts on $J^{r,s}(n,p)$ and two jets are equivalent under this action, that is $j^{r,s}g \sim_G j^{r,s}f$, means that $g \sim_{\mathcal{G}} f + \tilde{f}$ for $\tilde{f} \in M_{r,s}(L)$, as required in a classification. This should clarify the (strictly incorrect) notation $\sim_{J^r\mathcal{G}}$ and $\sim_{J^{r+1}\mathcal{G}}$ used in the statement of the theorem.

Proof. $J^r\mathcal{G}$ $(= J^{r,s+1}\mathcal{G})$ is a Lie group and acts on $m_n.\mathcal{E}(n,p)/m_n^{r+1}.\mathcal{E}(n,p)$, the standard r-jet-space. Now, by hypothesis, J^rL $(= J^{r,s+1}L)$ is a Lie subalgebra of $L(J^r\mathcal{G})$ so there exists a (unique, connected) Lie subgroup G of $J^r\mathcal{G}$ with Lie algebra $LG = J^rL$; see [War, Theorem 3.19]. As in the proof of Proposition 2.21, with A denoting $m_n.\mathcal{E}(n,p)/m_n^{r+1}.\mathcal{E}(n,p)$ and B denoting the image of $M_{r,s+1}(L)$ under the natural projection into A, we have

$$j^rl \cdot (j^rf + j^rh) - j^rl \cdot (j^rf) = j^r(l \cdot (f+h) - l \cdot f) \in B$$

for all $j^rl \in J^rL$, $j^rf \in A$ and $j^rh \in B$, so the action of G induces an action on $J^{r,s+1}(n,p)$. The result now follows as in the generalised complete transversal theorem 2.10 once we have shown condition 2.2 holds. But for $j^{r,s+1}h \in H^{r,s+1}$ (that is $h \in M_{r,s}$)

$$j^rl \cdot (j^{r,s+1}f + j^{r,s+1}h) - j^rl \cdot (j^{r,s+1}f) = j^{r,s+1}(l \cdot (f+h) - l \cdot f) = 0$$

since $l \cdot (f+h) - l \cdot f \in M_{r,s+1}$ by Lemma 2.20, giving the required result. Thus any (r,s)-jet $j^{r,s}g$ with $j^{r,s}g \sim_G j^{r,s}f$ has $(r,s+1)$-jet $j^{r,s+1}g \sim_G j^{r,s+1}f + t$ for some $t \in T$. Now G is a subgroup of $J^r\mathcal{G}$ whose action on $J^{r,s+1}(n,p)$ is induced from the action of $\mathcal{G}$ on $m_n.\mathcal{E}(n,p)$ and the result follows. □

In applications the above complete transversal theorem is used to enumerate the orbits up to a given (standard) jet-level. Determinacy is then given with respect to the standard filtration by degree as this is generally more convenient. The nilpotent methods can be exploited in determinacy calculations to give what proves to be excellent determinacy estimates. We give the following determinacy result for completeness. It is equivalent to results found in [BduPW], only formulated in terms of subspaces of $L\mathcal{G}$ as in our previous theorems.

Corollary 2.23 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $L \subset L\mathcal{G}$ satisfying conditions 1 and 2 of Theorem 2.22. Suppose further that $\mathcal{G}$ satisfies the Mather condition, that is the following are equivalent for $f \in m_n.\mathcal{E}(n,p)$:*

1. *f is finitely $\mathcal{G}$-determined — i.e., there exists $N < \infty$ such that for $g \in m_n.\mathcal{E}(n,p)$, $f - g \in m_n^N.\mathcal{E}(n,p)$ implies f, g are $\mathcal{G}$-equivalent;*
2. *$dim_{\mathbf{R}}(m_n.\mathcal{E}(n,p)/L\mathcal{G} \cdot f) < \infty$;*
3. *there exists $N < \infty$ such that $L\mathcal{G} \cdot f \supset m_n^N.\mathcal{E}(n,p)$.*

Then a map-germ $f \in m_n.\mathcal{E}(n,p)$ is k-$\mathcal{G}$-determined if

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f.$$

Proof. From the Mather condition and the inclusion $m_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f$ it follows that f must be finitely $\mathcal{G}$-determined, N-$\mathcal{G}$-determined say. We can construct the nilpotent filtration $M_{r,s}(L)$ and associated jet-spaces. Suppose g satisfies $f-g \in m_n^{k+1}.\mathcal{E}(n,p)$, thus g has the same $(k+1,0)$-jet as f. Now, for $r \geq k+1$, $s \geq 0$ and any $\tilde{f}$ with the same (r,s)-jet as f, since $L \cdot f \supset m_n^{k+1}.\mathcal{E}(n,p)$ it follows that the $(r, s+1)$-transversal of f is empty and $j^{r,s+1}\tilde{f} \sim_{\mathcal{G}} j^{r,s+1}f$. Hence, by induction, f and g must have the same N-jet ($f - g \in m_n^{N+1}.\mathcal{E}(n,p)$) and since *a priori* f is N-determined we see that g is $\mathcal{G}$-equivalent to f. □

The Mather condition holds for all the usual cases, such as the standard Mather groups, of course; see [MathIII], [Wal], [duP], for example. More generally, it holds for the 'geometric subgroups' of Damon, these include subgroups which preserve varieties living in the source and target for example; see Chapter 5. We refer to [D] for more on geometric subgroups; also to [BduPW, Section 4] for further discussion on the Mather condition.

2.4 Weighted Filtrations

Although the results of Section 2.3 provide powerful classification techniques, some classifications lend themselves more naturally to the use of weighted filtrations. We take this opportunity to review the basic results concerning weighted filtrations and the associated jet-spaces and jet-groups. The original ideas were introduced by Arnol'd for the $\mathcal{R}$ case, [A1]. These extend to the other subgroups of $\mathcal{K}$, a concise treatment is given in [BduPW, Section 5]. We use these as our two basic references, filling in the details required by the complete transversal theorems. For the case of function-germs (under $\mathcal{R}$) equivalent results were given in [A1]. For other subgroups of $\mathcal{K}$ there has been much work on weighted determinacy but we do not know of any complete transversal theorems. So our results, together with the determinacy results, provide a workable classification method for weighted filtrations.

We firstly recall the basics.

Definition 2.24 Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a given sequence of positive integers referred to as *weights*. A function-germ $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ is said to be *weighted homogeneous of weight* r (with respect to α) if

$$f(t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n) = t^r f(x_1, \ldots, x_n)$$

for all $t \in \mathbf{R}$. For a monomial function $x_1^{k_1} \cdots x_n^{k_n}$ an equivalent condition is that $\alpha_1 k_1 + \cdots + \alpha_n k_n = r$. And a polynomial function is weighted homogeneous if and only if all its constituent monomials are.

Remark. For theoretical considerations one often allows rational weights and can assume a weighted homogeneous polynomial is of weight 1.

Definition 2.25 The ideal in $m_n.\mathcal{E}_n$ generated by the monomials of weight r and greater is denoted $F^r_\alpha\mathcal{E}_n$, or just $F^r\mathcal{E}_n$ for short. The sequence of ideals $\{F^r_\alpha\mathcal{E}_n\}$, for $r \geq 1$, defines a filtration of the ring $m_n.\mathcal{E}_n$.

Note that $F^1_\alpha\mathcal{E}_n = m_n$ (for all α). Also, when dealing with function-germs and $\mathcal{R}$ equivalence one sometimes works with $\mathcal{E}_n$ and the filtration $\{F^r_\alpha\mathcal{E}_n\}$ for $r \geq 0$. This was the case in the work of Arnol'd.

For map-germs $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ we must introduce weights for each target coordinate too.

Definition 2.26 For α as above, let $\beta = (\beta_1, \ldots, \beta_p)$ be a sequence of non-negative integers, and let $f = (f_1, \ldots, f_p)$ be a map-germ $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$. Then f is said to be *weighted homogeneous of weight* r (with respect to (α, β)) if f_i is weighted homogeneous of weight $r + \beta_i$ (as in definition 2.24), for $i = 1, \ldots, p$.

Definition 2.27 The submodule of $m_n.\mathcal{E}(n,p)$ generated by the map-germs $f = (f_1, \ldots, f_p)$ such that $f_i \in F_\alpha^{r+\beta_i}\mathcal{E}_n$, for $1 \le i \le p$, is denoted $F_{\alpha,\beta}^r\mathcal{E}(n,p)$, or just $F^r\mathcal{E}(n,p)$ for short. The sequence of submodules $\{F_{\alpha,\beta}^r\mathcal{E}(n,p)\}$, for $r \ge 0$, defines a filtration of the module $F_{\alpha,\beta}^0\mathcal{E}(n,p)$. (Note that generally $F_{\alpha,\beta}^0\mathcal{E}(n,p) \neq m_n.\mathcal{E}(n,p)$ but this causes no problems.)

We recall the following results from [BduPW, Section 5].

Lemma 2.28

1. *$f \in F_{\alpha,\beta}^r\mathcal{E}(n,p)$ if and only if, for all k and for all $h \in F_\beta^k\mathcal{E}_p$, $h \circ f \in F_\alpha^{r+k}\mathcal{E}_n$.*
2. *If $f, g \in F_{\alpha,\beta}^0\mathcal{E}(n,p)$ with $(f - g) \in F_{\alpha,\beta}^r\mathcal{E}(n,p)$ and $h \in F_{\beta,\gamma}^s\mathcal{E}(p,q)$, then $(h \circ f - h \circ g) \in F_{\alpha,\gamma}^{r+s}\mathcal{E}(n,q)$.*

In this section we will take $\mathcal{G}$ to be one of the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$, though the methods do extend to other subgroups of $\mathcal{K}$. We begin by defining a filtration of such groups compatible with the weighted filtration of $m_n.\mathcal{E}(n,p)$; not all elements of $\mathcal{K}$ will respect the weighted filtration. For example, for function germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$ with weights $(2,3)$ we observe that $x^3 + y^2$ is weighted homogeneous of weight 6, but the diffeomorphism $(x,y) \mapsto (y,x)$ in $\mathcal{R}$ maps this to an element of $F^4\mathcal{E}_2$ therefore not respecting the filtration. The required definition in the $\mathcal{R}$ case is due to Arnol'd.

Definition 2.29 For $r \ge 0$ define

$$F^r\mathcal{R} = \{\, \phi \in \mathcal{R} : f \circ \phi - f \in F^{r+t}\mathcal{E}_n \quad \text{for all} \quad f \in F^t\mathcal{E}_n \quad \text{and for all } t \,\}.$$

(Arnol'd calls this the *group of diffeomorphisms of filtration* r.)

This can be put in more concrete terms.

Proposition 2.30 *The above property in definition 2.29 characterises the set (a group in fact)*

$$(1_n + F^r_{\alpha,\alpha}\mathcal{E}(n,n)) \cap \mathcal{R}.$$

Proof. If ϕ satisfies 2.29 then putting $f = x_i \in F^{\alpha_i}\mathcal{E}_n$, $i = 1, \ldots, n$ (the source coordinate functions) we have $x_i \circ \phi - x_i \in F^{r+\alpha_i}\mathcal{E}_n$ so

$$\phi \in (1_n + F^r_{\alpha,\alpha}\mathcal{E}(n,n)) \cap \mathcal{R}.$$

Conversely, suppose $\phi \in (1_n + F^r_{\alpha,\alpha}\mathcal{E}(n,n)) \cap \mathcal{R}$. Then for $f = x_1^{k_1} \ldots x_n^{k_n} \in F^t\mathcal{E}_n$, that is $\alpha_1 k_1 + \cdots + \alpha_n k_n \geq t$, we have

$$f \circ \phi - f = (x_1 + g_1)^{k_1} \ldots (x_n + g_n)^{k_n} - x_1^{k_1} \ldots x_n^{k_n}$$

where $g_i \in F^{r+\alpha_i}\mathcal{E}_n$. So

$$f \circ \phi - f = \left(\sum_{i=0}^{k_1} \binom{k_1}{i} x_1^{k_1-i} g_1^i\right) \ldots \left(\sum_{i=0}^{k_n} \binom{k_n}{i} x_n^{k_n-i} g_n^i\right) - x_1^{k_1} \ldots x_n^{k_n}$$

which is therefore a sum of monomial terms of the form $x_1^{k_1-i_1} g_1^{i_1} \ldots x_n^{k_n-i_n} g_n^{i_n}$ where $i_1, \ldots, i_n$ are all *strictly* greater than zero. Now $x_j^{k_j-i_j}$ has weight $\alpha_j(k_j - i_j)$ and $g_j^{i_j} \in F^{i_j(r+\alpha_j)}\mathcal{E}_n$ so $x_j^{k_j-i_j} g_j^{i_j} \in F^{\alpha_j k_j + i_j r}\mathcal{E}_n$ and therefore

$$x_i^{k_1-i_1} g_1^{i_1} \ldots x_n^{k_n-i_n} g_n^{i_n} \in F^{\alpha_1 k_1 + \cdots + \alpha_n k_n + (i_1 + \cdots + i_n) r}\mathcal{E}_n.$$

Now $\alpha_1 k_1 + \cdots + \alpha_n k_n \geq t$ and $i_1, \ldots, i_n$ are all strictly greater than zero so

$$f \circ \phi - f \in F^{r+t}\mathcal{E}_n.$$

Finally, $F^t\mathcal{E}_n$ is, by definition, generated by such monomials f so the result holds for all elements of $F^t\mathcal{E}_n$. □

Thus $F^r\mathcal{R} = (1_n + F^r_{\alpha,\alpha}\mathcal{E}(n,n)) \cap \mathcal{R}$. Likewise, we extend the definition to the group $\mathcal{L}$ and have $F^r\mathcal{L} = (1_p + F^r_{\beta,\beta}\mathcal{E}(p,p)) \cap \mathcal{L}$. However, we will concentrate on the filtration of $\mathcal{C}$. The filtrations of $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$ induce filtrations on $\mathcal{A}$ and $\mathcal{K}$. Also, since $\mathcal{L} \subset \mathcal{C}$ we need only establish the results for the cases $\mathcal{R}$ and $\mathcal{C}$.

Recall that $\mathcal{C}$ consists of diffeomorphisms of $(\mathbf{R}^{n+p}, 0)$ which project to the identity on $\mathbf{R}^n$ and leave fixed the subspace $\mathbf{R}^n \times \{0\}$, that is

$$\mathcal{C} = 1_{n+p} + m_p.\mathcal{E}(n+p,p),$$

where we consider $\mathcal{E}(n+p,p)$ as a subspace of $\mathcal{E}(n+p,n+p)$ (by associating $f \in \mathcal{E}(n+p,p)$, $(x,y) \mapsto f(x,y)$, with the germ $(x,y) \mapsto (0, f(x,y))$ in $\mathcal{E}(n+p,n+p)$) and consider m_p as an ideal of $\mathcal{E}_{n+p}$ (via $\pi^* : \mathcal{E}_p \longrightarrow \mathcal{E}_{n+p}$ where $\pi : \mathbf{R}^{n+p} \longrightarrow \mathbf{R}^p$, $(x,y) \mapsto y$). We define $F^r\mathcal{C}$ analogously to the $\mathcal{R}$ case.

Definition 2.31 For $r \geq 0$ define

$$F^r\mathcal{C} = (1_{n+p} + F^r_{\alpha\cup\beta,\alpha\cup\beta}\mathcal{E}(n+p, n+p)) \cap \mathcal{C},$$

where $\alpha \cup \beta$ denotes the sequence $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p)$. Note that if $\mathcal{R}_{n+p}$ denotes the group of diffeomorphisms of $\mathbf{R}^{n+p}$ at 0 then

$$F^r\mathcal{C} = F^r\mathcal{R}_{n+p} \cap \mathcal{C}.$$

(Here we use the subscript '$n+p$' to distinguish $\mathcal{R}_{n+p} \subset \mathcal{E}(n+p, n+p)$ from the group of diffeomorphisms of $\mathbf{R}^n$ at 0, $\mathcal{R} \subset \mathcal{E}(n, n)$.)

We will need a characterisation for $F^r\mathcal{C}$ analogous to that of $F^r\mathcal{R}$ in 2.29.

Proposition 2.32

1. *If $f, g \in F^0_{\alpha,\beta}\mathcal{E}(n,p)$ with $(f-g) \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ and $\phi \in F^r\mathcal{C}$, then the action $(\cdot)$ of $F^r\mathcal{C}$ on $F^0_{\alpha,\beta}\mathcal{E}(n,p)$ defined by $(1_n, \phi \cdot f) = \phi \circ (1_n, f)$ satisfies*

$$(\phi \cdot f - f) - (\phi \cdot g - g) \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p).$$

2. *We have*

$$\begin{aligned} F^r\mathcal{C} \;=\; & \{\phi \in \mathcal{C} : \phi \cdot f - f \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p) \\ & \textit{for all} \quad f \in F^t_{\alpha,\beta}\mathcal{E}(n,p) \quad \textit{and for all } t\}. \end{aligned}$$

Proof. Part 1 follows from Lemma 2.28, part 2. We refer to [BduPW, Section 5] for more details.
In part 2, to see that $F^r\mathcal{C}$ satisfies the given property just use part 1 with $g = 0$. Similarly one checks, if $\phi \in \mathcal{C}$ satisfies the given property then $\phi \in F^r\mathcal{C}$. □

It is an easy matter to establish the Lie algebras of the above groups.

Lemma 2.33

1. *The Lie algebra of $F^r\mathcal{R}$ can be identified with $F^r_{\alpha,\alpha}\mathcal{E}(n,n)$; and that of $F^r\mathcal{L}$ with $F^r_{\beta,\beta}\mathcal{E}(p,p)$.*

2. *The Lie algebra of $F^r\mathcal{C}$ can be identified with $F^r_{\alpha\cup\beta,\beta}\mathcal{E}(n+p,p)$.*

3. *The Lie algebra action satisfies the following approximation lemma. If* $f, g \in F^0_{\alpha,\beta}\mathcal{E}(n,p)$ *with* $f - g \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ *and* $l \in L(F^r\mathcal{G})$ *then*

$$l \cdot f - l \cdot g \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p).$$

Proof. The details are standard and we omit them, making only the following comments. 1 and 2 follow directly from the definitions of the groups $F^r\mathcal{G}$. 3 follows as in the proofs of the approximation lemmas using standard coordinates and identifying vectors $l \cdot f$ as elements of $m_n.\mathcal{E}(n,p)$; see, for example, [duP, Sublemma 2.2] or [Wal, Section 1]. The $\mathcal{R}$ case is straight forward, as is the $\mathcal{C}$ case via Lemma 2.28. □

From Lemma 2.33 part 3 it follows that the Lie algebra action of such groups respects the filtration, in that for $f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ and $l \in L(F^r\mathcal{G})$ ($\mathcal{G}$ one of the standard Mather groups) we have $l \cdot f \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p)$. (We say l is *weighted homogeneous* of weight r.)

We now discuss the quotient groups associated to a filtration $\{F^r\mathcal{G}\}$. The importance of 'weighted filtrations' comes from the fact that the subgroups $F^r\mathcal{G}$ respect the filtration and there exists an induced action at the jet-level. We make this precise below. Firstly we note the following.

Proposition 2.34 *Each* $F^r\mathcal{G}$ *is a normal subgroup of* $F^0\mathcal{G}$.

Proof. We use the characterisation of $F^r\mathcal{R}$ and $F^r\mathcal{C}$ given in Proposition 2.30 and Proposition 2.32 part (2). Firstly show $F^r\mathcal{G}$ is a subgroup of $F^0\mathcal{G}$. Suppose ϕ_1, $\phi_2 \in F^r\mathcal{R}$ and $f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ for some t then

$$\begin{aligned}
(\phi_1\phi_2) \cdot f - f &= \phi_1 \cdot (\phi_2 \cdot f) - f \\
&= \phi_1 \cdot (f + f_1) - f \quad \text{for} \quad f_1 \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p) \\
&= f + f_1 + f_2 - f \quad \text{for} \quad f_2 \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p) \\
&\in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p).
\end{aligned}$$

Thus $\phi_1\phi_2 \in F^r\mathcal{R}$, proving closure. The argument is the same in the $\mathcal{C}$ case. We now observe that for $\phi \in F^r\mathcal{R}$ and for any t, $\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p)) = F^t_{\alpha,\beta}\mathcal{E}(n,p)$. Firstly note that the action of ϕ induces an automorphism $\phi : \mathcal{E}_n \longrightarrow \mathcal{E}_n$ and by the first isomorphism theorem we see that $\mathcal{E}_n/F^t_{\alpha,\beta}\mathcal{E}(n,p)$ is isomorphic to $\mathcal{E}_n/\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p))$. Now consider the natural surjection $\rho : \mathcal{E}_n/\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p)) \longrightarrow \mathcal{E}_n/F^t_{\alpha,\beta}\mathcal{E}(n,p)$ induced by the identity map; this is well-defined since $\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p)) \subset F^t_{\alpha,\beta}\mathcal{E}(n,p)$. We have already established that these quotient spaces are isomorphic, and since $\mathcal{E}_n/F^t_{\alpha,\beta}\mathcal{E}(n,p)$ is finite dimensional so is $\mathcal{E}_n/\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p))$. It therefore follows that ρ is an isomorphism and thus $\phi \cdot (F^t_{\alpha,\beta}\mathcal{E}(n,p)) = F^t_{\alpha,\beta}\mathcal{E}(n,p)$. In particular, if $\phi \in F^0\mathcal{R}$ then $\phi^{-1} \in F^0\mathcal{R}$. In the case $\mathcal{G} = \mathcal{C}$ recall from definition 2.31 that $F^r\mathcal{C} = F^r\mathcal{R}_{n+p} \cap \mathcal{C}$. So if $\phi \in F^0\mathcal{C}$ then $\phi^{-1} \in F^0\mathcal{R}_{n+p}$ (from the previous argument) and $\phi^{-1} \in \mathcal{C}$ so $\phi^{-1} \in F^0\mathcal{C}$.

We now finish the proof that $F^r\mathcal{R}$ is a group, the argument holds for the case $F^r\mathcal{C}$ too. Given $\phi \in F^r\mathcal{R}$ suppose $f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ for some t. Then

$$\phi \cdot (\phi^{-1} \cdot f) - f = (\phi\phi^{-1}) \cdot f - f = 0.$$

Now, $\phi \in F^r\mathcal{R} \subset F^0\mathcal{R}$ so from the preceding comments $\phi^{-1} \in F^0\mathcal{R}$ so $\phi^{-1} \cdot f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ and

$$\phi \cdot (\phi^{-1} \cdot f) - f = \phi^{-1} \cdot f + f_1 - f \quad \text{for} \quad f_1 \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p).$$

Thus $\phi^{-1} \cdot f - f \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p)$ and it follows that $\phi^{-1} \in F^r\mathcal{R}$.

$F^r\mathcal{R}$ and $F^r\mathcal{C}$ are therefore subgroups of $\mathcal{K}$. We now show they are normal subgroups of $F^0\mathcal{R}$ and $F^0\mathcal{C}$, respectively. Suppose $\theta \in F^0\mathcal{R}$, $\phi \in F^r\mathcal{R}$ and $f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ for some t. Then

$$\begin{aligned}
(\theta^{-1}\phi\theta) \cdot f - f &= \theta^{-1} \cdot (\theta \cdot f + f_1) - f \quad \text{for} \quad f_1 \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p) \\
&= \theta^{-1} \cdot f_1 \quad (\text{since } \mathcal{R} \text{ acts linearly}) \\
&\in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p).
\end{aligned}$$

This requires θ and θ^{-1} belong to $F^0\mathcal{R}$. Now suppose $\theta \in F^0\mathcal{C}$, $\phi \in F^r\mathcal{C}$ and $f \in F^t_{\alpha,\beta}\mathcal{E}(n,p)$ for some t. Then

$$\begin{aligned}
(\theta^{-1}\phi\theta) \cdot f - f &= \theta^{-1} \cdot (\theta \cdot f + f_1) - f \quad \text{for} \quad f_1 \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p) \\
&= \theta^{-1} \cdot (\theta \cdot f + f_1) - \theta^{-1} \cdot (\theta \cdot f).
\end{aligned}$$

From Proposition 2.32 part (1) we have

$$\left(\theta^{-1} \cdot (\theta \cdot f + f_1) - \theta^{-1} \cdot (\theta \cdot f)\right) - \left((\theta \cdot f + f_1) - \theta \cdot f\right) \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p)$$

so $(\theta^{-1}\phi\theta) \cdot f - f \in F^{r+t}_{\alpha,\beta}\mathcal{E}(n,p)$. This requires θ and θ^{-1} belong to $F^0\mathcal{C}$. The results now follow. □

It was noted in [BduPW] that the groups $F^r\mathcal{G}$ are strongly closed subgroups of $\mathcal{G}$ — this is useful for applications to determinacy. It also follows that each group is jet-closed — this is helpful in the complete transversal theorems.

Proposition 2.35

1. *Each subgroup* $F^r\mathcal{G}$ *respects the filtration* $\{F^r_{\alpha,\beta}\mathcal{E}(n,p)\}$ *of* $m_n.\mathcal{E}(n,p)$.

2. *For* $s, \lambda, r \geq 0$ *the action of* $F^r\mathcal{G}$ *on* $m_n.\mathcal{E}(n,p)$ *induces an action on* $F^s_{\alpha,\beta}\mathcal{E}(n,p)/F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$.

3. *In addition, if* $\mu \geq 0$ *and* $\lambda \leq r + \mu$ *then the action in part 2 induces an action of* $F^r\mathcal{G}/F^{r+\mu}\mathcal{G}$ *on* $F^s_{\alpha,\beta}\mathcal{E}(n,p)/F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$.

Proof. We need only consider the cases $\mathcal{G} = \mathcal{R}$ and $\mathcal{G} = \mathcal{C}$.

1. For $\phi \in F^r\mathcal{R}$ or $F^r\mathcal{C}$ and $f \in F^s_{\alpha,\beta}\mathcal{E}(n,p)$ we have $\phi \cdot f - f \in F^{s+r}_{\alpha,\beta}\mathcal{E}(n,p) \subset F^s_{\alpha,\beta}\mathcal{E}(n,p)$. Thus $\phi \cdot f \in F^s_{\alpha,\beta}\mathcal{E}(n,p)$.

2. Consider $f, g \in F^s_{\alpha,\beta}\mathcal{E}(n,p)$ with $f - g \in F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$. Then for $\phi \in F^r\mathcal{R}$ we have $\phi \cdot f - \phi \cdot g = \phi \cdot (f - g) \in F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$ by part 1 and the fact that $\mathcal{R}$ acts linearly. For $\phi \in F^r\mathcal{C}$ we have $(\phi \cdot f - \phi \cdot g) - (f - g) \in F^{s+\lambda+r}_{\alpha,\beta}\mathcal{E}(n,p)$ by Proposition 2.32 part 1, thus $\phi \cdot f - \phi \cdot g \in F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$.

3. By part 2 there is an action of $F^r\mathcal{G}$ on $F^s_{\alpha,\beta}\mathcal{E}(n,p)/F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p)$. Consider $F^r\mathcal{G}/F^{r+\mu}\mathcal{G}$ and suppose $\theta, \theta' \in F^r\mathcal{G}$ define the same coset, so $\theta' = \theta\phi$ for $\phi \in F^{r+\mu}\mathcal{G}$. For $f \in F^s_{\alpha,\beta}\mathcal{E}(n,p)$ we have $\phi \cdot f = f + \tilde{f}$ with $\tilde{f} \in F^{s+r+\mu}_{\alpha,\beta}\mathcal{E}(n,p)$. So

$$(\theta\phi) \cdot f - \theta \cdot f = \theta \cdot (f + \tilde{f}) - \theta \cdot f \in F^{s+r+\mu}_{\alpha,\beta}\mathcal{E}(n,p),$$

(for both cases, $\mathcal{R}$ and $\mathcal{C}$, as in part 2). Now from the hypothesis $\lambda \leq r + \mu$ it follows that $F^{s+\lambda}_{\alpha,\beta}\mathcal{E}(n,p) \supset F^{s+r+\mu}_{\alpha,\beta}\mathcal{E}(n,p)$ and the action of the quotient group is well-defined. □

Corollary 2.36 *In particular, for* $s = 0$, $\lambda = k + 1$, $r \geq 0$ *and* $\mu = k + 1 - r \geq 0$, *there is an action of the quotient group (Lie group, in fact)* $F^r\mathcal{G}/F^{k+1}\mathcal{G}$ *on* $F^0_{\alpha,\beta}\mathcal{E}(n,p)/F^{k+1}_{\alpha,\beta}\mathcal{G}$.

The complete transversal theorem requires $r > 0$, the most significant case being $r = 1$ giving the group $F^1\mathcal{G}/F^{k+1}\mathcal{G}$. This is analogous to the $\mathcal{G}_1$ complete transversal theorem, Corollary 2.4, where the standard filtration by degree is used, only now we have the weaker criterion of respecting a weighted filtration and use a 'larger' group, $F^1\mathcal{G}$ (that is we use $L = L(F^1\mathcal{G})$). To establish the complete transversal theorem for weighted filtrations it is simply a matter of putting together the above results. In the following $\mathcal{G}$ is one of the standard Mather groups; $J^k(n,p)$ denotes the weighted jet-space $F^0_{\alpha,\beta}\mathcal{E}(n,p)/F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p)$;

H^k denotes the subspace of weighted homogeneous jets of degree k, that is the projection of $F^k_{\alpha,\beta}\mathcal{E}(n,p)$ into $J^k(n,p)$; and $J^k\mathcal{G}$ denotes the weighted jet-group $F^1\mathcal{G}/F^{k+1}\mathcal{G}$.

Theorem 2.37 *Let $\mathcal{G}$ be one of the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_p)$ be sequences of weights for $m_n.\mathcal{E}(n,p)$. Then $(\{F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p)\}, \{F^{k+1}_{\alpha,\beta}\mathcal{G}\})$ is a jet filtration. In addition, if $f \in F^0_{\alpha,\beta}\mathcal{E}(n,p)$, $k \geq 1$ and T a subspace of $H^{k+1} \subset J^{k+1}(n,p)$ such that*

$$J^{k+1}(L(F^1\mathcal{G})) \cdot j^{k+1}f + T \supset H^{k+1},$$

we have any (weighted) k-jet j^kg with $j^kg \sim_{J^k\mathcal{G}} j^kf$ has $(k+1)$-jet $j^{k+1}g \sim_{J^{k+1}\mathcal{G}} j^{k+1}f + t$ for some $t \in T$. Such a space T will be referred to as a complete transversal.

Proof. We will sketch the proof. $J^k(n,p) = F^0_{\alpha,\beta}\mathcal{E}(n,p)/F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p)$ is a finite dimensional vector space since $F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p)$ contains $m_n^N.\mathcal{E}(n,p)$ for large enough N; this quotient is therefore just a 'refinement' of the space of map-germs truncated by degree to one truncated by weight. Similarly $J^k\mathcal{G} = F^1\mathcal{G}/F^{k+1}\mathcal{G}$ is a Lie group of weighted homogeneous diffeomorphisms truncated by weight (refer to definition 2.29 and definition 2.31). That $J^k\mathcal{G}$ acts on $J^k(n,p)$ follows from Corollary 2.36. This action is smooth, being described locally by the composition of polynomial map-germs truncated by weight. We omit the details of the above remarks; the case $\mathcal{G} = \mathcal{R}$ has already been dealt with by Arnol'd, [A1]. Since $\mathcal{G}$ is jet-closed we have $J^s(L\mathcal{G}) \subset L(J^s\mathcal{G})$ for all s. (Earlier we stated that $\mathcal{G}$ is jet-closed with respect to the standard filtration by degree, however, the result holds for weighted filtrations. Alternatively, note that $J^s(L\mathcal{G}) \subset L(J^s\mathcal{G})$ is the important property and follows in the same way as Lemma 5.12.) Define L to be the Lie algebra $L(F^1\mathcal{G}) \subset L\mathcal{G}$. Now for all s, $J^sL \subset L(J^s\mathcal{G})$ is a Lie subalgebra, and for $f \in F^0_{\alpha,\beta}\mathcal{E}(n,p)$, $h \in F^s_{\alpha,\beta}\mathcal{E}(n,p)$ and $l \in L$ we have

$$l \cdot (f + h) - l \cdot f \in F^{s+1}_{\alpha,\beta}\mathcal{E}(n,p)$$

from the approximation lemma, Lemma 2.33 part (3). The hypotheses of the generalised complete transversal theorem 2.10 hold and the result follows. □

The following weighted determinacy result follows as a corollary.

Corollary 2.38 *Let $\mathcal{G}$, α and β be as in Theorem 2.37. Then a map-germ $f \in F^0_{\alpha,\beta}\mathcal{E}(n,p)$ is (weighted) k-$\mathcal{G}$-determined (i.e., $F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p)$-$\mathcal{G}$-determined) if*

$$F^{k+1}_{\alpha,\beta}\mathcal{E}(n,p) \subset L(F^1\mathcal{G}) \cdot f.$$

Proof. The proof follows along the same lines as that of Corollary 2.23. We merely observe the hypothesis implies f is finitely $\mathcal{G}$-determined (with respect to standard degree). Also that, with respect to the weighted filtration, the $(k+1)$ and higher complete transversals are all empty. The result therefore follows by induction on weighted degree. □

Chapter 3

Classification of Map-Germs from Surfaces to Four-Space

3.1 Introduction

In this chapter we are concerned with the classification of singularities $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ under $\mathcal{A}$ equivalence. We use the 'nilpotent' classification methods discussed in [BduPW] and Chapter 2; that is the use of nilpotent Lie algebras in the determinacy and complete transversal calculations. The majority of the calculations were done by computer using a package written in Maple. This package can deal with many different types of classification problem and is discussed in detail in Chapter 6.

Our original aim was the classification of the simple singularities. This was a joint project with J.M. West who was responsible for most of the initial results in this area, performing the classification using the same methods, but by hand calculations, [We]. However, the complexity of the calculations became such that our computer techniques were more suitable, and it was then easy to extend the classification well beyond the simple singularities. To this extent it is hard to find a natural stopping point for the classification; our criterion is to discover stems and 'series'. Such objects are well known in singularity theory and date back to Arnold, see [AGV] for a general reference; they are of great interest as the work of Ratcliffe on stems of map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ has shown, [Rat1, Rat2]. So, once a non-simple jet was been discovered in the classification, we will continue and classify its orbits in higher jet-spaces in the search for series. Some jets are very amenable at providing series, while others give rise to an extremely complicated structure of orbits at a higher jet level — such jets are then excluded from further consideration once we know they are not simple. For example, there are *fifteen* $J^5\mathcal{A}$-orbits over the 4-jet $(x, y^2, 0, 0)$, excluding factors of ± 1 (an $\mathcal{A}_1$-5-transversal contains *six* terms). Having identified all these orbits it is easy to show they are non-simple, and we prefer not to investigate them further.

Our classification provides a list of all the simple singularities; an extensive list of series; a complete stratification of the jet-space and a list of all the singularities with corank 1 and codimension less than or equal to 11. Our general interest was in identifying the simple singularities and series. In several cases, where it was appropriate to halt the classification, the $J^k\mathcal{A}$-codimension was 11 — this is where the number comes from. Although 11 is a general upper bound on the codimension, note that many of the cases were pursued to a lot higher value. The classification of the corank 1 singularities up to codimension 11 gives a wealth of examples of series and stems (one of our main objectives) and includes a classification of strata up to codimension 9, taking into account modality. For completeness we also consider the corank 2 case. Taking the classification beyond the 2-jet level leads to extremely complicated orbits and, with no specific applications to guide us, we do not discuss the classification past this stage. Overall we obtain a classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ up to codimension 8; this should be sufficient to cover most applications in geometry.

It was decided to perform all of the calculations using the computer; this not only provides an independent check and comparison with the results of [We], but also demonstrates that the computer methods are suitable for performing the classification in its entirety. The results below just give a summary of the computer calculations and how the classification proceeds. We remark that when we obtain a series of singularities using the computer results, we actually just obtain specific members of the series. That is, the computer can only allow us to conjecture the existence of a general series. To prove the series exists we must resort to hand calculation; though the computer results allow us to guess at the form of the tangent space, making the hand calculations a lot easier. We reproduce some of these calculations below. Several of the calculations are due to West, when this is the case we simply refer there for the details. Since our classification mainly consists of a summary of computer results, such hand calculations are our only description of how the classification method works. For a more detailed description of the classification method we refer to [We] where all the calculations were done by hand.

3.2 Classification Techniques

Consider map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ where we shall denote the source coordinates by (x, y) and the target coordinates by (u_1, u_2, u_3, u_4). Throughout the classification we shall use the subgroup $\mathcal{G}$ of $\mathcal{A}$ with Lie algebra

$$\begin{aligned} L\mathcal{A}_1 &\oplus Sp\{x\partial/\partial y\} \\ &\oplus Sp\{u_2\partial/\partial u_1, u_3\partial/\partial u_1, u_4\partial/\partial u_1, u_3\partial/\partial u_2, u_4\partial/\partial u_2, u_4\partial/\partial u_3\} \end{aligned}$$

which we shall denote $L\mathcal{G}$. This Lie algebra is nilpotent and $\mathcal{G}$ is unipotent (in the sense of Section 2.3) and we may apply the determinacy criterion of [BduPW] and the complete transversal techniques of Section 2.3. (Using these techniques we can work solely with the Lie algebra L and need not concern ourselves with the existence of the unipotent group $\mathcal{G}$. However, one can easily construct such a $\mathcal{G}$, hence the occurrence of the term $L\mathcal{G}$ in place of L throughout these calculations.)

Theorem 3.1 *A map-germ $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ is k-$\mathcal{G}$-determined if and only if*

$$m_2^{k+1}.\mathcal{E}(2,4) \subset L\mathcal{G} \cdot f + m_2^{k+1}.f^*(m_4).\mathcal{E}(2,4) + m_2^{2k+2}.\mathcal{E}(2,4).$$

Proof. This is just Lemma (2.6) applied to Theorem (2.1) of [BduPW]. See also Lemma 1.2 and Corollary 2.23 in this thesis. □

The $m_2^{k+1}.f^*(m_4).\mathcal{E}(2,4)$ terms allow us to reduce the degree of the jet-space needed for the determinacy calculation from $2k+1$; this is extremely important when using the computer.

For the complete transversal techniques we apply Theorem 2.22 using the above Lie algebra $L\mathcal{G}$ in the calculations. That is, we need to calculate $L\mathcal{G} \cdot f$ in the jet-spaces defined by the filtration

$$M_{r,s}(\mathcal{G}) = \sum_{i \geq s} (L\mathcal{G})^i \cdot (m_2^r.\mathcal{E}(2,4)) + m_2^{r+1}.\mathcal{E}(2,4).$$

Then, by the (r, s)-jet-space we mean the quotient $m_2.\mathcal{E}(2,4)/M_{r,s}(\mathcal{G})$; and by the space of homogeneous terms of degree (r, s) we mean the image of the space $M_{r,s-1}(\mathcal{G})$ in this quotient, and will denote it $H^{r,s}$. A complete transversal is then a subspace of T of $H^{r,s}$ satisfying $J^{r,s}(L\mathcal{G}) \cdot j^{r,s}f + T \supset H^{r,s}$. An example for the lower degree jet-spaces should clarify this. Each (r, s)-jet-space is just a refinement of the standard r-jet-space (by degree) and the best way to describe these spaces is to list the generators for each of the spaces $H^{r,s}$ (these just give the extra monomials which arise when passing from the $(r, s-1)$-jet-space to the (r, s)-jet-space). Calculation of $M_{r,s}$ gives the following.

(r,s)	Basis for $H^{r,s}$	Weight†
$(1,0)$	$\{0\}$	
$(1,1)$	$\{(0,0,0,y)\}$	1
$(1,2)$	$\{(0,0,y,0),(0,0,0,x)\}$	2
$(1,3)$	$\{(0,y,0,0),(0,0,x,0)\}$	3
$(1,4)$	$\{(y,0,0,0),(0,x,0,0)\}$	4
$(1,5)$ or $(2,0)$	$\{(x,0,0,0)\}$	5
$(2,1)$	$\{(0,0,0,y^2)\}$	2
$(2,2)$	$\{(0,0,y^2,0),(0,0,0,xy)\}$	3
$(2,3)$	$\{(0,y^2,0,0),(0,0,xy,0),(0,0,0,x^2)\}$	4
$(2,4)$	$\{(y^2,0,0,0),(0,xy,0,0),(0,0,x^2,0)\}$	5
$(2,5)$	$\{(xy,0,0,0),(0,x^2,0,0)\}$	6
$(2,6)$ or $(3,0)$	$\{(x^2,0,0,0)\}$	7

(†) The 'weight' column refers to the use of weights to partition the monomial vectors into their (r,s)-levels; see Section 6.7.5.

The pattern continues as one would expect for the higher degree jet-spaces, the generators for the $H^{r,s}$ spaces starting with $\{(0,0,0,y^r)\}$ for $s=1$ and finishing with $\{(x^r,0,0,0)\}$ for large enough s. This table should be kept in mind in the classification as we will no longer explicitly refer to the $H^{r,s}$ spaces and list their generators, but just state the (r,s)-transversal.

Remark. In this and later chapters we will specify a complete transversal by listing a set of basis vectors for the transversal.

A few other classification techniques will be required. The Mather Lemma, [MathIV, Lemma 3.1] or [BduPW, Lemma 1.1], can be applied to check equivalence of jets in a given family. The $M_{r,s}$ filtration always proved fine enough, but sometimes applying the Mather Lemma provides a useful check. (The conditions of the Mather Lemma are hard to check by hand, but it is one of the functions of our Maple classification package to perform these checks.) Explicit 'scaling' coordinate changes in the source and target are often required to reduce a whole family of jets to one of a few possible normal forms. Sometimes these are trivial, while in many cases it is best to apply the technique discussed in

Proposition 1.11 which reduces the problem to a simple linear one. Finally, in some cases where it is impossible to reduce a family of germs to a finite number of normal forms using scaling coordinate changes, one suspects the presence of a modulus. To prove there is a modulus we use the moduli detection criterion, Theorem 1.9. Once again, the relevant conditions are hard to check by hand but can be checked using a function from our classification package.

Note that our classification only determines a list of the possible simple singularities; 'simple' being in the sense of Arnol'd, see Section 1.3.6. Many of the singularities can be ruled out during the classification; but the question of which of the remaining are actually simple, and the search for invariants for these, is postponed until Chapter 4.

3.3 Classification

The notation used below will be as follows. We will denote a germ $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ by f and use (x, y) as source coordinates. The jet-space $J^r(2,4)$ will be identified with the space of all quadruples of polynomials in $\mathbf{R}[x, y]$ truncated to degree r; the (r, s)-jet-spaces induced by the $M_{r,s}(\mathcal{G})$-filtration being identified with quadruples of polynomials similarly, only we now truncate at some sub-level within the homogeneous polynomials of degree r according to the value of s. We will often use the notation f to refer to both a germ, and its image in some jet-space, as the context should always be clear.

Our tangent spaces will be subspaces of θ_f (the space of all germs at 0 of vector fields along f; [MathIII]). In particular, we consider the group $\mathcal{A}$, its subgroup consisting of germs with 1-jet the identify $\mathcal{A}_1$, and their tangent spaces $L\mathcal{A}$ and $L\mathcal{A}_1$. The tangent spaces to the orbit of f under these groups are denoted by $L\mathcal{A} \cdot f$ and $L\mathcal{A}_1 \cdot f$ (here we are adopting the notation of [BduPW], which places emphasis on the Lie algebras, instead of the older notation $T\mathcal{A} \cdot f$ and $T\mathcal{A}_1 \cdot f$). In the notation of [MathIII] we have

$$\begin{aligned} L\mathcal{A} \cdot f &= tf(m_2.\theta_2) + wf(m_4.\theta_4), \\ L\mathcal{A}_1 \cdot f &= tf(m_2^2.\theta_2) + wf(m_4^2.\theta_4), \end{aligned}$$

where m_2, m_4 are the maximal ideals of the local rings $\mathcal{E}_2, \mathcal{E}_4$ respectively, and θ_2, θ_4 denote the spaces $\theta_{1_{R^2}}, \theta_{1_{R^4}}$ of vector fields along the identity maps in the source and target respectively. However, referring to our remarks on the basics of singularity theory in Chapter 1, we see that in this local theory, θ_f is a free

$\mathcal{E}_2$-module of rank 4 and we can (and will) identify θ_f with $\mathcal{E}(2,4)$. In coordinate form our tangent spaces then become the following subspaces of $\mathcal{E}(2,4)$.

$$\begin{aligned} L\mathcal{A}\cdot f &= m_2.\left\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\}+m_4.\{e_1,e_2,e_3,e_4\}, \\ L\mathcal{A}_1\cdot f &= m_2^2.\left\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\}+m_4^2.\{e_1,e_2,e_3,e_4\}, \end{aligned}$$

where the first summand is an $\mathcal{E}_2$-module, $\{e_1,e_2,e_3,e_4\}$ denote the standard basis vectors in $\mathbf{R}^4$ (considered as a subspace of $\mathcal{E}(2,4)$), and the second summand is an $\mathcal{E}_4$-module via $f^*:\mathcal{E}_4\longrightarrow\mathcal{E}_2$. The tangent space $L\mathcal{G}\cdot f$ is calculated using the Lie algebra $L\mathcal{G}$ given in Section 3.2 and is just an extension of the $L\mathcal{A}_1\cdot f$ tangent space by vectors from $L\mathcal{A}\cdot f$.

The jet-groups are Lie groups and act smoothly on the jet-spaces. Projecting the above tangent spaces onto the jet-spaces gives the corresponding tangent spaces to the orbits of the jet-groups. Since $J^r(2,4)$ is a vector space we can identify it with its tangent space and the above tangent spaces are then just subspaces of $J^r(2,4)$. We can (and will) identify these tangent spaces with subspaces of $J^r(2,4)$ and consider then as a subspace of quadruples of truncated polynomials in $\mathbf{R}[x,y]$.

The above remarks and identifications will be used without further qualification in the calculations below. Our computer package makes use of these identifications, essentially carrying out extensive polynomial manipulations.

We will use the term 'series' below loosely, the specific cases qualifying the situation. In each case we prove the existence of a stem f and give a list of all the jets over this jet (the exception is, of course, the set of all jets with the same ∞-jet as f; these jets are of infinite codimension and are implicitly excluded from further consideration).

The classification below generally applies to the $\mathcal{A}$-classification of map-germs $(\mathbf{C}^2,0)\longrightarrow(\mathbf{C}^4,0)$. There is some collapsing of the orbits; in particular, many of the orbits which differ just by a $\pm$ sign in a term become a single orbit (after simple 'scaling' coordinate changes in the source and target).

The following theorem summarises our classification. In particular, the extent to which the jet spaces have been stratified is described, together with a list of all the singularities (finitely determined jets) obtained. We would like to summarise the stratification using a 'tree diagram' akin to those found in the classifications of [Mol, Mo2, Hob, Rie], etc.; however, the extent of our results make this infeasible and we prefer to list the $J^k(2,4)$-orbits in a suitable tabular format for each successive k.

Theorem 3.2 *The following tables give, for each k, a list of $\mathcal{A}$-invariant strata for the jet-space $J^k(2,4)$. These are the $J^k\mathcal{A}$-orbits, or unions of such orbits when moduli occur, which project down onto the $J^{k-1}\mathcal{A}$-orbits obtained at the previous level. For ease of reference between the $J^{k-1}\mathcal{A}$-orbits and the $J^k\mathcal{A}$-orbits we employ the following system. Each $J^{k-1}\mathcal{A}$-orbit is labelled with a capital letter, for example 'A'. This label will appear in the first column of the table of $J^k\mathcal{A}$-orbits (this immediately follows the table of $J^{k-1}\mathcal{A}$-orbits), thus highlighting the $J^k\mathcal{A}$-orbits over 'A'.*

As well as for determined jets, there are several other occasions when we shall not list the $J^k\mathcal{A}$-orbits over a given $(k-1)$-jet. These include cases where the jet is a stem and the orbits of higher degree form a natural 'series' — we just state the series rather than including each individual jet in the succeeding tables. In several cases we choose to stop the classification for a jet at a given level. If the higher degree orbits for a given jet are not listed in the subsequent table the following notation is employed. Either the capital letter which labels the jet will be appended with the symbol $*$ to refer the reader to comments immediately following the table; or the symbol - will appear in the 'label' column to indicate that the classification was taken no further for the particular jet.

Determined jets are indicated by the appearance of the determinacy degree in the column marked 'det'. In the cases where the orbits over a jet form a series and are commented upon immediately after the table we indicate this with the symbol • in the 'det' column. Looking down the 'det' columns of the tables therefore provides a list of all the finitely determined jets obtained and we will not repeat such a list elsewhere.

For each $J^k\mathcal{A}$-orbit we state the $J^k\mathcal{A}$-codimension; when the corresponding jet is k-determined this is the $\mathcal{A}$-codimension of the singularity. The coefficients a, b, c, appearing in some jets are moduli (that these are genuine moduli is checked using computer calculation and Theorem 1.9). In such cases the $J^k\mathcal{A}$-codimension of a representative of the stratum is given; the $J^k\mathcal{A}$-codimension of the stratum is just this stated codimension minus the number of moduli.

The following tables provide a list of all the singularities of codimension less than or equal to 8, together with an $\mathcal{A}$-invariant stratification of the space of all jets of codimension less than or equal to 8. If one restricts to the corank 1 case *this extends to codimension 11.*

1-Jets				
	det	stratum	codim	label
	1	$(x, y, 0, 0)$	0	-
	-	$(x, 0, 0, 0)$	3	A
	-	$(0, 0, 0, 0)$	8	B

2-Jets				
	det	stratum	codim	label
A	•	$(x, y^2, xy, 0)$	3	A*
	-	$(x, y^2, 0, 0)$	5	B
	-	$(x, xy, 0, 0)$	6	C
	-	$(x, 0, 0, 0)$	9	D
B	-	$(y^2, xy, x^2, 0)$	8	-
	-	$(y^2, x^2, 0, 0)$	10	-
	-	$(y^2, xy, 0, 0)$	11	-
	-	$(xy, 0, 0, 0)$	14	-
	-	$(y^2, 0, 0, 0)$	15	-
	-	$(0, 0, 0, 0)$	20	-

A: *classification of the jets over* $(x, y^2, xy, 0)$ *gives the series*

(x, y^2, xy, y^{2k+1}) $(2k+1)$-*determined*, $k \geq 1$,
codim $= k + 2$.

3-Jets				
	det	stratum	codim	label
B	3	(x, y^2, y^3, x^2y)	5	-
	•	$(x, y^2, y^3 \pm x^2y, 0)$	6	A*
	•	$(x, y^2, y^3, 0)$	7	B*
	-	$(x, y^2, x^2y, 0)$	7	C
	-	$(x, y^2, 0, 0)$	9	D
C	•	(x, xy, y^3, xy^2)	6	E*
	-	$(x, xy, y^3, 0)$	7	F
	-	$(x, xy + y^3, xy^2, 0)$	8	G
	-	$(x, xy + y^3, 0, 0)$	10	H
	-	$(x, xy, xy^2, 0)$	9	I
	-	$(x, xy, 0, 0)$	11	J

3-Jets (continued)				
	det	stratum	codim	label
D	-	(x, y^3, xy^2, x^2y)	9	K
	-	$(x, y^3 \pm x^2y, xy^2, 0)$	10	L
	-	$(x, y^3, xy^2, 0)$	11	-
	-	$(x, y^3, x^2y, 0)$	11	-
	-	$(x, y^3 \pm x^2y, 0, 0)$	13	-
	-	$(x, y^3, 0, 0)$	14	-
	-	$(x, xy^2, x^2y, 0)$	12	-
	-	$(x, xy^2, 0, 0)$	14	-
	-	$(x, x^2y, 0, 0)$	15	-
	-	$(x, 0, 0, 0)$	18	-

A: *classification of the jets over* $(x, y^2, y^3 \pm x^2y, 0)$ *gives the series*

$(x, y^2, y^3 \pm x^2y, x^ky)$ — $(k+1)$-*determined,* $k \geq 3$, *codim* $= k+3$.

B: *classification of jets over* $(x, y^2, y^3, 0)$ *gives the series*

(x, y^2, y^3, x^ky) — $(k+1)$-*determined,* $k \geq 2$, *codim* $= 2k+1$,

$(x, y^2, y^3 \pm x^jy, x^ky)$ — $(k+1)$-*determined,* $j \geq 2, \quad k \geq j+1$, *codim* $= j+k+1$.

E: *classification of jets over* (x, xy, y^3, xy^2) *gives the series*

$(x, xy, y^3, xy^2 + y^{3k+1})$ — $(3k+1)$-*determined,* $k \geq 1$, *codim* $= 3k+3$,

$(x, xy, y^3, xy^2 + y^{3k+2})$ — $(3k+2)$-*determined,* $k \geq 1$, *codim* $= 3k+4$,

$(x, xy + y^{3k+2}, y^3, xy^2)$ — $(3k+2)$-*determined,* $k \geq 1$, *codim* $= 3k+5$.

4-Jets				
	det	stratum	codim	label
C	•	(x, y^2, x^2y, xy^3)	7	A*
	-	$(x, y^2, x^2y, 0)$	8	B
D	•	(x, y^2, xy^3, x^3y)	9	C*
	•	$(x, y^2, xy^3 \pm x^3y, 0)$	10	D*
	-	$(x, y^2, xy^3, 0)$	11	E
	-	$(x, y^2, x^3y, 0)$	11	F
	-	$(x, y^2, 0, 0)$	13	G
F	4	(x, xy, y^3, y^4)	7	-
	-	$(x, xy, y^3, 0)$	8	H
G	•	$(x, xy + y^3, xy^2, y^4)$	8	I*
	-	$(x, xy + y^3, xy^2 + ay^4, xy^3)$	10	J
	-	$(x, xy + y^3, xy^2 + ay^4, 0)$	11	K
H	-	$(x, xy + y^3, y^4, xy^3)$	10	L
	-	$(x, xy + y^3, y^4, 0)$	11	M
	-	$(x, xy + y^3, xy^3, 0)$	12	N
	-	$(x, xy + y^3, 0, 0)$	14	-
I	-	(x, xy, xy^2, y^4)	9	O
	-	$(x, xy, xy^2 + y^4, xy^3)$	10	P
	-	$(x, xy, xy^2 + y^4, 0)$	11	Q
	-	$(x, xy + y^4, xy^2, xy^3)$	11	R
	-	(x, xy, xy^2, xy^3)	12	S
	-	$(x, xy + y^4, xy^2, 0)$	12	T
	-	$(x, xy, xy^2, 0)$	13	U
J	-	(x, xy, y^4, xy^3)	11	-
	-	$(x, xy, y^4, 0)$	12	-
	-	$(x, xy + y^4, xy^3, 0)$	13	-
	-	$(x, xy + y^4, 0, 0)$	15	-
	-	$(x, xy, xy^3, 0)$	14	-
	-	$(x, xy, 0, 0)$	16	-
K	-	$(x, y^3, xy^2, x^2y + y^4)$	9	V
	-	$(x, y^3, xy^2 + y^4, x^2y)$	10	W
	-	(x, y^3, xy^2, x^2y)	11	-
L	-	$(x, y^3 \pm x^2y, xy^2, y^4 + ax^3y)$	11	-
	-	$(x, y^3 \pm x^2y, xy^2 + y^4, x^3y)$	11	-
	-	$(x, y^3 \pm x^2y, xy^2 + y^4, 0)$	12	-
	-	$(x, y^3 \pm x^2y, xy^2, x^3y)$	12	-
	-	$(x, y^3 \pm x^2y, xy^2, 0)$	13	-

A: *classification of jets over* (x, y^2, x^2y, xy^3) *gives the series*

$(x, y^2, x^2y \pm y^{2k+1}, xy^3)$ — $(2k+1)$-*determined*, $k \geq 2$, *codim* $= k+5$.

C: *classification of jets over* (x, y^2, xy^3, x^3y) *gives the series*

$(x, y^2, xy^3, x^3y + y^{2k+1})$ — $(2k+1)$-*determined*, $k \geq 2$, *codim* $= k+7$.

D: *classification of jets over* $(x, y^2, xy^3 \pm x^3y, 0)$ *gives the series*

$$
\begin{array}{ll}
(x, y^2, xy^3 \pm x^3y, y^{2k+1} + ax^{2k}y) & (i), \\
(x, y^2, xy^3 \pm x^3y, y^{2k+1} + ax^{2k}y \pm x^jy) & (ii), \\
(x, y^2, xy^3 \pm x^3y, x^{2k}y \pm y^{2j+1}) & (iii), \\
(x, y^2, xy^3 \pm x^3y, x^{2k+1}y + y^{2j+1}) & (iv),
\end{array}
$$

where

- *(i)* $(2k+1)$-*determined*, $k \geq 2$,
  $a \neq -1$ *(k even)*, $a \neq \pm 1$ *(*± 1 *respective of* $\pm x^3y$*) (k odd)*,
  codim $= 3k+5$;
- *(ii)* $(j+1)$-*determined*, $k \geq 2$, $j \geq 2k+1$,
  $a = -1$ *(k even)*, $a = \pm 1$ *(*± 1 *respective of* $\pm x^3y$*) (k odd)*,
  codim $= k+j+4$;
- *(iii)* $(2j+1)$-*determined*, $k \geq 2$, $j \geq k+1$,
  codim $= 2k+j+4$;
- *(iv)* $(2j+1)$-*determined*, $k \geq 2$, $j \geq k+1$,
  codim $= 2k+j+5$.

Note that in case (ii) the codimension is just $(3k+5)+(j-(2k+1)) = k+j+4$.

I: *classification of jets over* $(x, xy+y^3, xy^2, y^4)$ *gives the series*

$(x, xy+y^3, xy^2+y^{2k+1}, y^4)$ — $(2k+1)$-*determined*, $k \geq 2$, *codim* $= k+6$.

5-Jets				
	det	stratum	codim	label
B	5	(x, y^2, x^2y, y^5)	8	-
	•	$(x, y^2, x^2y \pm y^5, 0)$	9	A*
	-	$(x, y^2, x^2y, 0)$	10	B
E	5	$(x, y^2, xy^3, y^5 \pm x^4y)$	11	-
	-	$(x, y^2, xy^3 + x^4y, y^5)$	12	C
	•	(x, y^2, xy^3, y^5)	13	D*
	•	(x, y^2, xy^3, x^4y)	12	E*
	-	$(x, y^2, xy^3 + x^4y, 0)$	13	F
	-	$(x, y^2, xy^3, 0)$	14	G
F	5	$(x, y^2, x^3y, y^5 \pm x^2y^3)$	11	-
	5	(x, y^2, x^3y, y^5)	12	-
	5	$(x, y^2, x^3y + y^5, x^2y^3)$	12	-
	•	$(x, y^2, x^3y + y^5, 0)$	13	H*
	-	(x, y^2, x^3y, x^2y^3)	13	I
	-	$(x, y^2, x^3y, 0)$	14	J
G	5	$(x, y^2, y^5 + ax^4y, x^2y^3 \pm x^4y)$ $a \neq 0, -1$	14	-
	-	$(x, y^2, y^5, x^2y^3 \pm x^4y)$	14	-
	-	$(x, y^2, y^5 - x^4y, x^2y^3 \pm x^4y)$	14	-
	5	$(x, y^2, y^5 \pm x^4y, x^2y^3)$	14	-
	-	(x, y^2, y^5, x^2y^3)	15	-
	5	$(x, y^2, y^5 \pm x^2y^3, x^4y)$	14	-
	-	$(x, y^2, y^5 \pm x^2y^3 + ax^4y, 0)$	16	-
	-	(x, y^2, y^5, x^4y)	15	-
	-	$(x, y^2, y^5 \pm x^4y, 0)$	16	-
	-	$(x, y^2, y^5, 0)$	17	-
	-	(x, y^2, x^2y^3, x^4y)	15	-
	-	$(x, y^2, x^2y^3 \pm x^4y, 0)$	16	-
	-	$(x, y^2, x^2y^3, 0)$	17	-
	-	$(x, y^2, x^4y, 0)$	17	-
	-	$(x, y^2, 0, 0)$	19	-
H	5	(x, xy, y^3, y^5)	8	-
	-	$(x, xy + y^5, y^3, 0)$	9	K
	-	$(x, xy, y^3, 0)$	10	L
J	5	$(x, xy + y^3, xy^2 + ay^4, xy^3 + by^5)$ for generic (a, b) — see Section 3.3.7 Part (14)	11	-
K	5	$(x, xy + y^3, xy^2 + ay^4, y^5)$ for generic a — see Section 3.3.7 Part (15)	11	-
	-	$(x, xy + y^3, xy^2 + ay^4, 0)$	12	-

5-Jets (continued)				
	det	stratum	codim	label
L	-	$(x, xy + y^3, y^4, xy^3 + ay^5)$	11	M
M	5	$(x, xy + y^3, y^4, y^5 \pm x^2y^3)$	11	-
	5	$(x, xy + y^3, y^4, y^5)$	12	-
	-	$(x, xy + y^3, y^4, x^2y^3)$	12	N
	-	$(x, xy + y^3, y^4, 0)$	13	O
N	-	$(x, xy + y^3, xy^3 + xy^4, y^5)$	12	-
	-	$(x, xy + y^3, xy^3, y^5)$	13	-
	-	$(x, xy + y^3, xy^3 + ay^5, xy^4)$	14	-
	-	$(x, xy + y^3, xy^3 + ay^5, 0)$	15	-
	-	$(x, xy + y^3, xy^3 + \frac{6}{5}y^5 + xy^4, 0)$	15	-
O	-	$(x, xy, xy^2 + y^5, y^4)$	9	P
	-	(x, xy, xy^2, y^4)	10	Q
P	-	$(x, xy, xy^2 + y^4, xy^3 + ay^5)$	11	R
Q	-	$(x, xy, xy^2 + y^4, y^5)$	11	S
	-	$(x, xy, xy^2 + y^4, 0)$	12	-
R	-	$(x, xy + y^4, xy^2, xy^3 + y^5)$	11	T
	-	$(x, xy + y^4, xy^2 + ay^5, xy^3)$ $a \neq -\frac{1}{2}$	13	-
	-	$(x, xy + y^4, xy^2 - \frac{1}{2}y^5, xy^3)$	14	-
S	-	$(x, xy, xy^2, xy^3 + y^5)$	12	-
	-	$(x, xy, xy^2 + y^5, xy^3)$	13	-
	-	$(x, xy + y^5, xy^2, xy^3)$	14	-
	-	(x, xy, xy^2, xy^3)	15	-
T	-	$(x, xy + y^4, xy^2, y^5)$	12	U
	-	$(x, xy + y^4, xy^2 + ay^5, xy^4)$ $a \neq -2$	14	-
	-	$(x, xy + y^4, xy^2 - 2y^5, xy^4)$	15	-
	-	$(x, xy + y^4, xy^2 + ay^5, 0)$ $a \neq -2$	15	-
	-	$(x, xy + y^4, xy^2 - 2y^5, 0)$	16	-
U	-	(x, xy, xy^2, y^5)	13	-
	-	$(x, xy, xy^2 + y^5, xy^4)$	14	-
	-	$(x, xy, xy^2 + y^5, 0)$	15	-
	-	$(x, xy + y^5, xy^2, xy^4)$	15	-
	-	(x, xy, xy^2, xy^4)	16	-
	-	$(x, xy + y^5, xy^2, 0)$	16	-
	-	$(x, xy, xy^2, 0)$	17	-

5-Jets (continued)				
	det	stratum	codim	label
V	5	$(x, y^3, xy^2, x^2y + y^4 \pm y^5)$	9	-
	5	$(x, y^3, xy^2, x^2y + y^4)$	10	-
W	5	$(x, y^3, xy^2 + y^4, x^2y + ay^5)$ $a \neq 1$	11	-
	-	$(x, y^3, xy^2 + y^4, x^2y + y^5)$	12	-

A: *classification of jets over* $(x, y^2, x^2y \pm y^5, 0)$ *gives the series*

$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}) \qquad (i),$$
$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3}) \qquad (ii),$$
$$(x, y^2, x^2y \pm y^5, y^{2k+1}) \qquad (iii),$$

where

(i) $(2k+3)$-*determined*, $k \geq 2$, $a^2 \pm 1 \neq 0$, *codim* $= 2k + 6$;

(ii) $(2j+3)$-*determined*, $k \geq 2$, $j \geq k+1$, $a^2 \pm 1 = 0$, *codim* $= k + j + 5$;

(iii) $(2k+1)$-*determined*, $k \geq 3$, *codim* $= 2k + 4$.

Note that in case (ii) the codimension is just $(2k+6) + (j - (k+1)) = k + j + 5$.

D: *classification of jets over* (x, y^2, xy^3, y^5) *gives the series*

$(x, y^2, xy^3, y^5 \pm x^k y)$	$(k+1)$-*determined*, *codim* $= 2k+3$,
$(x, y^2, xy^3 \pm x^k y, y^5 \pm x^{k+1} y)$	$(k+2)$-*determined*, *codim* $= 2k+4$,
$(x, y^2, xy^3 \pm x^k y, y^5 \pm x^{k+2} y)$	$(k+3)$-*determined*, *codim* $= 2k+5$,
$\vdots$	$\vdots$
$(x, y^2, xy^3 \pm x^k y, y^5 \pm x^{2k-3} y)$	$(2k-2)$-*determined*, *codim* $= 3k$,
$(x, y^2, xy^3 \pm x^k y, y^5 + ax^{2k-2} y)$	$(2k-1)$-*determined*, $a \neq -1$, *codim* $= 3k+2$,
$(x, y^2, xy^3 \pm x^k y, y^5 - x^{2k-2} y \pm x^j y)$	$(j+1)$-*determined*, $j \geq 2k-1$, *codim* $= k+j+3$,

where $k \geq 4$. *In the final case the codimension is just* $(3k+2)+(j-(2k-1)) = k+j+3$.

E: *classification of jets over* (x, y^2, xy^3, x^4y) *gives the series*

$(x, y^2, xy^3, x^4y \pm y^{2k+1})$	$(2k+1)$-*determined*, $k \geq 3$, *codim* $= k+9$.

H: *classification of jets over* $(x, y^2, x^3y + y^5, 0)$ *gives the series*

$(x, y^2, x^3y + y^5, xy^{2k+1} \pm y^{2k+3})$	$(2k+3)$-*determined*, $k \geq 2$, *codim* $= 3k+7$,
$(x, y^2, x^3y + y^5, xy^{2k+1})$	$(2k+3)$-*determined*, $k \geq 2$, *codim* $= 3k+8$,
$(x, y^2, x^3y + y^5, y^{2k+3} \pm x^2y^{2k+1})$	$(2k+3)$-*determined*, $k \geq 2$, *codim* $= 3k+8$,
$(x, y^2, x^3y + y^5, y^{2k+3})$	$(2k+3)$-*determined*, $k \geq 2$, *codim* $= 3k+9$,
$(x, y^2, x^3y + y^5, x^2y^{2k+1} \pm xy^{2k+3})$	$(2k+4)$-*determined*, $k \geq 2$, *codim* $= 3k+9$,
$(x, y^2, x^3y + y^5, x^2y^{2k+1})$	$(2k+4)$-*determined*, $k \geq 2$, *codim* $= 3k+10$.

6-Jets				
	det	stratum	codim	label
B	-	(x, y^2, x^2y, xy^5)	10	A
	-	$(x, y^2, x^2y, 0)$	11	B
C	6	$(x, y^2, xy^3 + x^4y, y^5 \pm x^5y)$	12	-
	-	$(x, y^2, xy^3 + x^4y, y^5)$	13	C
F	•	$(x, y^2, xy^3 + x^4y, x^5y)$	13	D*
	-	$(x, y^2, xy^3 + x^4y, 0)$	14	E
G	•	(x, y^2, xy^3, x^5y)	14	F*
	-	$(x, y^2, xy^3 \pm x^5y, 0)$	15	G
	-	$(x, y^2, xy^3, 0)$	16	H
I	-	$(x, y^2, x^3y \pm xy^5, x^2y^3)$	13	I
	-	(x, y^2, x^3y, x^2y^3)	14	J
J	-	(x, y^2, x^3y, xy^5)	14	K
	-	$(x, y^2, x^3y \pm xy^5, 0)$	15	-
	-	$(x, y^2, x^3y, 0)$	16	-
K	-	$(x, xy + y^5, y^3, xy^5)$	9	L
	-	$(x, xy + y^5, y^3, 0)$	10	M
L	-	(x, xy, y^3, xy^5)	10	N
	-	$(x, xy, y^3, 0)$	11	O
M	-	$(x, xy + y^3, y^4, xy^3 + ay^5)$ $a \neq -1, \frac{6}{5}$	11	P
	-	$(x, xy + y^3, y^4, xy^3 - y^5)$	12	P
	-	$(x, xy + y^3, y^4, xy^3 + \frac{6}{5}y^5)$	12	P
	-	$(x, xy + y^3, y^4, xy^3 - y^5 + y^6)$	11	-
	-	$(x, xy + y^3, y^4, xy^3 + \frac{6}{5}y^5 + y^6)$	11	-
N	-	$(x, xy + y^3, y^4, x^2y^3 + y^6)$	12	Q
	-	$(x, xy + y^3, y^4, x^2y^3)$	13	-
O	•	$(x, xy + y^3, y^4, y^6)$	13	R*
	-	$(x, xy + y^3, y^4, 0)$	14	S
P	-	$(x, xy + ay^6, xy^2 + y^5 + y^6, y^4)$	10	T
	-	$(x, xy \pm y^6, xy^2 + y^5, y^4)$	10	U
	-	$(x, xy, xy^2 + y^5, y^4)$	11	V
Q	-	$(x, xy + y^6, xy^2 + y^6, y^4)$	10	-
	-	$(x, xy, xy^2 + y^6, y^4$	11	-
	-	$(x, xy + y^6, xy^2, y^4)$	11	-
	-	(x, xy, xy^2, y^4)	12	-

6-Jets (continued)				
	det	stratum	codim	label
R	-	$(x, xy, xy^2 + y^4, xy^3 + ay^5)$ $a \neq 0, \frac{4}{3}$	11	-
	-	$(x, xy, xy^2 + y^4, xy^3)$	13	-
	-	$(x, xy, xy^2 + y^4, xy^3 + \frac{4}{3}y^5)$	12	
	-	$(x, xy, xy^2 + y^4, xy^3 + y^6)$	11	-
	-	$(x, xy + y^6, xy^2 + y^4, xy^3)$	12	-
	-	$(x, xy, xy^2 + y^4 \pm y^6, xy^3 + \frac{4}{3}y^5)$	11	-
S	-	$(x, xy, xy^2 + y^4, y^5)$	11	W
T	-	$(x, xy + y^4, xy^2 + ay^6, xy^3 + y^5)$	12	-
U	-	$(x, xy + y^4, xy^2 + y^6, y^5)$	12	X
	-	$(x, xy + y^4, xy^2, y^5)$	13	-

D: *classification of jets over* $(x, y^2, xy^3 + x^4y, x^5y)$ *gives the series*

$(x, y^2, xy^3 + x^4y, x^5y \pm y^{2k+1})$ $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 10$.

F: *classification of jets over* (x, y^2, xy^3, x^5y) *gives the series*

$(x, y^2, xy^3, x^5y + y^{2k+1})$ $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 11$.

R: *classification of jets over* $(x, xy + y^3, y^4, y^6)$ *gives the series*

$(x, xy + y^3, y^4, y^6 + y^{2k+1})$ $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 10$.

7-Jets				
	det	stratum	codim	label
A	7	$(x, y^2, x^2y, xy^5 + y^7)$	10	-
	7	$(x, y^2, x^2y \pm y^7, xy^5)$	11	-
	•	(x, y^2, x^2y, xy^5)	12	A*
B	7	(x, y^2, x^2y, y^7)	11	-
	-	$(x, y^2, x^2y \pm y^7, 0)$	12	-
	-	$(x, y^2, x^2y, 0)$	13	-
C	7	$(x, y^2, xy^3 + x^4y, y^5 + ax^6y)$ $a \neq -1$	14	-
	•	$(x, y^2, xy^3 + x^4y, y^5 - x^6y)$	14	B*
E	7	$(x, y^2, xy^3 + x^4y, y^7 \pm x^6y)$	14	-
	-	$(x, y^2, xy^3 + x^4y, y^7)$	15	C
	•	$(x, y^2, xy^3 + x^4y, x^6y)$	15	D*
	-	$(x, y^2, xy^3 + x^4y, 0)$	16	E
G	7	$(x, y^2, xy^3 \pm x^5y, y^7 \pm x^6y)$	15	-
	-	$(x, y^2, xy^3 \pm x^5y, y^7)$	16	F
	•	$(x, y^2, xy^3 \pm x^5y, x^6y)$	16	G*
	-	$(x, y^2, xy^3 \pm x^5y, 0)$	17	H
H	7	$(x, y^2, xy^3, y^7 \pm x^6y)$	16	-
	-	$(x, y^2, xy^3 + x^6y, y^7)$	17	-
	-	(x, y^2, xy^3, y^7)	18	-
	-	(x, y^2, xy^3, x^6y)	17	-
	-	$(x, y^2, xy^3 + x^6y, 0)$	18	-
	-	$(x, y^2, xy^3, 0)$	19	-
I	7	$(x, y^2, x^3y \pm xy^5 + by^7, x^2y^3 + ay^7)$ $a^3 \mp 2a^2 + b^2 + a \neq 0$	15	-
J	7	$(x, y^2, x^3y + ay^7, x^2y^3 \pm y^7)$ $a^2 \pm 1 \neq 0$	15	-
	•	$(x, y^2, x^3y + ay^7, x^2y^3 \pm y^7)$ $a^2 \pm 1 = 0$	15	I*
	7	$(x, y^2, x^3y + y^7, x^2y^3)$	15	-
	-	(x, y^2, x^3y, x^2y^3)	16	J
K	7	$(x, y^2, x^3y + ay^7, xy^5 + y^7)$ $a \neq 1$	15	-
	•	$(x, y^2, x^3y + y^7, xy^5 + y^7)$	15	K*
	7	$(x, y^2, x^3y + y^7, xy^5)$	15	-
	•	(x, y^2, x^3y, xy^5)	16	L*
L	-	$(x, xy + y^5, y^3, xy^5 \pm y^7)$	9	M
	-	$(x, xy + y^5, y^3, xy^5)$	10	-
M	-	$(x, xy + y^5, y^3, y^7)$	10	N
	-	$(x, xy + y^5, y^3, 0)$	11	-

7-Jets (continued)				
	det	stratum	codim	label
N	-	$(x, xy, y^3, xy^5 + y^7)$	10	O
	-	(x, xy, y^3, xy^5)	11	-
O	-	(x, xy, y^3, y^7)	11	P
	-	$(x, xy, y^3, 0)$	12	-
P	7	$(x, xy + y^3, y^4, xy^3 + ay^5 \pm y^7)$ $a \neq -1, \frac{6}{5}$	11	-
	7	$(x, xy + y^3, y^4, xy^3 - y^5 \pm y^7)$	12	-
	7	$(x, xy + y^3, y^4, xy^3 + \frac{6}{5}y^5 \pm y^7)$	12	-
	7	$(x, xy + y^3, y^4, xy^3 - y^5 \pm xy^6)$	13	-
	7	$(x, xy + y^3, y^4, xy^3 + ay^5)$ $a \neq \frac{6}{5}, \pm 1$	12	-
	7	$(x, xy + y^3, y^4, xy^3 + \frac{6}{5}y^5)$	13	-
	-	$(x, xy + y^3, y^4, xy^3 + y^5)$	12	-
	-	$(x, xy + y^3, y^4, xy^3 - y^5)$	14	-
Q	7	$(x, xy + y^3, y^4, x^2y^3 + y^6 + ay^7)$ $a \neq -1$	13	-
	-	$(x, xy + y^3, y^4, x^2y^3 + y^6 - y^7)$	13	-
S	-	$(x, xy + y^3, y^4, y^7 + xy^6)$	14	-
	-	$(x, xy + y^3, y^4, y^7)$	15	-
	-	$(x, xy + y^3, y^4, xy^6)$	15	-
	-	$(x, xy + y^3, y^4, 0)$	16	-
T	7	$(x, xy + ay^6 + by^7, xy^2 + y^5 + y^6 + cy^7, y^4)$ for all a, b, c	12	-
U	7	$(x, xy \pm y^6 + ay^7, xy^2 + y^5 + by^7, y^4)$ for all a, b	12	-
V	7	$(x, xy + ay^7, xy^2 + y^5 \pm y^7, y^4)$ for all a	12	-
	7	$(x, xy \pm y^7, xy^2 + y^5, y^4)$	12	-
	7	$(x, xy, xy^2 + y^5, y^4)$	13	-
W	7	$(x, xy + ay^7, xy^2 + y^4, y^5 \pm y^7)$ for all a	12	-
	7	$(x, xy \pm y^7, xy^2 + y^4, y^5)$	12	-
	7	$(x, xy, xy^2 + y^4, y^5)$	13	-
X	7	$(x, xy + y^4, xy^2 + y^6 + by^7, y^5 + ay^7)$ for all a, b	14	-

A: *classification of jets over* (x, y^2, x^2y, xy^5) *gives the series*

$(x, y^2, x^2y \pm y^{2k+1}, xy^5)$ *(2k+1)-determined,* $k \geq 3$, *codim* $= k + 8$.

B: *classification of jets over* $(x, y^2, xy^3 + x^4y, y^5 - x^6y)$ *gives the series*

$(x, y^2, xy^3 + x^4y, y^5 - x^6y \pm x^ky)$ — $(k+1)$-*determined*, $k \geq 7$, *codim* $= k + 7$.

D: *classification of jets over* $(x, y^2, xy^3 + x^4y, x^6y)$ *gives the series*

$(x, y^2, xy^3 + x^4y, x^6y \pm y^{2k+1})$ — $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 11$.

G: *classification of jets over* $(x, y^2, xy^3 \pm x^5y, x^6y)$ *gives the series*

$(x, y^2, xy^3 \pm x^5y, x^6y \pm y^{2k+1})$ — $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 12$.

I: *classification of jets over* $(x, y^2, x^3y + ay^7, x^2y^3 \pm y^7)$, *where* $a^2 \pm 1 = 0$, *gives the series*

$(x, y^2, x^3y + ay^7 \pm y^{2k+1}, x^2y^3 \pm y^7)$ — $(2k+1)$-*determined*, $k \geq 4$, *codim* $= k + 11$.

K: *classification of jets over* $(x, y^2, x^3y + y^7, xy^5 + y^7)$ *gives the series*

$(x, y^2, x^3y + y^7 \pm y^{2k+1}, xy^5 + y^7)$ — $(2k+1)$-*determined*, $k \geq 4$, *codim* $= k + 11$.

L: *classification of jets over* (x, y^2, x^3y, xy^5) *gives the series*

$(x, y^2, x^3y + y^{2k+1}, xy^5)$ — $(2k+1)$-*determined*, $k \geq 3$, *codim* $= k + 12$.

8-Jets				
	det	stratum	codim	label
C	8	$(x, y^2, xy^3 + x^4y, y^7 \pm x^7y)$	15	-
	•	$(x, y^2, xy^3 + x^4y, y^7)$	16	A*
E	•	$(x, y^2, xy^3 + x^4y, x^7y)$	16	B*
	•	$(x, y^2, xy^3 + x^4y, 0)$	17	C*
F	8	$(x, y^2, xy^3 \pm x^5y, y^7 + x^7y)$	16	-
	•	$(x, y^2, xy^3 \pm x^5y, y^7)$	17	D*
H	•	$(x, y^2, xy^3 \pm x^5y, x^7y)$	17	E*
	-	$(x, y^2, xy^3 \pm x^5y, 0)$	18	-
J	-	$(x, y^2, x^3y \pm xy^7, x^2y^3)$	16	-
	-	(x, y^2, x^3y, x^2y^3)	17	-
M	8	$(x, xy + y^5, y^3, xy^5 \pm y^7 + ay^8)$ for all a	10	-
N	8	$(x, xy + y^5, y^3, y^7 + y^8)$	10	-
	8	$(x, xy + y^5, y^3, y^7)$	11	-
O	8	$(x, xy, y^3, xy^5 + y^7 + y^8)$	10	-
	8	$(x, xy, y^3, xy^5 + y^7)$	11	-
P	8	$(x, xy, y^3, y^7 + y^8)$	11	-
	8	(x, xy, y^3, y^7)	12	-

A: *although this appears to be part of a series, it is interrupted at the 10-jet-level by the occurrence a unimodular family. In total we obtain the singularities:*

$(x, y^2, xy^3 + x^4y, y^7 \pm x^7y)$ — *8-determined*, *codim* $= 15$,

$(x, y^2, xy^3 + x^4y, y^7 \pm x^8y)$ — *9-determined*, *codim* $= 16$,

$(x, y^2, xy^3 + x^4y, y^7 + ax^9y)$ — *10-determined*, $a \neq 1$, *codim* $= 18$,

and the series:

$(x, y^2, xy^3 + x^4y, y^7 + x^9y \pm x^ky)$ — $(k+1)$*-determined*, $k \geq 10$, *codim* $= k + 8$.

B: *classification of jets over* $(x, y^2, xy^3 + x^4y, x^7y)$ *gives the series*

$(x, y^2, xy^3 + x^4y, x^7y \pm y^{2k+1})$ — $(2k+1)$*-determined*, $k \geq 4$, *codim* $= k + 12$.

C: *further classification of jets over $(x, y^2, xy^3 + x^4y, 0)$ indicates the existence of a more general series which includes the results of* **A** *and* **B** *as sub-branches. We have not investigated this fully but have found the following series*

$$(x, y^2, xy^3 + x^4y, x^{2k+1}y \pm y^{2j+1}) \qquad (2j+1)\text{-determined}, \; k \geq 2, \; j \geq k+1, \; codim = 2k + j + 6,$$

$$(x, y^2, xy^3 + x^4y, x^{2k}y \pm y^{2j+1}) \qquad (2j+1)\text{-determined}, \; k \geq 3, \; j \geq k, \; codim = 2k + j + 5,$$

(the cases $k = 2$ and $k = 3$ occurred in earlier examples). However, there are other series over the 5-jet $(x, y^2, xy^3 + x^4y, 0)$.

D: *although this appears to be part of a series, it is interrupted at the 13-jet-level by the occurrence a unimodular family. In total we obtain the singularities:*

$$(x, y^2, xy^3 \pm x^5y, y^7 + x^7y) \qquad 8\text{-determined}, \; codim = 16,$$

$$(x, y^2, xy^3 \pm x^5y, y^7 \pm x^8y) \qquad 9\text{-determined}, \; codim = 17,$$

$$\vdots \qquad\qquad \vdots$$

$$(x, y^2, xy^3 \pm x^5y, y^7 + x^{11}y) \qquad 12\text{-determined}, \; codim = 20,$$

$$(x, y^2, xy^3 \pm x^5y, y^7 + ax^{12}y) \qquad 13\text{-determined}, \quad a \neq \pm 1, \; codim = 22,$$

(where $a \neq \pm 1$ is respective of the term $\pm x^5y$ in the jet). We do not investigate this further.

E: *classification of jets over $(x, y^2, xy^3 \pm x^5y, x^7y)$ gives the series*

$$(x, y^2, xy^3 \pm x^5y, x^7y \pm y^{2k+1}) \qquad (2k+1)\text{-determined}, \; k \geq 4, \; codim = k + 13.$$

Proof. The proof will take up the rest of the chapter. It is easy to determine where a specific part of the classification is carried out. We refer to our numbering system described at the end of Section 3.3.2. To verify the bounds on the codimension given in the statement of the theorem we observe the following. For a map-germ f consider the map

$$\begin{array}{rcll} \pi : (J^{k+1}\mathcal{A}) \cdot j^{k+1}f & \longrightarrow & (J^k\mathcal{A}) \cdot j^k f & \\ j^{k+1}g & \mapsto & j^k g & \quad g \in \mathcal{A} \cdot f \end{array}$$

from the $J^{k+1}\mathcal{A}$ orbit of f to the $J^k\mathcal{A}$ orbit of f. Any $j^kg \in (J^k\mathcal{A}) \cdot j^kf$ may be written in the form $j^k\phi \cdot j^kf$ for some $\phi \in \mathcal{A}$. But

$$\pi(j^{k+1}\phi \cdot j^{k+1}f) = j^k(\phi \cdot f) = j^k\phi \cdot j^kf$$

so π is surjective and $\pi^{-1}((J^k\mathcal{A})\cdot j^kf)$ is a submanifold of $J^{k+1}(n,p)$ with codimension equal to the codimension of $(J^k\mathcal{A}) \cdot j^kf$ in $J^k(n,p)$. But $\pi^{-1}((J^k\mathcal{A}) \cdot j^kf) \supset (J^{k+1}\mathcal{A}) \cdot j^{k+1}f$ and it follows that $J^k\mathcal{A}$-Codim$(f) \leq J^{k+1}\mathcal{A}$-Codim$(f)$. Now if a map-germ f is k-$\mathcal{A}$-determined then, by the determinacy theorems of [BduPW], there exists some unipotent subgroup $\mathcal{G}$ of $\mathcal{A}$ such that

$$m_2^{k+1}.\mathcal{E}(2,4) \subset L\mathcal{G} \cdot f,$$

and in particular

$$m_2^{k+1}.\mathcal{E}(2,4) \subset L\mathcal{A} \cdot f,$$

hence, $J^k\mathcal{A}$-Codim$(f) = \mathcal{A}$-Codim(f). Note that this last result also allows one to calculate $\mathcal{A}$-Codim(f) using the computer, this was of course exploited. □

The question of determining the simple singularities will be postponed until Chapter 4, where a more detailed study of such map-germs will be carried out.

3.3.1 The 1-Jets

We will restrict our attention to germs of corank ≤ 1 at present.

Lemma 3.3 (Rank Theorem) *Let $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ be a smooth map-germ.*

1. *If the rank of df at 0 is k then there exists germs of diffeomorphisms $\phi : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ and $\psi : (\mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$ such that $\psi \circ f \circ \phi$ is of the form*
$$x = (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, f_{k+1}(x), \ldots, f_p(x)),$$
where $f_{k+1}, \ldots, f_p \in m_n^2.\mathcal{E}(n,p)$.

2. *If df has constant rank k on a neighbourhood of 0 then we can choose ϕ, ψ so that the f_i are identically zero.*

We need part (1) of the Rank Theorem: assuming f is of corank 0 or 1 we see that, up to $\mathcal{A}$-equivalence, it is of the form (x, y, f_3, f_4) or (x, f_2, f_3, f_4) respectively. So the $J^1\mathcal{A}$-orbits of f are

$$\begin{array}{ll} (x, y, 0, 0) & \text{1-determined}, \\ (x, 0, 0, 0) & \text{(A)}. \end{array}$$

The reduction of f to these forms is obtained using the $M_{1,s}(\mathcal{G})$ filtration and is trivial in each case; though the computer also gives these orbits — the use of this filtration demonstrates the automation of our classification techniques. The determinacy calculation for the immersion is straight forward or can be seen from part (2) of the Rank Theorem, so we now consider the 1-jet $(x, 0, 0, 0)$.

3.3.2 The 2-Jets

There are four $J^2\mathcal{A}$-orbits over $(x, 0, 0, 0)$:

$$\begin{array}{ll} (x, y^2, xy, 0) & \text{(B)}, \\ (x, y^2, 0, 0) & \text{(C)}, \\ (x, xy, 0, 0) & \text{(D)}, \\ (x, 0, 0, 0) & \text{(A)}. \end{array}$$

Remark. The above is easy to show. We can apply the $\mathcal{A}_1$-complete transversal methods, the resulting transversal giving the $J^3\mathcal{A}$-orbits to be of the form

$$(x, a_1y^2 + b_1xy, a_2y^2 + b_2xy, a_3y^2 + b_3xy).$$

Then, by standard linear algebra, if the rank of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

is 2 we can change coordinates to obtain $(x, y^2, xy, 0)$. If it is of rank 1 we may suppose (after change of coordinates) that either $a_1 \neq 0$ or $b_1 \neq 0$. Then, if $a_1 \neq 0$ we obtain $(x, y^2, 0, 0)$, and if $b_1 \neq 0$ we obtain $(x, xy, 0, 0)$. Finally, if it is or rank 0 we have $(x, 0, 0, 0)$. However, it is our intention to cut out such *ad hoc* calculations, no matter how trivial. Our computer methods are more automated but can give redundant jets if applied blindly. For example, at the $(2, 1)$-level we obtain $(x, 0, 0, ay^2), a \in \mathbf{R}$, as a family of jets; and then at the $(2, 3)$-level obtain $(x, 0, bxy, ay^2)$. However, jets of this form are included in our 'simpler' list of orbits and are redundant. To get the list of orbits stated above consider $(x, 0, 0, 0)$

as a $(2,2)$-jet. Then the $(2,3)$-transversal gives the family of jets $(x, ay^2, bxy, 0)$. Scaling gives: (i) $(x, y^2, xy, 0)$, (ii) $(x, y^2, 0, 0)$, (iii) $(x, 0, xy, 0)$, or (iv) $(x, 0, 0, 0)$, depending on whether a and b are non-zero. The $(2, s)$-transversals, for $s > 3$, are empty for (i),(ii) and (iii) and we can now work with them as 3-jets. In (iv), the $(2,4)$-transversal gives $(x, axy, 0, 0)$ and scaling gives the orbits: (v) $(x, xy, 0, 0)$ and (vi) $(x, 0, 0, 0)$; again the higher $(2, s)$-transversals are empty. However, (v) and (iii) are clearly in the same orbit and we need only consider (i), (ii), (v) and (vi) as 3-jets. (This demonstrates the redundant jets that can appear, as mentioned above.) These calculations are tedious but are easily handled, especially by computer. The method may be applied at all stages of the classification and is preferable to the *ad hoc* methods which are often employed (such as linear transformations, Tschirnhaus's transformation, completing squares, etc.) which have limited scope.

The numbering system used throughout the classification will be as follows. Each of the jets (A), (B), (C) and (D) above will be considered in its own section below. The jets which branch off at each higher jet-level will be numbered consecutively; this numbering only applies within the specific section, starting from (1) for each new section. Each branch will be pursued as far as necessary, that is until a non-simple jet arises, a series arises, or in some cases (for example where we choose to pursue the non-simples and families involving moduli) where the complexity of the higher orbit structures suggests a natural stopping point (further investigation should be dictated by future considerations with specific applications in mind). This method is preferred to the method sometimes employed where *all* the 2-jets are considered, followed by *all* the resulting 3-jets, and so on. We feel it is easier for the reader to follow the classification though by pursuing each jet, in turn, until it is finished with.

3.3.3 The 2-Jet $(x, y^2, xy, 0)$

A $(3,1)$-transversal for $(x, y^2, xy, 0)$ is $\{(0,0,0,y^3)\}$; the higher $(3, s)$-transversals are empty. There are two $J^3\mathcal{A}$-orbits over $(x, y^2, xy, 0)$:

(x, y^2, xy, y^3)	3-determined,
$(x, y^2, xy, 0)$	(B).

The 4-transversal for (B) is empty. A $(5,1)$-transversal is $\{(0,0,0,y^5)\}$; the higher $(5,s)$-transversals are empty. So there are two $J^5\mathcal{A}$-orbits over $(x,y^2,xy,0)$:

$$\begin{array}{ll} (x,y^2,xy,y^5) & \text{5-determined,} \\ (x,y^2,xy,0) & \text{(B).} \end{array}$$

Further calculation by the computer suggests the series of singularities:

$$(x,y^2,xy,y^{2k+1}) \qquad (2k+1)\text{-determined,} \quad k \geq 1.$$

This is indeed the case. Since this is one of the easier examples we take this opportunity to demonstrate the calculations involved behind the classification method. The calculations needed to produce this series have also been carried out by West, [We].

Consider $f = (x,y^2,xy,0)$ as a $2k$-jet, for some $k \geq 1$. Now

$$\partial f/\partial x = (1,0,y,0), \quad \partial f/\partial y = (0,2y,x,0).$$

The $\mathcal{L}_1$-tangent space contains vectors of the form

$$\{x^a y^b e_i : a+b = 2k+1, a \geq 1, i = 1,\ldots,4\}$$

and the vectors $(y^{2k+1},0,0,0),(0,y^{2k+1},0,0)$ follow (modulo $m_2^{2k+2}.\mathcal{E}(2,4)$) from $\partial f/\partial x$, $\partial f/\partial y$, respectively. Finally,

$$\begin{aligned} (0,0,y^{2k+1},0) &= y^{2k}\partial f/\partial x - (y^2)^k e_1 \\ &= y^{2k}\partial/\partial x(f) - u_2^k \partial/\partial u_1(f). \end{aligned}$$

(Note that for $k=1$, $u_2\partial/\partial u_1 \in L\mathcal{G}$ but $u_2\partial/\partial u_1 \notin L\mathcal{A}_1$, and the $\mathcal{A}_1$-transversal contains y^3e_3 as a redundant term — see the following Remark.) Thus, a $(2k+1)$-transversal is $\{(0,0,0,y^{2k+1})\}$; strictly speaking a $(2k+1,1)$-transversal is $\{(0,0,0,y^{2k+1})\}$ but then all the $(2k+1,s)$-transversals, for $s > 1$, are found to be empty. So (after scaling) the two $J^{2k+1}\mathcal{A}$-orbits over f are

$$\begin{array}{ll} (x,y^2,xy,y^{2k+1}) & (2k+1)\text{-determined,} \\ (x,y^2,xy,0). & \end{array}$$

For the second of these, consider f as a $(2k+1)$-jet. We now have

$$L\mathcal{G}\cdot f \supset m_2^{2k+2}.\mathcal{E}(2,4)$$

and the $(2k+2)$-transversal is empty. Thus, all germs over f are equivalent to one of the form (x,y^2,xy,y^{2l+1}) for some $l \geq 1$, or have the same l-jet as f for all $l \geq 2$. That is, f is a stem.

For the determinacy calculation we need to show that

$$m_2^{2k+2}.\mathcal{E}(2,4) \subset L\mathcal{G}\cdot f + m_2^{2k+2}.f^*(m_4).\mathcal{E}(2,4) + m_2^{4k+4}.\mathcal{E}(2,4),$$

for $f = (x, y^2, xy, y^{2k+1})$. Now, $f^*(m_4).\mathcal{E}_2 = \{x, y^2\}.\mathcal{E}_2$ and $m_2^{2k+2}.f^*(m_4).\mathcal{E}_2 \supset m_2^{2k+4}$ so we need only work modulo $m_2^{2k+4}.\mathcal{E}(2,4)$; we also obtain every monomial of degree $2k+3$ which involves x. Finally,

$$y^{2k+3}e_i = u_2u_4\partial/\partial u_i(f) \in L\mathcal{L}_1\cdot f.$$

The monomials of degree $2k+2$ can be obtained easily from the $\mathcal{L}_1$-tangent space.

Remark. To stress further the power of our classification method and the possible automation of the process, we compare it with the $\mathcal{A}_1$-complete transversal method. An $\mathcal{A}_1$-3-transversal of $(x, y^2, xy, 0)$ is $\{(0,0,y^3,0),(0,0,0,y^3)\}$ giving the family of 3-jets $(x, y^2, xy + ay^3, by^3)$. Clearly, if $b \neq 0$ then this is equivalent to (x, y^2, xy, y^3), and if $b = 0$ then it is possible to find explicit changes of coordinates which reduce the jet to $(x, y^2, xy, 0)$. In more complicated examples this can be a problem and we must resort to use of the Mather Lemma to simplify the orbits. However, by extending the $\mathcal{A}_1$ group to the unipotent group $\mathcal{G}$, we demonstrated above that such considerations were unnecessary.

3.3.4 The 2-Jet $(x, y^2, 0, 0)$

The $J^3\mathcal{A}$-orbits over $(x, y^2, 0, 0)$ are:

(x, y^2, y^3, x^2y)	3-determined,
$(x, y^2, y^3 \pm x^2y, 0)$	(1),
$(x, y^2, y^3, 0)$	(2),
$(x, y^2, x^2y, 0)$	(3),
$(x, y^2, 0, 0)$	(C).

This is straight forward using the same methods as for $(x, 0, 0, 0)$ in Section 3.3.2. Either note that an $\mathcal{A}_1$-3-transversal is

$$\{(0,0,0,y^3),(0,0,y^3,0),(0,0,0,x^2y),(0,0,x^2y,0)\}$$

and then use linear algebra to reduce to the stated forms, or use the $M_{r,s}(\mathcal{G})$-filtration. In the latter case, considering $(x, y^2, 0, 0)$ as a $(3,1)$-jet gives the following 3-jets: (x, y^2, y^3, x^2y), $(x, y^2, y^3 \pm x^2y, 0)$, $(x, y^2, y^3, 0)$, (the redundant jet $(x, y^2, 0, x^2y)$), and finally, when we reach the stage of considering $(x, y^2, 0, 0)$ as a

(3, 3)-jet, the (3, 4) transversal gives $(x, y^2, x^2y, 0)$ and $(x, y^2, 0, 0)$. The reason we need not consider $(x, y^2, 0, 0)$ as a (3, 0)-jet is that the resulting jets are all equivalent to ones in the above 'simpler' list. (The (3, 1)-orbits are $(x, y^2, 0, y^3)$ and $(x, y^2, 0, 0)$. But continuing with $(x, y^2, 0, y^3)$ gives the 3-jets $(x, y^2, x^2y, y^3 \pm x^2y)$, $(x, y^2, 0, y^3 \pm x^2y)$, (x, y^2, x^2y, y^3) and $(x, y^2, 0, y^3)$; the first and third of these being $\mathcal{A}$-equivalent.Then each of these is $\mathcal{A}$-equivalent to one in our previous list.)

(1) $(x, y^2, y^3 \pm x^2y, 0)$

The only non-empty $(4, s)$-transversal of $(x, y^2, y^3 \pm x^2y, 0)$ is $\{(0, 0, 0, x^3y)\}$ and the $J^4\mathcal{A}$-orbits over $(x, y^2, y^3 \pm x^2y, 0)$ are:

$(x, y^2, y^3 \pm x^2y, x^3y)$	4-determined,
$(x, y^2, y^3 \pm x^2y, 0)$	(1).

Continuing with (1) we obtain the following series; the general calculation can be found in [We]:

$$(x, y^2, y^3 \pm x^2y, x^ky) \qquad (k+1)\text{-determined}, \quad k \geq 3.$$

(2) $(x, y^2, y^3, 0)$

Working with the $(4, s)$-transversals of $(x, y^2, y^3, 0)$, the first non-empty one is $\{(0, 0, 0, x^3y)\}$ giving the jets (x, y^2, y^3, x^3y) and $(x, y^2, y^3, 0)$, and the rest are empty in the former case and just $\{(0, 0, x^3y, 0)\}$ in the later. So there are three $J^4\mathcal{A}$-orbits over $(x, y^2, y^3, 0)$:

(x, y^2, y^3, x^3y)	4-determined,
$(x, y^2, y^3 + x^3y, 0)$	(4),
$(x, y^2, y^3, 0)$	(2).

The only $(5, s)$-transversal of (4) is $\{(0, 0, 0, x^4y)\}$ giving the two $J^5\mathcal{A}$-orbits:

$(x, y^2, y^3 + x^3y, x^4y)$	5-determined,
$(x, y^2, y^3 + x^3y, 0)$	(4).

Continuing with (4) gives the series:

$$(x, y^2, y^3 + x^3y, x^ky) \qquad (k+1)\text{-determined}, \quad k \geq 4.$$

Now consider (2) as a 4-jet. The $J^5\mathcal{A}$-orbits over $(x, y^2, y^3, 0)$ are:

(x, y^2, y^3, x^4y)	5-determined,
$(x, y^2, y^3 \pm x^4y, 0)$	(5),
$(x, y^2, y^3, 0)$	(2).

Then, (5) gives a series as (4) did and (2) extends similarly to the 6-jet-space. In summary, we obtain two series of singularities:

(x, y^2, y^3, x^ky)	$(k+1)$-determined, $k \geq 2$,
$(x, y^2, y^3 \pm x^jy, x^ky)$	$(k+1)$-determined, $j \geq 2$, $k \geq j+1$.

The cases $k = 2$ in the former and $j = 2$ in the latter arose earlier. For these series, the general calculations can be found in [We].

(3) $(x, y^2, x^2y, 0)$

The only non-empty $(4, s)$-transversal of $(x, y^2, x^2y, 0)$ is $\{(0, 0, 0, xy^3)\}$ and the $J^4\mathcal{A}$-orbits are:

(x, y^2, x^2y, xy^3)	(6),
$(x, y^2, x^2y, 0)$	(3).

(6) (x, y^2, x^2y, xy^3)

The only non-empty $(5, s)$-transversal of (x, y^2, x^2y, xy^3) is $\{(0, 0, y^5, 0)\}$ and the $J^5\mathcal{A}$-orbits are:

$(x, y^2, x^2y \pm y^5, xy^3)$	5-determined,
(x, y^2, x^2y, xy^3)	(6).

Continuing with (6); the 6-transversal is empty, a 7-transversal is $\{(0, 0, y^7, 0)\}$ giving the $J^7\mathcal{A}$-orbits:

$(x, y^2, x^2y \pm y^7, xy^3)$	7-determined,
(x, y^2, x^2y, xy^3)	(6).

This continues and gives the following series; refer to [We] for details,

$$(x, y^2, x^2y \pm y^{2k+1}, xy^3) \qquad (2k+1)\text{-determined}, \quad k \geq 2.$$

(3) $(x, y^2, x^2y, 0)$

Now consider $(x, y^2, x^2y, 0)$ as a 4-jet. A $(5,1)$-transversal is $\{(0,0,0,y^5)\}$ giving the two orbits: (x, y^2, x^2y, y^5) and $(x, y^2, x^2y, 0)$. For the first, the higher $(5,s)$-transversals are empty, while for the second a $(5,2)$-transversal is $(0,0,y^5,0)$ and the rest are empty. So, in total, the $J^5\mathcal{A}$-orbits over $(x, y^2, x^2y, 0)$ are:

(x, y^2, x^2y, y^5)	5-determined,
$(x, y^2, x^2y \pm y^5, 0)$	(7),
$(x, y^2, x^2y, 0)$	(3).

(7) $(x, y^2, x^2y \pm y^5, 0)$

The only non-empty $(6,s)$-transversal of $(x, y^2, x^2y \pm y^5, 0)$ is $\{(0,0,0,xy^5)\}$ and $J^6\mathcal{A}$-orbits are:

$(x, y^2, x^2y \pm y^5, xy^5)$	(8),
$(x, y^2, x^2y \pm y^5, 0)$	(7).

(8) $(x, y^2, x^2y \pm y^5, xy^5)$

The only non-empty $(7,s)$-transversal of $(x, y^2, x^2y \pm y^5, xy^5)$ is $\{(0,0,0,y^7)\}$ and we have a one parameter family of $J^7\mathcal{A}$-orbits over $(x, y^2, x^2y \pm y^5, xy^5)$:

$$(x, y^2, x^2y \pm y^5, xy^5 + ay^7) \qquad \text{7-determined}, \quad a^2 \pm 1 \neq 0.$$

Here $a \in \mathbf{R}$ and the ± 1 is respective of the $\pm y^5$ term in the jet. (So in the real case $(x, y^2, x^2y + y^5, xy^5 + ay^7)$ is determined for all $a \in \mathbf{R}$.) Consider the 'scaling' coordinate changes

$$(x, y) \mapsto (e^{\lambda_1}x, e^{\lambda_2}y), \qquad (u_1, u_2, u_3, u_4) \mapsto (e^{\mu_1}u_1, e^{\mu_2}u_2, e^{\mu_3}u_3, e^{\mu_4}u_4),$$

for (fixed) $\lambda_i, \mu_i \in \mathbf{R}$ (c.f., Proposition 1.11). Then, reducing a to ± 1 is equivalent to solving the linear system

$$\begin{aligned} 2\lambda_1 + \lambda_2 + \mu_3 &= 0, & \lambda_1 + 5\lambda_2 + \mu_4 &= 0, \\ 5\lambda_2 + \mu_3 &= 0, & 7\lambda_2 + \mu_4 &= \log(1/|a|), \end{aligned}$$

that is,

$$\begin{aligned} 2\lambda_1 - 4\lambda_2 &= 0, \\ \lambda_1 - 2\lambda_2 &= -\log(1/|a|). \end{aligned}$$

This is consistent only for $a = \pm 1$. So we cannot reduce a using 'scaling'; this suggests it may be a modulus. However, to show a is a modulus we apply Theorem 1.9. Working with the group $\mathcal{A}$ in the standard 7-jet-space $J^7(2,4)$ with $f = (x, y^2, x^2y \pm y^5, xy^5 + ay^7)$, we have

$$(0,0,0,y^7) \notin L\mathcal{A} \cdot f \quad \text{modulo} \quad m_2^8.\mathcal{E}(2,4), \qquad \text{for all } a,$$

and it follows that a is a modulus. We remark that, in general, such calculations, where we must show that a vector is *not* in the tangent space, are extremely hard by hand. One of the functions of our computer package is to perform such calculations.

Remark (Non-Simple Germs). We can rule out (8) and any jets above it (for example those which occur when the modulus a satisfies $a^2 \pm 1 = 0$) from being simple. In addition, the 6-jet (7) is non-simple and any higher degree jet over (7) can be excluded from the list of possible simple singularities. For consider working in some higher degree jet-space and suppose some jet j has 6-jet $(x, y^2, x^2y \pm y^5, 0)$. Then any open neighbourhood of j must contain a jet with 6-jet $(x, y^2, x^2y \pm y^5, \epsilon xy^5)$ for some $\epsilon \neq 0$. But from the previous complete transversal calculations, this jet is equivalent to $(x, y^2, x^2y \pm y^5, xy^5 + ay^7)$ for some $a \in \mathbf{R}$. That is, any open neighbourhood of j contains a non-simple jet so, by definition, j is non-simple. Such arguments will be used several times in later sections to rule out simplicity. Note that the above argument proves that (3) is non-simple as a 6-jet (any neighbourhood contains $(x, y^2, x^2y + \epsilon_1 y^5, \epsilon_2 xy^5)$), but we cannot rule (3) out as a non-simple 5-jet. However, the 6-jets above the 5-jet (3) are indeed non-simple, as will be seen in Section (3) below.

Even though (7) and (8) can no longer provide examples of simple singularities, they do provide a rich supply of series. We firstly consider the degenerate values of the modulus a.

Consider the 7-jet $(x, y^2, x^2y \pm y^5, xy^5 + ay^7)$. This is 7-determined provided $a^2 \pm 1 \neq 0$ (± 1 respective of $\pm y^5$). If $a^2 \pm 1 = 0$ then the 8-transversal is still empty, whereas a 9-transversal is $\{(0,0,0,y^9)\}$. The $J^9\mathcal{A}$-orbits are:

$$(x, y^2, x^2y \pm y^5, xy^5 + ay^7 \pm y^9) \qquad \text{9-determined,}$$
$$(x, y^2, x^2y \pm y^5, xy^5 + ay^7).$$

Continuing gives the series:

$$(x, y^2, x^2y \pm y^5, xy^5 + ay^7 \pm y^{2k+1}) \qquad (2k+1)\text{-determined}, \quad k \geq 4, \quad a^2 \pm 1 = 0.$$

This will be proved in a more general setting for (7) below.

(7) $(x, y^2, x^2y \pm y^5, 0)$

The only non-empty $(7, s)$-transversal of $(x, y^2, x^2y \pm y^5, 0)$ is $\{(0,0,0,y^7)\}$ and the $J^7\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, x^2y \pm y^5, y^7) & \text{7-determined,} \\ (x, y^2, x^2y \pm y^5, 0) & (7). \end{array}$$

Continuing with (7) gives the following series:

$$\begin{array}{ll} (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}) & \text{(i),} \\ (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3}) & \text{(ii),} \\ (x, y^2, x^2y \pm y^5, y^{2k+1}) & \text{(iii),} \end{array}$$

where

$$\begin{array}{ll} \text{(i)} & (2k+3)\text{-determined, } k \geq 2,\ a^2 \pm 1 \neq 0; \\ \text{(ii)} & (2j+3)\text{-determined, } k \geq 2,\ j \geq k+1,\ a^2 \pm 1 = 0; \\ \text{(iii)} & (2k+1)\text{-determined, } k \geq 3. \end{array}$$

To see this consider $f = (x, y^2, x^2y \pm y^5, 0)$ as a $(2k+1)$-jet for $k \geq 2$. In the examples here, we can always obtain vectors of the form $\{\, x^a y^b e_i : a \geq 2 \,\}$ (modulo higher order terms) using the $\mathcal{L}_1$-tangent space. We therefore need only consider the vectors of the form $\{xy^b e_i\}$ and $\{\, y^b e_i \,\}$ below. Now

$$\partial f/\partial x = (1, 0, 2xy, 0), \quad \partial f/\partial y = (0, 2y, x^2 \pm 5y^4, 0),$$

and $xy^{2k+1}e_1$, $xy^{2k+1}e_2$ follow from these, respectively. Also,

$$2xy^{2k+1}e_3 = y^{2k}\partial/\partial x(f) - u_2^k \partial/\partial u_1(f)$$

and $y^{2k+2}e_i \in L\mathcal{L}_1 \cdot f$. A $(2k+2)$-transversal is therefore $\{(0,0,0,xy^{2k+1})\}$ and the $J^{2k+2}\mathcal{A}$-orbits are:

$$\begin{array}{l} (x, y^2, x^2y \pm y^5, xy^{2k+1}), \\ (x, y^2, x^2y \pm y^5, 0). \end{array}$$

Consider the first of these; put $f = (x, y^2, x^2y \pm y^5, xy^{2k+1})$. Now

$$\partial f/\partial x = (1, 0, 2xy, y^{2k+1}), \quad \partial f/\partial y = (0, 2y, x^2 \pm 5y^4, (2k+1)xy^{2k}),$$

and $y^{2k+3}e_1$, $y^{2k+3}e_2$ follow from these, respectively. Also, modulo $m_2^{2k+4}.\mathcal{E}(2,4)$

$$\pm 4y^{2k+3}e_3 = y^{2k-1}\partial/\partial y(f) - 2u_2^k \partial/\partial u_2(f) - u_2^{k-1}u_3\partial/\partial u_3(f)$$

and $xy^{2k+2}e_i \in L\mathcal{L}_1 \cdot f$. However, we cannot obtain $y^{2k+3}e_4$ so a $(2k+3)$-transversal is $\{(0,0,0,y^{2k+3})\}$ and there is a one-parameter family of $J^{2k+3}\mathcal{A}$-orbits:

$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}) \qquad (2k+3)\text{-determined}, \quad a^2 \pm 1 \neq 0.$$

Trying to 'scale' a to a unit leads to the same inconsistent linear system as encountered at the 7-jet-level. Indeed, with $f = (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3})$

$$\begin{aligned} \partial f/\partial x &= (1, 0, 2xy, y^{2k+1}), \\ \partial f/\partial y &= (0, 2y, x^2 \pm 5y^4, (2k+1)xy^{2k} + a(2k+3)y^{2k+2}), \end{aligned}$$

and it is not hard to convince oneself (though we prefer to omit the full-blown calculations) that

$$(0,0,0,y^{2k+3}) \notin L\mathcal{A} \cdot f \quad \text{modulo} \quad m_2^{2k+4}.\mathcal{E}(2,4)$$

so that a is a modulus.

For the determinacy calculation we must check that all monomial vectors of degree $2k+4$ and $\{y^{2k+5}e_i\}$ are in $L\mathcal{G}\cdot f$ modulo $m_2^{2k+6}.\mathcal{E}(2,4)$ (cf. the determinacy calculation in Section 3.3.3). Now, from earlier comments we need only check $\{y^{2k+4}e_i\}$, $\{xy^{2k+3}e_i\}$ together with $\{y^{2k+5}e_i\}$. The first is trivial. For the second note that

$$xy^{2k+3}e_i = u_2u_4\partial/\partial u_i(f) - ay^{2k+5}e_i,$$

so we need only check $\{y^{2k+5}e_i\}$. Now

$$\begin{aligned} u_1u_4\partial/\partial u_i(f) &= (x^2y^{2k+1} + axy^{2k+3})e_i, \\ u_2^k u_3\partial/\partial u_i(f) &= (x^2y^{2k+1} \pm y^{2k+5})e_i, \end{aligned}$$

so

$$(axy^{2k+3} \mp y^{2k+5})e_i \in L\mathcal{L}_1 \cdot f.$$

But

$$u_2u_4\partial/\partial u_i(f) = (xy^{2k+3} + ay^{2k+5})e_i$$

so

$$(a^2 \pm 1)y^{2k+5}e_i \in L\mathcal{L}_1 \cdot f$$

and f is $(2k+3)$-determined for $a^2 \pm 1 \neq 0$.

Next, if $a^2 \pm 1 = 0$ then a $(2k+5)$-transversal is $\{(0,0,0,y^{2k+5})\}$. Generally, if $(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3})$ is a $(2j+1)$-jet for $j \geq k+1$, then from the above calculations, we see that the $(2j+2)$-transversal is empty. At the $(2j+3)$-level: $xy^{2j+2}e_i \in L\mathcal{L}_1 \cdot f$; $y^{2j+3}e_1$ and $y^{2j+3}e_2$ follow from $\partial f/\partial x$ and $\partial f/\partial y$, respectively; and, modulo $m_2^{2j+4}.\mathcal{E}(2,4)$

$$\pm 4y^{2j+3}e_3 = y^{2j-1}\partial/\partial y(f) - 2u_2^j\partial/\partial u_2(f) - u_2^{j-1}u_3\partial/\partial u_3(f).$$

However, we cannot obtain $y^{2j+3}e_4$ so a $(2j+3)$-transversal is $\{(0,0,0,y^{2j+3})\}$ and the $J^{2j+3}\mathcal{A}$-orbits are:

$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3}),$$
$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}).$$

The first is $(2j+3)$-determined; the second is of the stated form. For the determinacy calculation we can work modulo $m_2^{2j+6}.\mathcal{E}(2,4)$ and, from earlier comments, need only check $\{y^{2j+4}e_i\}$, $\{xy^{2j+3}e_i\}$ and $\{y^{2j+5}e_i\}$. The first is trivial. For the second and third note that with

$$f = (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3})$$

we have

$$u_2^{j-k+1}u_4\partial/\partial u_i(f) = (xy^{2j+3} + ay^{2j+5})e_i \quad \text{modulo} \quad m_2^{2j+6}.\mathcal{E}(2,4),$$

and the same trick as used above involving $u_1u_4\partial/\partial u_i(f)$, $u_2^k u_3\partial/\partial u_i(f)$ and $u_2u_4\partial/\partial u_i(f)$ gives

$$(\pm ay^{2j+5} \mp xy^{2j+3})e_i \in L\mathcal{L}_1 \cdot f,$$

since $a^2 \pm 1 = 0$. Combining these we obtain $y^{2j+5}e_i, xy^{2j+3}e_i \in L\mathcal{L}_1 \cdot f$ modulo $m_2^{2j+6}.\mathcal{E}(2,4)$, as $a \neq 0$.

It remains to consider $(x, y^2, x^2y \pm y^5, 0)$ as a $(2k+2)$-jet. From $\partial f/\partial x$ and $\partial f/\partial y$ we get $y^{2k+3}e_1$ and $y^{2k+3}e_2$; also $y^{2k+3}e_3$ follows (as in the $(2k+3)$-transversal calculation for $(x, y^2, x^2y \pm y^5, xy^{2k+1})$ above). We cannot obtain $y^{2k+3}e_4$ and a $(2k+3)$-transversal is $\{(0,0,0,y^{2k+3})\}$. The $J^{2k+3}\mathcal{A}$-orbits are:

$$(x, y^2, x^2y \pm y^5, y^{2k+3}) \qquad (2k+3)\text{-determined},$$
$$(x, y^2, x^2y \pm y^5, 0).$$

The determinacy calculation is easy. We need only check $xy^{2k+3}e_i$, $y^{2k+4}e_i$ and $y^{2k+5}e_i$, but these clearly lie in $L\mathcal{L}_1 \cdot f$.

(3) $(x, y^2, x^2y, 0)$

Consider the 5-jet $(x, y^2, x^2y, 0)$. The only $(6, s)$-transversal is $\{(0, 0, 0, xy^5)\}$ and there are two $J^6\mathcal{A}$-orbits:

$$\begin{array}{ll} (x, y^2, x^2y, xy^5) & (9), \\ (x, y^2, x^2y, 0) & (3). \end{array}$$

As stated in Section (8), the 6-jet (3) is non-simple; the same argument shows the 6-jet (9) is non-simple. Any higher degree jet with 6-jet (3) or (9) can be excluded from our list of possible simple singularities. We will investigate briefly the behaviour of the higher degree jets over (9) and (3).

The $J^7\mathcal{A}$-orbits over (9) are:

$$\begin{array}{ll} (x, y^2, x^2y, xy^5 + y^7) & \text{7-determined}, \\ (x, y^2, x^2y \pm y^7, xy^5) & \text{7-determined}, \\ (x, y^2, x^2y, xy^5) & (9). \end{array}$$

Continuing with (9) gives the series:

$$(x, y^2, x^2y \pm y^{2k+1}, xy^5) \qquad \text{(2k+1)-determined}, \quad k \geq 3.$$

The $J^7\mathcal{A}$-orbits over (3) are:

$$\begin{array}{ll} (x, y^2, x^2y, y^7) & \text{7-determined}, \\ (x, y^2, x^2y \pm y^7, 0), & \\ (x, y^2, x^2y, 0) & (3). \end{array}$$

Remark. Continuing with (3) gives series which resemble those discovered earlier. For instance, (9) can be likened with the 4-jet (6): (x, y^2, x^2y, xy^3); and the $J^7\mathcal{A}$-orbits over (3) can be likened with the 5-jets (x, y^2, x^2y, y^5), (7): $(x, y^2, x^2y \pm y^5, 0)$ and (3): $(x, y^2, x^2y, 0)$. This suggests the existence of a general, larger class of 'series', starting from the 3-jet $(x, y^2, x^2y, 0)$ and including the previous series as 'sub-branches'. That is, the 3-jet $(x, y^2, x^2y, 0)$ appears to be a stem. Although we have made some progress taking the classification further, there is no obvious stopping point and we choose to stop here. Further classification of jets over $(x, y^2, x^2y, 0)$ is complicated by the fact that at each stage a finite number of extra branches occur. This behaviour is exhibited in (6) and (9) above. These are part of the branch $(x, y^2, x^2y, xy^{2k+1})$ for $k = 1, 2$, respectively. Although a general series branches off in each case (involving the terms $(0, 0, y^{2j+1}, 0)$), for (6) there are no 'extra jets' but for (9) we obtain the extra 7-determined jet $(x, y^2, x^2y, xy^5 + y^7)$. Similarly, for $k = 3$ the extra jets $(x, y^2, x^2y, xy^7 + y^9)$ and $(x, y^2, x^2y, xy^7 + y^{11})$ crop up (these are 9- and 11-determined, respectively). Such behaviour is exhibited by all the cases, not just $(x, y^2, x^2y, xy^{2k+1})$.

3.3.5 The 3-Jet $(x, y^2, 0, 0)$

The $J^4\mathcal{A}$-orbits over $(x, y^2, 0, 0)$ are:

$$\begin{array}{ll} (x, y^2, xy^3, x^3y) & (1), \\ (x, y^2, xy^3 \pm x^3y, 0) & (2), \\ (x, y^2, xy^3, 0) & (3), \\ (x, y^2, x^3y, 0) & (4), \\ (x, y^2, 0, 0) & (C). \end{array}$$

To see this use the same methods as Section 3.3.2 and Section 3.3.4: that is, either note that an $\mathcal{A}_1$-4-transversal is

$$\{(0,0,0,xy^3), (0,0,xy^3,0), (0,0,0,x^3y), (0,0,x^3y,0)\}$$

or use the $M_{r,s}(\mathcal{G})$-filtration (starting with $(x, y^2, 0, 0)$ as a $(4,2)$-jet).

(1) (x, y^2, xy^3, x^3y)

The $J^5\mathcal{A}$-orbits over (x, y^2, xy^3, x^3y) are:

$$\begin{array}{ll} (x, y^2, xy^3, x^3y + y^5) & \text{5-determined}, \\ (x, y^2, xy^3, x^3y) & (1), \end{array}$$

and continuing with (1) gives the series

$$(x, y^2, xy^3, x^3y + y^{2k+1}) \qquad (2k+1)\text{-determined}, \quad k \geq 2.$$

Details of the calculation may be found in [We].

(2) $(x, y^2, xy^3 \pm x^3y, 0)$

A $(5,1)$-transversal of $(x, y^2, xy^3 \pm x^3y, 0)$ is $\{(0,0,0,y^5)\}$; then, for all resulting $(5,1)$-orbits, the only higher non-empty $(5,s)$-transversal is the $(5,5)$-transversal: $\{(0,0,0,x^4y)\}$. So the $J^5\mathcal{A}$-orbits over $(x, y^2, xy^3 \pm x^3y, 0)$ are:

$$\begin{array}{lll} (x, y^2, xy^3 \pm x^3y, y^5 + ax^4y) & \text{5-determined}, & a \neq -1, \\ (x, y^2, xy^3 \pm x^3y, x^4y) & (5), & \\ (x, y^2, xy^3 \pm x^3y, 0) & (2). & \end{array}$$

Trying to reduce a to a unit in the above 1-parameter family leads to an inconsistent system of linear equations (cf. Section 3.3.4, Part (8)). Indeed, working with the group $\mathcal{A}$ in the 5-jet-space $J^5(2,4)$ we find that

$$(0,0,0,x^4y) \notin L\mathcal{A} \cdot f \quad \text{modulo} \quad m_2^6.\mathcal{E}(2,4),$$

so that by Theorem 1.9 a is a modulus.

We now investigate the exceptional value of the modulus. Further calculation by computer predicts the series:

$$(x, y^2, xy^3 \pm x^3y, y^5 - x^4y \pm x^ky) \qquad (k+1)\text{-determined}, \; k \geq 5.$$

Continuing with (5) we find that the 6-transversal is empty, so (5) represents the only $J^6\mathcal{A}$-orbit(s). Then the $J^7\mathcal{A}$-orbits are found to be:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^3y, x^4y \pm y^7) & 7\text{-determined}, \\ (x, y^2, xy^3 \pm x^3y, x^4y) & (5). \end{array}$$

Further calculation by computer predicts the series:

$$(x, y^2, xy^3 \pm x^3y, x^4y \pm y^{2k+1}) \qquad (2k+1)\text{-determined}, \; k \geq 3.$$

Continuing with (2) we find that the $J^6\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^3y, x^5y) & (6), \\ (x, y^2, xy^3 \pm x^3y, 0) & (2). \end{array}$$

For (6), further calculation by computer predicts the series:

$$(x, y^2, xy^3 \pm x^3y, x^5y + y^{2k+1}) \qquad (2k+1)\text{-determined}, \; k \geq 3.$$

The three series predicted above occur as branches of the following, more general, series:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^3y, y^{2k+1} + ax^{2k}y) & \text{(i)}, \\ (x, y^2, xy^3 \pm x^3y, y^{2k+1} + ax^{2k}y \pm x^jy) & \text{(ii)}, \\ (x, y^2, xy^3 \pm x^3y, x^{2k}y \pm y^{2j+1}) & \text{(iii)}, \\ (x, y^2, xy^3 \pm x^3y, x^{2k+1}y + y^{2j+1}) & \text{(iv)}, \end{array}$$

where

(i) $(2k+1)$-determined, $k \geq 2$,
$a \neq -1$ (k even), $a \neq \pm 1$ (± 1 respective of $\pm x^3y$) (k odd),

(ii) $(j+1)$-determined, $k \geq 2$, $j \geq 2k+1$,
$a = -1$ (k even), $a = \pm 1$ (± 1 respective of $\pm x^3y$) (k odd),

(iii) $(2j+1)$-determined, $k \geq 2$, $j \geq k+1$,

(iv) $(2j+1)$-determined, $k \geq 2$, $j \geq k+1$.

We will describe the calculations for these four series. They were discovered by extending the above classification of $(x, y^2, xy^3 \pm x^3y, 0)$ into the 7-jet-space, and higher, using our computer classification package.

Consider $f = (x, y^2, xy^3 \pm x^3y, 0)$ as a $2k$-jet, for $k \geq 2$. The following preliminary observations will be of use throughout the whole argument. Consider the monomial vectors of degree $l+1$ for some l; the cases l even and l odd will be considered separately. The monomials of degree $l+1$ are

$$\begin{matrix} y^{l+1} & & x^2y^{l-1} & & x^4y^{l-3} & & x^6y^{l-5} & \\ & xy^l & & x^3y^{l-2} & & x^5y^{l-4} & & \cdots \end{matrix}$$

For l even the bottom row clearly lies in $L\mathcal{L}_1 \cdot f$, and for l odd the top row clearly lies in $L\mathcal{L}_1 \cdot f$. Now, by taking combinations of $u_1\partial/\partial u_i(f)$, $u_2\partial/\partial u_i(f)$ and $u_3\partial/\partial u_i(f)$, we see that the following are in $L\mathcal{L}_1 \cdot f$.

l even:

$$\begin{matrix} (x^2y^{l-1} \pm x^4y^{l-3})e_i \\ (x^4y^{l-3} \pm x^6y^{l-5})e_i \\ (x^6y^{l-5} \pm x^8y^{l-7})e_i \\ \vdots \\ (x^{l-4}y^5 \pm x^{l-2}y^3)e_i \\ (x^{l-2}y^3 \pm x^ly)e_i \end{matrix} \qquad (\dagger).$$

So to obtain all monomial vectors of degree $l+1$, it is sufficient to obtain x^lye_i and $y^{l+1}e_i$.

l odd:

$$\begin{matrix} (xy^l \pm x^3y^{l-2})e_i \\ (x^3y^{l-2} \pm x^5y^{l-4})e_i \\ (x^5y^{l-4} \pm x^7y^{l-6})e_i \\ \vdots \\ (x^{l-4}y^5 \pm x^{l-2}y^3)e_i \\ (x^{l-2}y^3 \pm x^ly)e_i \end{matrix} \qquad (\ddagger).$$

So to obtain all monomial vectors of degree $l+1$, it is sufficient to obtain x^lye_i.

Now, f is a $2k$-jet and

$$\partial f/\partial x = (1, 0, y^3 \pm 3x^2y, 0), \quad \partial f/\partial y = (0, 2y, 3xy^2 \pm x^3, 0),$$

so we can obtain (modulo $m_2^{2k+2}.\mathcal{E}(2,4)$) $x^{2k}ye_1$, $y^{2k+1}e_1$, $x^{2k}ye_2$ and $y^{2k+1}e_2$. Also,

$$\pm 2x^{2k}ye_3 = x^{2k-2}\partial/\partial x(f) - u_1^{2k-2}\partial/\partial u_1(f) - u_1^{2k-3}\partial/\partial u_3(f),$$

and then, from (†), we obtain $x^2y^{2k-1}e_3$, and since

$$y^{2k-2}\partial/\partial x(f) - u_2^{k-1}\partial/\partial u_1(f) = (0,0,y^{2k+1} \pm 3x^2y^{2k-1},0)$$

we obtain $y^{2k+1}e_3$ also. We cannot obtain $y^{2k+1}e_4$ or $x^{2k}ye_4$ so the complete transversal theory gives the $J^{2k+1}\mathcal{A}$-orbits:

$$\begin{array}{l}(x,y^2,xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y),\\ (x,y^2,xy^3 \pm x^3y, x^{2k}y),\\ (x,y^2,xy^3 \pm x^3y, 0).\end{array}$$

Consider the first of these; put $f = (x,y^2,xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y)$. Trying to 'scale' a to a unit leads to the inconsistent system of linear equations

$$\begin{aligned} 2\lambda_1 - 2\lambda_2 &= 0,\\ 2k\lambda_1 - 2k\lambda_2 &= \log(1/|a|).\end{aligned}$$

Now,

$$\begin{aligned} \partial f/\partial x &= (1,0,y^3 \pm 3x^2y, 2kax^{2k-1}y),\\ \partial f/\partial y &= (0,2y,3xy^2 \pm x^3,(2k+1)y^{2k}+ax^{2k}),\end{aligned}$$

and working with the group $\mathcal{A}$ in $J^{2k+1}(2,4)$, it can be seen that

$$(0,0,0,x^{2k}y) \notin L\mathcal{A}\cdot f \quad \text{modulo} \quad m_2^{2k+2}.\mathcal{E}(2,4),$$

though we prefer to omit the technical details. Thus, a is indeed a modulus. We now consider the determinacy calculation. Since

$$m_2^{2k+2}.f^*(m_2).\mathcal{E}(2,4) \supset m_2^{2k+4}.\mathcal{E}(2,4)$$

we need only check the terms of degree $2k+2$ and $2k+3$. From (‡) we need only check $x^{2k+1}ye_i$, and from (†) we need only check $x^{2k+2}ye_i$ and $y^{2k+3}e_i$. Now (‡) gives

$$\begin{aligned} (xy^{2k+1} - x^5y^{2k-3})e_i &\in L\mathcal{L}_1\cdot f, \quad \text{and then}\\ (xy^{2k+1} \pm x^7y^{2k-5})e_i &\in L\mathcal{L}_1\cdot f,\\ (xy^{2k+1} - x^9y^{2k-7})e_i &\in L\mathcal{L}_1\cdot f,\\ &\vdots\end{aligned}$$

and finally (there are $k-1$ such expressions) we obtain

$$\begin{cases} (xy^{2k+1} \pm x^{2k+1}y)e_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is odd,} \\ (xy^{2k+1} - x^{2k+1}y)e_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is even.} \end{cases}$$

But,

$$u_1u_4\partial/\partial u_i(f) = (xy^{2k+1} + ax^{2k+1}y)e_i$$

so

$$\begin{cases} (a \mp 1)x^{2k+1}ye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is odd,} \\ (a+1)x^{2k+1}ye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is even,} \end{cases}$$

and we have $x^{2k+1}ye_i \in L\mathcal{L}_1 \cdot f$ provided $a \neq \pm 1$ (sign respective of $\pm x^3y$) for odd k, and provided $a \neq -1$ for even k. Similarly (†) gives

$$\begin{cases} (a \mp 1)x^{2k+2}ye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is odd,} \\ (a+1)x^{2k+2}ye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is even,} \end{cases}$$

and we have $x^{2k+2}ye_i \in L\mathcal{L}_1 \cdot f$ for the same conditions on a. Finally,

$$u_2u_4\partial/\partial u_i(f) = (y^{2k+3} + ax^{2k}y^3)e_i,$$

and it follows that $y^{2k+3}e_i \in L\mathcal{L}_1 \cdot f$ and that f is $(2k+1)$-determined for the given conditions on a.

Now suppose that a satisfies the degenerate conditions described above. This amounts to saying that the expression in xy^{2k+1} and $x^{2k+1}y$ obtained above from (‡) is just

$$(xy^{2k+1} + ax^{2k+1}y)e_i \in L\mathcal{L}_1 \cdot f.$$

Consider $f = (x, y^2, xy^3 \pm x^3y, y^{2k+1} + ax^{2k}y)$ as a j-jet, for $j \geq 2k+1$. Then using (†) or (‡) according to the parity of j, we find that a $(j+1)$-transversal is $\{(0,0,0,x^jy)\}$. For example, when j is even we need x^jye_i and $y^{j+1}e_i$. Now, x^jye_1, $y^{j+1}e_1$, x^jye_2, $y^{j+1}e_2$, x^jye_3, and $y^{j+1}e_3$ follow (modulo $m_2^{j+2}.\mathcal{E}(2,4)$) from $\partial f/\partial x$ and $\partial f/\partial y$ in the same manner as for the $(2k+1)$-transversal calculation above. Note that using (†) as before, when we obtained $x^{2k+2}ye_i$, gives

$$\begin{cases} (a \mp 1)x^jye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is odd,} \\ (a+1)x^jye_i \in L\mathcal{L}_1 \cdot f & \text{if } k \text{ is even.} \end{cases}$$

The important point is that these hold for *all* values of j. That is, for the given conditions on a, we cannot obtain x^jye_i in the same way that we could not obtain $x^{2k+2}ye_i$ and $x^{2k+1}ye_i$, and this is the case for every j; we omit the full details. It now follows that $\{(0,0,0,x^jy)\}$ is a $(j+1)$-transversal, since $x^ay^be_4$ (where

$a+b=j+1$ and $a \geq 1$) then follows by (†), and if $j=2m$ then $y^{j+1}e_4$ follows from

$$u_2^{m-k}u_4\partial/\partial u_4(f) = (y^{j+1}+ax^{2k}y^{j-2k+1})e_4.$$

The case j odd is similar. The $J^{j+1}\mathcal{A}$-orbits are of the form

$$(x, y^2, xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y+bx^jy)$$

where $b \in \mathbf{R}$. Trying to scale b to a unit is equivalent to solving the linear system

$$\begin{aligned} 2\lambda_1 - 2\lambda_2 &= 0, \\ j\lambda_1 - 2k\lambda_2 &= \log(1/|b|), \\ (j-2k)\lambda_1 &= \log(1/|b|). \end{aligned}$$

This is consistent and we can scale b to ± 1 provided it is non-zero. So the $J^{j+1}\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y \pm x^jy) & (j+1)\text{-determined}, \\ (x, y^2, xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y). & \end{array}$$

The second is of the previously considered form, so we need only consider the determinacy calculation. We can work modulo $m_2^{j+4}.\mathcal{E}(2,4)$ and need only check $x^{j+1}ye_i$, $y^{j+2}e_i$, $x^{j+2}ye_i$ and $y^{j+3}e_i$. Now, with $f=(x, y^2, xy^3 \pm x^3y, y^{2k+1}+ax^{2k}y \pm x^jy)$,

$$u_1u_4\partial/\partial u_i(f) = (xy^{2k+1}+ax^{2k+1}y \pm x^{j+1}y)e_i$$

and, as noted above, from (‡) we have

$$(xy^{2k+1}+ax^{2k+1}y)e_i \in L\mathcal{L}_1 \cdot f$$

(since a satisfies the degenerate conditions) so $x^{j+1}ye_i \in L\mathcal{L}_1 \cdot f$. If j is even then clearly $y^{j+2}e_i \in L\mathcal{L}_1 \cdot f$. For odd j, $j=2m+1$ say, we have

$$u_2^{m-k+1}u_4\partial/\partial u_i(f) = (y^{j+2}+ax^{2k}y^{j-2k+2})e_i,$$

modulo $m_2^{j+3}.\mathcal{E}(2,4)$. But we already have $x^{j+1}ye_i$ so it follows from (†) (as $j+1$ is even) that $x^{2k}y^{j-2k+2}e_i \in L\mathcal{L}_1 \cdot f$ and therefore that $y^{j+2}e_i \in L\mathcal{L}_1 \cdot f$. The terms of degree $j+3$ follow similarly; now we start with $u_1^2u_4\partial/\partial u_i(f)$ and use (†).

Now consider the $(2k+1)$-jet $f=(x, y^2, xy^3 \pm x^3y, x^{2k}y)$. Suppose this is a $(2j+1)$-jet, for some $j \geq k$. Now, since $x^{2j+1}ye_i \in L\mathcal{L}_1 \cdot f$, using (‡) we see that the $(2j+2)$-transversal is empty. So now consider f as a $(2j+2)$-jet. We can obtain $x^{2j+2}ye_i$ and $y^{2j+3}e_i$ for $i = 1, 2, 3$ in the same manner as for the $(2k+1)$-transversal calculation; in fact, it is clear that $x^{2j+2}ye_i \in L\mathcal{L}_1 \cdot f$ for all i. So, from (†), a $(2j+3)$-transversal is $\{(0, 0, 0, y^{2j+3})\}$. The $J^{2j+3}\mathcal{A}$-orbits are of the form

$$(x, y^2, xy^3 \pm x^3y, x^{2k}y + ay^{2j+3})$$

where $a \in \mathbf{R}$. Trying to scale a to a unit is equivalent to solving the linear system

$$\begin{aligned} 2\lambda_1 - 2\lambda_2 &= 0, \\ 2k\lambda_1 - (2j+2)\lambda_2 &= -\log(1/|a|). \end{aligned}$$

This is consistent for the given j and we can scale a to ± 1 provided it is non-zero. So the $J^{2j+3}\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^3y, x^{2k}y \pm y^{2j+3}) & (2j+3)\text{-determined}, \\ (x, y^2, xy^3 \pm x^3y, x^{2k}y). & \end{array}$$

The second is of the stated form, so we need only consider the determinacy calculation of the first. We can work modulo $m_2^{2j+6}.\mathcal{E}(2,4)$ and need only check $x^{2j+3}ye_i$, $y^{2j+4}e_i$, $x^{2j+4}ye_i$ and $y^{2j+5}e_i$. Put $f = (x, y^2, xy^3 \pm x^3y, x^{2k}y \pm y^{2j+3})$. Now, modulo $m_2^{2j+6}.\mathcal{E}(2,4)$,

$$x^{2j+3}ye_i = u_1^{2j-2k+3}u_4\partial/\partial u_i(f)$$

and $x^{2j+4}ye_i$ is obtained similarly. Clearly $y^{2j+4}e_i \in L\mathcal{L}_1 \cdot f$, and it remains to consider $y^{2j+5}e_i$. We have

$$\begin{aligned} u_2u_4\partial/\partial u_i(f) &= (x^{2k}y^3 \pm y^{2j+5})e_i, \\ u_1^{2k-1}u_3\partial/\partial u_i(f) &= (x^{2k}y^3 \pm x^{2k+2}y)e_i, \\ u_1^2u_4\partial/\partial u_i(f) &= (x^{2k+2}y \pm x^2y^{2j+3})e_i. \end{aligned}$$

Since we have already shown that $x^{2j+4}ye_i \in L\mathcal{L}_1 \cdot f$ (modulo $m_2^{2j+6}.\mathcal{E}(2,4)$) it follows from (†) that we obtain $x^2y^{2j+3}e_i$, and therefore $y^{2j+5}e_i$, as well.

Finally we must consider the $(2k+1)$-jet $f = (x, y^2, xy^3 \pm x^3y, 0)$. From (‡) we need only check the terms $x^{2k+1}ye_i$. For $i = 1, 2$ these easily follow from $\partial f/\partial x$ and from $\partial f/\partial y$, and

$$\pm 2x^{2k+1}ye_3 = x^{2k-1}\partial/\partial x(f) - u_1^{2k-1}\partial/\partial u_1(f) - u_1^{2k-2}u_3\partial/\partial u_3(f).$$

So a $(2k+2)$-transversal is $\{(0, 0, 0, x^{2k+1}y)\}$ and the $J^{2k+2}\mathcal{A}$-orbits are:

$$\begin{array}{l} (x, y^2, xy^3 \pm x^3y, x^{2k+1}y), \\ (x, y^2, xy^3 \pm x^3y, 0). \end{array}$$

For the second we are back to our original consideration of $(x, y^2, xy^3 \pm x^3y, 0)$ as a jet of even degree. Consider the first as a $(2j+2)$-jet, for some $j \geq k$. From (†) we need only check $x^{2j+2}ye_i$ and $y^{2j+3}e_i$. The first clearly lies in $L\mathcal{L}_1 \cdot f$ and for $i = 1, 2, 3$ we obtain the second using the usual arguments, as in the previous calculations. A $(2j+3)$-transversal is $\{(0, 0, 0, y^{2j+3})\}$ and after 'scaling' coordinate changes we find that the $J^{2j+3}\mathcal{A}$-orbits are:

$(x, y^2, xy^3 \pm x^3y, x^{2k+1}y \pm y^{2j+3})$	$(2j+3)$-determined,
$(x, y^2, xy^3 \pm x^3y, x^{2k+1}y)$.	

Using (‡) we see that the $(2j+4)$-transversal of the second is empty so the resulting $J^{2j+4}\mathcal{A}$-orbits have already been considered. The determinacy calculation for the first is almost identical to the previous determinacy calculation.

Remark (Non-Simple Germs). All the series obtained above from the 5-jet (2): $(x, y^2, xy^3 \pm x^3y, 0)$ are non-simple. To see this we use the same argument as Section 3.3.4, (8): that is, any open neighbourhood of the given jet contains a jet equivalent to one in the unimodular family $(x, y^2, xy^3 \pm x^3y, y^5 + ax^4y)$. In addition, the jets (3): $(x, y^2, xy^3, 0)$, (4): $(x, y^2, x^3y, 0)$ and (C): $(x, y^2, 0, 0)$ are non-simple as 5-jets. However, we cannot rule them out as 4-jets and consider this next.

(3) $(x, y^2, xy^3, 0)$

The $J^5\mathcal{A}$-orbits over $(x, y^2, xy^3, 0)$ are

$(x, y^2, xy^3, y^5 \pm x^4y)$	5-determined,
$(x, y^2, xy^3 + x^4y, y^5)$	(7),
(x, y^2, xy^3, y^5)	(8),
(x, y^2, xy^3, x^4y)	(9),
$(x, y^2, xy^3 + x^4y, 0)$	(10),
$(x, y^2, xy^3, 0)$	(3).

Consider $(x, y^2, xy^3, 0)$ as a $(5, 0)$-jet. A $(5, 1)$-transversal is $\{(0, 0, 0, y^5)\}$ giving (after scaling) the $(5, 1)$-orbits (x, y^2, xy^3, y^5) and $(x, y^2, xy^3, 0)$. Consider the first of these. The $(5, s)$-transversals are empty for $2 \leq s \leq 4$ but a $(5, 5)$-transversal is $\{(0, 0, 0, x^4y)\}$ giving the $(5, 5)$-orbits $(x, y^2, xy^3, y^5 \pm x^4y)$ and (x, y^2, xy^3, y^5). For the first, all higher $(5, s)$-transversals are empty; for the second, the only higher non-empty transversal is the $(5, 6)$-transversal: $\{(0, 0, x^4y, 0)\}$. In total, we obtain the $J^5\mathcal{A}$-orbits $(x, y^2, xy^3, y^5 \pm x^4y)$, $(x, y^2, xy^3 + x^4y, y^5)$ and (x, y^2, xy^3, y^5).

Now consider the (5, 1)-jet $(x, y^2, xy^3, 0)$. In a similar fashion, the (5, 5)- and (5, 6)-transversals lead to the $J^5\mathcal{A}$-orbits (x, y^2, xy^3, x^4y), $(x, y^2, xy^3 + x^4y, 0)$ and $(x, y^2, xy^3, 0)$.

Remark (Non-Simple Germs). By comparing $(x, y^2, xy^3, y^5 \pm x^4y)$, (8), (9) and (3) with the unimodular family $(x, y^2, xy^3 \pm x^3y, y^5 + ax^4y)$ we can use the argument of Section 3.3.4(8) to show that these are non-simple. For (7) note that any open neighbourhood of the 5-jet $(x, y^2, xy^3 + x^4y, y^5)$ contains a jet of the form $(x, y^2, xy^3 + \epsilon x^3y + x^4y, y^5)$ for $\epsilon \neq 0$. Applying scaling coordinate changes we can write this as $(x, y^2, xy^3 \pm x^3y + x^4y, y^5)$. But this jet has (5, 1)-jet $(x, y^2, xy^3 \pm x^3y, y^5)$ so, by the complete transversal calculation in Section 3.3.5(2) above, must be equivalent as a 5-jet to a member of the unimodular family $(x, y^2, xy^3 \pm x^3y, y^5 + ax^4y)$. A similar argument shows (10) is non-simple.

(7) $(x, y^2, xy^3 + x^4y, y^5)$

The only non-empty $(6, s)$-transversal of $(x, y^2, xy^3 + x^4y, y^5)$ is $\{(0, 0, 0, x^5y)\}$ and the $J^6\mathcal{A}$-orbits are:

$(x, y^2, xy^3 + x^4y, y^5 \pm x^5y)$	6-determined,
$(x, y^2, xy^3 + x^4y, y^5)$	(7).

The only non-empty $(7, s)$-transversal of $(x, y^2, xy^3 + x^4y, y^5)$ is $\{(0, 0, 0, x^6y)\}$ and the $J^7\mathcal{A}$-orbits are:

$$(x, y^2, xy^3 + x^4y, y^5 + ax^6y) \qquad \text{7-determined}, \quad a \neq -1.$$

We cannot 'scale' a to a unit; indeed, using Theorem 1.9 and computer calculation, we verify that a is a modulus. For $a = -1$, further calculation by computer predicts the series:

$$(x, y^2, xy^3 + x^4y, y^5 - x^6y \pm x^ky) \qquad (k+1)\text{-determined}, \; k \geq 7.$$

This has been checked by hand calculation; the details are similar to those given below for the series over (8).

(8) (x, y^2, xy^3, y^5)

A (6, 6)-transversal of (x, y^2, xy^3, y^5) is $\{(0, 0, 0, x^5y)\}$. The resulting orbits are $(x, y^2, xy^3, y^5 + x^5y)$ and (x, y^2, xy^3, y^5). All higher $(6, s)$-transversals are empty for the first. For the second, a $(6,7)$-transversal is $\{(0,0,x^5y,0)\}$; the higher $(6,s)$-transversals are then empty for all of the resulting orbits. So, the $J^6\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x,y^2,xy^3,y^5+x^5y) & \text{6-determined,} \\ (x,y^2,xy^3\pm x^5y,y^5) & (11), \\ (x,y^2,xy^3,y^5) & (8). \end{array}$$

(11) $(x,y^2,xy^3\pm x^5y,y^5)$

Continuing gives the 7-, 8- and 9-determined jets:

$$\begin{array}{ll} (x,y^2,xy^3\pm x^5y,y^5\pm x^6y) & \text{7-determined,} \\ (x,y^2,xy^3\pm x^5y,y^5\pm x^7y) & \text{8-determined,} \\ (x,y^2,xy^3\pm x^5y,y^5+ax^8y) & \text{9-determined,} \quad a\neq -1. \end{array}$$

It is verified that a is a modulus, and that for $a=-1$ the following series arises:

$$(x,y^2,xy^3\pm x^5y,y^5-x^8y\pm x^ky) \qquad (k+1)\text{-determined, } k\geq 9.$$

The details are discussed in a more general setting next.

(8) (x,y^2,xy^3,y^5)

Continuing with (x,y^2,xy^3,y^5) we obtain the following series:

$$\begin{array}{ll} (x,y^2,xy^3,y^5\pm x^ky) & (k+1)\text{-determined,} \\ (x,y^2,xy^3\pm x^ky,y^5\pm x^{k+1}y) & (k+2)\text{-determined,} \\ (x,y^2,xy^3\pm x^ky,y^5\pm x^{k+2}y) & (k+3)\text{-determined,} \\ \vdots & \vdots \\ (x,y^2,xy^3\pm x^ky,y^5\pm x^{2k-3}y) & (2k-2)\text{-determined,} \\ (x,y^2,xy^3\pm x^ky,y^5+ax^{2k-2}y) & (2k-1)\text{-determined,} \\ & \quad a\neq -1, \\ (x,y^2,xy^3\pm x^ky,y^5-x^{2k-2}y\pm x^jy) & (j+1)\text{-determined,} \\ & \quad j\geq 2k-1, \end{array}$$

where $k\geq 4$ (the case $k=4$ occurred earlier). Consider $f=(x,y^2,xy^3,y^5)$ as a k-jet, for $k\geq 5$. It is clear that all monomial vectors of degree $k+1$ lie in $L\mathcal{L}_1\cdot f$, except those of the form x^kye_i. Now

$$\partial f/\partial x=(1,0,y^3,0), \quad \partial f/\partial y=(0,2y,3xy^2,5y^4),$$

so $x^k y e_1$ and $x^k y e_2$ follow, modulo $m_2^{k+2}.\mathcal{E}(2,4)$. We cannot obtain $x^k y e_4$ so a $(k+1,k+1)$-transversal is $\{(0,0,0,x^k y)\}$ giving the orbits $(x,y^2,xy^3,y^5 \pm x^k y)$ and (x,y^2,xy^3,y^5). Denoting the first by f we have

$$u_1 u_4 \partial/\partial u_i(f) - u_2 u_3 \partial/\partial u_i(f) = \pm x^{k+1} y e_i,$$

so that all higher $(k+1,s)$-transversals are empty. For the second the only non-empty transversal is the $(k+1,k+2)$-transversal: $\{(0,0,x^k y,0)\}$. The $J^{k+1}\mathcal{A}$-orbits over (x,y^2,xy^3,y^5) are:

$$\begin{array}{ll} (x,y^2,xy^3,y^5 \pm x^k y) & (k+1)\text{-determined}, \\ (x,y^2,xy^3 \pm x^k y,y^5), & \\ (x,y^2,xy^3,y^5) & (8). \end{array}$$

For the determinacy calculation we can work modulo $m_2^{k+4}.\mathcal{E}(2,4)$ and need only check the terms $x^{k+1} y e_i$ and $x^{k+2} y e_i$. This is straight forward using the $\mathcal{L}_1$-tangent space.

Now consider $f = (x,y^2,xy^3 \pm x^k y,y^5)$ as a j-jet, for $j \geq k+1$. As above, we see that the first non-empty transversal is the $(j+1,j+1)$-transversal: $\{(0,0,0,x^j y)\}$. The resulting orbits are of the form

$$(x,y^2,xy^3 \pm x^k y, y^5 + a x^j y) \quad \text{for} \quad a \in \mathbf{R}.$$

We will address the question of 'scaling' shortly. Let f denote this family, then

$$\begin{aligned} \partial f/\partial x &= (1,0,y^3 \pm k x^{k-1} y, j a x^{j-1} y), \\ \partial f/\partial y &= (0,2y,3xy^2 \pm x^k, 5y^4 + a x^j). \end{aligned}$$

Now,

$$x^{j-k+1}\partial/\partial x(f) - u_1^{j-k+1}\partial/\partial u_1(f) = (0,0,x^{j-k+1}y^3 \pm k x^j y, 0)$$

modulo $m_2^{j+2}.\mathcal{E}(2,4)$, and

$$u_1^{j-k} u_3 \partial/\partial u_3(f) = (x^{j-k+1}y^3 \pm x^j y)e_3$$

so we have $x^j y e_3$. It follows that all higher $(j+1,s)$-transversals are empty and the $J^{j+1}\mathcal{A}$-orbits are given by the above family. Trying to 'scale' a to a unit is equivalent to solving the linear system

$$\begin{aligned} (k-1)\lambda_1 - 2\lambda_2 &= 0, \\ j\lambda_1 - 4\lambda_2 &= \log(1/|a|), \end{aligned}$$

which is consistent (for $a \neq \pm 1$) provided $j \neq 2k-2$. So, for $k+1 \leq j < 2k-2$ the orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^k y, y^5 \pm x^{k+1} y) & (k+2)\text{-determined}, \\ \vdots & \vdots \\ (x, y^2, xy^3 \pm x^k y, y^5 \pm x^{2k-3} y) & (2k-2)\text{-determined}. \end{array}$$

For the determinacy calculation we can work modulo $m_2^{j+4}.\mathcal{E}(2,4)$ and need only check $x^{j+1}ye_i$ and $x^{j+2}ye_i$. Then,

$$\begin{aligned} u_1 u_4 \partial/\partial u_i(f) &= (xy^5 \pm x^{j+1}y)e_i, \\ u_2 u_3 \partial/\partial u_i(f) &= (xy^5 \pm x^k y^3)e_i, \\ u_1^{k-1} u_3 \partial/\partial u_i(f) &= (x^k y^3 \pm x^{2k-1} y)e_i, \end{aligned}$$

and since $j < 2k-2$, $x^{2k-1}ye_i \in m_2^{j+3}.\mathcal{E}(2,4)$ and $x^{j+1}ye_i$ follows modulo $m_2^{j+3}.\mathcal{E}(2,4)$. Similarly, $x^{j+2}ye_i$ follows modulo $m_2^{j+4}.\mathcal{E}(2,4)$, and this completes the calculation.

Now consider the case $j = 2k-2$, and put $f = (x, y^2, xy^3 \pm x^k y, y^5 + ax^{2k-2}y)$. This is $(2k-1)$-determined, for $a \neq -1$, and a is indeed a modulus. (To prove the latter we must show that $(0,0,0,x^{2k-2}y) \notin L\mathcal{A} \cdot f$ modulo $m_2^{2k}.\mathcal{E}(2,4)$ — the details are extremely tedious and are omitted). Now

$$\begin{aligned} \partial f/\partial x &= (1, 0, y^3 \pm kx^{k-1}y, (2k-2)ax^{2k-3}y), \\ \partial f/\partial y &= (0, 2y, 3xy^2 \pm x^k, 5y^4 + ax^{2k-2}), \end{aligned}$$

and, as above, $u_1u_4\partial/\partial u_i(f)$, $u_2u_3\partial/\partial u_i(f)$, $u_1^{k-1}u_3\partial/\partial u_i(f)$ give us

$$(xy^5 + ax^{2k-1}y)e_i, \quad (xy^5 \pm x^k y^3)e_i, \quad (x^k y^3 \pm x^{2k-1}y)e_i,$$

and it follows that $(a+1)x^{2k-1}ye_i \in L\mathcal{G} \cdot f$. It is then clear that f is $(2k-1)$-determined for $a \neq -1$. Now suppose $a = -1$ and consider f as a j-jet, for $j \geq 2k-1$. As before, we need only check the terms $x^j ye_i$. These follow for $i = 1$ and $i = 2$ from $\partial f/\partial x$ and $\partial f/\partial y$, respectively. Also,

$$x^{j-k+1}\partial/\partial x(f) - u_1^{j-k+1}\partial/\partial u_1(f) = (0, 0, x^{j-k+1}y^3 \pm kx^j y, 0)$$

modulo $m_2^{j+2}.\mathcal{E}(2,4)$, and

$$u_1^{j-k}u_3\partial/\partial u_3(f) = (x^{j-k+1}y^3 \pm x^j y)e_3$$

so $x^j ye_3 \in L\mathcal{G} \cdot f$. A $(j+1)$-transversal is therefore $\{(0,0,0,x^j y)\}$ and the $J^{j+1}\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^k y, y^5 - x^{2k-2}y \pm x^j y) & (j+1)\text{-determined}, \\ (x, y^2, xy^3 \pm x^k y, y^5 - x^{2k-2}y). & \end{array}$$

We need only consider the determinacy calculation. Again, we can work modulo $m_2^{j+4}.\mathcal{E}(2,4)$ and need only check $x^{j+1}ye_i$ and $x^{j+2}ye_i$. But,

$$u_1u_4\partial/\partial u_i(f) - u_2u_3\partial/\partial u_i(f) \pm u_1^{k-1}u_3\partial/\partial u_i(f) = (a+1)x^{2k-1}ye_i \pm x^{j+1}ye_i,$$

and then $x^{j+1}ye_i \in L\mathcal{G}\cdot f$ since $a=-1$. Similarly $x^{j+2}ye_i \in L\mathcal{G}\cdot f$ and it follows that f is $(j+1)$-determined.

(9) (x, y^2, xy^3, x^4y)

The 6-transversal of (x, y^2, xy^3, x^4y) is empty, and the only non-empty $(7,s)$-transversal is $\{(0,0,0,y^7)\}$. The $J^7\mathcal{A}$-orbits are:

$(x, y^2, xy^3, x^4y \pm y^7)$	7-determined,
(x, y^2, xy^3, x^4y)	(9).

Further calculation by computer for (9) predicts the series:

$$(x, y^2, xy^3, x^4y \pm y^{2k+1}) \qquad (2k+1)\text{-determined}, \quad k \geq 3.$$

This is indeed the case, as shown by West.

(10) $(x, y^2, xy^3 + x^4y, 0)$

The only non-empty $(6,s)$-transversal of $(x, y^2, xy^3+x^4y, 0)$ is $\{(0,0,0,x^5y)\}$ and the J^6-orbits are:

$(x, y^2, xy^3 + x^4y, x^5y)$	(12),
$(x, y^2, xy^3 + x^4y, 0)$	(10).

(12) $(x, y^2, xy^3 + x^4y, x^5y)$

The only non-empty $(7,s)$-transversal of $(x, y^2, xy^3 + x^4y, x^5y)$ is $\{(0,0,0,y^7)\}$ and the $J^7\mathcal{A}$-orbits are:

$(x, y^2, xy^3 + x^4y, x^5y \pm y^7)$	7-determined,
$(x, y^2, xy^3 + x^4y, x^5y)$	(12).

Further calculation by computer for (12) predicts the series:

$$(x, y^2, xy^3 + x^4y, x^5y \pm y^{2k+1}) \qquad (2k+1)\text{-determined}, \quad k \geq 3.$$

Again, this calculation has been done by West.

(10) $(x, y^2, xy^3 + x^4y, 0)$

We merely summarise the findings in this case. The $J^7\mathcal{A}$-orbits over $(x, y^2, xy^3 + x^4y, 0)$ are:

$(x, y^2, xy^3 + x^4y, y^7 \pm x^6y)$	7-determined,
$(x, y^2, xy^3 + x^4y, y^7)$	(i),
$(x, y^2, xy^3 + x^4y, x^6y)$	(ii),
$(x, y^2, xy^3 + x^4y, 0)$	(10).

Continuing with (i) gives the following singularities:

$(x, y^2, xy^3 + x^4y, y^7 \pm x^7y)$	8-determined,
$(x, y^2, xy^3 + x^4y, y^7 \pm x^8y)$	9-determined,
$(x, y^2, xy^3 + x^4y, y^7 + ax^9y)$	10-determined, $a \neq 1$,

and the series:

$(x, y^2, xy^3 + x^4y, y^7 + x^9y \pm x^ky)$ $(k+1)$-determined, $k \geq 10$.

Continuing with (ii) gives the series:

$(x, y^2, xy^3 + x^4y, x^6y \pm y^{2k+1})$ $(2k+1)$-determined, $k \geq 3$.

(The case $k = 3$ occurred above.)

The $J^8\mathcal{A}$-orbits over $(x, y^2, xy^3 + x^4y, 0)$ are:

$(x, y^2, xy^3 + x^4y, x^7y)$	(iii),
$(x, y^2, xy^3 + x^4y, 0)$	(10).

Continuing with (iii) gives the series:

$(x, y^2, xy^3 + x^4y, x^7y \pm y^{2k+1})$ $(2k+1)$-determined, $k \geq 4$.

The $J^9\mathcal{A}$-orbits over $(x, y^2, xy^3 + x^4y, 0)$ are:

$(x, y^2, xy^3 + x^4y, y^9 \pm x^8y)$	9-determined,
$(x, y^2, xy^3 + x^4y, y^9)$,	
$(x, y^2, xy^3 + x^4y, x^8y)$,	
$(x, y^2, xy^3 + x^4y, 0)$	(10).

These extend analogously to the 7-jets over $(x, y^2, xy^3 + x^4y, 0)$. Generally we obtain four orbits (modulo a factor of ± 1 in the terms) of degree $2k+1$, for $k \geq 3$, over $(x, y^2, xy^3 + x^4y, 0)$, and the second extends to a unimodular family in a higher degree jet-space (cf. (i) above). Such complications make the general series over $(x, y^2, xy^3 + x^4y, 0)$ hard to obtain. However, we have obtained the two series:

$$(x, y^2, xy^3 + x^4y, x^{2k+1}y \pm y^{2j+1}) \qquad (2j+1)\text{-determined}, \quad k \geq 2, \quad j \geq k+1,$$
$$(x, y^2, xy^3 + x^4y, x^{2k}y \pm y^{2j+1}) \qquad (2j+1)\text{-determined}, \quad k \geq 3, \quad j \geq k,$$

(the cases $k = 2$ and $k = 3$ were shown above) which occur naturally during further classification. The details are similar to previous calculations and are omitted.

(3) $(x, y^2, xy^3, 0)$

The first non-empty $(6, s)$-transversal of $(x, y^2, xy^3, 0)$ is the $(6, 6)$-transversal: $\{(0, 0, 0, x^5y)\}$. The corresponding orbits are (x, y^2, xy^3, x^5y) and $(x, y^2, xy^3, 0)$. The higher $(6, s)$-transversals are empty for the first; the only non-empty transversal for the second is the $(6, 7)$-transversal: $\{0, 0, x^5y, 0)\}$. Altogether, there are four $J^6\mathcal{A}$-orbits:

$$(x, y^2, xy^3, x^5y) \qquad (13),$$
$$(x, y^2, xy^3 \pm x^5y, 0) \qquad (14),$$
$$(x, y^2, xy^3, 0) \qquad (3).$$

(13) (x, y^2, xy^3, x^5y)

Further calculation by computer predicts the series:

$$(x, y^2, xy^3, x^5y + y^{2k+1}) \qquad (2k+1)\text{-determined}, \quad k \geq 3.$$

This is indeed the case, as shown by West.

(14) $(x, y^2, xy^3 \pm x^5y, 0)$

A $(7, 1)$-transversal of $(x, y^2, xy^3 \pm x^5y, 0)$ is $\{(0, 0, 0, y^7)\}$. For all the resulting orbits, a $(7, 7)$-transversal is $\{(0, 0, 0, x^6y)\}$; all higher $(7, s)$-transversals are empty.

The $J^7\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^5y, y^7 \pm x^6y) & \text{7-determined,} \\ (x, y^2, xy^3 \pm x^5y, y^7) & (15), \\ (x, y^2, xy^3 \pm x^5y, x^6y) & (16), \\ (x, y^2, xy^3 \pm x^5y, 0) & (14). \end{array}$$

(15) $(x, y^2, xy^3 \pm x^5y, y^7)$

Further calculation gives the singularities:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^5y, y^7 + x^7y) & \text{8-determined,} \\ (x, y^2, xy^3 \pm x^5y, y^7 \pm x^8y) & \text{9-determined,} \\ \vdots & \\ (x, y^2, xy^3 \pm x^5y, y^7 + x^{11}y) & \text{12-determined.} \end{array}$$

However, this is interrupted at the 13-jet-level by the occurrence of the unimodular family

$$(x, y^2, xy^3 \pm x^5y, y^7 + ax^{12}y) \qquad \text{13-determined,} \quad a \neq \pm 1,$$

(where $a \neq \pm 1$ is respective of the term $\pm x^5y$ in the jet) and we choose not to investigate this further.

(16) $(x, y^2, xy^3 \pm x^5y, x^6y)$

Further calculation by computer predicts the series:

$$(x, y^2, xy^3 \pm x^5y, x^6y \pm y^{2k+1}) \qquad (2k+1)\text{-determined, } k \geq 3.$$

This is indeed the case; though we omit the details, they are similar to previous cases.

(14) $(x, y^2, xy^3 \pm x^5y, 0)$

The only non-empty $(8, s)$-transversal of $(x, y^2, xy^3 \pm x^5y, 0)$ is $\{(0, 0, 0, x^7y)\}$ and the $J^8\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^5y, x^7y), & \\ (x, y^2, xy^3 \pm x^5y, 0) & (14). \end{array}$$

Further calculation with the first of these gives the series:

$$(x, y^2, xy^3 \pm x^5y, x^7y \pm y^{2k+1}) \qquad (2k+1)\text{-determined},\ k \geq 4.$$

Again, we omit the details.

(14) $(x, y^2, xy^3 \pm x^5y, 0)$ **(3)** $(x, y^2, xy^3, 0)$

We still have to consider the $J^9\mathcal{A}$-orbits over (14) and the $J^7\mathcal{A}$-orbits over (3), but we choose to halt the classification for both of these. Further results suggest what has already gone by is just part of a more general series. For example, the $J^9\mathcal{A}$-orbits over $(x, y^2, xy^3 \pm x^5y, 0)$ are:

$$\begin{array}{ll} (x, y^2, xy^3 \pm x^5y, y^9 \pm x^8y) & \text{9-determined}, \\ (x, y^2, xy^3 \pm x^5y, y^9), & \\ (x, y^2, xy^3 \pm x^5y, x^8y), & \\ (x, y^2, xy^3 \pm x^5y, 0) & (14). \end{array}$$

Compare with the $J^7\mathcal{A}$-orbits over $(x, y^2, xy^3 \pm x^5y, 0)$. More generally, the $J^7\mathcal{A}$-orbits over $(x, y^2, xy^3, 0)$ are:

$$\begin{array}{ll} (x, y^2, xy^3, y^7 \pm x^6y) & \text{7-determined}, \\ (x, y^2, xy^3 + x^6y, y^7), & \\ (x, y^2, xy^3, y^7), & \\ (x, y^2, xy^3, x^6y), & \\ (x, y^2, xy^3 + x^6y, 0), & \\ (x, y^2, xy^3, 0) & (3). \end{array}$$

Compare with the $J^5\mathcal{A}$-orbits over $(x, y^2, xy^3, 0)$.

Remark. The above suggests that if we continue we will find that all our earlier series are incorporated in a larger, more general, series. That is, the 4-jet $(x, y^2, xy^3, 0)$ appears to be a stem. So there is no obvious stopping point with the classification of jets over $(x, y^2, xy^3, 0)$, but we prefer to stop here. We do remark that our computer methods *are* capable of pursuing the classification further, as future considerations, such as specific applications to geometry, dictate.

We now return to our penultimate 4-jet.

(4) $(x, y^2, x^3y, 0)$

The $J^5\mathcal{A}$-orbits over $(x, y^2, x^3y, 0)$ are:

$(x, y^2, x^3y, y^5 \pm x^2y^3)$	5-determined,
(x, y^2, x^3y, y^5)	5-determined,
$(x, y^2, x^3y + y^5, x^2y^3)$	5-determined,
$(x, y^2, x^3y + y^5, 0)$	(17),
(x, y^2, x^3y, x^2y^3)	(18),
$(x, y^2, x^3y, 0)$	(4).

To see this note that a $(5,1)$-transversal of $(x, y^2, x^3y, 0)$ is $\{(0,0,0,y^5)\}$ giving the $(5,1)$-orbits (x, y^2, x^3y, y^5) and $(x, y^2, x^3y, 0)$. For the first, the only higher non-empty $(5,s)$-transversal is the $(5,3)$-transversal: $\{(0,0,0,x^2y^3)\}$, and we obtain the $J^5\mathcal{A}$-orbits $(x, y^2, x^3y, y^5 \pm x^2y^3)$ and (x, y^2, x^3y, y^5). For the second, a $(5,2)$-transversal is $\{(0,0,y^5,0)\}$ giving the $(5,2)$-orbits $(x, y^2, x^3y + y^5, 0)$ and $(x, y^2, x^3y, 0)$. The only higher $(5,s)$-transversal in both cases is the $(5,3)$-transversal: $\{(0,0,0,x^2y^3)\}$, and the resulting $J^5\mathcal{A}$-orbits are as listed above.

Remark (Non-Simple Germs). The 5-jet (x, y^2, x^3y, y^5) and all jets with 5-jet $(x, y^2, x^3y, 0)$ are clearly non-simple. Any open neighbourhood of such a jet contains a jet equivalent to one in the unimodular family $(x, y^2, xy^3 + x^3y, y^5 + ax^4y)$. We can rule out the others as being non-simple as well. For example, any open neighbourhood of the 5-jet $(x, y^2, x^3y, y^5 \pm x^2y^3)$ contains a jet of the form $(x, y^2, x^3y + \epsilon xy^3, y^5 \pm x^2y^3)$ for $\epsilon \neq 0$. Applying scaling coordinate changes we can write this as $(x, y^2, xy^3 \pm x^3y, y^5 + bx^2y^3)$ for some $b \in \mathbf{R}$. But this jet has $(5,1)$-jet $(x, y^2, xy^3 \pm x^3y, y^5)$ so, by the complete transversal calculation in Section 3.3.5(2), must be equivalent as a 5-jet to a member of the unimodular family $(x, y^2, xy^3 \pm x^3y, y^5 + ax^4y)$. A similar argument shows the remaining 5-jets are non-simple.

(17) $(x, y^2, x^3y + y^5, 0)$

The only non-empty $(6,s)$-transversal of $(x, y^2, x^3y + y^5, 0)$ is $\{(0,0,0,xy^5)\}$ and the $J^6\mathcal{A}$-orbits are:

$(x, y^2, x^3y + y^5, xy^5)$	(19),
$(x, y^2, x^3y + y^5, 0)$	(17).

(19) $(x, y^2, x^3y + y^5, xy^5)$

The only non-empty $(7, s)$-transversal of $(x, y^2, x^3y + y^5, xy^5)$ is $\{(0, 0, 0, y^7)\}$. It is easily checked that the parameter in the resulting 1-parameter family of jets can be scaled to ± 1 if it is non-zero (cf. earlier cases) and the $J^7\mathcal{A}$-orbits are:

$(x, y^2, x^3y + y^5, xy^5 \pm y^7)$	7-determined,
$(x, y^2, x^3y + y^5, xy^5)$	7-determined.

(17) $(x, y^2, x^3y + y^5, 0)$

A $(7, 1)$-transversal of $(x, y^2, x^3y + y^5, 0)$ is $\{(0, 0, 0, y^7)\}$ and the resulting $(7, 1)$-orbits are (x, y^2, x^3y+y^5, y^7) and $(x, y^2, x^3y+y^5, 0)$. In both cases the only higher $(7, s)$-transversal is the $(7, 3)$-transversal: $\{(0, 0, 0, x^2y^5)\}$. The $J^7\mathcal{A}$-orbits (after scaling) are therefore:

$(x, y^2, x^3y + y^5, y^7 \pm x^2y^5)$	7-determined,
$(x, y^2, x^3y + y^5, y^7)$	7-determined,
$(x, y^2, x^3y + y^5, x^2y^5)$	(20),
$(x, y^2, x^3y + y^5, 0)$	(17).

(20) $(x, y^2, x^3y + y^5, x^2y^5)$

The only non-empty $(8, s)$-transversal of $(x, y^2, x^3y + y^5, x^2y^5)$ is $\{(0, 0, 0, xy^7)\}$ and (after scaling) there are two $J^8\mathcal{A}$-orbits:

$(x, y^2, x^3y + y^5, x^2y^5 \pm xy^7)$	8-determined,
$(x, y^2, x^3y + y^5, x^2y^5)$	8-determined.

(17) $(x, y^2, x^3y + y^5, 0)$

Continuing gives the rather intricate series:

$(x, y^2, x^3y + y^5, xy^{2k+1} \pm y^{2k+3})$	$(2k+3)$-determined, $k \geq 2$,
$(x, y^2, x^3y + y^5, xy^{2k+1})$	$(2k+3)$-determined, $k \geq 2$,
$(x, y^2, x^3y + y^5, y^{2k+3} \pm x^2y^{2k+1})$	$(2k+3)$-determined, $k \geq 2$,
$(x, y^2, x^3y + y^5, y^{2k+3})$	$(2k+3)$-determined, $k \geq 2$,
$(x, y^2, x^3y + y^5, x^2y^{2k+1} \pm xy^{2k+3})$	$(2k+4)$-determined, $k \geq 2$,
$(x, y^2, x^3y + y^5, x^2y^{2k+1})$	$(2k+4)$-determined, $k \geq 2$.

This series was first encountered by West; however, its full nature only became apparent after classification using the computer. The details have also been verified by hand calculation — the original calculation was due to West so full details appear in [We].

(18) (x, y^2, x^3y, x^2y^3)

The only non-empty $(6, s)$-transversal of (x, y^2, x^3y, x^2y^3) is $\{(0, 0, xy^5, 0)\}$ and the $J^6\mathcal{A}$-orbits are:

$(x, y^2, x^3y \pm xy^5, x^2y^3)$	(21),
(x, y^2, x^3y, x^2y^3)	(18).

(21) $(x, y^2, x^3y \pm xy^5, x^2y^3)$

The 7-transversal gives the $J^7\mathcal{A}$-orbits over $(x, y^2, x^3y \pm xy^5, x^2y^3)$ as the bimodular family:

$$(x, y^2, x^3y \pm xy^5 + by^7, x^2y^3 + ay^7) \qquad \text{7-determined, } a^3 \mp 2a^2 + b^2 + a \neq 0.$$

Indeed, working with the group $\mathcal{A}$ in the 7-jet-space $J^7(2,4)$, it is verified by computer that $\{(0,0,y^7,0), (0,0,0,y^7)\}$ forms an independent set to $L\mathcal{A} \cdot f$, where f is the above family. That is,

$$\Big\{ \text{basis for } L\mathcal{A} \cdot f \text{ in } J^7(2,4) \Big\} \cup \Big\{ (0,0,y^7,0), (0,0,0,y^7) \Big\}$$

is an independent set, so by Theorem 1.9 a and b are both moduli. The determinacy calculation was performed by computer and shows that f is 7-determined for generic a and b, that is, (a, b) not lying on the cubic curve $a^3 \mp 2a^2 + b^2 + a = 0$ (where $\mp 2a^2$ is respective of $\pm xy^5$ in f). For the obvious reasons we will not investigate the non-generic behaviour.

(18) (x, y^2, x^3y, x^2y^3)

Pursuing this jet further gives the following singularities:

$$\begin{array}{ll} (x, y^2, x^3y + ay^7, x^2y^3 \pm y^7) & \text{(i)}, \\ (x, y^2, x^3y + ay^7 \pm y^{2k+1}, x^2y^3 \pm y^7) & \text{(ii)}, \\ (x, y^2, x^3y + y^7, x^2y^3) & \text{(iii)}, \\ (x, y^2, x^3y, x^2y^3) & \text{(18)}. \end{array}$$

where

(i) 7-determined, $a^2 \pm 1 \neq 0$ (± 1 respective of $\pm y^7$),
(ii) $(2k+1)$-determined, $k \geq 4$, $a^2 \pm 1 = 0$,
(iii) 7-determined.

The $J^8\mathcal{A}$-orbits over (x, y^2, x^3y, x^2y^3) are:

$$\begin{array}{ll} (x, y^2, x^3y \pm xy^7, x^2y^3), & \\ (x, y^2, x^3y, x^2y^3) & \text{(18)}. \end{array}$$

We do not consider these further; compare with (21).

(4) $(x, y^2, x^3y, 0)$

As noted in Section 3.3.5 Part (2), any jet with 5-jet $(x, y^2, x^3y, 0)$ is non-simple. We will not consider this jet further, apart from remarking on the following two stems which are easily obtained. Further calculation gives the $J^6\mathcal{A}$-orbits over $(x, y^2, x^3y, 0)$ as :

$$\begin{array}{ll} (x, y^2, x^3y, xy^5), & \\ (x, y^2, x^3y \pm xy^5, 0), & \\ (x, y^2, x^3y, 0) & \text{(4)}. \end{array}$$

We shall only consider the first of these. The $J^7\mathcal{A}$-orbits over (x, y^2, x^3y, xy^5) are:

$$\begin{array}{lll} (x, y^2, x^3y + ay^7, xy^5 + y^7) & \text{7-determined}, & a \neq 1, \\ (x, y^2, x^3y + y^7, xy^5) & \text{7-determined}, & \\ (x, y^2, x^3y, xy^5). & & \end{array}$$

In the first the parameter a is a modulus. By considering the exceptional value of a and also continuing with the jet (x, y^2, x^3y, xy^5) we then obtain the following two series:

$(x, y^2, x^3y + y^7 \pm y^{2k+1}, xy^5 + y^7)$	$(2k+1)$-determined, $k \geq 4$,
$(x, y^2, x^3y + y^{2k+1}, xy^5)$	$(2k+1)$-determined, $k \geq 3$.

We prefer to omit the details of the calculations.

3.3.6 The 4-Jet $(x, y^2, 0, 0)$

Previous calculations do not allow us to immediately say that any jet with 4-jet $(x, y^2, 0, 0)$ is non-simple. Further calculation shows the $J^4\mathcal{A}$-orbits over $(x, y^2, 0, 0)$ are of an extremely complex nature; an $\mathcal{A}_1$-5-transversal contains *six* terms (using the unipotent group $\mathcal{G}$ does improve matters) and moduli are present immediately. However, this now allows us to rule out all such jets, without resorting to arguments on counting codimensions' of orbits and families as has often been employed in the past. By appealing to the complete transversal method we not only rule out the non-simples but also obtain the $J^5\mathcal{A}$-orbits over $(x, y^2, 0, 0)$ and therefore extend the classification as a bonus!

Using the $M_{r,s}(\mathcal{G})$-filtration we proceed to to classify the $(5, s)$-orbits over the 4-jet $(x, y^2, 0, 0)$. As in Section 3.3.2 and Section 3.3.4 we must be careful and avoid redundant orbits (for example, all of the orbits over the $(5, 1)$-jet $(x, y^2, 0, y^5)$ are redundant; they are equivalent to orbits over the $(5, 2)$-jet $(x, y^2, y^5, 0)$). The calculation was performed by computer. Alternatively, we can resort to the $\mathcal{A}_1$-transversal techniques, but, as before, the simplification of the orbits must be performed by hand and is fairly tedious. For reference we note that an $\mathcal{A}_1$-5-transversal for $(x, y^2, 0, 0)$ is

$$\{(0,0,0,y^5), (0,0,y^5,0), (0,0,0,x^2y^3), (0,0,x^2y^3,0), (0,0,0,x^4y), (0,0,x^4y,0)\},$$

and, with respect to the $M_{r,s}(\mathcal{G})$-filtration method, the same vectors occur, being part of the $(5,1)$-, $(5,2)$-, $(5,3)$-, $(5,4)$-, $(5,5)$- and $(5,6)$-transversals, respectively. In summary, the $J^5\mathcal{A}$-orbits over $(x, y^2, 0, 0)$ are:

$$\begin{array}{ll}
(x, y^2, y^5 + ax^4y, x^2y^3 \pm x^4y), & \text{5-determined,} \\
 & \quad a \neq 0, -1, \\
(x, y^2, y^5 \pm x^4y, x^2y^3), & \text{5-determined,} \\
(x, y^2, y^5, x^2y^3), & \\
(x, y^2, y^5 \pm x^2y^3, x^4y) & \text{5-determined,} \\
(x, y^2, y^5 \pm x^2y^3 + ax^4y, 0), & \\
(x, y^2, y^5, x^4y), & \\
(x, y^2, y^5 \pm x^4y, 0), & \\
(x, y^2, y^5, 0), & \\
(x, y^2, x^2y^3, x^4y), & \\
(x, y^2, x^2y^3 \pm x^4y, 0), & \\
(x, y^2, x^2y^3, 0), & \\
(x, y^2, x^4y, 0), & \\
(x, y^2, 0, 0) & \text{(C).}
\end{array}$$

Now, using the same argument as in Section 3.3.4 Part (8), say, it follows that any jet with 5-jet one of the above is non-simple. For example, any open neighbourhood of $(x, y^2, y^5 \pm x^2y^3, x^4y)$ contains the 5-jet $(x, y^2, y^5 \pm x^2y^3, \epsilon x^2y^3 + x^4y)$ for some $\epsilon \neq 0$. The latter jet is $J^5\mathcal{A}$-equivalent to a jet of the form $(x, y^2, y^5 + ax^4y, x^2y^3 \pm x^4y)$ and is therefore not simple.

We do not investigate the above jets further, but now turn our attention to the 2-jet (D).

3.3.7 The 2-Jet $(x, xy, 0, 0)$

We now return to the 2-jet (D): $(x, xy, 0, 0)$. The classification of jets over this 2-jet is far more intensive than the classification of jets over $(x, y^2, 0, 0)$ performed above. In particular, our computer package performed most of the work below. The search for series over $(x, xy, 0, 0)$ is not as fruitful as in the previous sections. In this section we mainly content ourselves with ruling out the non-simples over $(x, xy, 0, 0)$. The classification was taken a lot further using the computer than suggested below, looking for series at higher jet-levels. In most cases the results did not suggest series and were not of enough general interest to include below. When a non-simple is encountered we will often just say that this jet is not considered further; the classification was taken further by computer and the results were used as our criterion for stopping.

There are six $J^3\mathcal{A}$-orbits over $(x, xy, 0, 0)$:

$$\begin{array}{ll} (x, xy, y^3, xy^2) & (1), \\ (x, xy, y^3, 0) & (2), \\ (x, xy + y^3, xy^2, 0) & (3), \\ (x, xy + y^3, 0, 0) & (4), \\ (x, xy, xy^2, 0) & (5), \\ (x, xy, 0, 0) & (D). \end{array}$$

To see this use the same methods as in Section 3.3.2. An $\mathcal{A}_1$-3-transversal is

$$\{(0,0,0,y^3), (0,0,y^3,0), (0,0,0,xy^2), (0,y^3,0,0), (0,0,xy^2,0)\},$$

then use linear algebra and elementary coordinate changes to reduce the resulting family of jets to the given forms. Alternatively, use the $M_{r,s}(\mathcal{G})$-filtration. Consider $(x, xy, 0, 0)$ as a $(3,1)$-jet. A $(3,2)$-transversal is $\{(0,0,y^3,0), (0,0,0,xy^2)\}$ and the resulting $(3,2)$-orbits are (x, xy, y^3, xy^2), $(x, xy, y^3, 0)$, $(x, xy, 0, xy^2)$ and $(x, xy, 0, 0)$. The higher $(3,s)$-transversals are empty in the first and second cases and we therefore obtain the corresponding $J^3\mathcal{A}$-orbits. The only non-empty transversal for the third is the $(3,3)$-transversal: $\{(0, y^3, 0, 0)\}$, but all the resulting orbits will be equivalent to ones stated above and we may therefore rule out this case. So now consider $(x, xy, 0, 0)$ as a $(3,2)$-jet. The only non-empty $(3,s)$-transversal is the $(3,3)$ transversal: $\{(0, y^3, 0, 0), (0, 0, xy^2, 0)\}$ and we obtain the $(3,3)$-orbits $(x, xy + y^3, xy^2, 0)$, $(x, xy + y^3, 0, 0)$, $(x, xy, xy^2, 0)$ and $(x, xy, 0, 0)$. The higher $(3,s)$-transversals are empty in each case so we obtain the stated $J^3\mathcal{A}$-orbits.

We remark that the above calculation, using the $M_{r,s}(\mathcal{G})$-filtration, can be performed in its entirety using the computer and is extremely slick. However, for the $\mathcal{A}_1$-transversal method we must resort to hand calculations and it is a lot more cumbersome to reduce the resulting orbits to those stated above.

Remark (Determinacy Calculations Using the Computer).
For $f \in m_2.\mathcal{E}(2,4)$, if

$$m_2^{k+1}.\mathcal{E}(2,4) \subset L\mathcal{G} \cdot f + m_2^{k+1}.f^*(m_4).\mathcal{E}(2,4) + m_2^{2k+2}.\mathcal{E}(2,4)$$

then f is k-determined. The determinacy calculation can therefore be performed in the jet-space $J^{2k+1}(2,4)$ — this is necessary for computer calculation. However, the calculations can be very intensive, and the smaller the degree of the jet-space needed, the better. In the earlier cases, where f had 2-jet $(x, y^2, 0, 0)$, since

$m_2^{k+1}.f^*(m_4).\mathcal{E}_2 \supset m_2^{k+3}$ we need only work in the jet-space $J^{k+2}(2,4)$. This no longer applies to germs with 2-jet $(x, xy, 0, 0)$. However, for the 3-jets (1), (2), (3) and (4) above we can work in $J^{k+3}(2,4)$; and the determinacy calculations performed below for the 4-jets over (5) may be done in $J^{k+4}(2,4)$, as will be seen. The majority of the determinacy calculations below were performed by computer using these observations.

(1) (x, xy, y^3, xy^2)

The only non-empty $(4,s)$-transversal of (x, xy, y^3, xy^2) is $\{(0,0,0,y^4)\}$ and there are two $J^4\mathcal{A}$-orbits:

$(x, xy, y^3, xy^2 + y^4)$	4-determined,
(x, xy, y^3, xy^2)	(1).

Continuing with (1), we find that a $(5,1)$-transversal is $\{(0,0,0,y^5)\}$ and the resulting orbits are $(x, xy, y^3, xy^2 + y^5)$ and (x, xy, y^3, xy^2). The higher $(5,s)$-transversals are empty in the first case; for the second a $(5,3)$-transversal is $\{(0, y^5, 0, 0)\}$ and the orbits are $(x, xy+y^5, y^3, xy^2)$ and (x, xy, y^3, xy^2). All higher $(5,s)$-transversals are empty and the $J^5\mathcal{A}$-orbits are:

$(x, xy, y^3, xy^2 + y^5)$	5-determined,
$(x, xy + y^5, y^3, xy^2)$	5-determined,
(x, xy, y^3, xy^2)	(1).

Next, the 6-transversal of (x, xy, y^3, xy^2) is empty, so this jet represents the only $J^6\mathcal{A}$-orbit. Continuing with (1) gives the following series:

$(x, xy, y^3, xy^2 + y^{3k+1})$	$(3k+1)$-determined,	$k \geq 1$,
$(x, xy, y^3, xy^2 + y^{3k+2})$	$(3k+2)$-determined,	$k \geq 1$,
$(x, xy + y^{3k+2}, y^3, xy^2)$	$(3k+2)$-determined,	$k \geq 1$.

To see this consider $f = (x, xy, y^3, xy^2)$ as a $3k$-jet, for some $k \geq 1$. Using the $\mathcal{L}_1$-tangent space we can obtain the vectors $x^a y^b e_i$ of degree $3k+1$, where $a \geq 1$ (because the components of f include xy^l, for $l = 0$, 1, 2, and y^{3m}, for $m \geq 1$). Now

$$\partial f/\partial x = (1, y, 0, y^2), \quad \partial f/\partial y = (0, x, 3y^2, 2xy),$$

and we can can obtain $y^{3k+1}e_1$ modulo $m_2^{3k+2}.\mathcal{E}(2,4)$. Also,

$$\begin{aligned} y^{3k+1}e_2 &= y^{3k}\partial/\partial x(f) - u_3^k \partial/\partial u_1(f), \\ 3y^{3k+1}e_3 &= y^{3k-1}\partial/\partial y(f) - u_3^{k-1}u_4\partial/\partial u_2(f) - 2u_1u_3^k\partial/\partial u_4(f), \end{aligned}$$

again working modulo $m_2^{3k+2}.\mathcal{E}(2,4)$. A little thought shows we cannot obtain $y^{3k+1}e_4$, so a $(3k+1)$-transversal is $\{(0,0,0,y^{3k+1})\}$ and the $J^{3k+1}\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x,xy,y^3,xy^2+y^{3k+1}) & (3k+1)\text{-determined}, \\ (x,xy,y^3,xy^2) & (1). \end{array}$$

For the determinacy calculation we can work modulo $m_2^{3k+5}.\mathcal{E}(2,4)$. Put $f = (x,xy,y^3,xy^2+y^{3k+1})$. Firstly consider the terms of degree $3k+2$ and work modulo $m_2^{3k+3}.\mathcal{E}(2,4)$. As before we can obtain the vectors $x^a y^b e_i$, for $a \geq 1$, using the $\mathcal{L}_1$-tangent space. Now

$$\partial f/\partial x = (1,y,0,y^2), \quad \partial f/\partial y = (0,x,3y^2,2xy+(3k+1)y^{3k}),$$

and $y^{3k+2}e_1$ follows from $\partial f/\partial x$. Also,

$$\begin{aligned} (y^{3k+1},0,0,0) &= -xy^2\partial/\partial x(f) + u_4\partial/\partial u_1(f) + \\ &\quad\; u_1u_3\partial/\partial u_2(f) + u_2u_3\partial/\partial u_4(f), \\ (y^{3k+1},y^{3k+2},0,y^{3k+3}) &= y^{3k+1}\partial/\partial x(f), \end{aligned}$$

so $y^{3k+2}e_2 \in L\mathcal{G}\cdot f$ modulo $m_2^{3k+3}.\mathcal{E}(2,4)$. (Note that $u_4\partial/\partial u_1 \in L\mathcal{G}$ but $u_4\partial/\partial u_1 \notin L\mathcal{A}_1$.) Finally,

$$3y^{3k+2}e_3 = y^{3k}\partial/\partial y(f) - u_1u_3^k\partial/\partial u_2(f) - 2u_2u_3^k\partial/\partial u_4(f),$$

and

$$\begin{aligned} (0,y^{3k+1},0,0) &= -xy\partial/\partial x(f) + u_2\partial/\partial u_1(f) + \\ &\quad\; u_4\partial/\partial u_2(f) + u_1u_3\partial/\partial u_4(f), \\ (0,y^{3k+1},0,y^{3k+2}) &= y^{3k}\partial/\partial x(f) - u_3^k\partial/\partial u_1(f), \end{aligned}$$

so $y^{3k+2}e_3$ and $y^{3k+2}e_4$ are both in $L\mathcal{G}\cdot f$. (Again it is worth noting that whereas $u_2\partial/\partial u_1$, $u_4\partial/\partial u_2$ and $u_3^k\partial/\partial u_1$ are contained in $L\mathcal{G}$ they are not contained in $L\mathcal{A}_1$, for $k=1$.) We now consider the terms of degree $3k+3$ and work modulo $m_2^{3k+4}.\mathcal{E}(2,4)$. As before we can obtain the vectors $x^a y^b e_i$, for $a \geq 1$, using the $\mathcal{L}_1$-tangent space. Clearly, $y^{3k+3}e_i \in L\mathcal{L}_1\cdot f$, so now consider the terms of degree $3k+4$; again we need only consider the vectors $y^{3k+4}e_i$. We can obtain $y^{3k+4}e_1$ from $\partial f/\partial x$, and

$$y^{3k+4}e_2 = y^{3k+3}\partial/\partial x(f) - u_3^{k+1}\partial/\partial u_1(f),$$

modulo $m_2^{3k+5}.\mathcal{E}(2,4)$. Now

$$u_3u_4\partial/\partial u_i(f) = (xy^5+y^{3k+4})e_i,$$

and

$$\begin{aligned}(0,0,3xy^5,0) &= xy^3\partial/\partial y(f) - u_1^2u_3\partial/\partial u_2(f) - \\ &\quad 2u_1u_2u_3\partial/\partial u_4(f) - (3k+1)u_1u_3^{k+1}\partial/\partial u_4(f), \\ (0,0,0,xy^5) &= xy^3\partial/\partial x(f) - u_1u_3\partial/\partial u_1(f) - u_2u_3\partial/\partial u_2(f),\end{aligned}$$

so we obtain $y^{3k+4}e_3$ and $y^{3k+4}e_4$. It now follows that f is $(3k+1)$-determined.

Next consider the $(3k+1)$-jet $f = (x, xy, y^3, xy^2)$ and compute the $(3k+2)$-transversals. As before, we need only concern ourselves with the vectors $y^{3k+2}e_i$. We cannot obtain $y^{3k+2}e_4$ so a $(3k+2,1)$-transversal is $\{(0,0,0,y^{3k+2})\}$ giving the $(3k+2,1)$-orbits $(x, xy, y^3, xy^2 + y^{3k+2})$ and (x, xy, y^3, xy^2). Firstly consider $f = (x, xy, y^3, xy^2 + y^{3k+2})$. Then,

$$\partial f/\partial x = (1, y, 0, y^2), \quad \partial f/\partial y = (0, x, 3y^2, 2xy + (3k+2)y^{3k+1}),$$

and, modulo $m_2^{3k+3}.\mathcal{E}(2,4)$,

$$\begin{aligned}3y^{3k+2}e_3 &= y^{3k}\partial/\partial y(f) - u_1u_3^k\partial/\partial u_2(f) - 2u_2u_3^k\partial/\partial u_4(f), \\ y^{3k+2}e_2 &= -xy\partial/\partial x(f) + u_2\partial/\partial u_1(f) + u_4\partial/\partial u_2(f) + u_1u_3\partial/\partial u_4(f),\end{aligned}$$

(note that $u_2\partial/\partial u_1$ and $u_4\partial/\partial u_2$ are in $L\mathcal{G}$) and finally $y^{3k+2}e_1$ follows from $\partial f/\partial x$. So all the higher $(3k+2,s)$-transversals are empty. Now consider the $(3k+2,1)$-jet (x, xy, y^3, xy^2). We can obtain $y^{3k+2}e_3$ and $y^{3k+2}e_1$ as above, but not $y^{3k+2}e_2$. A $(3k+2,3)$-transversal is $\{(0, y^{3k+2}, 0, 0)\}$; for both of the resulting orbits the higher $(3k+2,s)$-transversals are empty. So the $J^{3k+2}\mathcal{A}$-orbits over (x, xy, y^3, xy^2) are:

$$\begin{array}{ll}(x, xy, y^3, xy^2 + y^{3k+2}) & \text{(i)}, \\ (x, xy + y^{3k+2}, y^3, xy^2) & \text{(ii)}, \\ (x, xy, y^3, xy^2) & (1).\end{array}$$

Consider (i) and (ii), these are both $(3k+2)$-determined. For the determinacy calculations we can work modulo $m_2^{3k+6}.\mathcal{E}(2,4)$. Firstly consider the terms of degree $3k+3$ and work modulo $m_2^{3k+4}.\mathcal{E}(2,4)$. Then for both (i) and (ii), the vectors $x^ay^be_i$, for $a \geq 1$, follow as before, and the vectors $y^{3k+3}e_i$ clearly lie in $L\mathcal{L}_1 \cdot f$. Now consider the terms of degree $3k+4$ and work modulo $m_2^{3k+5}.\mathcal{E}(2,4)$. As before we need only consider the vectors $y^{3k+4}e_i$. For $y^{3k+4}e_1$, $y^{3k+4}e_2$ and $y^{3k+4}e_3$, the following observations apply to (i) and (ii). The vector $y^{3k+4}e_1$ follows from $\partial f/\partial x$ and

$$\begin{aligned}y^{3k+4}e_2 &= y^{3k+3}\partial/\partial x(f) - u_3^{k+1}\partial/\partial u_1(f), \\ 3y^{3k+4}e_3 &= y^{3k+2}\partial/\partial y(f) - u_3^ku_4\partial/\partial u_2(f) - 2u_1u_3^{k+1}\partial/\partial u_4(f).\end{aligned}$$

Then, for case (i) we have

$$\begin{aligned}(-xy^2,0,0,y^{3k+4}) &= y^{3k+2}\partial/\partial x(f) - u_3^{k+1}\partial/\partial u_2(f) - u_4\partial/\partial u_1(f),\\ (xy^2,0,0,0) &= xy^2\partial/\partial x(f) - u_1u_3\partial/\partial u_2(f) - u_2u_3\partial/\partial u_4(f);\end{aligned}$$

and for case (ii) we have

$$\begin{aligned}(-xy,0,0,y^{3k+4}) &= y^{3k+2}\partial/\partial x(f) - u_3^{k+1}\partial/\partial u_2(f) - u_2\partial/\partial u_1(f),\\ (xy,0,0,0) &= xy\partial/\partial x(f) - u_4\partial/\partial u_2(f) - u_1u_3\partial/\partial u_4(f);\end{aligned}$$

so that $y^{3k+4}e_4$ follows in both cases. Finally consider the terms of degree $3k+5$ and work modulo $m_2^{3k+6}.\mathcal{E}(2,4)$. As before we need only consider the vectors $y^{3k+5}e_i$. For (i) and (ii) $y^{3k+5}e_1$ follows from $\partial f/\partial x$. Now, for case (i)

$$\begin{aligned}(xy^5+y^{3k+5})e_2 &= u_3u_4\partial/\partial u_2(f),\\ (0,xy^5,0,0) &= xy^4\partial/\partial x(f) - u_2u_3\partial/\partial u_1(f) - u_1u_3^2\partial/\partial u_4(f),\end{aligned}$$

and for case (ii)

$$\begin{aligned}(xy^4+y^{3k+5})e_2 &= u_2u_3\partial/\partial u_2(f),\\ (0,xy^4,0,0) &= xy^3\partial/\partial x(f) - u_1u_3\partial/\partial u_1(f) - u_3u_4\partial/\partial u_4(f),\end{aligned}$$

and we obtain $y^{3k+5}e_2$ in both cases. Similarly, $y^{3k+5}e_3$ and $y^{3k+5}e_4$ follow from

$$\begin{aligned}u_3u_4\partial/\partial u_i(f) &= (xy^5+y^{3k+5})e_i,\\ xy^3\partial/\partial y(f) &= (0,x^2y^3,3xy^5,2x^2y^4+(3k+2)xy^{3k+4}),\\ xy^3\partial/\partial x(f) &= (xy^3,xy^4,0,xy^5),\end{aligned}$$

for case (i); and from

$$\begin{aligned}u_2u_3\partial/\partial u_i(f) &= (xy^4+y^{3k+5})e_i,\\ xy^2\partial/\partial y(f) &= (0,x^2y^2+(3k+2)xy^{3k+3},3xy^4,2x^2y^3),\\ xy^2\partial/\partial x(f) &= (xy^2,xy^3,0,xy^4),\end{aligned}$$

for case (ii). We can therefore deduce that (i) and (ii) are $(3k+2)$-determined.

It remains to consider the $(3k+2)$-jet (x,xy,y^3,xy^2). In this case

$$m_2^{3k+3}.\mathcal{E}(2,4) \subset L\mathcal{G}\cdot f$$

so that the $(3k+3)$-transversal is empty. We are therefore back to our original consideration of (x,xy,y^3,xy^2) as a jet of degree a multiple of 3.

(2) $(x, xy, y^3, 0)$

The only non-empty $(4, s)$-transversal of $(x, xy, y^3, 0)$ is $\{(0, 0, 0, y^4)\}$ and the $J^4\mathcal{A}$-orbits are:

(x, xy, y^3, y^4)	4-determined,
$(x, xy, y^3, 0)$	(2).

(2) $(x, xy, y^3, 0)$

A $(5, 1)$-transversal is $\{(0, 0, 0, y^5)\}$ and the resulting orbits are (x, xy, y^3, y^5) and $(x, xy, y^3, 0)$. The higher $(5, s)$-transversals are empty in the first case; while in the second only the $(5, 3)$-transversal contributes. In total, we obtain the $J^5\mathcal{A}$-orbits:

(x, xy, y^3, y^5)	5-determined,
$(x, xy + y^5, y^3, 0)$	(6),
$(x, xy, y^3, 0)$	(2).

(6) $(x, xy + y^5, y^3, 0)$

The only non-empty $(6, s)$-transversal is $\{(0, 0, 0, xy^5)\}$ and the $J^6\mathcal{A}$-orbits are:

$(x, xy + y^5, y^3, xy^5)$	(7),
$(x, xy + y^5, y^3, 0)$	(6).

(7) $(x, xy + y^5, y^3, xy^5)$

Continuing we find that $J^7\mathcal{A}$-orbits are:

$(x, xy + y^5, y^3, xy^5 \pm y^7)$	(8),
$(x, xy + y^5, y^3, xy^5)$	(7).

Next, the $J^8\mathcal{A}$-orbits over (8) are:

$(x, xy + y^5, y^3, xy^5 \pm y^7 + ay^8)$	8-determined, for all $a \in \mathbf{R}$.

Computer calculation using the group $\mathcal{A}$ in $J^8(2, 4)$ shows that a is indeed a modulus. It follows from the standard arguments (see Section 3.3.4) that any jet with 7-jet (7): $(x, xy + y^5, y^3, xy^5)$ is non-simple. The jet $(x, xy + y^5, y^3, xy^5)$ is not considered further.

(6) $(x, xy + y^5, y^3, 0)$

The only non-empty $(7, s)$-transversal of $(x, xy+y^5, y^3, 0)$ is $\{(0, 0, 0, y^7)\}$ and the $J^7\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, xy + y^5, y^3, y^7) & (9), \\ (x, xy + y^5, y^3, 0) & (6). \end{array}$$

It now follows that any jet with 7-jet $(x, xy + y^5, y^3, y^7)$ or $(x, xy + y^5, y^3, 0)$ is non-simple since a neighbourhood of such a jet contains a jet equivalent to one from the unimodular family $(x, xy + y^5, y^3, xy^5 \pm y^7 + ay^8)$. Further classification with (9) gives the singularities:

$$\begin{array}{ll} (x, xy + y^5, y^3, y^7 + y^8) & \text{8-determined}, \\ (x, xy + y^5, y^3, y^7) & \text{8-determined}. \end{array}$$

Continuing with (6) suggests the existence of an intricate series involving moduli; however, (6) is non-simple as a 7-jet and we will not consider it further.

(2) $(x, xy, y^3, 0)$

The only non-empty $(6, s)$-transversal of $(x, xy, y^3, 0)$ is $\{(0, 0, 0, xy^5)\}$ and the $J^6\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, xy, y^3, xy^5) & (10), \\ (x, xy, y^3, 0) & (2). \end{array}$$

(10) (x, xy, y^3, xy^5)

The only non-empty $(7, s)$-transversal of (x, xy, y^3, xy^5) is $\{(0, 0, 0, y^7)\}$ and the $J^7\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, xy, y^3, xy^5 + y^7) & (11), \\ (x, xy, y^3, xy^5) & (10). \end{array}$$

Comparison with the unimodular family $(x, xy+y^5, y^3, xy^5 \pm y^7 + ay^8)$ shows that any jet with 7-jet $(x, xy, y^3, xy^5 + y^7)$ or (x, xy, y^3, xy^5) is non-simple. We will not consider these jets further, apart from noting that there are two $J^8\mathcal{A}$-orbits over $(x, xy, y^3, xy^5 + y^7)$, both 8-determined:

$$\begin{array}{ll} (x, xy, y^3, xy^5 + y^7 + y^8) & \text{8-determined}, \\ (x, xy, y^3, xy^5 + y^7) & \text{8-determined}. \end{array}$$

(2) $(x, xy, y^3, 0)$

The only non-empty $(7, s)$-transversal of $(x, xy, y^3, 0)$ is $\{(0, 0, 0, y^7)\}$ and the $J^7\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, xy, y^3, y^7) & (12), \\ (x, xy, y^3, 0) & (2). \end{array}$$

Again, these jets are non-simple and we can rule them out from further consideration. We will just note that the $J^8\mathcal{A}$-orbits over (x, xy, y^3, y^7) are determined:

$$\begin{array}{ll} (x, xy, y^3, y^7 + y^8) & \text{8-determined}, \\ (x, xy, y^3, y^7) & \text{8-determined}. \end{array}$$

Having dealt with the (2): $(x, xy, y^3, 0)$, we now return to the 3-jet (3).

(3) $(x, xy + y^3, xy^2, 0)$

A $(4, 1)$-transversal of $(x, xy+y^3, xy^2, 0)$ is $\{(0, 0, 0, y^4)\}$ giving the orbits $(x, xy+y^3, xy^2, y^4)$ and $(x, xy + y^3, xy^2, 0)$. In the first case, the higher $(4, s)$-transversals are empty so we may now consider this as a 5-jet. In the second case, the only non-empty $(4, s)$-transversal is the $(4, 2)$-transversal: $\{(0, 0, y^4, 0), (0, 0, 0, xy^3)\}$. In total, we obtain the $J^4\mathcal{A}$-orbits:

$$\begin{array}{ll} (x, xy + y^3, xy^2, y^4) & (13), \\ (x, xy + y^3, xy^2 + ay^4, xy^3) & (14), \\ (x, xy + y^3, xy^2 + ay^4, 0) & (15). \end{array}$$

For cases (14) and (15) we cannot scale a to a unit. Computer calculation verifies that a is a modulus in both of these cases.

(13) $(x, xy + y^3, xy^2, y^4)$

The only non-empty $(5, s)$-transversal of $(x, xy + y^3, xy^2, y^4)$ is $\{(0, 0, y^5, 0)\}$ and the $J^5\mathcal{A}$-orbits are:

$$\begin{array}{ll} (x, xy + y^3, xy^2 + y^5, y^4) & \text{5-determined}, \\ (x, xy + y^3, xy^2, y^4) & (13). \end{array}$$

The 6-transversal of $(x, xy + y^3, xy^2, y^4)$ is empty. Continuing leads to the series:

$$(x, xy + y^3, xy^2 + y^{2k+1}, y^4) \qquad (2k + 1)\text{-determined}, \quad k \geq 2.$$

The details are similar to previous cases and we omit them.

(14) $(x, xy + y^3, xy^2 + ay^4, xy^3)$

The only non-empty $(5, s)$-transversal is $\{(0, 0, 0, y^5)\}$; this is the case for all values of a. The $J^5\mathcal{A}$-orbits form the bimodular family:

$$(x, xy + y^3, xy^2 + ay^4, xy^3 + by^5) \qquad \text{5-determined } (\dagger).$$

($\dagger$) The conditions on the moduli a and b which are required for determinacy are given by the computer determinacy check and are pretty complicated. However, we can deduce from the computer calculation that the family is 5-determined for generic (a, b). To see this we merely observe that the computer determines a finite set of algebraic curves, and the condition that (a, b) does not lie on any of these curves, is sufficient for determinacy. Each curve determines a submanifold of $\mathbf{R}^2$ (or $\mathbf{C}^2$, depending on context) of codimension 1, except at a finite number of singular points. The complement is therefore open and dense (in fact, is the complement of a null set). We will not consider this jet further.

(15) $(x, xy + y^3, xy^2 + ay^4, 0)$

For $a \neq \frac{3}{2}, \frac{1}{2}$ or 2, the only non-empty $(5, s)$-transversal of $(x, xy+y^3, xy^2+ay^4, 0)$ is $\{(0, 0, 0, y^5)\}$. For the exceptional values of a the $J^5\mathcal{A}$-orbits have a more complicated structure. We will not discuss these, but just note that for generic a (specifically, a not equal to the above values) the $J^5\mathcal{A}$-orbits are:

$$(x, xy + y^3, xy^2 + ay^4, y^5) \qquad \text{5-determined } (\dagger),$$
$$(x, xy + y^3, xy^2 + ay^4, 0) \qquad (15).$$

($\dagger$) The remarks in (14) apply. The computer determinacy check gives complicated conditions on the modulus a, but allows us to say that the family is 5-determined for generic a. (That is, for all a except for the roots of a finite number of algebraic equations.) We will not discuss the exceptional values, nor the $J^6\mathcal{A}$-orbits over (15). Having dealt with the orbits over (3) we now return to the 3-jet (4).

(4) $(x, xy + y^3, 0, 0)$

The first non-empty $(4, s)$-transversal of $(x, xy + y^3, 0, 0)$ is the $(4, 1)$-transversal: $\{(0, 0, 0, y^4)\}$. However, the resulting $J^4\mathcal{A}$-orbits are redundant. Consider $(x, xy+$

$y^3, 0, 0)$ as a $(4,1)$-jet; a $(4,2)$-transversal is $\{(0,0,y^4,0),(0,0,0,xy^3)\}$. The resulting $(4,2)$-orbits are $(x, xy+y^3, y^4, xy^3)$, $(x, xy+y^3, y^4, 0)$, $(x, xy+y^3, 0, xy^3)$ (which is redundant and need not be considered) and $(x, xy+y^3, 0, 0)$. The higher $(4,s)$-transversals are empty in all but the last case, where a $(4,3)$-transversal is $\{(0,0,xy^3,0)\}$. All higher transversals are then empty and the $J^4\mathcal{A}$-orbits are:

$$
\begin{array}{ll}
(x, xy+y^3, y^4, xy^3) & (16), \\
(x, xy+y^3, y^4, 0) & (17), \\
(x, xy+y^3, xy^3, 0) & (18), \\
(x, xy+y^3, 0, 0) & (4).
\end{array}
$$

Any open neighbourhood of such a jet will contain a jet equivalent to one in the unimodular family (14): $(x, xy+y^3, xy^2+ay^4, xy^3)$. So these four jets are non-simple. The full nature of the orbits above these is complicated; we will just look at each briefly.

(16) $(x, xy+y^3, y^4, xy^3)$

The only non-empty $(5,s)$-transversal of $(x, xy+y^3, y^4, xy^3)$ is $\{(0,0,0,y^5)\}$ and the $J^5\mathcal{A}$-orbits form the family:

$$(x, xy+y^3, y^4, xy^3+ay^5).$$

Computer calculations verify that a is a modulus. The 6-transversal for this family is empty provided $a \neq -1$ and $a \neq \frac{6}{5}$; for such values the only non-empty $(6,s)$-transversal is $\{(0,0,0,y^6)\}$. The $J^6\mathcal{A}$-orbits are therefore:

$$
\begin{array}{l}
(x, xy+y^3, y^4, xy^3+ay^5), \\
(x, xy+y^3, y^4, xy^3-y^5+y^6), \\
(x, xy+y^3, y^4, xy^3+\frac{6}{5}y^5+y^6).
\end{array}
$$

We will only consider the first; but even this one is complicated by the fact that the 7-transversal is $\{(0,0,0,y^7)\}$ only for $a \neq -1$ and contains further terms otherwise. We just note that the following are the $J^7\mathcal{A}$-orbits:

$$
\begin{array}{lll}
(x, xy+y^3, y^4, xy^3+ay^5 \pm y^7) & \text{7-determined}, & \text{all } a \in \mathbf{R}, \\
(x, xy+y^3, y^4, xy^3-y^5 \pm xy^6) & \text{7-determined}, & \\
(x, xy+y^3, y^4, xy^3+ay^5) & \text{7-determined}, & a \neq \pm 1.
\end{array}
$$

(17) $(x, xy+y^3, y^4, 0)$

A $(5,1)$-transversal for $(x, xy+y^3, y^4, 0)$ is $\{(0,0,0,y^5)\}$. For both of the resulting orbits the only higher $(5,s)$-transversal is the $(5,3)$-transversal: $\{(0,0,0,x^2y^3)\}$.

The $J^5\mathcal{A}$-orbits are therefore:

$$\begin{array}{ll} (x, xy+y^3, y^4, y^5 \pm x^2y^3) & \text{5-determined,} \\ (x, xy+y^3, y^4, y^5) & \text{5-determined,} \\ (x, xy+y^3, y^4, x^2y^3), & \\ (x, xy+y^3, y^4, 0) & (17). \end{array}$$

The $J^6\mathcal{A}$-orbits over $(x, xy+y^3, y^4, x^2y^3)$ are then:

$$\begin{array}{l} (x, xy+y^3, y^4, x^2y^3+y^6), \\ (x, xy+y^3, y^4, x^2y^3), \end{array}$$

and the first determined jet arises from $(x, xy+y^3, y^4, x^2y^3+y^6)$ at the 7-jet-level:

$$(x, xy+y^3, y^4, x^2y^3+y^6+ay^7) \qquad \text{7-determined,} \quad a \neq -1.$$

Here a is a modulus. We will not discuss the exceptional value of a nor the orbits over the 6-jet $(x, xy+y^3, y^4, x^2y^3)$.

Of more interest is (17); the $J^6\mathcal{A}$-orbits over $(x, xy+y^3, y^4, 0)$ are:

$$\begin{array}{ll} (x, xy+y^3, y^4, y^6), & \\ (x, xy+y^3, y^4, 0) & (17). \end{array}$$

Then continuing with $(x, xy+y^3, y^4, y^6)$ gives the series:

$$(x, xy+y^3, y^4, y^6+y^{2k+1}) \qquad (2k+1)\text{-determined,} \quad k \geq 3.$$

We omit the details of the calculation. The $J^7\mathcal{A}$-orbits over (17) are:

$$\begin{array}{ll} (x, xy+y^3, y^4, y^7+xy^6), & \\ (x, xy+y^3, y^4, y^7), & \\ (x, xy+y^3, y^4, xy^6), & \\ (x, xy+y^3, y^4, 0) & (17). \end{array}$$

These are not considered further.

(18) $(x, xy+y^3, xy^3, 0)$

Further calculation shows that the $J^5\mathcal{A}$-orbits over $(x, xy+y^3, xy^3, 0)$ are quite intricate:

$$\begin{array}{l} (x, xy+y^3, xy^3+xy^4, y^5), \\ (x, xy+y^3, xy^3, y^5), \\ (x, xy+y^3, xy^3+ay^5, xy^4), \\ (x, xy+y^3, xy^3+ay^5, 0), \\ (x, xy+y^3, xy^3+\frac{6}{5}y^5+xy^4, 0), \end{array}$$

where the parameter a appearing in two of the above cases is a modulus. These must be taken into the 6- and 7-jet-spaces and even the first case leads to complicated structures for the orbits. We do not pursue these further.

(4) $(x, xy + y^3, 0, 0)$

We do not consider this 4-jet further. To hint at some of the difficulties now present, we just remark that if we were using the $\mathcal{A}_1$-complete transversal methods then the resulting 5-transversal has *six* terms. Even using the unipotent group $\mathcal{G}$ does not improve matters much. We now return to the 3-jets.

(5) $(x, xy, xy^2, 0)$

The rest of this section will just be a brief excursion, studying the jets over $(x, xy, xy^2, 0)$. Our main aim is to show the remaining jets are non-simple so that they may be excluded from further consideration.

Now, a $(4, 1)$-transversal of $(x, xy, xy^2, 0)$ is $\{(0, 0, 0, y^4)\}$ and the resulting orbits are (x, xy, xy^2, y^4) and $(x, xy, xy^2, 0)$. All higher $(4, s)$-transversals are empty in the first case. For the second a $(4, 2)$-transversal is $\{(0, 0, y^4, 0), (0, 0, 0, xy^3)\}$ and, after scaling, we obtain the orbits $(x, xy, xy^2 + y^4, xy^3)$, $(x, xy, xy^2 + y^4, 0)$, (x, xy, xy^2, xy^3) and $(x, xy, xy^2, 0)$. For the first two cases the higher $(4, s)$-transversals are empty. For the latter two, only the $(4, 3)$-transversal is non-empty: $\{(0, y^4, 0, 0)\}$. The $J^4\mathcal{A}$-orbits over $(x, xy, xy^2, 0)$ are therefore:

(x, xy, xy^2, y^4)	(19),
$(x, xy, xy^2 + y^4, xy^3)$	(20),
$(x, xy, xy^2 + y^4, 0)$	(21),
$(x, xy + y^4, xy^2, xy^3)$	(22),
(x, xy, xy^2, xy^3)	(23),
$(x, xy + y^4, xy^2, 0)$	(24),
$(x, xy, xy^2, 0)$	(5).

The usual arguments show that, by comparison with the unimodular family (14): $(x, xy + y^3, xy^2 + ay^4, xy^3)$, any jet with 4-jet (20), (21), (23) or (5) must be non-simple. We will briefly discuss the structure of the orbits over these jets and also rule out the others as being non-simple.

(19) (x, xy, xy^2, y^4)

The only non-empty $(5, s)$-transversal of (x, xy, xy^2, y^4) is $\{(0, 0, y^5, 0)\}$ and the $J^5\mathcal{A}$-orbits are:

$(x, xy, xy^2 + y^5, y^4)$	(25),
(x, xy, xy^2, y^4)	(19).

(25) $(x, xy, xy^2 + y^5, y^4)$

The first non-empty $(6, s)$-transversal is the $(6, 2)$-transversal: $\{(0, 0, y^6, 0)\}$. For both of the resulting orbits the only other non-empty transversal is the $(6, 3)$-transversal: $\{(0, y^6, 0, 0)\}$. After scaling the $J^6\mathcal{A}$-orbits are therefore:

$$\begin{array}{ll} (x, xy + ay^6, xy^2 + y^5 + y^6, y^4), & \\ (x, xy \pm y^6, xy^2 + y^5, y^4), & \\ (x, xy, xy^2 + y^5, y^4) & (25). \end{array}$$

Computer calculation proves a is a modulus. It can now be seen that all of these jets are non-simple. Continuing, the $J^7\mathcal{A}$-orbits over the first and second form the trimodular and bimodular families:

$$\begin{array}{ll} (x, xy + ay^6 + by^7, xy^2 + y^5 + y^6 + cy^7, y^4) & \text{7-determined, for all } a,\ b,\ c, \\ (x, xy \pm y^6 + ay^7, xy^2 + y^5 + by^7, y^4) & \text{7-determined, for all } a,\ b, \end{array}$$

respectively. The $J^7\mathcal{A}$-orbits over (25) are:

$$\begin{array}{ll} (x, xy + ay^7, xy^2 + y^5 \pm y^7, y^4) & \text{7-determined, for all } a, \\ (x, xy \pm y^7, xy^2 + y^5, y^4) & \text{7-determined,} \\ (x, xy, xy^2 + y^5, y^4) & \text{7-determined.} \end{array}$$

(19) (x, xy, xy^2, y^4)

Further calculation gives the $J^6\mathcal{A}$-orbits over (x, xy, xy^2, y^4) as follows:

$$\begin{array}{ll} (x, xy + y^6, xy^2 + y^6, y^4), & \\ (x, xy, xy^2 + y^6, y^4), & \\ (x, xy + y^6, xy^2, y^4), & \\ (x, xy, xy^2, y^4) & (19). \end{array}$$

Then, due to the unimodular family $(x, xy+ay^6, xy^2+y^5+y^6, y^4)$ obtained above, any jet with 6-jet one of the above four is not simple.

(20) $(x, xy, xy^2 + y^4, xy^3)$

The only non-empty $(5, s)$-transversal is $\{(0, 0, 0, y^5)\}$ and the $J^5\mathcal{A}$-orbits over $(x, xy, xy^2 + y^4, xy^3)$ form the family:

$$(x, xy, xy^2 + y^4, xy^3 + ay^5).$$

Computer calculations verify that a is a modulus. We will not consider this further, only note that the 6-transversal is empty for generic a, and that the exceptional values complicate the situation. In fact, the $J^6\mathcal{A}$-orbits are:

$$\begin{array}{l}(x, xy, xy^2 + y^4, xy^3 + ay^5),\\ (x, xy, xy^2 + y^4, xy^3 + y^6),\\ (x, xy + y^6, xy^2 + y^4, xy^3),\\ (x, xy, xy^2 + y^4 \pm y^6, xy^3 + \frac{4}{3}y^5),\end{array}$$

but then, even for the first, there are several $J^7\mathcal{A}$-orbits.

(21) $(x, xy, xy^2 + y^4, 0)$

Again, the only non-empty $(5, s)$-transversal is $\{(0, 0, 0, y^5)\}$ and the resulting $J^5\mathcal{A}$-orbits are:

$$\begin{array}{ll}(x, xy, xy^2 + y^4, y^5), & \\ (x, xy, xy^2 + y^4, 0), & (21).\end{array}$$

We already know that any jet with 4-jet $(x, xy, xy^2 + y^4, 0)$ is not simple so these are both excluded from our list of simple singularities. We just note that the 6-transversal of the first, $(x, xy, xy^2 + y^4, y^5)$, is empty, and then the $J^7\mathcal{A}$-orbits provide the following 7-determined jets:

$$\begin{array}{lll}(x, xy + ay^7, xy^2 + y^4, y^5 \pm y^7) & \text{7-determined,} & \text{for all } a,\\ (x, xy \pm y^7, xy^2 + y^4, y^5) & \text{7-determined,} & \\ (x, xy, xy^2 + y^4, y^5) & \text{7-determined.} & \end{array}$$

(22) $(x, xy + y^4, xy^2, xy^3)$

The first non-empty $(5, s)$-transversal is the $(5, 1)$-transversal: $\{(0, 0, 0, y^5)\}$, and the resulting orbits are $(x, xy + y^4, xy^2, xy^3 + y^5)$ and $(x, xy + y^4, xy^2, xy^3)$. The higher $(5, s)$-transversals are empty in the first case, and in the second case only the $(5, 2)$-transversal: $\{(0, 0, y^5, 0)\}$, contributes. The resulting $J^5\mathcal{A}$-orbits are:

$$\begin{array}{l}(x, xy + y^4, xy^2, xy^3 + y^5),\\ (x, xy + y^4, xy^2 + ay^5, xy^3).\end{array}$$

It is verified that a is a modulus in the second. We do not consider these further, just eliminate the first from being simple. A 6-transversal is $\{(0, 0, y^6, 0)\}$ and the $J^6\mathcal{A}$-orbits over $(x, xy + y^4, xy^2, xy^3 + y^5)$ form the family:

$$(x, xy + y^4, xy^2 + ay^6, xy^3 + y^5),$$

where a is a modulus (as usual, verified by computer).

(23) (x, xy, xy^2, xy^3)

We already know that any jet with 4-jet (x, xy, xy^2, xy^3) is non-simple. We do not consider this jet further; the structure of the higher orbits is complicated by extreme branching. This is not surprising if one considers the terms present in the jet. For instance, the $J^5\mathcal{A}$-orbits form several branches:

$$\begin{array}{ll} (x, xy, xy^2, xy^3 + y^5), & \\ (x, xy, xy^2 + y^5, xy^3), & \\ (x, xy + y^5, xy^2, xy^3), & \\ (x, xy, xy^2, xy^3) & \qquad (23), \end{array}$$

and such branching occurs for several higher levels.

(24) $(x, xy + y^4, xy^2, 0)$

The first non-empty transversal is the $(5,1)$-transversal: $\{(0,0,0,y^5)\}$, giving the $(5,1)$-orbits $(x, xy + y^4, xy^2, y^5)$ and $(x, xy + y^4, xy^2, 0)$. The higher $(5,s)$-transversals are empty in the first case. In the second case a $(5,2)$-transversal is $\{(0,0,y^5,0),(0,0,0,xy^4)\}$; all higher transversals are empty. The $J^5\mathcal{A}$-orbits over $(x, xy + y^4, xy^2, 0)$ are:

$$\begin{array}{l} (x, xy + y^4, xy^2, y^5), \\ (x, xy + y^4, xy^2 + ay^5, xy^4), \\ (x, xy + y^4, xy^2 + ay^5, 0). \end{array}$$

In the second and third cases a is a modulus and we do not consider these further. To show the first is non-simple we note that a 6-transversal is $\{(0,0,y^6,0)\}$ and the resulting $J^6\mathcal{A}$-orbits are:

$$\begin{array}{l} (x, xy + y^4, xy^2 + y^6, y^5), \\ (x, xy + y^4, xy^2, y^5). \end{array}$$

Then, further calculation gives the $J^7\mathcal{A}$-orbits over $(x, xy + y^4, xy^2 + y^6, y^5)$ as:

$$(x, xy + y^4, xy^2 + y^6 + by^7, y^5 + ay^7) \qquad \text{7-determined, for all } a,\ b.$$

The calculation to show a and b are moduli was performed by computer (together with the determinacy calculation, noted on passing). Then, by the usual arguments, it follows that any jet with 6-jet $(x, xy + y^4, xy^2 + y^6, y^5)$ or $(x, xy + y^4, xy^2, y^5)$ is non-simple, and we will not consider these further.

(5) $(x, xy, xy^2, 0)$

We already know that any jet with 4-jet $(x, xy, xy^2, 0)$ is non-simple. Branching occurs for several higher levels and we will just list the $J^5\mathcal{A}$-orbits over $(x, xy, xy^2, 0)$ to give an indication of this:

$$\begin{array}{l} (x, xy, xy^2, y^5), \\ (x, xy, xy^2 + y^5, xy^4), \\ (x, xy, xy^2 + y^5, 0), \\ (x, xy + y^5, xy^2, xy^4), \\ (x, xy, xy^2, xy^4), \\ (x, xy + y^5, xy^2, 0), \\ (x, xy, xy^2, 0). \end{array}$$

3.3.8 The 3-Jet $(x, xy, 0, 0)$

We now return to the 3-jet (D): $(x, xy, 0, 0)$. There are six $J^4\mathcal{A}$-orbits over this jet:

$$\begin{array}{ll} (x, xy, y^4, xy^3) & (1), \\ (x, xy, y^4, 0) & (2), \\ (x, xy + y^4, xy^3, 0) & (3), \\ (x, xy + y^4, 0, 0) & (4), \\ (x, xy, xy^3, 0) & (5), \\ (x, xy, 0, 0) & (D). \end{array}$$

To see this use the same methods as in Section 3.3.2. An $\mathcal{A}_1$-3-transversal is

$$\{(0,0,0,y^4), (0,0,y^4,0), (0,0,0,xy^3), (0,y^4,0,0), (0,0,xy^3,0)\},$$

then use linear algebra and elementary coordinate changes to reduce the resulting family of jets to the given forms. Alternatively, use the $M_{r,s}(\mathcal{G})$-filtration; the procedure is entirely analogous to the calculation of the $J^3\mathcal{A}$-orbits over $(x, xy, 0, 0)$ using the $(3, s)$-transversals — see Section 3.3.7. Continuing the classification leads to rather complex structures and we will just concentrate on showing that any jet with 4-jet one of the above is non-simple.

Consider some jet j with 4-jet either (1), (2), (5) or (D). Then any open neighbourhood of j must contain a jet equivalent, as 4-jets, to $(x, xy+y^3, xy^2+ay^4, xy^3)$ and by Section 3.3.7 Part (3) we see that j is therefore not simple. Whereas, if j has 4-jet either (3) or (4) then any open neighbourhood of j must contain a jet equivalent, as 4-jets, to $(x, xy + y^3, y^4, xy^3)$. But then, by Section 3.3.7 Part(16), this jet must be equivalent, as 5-jets, to $(x, xy+y^3, y^4, xy^3+ay^5)$ and is therefore not simple.

We do not consider these 4-jets further, but now return to the final 2-jet.

3.3.9 The 2-Jet $(x, 0, 0, 0)$

The $J^3\mathcal{A}$-orbits over $(x, 0, 0, 0)$ are:

$$
\begin{array}{ll}
(x, y^3, xy^2, x^2y) & (1),\\
(x, y^3 \pm x^2y, xy^2, 0) & (2),\\
(x, y^3, xy^2, 0) & (3),\\
(x, y^3, x^2y, 0) & (4),\\
(x, y^3 \pm x^2y, 0, 0) & (5),\\
(x, y^3, 0, 0) & (6),\\
(x, xy^2, x^2y, 0) & (7),\\
(x, xy^2, 0, 0) & (8),\\
(x, x^2y, 0, 0) & (9),\\
(x, 0, 0, 0) & (A).
\end{array}
$$

This follows from the usual techniques, see, for example, Section 3.3.2. For reference we will state an $\mathcal{A}_1$-6-transversal; the same terms appear when we use the unipotent group $\mathcal{G}$, the corresponding $(3, s)$-level is indicated below too:

$$
\begin{array}{ll}
(0, 0, 0, y^3), & (3,1),\\
(0, 0, y^3, 0), (0, 0, 0, xy^2), & (3,2),\\
(0, y^3, 0, 0), (0, 0, xy^2, 0), (0, 0, 0, x^2y), & (3,3),\\
(0, xy^2, 0, 0), (0, 0, x^2y, 0), & (3,4),\\
(0, x^2y, 0, 0) & (3,5).
\end{array}
$$

The details are similar to previous cases. Note that the $(3,1)$-jet $(x, 0, 0, y^3)$ and the $(3,2)$-jets $(x, 0, y^3, xy^2)$, $(x, 0, y^3, 0)$ and $(x, 0, 0, xy^2)$ are all redundant — they are equivalent to jets obtained at the $(3,3)$-, $(3,4)$- and $(3,5)$-levels. There is a slight abnormality in the proceedings which we include for completeness, however.

Remark (Simplifying Families). Continuing the classification for the $(3,1)$-jet $(x, 0, 0, y^3)$ we obtain, at the $(3,5)$-level, the jets

$$(x, x^2y, xy^2 + ax^2y, y^3 \pm x^2y) \quad \text{and} \quad (x, 0, xy^2 + ax^2y, y^3 \pm x^2y).$$

The first is equivalent to (x, x^2y, xy^2, y^3) and poses no problems. However, in the second we cannot apply 'scaling' type coordinate changes to reduce a to a unit, yet it is not a modulus. Indeed, working with the group $\mathcal{A}$ in $J^3(2,4)$ with $f = (x, 0, xy^2 + ax^2y, y^3 \pm x^2y)$ we find that the tangent space $L\mathcal{A} \cdot f$ is of dimension 26 and contains the vector $(0, 0, x^2y, 0)$ provided $3a^2 \pm 4 \neq 0$ (the ± 4 is respective of the sign of the x^2ye_4 term in the jet). When $3a^2 \pm 4 = 0$ this fails and the dimension drops to 25. Thus, by the Mather Lemma, this family collapses into a finite number of orbits. For example, consider the case $(x, 0, xy^2 + ax^2y, y^3 - x^2y)$. In the real case we obtain the orbits

$$\begin{array}{l}(x, 0, xy^2, y^3 - x^2y),\\ (x, 0, xy^2 + \frac{2}{\sqrt{3}}x^2y, y^3 - x^2y),\\ (x, 0, xy^2 + 2x^2y, y^3 - x^2y).\end{array}$$

In the complex case the first and the last are certainly equivalent. The above forms are rather cumbersome but this at least suggests that simplification is possible and motivates us to try and reduce $(x, 0, xy^2 + ax^2y, y^3 - x^2y)$ to some normal form, in particular, one of the forms listed as the $J^3\mathcal{A}$-orbits above. Now, the source coordinate change

$$(x, y) \mapsto (x, y - \tfrac{1}{2}ax)$$

reduces the $xy^2 + ax^2y$ term to $x(y^2 - \frac{1}{4}a^2x^2)$. So f is $\mathcal{A}$-equivalent to

$$(x, 0, xy^2 - \tfrac{1}{4}a^2x^3, y^3 - \tfrac{3}{2}axy^2 + (\tfrac{3}{4}a^2 \pm 1)x^2y + (-\tfrac{1}{8}a^3 \mp \tfrac{1}{2}a)x^3)$$

which is equivalent to

$$(x, 0, xy^2, y^3 + (\tfrac{3}{4}a^2 \pm 1)x^2y).$$

So, we can reduce f to one of the stated normal forms, namely $(x, y^3 \pm x^2y, xy^2, 0)$ if $3a^2 \pm 4 \neq 0$, and $(x, y^3, xy^2, 0)$ otherwise. Apart from these extra considerations for the family $(x, 0, xy^2 + ax^2y, y^3 \pm x^2y)$, everything follows though smoothly to produce the list of $J^3\mathcal{A}$-orbits stated above.

(1) (x, y^3, xy^2, x^2y)

A $(4,1)$-transversal for (x, y^3, xy^2, x^2y) is $\{(0, 0, 0, y^4)\}$ and the resulting orbits are $(x, y^3, xy^2, x^2y + y^4)$ and (x, y^3, xy^2, x^2y). All higher transversals are empty in the first case, and just the $(4,2)$-transversal $\{(0, 0, y^4, 0)\}$ in the second. The resulting $J^4\mathcal{A}$-orbits are:

$$\begin{array}{ll}(x, y^3, xy^2, x^2y + y^4), & \\ (x, y^3, xy^2 + y^4, x^2y), & \\ (x, y^3, xy^2, x^2y) & (1).\end{array}$$

The only non-empty $(5,s)$-transversal for the first two is $\{(0,0,0,y^5)\}$ and, in total, we obtain the $J^5\mathcal{A}$-orbits:

$$\begin{array}{lll}
(x, y^3, xy^2, x^2y + y^4 \pm y^5) & \text{5-determined,} & \\
(x, y^3, xy^2, x^2y + y^4) & \text{5-determined,} & \\
(x, y^3, xy^2 + y^4, x^2y + ay^5) & \text{5-determined,} & a \neq 1.
\end{array}$$

The parameter a appearing above is a modulus. It now follows that any jet with 4-jet (1) is non-simple. We do not consider these further.

(2) $\quad (x, y^3 \pm x^2y, xy^2, 0)$

The $J^4\mathcal{A}$-orbits over $(x, y^3 \pm x^2y, xy^2, 0)$ are:

$$\begin{array}{ll}
(x, y^3 \pm x^2y, xy^2, y^4 + ax^3y), & \\
(x, y^3 \pm x^2y, xy^2 + y^4, x^3y), & \\
(x, y^3 \pm x^2y, xy^2 + y^4, 0), & \\
(x, y^3 \pm x^2y, xy^2, x^3y), & \\
(x, y^3 \pm x^2y, xy^2, 0) & (2).
\end{array}$$

A $(4,1)$-transversal for $(x, y^3 \pm x^2y, xy^2, 0)$ is $\{(0,0,0,y^4)\}$ and the resulting orbits are $(x, y^3 \pm x^2y, xy^2, y^4)$ and $(x, y^3 \pm x^2y, xy^2, 0)$. The only higher non-empty transversal for the first is the $(4,4)$-transversal: $\{(0,0,0,x^3y)\}$, giving the unimodular family $(x, y^3 \pm x^2y, xy^2, y^4 + ax^3y)$. For the second, a $(4,2)$-transversal is $\{(0,0,y^4,0)\}$ giving the orbits $(x, y^3 \pm x^2y, xy^2 + y^4, 0)$ and $(x, y^3 \pm x^2y, xy^2, 0)$. In both cases the only higher non-empty transversal is the $(4,4)$-transversal: $\{(0,0,0,x^3y)\}$, and the resulting $J^4\mathcal{A}$-orbits are as listed above. This takes us to codimension 11 and we do not consider these further.

Further calculations and the usual arguments show that any jet with 3-jet (2), (3), ..., (9) or (A) is non-simple. We shall not reproduce the details. All that remains now is to investigate the corank 2 case.

3.3.10 The Corank 2 Case

A germ $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ of corank 2 has 1-jet the zero-jet. Applying the usual classification techniques one finds that the $J^2\mathcal{A}$-orbits over $(0,0,0,0)$ are:

$$(y^2, xy, x^2, 0), \quad (y^2 \pm x^2, xy, 0, 0), \quad (y^2, xy, 0, 0),$$

$$(y^2, x^2, 0, 0), \quad (y^2 \pm x^2, 0, 0, 0), \quad (y^2, 0, 0, 0),$$

$$(xy, x^2, 0, 0), \quad (xy, 0, 0, 0), \quad (x^2, 0, 0, 0), \quad (0, 0, 0, 0).$$

Of these, the ones of equal $J^2\mathcal{A}$-codimension are equivalent, namely

$$\begin{aligned}
(y^2 \pm x^2, xy, 0, 0) &\text{ and } (y^2, x^2, 0, 0), \\
(y^2, xy, 0, 0) &\text{ and } (xy, x^2, 0, 0), \\
(y^2 \pm x^2, 0, 0, 0) &\text{ and } (xy, 0, 0, 0), \\
(y^2, 0, 0, 0) &\text{ and } (x^2, 0, 0, 0),
\end{aligned}$$

as one can easily check. The $J^2\mathcal{A}$-orbits over $(0, 0, 0, 0)$ are therefore as given in Theorem 3.2. Since the $J^2\mathcal{A}$-codimensions differ for all of these we have a minimal set. The higher orbits over these jets are extremely complicated and, with no specific applications to guide us, we stop the classification here. The overall result is a classification of all the singularities $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ up to codimension 8; we feel that this should be easily sufficient for most applications. It remains to prove that none of the corank 2 singularities are simple.

Proposition 3.4 *Any germ $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ with zero 1-jet is non-simple.*

Proof. (J.W. Bruce.) The space X of 3-jets of such germs is of dimension $4(3 + 4) = 28$. We have a group action of $J^3\mathcal{A} = J^3\mathcal{R} \times J^3\mathcal{L}$ on X. However, some elements of $J^3\mathcal{A}$ act trivially on X and, as a little thought shows, the orbits of this group coincide precisely with those of $J^2\mathcal{R} \times J^1\mathcal{L}$. The latter has dimension $10 + 16 = 26$, and so X has a 2-parameter family of moduli. □

Chapter 4

Geometry of Map-Germs from Surfaces to Four-Space

For the geometrical considerations discussed in this section it is more natural to work over the field of complex numbers and consider singularities $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$. Our previous classification provides a list of such singularities, except that the orbits which split into several distinct cases due to $\pm$ terms now collapse into one. Although the concepts apply in general, we take the more plausible line of restricting to the simple singularities. We describe the possible deformations and prove the simplicity of the given map-germs using adjacency diagrams. One of the main concerns in Chapter 3 was to discover stems. The study of stems and their geometry is left for future work, though we show that the results of Chapter 3 do provide an abundant supply of examples. We conclude the current chapter by considering several geometrical invariants associated with map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ and the calculation of such invariants by computer. We remark that all the invariants are defined for real map-germs $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$, but they are only defined algebraically, not geometrically. If f is a polynomial or real analytic map-germ then these invariants take the same values as their complex counterparts defined for the complexification $f_{\mathbf{C}}$ of f. The geometrical interpretation of, say, the double point number $D(f)$ defined in Section 4.3 does not apply in the real case, though one would expect it to reflect some information on the real double points; c.f., [Mo3, Section 6].

Label	Singularity	Determinacy Degree
I	$(x, y, 0, 0)$	1
II_k	(x, y^2, xy, y^{2k+1})	$2k+1, \quad k \geq 1$
III_k	$(x, y^2, y^3, x^k y)$	$k+1, \quad k \geq 2$
$\text{IV}_{j,k}$	$(x, y^2, y^3 + x^j y, x^k y)$	$k+1, \quad j \geq 2, \quad k \geq j+1$
V_k	$(x, xy, y^3, xy^2 + y^{3k+1})$	$3k+1, \quad k \geq 1$
VI_k	$(x, xy, y^3, xy^2 + y^{3k+2})$	$3k+2, \quad k \geq 1$
VII_k	$(x, xy + y^{3k+2}, y^3, xy^2)$	$3k+3, \quad k \geq 1$
VIII_k	$(x, y^2, x^2 y + y^{2k+1}, xy^3)$	$2k+1, \quad k \geq 2$
IX	$(x, y^2, xy^3, x^3 y + y^5)$	5
X	(x, xy, y^3, y^4)	4
XI_k	$(x, xy + y^3, xy^2 + y^{2k+1}, y^4)$	$2k+1, \quad k \geq 2$
XII	$(x, y^2, x^2 y, y^5)$	5
XIII	(x, xy, y^3, y^5)	5
XIV	$(x, y^3, xy^2, x^2 y + y^4 + y^5)$	5
XV	$(x, y^3, xy^2, x^2 y + y^4)$	5

Table 4.1: Simple Singularities $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$

4.1 The Simple Singularities and Stems

Table 4.1 lists all the simple singularities $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$. Each singularity is labelled with a Roman numeral for reference in later sections.

The fact that all of these are simple can be proved from the adjacencies calculated in Section 4.2. We will show that the list is exhaustive. If we follow the classification carried out in Chapter 3 carefully we rule out all map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ from being simple, apart from those in the above table and the 4-jet

$$(x, y^2, xy^3, x^3 y) \qquad \text{see Section 3.3.5(1).}$$

This gives rise to the series

$$(x, y^2, xy^3, x^3 y + y^{2k+1}) \qquad k \geq 2.$$

The member of the series corresponding to $k = 2$ is simple as will be seen. In Chapter 3 it was easy to rule out certain germs from being simple — see Remark 3.3.4(8). To show the germs in the above series for which $k \geq 3$ are non-simple is a bit more involved, requiring explicit coordinate changes. However, the failure of simplicity becomes apparent on unfolding such germs to determine the adjacencies. To show simplicity fails we need only find one adjacency with a member of a modular family. Consider the 6-jet $j = (x, y^2, xy^3, x^3y)$. Any open neighbourhood of j must contain a jet of the form $j_\epsilon = (x, y^2, xy^3 + \epsilon x^2y, x^3y)$ for $\epsilon \neq 0$. Applying 'scaling' coordinate changes we can write this as $(x, y^2, x^2y + xy^3, x^3y)$. The source coordinate change $(x, y) \mapsto (x - \frac{1}{2}y^2, y)$ then gives

$$\begin{aligned} j_\epsilon &\sim_{J^6\mathcal{A}} (x - \tfrac{1}{2}y^2, y^2, x^2y - \tfrac{1}{4}y^5, x^3y - \tfrac{3}{2}x^2y^3 + \tfrac{3}{4}xy^5) \\ &\sim_{J^6\mathcal{A}} (x, y^2, x^2y + y^5, xy^5) \end{aligned}$$

after changes of coordinates in the target followed by further scaling. But referring to Section 3.3.4(8), any germ with 6-jet $(x, y^2, x^2y + y^5, xy^5)$ is $J^7\mathcal{A}$-equivalent to a member of the unimodular family $(x, y^2, x^2y + y^5, xy^5 + ay^7)$. Thus, any germ with 6-jet j is non-simple. Note that at the 5-jet-level, with $k = 2$, we do not encounter this problem. Applying the same coordinate changes we obtain

$$\begin{aligned} (x - \tfrac{1}{2}y^2, y^2, xy^3 + \epsilon x^2y, x^3y + y^5) &\sim_{J^5\mathcal{A}} (x, y^2, x^2y - \tfrac{1}{4}y^5, x^3y - \tfrac{3}{2}x^2y^3 + y^5) \\ &\sim_{J^5\mathcal{A}} (x, y^2, x^2y, y^5) \end{aligned}$$

after changes of coordinates in the target, and this is of type XII.

The aim of Chapter 3 was to discover the simple singularities and stems for map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$. Recall that in the general scenario where we consider germs $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ with $\mathcal{G}$ a subgroup of $\mathcal{K}$, a map-germ f is said to be a *$\mathcal{G}$-stem* if it is not finitely $\mathcal{G}$-determined and there is an integer k_0 such that any map-germ g with the same k_0-jet as f is either $\mathcal{G}$-equivalent to f or is finitely $\mathcal{G}$-determined. Significant results on stems have been obtained by Ratcliffe, concentrating mainly on map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$, [Rat1, Rat2]. However, one general problem is that complete transversal classification techniques, such as those used in Chapter 3, provide us with examples of *weak stems*. A map-germ f is said to be a *$\mathcal{G}$-weak stem* if it is not finitely $\mathcal{G}$-determined and there is an integer k_0 such that any map-germ g with the same k_0-jet as f is either $J^k\mathcal{G}$-equivalent to f for all k or is finitely $\mathcal{G}$-determined. Ratcliffe has made significant progress in showing weak stems are stems, but the problem is still open. However, for a 'series' produced by the complete transversal theorems this problem does not arise. Formally, we have a map-germ f and an integer k_0 such that any map-germ g with the same k_0-jet as f either has the same ∞-jet as f (that is $j^kg = j^kf$ for all k) or is finitely $\mathcal{G}$-determined. If $j^\infty g = j^\infty f$ then the Taylor expansions of g and f at 0 agree, so for analytic map-germs we have $g = f$. It follows that f is a stem. For example, consider the simple singularity of type

II_k and define $f = (x, y^2, xy, 0)$, $k_0 = 2$. The complete transversal calculations in Section 3.3.3 showed that any map-germ g with the same 2-jet as f is either equivalent to a jet of the form (x, y^2, xy, y^{2k+1}), for some $k \geq 2$, and is therefore finitely $\mathcal{A}$-determined, or has the same k-jet as f for all k. We conclude this section by recording this formally.

Remark 4.1 *Given a map-germ f, suppose f is not finitely $\mathcal{G}$-determined and there is an integer k_0 such that any map-germ g with the same k_0-jet as f either has the same ∞-jet as f or is finitely $\mathcal{G}$-determined, then f is a $\mathcal{G}$-stem. In particular, Table 4.1 and the results of Chapter 3 provide a large number of $\mathcal{A}$-stems for map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$.*

4.2 Adjacency Diagrams

We recall the following definition.

Definition 4.2 Let X and Y be two classes of germs $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ under $\mathcal{G}$-equivalence ($\mathcal{G}$ some subgroup of $\mathcal{K}$). We say class X is *adjacent* to class Y, and denote this $X \longrightarrow Y$, if for some representative f of X and some $\mathcal{G}$-unfolding of f, $F : (\mathbf{C}^n \times \mathbf{C}^s, 0) \longrightarrow (\mathbf{C}^p \times \mathbf{C}^s, 0)$, $F(x, u) = (f_u(x), u)$, (so $f_0 = f$ and $f_u(0) = 0$ for small u), the (germ of the) set $\{ u \in \mathbf{C}^s : f_u \in Y \}$ contains $\{u = 0\}$ in its closure.

To calculate all the adjacencies of a class X we can take any representative f of X and need only work with a $\mathcal{G}$-*versal* unfolding F of f.

Remark. In such calculations one occasionally finds that for isolated values of u, f_u belongs to some class Y. This does *not* constitute an adjacency $X \longrightarrow Y$ since the deformation f_u of f does not lie in Y for arbitrarily small values of u (the 'closure' clause in the above definition).

We have calculated the adjacencies for the map-germs in Table 4.1. In particular, we see that each class in the table is adjacent to only a finite number of other classes; this is enough to prove the class is simple (in the sense of Chapter 1). These calculations were extremely involved and we omit the details.

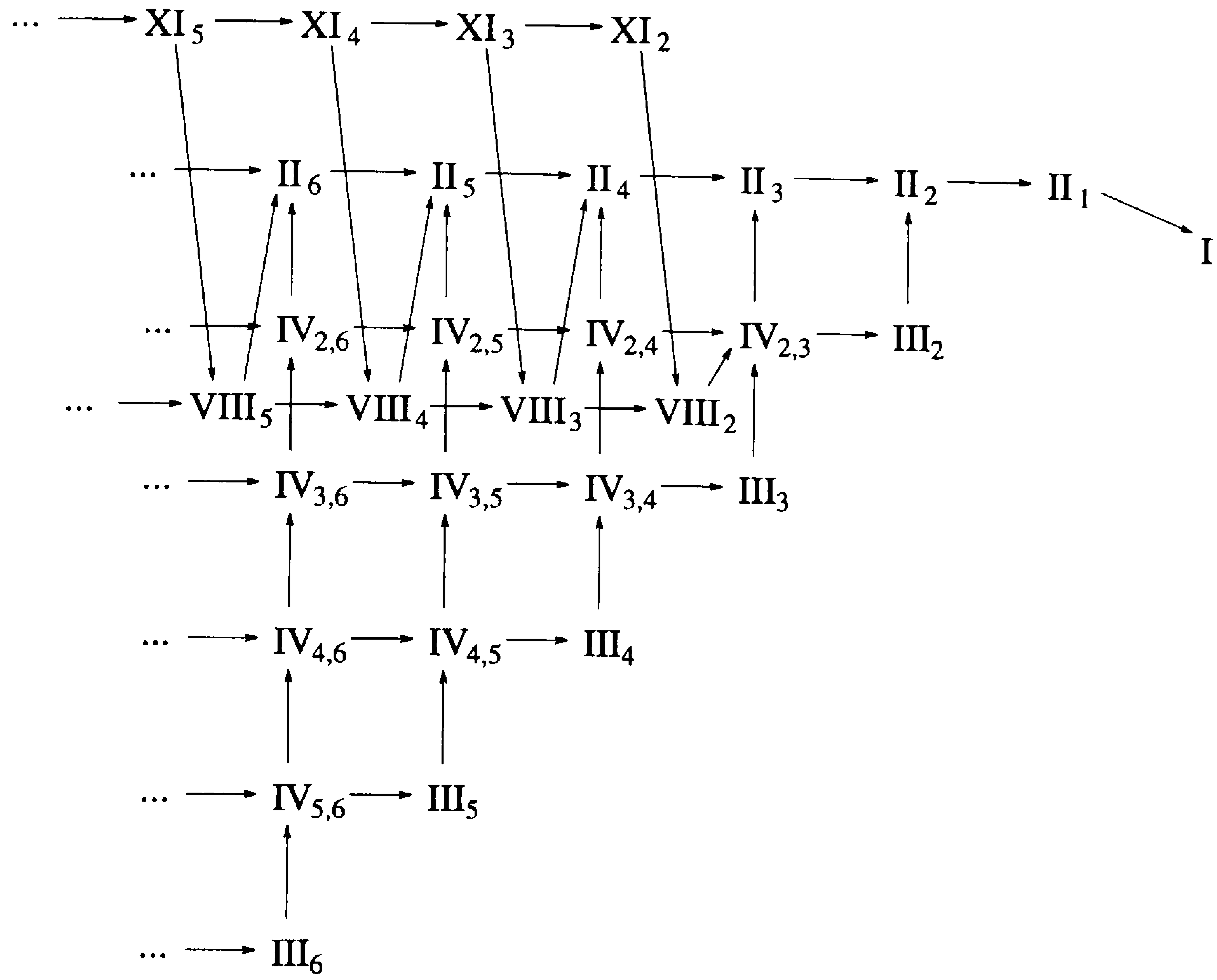


Figure 4.1: Adjacencies

For each of the map-germs in question we must calculate an $\mathcal{A}$-versal unfolding $F(x,u) = (f_u(x), u)$ and recognise the class of f_u for the different possibilities of u. Our computer classification package `Transversal` helped in both of these tasks. If we were just concerned with proving simplicity then the calculations are greatly simplified. For the different possibilities of u we need only identify a finite number of *possible* adjacencies (and these are usually clear from the 'stratification tree' obtained in the classification), as opposed to establishing *exactly* which of these adjacencies exist. However, adjacencies are important in understanding the geometry of the singularities and the possibilities under deformation. This is especially so for surface singularities in 4-space, where our geometric intuition can be somewhat lacking.

The adjacencies diagrams for the map-germs given in Table 4.1 appear below. The first two diagrams, Figure 4.1 and Figure 4.2, show how the series are related. The full set of adjacencies, including all the sporadic classes, appears in the third diagram, Figure 4.3.

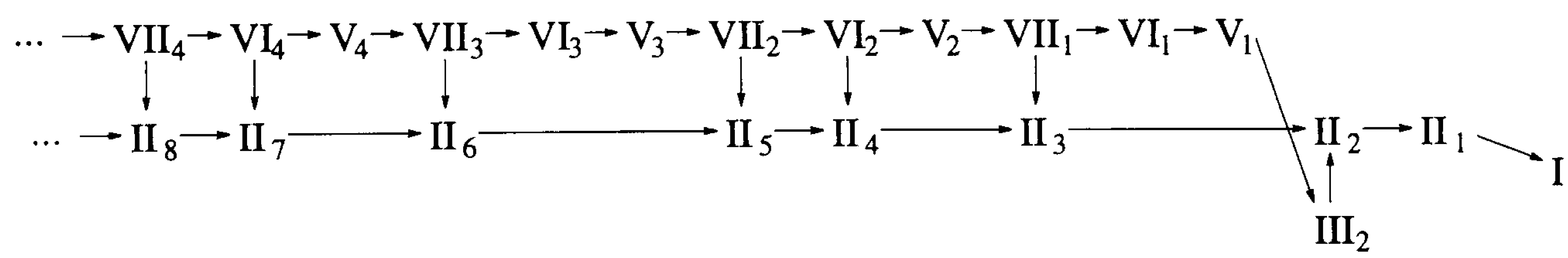


Figure 4.2: Adjacencies

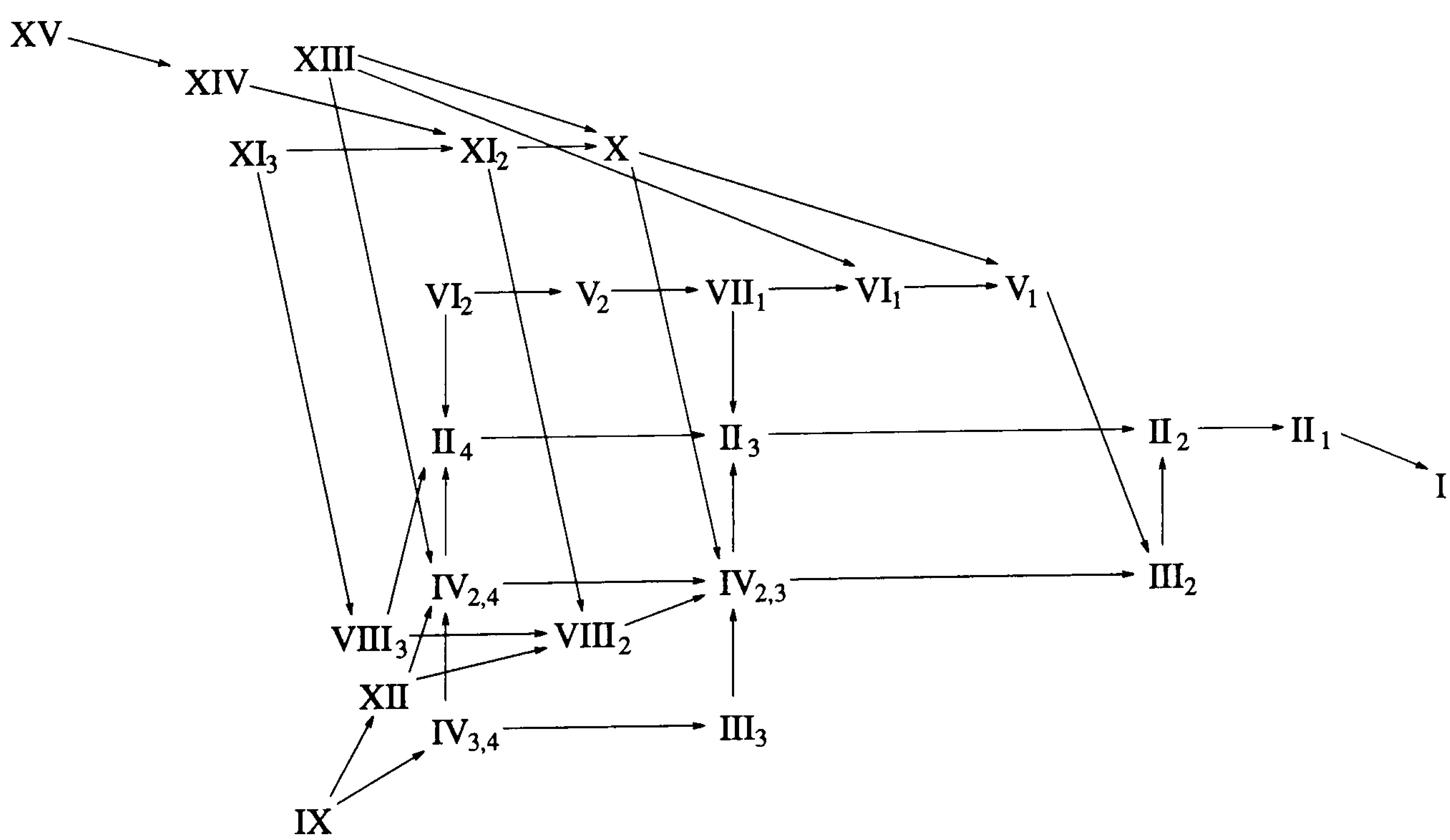


Figure 4.3: Adjacencies

4.3 Geometrical Invariants from Multiple Point Schemes, Singular Algebras and the $\mathcal{L}$ Group

In geometrical considerations the stable singularities play an important role. For map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ there are no stable singularities except the immersion. This can be seen from the classification carried out in Chapter 3 where all the orbits (after the immersion) have $\mathcal{A}_e$-codimension ≥ 1, or from the Whitney Immersion Theorem (see [GG, Theorem II.5.6]). Considering multi-germs, we find there is only one stable multi-germ, the transverse double point

$$(x_1, y_1, 0, 0;\ 0, 0, x_2, y_2).$$

(Our classification package `Transversal` is not yet capable of dealing with multi-germs, though the calculations in low codimension are feasible by hand anyway.) We recall some results on double point schemes and show that under generic deformation a singularity $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ splits up into a finite number, $\mathrm{D}(f)$, of double points which we can count (for corank 1 maps at least). This can be thought of as the analogue of a 'Morsification' of a function-germ. Double points provide the main geometric invariants, though triple points and singular algebras provide invariants which are not as trivial as one might at first expect. The geometrical interpretation of these latter invariants is somewhat vague for maps from $\mathbf{C}^2$ to $\mathbf{C}^4$, but they are perfectly good invariants so we include them below. Our final invariant is, essentially, the $\mathcal{L}_e$-codimension of the given map-germ. The important point is that this number is finite for $\mathcal{A}$-finite map-germs with the source and target dimension under consideration. We finish by noting some relations between the invariants.

The results of the calculations are shown in the Table 4.2. We include the $\mathcal{A}_e$-codimension as well, this being a fundamental invariant for singularities. The $\mathcal{A}_e$-codimension and $\mathcal{A}$-codimension are related so either may be used. For a finitely $\mathcal{A}$-determined, non-stable multi-germ $f : (\mathbf{C}^n, S) \longrightarrow (\mathbf{C}^p, 0)$, where $S = \{s_1, \ldots, s_r\}$ we have

$$\mathcal{A}_e\text{-codim}(f) = \mathcal{A}\text{-codim}(f) + r(p-n) - p.$$

(The case $r = 1$ was noted in [Wal]; a full proof due to Wilson appears in the unpublished notes [Wil].)

Label	Singularity	$\mathcal{A}_e$-Codim	D	T	C	L
I	$(x, y, 0, 0)$	0	0	0	0	0
II_k	(x, y^2, xy, y^{2k+1})	k	k	0	1	k
III_k	$(x, y^2, y^3, x^k y)$	$2k-1$	k	0	k	k
$\mathrm{IV}_{j,k}$	$(x, y^2, y^3 + x^j y, x^k y)$	$j+k-1$	k	0	j	k
V_k	$(x, xy, y^3, xy^2 + y^{3k+1})$	$3k+1$	$3k$	k	2	$4k$
VI_k	$(x, xy, y^3, xy^2 + y^{3k+2})$	$3k+2$	$3k+1$	k	2	$4k+1$
VII_k	$(x, xy + y^{3k+2}, y^3, xy^2)$	$3k+3$	$3k+2$	k	2	$4k+2$
VIII_k	$(x, y^2, x^2 y + y^{2k+1}, xy^3)$	$k+3$	$k+2$	0	2	$k+2$
IX	$(x, y^2, xy^3, x^3 y + y^5)$	7	5	0	3	5
X	(x, xy, y^3, y^4)	5	3	1	2	4
XI_k	$(x, xy + y^3, xy^2 + y^{2k+1}, y^4)$	$k+4$	$k+3$	$k-1$	3	$k+4$
XII	$(x, y^2, x^2 y, y^5)$	6	4	0	2	4
XIII	(x, xy, y^3, y^5)	6	4	1	2	5
XIV	$(x, y^3, xy^2, x^2 y + y^4 + y^5)$	7	5	1	3	6
XV	$(x, y^3, xy^2, x^2 y + y^4)$	8	5	1	3	6

Table 4.2: Geometric Invariants

4.3.1 Multiple Point Schemes

To begin with we recall some basic results and definitions for multi-point schemes. A comprehensive treatment can be found in [MarMo] where the emphasis is toward local singularity theory. Consider the general case where $f : \mathbf{C}^n \longrightarrow \mathbf{C}^p$; the basic notion of the double point locus $\tilde{D}^2(f)$ is the closure in $\mathbf{C}^n \times \mathbf{C}^n$ of the set $\{ (x, x') \in \mathbf{C}^n \times \mathbf{C}^n : f(x) = f(x'), x \neq x' \}$. This definition proves to be inadequate — it is more appropriate to give $\tilde{D}^2(f)$ a scheme structure.

Definition 4.3 Denote the diagonals in $\mathbf{C}^n \times \mathbf{C}^n$ and $\mathbf{C}^p \times \mathbf{C}^p$ by Δ_n and Δ_p, respectively. Denote the sheaves of ideals defining them by $\mathcal{I}_n$ and $\mathcal{I}_p$. We define the *double point scheme* $\tilde{D}^2(f)$ by means of one of the following two sheaves of ideals:

1. $\mathcal{I}_2(f) = \mathrm{Ann}_{\mathcal{O}_{\mathbf{C}^{2n}}} \mathcal{I}_n/(f \times f)^* \mathcal{I}_p$;
2. $\tilde{\mathcal{I}}_2(f) = (f \times f)^* \mathcal{I}_p + \mathcal{F}_0(\mathcal{I}_n/(f \times f)^* \mathcal{I}_p)$.

Here, we regard $\mathcal{I}_n/(f \times f)^* \mathcal{I}_p$ as an $\mathcal{O}_{\mathbf{C}^{2n}}$-module and $\mathcal{F}_0$ is its 0th Fitting ideal sheaf, (see [L, Section 4.D], [Tou, Chapter 1]). Ann denotes the annihilator of a sheaf so $\mathcal{I}_2(f)$ is equivalently the quotient ideal sheaf $(\,(f \times f)^*\mathcal{I}_p : \mathcal{I}_n\,) = \{\, g \in \mathcal{O}_{\mathbf{C}^{2n}} : g.\mathcal{I}_n \subset (f \times f)^*\mathcal{I}_p \,\}$, also known as the 'transporter ideal'.

This provides a general definition which we have included for completeness. For a large class of maps (including those whose germs are finitely $\mathcal{A}$-determined) $\mathcal{I}_2(f)$ and $\tilde{\mathcal{I}}_2(f)$ coincide (they always coincide away from Δ_n, being equal to $(f \times f)^*\mathcal{I}_p$). The following observation from [Mo3] provides an easy method for calculating $\tilde{\mathcal{I}}_2(f)$ and is considerably more intuitive. For $f = (f_1, \dots, f_p)$ and for each i the function germ $(x, x') \mapsto f_i(x) - f_i(x')$ belongs to $\mathcal{I}_n$ so by the Hadamard Lemma there exists germs $\alpha_{ij} \in \mathcal{O}_{\mathbf{C}_{2n}}$, $1 \le i \le p$, $1 \le j \le n$, such that

$$\sum_j \alpha_{ij}(x, x')(x_j - x'_j) = f_i(x) - f_i(x').$$

Now, if $f(x) = f(x')$ and $x \neq x'$ then the matrix $\alpha(x, x') = (\alpha_{ij}(x, x'))$ has non-zero kernel so every n by n minor must vanish. Denoting the ideal in $\mathcal{O}_{\mathbf{C}^{2n}}$ generated by the n by n minors of α by $\mathrm{Min}_n(\alpha)$ one can show that $\tilde{D}^2(f)$ is defined by means of the sheaf of ideals $(f \times f)^*\mathcal{I}_p + \mathrm{Min}_n(\alpha)$. Indeed, Mond showed that

$$\tilde{\mathcal{I}}_2(f) = (f \times f)^*\mathcal{I}_p + \mathrm{Min}_n(\alpha).$$

He also showed that α is the Jacobian matrix of f so that away from Δ_n, $\tilde{D}^2(f)$ consists of the double points of f, while the point $(x, x) \in \Delta_n$ belongs to $\tilde{D}^2(f)$ precisely when f fails to be an immersion at x.

The above sketches the general theory. Having dealt with this we now discuss the calculations for map-germs $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$. We are interested in germs of corank 1. (For such maps it can be shown that $\mathcal{I}_2(f)$ and $\tilde{\mathcal{I}}_2(f)$ coincide, see [Mo3, Remark 3.2.(i)].) Now, by the Rank Theorem (see Lemma 3.3, for example) f may be written in the form

$$f(x, y) = (x, f_1(x, y), f_2(x, y), f_3(x, y))$$

after suitable coordinate changes. The matrix α can be taken to be

$$\begin{pmatrix} 1 & 0 \\ \frac{f_1(x,y')-f_1(x',y')}{x-x'} & \frac{f_1(x,y)-f_1(x,y')}{y-y'} \\ \frac{f_2(x,y')-f_2(x',y')}{x-x'} & \frac{f_2(x,y)-f_2(x,y')}{y-y'} \\ \frac{f_3(x,y')-f_3(x',y')}{x-x'} & \frac{f_3(x,y)-f_3(x,y')}{y-y'} \end{pmatrix}$$

so that

$$\begin{aligned} \mathcal{I}_2(f) &= (f \times f)^*\mathcal{I}_4 + \mathrm{Min}_2(\alpha) \\ &= \langle x - x', \frac{f_1(x,y) - f_1(x,y')}{y - y'}, \frac{f_2(x,y) - f_2(x,y')}{y - y'}, \frac{f_3(x,y) - f_3(x,y')}{y - y'} \rangle. \end{aligned}$$

Moreover, since $x - x' \in \mathcal{I}_2(f)$ there is a natural embedding of $\tilde{D}^2(f)$ in $\mathbf{C} \times \mathbf{C}^2$ given by

$$\begin{array}{ccc} \mathbf{C}^2 \times \mathbf{C}^2 & \longrightarrow & \mathbf{C} \times \mathbf{C}^2 \\ (x, y, x', y') & \mapsto & (x, y, y'). \end{array}$$

For the corank 1 case we will consider this embedding, which we will also denote by $\tilde{D}^2(f)$; it is defined by the sheaf of ideals

$$\langle g_1(x, y, y'), g_2(x, y, y'), g_3(x, y, y') \rangle_{\mathcal{O}_3},$$

where

$$g_i(x, y, y') = \frac{f_i(x, y) - f_i(x, y')}{y - y'},$$

and will be denoted by $\mathcal{I}_2(f)$. Higher multiple point schemes $\tilde{D}^k(f)$ may be defined inductively; for the corank 1 case they admit a natural embedding into $\mathbf{C}^{n-1} \times \mathbf{C}^k$ (for $n < p$). The sheaf of ideals used to define $\tilde{D}^k(f)$ is a subsheaf of $\mathcal{O}_{\mathbf{C}^{n-1+k}}$ and is denoted $\mathcal{I}_k(f)$. We will just note that $\mathcal{I}_3(f)$ is defined by the sheaf of ideals

$$\langle g_i(x, y, y'), h_i(x, y, y', y'') \rangle_{\mathcal{O}_4} \quad \text{for} \quad i = 1, \ldots, 3$$

where

$$h_i(x, y, y', y'') = \frac{g_i(x, y, y') - g_i(x, y, y'')}{y' - y''}.$$

We refer to [Mo2, Section 3] and [MarMo, Section 1.2] for more details. These observations will be used throughout the rest of this section.

Remarks. We note that there is a natural action of the symmetric group S_k on $\tilde{D}^k(f)$ (via permutation of the coordinates on $\mathbf{C}^k$) and that $\mathcal{I}_k(f)$ can be described in terms of S_k-invariant generators. This is useful for many technical considerations but for our purposes the generators described previously (for $\mathcal{I}_2(f)$ and $\mathcal{I}_3(f)$) will suffice. For further details on all of these points we refer to [MarMo].

Our earlier remarks suggesting that the double point scheme of a finitely-determined map-germ $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ carries most of the geometrical information is verified by the following theorems. Firstly we observe that $\tilde{D}^2(f)$ consists of isolated points, while the higher multiple point schemes are trivial.

Theorem 4.4 *Let $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ $(n < p)$ be a finite mapping of corank 1. Then*

1. *f is stable if and only if for each $k \geq 2$, $\tilde{D}^k(f)$ is smooth of dimension $p - k(p-n)$, or empty;*

2. *f is finitely determined if and only if for each k with $p - k(p-n) \geq 0$, $\tilde{D}^k(f)$ is either an isolated complete intersection singularity of dimension $p - k(p-n)$ or empty, and if, furthermore, for those k with $p-k(p-n) < 0$, $\tilde{D}^k(f)$ consists at most of the point $\{0\}$.*

Proof. [MarMo, Theorem 2.14] □

This does not guarantee $\tilde{D}^2(f)$ is non-trivial (it too may just consist of the point $\{0\}$ or be empty), but the following theorem shows that double points are exhibited in a generic deformation of f.

Theorem 4.5 *If $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ has corank 1 and $k = p/(p-n)$ is a positive integer, then a generic deformation of f has*

$$1/k! \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n-1} \times \mathbf{C}^k} / \mathcal{I}_k(f)$$

ordinary k-tuple points.

Proof. See [Mo3, Proposition 3.7] and the remarks which follow it. □

Thus a generic deformation of a corank 1 germ $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ has

$$D(f) = 1/2 \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^3} / \mathcal{I}_2(f)$$

ordinary double points. This number provides the main geometrical invariant for such map-germs.

The triple point schemes provide non-trivial invariants. We define the triple point number

$$T(f) = 1/6 \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^4} / \mathcal{I}_3(f).$$

From Theorem 4.5, a map-germ $\tilde{f} : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ splits up into $T(\tilde{f})$ triple points under a generic deformation. For a map-germ $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$, $T(f)$

appears to be the minimum number of triple points $T(\tilde{f})$ among all map-germs $\tilde{f} = \pi \circ f$ where $\pi : (\mathbf{C}^4, 0) \longrightarrow (\mathbf{C}^3, 0)$ is a surjection onto $\mathbf{C}^3$. One would not expect $T(f)$ to reflect the geometry of map-germs $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ as well as $D(f)$. The results of our calculations indicate this. However, $T(f)$ does give a non-trivial invariant, whereas the algebras associated to the higher multiple point schemes ($\mathcal{O}_{\mathbf{C}^{n-1}\times\mathbf{C}^k}/\mathcal{I}_k(f)$ for $k \geq 4$) all collapse to 0 (at least for the simple singularities).

We have calculated $D(f)$ and $T(f)$ for many of the singularities arising in our classification, the results are given for the simple singularities in Table 4.2. The calculation of the above quotient ring can be fairly tedious in most cases. We have developed a way of performing these calculations using our `Transversal` classification package discussed in Chapter 6; see Section 4.4 (for series this only applies to specific members, but indicates the general result). We do not calculate the double point schemes $\tilde{D}^2(f)$. Again, these may be obtained using symbolic algebra (Gröebner bases and elimination theory, for example). An investigation into the double point schemes of map-germs and their unfoldings promises to give useful geometric information but we content ourselves with the invariants $D(f)$ and $T(f)$ and postpone such work for the future.

4.3.2 Singular Algebras

Consider map-germs $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ and suppose $W \subset J^k(n,p)$ is an $\mathcal{A}$-invariant submanifold. Given a map germ f such that $j^k f(0) \in W$ denote the ideal in $\mathcal{O}_{J^k(n,p), j^k f(0)}$ consisting of germs which vanish on W by $I_{W, j^k f(0)}$. We define the *algebra of contact* of $j^k f$ with W to be

$$Q_W f(0) = \mathcal{O}_n / (j^k f^* (I_{W, j^k f(0)})).$$

These algebras provide interesting $\mathcal{A}$-invariants. We require the following result; see [Mo3, Lemma 2.2].

Lemma 4.6 *If $f \sim_{\mathcal{A}} g$ and then $Q_W f(0)$ is isomorphic to $Q_W g(0)$.*

Consider $W = \Sigma^k \subset J^1(n,p)$, the set of linear maps of corank k, and a map-germ f of corank k. Let $L(n,p)$ denote the space of linear maps $\mathbf{C}^n \longrightarrow \mathbf{C}^p$, then $df : (\mathbf{C}^n, 0) \longrightarrow L(n,p)$ and $df(0) \in \Sigma^k$. The corresponding contact algebra is called the *singular algebra* of f:

$$Q_{\Sigma^k} f(0) = \mathcal{O}_n / (df^* (I_{\Sigma^k, df(0)})).$$

In the present scenario $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ is of corank 1 and we will look at the invariants $Q_{\Sigma^1} f(0)$. We can write f in the form

$$f(x, y) = (x, f_1(x, y), f_2(x, y), f_3(x, y))$$

so

$$df(x, y) = \begin{pmatrix} 1 & 0 \\ \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \\ \partial f_3/\partial x & \partial f_3/\partial y \end{pmatrix}.$$

The ideal of germs vanishing on Σ^1 is generated by the 2×2 minors considered as functions $L(2, 4) \longrightarrow \mathbf{C}$. For corank 1 map-germs f one sees that

$$df^*(I_{\Sigma^1, df(0)}) = \langle \partial f_1/\partial y, \partial f_2/\partial y, \partial f_3/\partial y \rangle.$$

Using these observations we have calculated the dimension

$$C(f) = \dim_{\mathbf{C}} Q_{\Sigma^1} f(0)$$

for all the simple singularities $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$. The results are shown in Table 4.2.

$C(f)$ provides a non-trivial invariant but like $T(f)$ it reflects the geometry of map-germs $\tilde{f} : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$. From [Mo2, Section 2] we see that $C(\tilde{f})$ counts the number of cross-caps in a generic deformation of $\tilde{f}$. The important point is that a cross-cap is characterised by being the only non-immersive germ $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ whose 1-jet meets Σ^1 transversally. In the $\mathbf{C}^4$ case there are no such map-germs; the following remarks provide our best geometrical interpretation. For a map-germ $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$, $C(f)$ appears to be the *minimum* number of cross-caps $C(\tilde{f})$ among all map-germs $\tilde{f} = \pi \circ f$ where $\pi : (\mathbf{C}^4, 0) \longrightarrow (\mathbf{C}^3, 0)$ is a surjection onto $\mathbf{C}^3$. Consider the simple singularity $f = (x, y^2, y^3 + x^j y, x^k y)$, for example. Then

$$df^*(I_{\Sigma^1, df(0)}) = \langle y, 3y^2 + x^j, x^k \rangle = \langle y, x^j \rangle$$

and $C(f) = C(\tilde{f})$ where $\tilde{f} = (x, y^2, y^3 + x^j y)$.

4.3.3 The $\mathcal{L}$ Group

Our final invariant comes from the following observations.These results are due to Gaffney, [Ga1, Chapters 3 and 5]; see also [Wal, Section 2].

Theorem 4.7 *Consider map-germs* $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$.

1. *If* $2n \leq p$, *any* $\mathcal{A}$*-finite germ is* $\mathcal{L}$*-finite.*
2. *If map-germs* f, g *are* $\mathcal{A}$*-finite and* $n < p$, *then* $f \sim_{\mathcal{A}} g$ *if and only if the algebras* $f^*(\mathcal{O}_p)$, $g^*(\mathcal{O}_p)$ *are isomorphic.*

Note that $f^*(\mathcal{O}_p)$ is the subalgebra of $\mathcal{O}_n$ generated (as an algebra) by the components of f (and 1); this generally gives 'finer' invariants than those associated to the ideal generated by the components of f, $f^*(m_p).\mathcal{O}_n$. In standard coordinates, the $\mathcal{L}_e$-tangent space to the orbit of f is just p copies of this subalgebra

$$L\mathcal{L}_e \cdot f = f^*(\mathcal{O}_p).\{e_1, \ldots, e_p\},$$

so

$$\mathcal{L}_e\text{-Codim}(f) = \dim_{\mathbf{C}} (\mathcal{O}(n,p)/L\mathcal{L}_e \cdot f) = p \dim_{\mathbf{C}} (\mathcal{O}_n/f^*(\mathcal{O}_p)).$$

For $2n \leq p$ and any map-germ f we therefore define the invariant

$$L(f) = \dim_{\mathbf{C}} (\mathcal{O}_n/f^*(\mathcal{O}_p)).$$

The previous remarks ensure that this is indeed an invariant and is finite for $\mathcal{A}$-finite map-germs.

For map-germs $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^4, 0)$ the above observations apply and $L(f)$ is a natural invariant to work with. We have performed the calculations for the simple singularities, the results are shown in Table 4.2.

4.3.4 Some Remarks and Relations

Collectively, $\mathcal{A}_e$-Codim, D, T, C and L almost form a complete set of invariants for the simple singularities. One easily verifies that these invariants distinguish all of the normal forms in Table 4.2 except the pairs $\text{IV}_{2,k+2}$, VIII_k and $\text{IV}_{3,5}$, IX. This can be resolved by looking at their $\mathcal{A}$-orbits in the 3-jet-space. From Chapter 3 we find that $\text{IV}_{2,k+2}$ and VIII_k are of $J^3\mathcal{A}$-codimension 6 and 7, respectively, demonstrating that they are not $\mathcal{A}$-equivalent. Similarly, $\text{IV}_{3,5}$ and IX are of $J^3\mathcal{A}$-codimension 7 and 9, respectively. Our list of simple singularities therefore contains no redundancies.

From the results in Table 4.2 we observe that

$$L(f) = D(f) + T(f)$$

for all the simple singularities f except the series XI_k. The exceptional case is not weighted homogeneous — a possible clue to this abnormality! Also note that for a pleasingly large proportion of the singularities

$$\mathcal{A}_e\text{-codim}(f) = D(f) + C(f) - 1.$$

This is reminiscent of the codimension formula in [Mo2, Section 4] for map-germs $(\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ where $\mathcal{A}_e\text{-codim}(f) \leq 1/2N(f) + T(f) + C(f) - 1$ (bearing in mind $D(f)$ in the $\mathbf{C}^4$ case is analogous to $T(f)$ in the $\mathbf{C}^3$ case). The full relation may become clear with future work and the discovery of new invariants which reflect the geometry better (than $C(f)$, for example).

4.4 Computational Techniques for the Invariants

Most of the invariants described above were calculated by computer. (In the case of series the computer was used to calculate the invariants for initial members, this indicates the general result.) For $D(f)$ and $T(f)$ we need to calculate an ideal $\langle g_1, \ldots, g_r\rangle$ in $\mathcal{O}_s$ for some s and germs g_i, depending on the particular requirement. This can be performed by our `Transversal` package, in particular, using the function `jetcalc` to calculate $\langle g_1, \ldots, g_r\rangle$ to a given degree k, (i.e., in the k-jet-space, working modulo m_s^{k+1}). If

$$m_s^k \subset \langle g_1, \ldots, g_r\rangle + m_s^{k+1}$$

(one of the main functions of the package is to carry out such checks) then the Nakayama Lemma implies

$$m_s^k \subset \langle g_1, \cdots, g_r\rangle$$

and the dimension

$$\dim_{\mathbf{C}} \mathcal{O}_s / \langle g_1, \cdots, g_r\rangle$$

may be calculated by `Transversal` by working in the k-jet-space. In fact the $(k-1)$-jet-space is sufficient but `jetcalc` will have already calculated a basis for the complementary space to $\langle g_1, \ldots, g_r\rangle + m_s^{k+1}$ in $\mathcal{O}_s$ during the previous calculation.

The calculation of ideals $\langle g_1, \ldots, g_r\rangle$ was not one of the original intentions for `Transversal`. We have to 'customise' the process slightly. Technically, one must tell `jetcalc` to calculate the '$\mathcal{R}$' group using a source Lie algebra (specified by a `liealg` routine) which is generated by the g_i. These do not act on germs (as the standard $x_i\partial/\partial x_j$ do, for example; c.f., the function `stdjacobian`) and any map-germ passed to the function `jetcalc` is just a dummy variable and plays no part. We refer to Chapter 6 for a detailed discussion of how the function `jetcalc` works.

The invariant $L(f)$ is calculated by `jetcalc` simply by calculating the $\mathcal{L}_e$-codimension of f. This is not an inefficient process as `jetcalc` does not calculate the full $\mathcal{L}_e$-tangent space, just the ideal $f^*(\mathcal{O}_p)$; we refer to Section 6.7.5 ('The Pre-Tangent Space') for details. Since `jetcalc` calculates $\dim_{\mathbf{C}}(\mathcal{O}_n/f^*(\mathcal{O}_p))$ in a given jet-space we use Lemma 1.2 to check the result holds in $\mathcal{O}_n$ (just as the Nakayama Lemma had to be used in the previous calculation).

Generally, the calculations described here are best suited to computational methods from commutative algebra, such as the Gröbner basis techniques used by computer packages like `Macaulay` or `Singular`. However, it should be noted that all the calculations above were dealt with easily using `Transversal`. We also remark that `Transversal` played a major role in identifying the adjacencies in Section 4.2. In many cases explicit coordinate changes were extremely difficult to find and we managed to identify the germ f_u under consideration by calculating invariants or applying the Mather Lemma to show triviality in a family.

Chapter 5

Classification of Function-Germs on Discriminant Varieties

Up to now we have only considered the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$. We now turn our attention to another group of great interest in singularity theory, namely the subgroup $\mathcal{R}(X)$ of $\mathcal{R}$ consisting of germs of diffeomorphisms of $(\mathbf{C}^n, 0)$ which preserve a given variety X. Our main interest is when X is a discriminant variety $\mathcal{D}$ — we refer to Chapter 1 for a discussion on versal unfoldings and discriminants. For technical reasons it is best to work with the complex analytic case. In this case the module of vector fields tangent to $\mathcal{D}$, $\Theta(\mathcal{D})$, is finitely generated as a module over $\mathcal{O}_p$ (indeed, it is a free module and there is an algorithm due to Saito, [Sai], for calculating the generators; see Section 5.1 below.) However, in the real smooth case $\Theta(\mathcal{D})$ is not finitely generated as a module over $\mathcal{E}_p$ and the techniques of singularity theory are not suitable; see [A2, p.569]. Our results do provide a classification in the real smooth case (since the Saito vector fields still belong to $\Theta(\mathcal{D})$, they just do not generate it), that is, we obtain a list of normal forms up to $\mathcal{R}(\mathcal{D})$ equivalence. Classifications in the real smooth case are, of course, important in geometrical applications.

In this chapter we classify function-germs on the discriminants of the simple singularities: A_k, D_k and E_k; extending the lists found in [A2]. We need to calculate the basic vector fields of Saito, tangent to the given discriminant, and use the method described in [B2]. A classification method using weighted filtrations is developed and we demonstrate that such classifications can be performed efficiently using our computer classification package. An algorithm is developed which allows the Saito vector fields to be calculated by computer, thus automating the whole process and reducing the possibility of errors.

We begin by reviewing the results on Saito vector fields, determinacy and stability; these will be useful elsewhere too. Next the classification methods are developed, followed by a description of the computer calculations and a summary of the results. Such classifications have important applications in geometry. To date, the generic (stable) singularities have been studied in depth and the results applied to the generic evolution of wavefronts, [A2], for example. We do not consider map-germs $(\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ where the target dimension p is greater than one. We note, however, that the classification methods easily generalise to such cases and our computer methods are capable of performing the relevant calculations. Stable map-germs on discriminants have already been considered in [B4], and the classification of such map-germs carried out in [BG1]. Again there are several applications; for example, classifying the outlines of smooth surfaces in $\mathbf{R}^3$ and their duals [BG2], and the bifurcation of plane caustics by reflection [BG3].

5.1 Vector Fields on Discriminants

Throughout this section we will deal with the complex analytic case — this forms the natural setting for the results. $\mathcal{O}_p$ will denote the algebra of germs of analytic functions on $\mathbf{C}^p$ at 0.

Let $(X, 0) \subset (\mathbf{C}^p, 0)$ be the germ of a reduced analytic subvariety of $\mathbf{C}^p$ at 0. We shall consider analytic function germs $(\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ and say two germs are equivalent if one can be obtained from the other by source coordinate changes which preserve X.

Definition 5.1 Let $\mathcal{I}$ denote the (radical) ideal in $\mathcal{O}_p$ corresponding to X, that is the ideal of germs of functions vanishing on X.

1. A diffeomorphism $\phi : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}^p, 0)$ is said to *preserve* X if $\phi(X)$ is equal to X as germs at 0, $(\phi(X), 0) = (X, 0)$; equivalently the induced isomorphism $\phi^* : \mathcal{O}_p \longrightarrow \mathcal{O}_p$ satisfies $\phi^*(\mathcal{I}) = \mathcal{I}$. The group of such diffeomorphisms preserving X is denoted $\mathcal{R}(X)$.

2. Two function germs $f, g \in \mathcal{O}_p$ are $\mathcal{R}(X)$-*equivalent* if there exists $\phi \in \mathcal{R}(X)$ such that $g \circ \phi = f$.

3. Let δ be a germ of an analytic vector field on $\mathbf{C}^p$ at 0. Then δ is said to be *logarithmic* for $(X, 0)$ if, when considered as a derivation $\delta : \mathcal{O}_p \longrightarrow \mathcal{O}_p$,

$f \mapsto \delta \cdot f$, we have $\delta \cdot f \in \mathcal{I}$ for all $f \in \mathcal{I}$ (that is, $\delta \cdot f$ vanishes on X). The $\mathcal{O}_p$-module of logarithmic vector fields is denoted $\Theta(X)$ (the notation $\mathrm{Der}(\log X)$ is common too).

Intuitively, an infinitesimal approach is given by integrating vector fields tangent to X to yield diffeomorphisms which preserve X. The following proposition confirms this and shows that the logarithmic vector fields are precisely those vector fields tangent to X.

Proposition 5.2

1. *The germ at 0 of a vector field δ lies in $\Theta(X)$ if and only if at each smooth point x (sufficiently close to 0) of each irreducible component X_i of X the vector field δ is tangent to X_i at x.*
2. *If $X = \bigcup_{i=1}^{s} X_i$ is the irreducible decomposition of X then $\Theta(X) = \bigcap_{i=1}^{s} \Theta(X_i)$.*
3. *Suppose $\delta \in \Theta(X)$ vanishes at 0. Then the flow ϕ_t generated by δ preserves $(X, 0)$. Thus $\phi_t \in \mathcal{R}(X)$ for all t.*

Proof. See [BR, Section 1]. □

We now turn our attention to determinacy under $\mathcal{R}(X)$-equivalence.

Definition 5.3 A germ $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ is k-$\mathcal{R}(X)$-determined if for all $h_1 : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ with the same k-jet as h the germs h and h_1 are $\mathcal{R}(X)$-equivalent.

The group $\mathcal{R}(X)$ is one of Damon's 'geometric subgroups' of $\mathcal{A}$ and the following determinacy theorem holds; see [D].

Theorem 5.4 *A germ $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ is finitely $\mathcal{R}(X)$-determined if the ideal*

$$J_X(h) = \{\, \delta \cdot h : \delta \in \Theta(X) \,\}$$

in $\mathcal{O}_p$ contains some power of the maximal ideal m_p.

The above determinacy theorem can also be derived from a slight modification of the usual proofs. For example, see [BG5, Chapter 11]; only now we replace $\mathcal{R}$ by $\mathcal{R}(X)$ and need $\mathcal{R}(X)$-trivial families as our technical tool: the crucial difference being that the flows used preserve X — this follows from Proposition 5.2. In all our examples the vector fields $\delta \in \Theta(X)$ vanish at 0 ($J_X(h) \supset m_n$) making the classification considerably easier. For instance, we obtain the following determinacy estimate: "if $m_n^{k+1} \subset m_n.J_X(h)$ then h is k-$\mathcal{R}(X)$-determined". This follows in the same way as "$m_n^{k+1} \subset m_n^2.J(h)$ implies h is k-$\mathcal{R}$-determined" given, for example, in [BG5, Theorem 11.20], [G, Chapter IV, Section 3]. In this latter case the key point is that we are using the $\mathcal{R}_1$ group and working with vector fields in $m_n^2.\{\partial/\partial x_i\}$ which vanish to order 2; similarly in the $\mathcal{R}(X)$ case, only we now work with the module of vector fields $m_n.\Theta(X)$ since a vector field $\delta \in \Theta(X)$ already vanishes at 0. This gives a powerful determinacy criterion, though we can do better by introducing a weighted filtration and using 'more' of the module $\Theta(X)$ in the calculation. We discuss this in Section 5.2.2.

We are interested in the case X a discriminant variety $\mathcal{D}$ and restrict our attention to this now. Let $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ be a germ with an isolated singularity at 0. It follows that f is finitely $\mathcal{R}$-determined and the quotient space $\mathcal{O}_n/J(f)$ is a finite dimensional $\mathbf{C}$-vector space (where $J(f)$ denotes the Jacobian ideal $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle$). Let $g_1, \ldots, g_p$ form a basis for this vector space and define

$$\begin{aligned} F : (\mathbf{C}^n \times \mathbf{C}^p, 0) &\longrightarrow (\mathbf{C}, 0), \\ (x, u) &\mapsto f(x) + \sum_{i=1}^{p} u_i g_i(x), \end{aligned}$$

where (x, u) denote coordinates on $\mathbf{C}^n \times \mathbf{C}^p$. Then the map

$$(x, u) \mapsto (F(x, u), u)$$

is a versal unfolding of f. Let $\mathcal{D} \subset \mathbf{C}^p$ denote the discriminant of f. $\mathcal{O}_{n,p}$ is the algebra of analytic function germs in x, u; denote the ideal

$$\langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n \rangle \subset \mathcal{O}_{n,p}$$

by $J(F)$ (the context being clear). Now, $\mathcal{O}_n/J(f)$ is a finite dimensional vector space and $\mathcal{O}_{n,p}/J(F)$ a finitely generated $\mathcal{O}_{n,p}$-module, so that by the Preparation Theorem (a suitable version is [Mart1, Theorem 0.2.1]) it follows that the germs $g_1, \ldots, g_p$ generate $\mathcal{O}_{n,p}/J(F)$ as an $\mathcal{O}_p$-module via the natural projection map $(\mathbf{C}^n \times \mathbf{C}^p, 0) \longrightarrow (\mathbf{C}^p, 0)$. In fact, we have a stronger result.

Lemma 5.5 *The germs $g_1, \ldots, g_p$ form a free $\mathcal{O}_p$-basis of $\mathcal{O}_{n,p}/J(F)$.*

Thus, for $1 \leq j \leq p$ we can define $a_{ij} \in \mathcal{O}_p$ uniquely by

$$F.g_j = \sum_{i=1}^{p} a_{ij} g_i \bmod J(F).$$

The main result for constructing vector fields tangent to the discriminant $\mathcal{D}$ now follows.

Theorem 5.6 (Saito) *The vector fields*

$$\theta_j = \sum_{i=1}^{p} a_{ij} \partial/\partial u_i$$

are analytic vector fields and form a free $\mathcal{O}_p$-basis *for* $\Theta(\mathcal{D})$ $(= \mathrm{Der}(\log \mathcal{D})$, *the module of vector fields tangent to* $\mathcal{D})$.

Self-contained proofs of these results can be found in the appendix of [B2].

We have already mentioned that the case 'all vector fields $\delta \in \Theta(\mathcal{D})$ vanish at 0' is of importance in classification results. In [B2] it was shown that the vector fields θ_j all vanish at $0 \in \mathcal{D}$ if and only if f is right equivalent to a weighted homogeneous function. This has striking consequences on the existence and number of stable singularities under $\mathcal{R}(\mathcal{D})$-equivalence. The stable singularities are of immense importance in applications to generic geometry; we recall the following for completeness. Firstly note that the concepts of unfolding, $\mathcal{R}(\mathcal{D})$-isomorphic unfoldings and $\mathcal{R}(\mathcal{D})$-trivial unfoldings are defined in the usual way, but restricting the families of diffeomorphisms to germs which preserve the given variety $\mathcal{D}$ (see, for example, [BR, BG1, D]). Then following [BR] we define a germ $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ to be $\mathcal{R}(\mathcal{D})$-*stable* if $H(u,t) = h(u) + t$ is an $\mathcal{R}(\mathcal{D})$-versal unfolding of h. One then finds that h is $\mathcal{R}(\mathcal{D})$-stable if and only if 1 spans $\mathcal{O}_n/J_X(h)$ as a $\mathbf{C}$-vector space, that is, $J_X(h) = \mathcal{O}_n$ or m_n. We recall the following result from [B2].

Proposition 5.7 *The vector fields θ_j all vanish at $0 \in \mathcal{D}$ if and only if f is right equivalent to a weighted homogeneous function. In this case, there is at most one stable function, and it is equivalent to its linear part.*

The above limits the number of stable singularities; the existence of stable singularities is answered in [B4, Theorem 1.4]. In the case of functions (that is, map-germs $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}^q, 0)$ with target dimension $q = 1$) the result is as follows.

Theorem 5.8 *Let f and $\mathcal{D}$ be as above, with f right-equivalent to a weighted homogeneous function of weight d, and consider germs $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$. Let $g_1, \ldots, g_p$ form a basis of monomials for the $\mathbf{C}$-vector space $\mathcal{O}_n/J(f)$, as above, and define α_i to be the weight of g_i. Then there exists $\mathcal{R}(\mathcal{D})$-stable germs if and only if $\alpha_i \neq d$ for all i.*

A special case of this is $\alpha_i < d$ for all i. Equivalently, f must be one of the simple singularities A_k, D_k, E_6, E_7 or E_8, [A1, Theorem 10.3]. In the subsequent classifications we take f to be one of the simple singularities and indeed find that there exists a unique stable germ in each case, namely $h = u_1$.

5.2 Classification Techniques

In this section we discuss how to calculate the Saito vector fields θ_j which generate $\Theta(\mathcal{D})$, in particular, producing an algorithm for use on a computer. We then formulate powerful determinacy and complete transversal theorems which will be used in the following sections.

5.2.1 Computer Calculation of the Saito Vector Fields

Although the Saito vector fields can be calculated by hand using the results of the above section (see [B2], for example) such calculations can be at best tedious and at worst virtually impossible without the help of a computer to perform the symbolic algebra. As an indication we include some results of these calculations in Appendix A — note that some of the resulting vector fields span several pages. Not only do these need to be calculated, but they also need to be evaluated in complete transversal and determinacy calculations where they operate on a given germ — hence the advantages of our computer methods.

To calculate the Saito vector fields we need to find the coefficients $a_{ij} \in \mathcal{O}_p$ discussed in the last section. The preparation theorem assures us of their existence but we need some algorithm for calculating them which takes into account the appropriate quotient ring structure. We summarise our choices of bases $\{g_1, \dots, g_p\}$ and give the corresponding algorithms for calculating the a_{ij}, suitable for implementation by computer.

We first note that the normal forms used for the simple singularities are weighted homogeneous with respect to the following weights.

Singularity	Normal Form	Weight		Total Weight
		x	y	
A_k	x^{k+1}	1	-	$k+1$
D_k	x^2y+y^{k-1}	$k-2$	2	$2k-2$
E_6	x^3+y^4	4	3	12
E_7	x^3+xy^3	3	2	9
E_8	x^3+y^5	5	3	15

Theorem 5.9 *For each of the cases A_k, D_k and E_k, the products $F.g_j$ defined in the previous section are polynomials. They can therefore be written in the form $\sum a_{ij}g_i$, modulo $J(F)$, using the following monomial substitutions.*

$\mathbf{A_k}$. *Here $f(x) = x^{k+1}$, $J(f) = \langle x^k \rangle$ and we take*

$$g_1 = x^{k-1},\ g_2 = x^{k-2}, \dots, g_k = 1,$$

$$F(x, u_1, \dots, u_k) = x^{k+1} + u_1x^{k-1} + u_2x^{k-2} + \cdots + u_k.$$

Any polynomial in $\mathcal{O}_{n,p}$ may be written in the form $\sum a_ig_i$, modulo $J(F)$, with $a_i \in \mathcal{O}_p$ by using the following monomial substitutions repeatedly. Replace:

$$x^k \quad by \quad \frac{1}{k+1}\left((k-1)u_1x^{k-2} + (k-2)u_2x^{k-3} + \cdots + u_{k-1}\right).$$

$\mathbf{D_k}$. *Here $f(x,y) = x^2y + y^{k-1}$, $J(f) = \langle xy, x^2 + (k-1)y^{k-2} \rangle$ and we take*

$$g_1 = y^{k-2},\ g_2 = y^{k-3}, \dots, g_{k-2} = y,\ g_{k-1} = 1,\ g_k = x,$$

$$F(x, y, u_1, \dots, u_k) = x^2y + y^{k-1} + u_1y^{k-2} + u_2y^{k-3} + \cdots + u_{k-2}y + u_{k-1} + u_kx.$$

In this case we apply the following monomial substitutions. Replace:

$$\begin{array}{lll} xy & by & -\tfrac{1}{2}u_k \\ x^2 & by & -\left((k-1)y^{k-2} + (k-2)u_1y^{k-3} + (k-3)u_2y^{k-4} + \cdots + u_{k-2}\right) \\ y^{k-1} & by & -\dfrac{1}{k-1}\left(x^2y + (k-2)u_1y^{k-2} + (k-3)u_2y^{k-3} + \cdots + u_{k-2}y\right). \end{array}$$

$\mathbf{E_6}$. *Here* $f(x,y) = x^3 + y^4$, $J(f) = \langle x^2, y^3 \rangle$ *and we take*

$$g_1 = xy^2,\ g_2 = xy,\ g_3 = y^2,\ g_4 = x,\ g_5 = y,\ g_6 = 1,$$

$$F(x, y, u_1, \ldots, u_6) = x^3 + y^4 + u_1 xy^2 + u_2 xy + u_3 y^2 + u_4 x + u_5 y + u_6.$$

In this case we apply the following monomial substitutions. Replace:

$$\begin{array}{lll} x^2y & by & -\frac{1}{3}(u_1y^3 + u_2y^2 + u_4y) \\ x^2 & by & -\frac{1}{3}(u_1y^2 + u_2y + u_4) \\ y^3 & by & -\frac{1}{4}(2u_1xy + u_2x + 2u_3y + u_5). \end{array}$$

$\mathbf{E_7}$. *Here* $f(x,y) = x^3 + xy^3$, $J(f) = \langle 3x^2 + y^3, xy^2 \rangle$ *and we take*

$$g_1 = y^4,\ g_2 = y^3,\ g_3 = xy,\ g_4 = y^2,\ g_5 = x,\ g_6 = y,\ g_7 = 1,$$

$$F(x, y, u_1, \ldots, u_7) = x^3 + xy^3 + u_1 y^4 + u_2 y^3 + u_3 xy + u_4 y^2 + u_5 x + u_6 y + u_7.$$

In this case we apply the following monomial substitutions. Replace:

$$\begin{array}{lll} xy^2 & by & -\frac{1}{3}(4u_1y^3 + 3u_2y^2 + u_3x + 2u_4y + u_6) \\ x^2 & by & -\frac{1}{3}(y^3 + u_3y + u_5) \\ y^5 & by & -(3x^2y^2 + u_3y^3 + u_5y^2). \end{array}$$

$\mathbf{E_8}$. *Here* $f(x,y) = x^3 + y^5$, $J(f) = \langle x^2, y^4 \rangle$ *and we take*

$$g_1 = xy^3,\ g_2 = xy^2,\ g_3 = y^3,\ g_4 = xy,\ g_5 = y^2,\ g_6 = x,\ g_7 = y,\ g_8 = 1,$$

$$F(x, y, u_1, \ldots, u_8) = x^3 + y^5 + u_1 xy^3 + u_2 xy^2 + u_3 y^3 + u_4 xy + u_5 y^2 + u_6 x + u_7 y + u_8.$$

In this case we apply the following monomial substitutions. Replace:

$$\begin{array}{lll} x^2y & by & -\frac{1}{3}(u_1y^4 + u_2y^3 + u_4y^2 + u_6y) \\ x^2 & by & -\frac{1}{3}(u_1y^3 + u_2y^2 + u_4y + u_6) \\ y^4 & by & -\frac{1}{5}(3u_1xy^2 + 2u_2xy + 3u_3y^2 + u_4x + 2u_5y + u_7). \end{array}$$

Note: *in the event of more than one choice of monomial substitution being available, the first substitution (in the list of three) takes preference over the second, which takes preference over the third.*

Proof. The proof follows a similar argument for all the cases. We observe that the given substitutions are valid in the quotient ring. We then show that, with the choice of weights given above, after a finite number of substitutions the monomial is replaced by monomials of strictly smaller weight or one of the generators g_i. A monomial $m \in \mathbf{C}[x, y, u_1, \dots, u_p]$ can be written in the form $m = ux^a y^b \in \mathbf{C}[u_1, \dots, u_p][x, y]$ (so $u \in \mathbf{C}[u_1 \dots, u_p]$ is a monomial and a and b are non-negative integers). We then verify that for such monomials m, none of the substitutions apply if and only if $x^a y^b$ is one of the g_i. In this case the monomial m is eliminated from further consideration and added to a polynomial which stores the final (reduced) result. This guarantees the substitution process will terminate after a finite number of steps and when the process does terminate the resulting polynomial will be of the form $\sum a_i g_i$ with $a_i \in \mathcal{O}_p$. We shall give the proof for the cases D_k and E_7, the rest being similar.

$\mathbf{D_k}$. In this case

$$J(F) = \langle 2xy + u_k, x^2 + (k-1)y^{k-2} + (k-2)u_1 y^{k-3} + (k-3)u_2 y^{k-4} + \cdots + u_{k-2} \rangle$$

and modulo $J(F)$ we have

$$xy \sim -\tfrac{1}{2}u_k \tag{5.1}$$
$$x^2 \sim -\left((k-1)y^{k-2} + (k-2)u_1 y^{k-3} + (k-3)u_2 y^{k-4} + \cdots + u_{k-2}\right) \tag{5.2}$$
$$y^{k-1} \sim -\frac{1}{k-1}\left(x^2 y + (k-2)u_1 y^{k-2} + (k-3)u_2 y^{k-3} + \cdots + u_{k-2} y\right) \tag{5.3}$$

so that the substitutions are valid. Given a monomial $m \in \mathbf{C}[x, y, u_1, \dots, u_p]$ write it in the form $m = ux^a y^b$, as discussed in the preamble. If $xy \,/\, x^a y^b$ then substitution 5.1 will be applied, the weight of the resulting monomial being k less than that of of m. Otherwise m must be of the form ux^a or uy^b. If $m = ux^a$ and $a \geq 2$ then substitution 5.2 will apply. The weight does not decrease for *all* the resulting monomials, but they will either be divisible by xy and the weight will be decreased during the next substitution (5.1 takes preference and will be applied), or will be of the form ug_i — the ultimate goal. The same applies if $m = uy^b$ using substitution 5.3. Hence, after a finite number of steps we reduce the weight of the monomials or produce generators. Finally, suppose none of the substitutions 5.1, 5.2 and 5.3 apply. Then m must be of the form ux or uy^b with $0 \leq b \leq k-2$, that is of the form ug_i for some g_i, as required.

$\mathbf{E_7}$. In this case

$$J(F) = \langle 3x^2 + y^3 + u_3 y + u_5, 3xy^2 + 4u_1 y^3 + 3u_2 y^2 + u_3 x + 2u_4 y + u_6 \rangle$$

and modulo $J(F)$ we have

$$\begin{aligned} xy^2 &\sim -\tfrac{1}{3}(4u_1y^3 + 3u_2y^2 + u_3x + 2u_4y + u_6) && (5.4)\\ x^2 &\sim -\tfrac{1}{3}(y^3 + u_3y + u_5) && (5.5)\\ y^5 &\sim -(3x^2y^2 + u_3y^3 + u_5y^2) && (5.6)\end{aligned}$$

so that the substitutions are valid. Write a monomial $m \in \mathbf{C}[x, y, u_1, \ldots, u_p]$ in the form $m = ux^a y^b$. If $xy^2 \,/\, x^a y^b$ then substitution 5.4 will be applied, the weight of the resulting monomials being at least one less than that of m. Otherwise m must be of he form $ux^a y$, ux^a or uy^b. If $m = ux^a y$ and $a \geq 2$ then substitution 5.5 will apply. The weight decreases for all the resulting monomials except $-\frac{1}{3}ux^{a-2}y^4$, but this will either be a generator ($a = 2$) or will be divisible by xy^2 in which case the weight will be decreased during the next substitution (substitution 5.4 takes preference). Otherwise $m = uxy$ or $m = uy$ and m is of the form ug_i for some g_i, as required. Similarly, if $m = ax^a$ or $m = uy^b$, the substitutions will produce monomials of smaller weight or generators within a finite number of steps. They will fail to apply when m takes the form ux or uy^b with $0 \leq b \leq 4$, that is ug_i for some g_i. □

Remarks.

(1). The given substitutions are clearly allowed, working in the quotient ring $\mathcal{O}_{n,p}/J(F)$. The important point is that such substitutions can be applied repeatedly by a computer and will: (i) terminate after a finite number of substitutions, without developing into an infinite loop; (ii) will fail to apply only when the monomial under simplification is of the form ag_i with $a \in \mathcal{O}_p$. (i) and/or (ii) do not necessarily hold for other 'obvious' choices of substitutions (even for some of the above cases when applied in a different order to that stated). And in some of the longer calculations, in particular, we need to be certain that the choices will lead to a successful algorithm beforehand.

(2). Several of the calculations may be performed by hand. In the cases where f is weighted homogeneous it is convenient to simplify F (modulo $J(F)$) to begin with (c.f., [B2] where F is replaced by F_1). The resulting vector fields differ only by a scalar multiple from those calculated by computer. They will be used in the classifications for the A_2 and A_3 cases described below — these were done by hand, as well as by computer, to demonstrate the classification method.

(3). The choices of the g_i given above will be used in the A_k and E_k classifications below; however, a minor modification is made for the D_k's. The choice made above was convenient for the notation but, in practice, we will order the g_i (which are weighted homogeneous) so that

$$\mathrm{wt}(g_1) \geq \mathrm{wt}(g_2) \geq \cdots \geq \mathrm{wt}(g_{k-1}) > \mathrm{wt}(g_k) = 0.$$

In particular, we always choose $g_k = 1$ and as a result θ_k is always the 'Euler vector field' — the only Saito vector field of weight 0 (see below). This minor change has no consequence to the above proof; a permutation of the u_i results. We will state the specific ordering of the g_i for each case D_k — the resulting modifications to F and required substitutions should be clear.

The Computer Program

A major part of the work in this section was the writing of a program to calculate the Saito vector fields by implementing the above algorithm. The program was written in Maple; we shall not discuss the programming strategy nor the code itself — most of this was routine. We shall just describe what the program does.

There are several routines, the most important being the reduction algorithm which carries out the substitutions described above. The user creates a 'set-up' file which defines the unfolding F, the generators g_i, and the allowed relations as a table of monomials (`rel1`, say) and a table of the corresponding substitutions (`rel2`, say). (This file must also specify the coordinates being used on $\mathbf{R}^n$ so that these may be distinguished from those on $\mathbf{R}^p$. In our examples these are just (x) in the A_k cases, and (x, y) in the others.) When a polynomial f, say, is passed to the reduction routine it is expanded to a sum of monomials. Then each monomial is checked to see if it is divisible by a monomial `rel1[i]` in the table `rel1`. If this is the case then the entry `rel2[i]` in `rel2` is substituted in and the result stored for the next pass through the substitution loop. Otherwise, if the monomial is divisible by none of the `rel1[i]` then, by the theory, it must be of the form ag_i with $a \in \mathcal{O}_p$ and is added to the 'reduced form' of f — a polynomial to be returned at the end when all of the monomials have been reduced. (In this instance, the variable which stores the polynomial to be reduced in the next pass through the substitution loop is then equal to the zero polynomial.) Before terminating, the routine checks that the reduced form of f is indeed of the form $\sum a_i g_i$ with $a_i \in \mathcal{O}_p$ and returns an error otherwise. This is a 'safety check' to make sure the relations given by the user (in `rel1` and `rel2`) do indeed work. If an error is not spotted at this stage it is likely to be carried though to further calculations with little chance of being noticed.

Other routines perform functions such as extracting the coefficients a_i in the reduced form $\sum a_i g_i$ and saving them to a file. There is a routine which incorporates all these features, calculating the a_{ij} in $F.g_j = \sum a_{ij} g_i$ for all j, forming the corresponding Saito vector fields, and storing these in a format suitable for use with our 'Transversal' classification package (in particular, as a `liealg` routine for use with `wtcalc`) — see Chapter 6. This therefore minimises human involvement in the whole calculation (including subsequent classifications using the results) hopefully eliminating errors. One must be careful when defining F, the generators g_i, and the relations (`rel1` and `rel2`) in the 'set-up' file, of course.

Finally, we mention that a routine which calculates the vector fields tangent to the bifurcation varieties of the simple singularities has been written as well. This uses a similar algorithm, only due to Bruce, [B3].

5.2.2 Determinacy and Complete Transversal Theorems

Suppose we define a weighted filtration of m_n and $\mathcal{R}$ as in Section 2.4. Since $\mathcal{R}(\mathcal{D})$ is a subgroup of $\mathcal{R}$ this restricts to a filtration of $\mathcal{R}(\mathcal{D})$ and the results of Section 2.4 apply. (In the notation introduced there, we filter $\mathcal{R}(\mathcal{D})$ by the normal subgroups $(1_n + F^r\mathcal{O}(n,n)) \cap \mathcal{R}(\mathcal{D})$.) We shall not reproduce all the details again; however, we do need to define an appropriate subspace L of $L(F^1\mathcal{R}(\mathcal{D}))$ for which the complete transversal theorem holds. We then deduce, as a corollary, a determinacy result which is more powerful than that discussed in Section 5.1 above.

Proposition 5.10 *Let $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ be a finitely determined analytic germ. Suppose further that f is weighted homogeneous of weight d with respect to the system of weights $(\bar{w}_1, \ldots, \bar{w}_n)$. Let $\{g_1, \ldots, g_p\}$ be a basis of monomials for $\mathcal{O}_n/J(f)$ and ordered such that*

$$\mathrm{wt}(g_1) \geq \mathrm{wt}(g_2) \geq \cdots \geq \mathrm{wt}(g_{p-1}) > \mathrm{wt}(g_p) = 0$$

(so choose $g_p = 1$). Define α_i to be $\mathrm{wt}(g_i)$. Let $(u_1, \ldots, u_p)$ be coordinates on $\mathbf{C}^p$ and assign weights $w = (w_1, \ldots, w_p)$ where $w_i = d - \alpha_i$. Let a_{ij} be defined by

$$F.g_j = \sum_i a_{ij} g_i \bmod J(F)$$

and θ_j be the Saito vector fields which generate $\Theta(\mathcal{D})$

$$\theta_j = \sum_i a_{ij} \partial/\partial u_i,$$

as described above. Then

1. *the a_{ij} are weighted homogeneous and*
$$\mathrm{wt}(a_{ij}) = w_i + \alpha_j = d - \alpha_i + \alpha_j;$$
2. *the vector fields θ_j are weighted homogeneous and*
$$\mathrm{wt}(\theta_j) = \alpha_j.$$
(That is, $\theta_j \cdot g \in F_w^{t+\alpha_j}\mathcal{O}_p$ for all $g \in F_w^t\mathcal{O}_p$ and all t.)

Proof. 1. $F = f + \sum u_i g_i \in \mathcal{O}_{n,p}$ is weighted homogeneous of weight d with respect to $(\bar{w}_1, \ldots, \bar{w}_n, w_1, \ldots, w_p)$. So $F.g_j$ is weighted homogeneous of weight $d + \alpha_j$. Now $F.g_j$ can be written in the form
$$F.g_j = \sum_i a_{ij} g_i + \sum_i b_{ij} \partial F/\partial x_i,$$
and since $\partial F/\partial x_i$ is also weighted homogeneous (of weight $d - \bar{w}_i$, in fact) it follows that the a_{ij} are weighted homogeneous of weight $d - \alpha_i + \alpha_j$.

2. Observe that since a_{ij} is weighted homogeneous of weight $d - \alpha_i + \alpha_j$, $a_{ij}\partial/\partial u_i$ is weighted homogeneous of weight $(d-\alpha_i+\alpha_j)-(d-\alpha_i) = \alpha_j$. That is, for any monomial $g \in F_w^t\mathcal{O}_p$ we have $a_{ij}\partial/\partial u_i \cdot g \in F_w^{t+\alpha_j}\mathcal{O}_p$. This is independent of i so $\theta_j = \sum_i a_{ij}\partial/\partial u_i$ is weighted homogeneous of weight α_j. □

Remark. We will consider the cases where $\alpha_i < d$ for all i (this implies f must be one of the simple singularities) so that the weights w_i are positive.

It is natural to formulate complete transversal and determinacy results using a weighted filtration. We will require the following technical lemmas, the proofs were supplied by J.W. Bruce.

Lemma 5.11 *Let $(X, 0) \subset (\mathbf{C}^p, 0)$ be the germ of an analytic variety, then $L(\mathcal{R}(X)) = \Theta(X)$ (using definition 2.7).*

Proof. Let $\Phi : (\mathbf{C}^p \times \mathbf{C}, 0) \longrightarrow (\mathbf{C}^p, 0)$ be an analytic germ with $\Phi(x, 0) = x$ for all $x \in \mathbf{C}^p$ and $\Phi(X, t) = X$ for small t. Consider the vector field $x \mapsto \partial\Phi(x,t)/\partial t|_{t=0}$, also denoted $\partial\Phi_t/\partial t|_{t=0}$. Clearly this is tangent to X, i.e., lies in $\Theta(X)$. Conversely, any vector field in $\Theta(X)$ can be integrated to give a flow Φ as above. □

Lemma 5.12 *Let $(\mathcal{D},0) \subset (\mathbf{C}^p,0)$ be the discriminant of a function-germ f. Suppose further that f is right equivalent to a weighted homogeneous function. Then for all s we have $J^s(L\mathcal{R}(\mathcal{D})) \subset L(J^s\mathcal{R}(\mathcal{D}))$.*

Proof. If $\xi \in J^s(L\mathcal{R}(\mathcal{D}))$ then we can find some $\tilde{\xi} \in L\mathcal{R}(\mathcal{D})$ with $j^s\tilde{\xi} = \xi$. In other words for some flow $\Phi : (\mathbf{C}^p \times \mathbf{C},0) \longrightarrow (\mathbf{C}^p,0)$ as in the preceding proof we have $\tilde{\xi} = \partial\Phi_t/\partial t|_{t=0}$. By Proposition 5.7 $\Theta(\mathcal{D})$ is a pointed space, that is all vector fields which belong to $\Theta(\mathcal{D})$ vanish at the origin. So for all small t, $\Phi_t : (\mathbf{C}^p,0) \longrightarrow (\mathbf{C}^p,0)$ and $\Phi_t(X) = X$. That is $t \mapsto j^s(\Phi_t)$ is a path in $J^s\mathcal{R}(\mathcal{D})$ so $\partial(j^s\Phi_t)/\partial t|_{t=0}$ lies in $L(J^s\mathcal{R}(\mathcal{D}))$. But $j^s(\partial\Phi_t/\partial t|_{t=0}) = \partial(j^s\Phi_t)/\partial t|_{t=0}$ so the result holds. □

Theorem 5.13 *Assume the hypotheses and notation of Proposition 5.10. Apply the definition of weighted filtration given in Section 2.4. Let $\{F^r_w\mathcal{O}_p\}$ be the weighted filtration of m_p, which we will abbreviate to $\{F^r\mathcal{O}_p\}$. The weighted filtration $\{F^r\mathcal{R}\}$ induces a filtration of $\mathcal{R}(\mathcal{D})$ and the results of Section 2.4 hold. Define*

$$L = \langle\theta_1,\ldots,\theta_{p-1}\rangle + F^1\mathcal{O}_p.\langle\theta_p\rangle \subset L\mathcal{R}(\mathcal{D}).$$

Let $h \in m_p$, $k \geq 1$ and T be a subspace of $H^{k+1} \subset J^{k+1}(n,p)$ such that

$$J^{k+1}L \cdot j^{k+1}h + T \supset H^{k+1}.$$

Then any (weighted) k-jet j^kh_1 with $j^kh_1 \sim_{J^k\mathcal{R}(\mathcal{D})} j^kh$ satisfies $j^{k+1}h_1 \sim_{J^{k+1}\mathcal{R}(\mathcal{D})} j^{k+1}h + t$ for some $t \in T$. Note that here all 'jet' notation refers to weighted jet-spaces with respect to the system of weights, w — we refer to Section 2.4 for a summary of the notation. (The result can be paraphrased as 'any h_1 with $h_1 \sim_{\mathcal{R}(\mathcal{D})} h + \phi$ ($\phi \in F^{k+1}\mathcal{O}_p$) satisfies $h_1 \sim_{\mathcal{R}(\mathcal{D})} h + t + \psi$ for some $t \in T$ ($\psi \in F^{k+2}\mathcal{O}_p$)', as required in a classification with respect to $\mathcal{R}(\mathcal{D})$-equivalence.)

Proof. The proof follows from Theorem 2.10 in a similar way to that of Theorem 2.37. Again we will only sketch the proof, highlighting the main points. We work with the Lie subalgebra L of $L\mathcal{R}(\mathcal{D})$. The way in which the general complete transversal result Theorem 2.10 was formulated means we do not need to concern ourselves with a corresponding subgroup of $\mathcal{R}(\mathcal{D})$. However, the main problem now is that for a given s, J^sL is a subalgebra of $J^s(L\mathcal{R})$ and the resulting coordinate changes in $\mathbf{C}^p$ may *not* preserve $\mathcal{D}$. It is not clear if $\mathcal{R}(\mathcal{D})$ is jet-closed with respect to the weighted filtration, but either way, from Lemma 5.12 we have $J^s(L\mathcal{R}(\mathcal{D})) \subset L(J^s\mathcal{R}(\mathcal{D}))$ for all s — this is the important result. It follows that $J^{k+1}L \subset J^{k+1}(L\mathcal{R}(\mathcal{D}))$ is a Lie subalgebra of $L(J^{k+1}\mathcal{R}(\mathcal{D}))$ so there is a Lie subgroup G of $J^{k+1}\mathcal{R}(\mathcal{D})$ with Lie algebra $J^{k+1}L$. This subgroup G is used in the proof of the complete transversal theorem 2.10 so the equivalence used in the conclusion to the theorem will preserve $\mathcal{D}$. The result therefore follows from Theorem 2.10 once we have verified that for $f \in F^1\mathcal{O}_p$, $g \in F^s\mathcal{O}_p$ and $l \in L$

$$l \cdot (f+g) - l \cdot f \in F^{s+1}\mathcal{O}_p.$$

But this follows from the fact that the vector fields θ_j are weighted homogeneous of positive weight for $j = 1, \ldots, p-1$ (by Lemma 5.10) and they act linearly. □

The space L defined in the above theorem is used for complete transversal and determinacy calculations. We must multiply the Euler vector field θ_p by $F^1\mathcal{O}_p$ to obtain vector fields of positive weight. The following determinacy theorem is a corollary to the complete transversal theorem. The proof follows the same lines as Corollary 2.23. We just need to note that the hypotheses imply that h is finitely determined with respect to standard degree, but this follows as $\mathcal{R}(\mathcal{D})$ is one of Damon's 'geometric subgroups' — see Theorem 5.4. We remark that the following determinacy theorem is, in practice, an immense improvement over the determinacy results mentioned after Theorem 5.4 where one needs to use the space $F^1\mathcal{O}_p.\langle\theta_1, \ldots, \theta_p\rangle$ in place of L.

Corollary 5.14 *With the notation of Theorem 5.13, a germ $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ is k-$\mathcal{R}(\mathcal{D})$-determined if*

$$F^{k+1}\mathcal{O}_p \subset L \cdot h.$$

To perform determinacy calculations on computer we must reduce this to a finite dimensional situation (the function `wtcalc` described in Chapter 6 calculates tangent spaces in a given weighted jet-space).

Corollary 5.15 *Assume the notation of Theorem 5.13 and define w_{max} to be the maximum of the weights $w_1, \ldots, w_p$. Then, a germ $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ is k-$\mathcal{R}(\mathcal{D})$-determined if*

$$F^{k+1}\mathcal{O}_p \subset L \cdot h + F^{k+1+w_{max}}\mathcal{O}_p.$$

Proof. Since $\mathcal{O}_p$ is a local ring we can apply the Nakayama Lemma to see that the condition in Corollary 5.14 is equivalent to

$$F^{k+1}\mathcal{O}_p \subset L \cdot h + m_p.F^{k+1}\mathcal{O}_p.$$

Now, by definition, the ideal $F^{k+1+w_{max}}\mathcal{O}_p$ is generated by the monomials in $\mathcal{O}_p$ of weight $\geq k+1+w_{max}$. Such monomials can therefore be written in the form $u_i m$ for $1 \leq i \leq k$ and m a monomial of weight $\geq k+1$. Thus $u_i m \in m_p.F^{k+1}\mathcal{O}_p$ so $F^{k+1+w_{max}}\mathcal{O}_p \subset m_p.F^{k+1}\mathcal{O}_p$ and the result follows. □

To check determinacy we can therefore work in the $(k+w_{max})$-jet-space and perform the calculation using a computer. Indeed, suppose we consider h as a k-jet and calculate the complete transversals of weight $k+1$ to weight $k+w_{max}$ using the computer. If these are all found to be empty then the condition in Corollary 5.15 holds and h is k-$\mathcal{R}(\mathcal{D})$-determined. (We can, of course, improve matters by using $m_p.F^{k+1}\mathcal{O}_p$ instead of $F^{k+1+w_{max}}\mathcal{O}_p$, but this was found to be unnecessary in practice, and hard to implement on a computer.)

In a classification the standard 'scaling' coordinate changes of the form

$$(u_1, \ldots, u_p) \mapsto (\lambda_1 u_1, \ldots, \lambda_p u_p),$$

where $\lambda_i \in \mathbf{C}$, $\lambda_i \neq 0$, are extremely useful for simplifying orbits. Such coordinate changes cannot be used in the present situation since they do not necessarily preserve the discriminant $\mathcal{D}$. We have the following restricted version though.

Proposition 5.16 *Assume the hypotheses and conventions of Proposition 5.10. Then for $t \in \mathbf{C}$, $t \neq 0$, the map-germ $(\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}^p, 0)$ defined by*

$$(u_1, \ldots, u_p) \mapsto (t^{w_1} u_1, \ldots, t^{w_p} u_p)$$

is an element of the group $\mathcal{R}(\mathcal{D})$.

Proof. The map-germ is a diffeomorphism so belongs to $\mathcal{R}$. We will show that it preserves $\mathcal{D}$. From Proposition 5.10 the versal unfolding $F = f + \sum u_i g_i \in \mathcal{O}_{n,p}$ is weighted homogeneous of weight d with respect to $(\bar{w}_1, \ldots, \bar{w}_n, w_1, \ldots, w_p)$. Hence

$$F(t^{\bar{w}_1} x_1, \ldots, t^{\bar{w}_n} x_n, t^{w_1} u_1, \ldots, t^{w_p} u_p) = t^d F(x_1, \ldots, x_n, u_1, \ldots, u_p).$$

Also note that $\partial F/\partial x_i$ is weighted homogeneous of weight $d - \bar{w}_i$ (for $1 \leq i \leq n$) so that an analogous result holds for $\partial F/\partial x_i$. Thus, since $t \neq 0$

$$\begin{aligned}
(u_1, \ldots, u_p) \in \mathcal{D} &\iff \exists\,(x_1, \ldots, x_n) : F = \partial F/\partial x_1 = \cdots = \partial F/\partial x_n = 0 \\
&\qquad \text{at} \quad (x_1, \ldots, x_n, u_1, \ldots, u_p) \\
&\iff \exists\,(x_1, \ldots, x_n) : F = \partial F/\partial x_1 = \cdots = \partial F/\partial x_n = 0 \\
&\qquad \text{at} \quad (t^{\bar{w}_1} x_1, \ldots, t^{\bar{w}_n} x_n, t^{w_1} u_1, \ldots, t^{w_p} u_p) \\
&\iff (t^{w_1} u_1, \ldots, t^{w_p} u_p) \in \mathcal{D}.
\end{aligned}$$

The diffeomorphism therefore belongs to $\mathcal{R}(\mathcal{D})$. □

Finally, for the A_2 and A_3 cases we list the monomials of a given weight. This will be a useful reference for the classifications in Sections 5.3 and 5.4 below. For the A_3 case the monomials of a given weight include those in both the A_3 column and the A_2 column.

Weight	A_2 Monomials	A_3 Monomials
2	u_1	-
3	u_2	-
4	u_1^2	u_3
5	u_1u_2	-
6	$u_1^3,\ u_2^2$	u_1u_3
7	$u_1^2u_2$	u_2u_3
8	$u_1^4,\ u_1u_2^2$	$u_1^2u_3,\ u_3^2$
9	$u_1^3u_2,\ u_2^3$	$u_1u_2u_3$
10	$u_1^5,\ u_1^2u_2^2$	$u_1^3u_3,\ u_2^2u_3,\ u_1u_3^2$
11	$u_1^4u_2,\ u_1u_2^3$	$u_1^2u_2u_3,\ u_2u_3^2$
12	$u_1^6,\ u_1^3u_2^2,\ u_2^4$	$u_1^4u_3,\ u_1^2u_3^2,\ u_1u_2^2u_3,\ u_3^3$
13	$u_1^5u_2,\ u_1^2u_2^3$	$u_1^3u_2u_3,\ u_1u_2u_3^2,\ u_2^3u_3$
14	$u_1^7,\ u_1^4u_2^2,\ u_1u_2^4$	$u_1^5u_3,\ u_1^3u_3^2,\ u_1^2u_2^2u_3,\ u_1u_3^3,\ u_2^2u_3^2$
15	$u_1^6u_2,\ u_1^3u_2^3,\ u_2^5$	$u_1^4u_2u_3,\ u_1^2u_2u_3^2,\ u_1u_2^3u_3,\ u_2u_3^3$
16	$u_1^8,\ u_1^5u_2^2,\ u_1^2u_2^4$	$u_1^6u_3,\ u_1^4u_3^2,\ u_1^3u_2^2u_3,\ u_1^2u_3^3,\ u_1u_2^2u_3^2,\ u_2^4u_3,\ u_3^4$

5.3 Classification of Function-Germs on the A_2 (Cusp) Discriminant

In this section we classify all function-germs on the cusp (A_2) discriminant variety $\mathcal{D}$ under $\mathcal{R}(\mathcal{D})$-equivalence and up to modality 2. This extends the results of [A2] where the same weighted filtration was employed. Some families of higher modality occur naturally in the classification and we include these too. The results are summarised in the following theorem; we only state the finitely determined germs, the list is short and does not warrant a full stratification diagram of the jet-space. The classification was performed by computer. In comparison, it was possible, though very tedious, to perform the calculations by hand. We did this and reproduce the details to demonstrate how the classification method proceeds. Generally, we will not resort to hand calculations in such classifications if we can avoid doing so. In this and later sections, we will use $a, b, c, \dots$ to denote moduli.

Theorem 5.17 *Every function-germ $h : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}, 0)$ on the cusp discriminant variety $\mathcal{D}$ of $\mathcal{R}(\mathcal{D})$-modality ≤ 2 is $\mathcal{R}(\mathcal{D})$-equivalent to one of the following finitely determined germs. The first germs of modality ≥ 3 to occur during the classification are given in the second section of the table; they occur as the initial singularities in the stated series. The codimension refers to the dimension of $\mathcal{O}_n/J_X(h)$ as a $\mathbf{C}$-vector space; u_1 is the only stable singularity.*

Singularity	Determinacy Degree	Codim
u_1	2	1
u_2	3	2
$u_1^2 + au_2^n$	$3n, \quad n \geq 2, \quad a \neq 0$	$n+2$
$u_1u_2 + au_1^3$	6	5
$u_1^3 + au_2^2 + bu_1u_2^2$	$8, \quad 4a \neq 27, \quad a \neq 0$	7
$u_1^3 + \frac{27}{4}u_2^2 + au_1^nu_2 + bu_1^{n+1}u_2$	$2n+5, \quad n \geq 2, \quad a \neq 0$	$2n+4$
$u_1^3 + \frac{27}{4}u_2^2 + au_1^n + bu_1^{n+1}$	$2n+2, \quad n \geq 4, \quad a \neq 0$	$2n+1$
$u_2^2 + au_1^n + bu_1^{n+1}$	$2n+2, \quad n \geq 4, \quad a \neq 0$	$n+4$
$u_1^3 + au_1u_2^n + bu_2^{n+1} + cu_2^{n+2}$	$3n+6, \quad n \geq 2, \quad a, b \neq 0\,(\dagger)$	$2n+5$
$u_1^3 + au_2^n + bu_1u_2^n$	$3n+2, \quad n \geq 3, \quad a \neq 0$	$2n+3$
$u_1^2u_2 + au_1^4 + bu_2^n + cu_2^{n+1}$	$3n+3, \quad n \geq 3, \quad b \neq 0$	$n+7$

($\dagger$) *For the case $n = 2$ the condition $b \neq 0$ needs to be replaced by $4a^3 + 27b^2 \neq 0$.*

The proof will take up the rest of this section.

The calculations from Section 5.2 show that the module of vector fields tangent to the cusp discriminant is generated by

$$\theta_1 = 9u_2\partial/\partial u_1 - 2u_1^2\partial/\partial u_2, \qquad \theta_2 = 2u_1\partial/\partial u_1 + 3u_2\partial/\partial u_2,$$

θ_2 being the Euler vector field. Let M denote the ideal $\langle\theta_1 \cdot h\rangle + F^1\mathcal{O}_2.\langle\theta_2 \cdot h\rangle$ where h is some germ in $\mathcal{O}_2$; this is used in the complete transversal and determinacy calculations.

The classification starts at the (weighted) 2-jet-level. The 2-jets take the form au_1 for $a \in \mathbf{C}$. Applying Proposition 5.16 we can use 'scaling' coordinate changes, namely

$$(u_1, u_2) \mapsto (t^2u_1, t^3u_3), \quad t \in \mathbf{C},$$

to give the J^2-orbits

u_1	2-determined (stable),
0	(1).

To show determinacy note that for $h = u_1$

$$\theta_1 \cdot h = 9u_2, \qquad \theta_2 \cdot h = 2u_1,$$

so $\langle u_2 \rangle \subset M$ and $F^1\mathcal{O}_2.\langle u_1 \rangle \subset M$ and therefore $F^3\mathcal{O}_2 \subset M + F^6\mathcal{O}_2$. Thus, u_1 is (weighted) 2-determined by Corollary 5.15. Note that in this determinacy calculation, and several others, the result follows because we can use $\langle \theta_1 \cdot h \rangle$. This is allowed only because we are using a weighted filtration (and θ_1 'increases weight').

Since 1 spans $\mathcal{O}_n/\langle \theta_1 \cdot h, \theta_2 \cdot h \rangle$ as a **C**-vector space, u_1 is stable. From Proposition 5.7 we see that u_1 is the only stable singularity.

(1) Continuing, a 3-transversal of 0 is $\{u_2\}$ and, after scaling, we obtain the J^3-orbits

u_2	3-determined,
0	(1).

Determinacy follows as before (here we have $F^4\mathcal{O}_2 \subset M + F^7\mathcal{O}_2$).

(1) A 4-transversal of 0 is $\{u_1^2\}$ and the J^4 orbits are

u_1^2	(2),
0	(1).

(2) Putting $h = u_1^2$ gives

$$\theta_1 \cdot h = 18u_1u_2, \qquad \theta_2 \cdot h = 4u_1^2,$$

so $F^5\mathcal{O}_2 \subset M + F^6\mathcal{O}_2$ and the 5-transversal is empty. However, a 6-transversal is $\{u_2^2\}$ and the 6-jets take the form $u_1^2 + au_2^2$. The scaling coordinate changes referred to above will not reduce a to 1. Indeed, with $h = u_1^2 + au_2^2$ we have

$$\theta_1 \cdot h = 18u_1u_2 - 4au_1^2u_2, \qquad \theta_2 \cdot h = 4u_1^2 + 6au_2^2,$$

and $L\mathcal{R}(\mathcal{D}) \cdot h = \langle \theta_1 \cdot h, \theta_2 \cdot h \rangle$. Working with the group $\mathcal{R}(\mathcal{D})$ in the (weighted) 6-jet-space we see that

$$u_2^2 \notin L\mathcal{R}(\mathcal{D}) \cdot h \quad \text{modulo} \quad F^7\mathcal{O}_2,$$

so that a is a modulus by Theorem 1.9. The J^6-orbits over u_1^2 are therefore

$u_1^2 + au_2^2$	6-determined, $a \neq 0$,
u_1^2	(2).

Continuing with (2) gives a series and we will consider the general case:

$$u_1^2 + au_2^n \qquad 3n\text{-determined}, \quad n \geq 2, \quad a \neq 0.$$

Consider $h = u_1^2$ as a $(3n-2)$-jet for $n \geq 2$. Then from the previous calculations of $\theta_1 \cdot h$ and $\theta_2 \cdot h$ we see that the $(3n-1)$-transversal is empty while a $3n$-transversal is $\{u_2^n\}$. The J^{3n}-orbits are therefore of the form $u_1^2 + au_2^n$. As before one easily checks that a is a modulus. For determinacy we show that $F^{3n+1}\mathcal{O}_2 \subset M + F^{3n+4}\mathcal{O}_2$ with $h = u_1^2 + au_2^n$. Now

$$\theta_1 \cdot h = 18u_1u_2 - 2nau_1^2u_2^{n-1}, \qquad \theta_2 \cdot h = 4u_1^2 + 3nau_2^n,$$

so from $\theta_1 \cdot h$ we see $18u_1^3u_2 - 2nau_1^4u_2^{n-1} \in M$. But $u_1^4u_2^{n-1}$ is of weight $3n+5$ so $\langle u_1^3u_2\rangle \subset M + F^{3n+4}\mathcal{O}_2$. Similarly we have $18u_1^2u_2 - 2nau_1^3u_2^{n-1} \in M$ so $\langle u_1^2u_2\rangle \subset M + F^{3n+4}\mathcal{O}_2$ and $18u_1u_2^2 - 2nau_1^2u_2^n \in M$ where $u_1^2u_2^n$ is of weight $3n+4$ so $\langle u_1u_2^2\rangle \subset M + F^{3n+4}\mathcal{O}_2$. From $\theta_2 \cdot h$ we have $4u_1^4 + 3nau_1^2u_2^n \in M$ so similarly $\langle u_1^4\rangle \subset M + F^{3n+4}\mathcal{O}_2$. Finally, $4u_1^2u_2 + 3nau_2^{n+1} \in M$ so $\langle u_2^{n+1}\rangle \subset M + F^{3n+4}\mathcal{O}_2$ for $a \neq 0$. It therefore follows that $F^{3n+1}\mathcal{O}_2 \subset M + F^{3n+4}\mathcal{O}_2$ and that h is $3n$-determined for $a \neq 0$. If $a = 0$ we see that the next non-empty transversal is $\{u_2^{n+1}\}$ and the series continues.

(1) Continuing, a 5-transversal of 0 is $\{u_1u_2\}$ and, after scaling, we obtain the J^5-orbits

$$\begin{array}{ll} u_1u_2 & (3), \\ 0 & (1). \end{array}$$

(3) Now, with $h = u_1u_2$ we have

$$\theta_1 \cdot h = 9u_2^2 - 2u_1^3, \qquad \theta_2 \cdot h = 5u_1u_2,$$

and one sees that a 6-transversal is $\{u_1^3\}$ giving the J^6-orbits

$$u_1u_2 + au_1^3 \qquad 6\text{-determined}, \quad \text{for all } a \in \mathbf{C}.$$

Putting $h = u_1u_2 + au_1^3$ gives

$$\theta_1 \cdot h = 9u_2^2 - 2u_1^3 + 27au_1^2u_2, \qquad \theta_2 \cdot h = 5u_1u_2 + 6au_1^3,$$

and it follows that a is a modulus as in the earlier case. For determinacy we show that $F^7\mathcal{O}_2 \subset M + F^{10}\mathcal{O}_2$. From $\theta_2 \cdot h$ we have $5u_1^3u_2 + 6au_1^5 \in M$, but u_1^5 is of weight 10 so $\langle u_1^3u_2\rangle \subset M + F^{10}\mathcal{O}_2$. Also $5u_1u_2^2 + 6au_1^3u_2 \in M$ so $\langle u_1u_2^2\rangle \subset M + F^{10}\mathcal{O}_2$. From $\theta_1 \cdot h$ we have $9u_2^3 - 2u_1^3u_2 + 27au_1^2u_2^2 \in M$ and it follows that $\langle u_2^3\rangle \subset M + F^{10}\mathcal{O}_2$. Similarly, $9u_1u_2^2 - 2u_1^4 + 27au_1^3u_2 \in M$ so $\langle u_1^4\rangle \subset M + F^{10}\mathcal{O}_2$. Finally, from $\theta_2 \cdot h$ we now have $5u_1^2u_2 + 6au_1^4 \in M$ so $\langle u_1^2u_2\rangle \subset M + F^{10}\mathcal{O}_2$. Referring to the table of monomials in Section 5.2.2 we observe that $F^7\mathcal{O}_2 \subset M + F^{10}\mathcal{O}_2$ and that h is therefore 6-determined.

(1) A 6-transversal of 0 is $\{u_1^3, u_2^2\}$ and we obtain the J^6-orbits

$$\begin{array}{ll} u_1^3 + au_2^2 & (4), \\ u_2^2 & (5), \\ 0 & (1). \end{array}$$

(4) Consider $h = u_1^3 + au_2^2$, then

$$\theta_1 \cdot h = (27 - 4a)u_1^2u_2, \qquad \theta_2 \cdot h = 6u_1^3 + 6au_2^2.$$

It follows that a is a modulus; in fact more moduli occur as we will show later. Now consider the calculation of a 7-transversal. If $4a \neq 27$ then the 7-transversal is empty; otherwise a 7-transversal is $\{u_1^2u_2\}$. (**Note:** this follows from the vector $\theta_1 \cdot h$ which we *can* use with the given weighted filtration.) The J^7-orbits are therefore

$$\begin{array}{ll} u_1^3 + au_2^2 & 4a \neq 27, \quad (4), \\ u_1^3 + \frac{27}{4}u_2^2 + bu_1^2u_2 & (6). \end{array}$$

(4) Consider $h = u_1^3 + au_2^2$; from the previous calculations of $\theta_1 \cdot h$ and $\theta_2 \cdot h$ we find that the tangent space M contains $6u_1^4 + 6au_1u_2^2$ so an 8-transversal is $\{u_1u_2^2\}$ and the J^8-orbits are

$$u_1^3 + au_2^2 + bu_1u_2^2 \qquad \text{8-determined}, \quad 4a \neq 27, \quad a \neq 0.$$

Now putting $h = u_1^3 + au_2^2 + bu_1u_2^2$ gives

$$\theta_1 \cdot h = (27 - 4a)u_1^2u_2 + 9bu_2^3 - 4bu_1^3u_2, \qquad \theta_2 \cdot h = 6u_1^3 + 6au_2^2 + 8bu_1u_2^2,$$

so working with the group $\mathcal{R}(\mathcal{D})$ in the 8-jet-space we find that $\{u_2^2, u_1u_2^2\}$ forms an independent set to $L\mathcal{R}(\mathcal{D}) \cdot h$. That is

$$\left\{ \text{basis for } L\mathcal{R}(\mathcal{D}) \cdot h \text{ in } J^8 \right\} \cup \left\{ u_2^2, u_1u_2^2 \right\}$$

is an independent set, so by Theorem 1.9 a and b are both moduli. For determinacy we show $F^9\mathcal{O}_2 \subset M + F^{12}\mathcal{O}_2$. From $\theta_1 \cdot h$ we have $(27-4a)u_1^4u_2 + 9bu_1^2u_2^3 - 4bu_1^5u_2 \in M$, but $u_1^2u_2^3$ and $u_1^5u_2$ are of weight 13 so $\langle u_1^4u_2\rangle \subset M + F^{12}\mathcal{O}_2$ since we have assumed $4a \neq 27$. Then using $\theta_2 \cdot h$ we obtain $6u_1^4u_2 + 6au_1u_2^3 + 8bu_1^2u_2^3 \in M$, and using the previous result and the fact that $u_1^2u_2^3$ is of weight 13 we have $\langle u_1u_2^3\rangle \subset M + F^{12}\mathcal{O}_2$ for $a \neq 0$. This deals with the monomials of weight 11, the monomials of weight 10, $u_1^2u_2^2$ and u_1^5, follow in the same way. Having obtained these we find that $\langle u_1^3u_2\rangle \subset M + F^{10}\mathcal{O}_2$ using $\theta_1 \cdot h$, and then $\langle u_2^3\rangle \subset M + F^{10}\mathcal{O}_2$ using $\theta_2 \cdot h$; that is we have $F^9\mathcal{O}_2 \subset M + F^{12}\mathcal{O}_2$ and the determinacy result follows.

The case $4a = 27$ is dealt with under (6), yet we still have to consider the case $a = 0$. From the above calculations we see that $M + F^{10}\mathcal{O}_2$ contains $\langle u_1^3u_2\rangle$ but not $\langle u_2^3\rangle$. A 9-transversal is therefore $\{u_2^3\}$ and the J^9-orbits are

$$u_1^3 + bu_1u_2^2 + cu_2^3.$$

The classification continues as follows; the details are similar to previous cases and we just provide a summary. The 10-transversal of $u_1^3 + bu_1u_2^2 + cu_2^3$ is empty. The 11-transversal is empty for $b \neq 0$, otherwise an 11-transversal is $\{u_1u_2^3\}$. We will consider the case $b \neq 0$ to begin with. A 12-transversal of $u_1^3 + bu_1u_2^2 + cu_2^3$ is $\{u_2^4\}$ and the J^{12}-orbits are

$$u_1^3 + bu_1u_2^2 + cu_2^3 + du_2^4 \qquad \text{12-determined,} \quad b \neq 0, \quad 4b^3 + 27c^2 \neq 0.$$

The determinacy result follows from the inclusion $F^{13}\mathcal{O}_2 \subset M + F^{16}\mathcal{O}_2$. Working with the group $\mathcal{R}(\mathcal{D})$ in the 12-jet-space, we find that b, c and d are moduli. We can therefore terminate this branch of the classification, having reached jets of modality greater than 2. Indeed, consider any higher jet j with 12-jet $u_1^3 + bu_1u_2^2 + cu_2^3 + du_2^4$ and such that $b = 0$ or $4b^3 + 27c^2 = 0$. Then any open neighbourhood of j must contain a jet $\tilde{j}$ whose 12-jet is of the form $u_1^3 + bu_1u_2^2 + cu_2^3 + du_2^4$ but where $b \neq 0$ and $4b^3 + 27c^2 \neq 0$; $\tilde{j}$ is therefore equivalent to such a trimodular jet. Thus, the modality of j is greater 2. The 12-determined jet $u_1^3 + bu_1u_2^2 + cu_2^3 + du_2^4$ provides the first example of a trimodular singularity in this classification.

We still have the case $b = 0$ in the 10-jet $u_1^3 + bu_1u_2^2 + cu_2^3$ above. Using the preceding argument this case can be ruled out as well. Consider any higher jet j with 9-jet $u_1^3 + cu_2^3$. Any open neighbourhood of j must contain a jet $\tilde{j}$ whose 9-jet is of the form $u_1^3 + bu_1u_2^2 + cu_2^3$ for some $b \neq 0$ and by the complete transversal calculations the 12-jet of $\tilde{j}$ must lie in the trimodular family $u_1^3 + bu_1u_2^2 + cu_2^3 + du_2^4$. However, the classification can be continued easily; we obtain two series involving moduli and will summarise the findings. As mentioned above, an 11-transversal of $u_1^3 + cu_2^3$ is $\{u_1u_2^3\}$ and the J^{11}-orbits are

$$u_1^3 + cu_2^3 + du_1u_2^3 \qquad \text{11-determined,} \quad c \neq 0.$$

When $c = 0$ this continues and gives the following trimodular and bimodular series:

$$u_1^3 + a_1 u_1 u_2^n + a_2 u_2^{n+1} + a_3 u_2^{n+2} \qquad (3n+6)\text{-determined}, \quad n \geq 2, \quad a_1, a_2 \neq 0 \ (\dagger),$$

$$u_1^3 + b_1 u_2^n + b_2 u_1 u_2^n \qquad (3n+2)\text{-determined}, \quad n \geq 3, \quad b_1 \neq 0.$$

($\dagger$) For the case $n = 2$ the condition $a_2 \neq 0$ needs to be replaced by $4a_1^3 + 27a_2^2 \neq 0$; see above.
The above calculations have also been verified by computer.

(6) Consider the 7-jet $h = u_1^3 + \frac{27}{4}u_2^2 + bu_1^2u_2$. Then

$$\theta_1 \cdot h = 18bu_1u_2^2 - 2bu_1^4, \qquad \theta_2 \cdot h = 6u_1^3 + \tfrac{6.27}{4}u_2^2 + 7bu_1^2u_2.$$

Working in the 8-jet-space $\theta_2 \cdot h$ gives $6u_1^4 + \frac{81}{2}u_1u_2^2 \in M + F^9\mathcal{O}_2$ and using $\theta_1 \cdot h$ we obtain the two monomials of weight 8, u_1^4 and $u_1u_2^2$, provided $b \neq 0$. If $b = 0$ we may take $\{u_1^4\}$ as an 8-transversal. The J^8-orbits are therefore

$$u_1^3 + \tfrac{27}{4}u_2^2 + bu_1^2u_2 \qquad b \neq 0,$$
$$u_1^3 + \tfrac{27}{4}u_2^2 + cu_1^4.$$

Consider the 8-jet $h = u_1^3 + \frac{27}{4}u_2^2 + bu_1^2u_2$ with $b \neq 0$. From the calculation of $\theta_2 \cdot h$ above we see that a 9-transversal is $\{u_1^3u_2\}$ and the J^9-orbits are

$$u_1^3 + \tfrac{27}{4}u_2^2 + bu_1^2u_2 + cu_1^3u_2.$$

One then verifies that $F^{10}\mathcal{O}_2 \subset M + F^{13}\mathcal{O}_2$ so that the germ is 9-determined; also that b and c are moduli.

Now, consider the 8-jet $u_1^3 + \frac{27}{4}u_2^2 + cu_1^4$. Here one finds that the 9-transversal is empty provided $c \neq 0$ and a 10-transversal is $\{u_1^5\}$, the resulting bimodular family $u_1^3 + \frac{27}{4}u_2^2 + cu_1^4 + du_1^5$ being 10-determined. If $c = 0$ then a 9-transversal is $\{u_1^3u_2\}$.

Continuing in this way using computer calculations suggests the two series of singularities

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^nu_2 + bu_1^{n+1}u_2 \qquad (2n+5)\text{-determined}, \quad n \geq 2, \quad a \neq 0,$$

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^n + bu_1^{n+1} \qquad (2n+2)\text{-determined}, \quad n \geq 4, \quad a \neq 0.$$

To see this consider $h = u_1^3 + \frac{27}{4}u_2^2$ as a $2r$-jet for some $r \geq 3$. Consider all monomials of weight $2r+1$. Since u_1^{r+1} has weight $2r+2$ any monomial of weight $2r+1$ must be of degree $\leq r$ in u_1. Such a monomial is therefore of the form $u_1^{r-s}u_2^t$ with s, t non-negative integers such that $2r - 2s + 3t = 2r + 1$. That is $s \equiv 1 \bmod 3$. The monomials of weight $2r + 1$ therefore take the form

$$u_1^{r-1}u_2,\ u_1^{r-4}u_2^3,\ u_1^{r-7}u_2^5,\ u_1^{r-10}u_2^7,\ \ldots$$

and finishing with one of the monomials u_2^t, $u_1u_2^t$ or $u_1^2u_2^t$ for appropriate t. (For example, see the monomials of weight 9, 11 and 13 respectively.) Now

$$\theta_1 \cdot h = 0, \qquad \theta_2 \cdot h = 6u_1^3 + \tfrac{81}{2}u_2^2,$$

so from $\theta_2 \cdot h$ we have

$$\begin{array}{rcl} 6u_1^{r-1}u_2 + \frac{81}{2}u_1^{r-4}u_2^3 & \in & M, \\ 6u_1^{r-4}u_2^3 + \frac{81}{2}u_1^{r-7}u_2^5 & \in & M, \\ 6u_1^{r-7}u_2^5 + \frac{81}{2}u_1^{r-10}u_2^7 & \in & M, \\ & \vdots & \end{array}$$

and so on. Thus, working in the $(2r+1)$-jet-space we see that a $(2r+1)$-transversal is $\{u_1^{r-1}u_2\}$ and the J^{2r+1}-orbits are

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^{r-1}u_2.$$

Now consider the monomials of weight $2r + 2$. Since u_1^{r+2} has weight $2r + 4$ any monomial of weight $2r + 2$ must be of degree $\leq r + 1$ in u_1. Such a monomial is therefore of the form $u_1^{r+1-s}u_2^t$ with s, t non-negative integers such that $2r + 2 - 2s + 3t = 2r + 2$. That is $s \equiv 0 \bmod 3$. The monomials of weight $2r + 2$ therefore take the form

$$u_1^{r+1},\ u_1^{r-2}u_2^2,\ u_1^{r-5}u_2^4,\ u_1^{r-8}u_2^6,\ \ldots$$

and finishing with one of the monomials u_2^t, $u_1u_2^t$ or $u_1^2u_2^t$ for appropriate t. (For example, see the monomials of weight 12, 14 and 16 respectively.) Putting $h = u_1^3 + \frac{27}{4}u_2^2 + au_1^{r-1}u_2$ we have

$$\begin{array}{rcl} \theta_1 \cdot h & = & 9(r-1)au_1^{r-2}u_2^2 - 2au_1^{r+1}, \\ \theta_2 \cdot h & = & 6u_1^3 + \frac{81}{2}u_2^2 + (2r+1)au_1^{r-1}u_2. \end{array}$$

Working in the $(2r + 2)$-jet-space, $\theta_2 \cdot h$ gives

$$6u_1^{r+1} + \tfrac{81}{2}u_1^{r-2}u_2^2 \in M + F^{2r+3}\mathcal{O}_2$$

which together with $\theta_1 \cdot h$ gives the monomials u_1^{r+1} and $u_1^{r-2}u_2^2$, provided $a \neq 0$. (**Note:** our weighted filtration *does* allow the use of the vector field $\theta_1 \cdot h$ in a complete transversal calculation.) Now $\theta_2 \cdot h$ gives

$$\begin{aligned} 6u_1^{r-2}u_2^2 + \tfrac{81}{2}u_1^{r-5}u_2^4 &\in M + F^{2r+3}\mathcal{O}_2, \\ 6u_1^{r-5}u_2^4 + \tfrac{81}{2}u_1^{r-8}u_2^6 &\in M + F^{2r+3}\mathcal{O}_2, \\ &\vdots \end{aligned}$$

and so on. It follows that $F^{2r+2}\mathcal{O}_2 \subset M + F^{2r+3}\mathcal{O}_2$ and the $(2r+2)$-transversal is empty for $a \neq 0$.

Consider the case $a \neq 0$ to begin with. Now $2r+3 = 2(r+1)+1$ and referring to the above comments concerning the calculation of a $2r+1$-transversal one similarly finds that a $(2r+3)$-transversal is $\{u_1^r u_2\}$ and the J^{2r+3}-orbits are

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^{r-1}u_2 + bu_1^r u_2.$$

(**Note:** the monomials in $\theta_1 \cdot h$ are of weight $2r+2$ in this case, $\theta_1 \cdot h$ does not contribute to the transversal, and we *do* require the term $u_1^r u_2$.) Finally we show $h = u_1^3 + \frac{27}{4}u_2^2 + au_1^{r-1}u_2 + bu_1^r u_2$ is $(2r+3)$-determined. We have

$$\begin{aligned} \theta_1 \cdot h &= 9(r-1)au_1^{r-2}u_2^2 - 2au_1^{r+1} + 9rbu_1^{r-1}u_2^2 - 2bu_1^{r+2}, \\ \theta_2 \cdot h &= 6u_1^3 + \tfrac{81}{2}u_2^2 + (2r+1)au_1^{r-1}u_2 + (2r+3)bu_1^r u_2. \end{aligned}$$

For determinacy we show that $F^{2r+4}\mathcal{O}_2 \subset M + F^{2r+7}\mathcal{O}_2$. Now $2r+6 = 2(r+2)+2$ and from above we therefore see that the monomials of weight $2r+6$ take the form

$$u_1^{r+3},\ u_1^r u_2^2,\ u_1^{r-3}u_2^4,\ u_1^{r-6}u_2^6,\ \ldots$$

but from $\theta_1 \cdot h$ and $\theta_2 \cdot h$ it follows that

$$\begin{aligned} 9(r-1)au_1^r u_2^2 - 2au_1^{r+3} &\in M + F^{2r+7}\mathcal{O}_2, \\ 6u_1^{r+3} + \tfrac{81}{2}u_1^r u_2^2 &\in M + F^{2r+7}\mathcal{O}_2, \end{aligned}$$

so that $u_1^{r+3}, u_1^r u_2^2 \in M + F^{2r+7}\mathcal{O}_2$ since $a \neq 0$. The rest of the monomials of weight $2r+6$ follow using $\theta_2 \cdot h$ as in previous calculations and $F^{2r+6}\mathcal{O}_2 \subset M + F^{2r+7}\mathcal{O}_2$. In a similar fashion we obtain the inclusions $F^{2r+5}\mathcal{O}_2 \subset M + F^{2r+6}\mathcal{O}_2$ and $F^{2r+4}\mathcal{O}_2 \subset M + F^{2r+5}\mathcal{O}_2$ and together these give the required determinacy result.

To finish off this part we note that a and b are indeed moduli. Working in the $(2r+3)$-jet-space and referring to $\theta_1 \cdot h$, $\theta_2 \cdot h$ calculated above, we see that

$\{u_1^{r-1}u_2, u_1^r u_2\}$ forms an independent set to $L\mathcal{R}(\mathcal{D}) \cdot h = \langle \theta_1 \cdot h, \theta_2 \cdot h \rangle$ and the result follows.

We now consider the case $a = 0$ where $h = u_1^3 + \frac{27}{4}u_2^2$ is a $(2r+1)$-jet. Referring to the calculations above we see that a $(2r+2)$-transversal is $\{u_1^{r+1}\}$ (it was previously empty) and the J^{2r+2}-orbits are

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^{r+1}.$$

Putting $h = u_1^3 + \frac{27}{4}u_2^2 + au_1^{r+1}$ we have

$$\theta_1 \cdot h = 9(r+1)au_1^r u_2, \qquad \theta_2 \cdot h = 6u_1^3 + \tfrac{81}{2}u_2^2 + 2(r+1)au_1^{r+1}.$$

Consider the case $a \neq 0$. Now $2r+3 = 2(r+1)+1$ and we refer to the above calculation of the monomials of weight $2r+1$. Since $a \neq 0$ the monomial $u_1^r u_2$ follows from $\theta_1 \cdot h$ and then the rest follow from $\theta_2 \cdot h$ as before. The $(2r+3)$-transversal is therefore empty. One finds that a $(2r+4)$-transversal is $\{u_1^{r+2}\}$, the rest of the monomials of weight $2r+4$ following from $\theta_2 \cdot h$. The J^{2r+4}-orbits are therefore

$$u_1^3 + \tfrac{27}{4}u_2^2 + au_1^{r+1} + bu_1^{r+2}.$$

In a similar fashion to the previous determinacy calculation one checks that this is $(2r+4)$-determined ($F^{2r+5}\mathcal{O}_2 \subset M + F^{2r+8}\mathcal{O}_2$) and that a and b are moduli.

Finally consider the case $a = 0$ above. Then $h = u_1^3 + \frac{27}{4}u_2^2$ is a $(2r+2)$-jet, that is a jet of weight a multiple of 2. This was our starting point and has already been dealt with.

(5) We now return to the 6-jet $h = u_2^2$. Now

$$\theta_1 \cdot h = -4u_1^2 u_2, \qquad \theta_2 \cdot h = 6u_2^2,$$

so the 7-transversal is empty while an 8-transversal is $\{u_1^4\}$ and the J^8-orbits are

$$u_2^2 + au_1^4.$$

Putting $h = u_2^2 + au_1^4$ we have

$$\theta_1 \cdot h = -4u_1^2 u_2 + 36au_1^3 u_2, \qquad \theta_2 \cdot h = 6u_2^2 + 8au_1^4.$$

The 9-transversal is empty while a 10-transversal is $\{u_1^5\}$ and the J^{10}-orbits are

$$u_2^2 + au_1^4 + bu_1^5 \qquad \text{10-determined}, \quad a \neq 0.$$

For the determinacy calculation we note that the 11-, 12- and 13-transversals are empty for $a \neq 0$. If $a = 0$ then one still finds that the 11-transversal is empty, while a 12-transversal is $\{u_1^6\}$. This is the start of the following series of singularities

$$u_2^2 + au_1^n + bu_1^{n+1} \qquad (2n+2)\text{-determined}, \quad n \geq 4, \quad a \neq 0,$$

and we shall discuss the calculations for the general case.

Consider $h = u_2^2$ as a $2r$-jet, for some $r \geq 3$. Recall from part (6) above that the monomials of weight $2r+1$ take the form

$$u_1^{r-1}u_2,\ u_1^{r-4}u_2^3,\ u_1^{r-7}u_2^5,\ u_1^{r-10}u_2^7,\ \ldots$$

finishing with one of the monomials u_2^t, $u_1u_2^t$ or $u_1^2u_2^t$ for appropriate t. Then from $\theta_1 \cdot h$ and $\theta_2 \cdot h$ (see above) we observe that the $(2r+1)$-transversal is empty. Similarly, it was shown that the monomials of weight $2r+2$ take the form

$$u_1^{r+1},\ u_1^{r-2}u_2^2,\ u_1^{r-5}u_2^4,\ u_1^{r-8}u_2^6,\ \ldots$$

so a $(2r+2)$-transversal is $\{u_1^{r+1}\}$ and the J^{2r+2}-orbits are

$$u_2^2 + au_1^{r+1}.$$

If $a = 0$ we are back to the initial consideration of u_2^2 as a jet of weight a multiple of 2. Suppose, therefore, that $a \neq 0$. We apply the same arguments at the $(2r+3)$- and $(2r+4)$-levels. The $(2r+3)$-transversal is empty while a $(2r+4)$-transversal is $\{u_1^{r+2}\}$ and the J^{2r+4}-orbits are

$$u_2^2 + au_1^{r+1} + bu_1^{r+2} \qquad (2r+4)\text{-determined}, \quad a \neq 0.$$

Putting $h = u_2^2 + au_1^{r+1} + bu_1^{r+2}$ gives

$$\begin{aligned} \theta_1 \cdot h &= -4u_1^2u_2 + 9(r+1)au_1^ru_2 + 9(r+2)bu_1^{r+1}u_2, \\ \theta_2 \cdot h &= 6u_2^2 + 2(r+1)au_1^{r+1} + 2(r+2)bu_1^{r+2}. \end{aligned}$$

The monomials of weight $2r+5$ take the form

$$u_1^{r+1}u_2,\ u_1^{r-2}u_2^3,\ u_1^{r-5}u_2^5,\ u_1^{r-8}u_2^7,\ \ldots$$

and it follows that $F^{2r+5}\mathcal{O}_2 \subset M + F^{2r+6}\mathcal{O}_2$. The monomials of weight $2r+6$ take the form

$$u_1^{r+3},\ u_1^ru_2^2,\ u_1^{r-3}u_2^4,\ u_1^{r-6}u_2^6,\ \ldots$$

and from $\theta_2 \cdot h$, $\theta_1 \cdot h$ we have

$$\begin{aligned} 6u_1^2u_2^2 + 2(r+1)au_1^{r+3} &\in M + F^{2r+7}\mathcal{O}_2, \\ -4u_1^2u_2^2 + 9(r+1)au_1^ru_2^2 &\in M + F^{2r+7}\mathcal{O}_2, \end{aligned}$$

respectively. But $\theta_2 \cdot h$ gives $u_1^r u_2^2 \in M + F^{2r+7}\mathcal{O}_2$ so $u_1^{r+3} \in M + F^{2r+7}\mathcal{O}_2$ provided $a \neq 0$. The other monomials of weight $2r+6$ then follow easily and $F^{2r+6}\mathcal{O}_2 \subset M + F^{2r+7}\mathcal{O}_2$. The inclusion $F^{2r+7}\mathcal{O}_2 \subset M + F^{2r+8}\mathcal{O}_2$ follows in the same way as that at the $2r+5$-level. Thus $F^{2r+5}\mathcal{O}_2 \subset M + F^{2r+8}\mathcal{O}_2$ and h is $(2r+4)$-determined for $a \neq 0$. Again, it is a routine matter to verify that a and b are moduli.

(1) We now come to the final 6-jet. A 7-transversal of 0 is $\{u_1^2u_2\}$ and we obtain the J^7-orbits

$u_1^2u_2$	(7),
0	(1).

(7) Consider $h = u_1^2u_2$; we have

$$\theta_1 \cdot h = 18u_1u_2^2 - 2u_1^4, \qquad \theta_2 \cdot h = 7u_1^2u_2.$$

So an 8-transversal is $\{u_1^4\}$ giving the J^8-orbits

$$u_1^2u_2 + au_1^4.$$

Putting $h = u_1^2u_2 + au_1^4$ we obtain

$$\theta_1 \cdot h = 18u_1u_2^2 - 2u_1^4 + 36au_1^3u_2, \qquad \theta_2 \cdot h = 7u_1^2u_2 + 8au_1^4,$$

and a 9-transversal is $\{u_2^3\}$. The J^9-orbits are

$$u_1^2u_2 + au_1^4 + bu_2^3.$$

Then $h = u_1^2u_2 + au_1^4 + bu_2^3$ gives

$$\begin{aligned} \theta_1 \cdot h &= 18u_1u_2^2 - 2u_1^4 + 36au_1^3u_2 - 6bu_1^2u_2^2, \\ \theta_2 \cdot h &= 7u_1^2u_2 + 8au_1^4 + 9bu_2^3. \end{aligned}$$

Now $\theta_2 \cdot h$ gives $u_1^2u_2^2 \in M + F^{11}\mathcal{O}_2$ and it then follows from $\theta_1 \cdot h$ that $u_1^5 \in M + F^{11}\mathcal{O}_2$, so the 10-transversal is empty. Similarly, $\theta_2 \cdot h$ gives $u_1^4u_2 \in M + F^{12}\mathcal{O}_2$ and then $u_1u_2^3 \in M + F^{12}\mathcal{O}_2$ from $\theta_1 \cdot h$, so the 11-transversal is empty. Likewise, from $\theta_2 \cdot h$ we obtain $u_1^3u_2^2$ and then u_1^6 from $\theta_1 \cdot h$. A 12-transversal is $\{u_2^4\}$ and the J^{12}-orbits are

$u_1^2u_2 + au_1^4 + bu_2^3 + cu_2^4$	12-determined,	$b \neq 0$.

Putting $h = u_1^2u_2 + au_1^4 + bu_2^3 + cu_2^4$, one verifies that $\{u_1^4, u_2^3, u_2^4\}$ forms an independent set to $L\mathcal{R}(\mathcal{D}) \cdot h$ in the 12-jet-space so that a, b and c are moduli. This example provides another trimodular singularity. The case $b = 0$ can, of course, be ruled out as well. We do note, however, that continuing the classification leads to the following trimodular series

$$u_1^2u_2 + au_1^4 + bu_2^n + cu_2^{n+1} \qquad (3n+3)\text{-determined}, \quad n \geq 3, \quad b \neq 0.$$

We prefer to omit the details (they were also verified by computer).

(1) Consider any higher jet j with 7-jet 0. Any open neighbourhood of j must contain a jet $\tilde{j}$ whose 7-jet is of the form $\epsilon u_1^2u_2$, for some $\epsilon \neq 0$. Then by the previous complete transversal calculation the 12-jet of $\tilde{j}$ must lie in the trimodular family $u_1^2u_2 + au_1^4 + bu_2^3 + cu_2^4$. That is, j has modality ≥ 3; all such jets being excluded, this concludes the classification.

5.4 Classification of Function-Germs on the A_3 (Swallowtail) Discriminant

In this section we classify all function-germs on the swallowtail (A_3) discriminant variety under $\mathcal{R}(\mathcal{D})$-equivalence and up to modality 2. This extends the results of [A2] where the same weighted filtration was employed. The results are summarised in the following theorem.

Theorem 5.18 *Every function-germ $h : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$ on the swallowtail discriminant variety $\mathcal{D}$ of $\mathcal{R}(\mathcal{D})$-modality ≤ 2 is $\mathcal{R}(\mathcal{D})$-equivalent to one of the following finitely determined germs. The first germ of modality ≥ 3 to occur during the classification is given in the second section of the table. The codimension refers to the dimension of $\mathcal{O}_n/J_X(h)$ as a $\mathbf{C}$-vector space; u_1 is the only stable singularity.*

Singularity	Determinacy Degree	Codim
u_1	2	1
$u_2 + au_1^n$	$2n, \quad n \geq 2, \quad a \neq 0$	$n+1$
$u_3 + au_1^n + bu_1^{n+1}$	$2n+2, \quad n \geq 2, \quad a \neq 0\,(\dagger)$	$n+3$
$u_3 - \frac{1}{4}u_1^2 + au_1^n + bu_1^{n+1}$	$2n+2, \quad n \geq 3, \quad a \neq 0$	$n+3$
$u_3 + \frac{1}{12}u_1^2 + au_1^nu_2 + bu_1^{n+1}u_2$	$2n+5, \quad n \geq 1, \quad a \neq 0$	$2n+4$
$u_3 + \frac{1}{12}u_1^2 + au_1^n + bu_1^{n+1}$	$2n+2, \quad n \geq 3, \quad a \neq 0$	$2n+1$
$u_1^2 + bu_2^2 + cu_2u_3 + du_3^2 + eu_3^3$	$12, \quad b, c, 4bd - c^2 \neq 0$	8

(†) *For the case* $n = 2$ *the extra conditions* $12a - 1 \neq 0$ *and* $4a + 1 \neq 0$ *are required.*

The proof uses the same techniques as for the cusp case. We omit a lot of the details — most of the classification was performed by computer. The calculations needed for series are more involved than in the cusp case, mainly due to the appearance of an extra coordinate, u_3. As an example we will discuss the calculations which lead to the series:

$$\begin{array}{l} u_3 + au_1^n + bu_1^{n+1}, \\ u_3 + \frac{1}{12}u_1^2 + au_1^n u_2 + bu_1^{n+1}u_2, \\ u_3 + \frac{1}{12}u_1^2 + au_1^n + bu_1^{n+1}. \end{array}$$

These indicate our approach to the problem.

The calculations from Section 5.2 show that the module of vector fields tangent to the swallowtail discriminant is generated by

$$\begin{aligned} \theta_1 &= (16u_3 - 4u_1^2)\partial/\partial u_1 - 8u_1u_2\partial/\partial u_2 - 3u_2^2\partial/\partial u_3, \\ \theta_2 &= 6u_2\partial/\partial u_1 + (8u_3 - 2u_1^2)\partial/\partial u_2 - u_1u_2\partial/\partial u_3, \\ \theta_3 &= 2u_1\partial/\partial u_1 + 3u_2\partial/\partial u_2 + 4u_3\partial/\partial u_3, \end{aligned}$$

θ_3 being the Euler vector field. Let M denote the ideal

$$\langle \theta_1 \cdot h, \theta_2 \cdot h \rangle + F^1\mathcal{O}_3.\langle \theta_3 \cdot h \rangle$$

where h is some germ in $\mathcal{O}_3$; this is used in the complete transversal and determinacy calculations.

Consider the 3-jet 0. A 4-transversal is $\{u_1^2, u_3\}$ and, applying the 'scaling' coordinate change

$$(u_1, u_2, u_3) \mapsto (t^2u_1, t^3u_2, t^4u_3), \quad t \in \mathbf{C},$$

discussed in Proposition 5.16, we obtain the J^4-orbits

$$\begin{array}{l} u_3 + au_1^2, \\ u_1^2, \\ 0. \end{array}$$

We shall consider $u_3 + au_1^2$ below. The 5-transversal is empty for $12a - 1 \neq 0$; otherwise a 5-transversal is $\{u_1u_2\}$. The J^5-orbits are therefore

$$\begin{array}{ll} u_3 + au_1^2 & 12a - 1 \neq 0, \\ u_3 + \frac{1}{12}u_1^2 + au_1u_2. & \end{array}$$

Firstly consider $u_3 + au_1^2$. A 6-transversal is $\{u_1^3\}$ giving the J^6-orbits

$$u_3 + au_1^2 + bu_1^3 \qquad \text{6-determined,} \quad a \neq 0, \quad 12a - 1 \neq 0, \quad 4a + 1 \neq 0.$$

The 7-transversal is empty. The 8-transversal is empty provided $4a + 1 \neq 0$ and $a \neq 0$ (as are the 9- and 10-transversals, leading to the above determinacy result). For $4a + 1 = 0$ or $a = 0$ an 8-transversal is $\{u_1^4\}$. The J^8-orbits are therefore

$$\begin{array}{ll} u_3 + bu_1^3 + cu_1^4 & \text{8-determined,} \quad b \neq 0, \\ u_3 - \frac{1}{4}u_1^2 + bu_1^3 + cu_1^4 & \text{8-determined,} \quad b \neq 0. \end{array}$$

Continuing (where $b = 0$) we obtain the series stated in the theorem; we describe the calculations only for the first case. Consider $h = u_3$ as a $(2r+1)$-jet for $r \geq 2$. Then

$$\theta_1 \cdot h = -3u_2^2, \qquad \theta_2 \cdot h = -u_1 u_2, \qquad \theta_3 \cdot h = 4u_3,$$

so M contains all monomials except those of the form u_1^s for some s. It follows that a $(2r + 2)$-transversal is $\{u_1^{r+1}\}$. Then the $(2r + 3)$-transversal is empty giving the J^{2r+3}-orbits of the form $u_3 + au_1^{r+1}$. If $a = 0$ we are back to the original situation so suppose $a \neq 0$. Similarly, a $(2r + 4)$-transversal is $\{u_1^{r+2}\}$ and the J^{2r+4}-orbits take the form

$$u_3 + au_1^{r+1} + bu_1^{r+2} \qquad (2r + 4)\text{-determined,} \quad a \neq 0.$$

Now, putting $h = u_3 + au_1^{r+1} + bu_1^{r+2}$ gives

$$\begin{aligned} \theta_1 \cdot h &= -3u_2^2 + (r+1)a(16u_3 - 4u_1^2)u_1^r + (r+2)b(16u_3 - 4u_1^2)u_1^{r+1}, \\ \theta_2 \cdot h &= -u_1u_2 + 6(r+1)au_1^r u_2 + 6(r+2)bu_1^{r+1}u_2, \\ \theta_3 \cdot h &= 4u_3 + 2(r+1)au_1^{r+1} + 2(r+2)bu_1^{r+2}. \end{aligned}$$

Working modulo $F^{2r+6}\mathcal{O}_3$ we obtain all of the monomials of weight $2r + 5$ as above (those of the form u_1^s are of even weight) so $F^{2r+5}\mathcal{O}_3 \subset M + F^{2r+6}\mathcal{O}_3$. Similarly all the of the monomials of weight $2r + 6$ follow modulo $F^{2r+7}\mathcal{O}_3$ once we have u_1^{r+3}. From $\theta_1 \cdot h$ we have

$$-3u_1u_2^2 + 16(r+1)au_1^{r+1}u_3 - 4(r+1)au_1^{r+3} \in M + F^{2r+7}\mathcal{O}_3,$$

but $\theta_2 \cdot h$ gives

$$-u_1u_2^2 + 6(r+1)au_1^r u_2^2 \in M + F^{2r+7}\mathcal{O}_3,$$

and $\theta_1 \cdot h$, $\theta_3 \cdot h$ give $u_1^r u_2^2, u_1^{r+1}u_3 \in M + F^{2r+7}\mathcal{O}_3$, respectively. Combining these it follows that $u_1^{r+3} \in M + F^{2r+7}\mathcal{O}_3$ for $a \neq 0$ so $F^{2r+6}\mathcal{O}_3 \subset M + F^{2r+7}\mathcal{O}_3$.

Continuing similarly, one eventually obtains $F^{2r+5}\mathcal{O}_3 \subset M + F^{2r+9}\mathcal{O}_3$ so that h is $(2r+4)$-determined. Finally, working in the $(2r+4)$-jet-space, we check that $\{u_1^{r+1}, u_1^{r+2}\}$ forms an independent set to $L\mathcal{R}(\mathcal{D}) \cdot h$ so that a and b are indeed moduli.

As our second example we return to the 5-jet $u_3 + \frac{1}{12}u_1^2 + au_1u_2$; in particular, we consider the two series stated above which stem from this. Consider $h = u_3 + \frac{1}{12}u_1^2$ as a $(2r+2)$-jet for $r \geq 1$. Then

$$\begin{aligned} \theta_1 \cdot h &= \tfrac{1}{6}(16u_3 - 4u_1^2)u_1 - 3u_2^2, \\ \theta_2 \cdot h &= 0, \\ \theta_3 \cdot h &= \tfrac{1}{3}u_1^2 + 4u_3, \end{aligned}$$

and combining $\theta_1 \cdot h$ and $\theta_3 \cdot h$ we obtain $\frac{8}{9}u_1^3 + 3u_2^2 \in M$. In any complete transversal calculation we can therefore replace any occurrence of u_2^2 by u_1^3, any occurrence of u_3 by u_1^2, and need only consider monomials of the form u_1^s and $u_1^s u_2$. It follows that a $(2r+3)$-transversal is $\{u_1^r u_2\}$ and the J^{2r+3}-orbits are $u_3 + \frac{1}{12}u_1^2 + au_1^r u_2$. Putting $h = u_3 + \frac{1}{12}u_1^2 + au_1^r u_2$, $\theta_1 \cdot h$, $\theta_2 \cdot h$ and $\theta_3 \cdot h$ are as above but with the addition of the following terms of higher weight

$$\begin{aligned} &ra(16u_3 - 4u_1^2)u_1^{r-1}u_2 - 8au_1^{r+1}u_2, \\ &6rau_1^{r-1}u_2^2 + a(8u_3 - 2u_1^2)u_1^r, \\ &(2r+3)au_1^r u_2, \end{aligned}$$

respectively. Using the above argument we need only consider the monomial u_1^{r+2} (modulo $F^{2r+5}\mathcal{O}_3$) when calculating a $(2r+4)$-transversal. From earlier calculations we see that

$$\tfrac{8}{9}u_1^{r+2} + 3u_1^{r-1}u_2^2 \in M + F^{2r+5}\mathcal{O}_3, \qquad \tfrac{1}{3}u_1^{r+2} + 4u_1^r u_3 \in M + F^{2r+5}\mathcal{O}_3$$

and it is then a routine matter to verify that $u_1^{r+2} \in M + F^{2r+5}\mathcal{O}_3$ using $\theta_2 \cdot h$, provided $a \neq 0$. We shall assume $a \neq 0$, then the $(2r+4)$-transversal is empty. Now consider h as a $(2r+4)$-jet. We need only consider the monomial $u_1^{r+1}u_2$ when calculating a $(2r+5)$-transversal. After further consideration we see that the monomial $u_1^{r+1}u_2$ cannot be obtained and a $(2r+5)$-transversal is $\{u_1^{r+1}u_2\}$ giving the J^{2r+5}-orbits

$$u_3 + \tfrac{1}{12}u_1^2 + au_1^r u_2 + bu_1^{r+1}u_2 \qquad (2r+5)\text{-determined}, \quad a \neq 0.$$

Putting $h = u_3 + \frac{1}{12}u_1^2 + au_1^r u_2 + bu_1^{r+1}u_2$ we see that the $\theta_i \cdot h$ are as above but with the addition of terms of higher weight which will be of no concern to us. As in the calculation for the $(2r+4)$-transversal, we see that the $(2r+6)$-transversal is empty so $F^{2r+6}\mathcal{O}_3 \subset M + F^{2r+7}\mathcal{O}_3$. When calculating a $(2r+7)$-transversal we need only consider the monomial $u_1^{r+2}u_2$. From $\theta_2 \cdot h$ we have

$$6rau_1^{r-1}u_2^3 + 8au_1^r u_2 u_3 - 2au_1^{r+2}u_2 \in M + F^{2r+8}\mathcal{O}_3$$

and from earlier calculations

$$\tfrac{8}{9}u_1^{r+2}u_2 + 3u_1^{r-1}u_2^3 \in M + F^{2r+8}\mathcal{O}_3, \qquad \tfrac{1}{3}u_1^{r+2}u_2 + 4u_1^r u_2 u_3 \in M + F^{2r+8}\mathcal{O}_3$$

and we find that $u_1^{r+2}u_2 \in M + F^{2r+8}\mathcal{O}_3$, provided $a \neq 0$. Hence, the $(2r+7)$-transversal is empty and $F^{2r+7}\mathcal{O}_3 \subset M + F^{2r+8}\mathcal{O}_3$. The calculation continues in this way and we obtain $F^{2r+6}\mathcal{O}_3 \subset M + F^{2r+10}\mathcal{O}_3$ proving that h is $(2r+5)$-determined.

It remains to consider the case $a = 0$. From the previous calculation we see that a $(2r+4)$-transversal for $u_3 + \frac{1}{12}u_1^2$ is $\{u_1^{r+2}\}$. Continuing we obtain the singularity

$$u_3 + \tfrac{1}{12}u_1^2 + au_1^{r+2} + bu_1^{r+3} \qquad (2r+6)\text{-determined}, \quad a \neq 0.$$

The calculation follows similar lines to those just described and we omit the details. If $a = 0$ we are back to the original consideration of $u_3 + \frac{1}{12}u_1^2$ as a jet of even weight. Finally, we remark that further calculations verify a and b are moduli in both of these series.

5.5 Classification of Function-Germs on the A_k, D_k and E_k Discriminants

Having dealt with the A_2 and A_3 cases above, we now consider extending the classification to function-germs on discriminants of the other simple singularities. We restrict to the first few cases of each series, namely A_4, D_4, D_5, D_6, E_6, E_7 and E_8. In each case the first singularity to occur after the stable singularity is of high modality. Continuing is viable in each case, though difficult — there being many degeneracy conditions on the moduli to consider. With no motivations such as specific applications to geometry we pursue the classifications no further. Our results demonstrate the scope of the computer classification package (all the calculations were performed by computer, including the far from trivial calculation of the vectors fields tangent to the given discriminant). These results also indicate the complexities involved in pursuing such classifications should one need to extend these lists in the future. One disturbing feature which comes out of this is that there is only one simple singularity in each case — the stable one!

The germs $g_1, \ldots, g_p$ which form a free $\mathcal{O}_p$-basis of $\mathcal{O}_{n,p}/J(F)$ are chosen as in Section 5.2.1. For the D_k cases we must reorder these g_i so that their weight is decreasing. This specific ordering is as follows.

D₄.

$$g_1 = y^2,\ g_2 = x,\ g_3 = y,\ g_4 = 1.$$

D₅.

$$g_1 = y^3,\ g_2 = y^2,\ g_3 = x,\ g_4 = y,\ g_5 = 1.$$

D₆.

$$g_1 = y^4,\ g_2 = y^3,\ g_3 = x,\ g_4 = y^2,\ g_5 = y,\ g_6 = 1.$$

Using the techniques of Section 5.2.2 we assign weights to the coordinates on $\mathbf{C}^p$, $(u_1, \ldots, u_p)$, as follows.

Discriminant	Weights										Total Weight
	x	y	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	
A_4	1	-	2	3	4	5	-	-	-	-	5
D_4	1	1	1	2	2	3	-	-	-	-	3
D_5	3	2	2	4	5	6	8	-	-	-	8
D_6	4	2	2	4	6	6	8	10	-	-	10
E_6	4	3	2	5	6	8	9	12	-	-	12
E_7	3	2	1	3	4	5	6	7	9	-	9
E_8	5	3	1	4	6	7	9	10	12	15	15

Theorem 5.19 *The following table shows the beginning of the classification of function germs $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ on the discriminant varieties $\mathcal{D}$ of the simple singularities A_p, D_p, and E_p under $\mathcal{R}(\mathcal{D})$-equivalence. The A_2 and A_3 cases were dealt with in earlier sections. The codimension refers to the dimension of $\mathcal{O}_n/J_X(h)$ as a $\mathbf{C}$-vector space; in each case u_1 is the only stable singularity. The a_i denote moduli and the determinacy degree for the non-simple cases holds for generic values of the moduli (more precisely, on the complement of an algebraic variety, i.e., the complement of a null set). The classification in the E_k cases was extremely complicated and was terminated just past the level shown below. In particular, the germs already have a high modality but are not determined at the given jet-level (because, for example, further moduli occur at higher levels).*

Discriminant	Singularity	Det	Codim
A_4	u_1	2	1
	$u_2 + a_1 u_1^2$	4	3
	$u_3 + a_1 u_1^2 + a_2 u_1 u_2 + a_3 u_1^3 + a_4 u_1^4$	8	7
D_4	u_1	1	1
	$u_2 + a_1 u_3 + a_2 u_1^2 + a_3 u_1^3 + a_4 u_1^4$	4	6
D_5	u_1	2	1
	$u_2 + a_1 u_1^2 + a_2 u_3 + a_3 u_1^3 + a_4 u_1^4 + a_5 u_1^5$	10	7
D_6	u_1	2	1
	$u_2 + a_1 u_1^2 + a_2 u_3 + a_3 u_1^3 + a_4 u_1^4 + a_5 u_1^5 + a_6 u_1^6$	12	8
E_6	u_1	2	1
	$u_1^2 + a_1 u_2 + a_2 u_3 + a_3 u_4 + a_4 u_5 + a_5 u_6 + a_6 u_3^2 +$ $a_7 u_3 u_5$	-	-
E_7	u_1	1	1
	$u_1^2 + a_1 u_2 + a_2 u_3 + a_3 u_4 + a_4 u_5 + a_5 u_6 + a_6 u_3^2 +$ $a_7 u_7 + a_8 u_3 u_5$	-	-
E_8	u_1	1	1
	$u_1^2 + a_1 u_2 + a_2 u_3 + a_3 u_4 + a_4 u_5 + a_5 u_6 + a_6 u_7 +$ $a_7 u_3^2 + a_8 u_8 + a_9 u_3 u_5$	-	-

Conjectures. From the above results we make the following natural conjectures. The first one, at least, seems pretty safe!

1. For functions germs $h : (\mathbf{C}^p, 0) \longrightarrow (\mathbf{C}, 0)$ on the discriminant varieties $\mathcal{D}$ of the simple singularities under $\mathcal{R}(\mathcal{D})$-equivalence there is only one simple germ, the stable one u_1. (The exception being the first example, A_2, where there are two simples, namely u_1 and u_2.)

2. For the D_k discriminant the singularity which follows the stable singularity u_1, that is, the singularity with non-zero u_2 coefficient, takes the form

$$u_2 + a_1 u_3 + a_2 u_1^2 + a_3 u_1^3 + \cdots + a_k u_1^k \qquad \begin{array}{l} 2k\text{-determined,} \\ \text{codim} = k+2. \end{array}$$

Chapter 6

Transversal: A Maple Package for Singularity Theory

The package consists of a collection of Maple procedures, the main purpose of these being to perform the symbolic algebra needed for classification problems in singularity theory. We will just describe the applications here, with only a brief reference to the technical machinery needed. (We shall use $\mathbf{F}$ to denote either of the fields $\mathbf{R}$ or $\mathbf{C}$.)

6.1 Description

The basic scenario is of some group $\mathcal{G}$ (usually one of the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$) acting on the space of germs $m_n.\mathcal{E}(n,p)$. We need to (i) list orbits of finitely determined germs and, (ii) calculate versal unfoldings of the normal forms of such germs. The crucial point is that we can reduce these problems to ones in linear algebra by working in suitable jet-spaces; the main objective being to calculate the tangent space to the orbit of the germ in the jet-space.

(i) Classification is done inductively at the jet-level via the method of complete transversals. That determinacy questions can be reduced to ones in terms of jet-spaces was shown in the work of Mather, [MathIII], but to form an efficient method we must also employ the 'unipotency' techniques of [BduPW].

(ii) *Exactly* the same (symbolic) methods can be used to calculate the versal unfolding of a finitely determined germ. We essentially just use the fundamental theorem for versal unfoldings, c.f., [Mart2] or [Wal].

We summarise the basic method below.

(a) For a given $f \in m_n.\mathcal{E}(n,p)$, jet-space degree k and group $\mathcal{G}$, calculate the tangent space to the orbit of f in the jet-space of degree k, that is

$$L(J^k\mathcal{G}) \cdot (j^k f) \qquad \text{in} \qquad J^k(n,p).$$

Specifically, calculate a spanning set for this tangent space.

(b) Reduce this spanning set to echelon form (after ordering the monomial vectors in $J^k(n,p)$) using Gaussian elimination. This gives a basis for the tangent space.

(c) Calculate a basis for the complementary space to the tangent space.

(d) Output the required results.

The function `jetcalc` does the main bulk of the work, namely parts (a)–(c), and stores the results for global access by other routines. There are a number of functions associated with (d), these act on the results of (b) or (c) as appropriate. For instance, a complete transversal can be obtained from (c), as can a versal unfolding and the appropriate codimension of f. Part (b) allows the hypotheses of the Mather Lemma ([MathIV, Lemma 3.1] or [BduPW, Lemma 1.1]) to be checked, as well as determinacy criteria. A method for detecting the presence of moduli also uses the results from (b).

All the user functions of the package will be described below, together with the global variables used to specify the action of the group $\mathcal{G}$. The following sections provide a document akin to a refence manual and, as a result, are somewhat technical in nature. Section 6.6 describes calculations for specific examples and provides and a gentler introduction.

6.2 Getting Started

The `Transversal` package is written in Maple V and can be run from Maple or X-Maple, whichever you prefer. (It will also run on the latest release, Maple

V.2., although modifications are required if it is to be installed as a library package since the Maple variable `_liblist` is now obsolete.) The Maple procedures which make up the package are (currently) stored on the Liverpool University Computer Laboratory's Sun system and on the Liverpool Singularities Group's Iris and Indigo workstations ('Whitney' and 'Thom'!). The library paths must be included in your Maple initialisation file. For example, if HOME denotes the full (absolute) path name of your home directory and the `Transversal` library has been installed in the subdirectory `maple/lib` then the following lines (or something equivalent) must be added to your `.mapleinit` file which is stored in your home directory. (Note the use of different types of quotes, this is important. HOME should be replaced by the absolute pathname of your home directory.)

```
neilslib := `HOME/maple/lib`:
_liblist := ['libname',neilslib]:
```

Here the first line should give the full directory path-name of the `Transversal` library, depending on where it is installed on your machine. The package can then be loaded by going into Maple in the usual way and entering the command:

```
with(transversal):
```

Alternatively, this command may be placed in your `.mapleinit` file so that `Transversal` is automatically loaded every time you use Maple.

If `Transversal` is not installed as a library package in this way the individual programs can be read into Maple using the `read` function instead.

6.3 Global Variables

The following global variables need to be set by the user. Their main purpose is to specify the action of the group $\mathcal{G}$. The procedures do not need the group $\mathcal{G}$ itself, rather a way of calculating the tangent space $L(J^k\mathcal{G}) \cdot (j^k f)$ for a given f and k. The global variables allow us to specify a large class of groups, as will be described below.

6.3.1 Brief Description

The global variable `equiv` may take the values R, L, A, C or K to specify the 'type' of equivalence (these correspond to $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$). Its value just tells `jetcalc` how to calculate $L(J^k\mathcal{G}) \cdot (j^k f)$; the actual group obtained depends on the other global variables. For example, if `equiv` = R then it is possible to obtain the groups $\mathcal{R}_e$, $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_2, \ldots$, as well as non-standard source coordinate change groups, as described next.

In many applications we have considered, the source coordinate changes are restricted (to preserve some variety in the source, say) so that the Lie algebra is not the standard one (here the module of vector fields tangent to the variety). We therefore allow the source coordinate changes to be user specified. The global variable `liealg` is a pointer to a Maple procedure which defines the Lie algebra of our required source group. Specifically, it must give a generating set of vector fields of the form

$$g_1\frac{\partial}{\partial x_1} + \cdots + g_n\frac{\partial}{\partial x_n} \qquad \text{with} \qquad g_i \in \mathcal{E}_n,$$

for the Lie algebra. The exact Maple syntax for these vector fields is discussed below, together with the other functions of the `liealg` procedures.

Several `liealg` procedures already exist. The standard one which specifies the 'pseudo right group', $\mathcal{R}_e$, is called `stdjacobian` and defines a generating set for the $\mathcal{E}_n$-module $tf(\theta_n)$; in coordinate form this is just the set of columns of the Jacobian matrix $(\partial f_i/\partial x_j)$. Others include `cusp`, `swallowtail` and `d4discrim` for the module of vector fields tangent to the respective discriminant varieties.

In many cases we can modify the group $\mathcal{G}$ without having to change the generating set specified by `liealg`. For instance, `liealg` := `stdjacobian` and `equiv` := R gives the $\mathcal{R}_e$ tangent space — not the usual $\mathcal{R}$ tangent space. This is needed for unfolding calculations but not for classification problems. We therefore modify the tangent space $L\mathcal{G}\cdot f$ by 'multiplying' its source and target components by powers of the maximal ideals m_n and m_p. The global variables `source_power` and `target_power` are used for this; a precise definition of the tangent space in terms of these variables, and how to obtain the standard groups, being given in the section below. For example, using `liealg`, `equiv`, `source_power` and `target_power` we can easily obtain the (tangent spaces for the) groups

$$\mathcal{R}_e,\ \mathcal{R},\ \mathcal{R}_1,\ \ldots\ ,\ \mathcal{A}_e,\ \mathcal{A},\ \mathcal{A}_1,\ \ldots\ ,\ \mathcal{K}_e,\ \mathcal{K},\ \mathcal{K}_1,\ \ldots\ ,\text{etc.}$$

One final point concerning $\mathcal{G}$ is that in standard classifications we must use the unipotency methods of [BduPW]; specifically we give a list of the 'nilpotent vectors' which must be added to the Lie algebra $L\mathcal{G}$. (Note: although *referred* to as the Lie algebra of $\mathcal{G}$, $L\mathcal{G}$ in general, may not be a Lie algebra. There are no known counter-examples at present; in all the standard cases $L\mathcal{G}$ *is* a Lie algebra.) The appropriate globals are `R_nilp` and `L_nilp`, together with the 'pseudo' Boolean variable `nilp` which tells `jetcalc` to include the nilpotent terms when set to `true`, and to ignore them when `false`. This allows us to perform calculations which do not require the nilpotent terms without having to reset them. It may also be set to a third value of `true_order` which will be discussed next.

To use unipotent groups in the complete transversal classification method we must add the nilpotent vectors to the Lie algebra as just described, but then must also tell `jetcalc` how to order the homogeneous jets of degree k; the required order being that induced by the 'nilpotent filtration', $M_{r,s}(\mathcal{G})$. We must use this order so that the basis for the complementary space calculated by `jetcalc` can be directly refined to give a complete transversal for the $M_{r,s}(\mathcal{G})$ filtration. In the particular case of $\mathcal{A}$ classifications, there are four *canonical* choices for the 'nilpotent vectors' which are added to the Lie algebra. It has been shown (see Section 6.7.5) that the required order can then be achieved by assigning weights to the source and target variables. For this we use the global variables `nilp_source_wt` and `nilp_target_wt` — each a list of weights dependent on the values of `R_nilp` and `L_nilp`.[1] Note that we have not developed a method for giving the order induced by the $M_{r,s}(\mathcal{G})$ filtration for general $\mathcal{G}$. This is a problem for future consideration — however, the use of weights gives the four standard '$\mathcal{A}$ nilpotent Lie algebras' and this has proved to be sufficient in this work (of course, this applies to $\mathcal{R}$ and $\mathcal{L}$ classifications too). To tell `jetcalc` to include the nilpotent terms and to order the homogeneous jets of degree k using the weights just described we set `nilp` equal to `true_order`.

6.3.2 The Individual Global Variables

We now describe all the user-defined global variables individually; their function, Maple data type, and so on.

[1]For the important case of $\mathcal{A}$ classifications, there is a procedure which defines all the global variables; in particular, all the nilpotent variables are assigned appropriately. See `setup_classn` in the next section.

liealg

This is a name (pointer) to another Maple procedure called from within `jetcalc`. Its main purpose is to define a generating set for the source Lie algebra together with the names to be used for the source coordinates. The latter is important so that these coordinates can be distinguished from any 'unfolding' or 'moduli' type parameters the user may wish to introduce into the germ f. In certain cases we always use a fixed set of nilpotent global variables and it is then convenient, from a users point of view, to assign these in the `liealg` procedure also. See the examples which define `liealg` procedures for the discriminant varieties (Appendix B). We remark on the following.

- The coordinate names defined in a `liealg` procedure must *always* be used when calling `jetcalc` with that particular setting of `liealg`. For instance, `stdjacobian` uses $x_1, \ldots, x_n$ as source coordinates and their actual Maple names are defined as $x1, \ldots, xn$, that is juxtaposition of x with a number (technically speaking we are using the Maple concept of concatenation of names). When called, `jetcalc` prints out the coordinate names it is using to clarify this.
- In the case `liealg` = `stdjacobian`, the source dimension n must also be specified. This is done by the global variable **source_dim**, a positive integer.

The rest of this section describes the technical details behind the `liealg` procedures and need only be read should the user want to write their own procedures.

A `liealg` procedure is defined with four formal parameters thus

`liealg_example` := proc(f,p,`tgtspace`);

where f, a list, stores the given jet passed to `jetcalc` and p, a positive integer, the target dimension deduced from the number of components of f. These are pre-determined in `jetcalc` before it calls the `liealg` procedure. The parameter `tgtspace` is of type 'table' and is assigned within the procedure.

The source coordinates must be specified by assigning a Maple name to each entry in the global list `coords`. This is another function which must be carried out by the `liealg` procedure. It is a good idea at this stage to check the required names are unassigned as Maple expressions and return an error otherwise (see the Maple code for syntax).

Next a generating set for the source Lie algebra must be specified using the table `tgtspace`. Each entry of `tgtspace` is *itself* of type 'table' with p components, and corresponds to a tangent vector v from the generating set. They are given in coordinate form, using the coordinates just defined, by specifying how v acts on the germ f. The precise syntax is as follows. Suppose the i-th vector in the generating set is of the form

$$v = g_1 \frac{\partial}{\partial x_1} + \cdots + g_n \frac{\partial}{\partial x_n} \qquad \text{with} \qquad g_j \in \mathcal{E}_n,$$

then the i-th entry of `tgtspace` specifies the p components of $v(f)$ and is defined in Maple by

$$\texttt{tgtspace}[i][1] := g_1{*}\text{diff}(f[1],\texttt{coords}[1]) + \cdots + g_n{*}\text{diff}(f[1],\texttt{coords}[n]);$$
$$\vdots$$
$$\texttt{tgtspace}[i][p] := g_1{*}\text{diff}(f[p],\texttt{coords}[1]) + \cdots + g_n{*}\text{diff}(f[p],\texttt{coords}[n]);$$

where f is given in Maple by a list of p entries, $f := [f_1, \ldots, f_p]$.

A warning is needed on the special case when the target dimension p is one. Since each entry `tgtspace`$[i]$ must itself be of type 'table', we must use expressions of the form

`tgtspace`$[i][1] := \ldots;$ and not
`tgtspace`$[i] := \ldots;$

Though the convenient shorthand

`tgtspace`$[i] := [\ldots];$

where the entry on the right hand side is of type 'list', works as well.

Finally, the global variables which define the nilpotent terms may be assigned, if required. We refer to the sections below for the description of the various nilpotent variables and the required syntax.

equiv

This specifies which 'type' of equivalence to use by taking a value from one of R, L, A, C or K. (Note that *capitals* must be used, and that if any of these letters are assigned Maple expressions then they must be evaluated to actual Maple names using single right-quotes. That is, using `equiv` := 'R' instead of `equiv` := R, in the $\mathcal{R}$ case, say.) A spanning set for the vector space $L(J^k\mathcal{G})\cdot(j^kf)$ is calculated by taking a generating set for the source and target components of $L\mathcal{G}\cdot f$, and then all non-zero k-jets which result from this using the appropriate module structure(s). This is done automatically by `jetcalc`, and `equiv` just tells `jetcalc` which module structure(s) to use. Any equivalence which uses source coordinate changes will use the `liealg` procedure to define the corresponding tangent space. However, any target coordinate changes are just the standard ones of $\mathcal{L}$ or $\mathcal{C}$ and can only be altered through the use of `target_power`.

source_power/target_power

These are non-negative integers which specify the power by which the appropriate maximal ideal, m_n or m_p, is to be raised. Note that the source component of the tangent space is always multiplied by a power of the ideal m_n, whereas the target component of the tangent space is multiplied by a power of the ideal m_p in the $\mathcal{L}$ and $\mathcal{A}$ cases, and by a power of m_n in the $\mathcal{C}$ and $\mathcal{K}$ cases. For example, consider the $\mathcal{A}$ case. Setting `equiv` := A; `liealg` := `stdjacobian`; gives the tangent space $L\mathcal{G}\cdot f$ (in the notation of Mather) as

$$tf(m_n^{\text{source\_power}}.\theta_n) + wf(m_p^{\text{target\_power}}.\theta_p).$$

The standard examples are therefore given by the following settings:

equivalence	`source_power`	`target_power`
$\mathcal{A}_e$	0	0
$\mathcal{A}$	1	1
$\mathcal{A}_1$	2	2
$\mathcal{A}_2$	3	3

Now consider the $\mathcal{K}$ case. Setting `equiv` := K; `liealg` := `stdjacobian`; gives the tangent space $L\mathcal{G}\cdot f$ as

$$tf(m_n^{\text{source\_power}}.\theta_n) + m_n^{\text{target\_power}}.f^*(m_p).\theta_f.$$

We can therefore obtain the following:

equivalence	source_power	target_power
$\mathcal{K}_e$	0	0
$\mathcal{K}$	1	0
$\mathcal{K}_1$	2	1
$\mathcal{K}_2$	3	2

compltrans

This is a Boolean variable. The function `pcomp`, described below, prints out the basis for the complementary space calculated by `jetcalc`. A complete transversal can be obtained from this basis just by extracting the terms of degree k. The fact that this works depends on the way `jetcalc` orders the monomial jets. The user need not concern themselves with the method — `jetcalc` does everything automatically. (Though the theory is discussed in Sections 6.7.3 — 6.7.5.) Setting `compltrans` := `true` causes `pcomp` to extract the complete transversal from the basis and then print it out. For the full complementary basis, say in unfolding theory, we set `compltrans` := `false` before calling `pcomp`.

nilp

This is a 'pseudo' Boolean variable. When set to `true` this tells `jetcalc` to include the 'nilpotent vectors' given by `R_nilp` and `L_nilp`; this would be the case for determinacy calculations. However, when set to `false` this tells `jetcalc` to ignore these variables; this would be the case for standard unfolding calculations, say, where the tangent space naturally contains all nilpotent terms anyway, and it would be wasteful to consider such terms twice. For complete transversal calculations using nilpotent terms the homogeneous jets of degree k must be ordered using the $M_{r,s}(\mathcal{G})$ filtration. We only employ methods for giving the four standard $\mathcal{A}$ orderings. To achieve this we set `nilp` equal to `true_order`, the 'nilpotent weights' `nilp_source_wt` and `nilp_target_wt` then being used to define the ordering. It is up to the user to make sure these weights are the correct weights to use with the 'nilpotent vectors' given by `R_nilp` and `L_nilp`; however, the function `setup_classn` discussed in Section 6.5 may be helpful.

Note that if `nilp` is set to `true_order` then `jetcalc` will include the 'nilpotent vectors' given by `R_nilp` and `L_nilp`, and use the 'nilpotent ordering' given by `nilp_source_wt` and `nilp_target_wt`. However, if `nilp` is set to `true` then `jetcalc` will include the 'nilpotent vectors' and use the default ordering, and the user need not worry about defining the lists `nilp_source_wt` and `nilp_target_wt`.

R_nilp/L_nilp

These are two variables of type 'list'; each entry in the lists being a list with two entries. These variables specify the extra 'nilpotent vectors' which we must include in the Lie algebra. 'Nilpotent vectors' in the source Lie algebra will take the form gv, where $g \in \mathcal{E}_n$ and v is in the source Lie algebra generating set. (For example, in the $\mathcal{A}$ case we will require some of the vectors of the form $x_i\partial/\partial x_j$ to be included; but in general v will be some combination of the standard vectors $\partial/\partial x_j$.) 'Nilpotent vectors' in the target Lie algebra will always be of the form $y_i\partial/\partial y_j$,[2] where $\{y_1, \ldots, y_p\}$ are coordinates in $\mathbf{F}^p$. The precise syntax is as follows. For $g \in \mathcal{E}_n$ and v_i the i-th vector specified by the table `tgtspace` (c.f., `liealg` above), the entry $[g, i]$ in the list `R_nilp` indicates that the vector gv_i is to be included in the source Lie algebra. (For example, with `liealg` = `stdjacobian`, to include $x_i\partial/\partial x_j$ in the source Lie algebra we have the entry $[xi, j]$ included in `R_nilp`.) Similarly, the entry $[i, j]$ in the list `L_nilp` indicates that the vector $y_i\partial/\partial y_j$ is to be included in the target Lie algebra.

We need only invoke the nilpotent variables for complete transversal or determinacy calculations. In this case the Lie algebra to work with takes the form $L\mathcal{G}_1$ + 'nilpotent vectors'. Thus, it is only the scalar multiples of the 'nilpotent vectors' which are not already included in $L\mathcal{G}_1$, and `jetcalc` need only extend the tangent space $L\mathcal{G}_1 \cdot f$ by considering the $\mathbf{F}$-span of all the 'nilpotent vectors' when they act on f; that is it does not need to invoke the whole of the module structure(s).

As an example consider the $\mathcal{A}$ classification of map-germs $(\mathbf{F}^2, 0) \to (\mathbf{F}^3, 0)$. We will use the unipotent subgroup of $\mathcal{A}$ with Lie algebra

$$L\mathcal{A}_1 \oplus Sp\{x_1\partial/\partial x_2\} \oplus Sp\{y_2\partial/\partial y_1, y_3\partial/\partial y_1, y_3\partial/\partial y_2\}$$

for the calculations. Only the scalar multiples of the 'nilpotent vectors'

$$(x_1\partial/\partial x_2) \cdot f,\ (y_2\partial/\partial y_1) \cdot f,\ (y_3\partial/\partial y_1) \cdot f,\ (y_3\partial/\partial y_2) \cdot f,$$

are not already present in $L\mathcal{A}_1 \cdot f$. We must therefore specify $\mathcal{A}_1$ and then add the 'nilpotent vectors' as follows. (Recall that we must also specify the source dimension when we are using `liealg` = `stdjacobian`.)

[2] In coordinate form this can be written $y_i.e_j$ where $\{e_1, \ldots, e_p\}$ are the canonical basis vectors of $\mathbf{F}^p$.

```
equiv := A;
liealg := stdjacobian;
source_dim := 2;
source_power := 2;
target_power := 2;
nilp := true;
R_nilp := [[x1,2]];
L_nilp := [[2,1],[3,1],[3,2]];
```

nilp_source_wt/nilp_target_wt

These are two variables of type 'list'; each entry in the lists being an integer. When using the 'nilpotent vectors' described above for complete transversal methods we must use the order induced from the 'nilpotent filtration' $M_{r,s}(\mathcal{G})$ on the homogeneous jets of degree k. The main point is that `jetcalc` always orders the jets so that a complete transversal is given by taking the vectors of degree k from the basis for the complementary space. By ordering the homogeneous jets of degree k using the order induced by $M_{r,s}(\mathcal{G})$ the user can, if necessary, use the 'unipotent complete transversal methods'. Even though `jetcalc` produces a complete transversal of degree k, we can obtain the appropriate $M_{r,s}(\mathcal{G})$ transversal by truncating the jets at that level — explicitly we just take the degree k terms in the complete transversal which belong to the required $M_{r,s}(\mathcal{G})$ jet-space; all the other degree k terms being ignored. (The modified germ f must then be fed back into `jetcalc` using the *same* value of k, but then truncating the resulting transversal at a higher $M_{r,s}(\mathcal{G})$ level than before.)

That we can construct complete transversals from the basis for the complementary space relies on how we order the monomial jets. This is discussed in Section 6.7.4. For the standard cases, the ordering required by the 'nilpotent filtration' can be achieved via a system of weights. We discuss this in Section 6.7.5.

The specific ordering related to given choices of `R_nilp` and `L_nilp` is defined by weights given by the lists `nilp_source_wt` and `nilp_target_wt`. (It is up to the user to make sure these weights are compatible with `R_nilp` and `L_nilp`; however, the function `setup_classn` discussed in Section 6.5 may be helpful.)

For the example $(\mathbf{F}^2, 0) \to (\mathbf{F}^3, 0)$ above, the choice of `R_nilp` and `L_nilp` requires the weights (see Section 6.7.5):

```
nilp := true_order;
nilp_source_wt := [2,1];
nilp_target_wt := [-2,-1,0];
```

6.4 Weighted Jet-Spaces

The function `jetcalc` calculates the tangent space to the orbit of a germ by working in a jet-space of some given degree. This uses the standard filtration of $m_n.\mathcal{E}(n,p)$ by degree, that is by the submodules $\{m_n^{k+1}.\mathcal{E}(n,p)\}$ for $k \geq 0$, and amounts to using truncated polynomials. A modified version of `jetcalc`, called `wtcalc`, implements filtrations given by weights and allows us to work with weighted jet-spaces. (Note that `jetcalc` *always* uses the standard filtration by degree. The methods which use the $M_{r,s}(\mathcal{G})$ filtrations may be implemented, but these just reorder the monomial terms in standard polynomials to allow us to calculate $M_{r,s}(\mathcal{G})$ complete transversals — the algorithm still truncates polynomials to the given (standard) degree.)

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a sequence of positive integers, $\beta = (\beta_1, \ldots, \beta_p)$ be a sequence of non-negative integers, and $\{F^r_{\alpha,\beta}.\mathcal{E}(n,p)\}$ be the corresponding filtration of $m_n.\mathcal{E}(n,p)$ by weight, which we will denote $\{F^r\}$. The function `wtcalc` then works in the same way as `jetcalc` except that it uses the filtration $\{F^r\}$ to define the (weighted) jet-spaces. That is, for a germ f and given weight it takes a generating set for the source and target components of $L\mathcal{G}\cdot f$; calculates all the non-zero weighted jets which result from this using the appropriate module structure(s); applies Gaussian elimination to produce a basis for the tangent space; and finally calculates a basis for the complementary space. This is the same algorithm that `jetcalc` follows, except now all polynomials are truncated by weight and not by standard degree.

The global variables which define $\mathcal{G}$ are used in *exactly* the same way as described in Section 6.3. As yet the $M_{r,s}(\mathcal{G})$ filtration has not been implemented; however, this is no problem in the applications considered so far where the weighted filtration is finer than the $M_{r,s}(\mathcal{G})$. So, in particular, the global variables `nilp_source_wt` and `nilp_target_wt` are *not* used by `wtcalc`, and `nilp` is a *real* Boolean variable, only taking the values `true` or `false`.

In addition to this we must specify the source and target weights, α and β. This is done with the global variables `source_wt` and `target_wt`, both of Maple data type 'list'.

As an example consider the classification of function-germs using a subgroup of the standard $\mathcal{R}$ group which preserves a given discriminant variety. For example, the classification of germs $(\mathbf{F}^3, 0) \to (\mathbf{F}, 0)$ using coordinate changes in the source which preserve the swallowtail discriminant, $\mathcal{D}$. This is commonly denoted $\mathcal{R}(\mathcal{D})$-equivalence. Suppose $\{\theta_1, \theta_2, \theta_3\}$ is a generating set for the module of vector fields tangent to $\mathcal{D}$; θ_3 being the Euler vector field. We refer to [B2] and Chapter 5 for the calculation of these. It is found that a natural system of weights is given by $\alpha = (2, 3, 4)$ (with $\beta = (0)$ as is standard in the case of functions). Now the Lie algebra is given by

$$L\mathcal{R}(\mathcal{D}) = \langle \theta_1, \theta_2, \theta_3 \rangle.$$

We recall that the θ_i vanish at 0; this explains why the above module is not multiplied by m_n, and for the coordinate changes which have 1-jet the identity, $\mathcal{R}(\mathcal{D})_1$, we need only multiply by m_n and not by m_n^2; the tangent space being given by

$$L\mathcal{R}(\mathcal{D})_1 \cdot f = m_n . \langle \theta_1 \cdot f, \theta_2 \cdot f, \theta_3 \cdot f \rangle.$$

For determinacy and complete transversal calculations we will use the 'nilpotent vectors' $\theta_1 \cdot f$ and $\theta_2 \cdot f$, and only the scalar multiples of these are not already present in $L\mathcal{R}(\mathcal{D})_1 \cdot f$. We therefore specify $\mathcal{R}(\mathcal{D})_1$ and then add the 'nilpotent vectors' as follows. (The value given to `target_power` is irrelevant in this example as we are just using coordinate changes in the source, but it must be assigned a non-negative integer.)

```
equiv := R;
liealg := swallowtail;
source_power := 1;
target_power := 0;
nilp := true;
R_nilp := [[1,1],[1,2]];
L_nilp := [];
source_wt := [2,3,4];
target_wt := [0];
```

The results produced by `wtcalc` may be inspected using various 'output' functions, in just the same way as for `jetcalc`. The same functions apply to `wtcalc` and `jetcalc` since both store their results (and other data needed by the functions, such as monomial reference tables) in common global variables.

6.5 Package Functions

Here we describe all the user functions; the standard Maple format for 'help' is used, except that examples will be given in a later section. The synopsis for each function is intended for reference, so there is inevitably some duplication of the notes for different functions.

Function: **intangent — test if a set of vectors is in the tangent space calculated by jetcalc (or wtcalc)**

Calling Sequence:

`intangent`$(v_1, v_2, \ldots)$

Parameters:

$v_1, v_2, \ldots$ — lists of expressions

Synopsis:

- Each parameter v_i specifies a jet in $J^k(n,p)$; its p entries being the components of the jet. The function tests to see if the set of vectors $\{v_1, v_2, \ldots\}$ together with the basis for the tangent space to the orbit of a germ (calculated by `jetcalc` or `wtcalc`) form a dependent set of vectors. It returns `true` when a dependent set results and `false` when an independent set results.

- For a single parameter v the function therefore returns `true` if v is in the tangent space, and `false` if not. This is useful for remembering which way round `intangent` works — that is `true` for 'in tangent space' (or 'dependent'). Also, in the case of a single v a simple method is sometimes worth exploiting. If v is in the basis for the complementary space, as given by `pcomp`, then v cannot be in the tangent space. However, if v is not in the basis this does not necessarily mean that v is in the tangent space — in this case `intangent` must be used.

- The function `jetcalc` (or `wtcalc`) must always have been called before using this function. Only the most recently calculated tangent space is used.

- A set of vectors by which the tangent space basis must be extended to give a basis for the whole space (spanned by the tangent space basis and the v_i) is calculated. To be precise, a matrix is calculated whose rows give these 'extension' vectors in coordinate form; this matrix is stored as the global variable ext_tangent. (Thus, if the rank of ext_tangent is less than the number of parameters v_i then true is returned; if these numbers are equal then false is returned.) In particular, ext_tangent may contain non-numeric elements (say, if the original germ passed to jetcalc involved 'moduli-type' parameters) and the rank may drop for certain values. In such cases, intangent prints a warning and outputs the matrix ext_tangent to allow the user to determine the degenerate situations. Since ext_tangent is a global variable it may be inspected at any later stage with the standard Maple commands for printing matrices. Note that printing ext_tangent will only give the vectors in coordinate form (and this depends on how jetcalc decided to order the monomial vectors in $J^k(n,p)$); however, this suffices for the above considerations.

Function: jetcalc — calculate the tangent space and complementary space to the orbit of a germ in a given jet-space

Calling Sequence:

jetcalc(f, k)

Parameters:

f – a list of expressions

k – a positive integer

Synopsis:

- The parameter f specifies a map-germ $(\mathbf{F}^n, 0) \to (\mathbf{F}^p, 0)$ by $f := [f_1, \dots, f_p]$ where f_i are Maple expressions in the source coordinates as given by coords. The parameter k specifies the jet-space degree.
- The $\mathcal{G}$ equivalence is described by the appropriate global variables; as are the nilpotent terms. This is discussed fully in Section 6.3.
- The tangent space to the orbit of f in the jet-space of degree k is calculated, together with the complementary space. Specifically, a basis for each of these spaces is obtained; these can be inspected by the various 'print' routines described in this section. (The actual data is stored in a specific format as the global variables coeffarray and compbasis respectively.) The '*See also*' note found at the end of this synopsis includes all the 'print' functions which may be used.

- When the tangent space spanning set is reduced to echelon form using Gaussian elimination (specifically, the vectors are ordered and a matrix of coefficients extracted; this matrix is then reduced to echelon form) some of the pivotal elements may not be numeric but polynomial expressions involving 'unfolding' or 'moduli' type parameters. For specific values of the parameters the pivotal elements will vanish and the tangent space degenerates . The calculation fails for such values and `jetcalc` must be called again using these values. A 'check list' of all non-numeric pivotal elements is created by `jetcalc` and may be accesed later. (The check list data is stored as the global variable `checklist`.) A warning is printed by `jetcalc` when the check list is non-empty.

- The dimension of the complementary space is given by the global variable `codim`; while the dimension of the tangent space is given by `basis_dim`. The jet-space degree used in the last call to `jetcalc` is given by `jetspace_deg`.

- The source and target dimensions are both calculated by `jetcalc` from the number of components of `coords` and of f respectively. They do not need to be user defined.

- See also: `intangent`, `pcomp`, `plist`, `ptangent`, `wtcalc`.

Function: pcomp — print basis for complementary space calculated by jetcalc (or wtcalc)

Calling Sequence:

`pcomp()`

Synopsis:

- Outputs a basis for the complementary space to the tangent space to the orbit of a jet.

- The function `jetcalc` (or `wtcalc`) must always have been called before using this function. Only the most recently calculated basis is stored.

- The dimension of the complementary space is given by the global variable `codim`.

- If the global variable `compltrans` is set to `false` the whole basis is output; if set to `true` only the degree k terms are output, where k was the degree passed to `jetcalc` (or `wtcalc`).

Function: **pdetterms — print terms which failed the determinacy criterion as calculated by Adetermined**

Calling Sequence:

`pdetterms()`

Synopsis:

- Outputs the terms which failed the determinacy criterion in the function `Adetermined`. For a full description see `Adetermined`.
- The function `Adetermined` must always be called before using this function.

Function: **plist — print check list calculated by jetcalc (or wtcalc)**

Calling Sequence:

`plist()`

`plist(`$flag$`)`

`plist(`$flag1, flag2$`)`

Parameters:

$flag, flag1, flag2$ – (optional) flags which may take the values 'P' or 'F'

Synopsis:

- Outputs the 'check list' calculated by `jetcalc` (or `wtcalc`). The 'check list' contains all the non-numeric pivotal elements which were formed when `jetcalc` performed Gaussian elimination. For a full description see `jetcalc`.
- The function `jetcalc` (or `wtcalc`) must always be called before using this function. Only the most recently calculated check list is stored.
- The data is stored globally as the table `checklist`. Each element in `checklist` represents a monomial vector in $J^k(n,p)$, that is a monomial term with indeterminates given by the list `coords`, multiplied by some coefficient. The precise format is as follows. For each entry in the table `checklist` the index number, coefficient and monomial term are output. The index number (which is proceeded by a '#') is output so that the corresponding entry in `checklist` may be explicitly obtained by the user at a later time. The most important

part is the coefficient which, by the definition of the check list, will be a non-numeric Maple expression. It will be desirable to, say, factor this expression to determine if it has any roots. Now each element of `checklist` is itself a table with two entries, the first giving the coefficient. Thus, to obtain the coefficient of the i-th term in `checklist` we use `checklist[i][1]`. Note that the coefficients `checklist[i][1]` are just the non-numeric pivotal elements in the echelon matrix `coeffarray` produced by `jetcalc` (or `wtcalc`). The entry `checklist[i][2]` just gives the corresponding monomial term for the i-th entry.

- A flag may be passed as a parameter to `plist`; this may take the values P or F or both (the later by passing P and F as two parameters $flag1$ and $flag2$). It may be necessary to use single right-quotes thus: 'P' or 'F'; if P or F is an assigned Maple expression then the use of quotes evaluates P and F to actual Maple names. If P is passed then the entries of the check list are printed out in turn with `plist` waiting for the user to type C;[RETURN] (C followed by a semi-colon followed by the [RETURN] key — for Continue) before proceeding with the next entry. Alternatively typing E;[RETURN] (for Exit) terminates the function. (Again right quotes may be required thus: 'C' or 'E'.) If the flag F is passed to `plist` then the coefficient entries in `checklist` (that is the entries `checklist[i][1]`) are factored by Maple before being output.

Function: pmons — print monomial vectors

Calling Sequence:

`pmons`(k)

`pmons`(k_1, k_2)

Parameters:

k, k_1, k_2 – non-negative integers (k_2 optional)

Synopsis:

- Outputs the homogeneous monomial vectors of degree k. If two parameters, k_1 and k_2, are passed then all homogeneous vectors of degree k_1 to degree k_2 are output. The vectors are output in order of increasing degree, with the actual order (for monomial vectors of the same degree) being that used by `jetcalc` when creating the matrix of coefficient vectors, `coeffarray`.

- If the global variable nilp is set to true_order then the order induced by the nilpotent filtration is used. This order is determined by the global lists nilp_source_wt and nilp_target_wt, with the homogeneous monomial vectors of each degree r (where $r = k$ or $k_1 \leq r \leq k_2$) being partitioned into their appropriate $M_{r,s}(\mathcal{G})$ jet-level, this level being output too. (Note that the possible 'nilpotent orders' are restricted to the standard ones obtained by weights, as mentioned in Section 6.7.5.) If the global variable nilp is set to false (or true) then the default order is used.

- This function requires no preliminary function calls or assigned global variables except those mentioned above and the use of coords to specify the coordinates. The global list coords may be assigned by the user; alternatively the liealg procedures will set coords when they are called (they are usually called by jetcalc but this requires all the global variables to be assigned).

Function: ptangent — print basis for tangent space calculated by jetcalc (or wtcalc)

Calling Sequence:

ptangent()

ptangent(v)

Parameters:

v – an (optional) list of expressions

Synopsis:

- Outputs a basis for the tangent space to the orbit of a germ. The basis is canonical in the sense that the coordinate form of each vector is just a row from the echelon matrix produced by jetcalc (or wtcalc), that is the matrix of coefficients, coeffarray.

- The function jetcalc (or wtcalc) must always have been called before using this function. Only the most recently calculated basis is stored.

- The dimension of this vector space is given by the global variable basis_dim.

- Each vector in the basis is output as a collection of monomial vectors together with a scalar coefficient. The vector is then given by the sum of such vectors.

- If the optional parameter v is given it must be a monomial vector; more precisely, v must be a list with one entry a monomial, the rest being zero. Vectors in the basis which contain v as a term are output. This is useful for a closer inspection of the tangent space. Note that v must be a monomial vector (otherwise nothing will be output).

Function: pvars — print global variables

Calling Sequence:

`pvars()`

Synopsis:

- Outputs all the user defined global variables which define the group $\mathcal{G}$. These are discussed fully in Section 6.3.

Function: pwtmons — print weighted monomial vectors

Calling Sequence:

`pwtmons(`w`)`

`pwtmons(`w_1, w_2`)`

Parameters:

w, w_1, w_2 – non-negative integers (w_2 optional)

Synopsis:

- Outputs the weighted homogeneous monomial vectors of weight w. If two parameters, w_1 and w_2, are passed then all weighted homogeneous vectors of weight w_1 to weight w_2 are output. The vectors are output in order of increasing weight, with the actual order (for monomial vectors of the same weight) being that used by `wtcalc` when creating the matrix of coefficient vectors, `coeffarray`.

- The weights must be specified for the source and target using the global lists `source_wt` and `target_wt`. In the case of weighted homogeneous functions (that is, where the target has dimension one) the standard 'source only' weights may be specified by `source_wt` with `target_wt` equal to the list `[0]`.

- This function requires no preliminary function calls or assigned global variables except `source_wt`, `target_wt` and the use of `coords` to specify the coordinates. The global list `coords` may be assigned by the user; alternatively the `liealg` procedures will set `coords` when they are called (they are usually called by `wtcalc` but this requires all the global variables to be assigned).

Function: setup_classn — set up global variables for $\mathcal{A}$ classification

Calling Sequence:

`setup_classn`(n)

`setup_classn`(n, p, l)

Parameters:

n, p – positive integers (p optional)

l – an (optional) list with two entries

Synopsis:

- This function provides an easy way of assigning the global variables. The actual settings are suitable for an $\mathcal{A}$ classification problem (determinacy and complete transversals) and specifically set up the group $\mathcal{A}_1$. The exact values used can of course be inspected with the `pvars` function.
- The parameter n specifies the source dimension to be used.
- The nilpotent variables `R_nilp`, `L_nilp`, `nilp_source_wt` and `nilp_target_wt` are not assigned unless the optional parameters p and l are given.
- The optional parameter p specifies the target dimension and is only needed if the nilpotent variables are to be assigned. Then the parameter l is required also — it provides an easy way of assigning the nilpotent variables, a choice of four standard possibilities being available. l may take the values $[x1, 0]$, $[0, x1]$, $[xn, 0]$ or $[0, xn]$. These symbols specify the *first* element in the required nilpotent ordering; the precise meaning being to give the orderings which start as:

$[x1^k, 0, \dots, 0], \dots;\ [0, \dots, 0, x1^k], \dots;\ [xn^k, 0, \dots, 0], \dots;$ or $[0, \dots, 0, xn^k], \dots;$

respectively. From this the function works out the standard nilpotent vectors and weights which give rise to such an order. For example, $l := [x1, 0]$ requires the Lie algebra

$$LA_1 \oplus Sp\{\, x_i\partial/\partial x_j : i > j \,\} \oplus Sp\{\, y_i\partial/\partial y_j : i < j \,\}$$

and so

nilp := true_order;
R_nilp := $[\,[x2,1],\ldots,[xn,1],[x3,2],\ldots,[xn,2],\ldots,[xn,n-1]\,]$;
L_nilp := $[\,[1,2],[1,3],[2,3],[1,4],[2,4],[3,4],\ldots,[1,p],\ldots,[p-1,p]\,]$;
nilp_source_wt := $[1,2,\ldots,n]$;
nilp_target_wt := $[0,-1,\ldots,-p+1]$;

Function: setup_unf — set up global variables for $\mathcal{A}$ unfolding

Calling Sequence:

setup_unf(n)

Parameters:

n – a positive integer

Synopsis:

- This function provides an easy way of assigning the global variables. The actual settings are suitable for an $\mathcal{A}$ unfolding problem and specifically set up the group $\mathcal{A}_e$. The exact values used can of course be inspected with the **pvars** function.
- The parameter n specifies the source dimension to be used.

Function: wtcalc — calculate the tangent space and complementary space to the orbit of a germ in a given weighted jet-space

Calling Sequence:

wtcalc(f, k)

Parameters:

f – a list of expressions

k – a positive integer

Synopsis:

- The parameter f specifies a map-germ $(\mathbf{F}^n, 0) \to (\mathbf{F}^p, 0)$ by $f := [f_1, \ldots, f_p]$ where f_i are Maple expressions in the source coordinates as given by `coords`. The parameter k specifies the weighted degree of the jet-space.

- The $\mathcal{G}$ equivalence is described by the appropriate global variables; as are the nilpotent terms. This is discussed fully in Section 6.3. The source and target weights which define the weighted filtration of $m_n.\mathcal{E}(n,p)$ are given by the global lists `source_wt` and `target_wt`.

- The tangent space to the orbit of f in the jet-space of weighted degree k is calculated, together with the complementary space. Specifically, a basis for each of these spaces is obtained; these can be inspected by the various 'print' routines described in this section. (The actual data is stored in a specific format as the global variables `coeffarray` and `compbasis` respectively.) The '*See also*' note found at the end of this synopsis includes all the 'print' functions which may be used. Note that `wtcalc` and `jetcalc` store all the data required by the 'print' routines in common global variables so that the same routines can be used by both.

- More detailed information can be found under `jetcalc`. The synopsis there also applies to `wtcalc`.

- See also: `intangent`, `pcomp`, `plist`, `ptangent`, `jetcalc`.

Function: **Aclassify — apply the method of complete transversals and find the first non-empty transversal for a given germ (this function is intended for $\mathcal{A}$-equivalence calculations)**

Calling Sequence:

`Aclassify`(f, k)

`Aclassify`(f, k, l)

Parameters:

f – a list of expressions

k, l – positive integers (l optional)

Synopsis:

- The parameter f specifies a map-germ $(\mathbf{F}^n,0) \to (\mathbf{F}^p,0)$ by $f := [f_1,\ldots,f_p]$ where f_i are Maple expressions in the source coordinates as given by `coords`.

- The group $\mathcal{A}_1$ and appropriate nilpotent terms must be defined via the global variables prior to calling this function.

- The function decides whether f is k-$\mathcal{A}$-determined and failing this outputs the *first* non-empty complete transversal. It essentially uses `jetcalc` repeatedly and is extremely useful in complicated cases where a number of lengthy calculations are required in order to check $\mathcal{A}$-determinacy. The determinacy criterion used is: if

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\cdot f + m_n^{2k+2}.\mathcal{E}(n,p),$$

where L is a Lie subalgebra of $L\mathcal{A}$ and J^1L is nilpotent on $\mathbf{F}^{n+p}$, then f is k-$\mathcal{A}$-determined, [BduPW]. The complete transversals of degree $k+1$ up to degree $2k+1$ are calculated in succession using `jetcalc`. If any of these are non-empty the calculation terminates and the transversal and jet-space degree are output, thus giving the *first* non-empty transversal for f. However, if all transversals are empty then the determinacy criterion holds and the function returns that f is k-$\mathcal{A}$-determined.

- All the usual functions which access the results of `jetcalc` may be used. The results only apply to the jet-space being used when the function terminated, the degree of this is stored as `jetspace_deg`.

- An important feature is the implementation of the 'extended' determinacy criterion. With reference to [BduPW]: if

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\cdot f + m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p) + m_n^{2k+2}.\mathcal{E}(n,p),$$

then f is k-$\mathcal{A}$-determined. The $m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p)$ terms usually allow us to reduce the upper degree of the transversals from $2k+1$. For suppose that $m_n^{k+1}.f^*(m_p) \supset m_n^{l+1}$ for some l, then the extended extended determinacy criterion reduces to

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\cdot f + m_n^{l+1}.\mathcal{E}(n,p).$$

So only the complete transversals of degree $k+1, k+2, \ldots, l$ need to be calculated, and if all are empty then f is k-$\mathcal{A}$-determined. For example, if $f \in m_n.\mathcal{E}(n,p)$ has as two of its coordinate functions x and y^2, then $m_n^{k+1}.f^*(m_p) \supset m_n^{k+3}$ and only the complete transversals of degree $k+1$ and $k+2$ need to be calculated. As a default, `Aclassify` will calculate all the complete transversals of degree $k+1$ up to degree $2k+1$. However, if a third (optional) parameter l is given, with l an integer, $k+1 \leq l < 2k+1$, then `Aclassify` will only calculate complete transversals up to degree l allowing the above method to be implemented.

- If either of the two methods mentioned above fails then the extended determinacy criterion may be applied in the following sense to test further for determinacy. Let l be either a third parameter passed to `Aclassify` (as just described) or the default value of $2k+1$. Suppose `Aclassify` produces a non-empty complete transversal of degree $r < l$ but all the terms in the transversal belong to $m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p)$ (this being easy to check). Using `jetcalc` the complete transversals of degree $r+1$ to degree l may be calculated. If they prove to be empty or contain terms in $m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p)$ then

$$m_n^s.\mathcal{E}(n,p) \subset L\mathcal{G} \cdot f + m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p) + m_n^{s+1}.\mathcal{E}(n,p),$$

  holds for s where $k+1 \leq s \leq l$. Hence, the 'extended' determinacy criterion holds proving f is k-$\mathcal{A}$-determined. If this method is used and the 'extended' criterion fails for some transversal of degree between $r+1$ and l then the classification may proceed by trying alternative determinacy criteria or (probably more appropriately) using the function `Adetermined` to check for determinacy. If the jet is thought, or known, not to be determined, the classification must proceed to the next jet-level using the first non-empty transversal for f. The appropriate transversal is the one obtained back at the degree r level and output by `Aclassify`. In many cases this 'extended' determinacy criterion proves to be useful. However, it may, in such circumstances, prove more efficient to use the `Adetermined` function. For instance, the choice of complete transversal used above is just one of many, and if some of the terms do *not* belong to $m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p)$ this does not necessarily mean that the determinacy criterion fails.

- See also: `Adetermined`.

Function: Adetermined — test for $\mathcal{A}$-determinacy of given germ

Calling Sequence:

`Adetermined`(f, k)

`Adetermined`(f, k, l)

Parameters:

f – a list of expressions

k, l – positive integers (l optional)

Synopsis:

- The parameter f specifies a map-germ $(\mathbf{F}^n, 0) \to (\mathbf{F}^p, 0)$ by $f := [f_1, \ldots, f_p]$ where f_i are Maple expressions in the source coordinates as given by `coords`.

- The group $\mathcal{A}_1$ and appropriate nilpotent terms must be defined via the global variables prior to calling this function.

- The function uses the following determinacy criterion: if

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f + m_n^{2k+2}.\mathcal{E}(n,p),$$

where L is a Lie subalgebra of $L\mathcal{A}$ and J^1L is nilpotent on $\mathbf{F}^{n+p}$, then f is k-$\mathcal{A}$-determined, [BduPW]. This condition is checked directly by working in the $(2k+1)$-jet-space with `jetcalc`. If it holds the conclusion that f is k-$\mathcal{A}$-determined is returned, otherwise the terms which fail the criterion are returned (these will be monomial vectors in $m_n^{k+1}.\mathcal{E}(n,p)$).

- All the usual functions which access the results of `jetcalc` may be used. In addition, any terms which fail the determinacy condition may be recalled with the function `pdetterms`. (The data is stored as the global variable `det_store`.)

- The following 'extended' determinacy criterion may be applied; with reference to [BduPW]: if

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f + m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p) + m_n^{2k+2}.\mathcal{E}(n,p),$$

then f is k-$\mathcal{A}$-determined. This can be used to significantly reduce the length of the calculation. The first point is that the required jet-space degree can be reduced from the general value of $2k+1$. For suppose that $m_n^{k+1}.f^*(m_p) \supset m_n^{l+1}$ for some l, then the extended extended determinacy criterion reduces to

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f + m_n^{l+1}.\mathcal{E}(n,p).$$

This condition may now be checked (using `jetcalc`) by working in the l-jet-space, instead of the $(2k+1)$-jet-space. For example, if $f \in m_n.\mathcal{E}(n,p)$ has as two of its coordinate functions x and y^2, then $m_n^{k+1}.f^*(m_p) \supset m_n^{k+3}$ and the calculation can be done in the $(k+2)$-jet-space instead of the $(2k+1)$-jet-space.

This is a clear improvement and reduces the computation time significantly, especially for large values of k. As a default `Adetermined` will work in the $(2k+1)$-jet-space (assuming the standard determinacy criterion). However, if a third (optional) parameter l is given, with l an integer, $k+1 \leq l < 2k+1$, then `Adetermined` will work in the l-jet-space allowing the above improvement to be implemented. The second point to note with the extended determinacy criterion is that `Adetermined` just checks that all the terms in $m_n^{k+1}.\mathcal{E}(n,p)$ are contained in $L\mathcal{G}\cdot f$ by working in the l-jet-space (where l is the third parameter passed to `Adetermined` or the default value of $2k+1$, as appropriate). Any terms which fail this test are output, but it is easy to decide if they belong to $m_n^{k+1}.f^*(m_p).\mathcal{E}(n,p)$ as the failed terms will be monomial vectors; this provides a full implementation of the extended determinacy criterion. (It is preferable to output these terms and let the user do the checking. If the computer were asked to do the checks the linear algebra it must perform becomes considerably more extensive.)

- In comparison with the `Aclassify` function, `Adetermined` is much more efficient for checking the determinacy criterion discussed above. For example, `Adetermined` is especially useful if it is already thought that f is k-$\mathcal{A}$-determined (say from rough calculations by hand) but with some terms in $m_n^{k+1}.\mathcal{E}(n,p)$ proving hard to check. However, if f fails the determinacy criterion then the results of `Adetermined` are of little use in comparison to those of `Aclassify` which then provide the next non-empty complete transversal for f.

- See also: `pdetterms`, `Aclassify`.

6.6 Examples

The following describes some standard calculations in singularity theory and how they may be carried out using the `Transversal` package. The appropriate Maple commands will be given together with the Maple response. However, for brevity, the full Maple response is not always indicated, just the final result.

To begin with run Maple or X-Maple; X-Maple is preferable, where possible, so that one may take advantage of the X-Windows environment. Now load the `Transversal` package by issuing the command `with(transversal)`. We will consider the classification of map-germs $(\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^4,0)$ under $\mathcal{A}$-equivalence.

6.6.1 Complete Transversals and Determinacy

In this example we consider the $J^3\mathcal{A}$-orbits over the 2-jet $(x, y^2, 0, 0)$. We use the 'unipotent' group $\mathcal{G}$ described in Section 3.2. The following command sets up the global variables with the corresponding 'nilpotent' Lie algebra and Maple responds by printing out the values.

```
> setup_classn(2,4,[0,x2]);
                        liealg = stdjacobian
                             equiv = A
                         compltrans = true
                          source_dim = 2
                         source_power = 2
                         target_power = 2
                        nilp = true_order
                              R_nilp:
                             [[x1, 2]]
                              L_nilp:
        [[2, 1], [3, 1], [4, 1], [3, 2], [4, 2], [4, 3]]
>
```

Recall that the function `setup_classn` assigns the global variables so that the $\mathcal{A}_1$ group is specified. The source dimension is a compulsory argument; while the target dimension and the 'Lie algebra flag' (`[0,x2]` above) are optional — in the above case the global variables `nilp`, `R_nilp`, `L_nilp` (and `nilp_source_wt` and `nilp_target_wt`, which are not printed out) are assigned and specify the required nilpotent Lie algebra. We now specify the germ $f = (x, y^2, 0, 0)$ and calculate the orbit in the 3-jet-space $J^3(2, 4)$.

```
> f := [x1,x2^2,0,0];
> jetcalc(f,3);
```

Maple responds with several lines reporting on the state of the calculation (which takes very little time in this case, of course) and finishes by displaying `Ready`. The matrix dimensions output should not raise too much concern; they give a good indication to the user about the complexity of the calculation in relation to other calculations performed by `jetcalc`; however, the matrices are very sparse and do not take long to reduce compared to usual matrices of such dimensions.

Important: `jetcalc` prints the map-germ it is using; it is always worthwhile checking this is the intended map, especially when several different maps have been defined, as it is a common mistake to pass the wrong one to `jetcalc`. All the data will now have been calculated and stored and may be viewed using the various 'print' routines. For example, to display a complete transversal we type

```
> pcomp();

                          3
            [0, 0, 0, x2 ]
                     3
            [0, 0, x2 , 0]
                      2
          [0, 0, 0, x1  x2]
                  2
          [0, 0, x1  x2, 0]

>
```

This is just the set of homogeneous terms of degree 3 from the basis for the complementary space to the orbit; since **compltrans** $=$ **true** the basis is truncated to produce a complete transversal. Recall that we are using the 'nilpotent filtration' induced by $L\mathcal{G}$ so that a $(3,1)$-transversal for $(x, y^2, 0, 0)$ is $\{(0,0,0,y^3)\}$, while a $(3,2)$-transversal is $\{(0,0,y^3,0)\}$, a $(3,3)$-transversal is $\{(0,0,0,x^2y)\}$, and a $(3,4)$-transversal is $\{(0,0,x^2y,0)\}$. Note that the partition of monomial vectors into their various $(3,s)$-levels can be displayed using the command

```
> pmons(3);
```

(This would *not* have worked before calling `jetcalc` unless the user specified the source coordinates by assigning the variable **coords** for themselves — this was done by the **liealg** routine, **stdjacobian**, called from within `jetcalc`.)

Although all of the degree 3 terms in the complete transversal are output, we should consider each separately at its own $(3,s)$-level. (This is fine since the degree 3 terms have been ordered according to their $(3,s)$-level — it is just more convenient to output the whole lot.) Thus, at the $(3,1)$-level we obtain the orbits $(x, y^2, 0, ay^3)$ for $a \in \mathbf{R}$ and, after scaling, these reduce to $(x, y^2, 0, y^3)$

and $(x, y^2, 0, 0)$. Continuing with the first gives the orbits (x, y^2, x^2y, y^3) and $(x, y^2, 0, y^3)$ but these are equivalent to ones obtained from the orbit $(x, y^2, 0, 0)$ as we shall now see — the details are therefore omitted. **Note:** one often finds that such redundancies occur; we have still to find a way of eliminating them without the need to consider every (r, s)-orbit like the $(3, 1)$-orbit in this case.

Consider the $(3, 1)$-jet $(x, y^2, 0, 0)$. From the earlier calculation, the $(3, 2)$-transversal gives, after scaling, the orbits $(x, y^2, y^3, 0)$ and $(x, y^2, 0, 0)$. Consider the first; we calculate the higher $(3, s)$-transversals by calling `jetcalc` again and specifying the same degree, 3.

```
> f := [x1,x2^2,x2^3,0];
> jetcalc(f,3);
```

Once `jetcalc` has finished we display the complete transversal using the function `pcomp`. Exactly the same vectors as before are output. We only consider the homogeneous $(3, 3)$ terms; the only one is $(0, 0, 0, x^2y)$ giving, after scaling, the $(3, 3)$-orbits (x, y^2, y^3, x^2y) and $(x, y^2, y^3, 0)$. Continuing with the first:

```
> f := [x1,x2^2,x2^3,x1^2*x2];
> jetcalc(f,3);
> pcomp();

                                    3
                        [0, 0, 0, x2 ]
                                 3
                        [0, 0, x2 , 0]
                                  2
                      [0, 0, 0, x1  x2]

>
```

This indicates the all the higher $(3, s)$-transversals for the $(3, 3)$-jet (x, y^2, y^3, x^2y) are empty and we have obtained a $J^3\mathcal{A}$-orbit. Returning to the other $(3, 3)$-jet $(x, y^2, y^3, 0)$, from the previous `jetcalc` calculation using this jet we see that the only higher non-empty $(3, s)$-transversal is the $(3, 4)$-transversal, $\{(0, 0, x^2y, 0)\}$. We obtain the $(3, 4)$-orbits $(x, y^2, y^3 \pm x^2y, 0)$ and $(x, y^2, y^3, 0)$. We already know the higher $(3, s)$-transversals are empty in the latter case, while in the former this is easily checked by a further call `jetcalc(f,3)` with $f = (x, y^2, y^3 \pm x^2y, 0)$. We therefore obtain two more $J^3\mathcal{A}$-orbits.

We now return to the (3,2)-jet $(x, y^2, 0, 0)$. From the original calculation for $(x, y^2, 0, 0)$ we see that a (3,3)-transversal is $\{(0, 0, 0, x^2y)\}$. For $f = (x, y^2, 0, x^2y)$ the higher $(3, s)$-transversals are empty. However, this $J^3\mathcal{A}$-orbit is redundant, for consider the other (3,3)-jet, $(x, y^2, 0, 0)$. The only higher transversal is the (3,4)-transversal, $\{(0, 0, x^2y, 0)\}$. For both (3,4)-orbits the higher transversals are empty so, in total, we obtain the $J^3\mathcal{A}$-orbits:

$$\begin{array}{l} (x, y^2, y^3, x^2y), \\ (x, y^2, y^3 \pm x^2y, 0), \\ (x, y^2, y^3, 0), \\ (x, y^2, x^2y, 0), \\ (x, y^2, 0, 0). \end{array}$$

We will demonstrate determinacy calculations by considering the first and second cases above. If a germ f has 2-jet $(x, y^2, 0, 0)$ we can appeal to the fact that $m_2^{k+1}.f^*(m_4).\mathcal{E}_2 \supset m_2^{k+3}$ and can work in the jet-space $J^{k+2}(2,4)$ to prove determinacy. Now (x, y^2, y^3, x^2y) is 3-determined. Even if we do not suspect this we still have to calculate the higher transversals, so may as well check determinacy in the process using the `Aclassify` procedure. This procedure simply calculates higher transversals, stopping if a non-empty transversal is produced or a given jet-level is reached. To check determinacy we must give `Aclassify` the determinacy degree k together with the jet-level it must proceed to, in this case $k+2$. If no limit degree is given `Aclassify` will use the default value of $2k+1$, which is always enough, and calculate the $(k+1), (k+2), \ldots, (2k+1)$-transversals. Such a calculation is very intensive and, in practice, we always try and reduce the limit from $2k+1$. For (x, y^2, y^3, x^2y) the appropriate commands are

```
> f := [x1,x2^2,x2^3,x1^2*x2];
> Aclassify(f,3,5);
```

Maple responds by calculating the 4- and 5-transversals; the state of the calculation is reported as for `jetcalc`. Both transversals are empty and Maple returns the conclusion

```
              2     3     2
        [x1, x2 , x2 , x1  x2]
```

```
                    germ is 3-A-determined
>
```

We could equally well use the Adetermined procedure. This checks the determinacy condition directly and can, in theory, show the condition holds even if Aclassify failed by producing a non-empty transversal. However, in practice, this have never been the case, and since Aclassify is faster and produces a useful result when the given germ is not finitely determined (namely the next non-empty transversal) it is the preferred procedure. If one wishes to use Adetermined the arguments are the same as for Aclassify, so in the previous case use Adetermined(f,3,5). Once the tangent space has been calculated and the determinacy condition checked, the same conclusion is returned.

We now consider the 3-jet $(x, y^2, y^3 \pm x^2y, 0)$. Using jetcalc we calculate the 4-transversal in the usual manner. The cases $(x, y^2, y^3 + x^2y, 0)$ and $(x, y^2, y^3 - x^2y, 0)$ must, of course, be considered separately; they produce identical results so we only discuss the first.

```
> f := [x1,x2^2,x2^3+x1^2*x2,0];
> jetcalc(f,4);
> pcomp();

                                    3
                      [0, 0, 0, x1   x2]

>
```

This demonstrates why jetcalc outputs all the homogeneous terms rather than restricting to each $(4, s)$-level individually. In the above case we see that all $(4, s)$-transversals are empty except the $(4,4)$-transversal, $\{(0, 0, 0, x^3y)\}$. So we immediately obtain the $J^4\mathcal{A}$-orbits $(x, y^2, y^3 + x^2y, x^3y)$ and $(x, y^2, y^3 + x^2y, 0)$. We could have used Aclassify to obtain this (Aclassify(f,3,5)), though it is clear that all the 4-transversals could not have been empty, anyway. It is appropriate to use Aclassify to investigate $(x, y^2, y^3 + x^2y, x^3y)$; this shows it is 4-determined. We then proceed with $(x, y^2, y^3 + x^2y, 0)$.

```
> g := [f[1],f[2],f[3],x1^3*x2];
> Aclassify(g,4,6);
```

```
                    2     3        2        3
            [x1, x2 , x2  + x1  x2, x1  x2]
                 germ is 4-A-determined

> jetcalc(f,5);
> pcomp();

                               4
               [0, 0, 0, x1  x2]

>
```

Further calculation shows that $(x, y^2, y^3 + x^2y, x^4y)$ is 5-determined, and proceeding further gives the first few germs in the series $(x, y^2, y^3 + x^2y, x^ky)$. We conjecture that this is indeed a series — the general proof now follows easily but we have to resort to hand calculations.

Remark ($\mathcal{A}_1$-Complete Transversals). If the $\mathcal{A}_1$-complete transversal methods are preferred, just follow the above but with the global variable `nilp` set equal to `false`. The command

```
> setup_classn(2);
```

will set up the global variables accordingly. Note, however, that the use of the $\mathcal{A}_1$ group occasionally gives redundant terms in the transversal; requires more work to simplify the resulting family of jets; and means that the determinacy calculations (for instance, if one were using the `Aclassify` procedure) will not incorporate the unipotent methods either.

6.6.2 Moduli Detection and the Mather Lemma

In this example we consider the 7-jet $f = (x, y^2, xy^3 + x^4y, y^5 + ax^6y)$. This was obtained by calculating the 7-transversal for the 6-jet $(x, y^2, xy^3 + x^4y, y^5)$; the only non-empty $(7, s)$-transversal was $\{(0, 0, 0, x^6y)\}$ giving rise to the above family. We cannot 'scale' a to a unit using simple 'scaling' coordinate changes and suspect it is a modulus. To prove a is a modulus we have to use the group $\mathcal{A}$ and work in the 7-jet-space to show

$$(0,0,0,x^6y) \notin L\mathcal{A} \cdot f \quad \text{modulo} \quad m_2^8.\mathcal{E}(2,4);$$

see Theorem 1.9. To begin with we must set up the global variables to define the $\mathcal{A}$ group.

Warning: if this done during a classification session where the group $\mathcal{G}$ is being used, it is easy to return to the complete transversal calculations while forgetting to reset the global variables to define $\mathcal{G}$. If the user is running X-Maple it is advisable to carry out these calculations, where we have changed the global variables to specify $\mathcal{A}$ instead of $\mathcal{G}$, by running a second version of Maple in a separate window.

Having entered Maple and loaded the `Transversal` package we can set up the $\mathcal{A}$ group as follows. Firstly set up the $\mathcal{A}_e$ group using the `setup_unf` procedure.

```
> setup_unf(2);
                        liealg = stdjacobian
                             equiv = A
                         compltrans = false
                          source_dim = 2
                         source_power = 0
                         target_power = 0
                           nilp = false
                              R_nilp:
                              R_nilp
                              L_nilp:
                              L_nilp
>
```

Then change this to the $\mathcal{A}$ group by setting the powers of the maximal ideals equal to 1.

```
> source_power := 1; target_power := 1;
```

It is extremely advisable to double check these settings by printing out the global variables.

```
> pvars();
```

(We have omitted the Maple response.) Now we calculate the tangent space $L\mathcal{A} \cdot f$ in $J^7(2,4)$ and determine whether the vector $(0,0,0,x^6y)$ belongs to this space.

```
> f := [x1,x2^2,x1*x2^3+x1^4*x2,x2^5+a*x1^6*x2];
> jetcalc(f,7);
> intangent([0,0,0,x1^6*x2]);
                                 false
>
```

The Maple response `false` indicates that a is a modulus.

Remark (Families of Higher Modality). To show that a and b are moduli in the family

$$g = (x, y^2, x^3y \pm xy^5 + by^7, x^2y^3 + ay^7)$$

we use the same Theorem and have to show that $\{(0,0,y^7,0),(0,0,0,y^7)\}$ forms an independent set to $L\mathcal{A} \cdot g$ in $J^7(2,4)$.

```
> g := [x1,x2^2,x1^3*x2+x1*x2^5+b*x2^7,x1^2*x2^3+a*x2^7];
> jetcalc(g,7);
> intangent([0,0,x2^7,0],[0,0,0,x2^7]);
                                 false
>
```

Again, the Maple response `false` indicates a and b are moduli.

We now return to determinacy calculations, in particular determinacy calculations for families. Remember to reset the group to $\mathcal{G}$ (using the `setup_classn` procedure as in Section 6.6.1) or move to an X-Maple window where $\mathcal{G}$ has been set up (cf. earlier remarks). Using `Aclassify` will show that f is 7-determined, but this only holds for generic a and does not indicate the exceptional values. For families it is more convenient to check determinacy by calculating each transversal separately using `jetcalc`.

```
> f := [x1,x2^2,x1*x2^3+x1^4*x2,x2^5+a*x1^6*x2];
> jetcalc(f,8);
```

Observe that when jetcalc has finished the calculation it responds with the warning:

```
WARNING: global variable 'checklist' is non-empty !!!
```

The Gaussian elimination routines in jetcalc try to choose the best pivotal elements. They give preference to numeric (non-symbolic) pivots but this is not always possible and the global variable checklist is used to store the non-numeric pivots. To display these we use the command plist.

```
> plist();

                                        7
             #1, 1 + a, [ 0, 0, 0, x1  x2 ]
                                    7
             #2, 1 + a, [ 0, 0, x1  x2, 0 ]
                                7
             #3, 1 + a, [ 0, x1  x2, 0, 0 ]
                            7
             #4, 1 + a, [ x1  x2, 0, 0, 0 ]

>
```

The first column indicates the index number of the pivotal element as an entry in the table checklist; the second column the actual pivotal element (this is the important bit); and the third column the monomial vector associated to this particular pivot (as an entry in the matrix of all coefficients of monomial vectors). In more complicated examples we can therefore access the pivotal element $\#i$ (for example, to factorise it and find the exceptional values — this is not necessary here, of course) using checklist[i][1]. (checklist[i][2] gives the monomial vector, but as a table, so the Maple command eval would have to be used.) Alternatively the command plist(F) will try and factorise all the pivotal elements as it displays them. Then,

```
> pcomp();
                    *** TRANSVERSAL EMPTY ***
>
```

shows that the 8-transversal is empty, but only provided $1+a \neq 0$. To investigate the exceptional value we must re-run `jetcalc`:

```
> h := subs(a=-1,f);
> jetcalc(h,8);
> pcomp();

                                   7
                     [0, 0, 0, x1  x2]


>
```

So we must consider the case $a = -1$ separately; we will return to this shortly. Continuing with the determinacy calculation:

```
> jetcalc(f,9);
      WARNING: global variable 'checklist' is non-empty !!!
>
```

Again, we must use `plist()` and display the non-numeric pivots. We find the conditions on a are the same as before, and then using `pcomp()` shows that the 9-transversal is empty. Thus, f is 7-determined for $a \neq -1$.

Note: sometimes the conditions on parameters such as a which are displayed by `plist` are redundant; re-running `jetcalc` with the 'exceptional' values substituted in causes no change to the complete transversal. The conditions only appeared in the first place because `jetcalc` had no other choice for the corresponding pivotal elements.

We now consider the 7-jet $h = (x, y^2, xy^3 + x^4y, y^5 - x^6y)$. From above, an 8-transversal is $\{(0, 0, 0, x^7y)\}$ giving the family of 8-jets

$$f = (x, y^2, xy^3 + x^4y, y^5 - x^6y + ax^7y).$$

We can apply 'scaling' coordinate changes to reduce this to $(x, y^2, xy^3 + x^4y, y^5 - x^6y \pm x^7y)$ and $(x, y^2, xy^3 + x^4y, y^5 - x^6y)$. Alternatively, this follows from the Mather Lemma and we take this opportunity to demonstrate the method. There are situations where the use of the Mather Lemma is more important, for instance, where 'scaling' will not work and applying the Mather Lemma shows triviality for the whole family. Such situations did not occur in our classification, however, so we use this example to demonstrate the general procedure.

Remember to begin by setting up the $\mathcal{A}$ group as described above, or using the X-Maple window which was reserved for such calculations, as appropriate. Next we calculate the tangent space $L\mathcal{A} \cdot f$ in $J^8(2,4)$, determine whether the vector $(0,0,0,x^7y)$ belongs to this space, and display the dimension of the space.

```
> f := [h[1],h[2],h[3],h[4]+a*x1^7*x2];
> jetcalc(f,8);
      WARNING: global variable 'checklist' is non-empty !!!
> plist();

                                              7
                   #1, 2/3 a, [ 0, 0, 0, x1  x2 ]

> intangent([0,0,0,x1^7*x2]);
   WARNING: original matrix contains non-numeric elements, check
                             checklist !!!
                                 true
> basis_dim;
                                 162
>
```

This tells us that $L\mathcal{A} \cdot f$ is of constant dimension 162 and contains $(0,0,0,x^7y)$ provided $a \neq 0$. The warning output from intangent also reminds us that non-numeric pivotal elements exist. On some occasions non-numeric terms are introduced by the elimination routine in intangent. When this happens intangent outputs a vector containing the offending term. This vector is a coordinate vector of coefficients but the specific interpretation does not matter. The important point is that the non-numeric terms should be treated in the same manner as those given by plist — the values for which they vanish should be investigated. Substituting $a = 0$ into f and following the above procedure shows $L\mathcal{A} \cdot f$ is of dimension 161 and does not contain $(0,0,0,x^7y)$. Thus, by the Mather Lemma, the family can be reduced to the orbits

$$(x, y^2, xy^3 + x^4y, y^5 - x^6y \pm x^7y),$$
$$(x, y^2, xy^3 + x^4y, y^5 - x^6y),$$

and in the complex case the two orbits $(x, y^2, xy^3 + x^4y, y^5 - x^6y + x^7y)$ and $(x, y^2, xy^3 + x^4y, y^5 - x^6y - x^7y)$ reduce to one. Further calculation gives determinacy and continuing with $(x, y^2, xy^3 + x^4y, y^5 - x^6y)$ produces the series $(x, y^2, xy^3 + x^4y, y^5 - x^6y \pm x^ky)$, which is $(k+1)$-determined for $k \geq 7$.

Remark (The 'Checklist' and Moduli Detection). We ignored the possibility of the global variable `checklist` being non-empty in moduli-detection calculations above. However, in such an eventuallity, the computer result (`false` returned by the function `intangent`) applies for all values of the parameters a_i, except those lying on a finite set of algebraic varieties (defined by the polynomial entries in `checklist`). So the result holds for all values of the a_i except those on a subset of stricly smaller dimension and by Theorem 1.9 the a_i are moduli.

6.6.3 Unfoldings and Codimension

We finish by showing how to calculate a versal unfolding for a finitely-determined map-germ. We shall consider the germ $f = (x, y^2, y^3, x^2y)$ discussed in Section 6.6.1. The same methods apply to more complicated examples. For instance, if we were to consider a family then we apply the same procedure but must inspect any non-numeric pivotal elements which arise using `plist`, and consider the exceptional values, as in Section 6.6.2.

We begin by setting up the $\mathcal{A}_e$ group using the **setup_unf** procedure as in Section 6.6.2. Next we calculate $L\mathcal{A}_e \cdot f$ in $J^3(2,4)$ and output the basis for its complementary space, this gives us the unfolding terms since f is 3-determined. Generally, if a germ $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ is k-$\mathcal{A}$-determined then, by the determinacy theorems of [BduPW], there exists some unipotent subgroup $\mathcal{G}$ of $\mathcal{A}$ such that

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{G} \cdot f,$$

and, in particular,

$$m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{A} \cdot f \quad \text{and} \quad m_n^{k+1}.\mathcal{E}(n,p) \subset L\mathcal{A}_e \cdot f,$$

so that we can calculate $\mathcal{A}_e$- (and $\mathcal{A}$-) unfoldings using `jetcalc` by working in the k-jet-space. Note that since `compltrans` $=$ `false`, when we call the `pcomp` procedure the whole basis for the complementary space will be output, as required.

```
> f := [x1,x2^2,x2^3,x1^2*x2];
> jetcalc(f,3);
> pcomp();

                    [0, 0, x2, 0]
                    [0, 0, 0, x2]
                   [0, 0, x1 x2, 0]

> codim;
                          3
>
```

Thus, f has $\mathcal{A}_e$-codimension 3 and a versal unfolding is

$$(x, y, u_1, u_2, u_3) \mapsto (x, y^2, y^3 + u_1 y + u_2 xy, x^2 y + u_3 y, u_1, u_2, u_3).$$

To consider, for example, adjacencies of finitely determined map-germs or use the codimension as an invariant at the jet-level (that is to consider non-sufficient strata as well as sufficient strata) we must use $\mathcal{A}$-unfoldings and the $\mathcal{A}$-codimension. Firstly modify the global variables to change the $\mathcal{A}_e$ group to the $\mathcal{A}$ group.

```
> source_power := 1; target_power := 1;
```

Again, it is always a good idea to re-check the settings with the pvars procedure before continuing. Now calculate the tangent space and print out the unfolding terms as before.

```
> jetcalc(f,3);
> pcomp();

                     [1, 0, 0, 0]
                     [0, 1, 0, 0]
                     [0, 0, 1, 0]
                     [0, 0, 0, 1]
                    [0, x2, 0, 0]
                    [0, 0, x2, 0]
                    [0, 0, 0, x2]
                   [0, 0, x1 x2, 0]
                   [0, 0, 0, x1 x2]
```

```
> codim;
                                  9
>
```

This highlights the fact that `jetcalc` always calculates the complementary space in $\mathcal{E}(n,p)$ and not $m_n.\mathcal{E}(n,p)$, so here we must ignore the constant terms. Thus, f has $\mathcal{A}$-codimension 5 and an $\mathcal{A}$-versal unfolding is

$$(x, y, u_1, u_2, u_3, u_4, u_5) \mapsto (x, y^2 + u_1 y, y^3 + u_2 y + u_3 xy, x^2 y + u_4 y + u_5 xy, u_1, u_2, u_3, u_4, u_5).$$

If one wishes to calculate the $\mathcal{A}$-codimension for a large number of examples then defining a Maple function to do this helps avoid silly arithmetic errors, and promotes the utmost in laziness!

```
> cod := () -> codim-4;
> cod();
                                  5
>
```

6.7 The Program Structure and Code

The algorithms presented in the section will be described using a 'pseudocode'. The syntax is based on that used by Maple (in particular, the symbol '#' is used to denote a comment). Many of the statements are self-explanatory and in several cases we use English descriptions in favor of following the precise syntax (which is often language specific and not the place in pseudocode). To achieve utmost clarity, bearing in mind the package is likely to be developed for future applications, we give a fully documented listing of the Maple code in Appendix B. The theory is described below for the real case, though the results apply equally well to the complex case.

We begin by describing the basic structure of the program. The algorithm is essentially the same as that followed if one were doing the calculation by hand, except that everything is calculated in full. For example, in a standard $\mathcal{A}$ classification we first obtain the generators for the $\mathcal{R}$ tangent space and multiply them by all monomials until we reach jets of degree greater than the given degree k, that is, zero k-jets. Similar observations apply to the $\mathcal{L}$ tangent space. We then look for the 'missing monomials' of the appropriate degree. In practice we usually spot whole families of monomials which are available very quickly; which tangent vectors will not really help us (and ignore them); and which will be the difficult monomials to find. The computer cannot do this and just grinds away producing every possible tangent vector in the jet space, that is, a spanning set for the tangent space as an **R**-vector subspace of the jet-space. The problem then reduces to taking real linear combinations of the tangent vectors. We therefore extract the monomial coefficients of the tangent vectors and create a matrix which is reduced to echelon form using Gaussian elimination (the order of the coefficients in the matrix is of significance and will be discussed later). Once we have an echelon matrix (not necessarily in *row reduced* echelon form) we have a basis for the tangent space and can generate a basis for the complementary space algorithmically. This is translated back into a jet format and the results stored for future access.

Remark. A vector (representing, say, a jet in $J^k(n,p)$ which is identified with the space of p-tuples of truncated polynomials in n indeterminates over **R**) will be of Maple data type 'table' (or occasionally an 'array', the difference here being unimportant) with indices 1 to p, where p is the target dimension and is determined from the number of components of f. The global variable `tgtspace` is a table of the tangent vectors (which grows as the program proceeds) and, explicitly, is therefore a table of tables! An exception to this is the original jet $f = (f_1, \ldots, f_p)$. It is more convenient from a user's point of view to use the data type 'list' here.

We start by discussing how to generate, and store in a suitable format, the monomials up to degree k.

6.7.1 The Index and Degree Reference Tables

There are several occasions in the program where we need to access multivariate monomials with the source coordinates as indeterminates; sometimes all the monomials up to degree k are needed, other times just the monomials of a given degree. For instance, all the generators of the $\mathcal{R}$ tangent space need to be multiplied by all monomials of degree `source_power` and higher until a zero k-jet results. We need to know all the monomials in the k-jet space when extracting coefficients and forming the coefficient matrix. Likewise, monomials with the target coordinates as indeterminates are needed when calculating the left tangent space.

It is therefore necessary to have a procedure which calculates all the appropriate monomials and stores them as a table. In fact, it is more useful just to store the integers which represent the monomial indices rather than the monomials themselves. We therefore have a table, each entry of which is a table of indices. The relevant procedure is called `get_ref_tables` and is discussed below.

We have two tables, namely `index_ref` used to store the indices, and `deg_ref` used to reference a given index degree in `index_ref`. More precisely, the index reference table contains all the indices (each given as a table) up to degree k for monomials in the source coordinates. (The order used to store them is graded lexicographic.) The ith entry in the degree reference table tells us whereabouts in the index reference table the indices of degree i begin. An example should clarify things.

Example. Consider monomials in the indeterminates x, y and z up to degree $k = 2$. The index (i_1, i_2, i_3) represents the monomial $x^{i_1} y^{i_2} z^{i_3}$.

i	`index_ref[i]`	monomial	degree
0	$(0,0,0)$	1	0
1	$(0,0,1)$	z	1
2	$(0,1,0)$	y	1
3	$(1,0,0)$	x	1
4	$(0,0,2)$	z^2	2
5	$(0,1,1)$	yz	2
6	$(0,2,0)$	y^2	2
7	$(1,0,1)$	xz	2
8	$(1,1,0)$	xy	2
9	$(2,0,0)$	x^2	2

`deg_ref[0]` = 0
`deg_ref[1]` = 1
`deg_ref[2]` = 4

It is also desirable to set `deg_ref[k+1]` so that we have a pointer to the end of the index reference table. So in this example: `deg_ref`$[3] = 10$.

Remark. For any $0 \leq d \leq k$ we can therefore access all the indices of degree d by a statement of the form

```
for i from deg_ref[d] to (deg_ref[d+1] - 1) do
```

```
    index := index_ref[i];
    ...
od
```

Similarly, some index `index_ref[i]` is of degree d if and only if

$$\texttt{deg\_ref}[\texttt{d}] \leq i < \texttt{deg\_ref}[\texttt{d}+1].$$

Such statements will be common place in the code.

Now, before we can create the full reference tables we must be able to construct all indices of degree **deg** in lexicographic order. There is a big problem here in that the number n (the source dimension) of indeterminates is not predetermined. One possible answer is to use a recursive procedure. A simpler and probably neater solution is as follows.

Acknowledgement. The method was suggested by Bruce Stephens of the Liverpool University Centre for Mathematical Software Research and Parallel Programming. We take this opportunity to express our warmest thanks.

The point to note is that the n-th index is essentially irrelevant to the problem, being determined by the $1, \ldots, (n-1)$ indices and the required degree. Restricting to these indices, the problem is simply to count from 0 upwards, in base $\texttt{deg}+1$, using $n-1$ digit numbers, until the left-most digit has reached the value **deg**. Thus, the method is to always try and increase the $(n-1)$ index (i.e., the right-most index) by one first. If this takes it past the value **deg** then we reset it to zero and try and increase the next index down (the $(n-2)$ index), and so on. We therefore obtain every index in $n-1$ indeterminates of degree $\leq$ **deg** and for each assign the n-th index accordingly to obtain indices in n indeterminates of degree exactly **deg**.

We create all the indices of degree **deg** as follows. Set the table **index** with the initial index $(0, 0, \ldots, 0, \texttt{deg})$, then store this in the table `index_ref`. Now call a separate procedure named **increment** which takes the table **index** as a parameter and tries to increase the index by one in the aforementioned fashion to obtain the next index of degree **deg**. When this is possible **increment** returns the Boolean value **true** . When this is not possible (i.e., the index of component 1 has passed the maximum value of **deg**) all the indices of degree **deg** will have been created and the Boolean value **false** is returned. The algorithm takes the simple form:

```
index := (0,0,...,0,deg);
store index in the table index_ref;
while increment(index,deg,n) = true do
   store new value of index in table index_ref;
od;
```

The `increment` procedure is called with the parameters `index`, `deg` and the source dimension n. It takes the form:

```
   # first try to increase (n-1) st. index
i  := n-1;
while i > 0 do
   index[i] := index[i]+1;
   if (total degree of indices 1 to (n-1)) > deg then
         # reset to zero
         # try to increase next index along to the left instead
      for j from i to n-1 do
         index[j] := 0;
      od;
      i := i-1;
   else
         # increment worked ok so leave the while loop
      i := -1;
   fi;
od;
   # i=0 means all indices of given degree obtained
   # so return 'false'
if i=0 then
   RETURN(false);
fi;
   # otherwise set the nth index to give indices of the required
   # total degree deg and return 'true'
index[n] := deg-(total degree of indices 1 to (n-1));
true;
```

We can now create indices of any degree `deg` and therefore form the reference tables. The place where a 'new' value of `deg` begins in the `index_ref` table is stored in `deg_ref[deg]`. We refer to the Maple code for full details. The procedure is named `get_ref_tables` and is called with four parameters thus:

```
get_ref_tables(k,num_indets,index_ref,deg_ref);
```

where k specifies the degree to go up to; num_indets the number of indeterminates (n above); and index_ref, deg_ref the two tables which are to be assigned the indices and degree reference points.

6.7.2 The Tangent Space Routines

We now describe how the source and target tangent spaces are created. Which of the following routines is invoked is dependent on the setting of the global variable equiv.

The Right Tangent Space

Recall that the liealg procedure will have given us a generating set for the right tangent space as an $\mathcal{E}_n$-module. We now multiply each generator by the monomials stored in the table index_ref (starting at degree source_power) until a zero k-jet is obtained. The resulting vectors are stored in the table tgtspace, this is a global variable.

```
copy liealg generating set into the table tgtcopy;
for i from 1 to (number of vectors in tgtcopy) do
      # check m(n)^source_power*tgtcopy[i] gives non zero k-jets
   least_deg := (least degree of all components of the
                 polynomial vector tgtcopy[i]);
   if source_power+least_deg <= k then
      for deg from deg_ref[source_power] to
                   deg_ref[k-least_deg+1]-1 do
         multiply tgtcopy[i] by the monomial with indices given
            by index_ref[deg];
         store result in the table tgtspace;
      od;
   fi;
od;
```

The Left Tangent Space

In standard coordinates the left tangent space is given by

$$f^*(m_p^{\texttt{target\_power}}).\{e_1,\ldots,e_p\}$$

where the e_i are the canonical basis vectors of $\mathbf{R}^p$. (Note that the right operand in this product is a set not an ideal.) Now use the fact that the left tangent space is an $\mathcal{E}_p$-submodule of θ_f and create monomials with the target coordinates as indeterminates. (Specifically, we create a reference table of indices as before.) The indeterminates are then substituted for the coordinate functions f_i of f. We start by using monomials of degree `target_power` and increase the degree until a zero k-jet results. We obtain polynomial jets in $J^k(n,1)$. For each non-zero jet we should multiply it by each of the canonical vectors e_i to get an $\mathcal{L}$ tangent vector in $J^k(n,p)$. For simplicity assume this is the case for now — the resulting jets being added to the table `tgtspace` — this will aid the description of the 'simplification algorithm' described in the next section. In practice, however, we will exploit the symmetry of the $\mathcal{L}$ tangent space when performing Gaussian elimination later and need only store the polynomial jet, a member of $J^k(n,1)$, at this stage. (Note that the index and degree reference tables for monomials with target coordinates as indeterminates are named `left_index_ref` and `left_deg_ref`.)

```
   # calculate the maximum degree required for the left indices
left_deg_lim := (integer quotient of k by (least degree of
                 the components of f));
   # check that the left tangent space gives non zero jets
if target_power <= left_deg_lim then
      # create left index and degree reference tables
   get_ref_tables(left_deg_lim,target_dim,
                  left_index_ref,left_deg_ref);
   for deg from left_deg_ref[target_power] to
                left_deg_ref[left_deg_lim+1]-1 do
         # substitute indeterminates for the components of f
      poly := f[1]^left_index_ref[deg][1]* ... *
              f[target_dim]^left_index_ref[deg][target_dim];
      if (poly is a non-zero k-jet) then
         for i from 1 to target_dim do
            store poly*e_i in table tgtspace;
         od;
```

```
      fi;
    od;
fi;
```

The $\mathcal{C}$ (Left Contact) Tangent Space

In standard coordinates the $\mathcal{C}$ tangent space is given by

$$m_n^{\texttt{target\_power}}.f^*(m_p).\mathcal{E}(n,p).$$

We now have an $\mathcal{E}_n$-module again and the methods of the right case apply. Each component function f_i of f is multiplied by monomials of degree `source_power` and higher. As in the $\mathcal{L}$ tangent space routine, in practice we just store the polynomial jets not the $\mathcal{C}$ jets themselves which are obtained by multiplying by the vectors e_i. Though, for simplicity, one may assume this to be the case, with the resulting $\mathcal{C}$ tangent vectors being added to the table `tgtspace`.

The code is similar to the $\mathcal{R}$ case. For all cases we refer to the Maple code for full details (the routine `jetcalc`; Appendix B) — it has been fully documented.

6.7.3 The Coefficient Matrix

We will continue to assume, for simplicity, that we have all the tangent vectors stored in the table `tgtspace`. That is, we have an $\mathbf{R}$-spanning set for the tangent space $L\mathcal{G}\cdot f$ in $J^k(n,p)$ and now proceed to reduce it to a basis using linear algebra techniques. Firstly, each vector in $J^k(n,p)$ is converted to coordinate form — we extract the coefficient of each constituent monomial vector to form a row vector of coefficients. This is done for each tangent vector to form a matrix `coeffarray`, say, which is subsequently reduced to echelon form using Gaussian elimination.

This description is helpful but, in practice, it is wasteful to physically create such a (huge) matrix and we have developed a method called *Indexed Gaussian Elimination* instead. This is fairly technical and it is advantageous to describe the general algorithm in terms matrices and use `coeffarray`. We will describe the indexed method and other technical points separately in Section 6.7.5.

The matrix `coeffarray` has dimensions

'number of vectors in `tgtspace`' by '$p *$ `deg_ref[k+1]`', where p in the target dimension. Each entry `coeffarray[i,j]` is a coefficient, where the row i gives the tangent vector `tgtspace[i]`, and the column j represents the monomial vector to which the coefficient belongs. The specific ordering of the monomial vectors is extremely important in relation to the calculation of complete transversals, as will be discussed in the following section. With this ordering the column value j represents the following monomial vectors. (Recall that there are `deg_ref[d]` to `deg_ref[d+1]-1` monomials of degree `d`.)

First set of p entries	—	the constant coefficients of successive components (1 to p) of the vector.
Next set of (`deg_ref`[2] − `deg_ref`[1]) $* p$ entries	—	the linear coefficients of the successive components.
⋮		⋮
Final set of (`deg_ref`[k + 1] − `deg_ref`[k]) $* p$ entries	—	coefficients for the degree k monomials of successive components.

The important point is that the degree does *not* decrease as we move along the row. In particular, all the coefficients for the degree k monomial vectors appear at the end of the row. This is required if we were calculating, say, $\mathcal{A}_1$ complete transversals. If a nilpotent filtration is to be used then the 'degree k coefficients' must still appear at the end of the row but must be ordered further, amongst themselves. This technical point is discussed in Section 6.7.5.

Remark (Fraction-Free Gaussian Elimination). Generally the coefficients of a jet will be integers. However, if the jet f is given as a family then the procedure above will give some coefficients as symbolic Maple expressions in further (that is, non-source) variables. (The program knows which variables are source coordinates — we told it in the `liealg` procedure.) We must therefore employ a type of *fraction-free* Gaussian elimination whereby no division is performed on the pivotal elements to reduce them to 1. However, if standard 'fraction-free techniques' (see, for example, [F, pp.82–87] for the case of working over $\mathbf{Z}$) are used then the entries in the matrix rapidly 'blow-up' to large expressions causing inefficient memory usage and increasing the performance time. Instead, we can get by using standard Gaussian elimination (now working over the field of real rational functions) except that *no* division is performed on a chosen pivotal element to reduce it to 1, but division *is* performed when using the pivot to reduce the rest of the column to zero (in contrast with fraction-free Gaussian elimination). So, if we have a row

$$(0, 0, \ldots, 0, a_i, a_{i+1}, \ldots, a_m),$$

where a_i is chosen to be a pivot (so is non-zero as a polynomial over $\mathbf{R}$), then we do not divide this row by a_i but reduce higher rows using the operation

$$\begin{gathered}(0, 0, \ldots, 0, b_i, b_{i+1}, \ldots, b_m) \mapsto \\ (0, 0, \ldots, 0, b_i, b_{i+1}, \ldots, b_m) - (b_i/a_i)(0, 0, \ldots, 0, a_i, a_{i+1}, \ldots, a_m).\end{gathered}$$

We then make a list of all the non-numeric pivotal elements; this is stored as the global table `checklist`, a table of rational functions. For most members of the family f, each corresponding to specific values of the unfolding parameters, the rational functions do not vanish and the elimination is valid. However, we must inspect each rational function in turn obtaining conditions on the unfolding parameters for which the elimination breaks down. To investigate the exceptional behaviour the solutions to these conditions must be substituted back into the family. The `checklist` is actually created by the 'complementary space' routine discussed next.

We employ a pivoting strategy where preference is given to numeric (non-symbolic) pivots where possible, otherwise to symbolic expressions of least length as a Maple expression. The sparse nature of the matrices is exploited by checking for a non-zero entry before trying to reduce a given row. Refer to the Maple code for full details of the pivoting technique (the routine `jetcalc`; Appendix B).

6.7.4 The Complementary Space and Complete Transversals

We currently have a basis for the tangent space stored in coordinate form as an echelon matrix. It is a simple matter to convert each coordinate/coefficient vector back to jet format when required — the result being a sum of monomial vectors. Next we calculate a basis for the complementary space to the tangent space $L\mathcal{G} \cdot f$ in $J^k(n, p)$. (For this we always work in the space of *all* jets at 0 and therefore include the constant jets). Specifically, we calculate a canonical basis consisting of monomial vectors.

The scenario, in general, is that of a finite dimensional (real) vector space V and a subspace W with a given basis. We extend this basis to a basis of V and in the process obtain a basis for the complementary space V/W to W in V. So far we have an echelon matrix whose rows form a basis for W. The complementary basis can be obtained algorithmically by 'scanning' through the echelon matrix looking for the pivotal elements and 'filling in the gaps'. For suppose we have an echelon matrix

$$(a_{ij}) = \begin{pmatrix} 0 & \dots & 0 & a_{1j_1} & & & & \dots & & a_{1n} \\ 0 & & \dots & & 0 & a_{2j_2} & & \dots & & a_{2n} \\ \vdots & & & & & & & & & \vdots \\ 0 & & & \dots & & & 0 & a_{rj_r} & \dots & a_{rn} \\ 0 & & & & \dots & & & & & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & & & & \dots & & & & & 0 \end{pmatrix}$$

with pivotal elements $a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$ (so these are non-zero elements and we have $1 \le j_1 < j_2 < \cdots < j_r \le n$), where $n = \dim(V)$ and $r = \dim(W)$. Then the canonical vectors

$$\{e_1, \ldots, \hat{e_{j_1}}, \ldots, \hat{e_{j_2}}, \ldots, \hat{e_{j_r}}, \ldots, e_n\}$$

(where $\hat{e_j}$ denotes the exclusion of e_j from this sequence of vectors) form a basis for the complementary space of W, since these vectors extend the basis of W to a set of n independent vectors in V.

It is easy to produce these vectors e_j algorithmically. Start with $i = j = 1$. If a_{ij} is zero then we include e_j in the complementary basis and increase j by one, thus moving to the next element in the given row. But if a_{ij} is non-zero (i.e., is a pivotal element) then we increase both the row i and column j by one, the vector e_j not being required. We continue in this way until either $j > n$ or $i > r$. If this loop terminates due to $j > n$ then we have obtained a basis for the complementary space. However, if $i > r$ then the last row of the echelon matrix (a_{ij}) must be of the form $(0, 0, \ldots, 0, a_{rj_r}, \ldots, a_{rn})$ and the remaining vectors $\{e_{j_r+1}, \ldots, e_n\}$ must also be included in our basis.

Now, going back to $V = J^k(n,p)$, the algorithm follows the same method but we must keep track of the column value a little more carefully, bearing in mind that the column number references both a vector component and a monomial index, as described in the previous section. Complications also arise in that, in practice, we do not actually create a matrix (a_{ij}) — see Section 6.7.5.

The above process of 'scanning' through (a_{ij}) looking for the pivotal elements gives us an ideal opportunity to pick off the non-numeric pivots and store them in `checklist`. For full details of the elimination algorithm we refer to the Maple code (the routine `jetcalc`; Appendix B).

Remark (Complete Transversals). The above algorithm will give a basis for the complementary space but in the 'complete transversal theory' we need all these basis terms to be contained in H^k, the space of homogeneous jets of degree k. More precisely, working with some group of equivalences $\mathcal{G}$, we are given a $(k-1)$-jet f and calculate a k-complete transversal $T \subset H^k$ satisfying

$$H^k \subset T + J^k L \cdot f,$$

where L is a Lie subalgebra of $L\mathcal{G}$ and $J^1 L$ is nilpotent on $\mathbf{F}^{n+p}$. We may therefore obtain a complete transversal using the above algorithm as follows. We 'scan' through (a_{ij}) in the same fashion until we reach the column corresponding to the start of the degree k terms. We then fill in the gaps for the final 'degree k' block in the above manner — including a vector in the complete transversal for every non-pivotal entry encountered. Since the monomial vectors in $J^k(n,p)$ were ordered so that those of degree k are represented by the latter columns of the matrix (a_{ij}) we do indeed obtain a complete transversal. In practice, all that is necessary is to calculate the whole basis for the complementary space, but to assign a flag to each basis element which indicates if it is of degree strictly less than k or of degree equal to k. The `pcomp` procedure which is used to print out this basis, outputs either all of the vectors or just those of degree k (and hence a complete transversal) according to the global Boolean variable `compltrans` being set to `false` or `true`, respectively. Of course, for this to work for a filtration $M_{k,s}$ induced by a nilpotent Lie algebra we must order the degree k terms according to the filtration, starting with the $H^{k,1}$ terms, followed by the $H^{k,2}$ terms, etc. This is discussed in more detail in Section 6.7.5.

6.7.5 Indexed Gaussian Elimination, Nilpotent Filtrations, the Pre-Tangent Space and Other X-Rated Features of Jetcalc Version GTi 1.9

The above procedure performs the required calculations fine. However, the matrices it produces can be large and the execution time somewhat larger than satisfactory — as discovered using earlier versions of `jetcalc`. We now describe improvements to the algorithm which are incorporated in the latest 'turbo' version of `jetcalc`. We also discuss a way of incorporating filtrations induced by nilpotent Lie algebras. The details below are of a more technical nature than before but the basic principles of the 'elimination algorithm' are as described in the previous sections; in particular, it is useful to keep the concept of the coefficient matrix in one's mind.

Indexed Gaussian Elimination

Physically creating the matrix of coefficients (`coeffarray` in the previous sections) is wasteful on memory and takes a long time relative to other routines. The data is already stored in a compact form as a table of tangent vectors `tgtspace`; each entry is just a table of polynomials and therefore stores the sparse data (the non-zero coefficients) in a minimalist way. We still use the concept of a matrix of coefficients so that Gaussian elimination (which, for our applications, is a very efficient numerical algorithm) can be used to produce a basis.

The idea is to use reference tables which, for a given row and column (i, j), point to the appropriate coefficient in the table `tgtspace`. The row i simply points to the ith entry of the table `tgtspace`, namely `tgtspace[i]`. The column j points to a coefficient of this tangent vector using a reference table called `coeffarray_ref`. For each j, `coeffarray_ref[j]` is itself a table with two entries. The first entry `coeffarray_ref[j][1]` is an integer between 1 and p which specifies the vector component; whereas the second entry `coeffarray_ref[j][2]` gives the monomial index by pointing to an entry in the table `index_ref`. Thus, the entry (i, j) of our (would be) matrix contains the coefficient of the monomial with indices `index_ref[coeffarray_ref[j][2]]` in the `coeffarray_ref[j][1]` component of the tangent vector `tgtspace[i]`. So every occurrence of an entry of the coefficient matrix, `coeffarray[i,j]`, as in previous sections, will now be replaced by the indexed version:

```
poly := tgtspace[i][coeffarray_ref[j][1]];
coefficient := get coefficient of the monomial in poly
               with indices index_ref[coeffarray_ref[j][2]];
```

(In the code, this process of extracting the coefficient from a table such as `tgtspace` is performed by the procedure `coeff_table`. This takes the table and the row and column (i, j) as parameters.)

This appears to be inefficient in that we must calculate each coefficient as it is required. However, in practice, this method proved to be two to three times faster, using three to four times less memory than methods which obtained the full matrix `coeffarray`. Also note that the row reduction operations performed in Gaussian elimination are now achieved by polynomial manipulations — a very efficient process in Maple which uses the internal functions (and replaces a large number of 'get coefficient' and '0 − (scale factor) $*$ 0 = 0' operations which inevitably occur during row reduction of a sparse matrix).

The use of reference tables will make the implementation of filtrations induced by nilpotent Lie algebras easy. A system of weights is used, as described later. Setting up the default version of `coeffarray_ref` so that the 'degree k coefficients' appear at the end of a row (but in lexicographic order, for example) is straightforward and we refer to the Maple code for more details.

The Pre-Tangent Space

Now we discuss a way of exploiting the symmetry present in the left tangent space. This applies to both the $\mathcal{L}$ and $\mathcal{C}$ cases; we will concentrate on the $\mathcal{L}$ case. In standard coordinates the left tangent space is given by

$$f^*(m_p^{\texttt{target\_power}}).\{e_1, \ldots, e_p\}$$

and once we have an $\mathbf{R}$-spanning set consisting of non-zero polynomial k-jets for the $\mathcal{E}_p$-ideal $f^*(m_p^{\texttt{target\_power}})$, we repeat this set p times, once for each canonical vector, and obtain a spanning set for the left tangent space in $J^k(n,p)$. So the coefficient matrix for the left tangent space is just a block matrix, the non-zero blocks being the coefficient matrix for the aforementioned ideal, repeated p times.

We begin by creating an $\mathbf{R}$-spanning set for the ideal $f^*(m_p^{\texttt{target\_power}})$ in $J^k(n,1)$, in practice this is what the left tangent space routine performs, storing the set of polynomials in the table `poly_table`. Gaussian elimination is then performed to reduce this to echelon form. As before, one can visualise this through the use of a matrix of coefficients whose rows correspond to the polynomial entries in the table `poly_table`, and whose columns correspond to each monomial index up to degree k (the entries 0 to `deg_ref[k+1]-1` in the table `index_ref`). Suppose the resulting echelon matrix is (a_{ij}), then the sort of argument we have in mind is as follows. If we were to 'stack' p copies of (a_{ij}) together diagonally, filling in the remaining upper and lower halves of the matrix with zeros then the resulting matrix represents the full left tangent space in $J^k(n,p)$ and is clearly still in echelon form, as required.

$$\begin{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rs} \end{pmatrix} & & & 0 \\ & \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rs} \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rs} \end{pmatrix} \end{pmatrix}$$

(r and s being appropriate, postitive integers). However, this corresponds to ordering the monomial vectors in $J^k(n,p)$ starting with the *all* the monomials (from degree 0 up to degree k) of the e_1 component, followed by *all* the monomials of the e_2 component, and so on. This is fine if we were only interested in finding a basis, but for complete transversal calculations we require all the coefficients for the degree k monomial vectors to appear at the end of the row and use the ordering described in Section 6.7.3. However, using this ordering we can still put together a matrix for the full left tangent space which is in echelon form. For each polynomial `poly`, say, in `poly_table` we include the jet `poly` $* e_1$ in the full matrix, followed by the jet `poly` $* e_2$, and so on, finishing with the jet `poly` $* e_p$. Now, using the ordering just described, the resulting coefficient matrix for the full left tangent space would then take the following form.

$$\left(\begin{array}{cccc|cccc|c|cccc} a_{11} & 0 & \dots & 0 & a_{12} & 0 & \dots & 0 & & a_{1s} & 0 & \dots & 0 \\ 0 & a_{11} & & \vdots & 0 & a_{12} & & \vdots & \dots & 0 & a_{1s} & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 & & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{11} & 0 & \dots & 0 & a_{12} & & 0 & \dots & 0 & a_{1s} \\ \hline a_{21} & 0 & \dots & 0 & a_{22} & 0 & \dots & 0 & & a_{2s} & 0 & \dots & 0 \\ 0 & a_{21} & & \vdots & 0 & a_{22} & & \vdots & \dots & 0 & a_{2s} & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 & & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{21} & 0 & \dots & 0 & a_{22} & & 0 & \dots & 0 & a_{2s} \\ \hline & & \vdots & & & & \vdots & & \ddots & & & \vdots & \\ \hline a_{r1} & 0 & \dots & 0 & a_{r2} & 0 & \dots & 0 & & a_{rs} & 0 & \dots & 0 \\ 0 & a_{r1} & & \vdots & 0 & a_{r2} & & \vdots & \dots & 0 & a_{rs} & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 & & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{r1} & 0 & \dots & 0 & a_{r2} & & 0 & \dots & 0 & a_{rs} \end{array}\right)$$

If the coefficient matrix (a_{ij}) of `poly_table` has already been reduced to echelon form (so 'the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain') then the above coefficient matrix for the full left tangent space is in echelon form as well.

The advantage of the above method is now obvious. We can create the full left tangent, in echelon form, simply by creating the table `poly_table` (which stores an **R**-spanning set for the ideal $f^*(m_p^{\texttt{target\_power}})$ in $J^k(n,1)$), reducing this to echelon form, and finally stacking together the result in the aforementioned fashion (the order in which the vectors are stored is crucial) to create the full echelon matrix. If the number of vectors in `poly_table` is x and the number of monomials of degree less than or equal to k is y, then the matrix corresponding to `poly_table` ((a_{ij}) above) is of dimensions (x, y). Whereas the full coefficient matrix for the left tangent space is of dimensions (px, py). Reducing `poly_table` to echelon form instead of reducing the full coefficient matrix (this is the intensive part of the calculation) therefore reduces the problem by an order of magnitude of p^2. Since the dimensions (x, y) involved can become large this is a major gain in efficiency; for example, the time taken for specific trial calculations has been reduced from hours to minutes using this technique.

A few more technical points need to be mentioned. We will only discuss these briefly — the full details can only be conveyed by referring to the Maple code. Firstly, it should be clear by now that we do not create the matrices such as (a_{ij}) mentioned above, but use the indexed techniques discussed in the previous section.

The resulting echelon matrix for the full left tangent space needs to be modified if we are using a nilpotent filtration. All the entries up to and including the degree $k-1$ coefficients are fine as they stand, but the last block of degree k coefficients must be reordered as dictated by the nilpotent filtration — see the following section. The last block may not be in echelon form any longer and must be reduced separately; this takes very little time however. (One can argue, heuristically, that this is due to the sparsity and the fact that the block is already a row permutation of an echelon matrix.)

Finally, we mention how to adjoin the left and right tangent spaces to create an echelon matrix for the whole tangent space in $J^k(n,p)$. The right tangent space matrix M_2, say, needs to be reduced to echelon form. However, we exploit the fact that the left tangent space matrix M_1, say, is already in echelon form, adjoining the matrices thus

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

The following efficiency improvements can then be made to the Gaussian elimination algorithm. Keep the current row and column pointer in the matrix M_1. If the corresponding entry is a pivot then reduce as usual; only the column in M_2 needs to be reduced to zero as the column in M_1 will already be zero. Otherwise, (if the entry in M_1 is zero) try and find a pivot in M_2. If this is possible, again only the column in M_2 needs to be reduced. *Important*: if we need to use M_2 to obtain a pivot then we do *not* swap the rows of M_1 and M_2 as in standard Gaussian elimination, but rather *insert* the row of M_2 into M_1 thus reserving the fact that M_1 is echelon. This is the basic idea at least. In the code it is more efficient to create a separate matrix which stores the final result: when a pivot is found the corresponding row is added to this 'result matrix' thus eliminating the need to physically insert a row of M_2 into M_1 (moving all the remaining rows of M_1 down). The table `tgtspace` is set aside to store the final (echelon) matrix as a table of polynomial vectors. This is a global table which will be used by several other procedures once `jetcalc` has terminated.

Nilpotent Filtrations

When the global variable `nilp` is set to `true_order` this tells `jetcalc` to include the 'nilpotent vectors' given by `R_nilp` and `L_nilp` and to order the monomial vectors of degree k according to the associated nilpotent filtration. The 'nilpotent vectors' are simply included at the end of the right tangent space table before Gaussian elimination is performed — there is nothing special here. One way to implement the nilpotent filtration is to order the degree k monomial vectors using a system of weights. This allows the use of the standard nilpotent Lie algebras, as we will describe. We use weights only as a means of telling the computer how to order the vectors; these weights do not satisfy the usual requirements in that the target weights may be negative integers. The usual notation is employed (see Section 2.4); $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_p)$ will denote the source and target weights respectively. The monomial vector $x_1^{k_1} \ldots x_n^{k_n} e_i$ is homogeneous of weight $k_1\alpha_1 + \cdots + k_n\alpha_n - \beta_i$. The $\mathcal{E}_n$-submodule of $m_n.\mathcal{E}(n,p)$ generated by such monomial vectors of weight $\geq k$ is denoted $F^k_{\alpha,\beta}\mathcal{E}(n,p)$.

For classification purposes one would prefer to use some nilpotent Lie algebra $L \subset L\mathcal{G}$ ($\mathcal{G}$ a subgroup of $\mathcal{K}$) which contains as many of the 'extra' vectors from

$L\mathcal{G} \setminus L\mathcal{G}_1$ as possible. We require the natural faithful representation of J^1L on $\mathbf{R}^{n+p}$

$$\rho : J^1L \subset gl(n, \mathbf{R}) \oplus gl(p, \mathbf{R}) \longrightarrow gl(n+p, \mathbf{R})$$

to be nilpotent (see Section 2.3). Then the maps

$$\pi_1 : J^1L \longrightarrow gl(n, \mathbf{R}), \qquad \pi_2 : J^1L \longrightarrow gl(p, \mathbf{R}),$$

can be considered as maps into the spaces of all strictly upper triangular n by n and p by p matrices, and as such the best possible scenario is when these maps are surjective. For our applications we can restrict to the four natural cases and make a formal definition along these lines. (There are many more examples, but the choice is not obvious and the following tend to be used exclusively in practice.)

Definition 6.1 Suppose L is a subalgebra of $L\mathcal{K}$ such that J^1L is nilpotent on $\mathbf{R}^{n+p}$. We will call this Lie algebra *canonical* if it is spanned by the vectors

$$\begin{array}{rcll} x_i\partial/\partial x_j & \in & L(J^1\mathcal{R}) & \text{for} \quad i \sim j, \\ y_i\partial/\partial y_j & \in & L(J^1\mathcal{L}) & \text{for} \quad i \sim j, \end{array}$$

where $(x_1, \ldots, x_n)$ denote source coordinates, $(y_1, \ldots, y_p)$ denote target coordinates and $\sim$ denotes either of the two order relations $<$ or $>$ (there being a total of four possibilities). In the $\mathcal{R}$ and $\mathcal{L}$ cases, L is a subalgebra of $L\mathcal{R}$ or $L\mathcal{L}$ and the definition is restricted to the appropriate vectors.

Now, for such Lie algebras we can assign source and target weights such that the partition of the monomial vectors of degree k via their weight corresponds to their partition into the (k, s)-jet-levels using the nilpotent filtration.

Example. In the classification of map-germs $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$ we used the canonical nilpotent Lie algebra

$$\begin{array}{rcl} L\mathcal{A}_1 & \oplus & Sp\{x\partial/\partial y\} \\ & \oplus & Sp\{u_2\partial/\partial u_1, u_3\partial/\partial u_1, u_4\partial/\partial u_1, u_3\partial/\partial u_2, u_4\partial/\partial u_2, u_4\partial/\partial u_3\}; \end{array}$$

see Section 3.2. The partition of the monomial vectors into the (k, s)-levels can be achieved using the weights $\alpha = (2, 1)$ and $\beta = (-3, -2, -1, 0)$. (This partition was shown for $k = 1$ and $k = 2$ in Section 3.2.) Note that the partitions only agree for fixed k and not for different levels (k_1, s_1) and (k_2, s_2) where $k_1 \neq k_2$; however, we only need to partition a given k-level. The general theory follows from the following proposition.

Proposition 6.2 *Suppose L is a canonical nilpotent Lie algebra, as in definition 6.1. Assign source and target weights, according to the case in question, as follows.*

Vectors			*Weight*
$x_i\partial/\partial x_j \in L(J^1\mathcal{R})$	*for*	$i<j$	$\alpha = (n,\ldots,2,1)$
$x_i\partial/\partial x_j \in L(J^1\mathcal{R})$	*for*	$i>j$	$\alpha = (1,2,\ldots,n)$
$y_i\partial/\partial y_j \in L(J^1\mathcal{L})$	*for*	$i<j$	$\beta = (0,-1,\ldots,-p+1)$
$y_i\partial/\partial y_j \in L(J^1\mathcal{L})$	*for*	$i>j$	$\beta = (-p+1,\ldots,-1,0)$

That is, assign x_i weight $n-i+1$ and i, respectively; assign y_i weight $1-i$ and $-p+i$, respectively. Then

$$\sum_{i\geq s} L^i\cdot(m_n^k.\mathcal{E}(n,p)) + m_n^{k+1}.\mathcal{E}(n,p) =$$
$$\left(F_{\alpha,\beta}^{k+s}\mathcal{E}(n,p)\cap m_n^k.\mathcal{E}(n,p)\right) + m_n^{k+1}.\mathcal{E}(n,p).$$

So for fixed k, the $M_{k,s}(L)$ filtration can be replaced by the weighted filtration modulo $m_n^{k+1}\mathcal{E}(n,p)$, that is the filtration on the right-hand side of the above expression. Recall that the space of 'homogeneous terms' of degree (k,s), $H^{k,s}$, is the image of $M_{k,s-1}(L)$ in the jet-space $m_n.\mathcal{E}(n,p)/M_{k,s}(L)$. In particular, the homogeneous monomial vectors of degree (k,s) are just those of degree k with weight $k+s-1$.

Proof. The proof is similar in all four cases. We shall consider the second and third combinations in the table, that is where $\alpha=(1,2,\ldots,n)$ and $\beta=(0,-1,\ldots,-p+1)$.
The $\subset$ inclusion is almost trivial. Observe that with the choice of weights, $m_n^r.\mathcal{E}(n,p)\subset F_{\alpha,\beta}^r\mathcal{E}(n,p)$ for any $r\geq 1$, also that the action of L increases weight, that is $L\cdot(m_n^r.\mathcal{E}(n,p))\subset m_n^{r+1}.\mathcal{E}(n,p)$. Thus

$$\sum_{i\geq s} L^i\cdot(m_n^k.\mathcal{E}(n,p))\subset F_{\alpha,\beta}^{k+s}\mathcal{E}(n,p)\cap m_n^k.\mathcal{E}(n,p).$$

For the reverse inclusion we need only consider monomial vectors of degree k in $F_{\alpha,\beta}^{k+s}\mathcal{E}(n,p)\cap m_n^k.\mathcal{E}(n,p)$; let $x_1^{k_1}\ldots x_n^{k_n}e_j$ be such a vector. Thus

$$k_1+2k_2+3k_3+\cdots+nk_n\geq k+s+(1-j) \tag{6.1}$$

Now

$$\begin{aligned}(x_2\partial/\partial x_1)^{k-k_1}\cdot(x_1^k) &= x_1^{k_1}x_2^{k-k_1}\\ (x_3\partial/\partial x_2)^{k-k_1-k_2}\cdot(x_1^{k_1}x_2^{k-k_1}) &= x_1^{k_1}x_2^{k_2}x_3^{k-k_1-k_2}\\ &\vdots\end{aligned}$$

and continuing in this fashion up to

$$(x_n\partial/\partial x_{n-1})^{k-k_1-k_2-\cdots-k_{n-1}}\cdot(x_1^{k_1}x_2^{k_2}\ldots x_{n-2}^{k_{n-2}}x_{n-1}^{k-k_1-k_2-\cdots-k_{n-2}}) = x_1^{k_1}x_2^{k_2}\ldots x_{n-2}^{k_{n-2}}x_{n-1}^{k_{n-1}}x_n^{k-k_1-k_2-\cdots-k_{n-1}}.$$

But $k=k_1+\cdots+k_n$ and $x_{i+1}\partial/\partial x_i\in L$ as L is canonical so

$$x_1^{k_1}\ldots x_n^{k_n}e_l\in L^r\cdot(m_n^k.\mathcal{E}(n,p))$$

where $l=1,\ldots,p$ and

$$\begin{aligned} r &= (n-1)k-(n-1)k_1-(n-2)k_2-(n-3)k_3-\cdots-k_{n-1} \\ &= k_2+2k_3+3k_4+\cdots+(n-2)k_{n-1}+(n-1)k_n. \end{aligned}$$

In particular, taking $l=1$ and applying the $j-1$ vector fields

$$y_1\partial/\partial y_2,\quad y_2\partial/\partial y_3,\quad \ldots\quad y_{j-1}\partial/\partial y_j$$

in sequence to $x_1^{k_1}\ldots x_n^{k_n}e_1$ shows that $x_1^{k_1}\ldots x_n^{k_n}e_j\in L^{r+j-1}\cdot(m_n^k.\mathcal{E}(n,p))$. But the original assumption on weights lead to 6.1 so

$$k_2+2k_3+3k_4+\cdots+(n-1)k_n+j-1\geq s$$

and

$$x_1^{k_1}\ldots x_n^{k_n}e_j\in\sum_{i\geq s}L^i\cdot(m_n^k.\mathcal{E}(n,p)).$$

□

Note: the full properties of a canonical nilpotent Lie algebra were not needed in the above proof and we could have weakened the hypotheses of the proposition accordingly. Though, in practice, one tends to work with such Lie algebras.

Using the above theory we order the degree k monomial vectors starting with those of weight k and finishing with those of weight $kn+p-1$ (the maximum weight, corresponding to the final $M_{k,s}$ jet-level). The specific ordering for vectors of equal weight is, of course, not important and is set by the computer. The entries `coeffarray_ref[col][1]` and `coeffarray_ref[col][2]`, where `col` runs through the last block of columns corresponding to the coefficients of the degree k monomial vectors, are assigned so that these monomial vectors are grouped in order of increasing weight. This means the correct ordering is automatically specified during Gaussian elimination by using `coeffarray_ref` to reference the coefficients and we need not worry further about this order.

Finally, suppose the jet f passed to `jetcalc` was a (k, s)-jet and the jet-space degree given was k. Although all of the degree k vectors in the basis for the complementary space are output when `pcomp` is called, since these vectors are ordered according to the nilpotent filtration, those in $H^{k,\tilde{s}}$, for $\tilde{s} > s$, are represented by the later columns of the matrix. The vectors which belong to $H^{k,s+1}$ must therefore form a $(k, s+1)$-transversal, as required.

Chapter 7

Profiles of Rotating Surfaces

We change the theme in this final chapter, considering the use of computer graphics to investigate projections of surfaces. The projection of a surface in $\mathbf{R}^3$ to a view plane can be considered locally as a map-germ $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$. The resulting profiles have been studied in great detail. Whitney [Wh] showed the only stable singularities are the fold (x, y^2) and the cusp $(x, xy + y^3)$ so, generically, we only expect the profiles to be smooth or have cusps. (There is a far stronger version of this result due to Mather, [Math].) This work has been taken further (for example, by allowing the view direction to vary to give a 2-parameter family of projections); see [A3, BG2, Ga4]. For a general discussion on profiles of surfaces we refer to [B1].

Instead of considering each profile in a family of projections separately, the approach we now follow is to consider a rigidly moving surface, the whole family of profiles and the resulting envelope. Such considerations are important in computer vision and in the reconstruction of the original surface from its profiles. This is because conventional reconstruction methods (see [GW, BC]) fail along the frontier. The geometry of such envelopes is hard to analyse — this is where the computer comes in, allowing us to conjecture results. We have developed programs which calculate and draw the family of profiles of a surface rotating about a fixed axis in $\mathbf{R}^3$, the envelope then becomes apparent — it is the curve 'picked out' by the eye; see the pictures in Section 7.3. For clarity, the envelope is also explicitly calculated by the computer and drawn in a separate window to the profiles. Several other geometrical objects associated to the surface are also displayed, via projection, in this window; we describe these below. Two versions of the program have been written. One takes a parametrized surface patch (specified by the user) as the original surface, while the other takes an ellipsoid in general position. In the ellipsoid case there is a method (due to P.J. Giblin) for parametrizing the critical set of the projection map and this leads to pictures of higher quality than in the general case.

The program has been used in recent research by Rycroft into the projection of surfaces, [Ryc]. We describe the basic algorithm used by the program below. Some of the results of [Ryc] are then recalled, in particular, the transitions of singular points on the envelope. We illustrate such transitions using the program. To begin with we shall review some of the geometrical features incorporated by the program. For more details we cite [Ryc] as a general reference, see also [GRP].

7.1 The Envelope of Profiles of a Rotating Surface

We restrict to the case of orthogonal (or parallel) projection onto the view plane. If the view plane is spanned by the orthonormal vectors u and v (which will be orthogonal to the view/projection direction), then the projection map is

$$\begin{aligned} \pi : \mathbf{R}^3 &\longrightarrow \mathbf{R}^2 \\ (x,y,z) &\mapsto ((x,y,z)\cdot u, (x,y,z)\cdot v). \end{aligned}$$

Let M be the surface under consideration with p a point on M. M can be written locally as an immersion $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^3, p)$. The composite map $\pi \circ f$ is sometimes called the *visual mapping.* The *critical set* of π, considered as a map from M to $\mathbf{R}^2$, will be denoted by Σ. This is just the set of critical points of the visual map $\pi \circ f$, mapped onto M by f. Equivalently, Σ consists of points $p \in M$ such that the tangent plane at p, T_pM, contains the view direction. The projection of Σ onto the view plane gives the *profile* of M (also known as the *apparent contour*, *outline* or *occluding contour* of M). Note that the profile is just the discriminant of $\pi \circ f$ (that is, the image of the set of critical points).

We now rotate M about a fixed axis in $\mathbf{R}^3$ by an angle ϕ. This gives a family of profiles Σ_ϕ, parametrized by ϕ, in the view plane and we can consider the envelope of this family. The standard way of defining an envelope applies to a family of submanifolds defined implicitly by a smooth map $F : \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$. For each $t \in \mathbf{R}$ we define $F_t(x) = F(t,x)$ and assume 0 is a regular value of F_t. Thus $F_t^{-1}(0)$ defines a family of $(n-1)$-submanifolds of $\mathbf{R}^n$. In our situation $n = 2$ and we have a family of curves. The assumption on F_t implies 0 is a regular value of F so that $F^{-1}(0)$ is an n-submanifold of $\mathbf{R}^{n+1}$. The envelope of the family F is then defined to be the profile of $F^{-1}(0)$; see [BG5, 5.1 - 5.3]. The formal definition is as follows.

Definition 7.1 The *envelope* of the family F is the set

$$\{ x \in \mathbf{R}^n : \exists t \in \mathbf{R} \quad \text{with} \quad F(t,x) = \partial F/\partial t(t,x) = 0 \} .$$

We refer to [BG5, Chapter 5] for other interpretations of the envelope. The above definition is too restrictive for a family of profiles in that the profiles $F_t^{-1}(0)$ have isolated singularities and are therefore not submanifolds of $\mathbf{R}^2$. However, if we apply definition 7.1 the resulting set is the envelope of the smooth parts of the curves $F_t^{-1}(0)$ together with the set of all singular points. We define the envelope to be the set given by definition 7.1 minus the set of singular points; see [Ryc, Remarks 3.2.2].

It is natural to ask which points of M contribute to the envelope of profiles. We define a curve on M consisting of the points whose normal line to M is coplanar with the axis of rotation. This curve will be called the S *curve*, it is independent of the view direction. Other characterisations of S are given in [Ryc, Proposition 3.4.5]. The importance of S is shown in [Ryc, Proposition 3.4.2]; we find that it is the points of S which contribute to the envelope of profiles.

As the surface M rotates we only see (as an outline) the part of the surface which is swept out by the critical sets. This leads to a very useful geometrical notion. We define a point $p \in M$ to be a *visible point* if it lies on a critical set Σ_ϕ for some value of ϕ, and define it to be a *non-visible point* otherwise. Such points form the *visible* and *non-visible regions* of the surface; the boundary between these regions is known as the *frontier* of the surface. (Strictly speaking, the frontier cannot be defined in this way. For circular motion the frontier is a simple curve (or pair of curves) and the 'boundary' idea applies, but in general it may intersect and a different definition must be used. However, we will not explicitly work with these ideas and can overlook them.)

It is the visible part of S which contributes to the envelope of profiles. We can be more precise about the relationship between the envelope of projections of S and the envelope of profiles. A *special point* of S is defined to be a point of S where the normal plane to S contains the axis of rotation. Any point on M describes a circle in $\mathbf{R}^3$ as M rotates. The projection of a circle described by a special point to the view plane is called a *special ellipse*; such ellipses form part of the envelope of projections of S. It is shown in [Ryc, Corollary 3.6.5] that

$$\begin{array}{c}\text{envelope of}\\ \text{projections of } S\end{array} = \text{envelope of profiles} + \text{special ellipses.}$$

Finally we note that the envelope of profiles is symmetric about the projection of the axis of rotation to the view plane (see [Ryc, Theorem 3.7.1]). This is important for technical considerations such as the recovery of the axis from the set of profiles, also geometrically in that certain transitions of singular points on the envelope only occur on the axis of symmetry. The symmetry is clearly depicted in the computer pictures.

All the features noted in this section are incorporated in our program. We discuss this now.

7.2 Computer Generation of the Family of Profiles and its Envelope

A brief description of the program and algorithm used to calculate the geometrical features is given. We restrict to the program which deals with the case M an ellipsoid. The version which deals with surface patches uses similar algorithms, we just need to modify the methods for obtaining the tangent plane and normal vectors to M and the S curve. The main difference is that we cannot parametrize the critical sets and the profile cannot be drawn as a set of points connected by line segments. The pictures are generally good, but we intend to improve the techniques in future work.

The program is written in C and runs on a Silicon Graphics machine. The Silicon Graphics GL graphics library is used — this is a powerful graphics library and allows us to take advantage of 'overlay screens'. This means the projection of the S curve can be superimposed over the envelope of profiles and by rotating M we produce an animated display where S is seen to trace out the envelope, as predicted. We also incorporate a window control panel library, written by R.J. Morris at the Liverpool University Department of Pure Mathematics. This allows parameters to be changed interactively through the use of windows and a mouse.

The ellipsoid M in general position is specified as follows. The parameters a,

b and c give the standard position ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which we will denote by E. An axis through the origin is specified (via its latitude and longitude in degrees) together with a rotation angle α. Let A_α denote the rotation transformation about this axis. Finally a translation vector s is specified. The ellipsoid E is moved to general position by applying A_α and translating by s, thus

$$M = A_\alpha(E) + s.$$

Similarly, the axis of rotation, also through the origin, about which we are to revolve M is specified by the user. If ϕ denotes the angle of rotation and B_ϕ the corresponding transformation then we will allow ϕ to vary and give the family of ellipsoids

$$B_\phi(M) = B_\phi(A_\alpha(E)) + B_\phi(s).$$

All these parameters may be changed interactively by the user via the control panel window. In addition, the family of profiles, the envelope of the profiles, and an animated display of the projections of the S curve for varying ϕ may be drawn by clicking the mouse on the appropriate button in the control window. Other parameters, such as the step length for ϕ used when drawing the family of profiles, may also be changed. The control window also incorporates a store/recall facility for saving the values of all parameters to a file for future use.

7.2.1 Calculating the Critical Sets and Profiles

The critical set $\Sigma \subset M$ of the projection map is calculated as follows. Throughout the rotation we will project M to the (x, y)-plane, the view direction is therefore $(0, 0, 1)$. The calculation is performed by firstly moving the ellipsoid to standard position. A translation gives the ellipsoid $B_\phi(A_\alpha(E))$, centred at the origin. We then rotate and work with E, the view direction becomes $u = (u_1, u_2, u_3) = A_\alpha^{-1}(B_\phi^{-1}(0, 0, 1))$. Let $\tilde{\Sigma}$ be the critical set of the projection map of E in direction u. $\tilde{\Sigma}$ is the set of points $(x, y, z) \in E$ such that the tangent plane to E at (x, y, z) contains the view direction u:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \\ \frac{xu_1}{a^2} + \frac{yu_2}{b^2} + \frac{zu_3}{c^2} &= 0 \end{aligned} \tag{7.1}$$

We can parametrize $\tilde{\Sigma}$ using the following observation made by P.J. Giblin. Change coordinates, thus

$$X = x/a, \quad Y = y/b, \quad Z = z/c,$$

$$U_1 = u_1/a, \quad U_2 = u_2/b, \quad U_3 = u_3/c,$$

and put $U = (U_1, U_2, U_3)$. Then equations 7.1 become

$$\begin{aligned} X^2 + Y^2 + Z^2 &= 1 \\ XU_1 + YU_2 + ZU_3 &= 0 \end{aligned} \tag{7.2}$$

The solution to equations 7.2 is the intersection of a unit sphere with the plane through 0 perpendicular to U. This is a unit circle and can be parametrized by

$$\theta \mapsto v\cos\theta + w\sin\theta,$$

where $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are unit vectors perpendicular to U. It is an easy matter to explicitly write down a vector v perpendicular to U and then define w using the cross product $w = U \times v$. (It is wise to normalise U before doing such calculations to reduce the possibility of numerical errors.) Changing coordinates back again gives the following parametrization of $\tilde{\Sigma}$.

$$\theta \mapsto (x, y, z)$$

where

$$\begin{aligned} x &= a(v_1\cos\theta + w_1\sin\theta) \\ y &= b(v_2\cos\theta + w_2\sin\theta) \\ z &= c(v_3\cos\theta + w_3\sin\theta) \end{aligned} \tag{7.3}$$

for $\theta \in [0, 2\pi)$. The critical set of M is obtained by applying the original transformations

$$\Sigma = B_\phi(A_\alpha(\tilde{\Sigma})) + B_\phi(s),$$

and the profile by applying the projection map

$$\pi : \mathbf{R}^3 \longrightarrow \mathbf{R}^2, \qquad (x, y, z) \mapsto (x, y)$$

to Σ.

The above process can be performed efficiently by computer. Firstly we calculate the matrix A_α and its inverse. For each value of ϕ we calculate the matrix B_ϕ and its inverse, then put $u = A_\alpha^{-1}(B_\phi^{-1}(0,0,1))$ and obtain the two vectors v and w perpendicular to u. Finally calculate $B_\phi(A_\alpha(x,y,z)) + B_\phi(s)$ for (x,y,z) given by equations 7.3 and plot the projection of these points to the (x,y)-plane, connected by line segments, by allowing θ to vary from 0 to 2π.

7.2.2 Calculating the Envelope of the Profiles

It is not obvious how to determine whether a point contributes to the envelope of profiles using definition 7.1. However, in the present scenario (where we are considering the envelope of profiles of a rotating surface) there is a characterisation of the envelope suitable for implementation by computer.

From [Ryc, Proposition 3.4.2] we see that for a given angle ϕ, a point $p \in B_\phi(M)$ contributes to the envelope (that is $\pi(p)$ lies on the envelope) if and only if p lies on S and belongs to the critical set Σ_ϕ of $B_\phi(M)$. We must therefore calculate the profile of each transformation $B_\phi(M)$ of M as ϕ is varied from 0 to 360 degrees. For a given angle ϕ, a set of points lying on the profile of $B_\phi(M)$ is calculated as described in the previous section. Suppose (x, y, z) lies on the critical set $\tilde{\Sigma}$ of the ellipsoid in standard position; (x, y, z) is calculated using equations 7.3. The corresponding point on M is $A_\alpha(x, y, z) + s$ and lies on S if and only if the normal line to M through this point is coplanar with the axis of rotation. This is an incidence property of affine subspaces of $\mathbf{R}^3$ and is invariant under affine transformation. The axis of rotation will be specified by the unit vector $r = (r_1, r_2, r_3)$ and $n = (n_1, n_2, n_3)$ will denote the unit normal to the ellipsoid E in standard position at the point (x, y, z). Applying the rotation A_α^{-1} the required condition is that the line

$$\{\, (x, y, z) + A_\alpha^{-1}(s) + \lambda n : \lambda \in \mathbf{R} \,\}$$

is coplanar with the line

$$\{\, \lambda A_\alpha^{-1}(r) : \lambda \in \mathbf{R} \,\}.$$

We can write down the vector n explicitly — it is the vector $(x/a^2, y/b^2, z/c^2)$ normalised to unit length. Now it is a standard result from linear algebra that a line l_1 through a point a_1 in direction v_1 is coplanar with a line l_2 through a point a_2 in direction v_2 if and only if the triple scalar product $[a_1 - a_2, v_1, v_2] = (a_1 - a_2) \cdot (v_1 \times v_2)$ vanishes. We can now check whether or not the point $A_\alpha(x, y, z) + s$ lies on S. If it does then its image under the rotation B_ϕ contributes to the envelope.

The computer performs the following process. Firstly, it calculates the matrix A_α, its inverse and the vectors $(r_1', r_2', r_3') = A_\alpha^{-1}(r)$ and $(s_1', s_2', s_3') = A_\alpha^{-1}(s)$. Then for each value of ϕ ($0 \leq \phi \leq 360$ degrees) it calculates the matrix B_ϕ, its inverse, and a set of points $\{(x, y, z)\} \subset E$ given by equations 7.3, as described in the previous section. For each of these points (x, y, z) the normal vector n is calculated and then the computer checks for the vanishing of the determinant

$$\begin{vmatrix} x+s'_1 & y+s'_2 & z+s'_3 \\ n_1 & n_2 & n_3 \\ r'_1 & r'_2 & r'_3 \end{vmatrix} \qquad (7.4)$$

When this determinant vanishes the point

$$B_\phi(A_\alpha(x,y,z)) + B_\phi(s)$$

is calculated and its projection under π, a point which lies on the envelope of profiles, is plotted.

Remark. Several comments must be made. Firstly note that ϕ is varied between 0 and 360 degrees using a finite number of steps, as is θ (used to parametrize the points (x, y, z) given by equations 7.3) for each value of ϕ. This depicts the envelope as a finite set of isolated points. The points are not produced in an order suitable for joining them up by line segments, as in the case of drawing the profiles. We must therefore be contented with isolated points. Generally the resolution produced is very good and depicts the envelope well. However, to ensure this is always the case the user is allowed to specify the step size for ϕ — the smaller the step size the more points used to draw the envelope (though the more computer time required). (The step size for θ could, in theory, be user specified too but, in practice, it takes a relatively small value fixed by the computer and the user need only specify the ϕ step size.) The next problem is that the determinant 7.4 will rarely be exactly zero for the isolated points (x, y, z) produced. We therefore use the criterion $|\det| < \epsilon_{env}$, for some (small) number ϵ_{env}, to define the envelope points. In practice, the default value is fine, but again we allow the user to modify this parameter to ensure a good picture is obtained. The parameters 'ϕ step' and ϵ_{env} may be modified interactively by the user via the control panel window and the mouse.

7.2.3 Calculating the S Curve, Special Points and Visible Points

The calculation of the S curve is now an easy matter. The method used to calculate the envelope in the previous section is used, only now we must consider all the points on the ellipsoid instead of restricting to points (x, y, z) on critical sets.

We will generate points (x, y, z) on the ellipsoid E in standard position by intersecting it with a family of planes parallel to one of the coordinate planes. The parameter S_{num} will be an integer, greater than 0, which determines the number of planes to slice E with. (The actual number used will be $2S_{num} - 1$.) We will use planes parallel to the (y, z)-plane, namely the planes

$$x = -a + \frac{ia}{S_{num}} \qquad \text{where} \qquad i = 1, 2, \ldots, 2S_{num} - 1.$$

(The extremities $\{x = \pm a\} \cap E = \{\pm a\}$ are of little use.) The intersection of such a plane with E is an ellipse which can be parametrized. One easily checks that the p oints (x, y, z) on the intersection are given by

$$\begin{aligned} x &= -a + \frac{ia}{S_{num}} \\ y &= mb\cos t \\ z &= mc\sin t \end{aligned} \tag{7.5}$$

where

$$m = \sqrt{\frac{i}{S_{num}}\left(2 - \frac{i}{S_{num}}\right)}$$

and $0 \leq t < 2\pi$.

The S curve is calculated using the process described for the envelope calculation, only now we try to find all the points of S (or, more specifically, a reasonable representation for the set of all points of S — those which lie on a family of planes parallel to a given coordinate plane). In the envelope calculation we restricted to points which lie on the critical set. The computer now performs the same calculations only generates the points (x, y, z) using equations 7.5 instead of equations 7.3. As before, the vanishing of the determinant 7.4 is replaced by the criterion $|\det| < \epsilon_S$. The parameters ϵ_S and S_{num} may be modified using the control panel window to improve the resolution of the picture. (The points lying on S are not produced in an order suitable for joining them together with line segments.) The computer actually performs two sweeps, the first using planes parallel to the (y, z)-plane, the second using planes parallel to the (x, z)-plane. This usually produces a good image for S.

The above process produces a set of points $\{A_\alpha(x, y, z) + s\}$ lying on $S \subset M$, it is the computationally demanding part of the calculation. Once calculated these points are stored in an array. Now S is invariant under rotation about the axis r, that is, the S curve for $B_\phi(M)$ is just $B_\phi(S)$. We obtain an image for $B_\phi(S)$ by applying B_ϕ to each of the previously stored points $A_\alpha(x, y, z) + s$; the computational aspects of this are minimal and the new curve $B_\phi(S)$ is produced almost instantaneously. This allows an animated sequence of the projections of the S curve to the (x, y)-plane to be viewed, one frame for each value of ϕ. The sequence is shown in the envelope window. Technically, it is displayed on an overlay screen for the window so that the S curve can be seen to 'sweep out' the envelope of profiles in real time without 'rubbing out' the envelope as it moves around. The sequence is displayed continually for $0 \leq \phi < 360$ degrees until the user clicks the mouse on the 'stop' button in the control window. Other features include a 'pause' option to freeze the display temporarily and the ability to change the delay time between frames and the step size for ϕ (again, via parameters in the control window).

Finally we discuss special points and visible points. The special points only occur on the S curve and we need only show the visible points of the ellipsoid which lie on S. These points will be shown under projection of S to the view plane (in the envelope window). Visible points are coloured white while non-visible points are coloured green. The special points are generally isolated so are exaggerated and displayed as pink 'blobs' (the 'blob' size may be changed interactively to suit the user's tastes!). These features are invariant under rotation about the axis r so the attributes are assigned during the algorithm which calculates the points on S — a fourth coordinate is assigned to each point which is a flag indicating whether the point should be displayed as visible, non-visible or special.

A point p on S is special if the normal plane to S at p contains the axis of rotation. Provided the normal to the surface M at p is not parallel to the axis of rotation this is equivalent to the tangent vector T to S at p being perpendicular to the axis of rotation. For clarity we recall that the normal plane to S at p is the plane *through* p spanned by two independent vectors perpendicular to T. Hence, if the normal plane contains the axis of rotation, the axis must be perpendicular to T. Provided the normal vector to M at p is not parallel to the axis of rotation, the normal plane to S at p must meet the axis of rotation (along the surface normal line through p, for example) since $p \in S$. Thus, if the axis of rotation is perpendicular to T then the axis must be contained in the normal plane. As we shall see, the condition 'T is perpendicular to the axis of rotation' is computationally viable and we will take this as our criterion for obtaining the special points. The view taken regarding the 'imposter' special points (where, in addition, the normal to M at p is parallel to the axis of rotation) is that such points are generally too unlikely to be of any concern, at least in what is just a graphical representation.

Let T_pS denote the tangent space to S at p, it is spanned by the vector T. Let N be the unit normal vector to M at p. Suppose $N \times r \neq 0$ then

$$N \times r \in T_pS \iff r \cdot T = 0.$$

(For the implication $\Rightarrow$ we just note that $N \times r$ is a non-zero vector perpendicular to N and r. The reverse implication then follows from the fact that $N \cdot T = 0$ always holds.) Now, $N \times r = 0$ just says that the normal vector to M at p is parallel to the axis of rotation. Having agreed to include such occurrences in our criterion for special points we are reduced to checking

$$N \times r \in T_pS.$$

Applying transformations which take $M = A_\alpha(E) + s$ onto the standard position ellipsoid E we find that S can be written as the zero set $F^{-1}(0)$ of the map

$$F : \mathbf{R}^3 \longrightarrow \mathbf{R}^2, \qquad F = (F_1, F_2),$$

where

$$\begin{aligned} F_1(x,y,z) &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \\ F_2(x,y,z) &= [(x,y,z) + A_\alpha^{-1}(s), n, A_\alpha^{-1}(r)], \end{aligned}$$

in the notation introduced for the envelope calculation earlier; in particular, n is the unit normal vector to E at (x,y,z) and may be explicitly written down. The tangent space to this curve at (x,y,z) is just $\mathrm{Ker}\,(dF(x,y,z))$. With $r' = A_\alpha^{-1}(r)$ the condition therefore reduces to

$$n \times r' \in \mathrm{Ker}\,(dF(x,y,z)) = \mathrm{Ker}\begin{pmatrix} dF_1(x,y,z) \\ dF_2(x,y,z) \end{pmatrix}.$$

But $\mathrm{Ker}\,(dF_1(x,y,z)) = T_{(x,y,z)}E$ and $n \times r' \in \mathrm{Ker}\,(dF_1(x,y,z))$ follows automatically. We therefore only need to check

$$dF_2(x,y,z)\,(n \times r') \;=\; 0 \qquad (7.6)$$

Expanding the above product 7.6 gives a very complex formula (after all, one must differentiate a triple scalar product) but the point now is that we have an explicit condition for the computer to check. As the computer calculates the S curve, considering points (x, y, z) produced by equations 7.5, it takes any point found to lie on S and checks

$$| \, dF_2(x, y, z) \, (n \times r') \, | < \epsilon_{Sp}$$

for some (small) number ϵ_{Sp}. When this occurs the flag associated to the point (x, y, z) is set to indicate a special point. As always, the parameter ϵ_{Sp} may be changed via the control window.

Finally we consider the visible points of S. Let β be the angle between the axis of rotation and the view plane, this is called the *slant* of the axis. We can assume $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ without loss of generality. From [Ryc, Proposition 3.5.3] we have a point $p \in M$ is visible if and only if $|N \cdot r| \leq \cos\beta$, where N is the normal to M at p. Now $M = A_\alpha(E) + s$ and $p = A_\alpha(x, y, z) + s$ for $(x, y, z) \in E$. Let n be the unit normal to E at (x, y, z), $r' = A_\alpha^{-1}(r)$ and β' the angle between r' and the plane obtained by applying A_α^{-1} to the (x, y)-plane. Then an equivalent condition is $|n \cdot r'| \leq \cos\beta'$ or $|n \cdot r'|^2 \leq \cos^2\beta'$. But $\beta' = \beta$ and β can be obtained using $(0, 0, 1) \cdot r = \sin\beta$ (since $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$). Thus, for each point (x, y, z) given by equations 7.5 which lies on the S curve, the computer checks

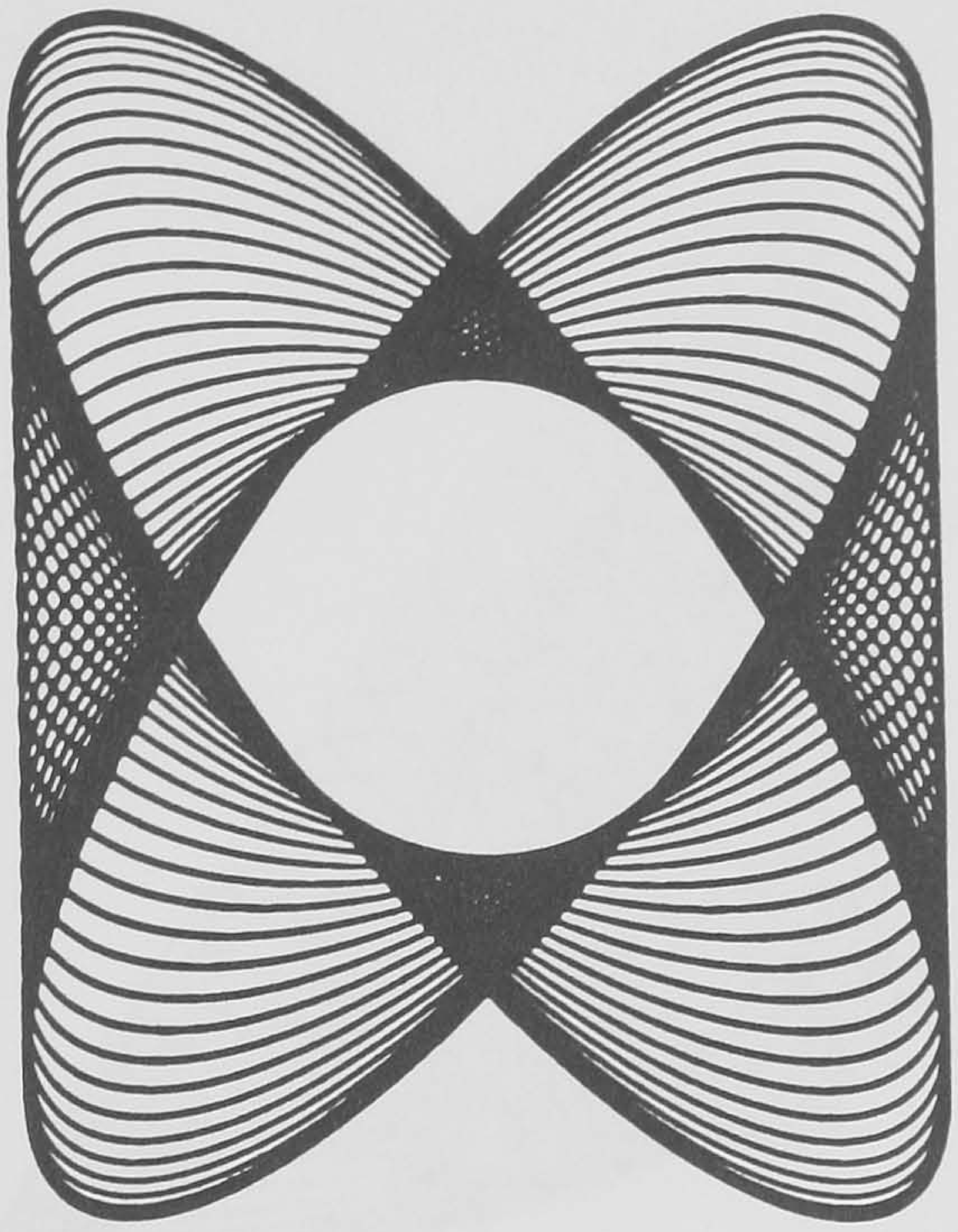
$$|n \cdot r'|^2 + |(0, 0, 1) \cdot r|^2 \leq 1.$$

When this occurs the flag associated to the point (x, y, z) is set to indicate a visible point, otherwise it indicates a non-visible point. One therefore sees the projections of S divided into white and green (i.e., visible and non-visible) regions with isolated special points.

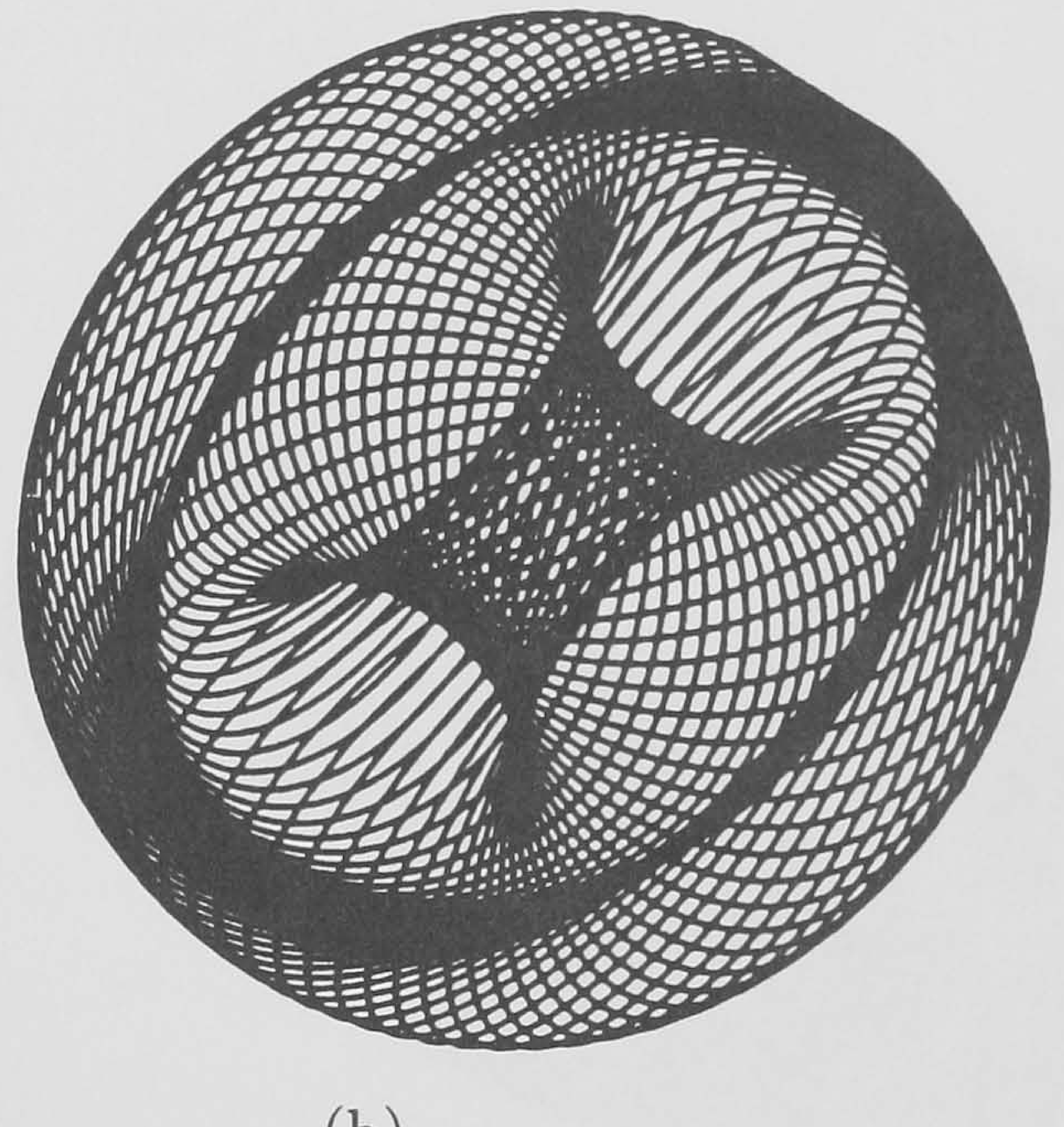
7.3 Examples; Transitions of the Singular Points on the Envelope

We now give some examples produced by the program together with the relevant mathematical background. We just give pictures of the profiles for a rotating ellipsoid, the resulting envelope is generally clear; see Figures 7.1 and 7.2. (In practice, we get the computer to draw the envelopes as well. This helps clarify the resulting singularities and their transitions.)

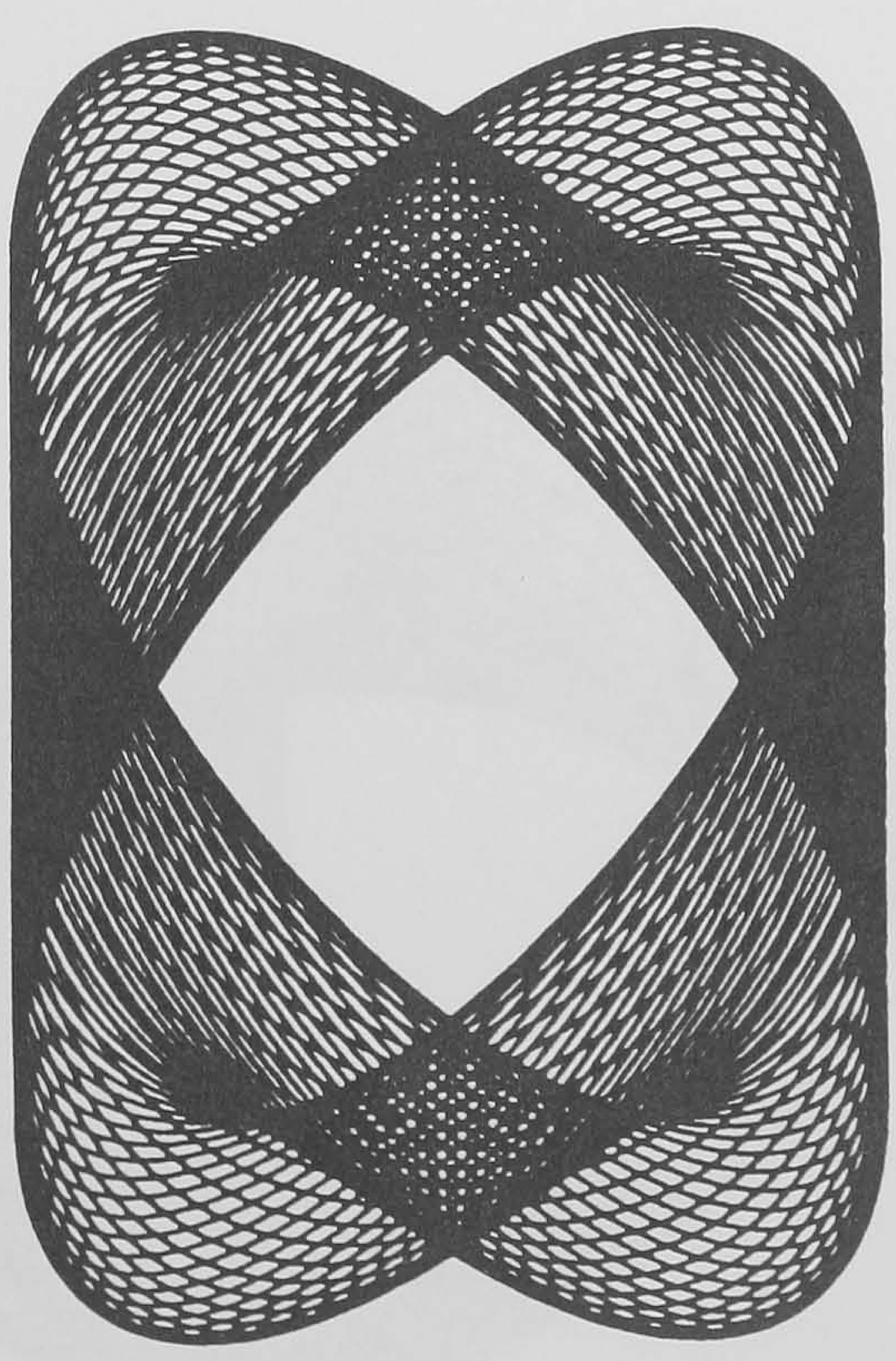
Many of the results proved by Rycroft should be apparent, for example, the symmetry of the envelope and the transitions in the singularities. The animation



(a)



(b)



(c)

Figure 7.1: Profiles I

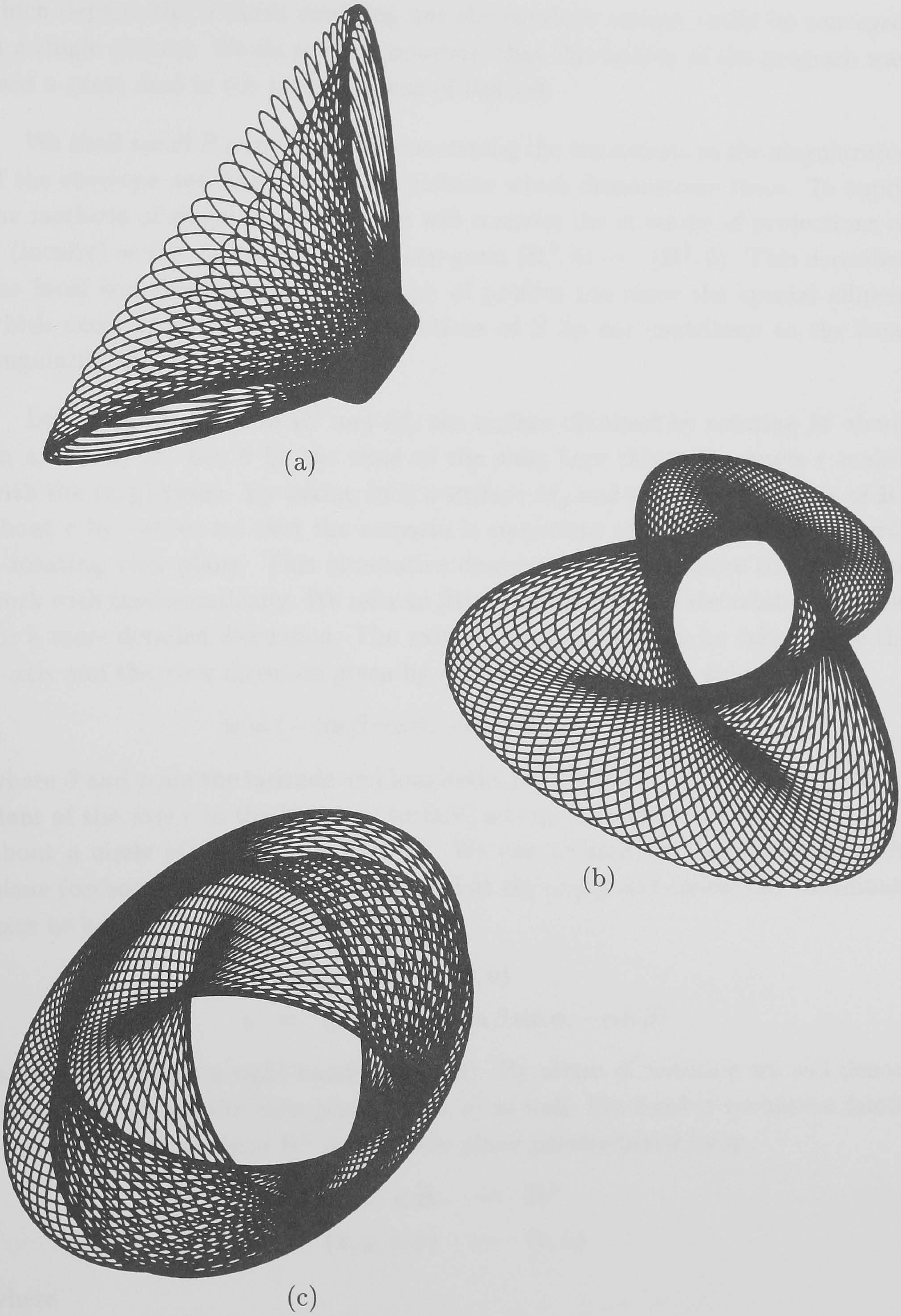


Figure 7.2: Profiles II

which depicts the S curve sweeping out the envelope cannot really be conveyed in a single picture. We do remark, however, that this facility of the program was used a great deal in the investigations of Rycroft.

We shall recall Rycroft's results concerning the transitions in the singularities of the envelope and finish by giving pictures which demonstrate these. To apply the methods of singularity theory we will consider the envelope of projections of S (locally) as the discriminant of a map-germ $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$. This describes the local singularities on the envelope of profiles too since the special ellipses which occur in the envelope of projections of S do not contribute to the local singularity theory.

Let M be a surface in $\mathbf{R}^3$ and M_ϕ the surface obtained by rotating M about an axis r by ϕ. Let β be the slant of the axis; here this is the angle r makes with the (x, y)-plane. By taking such a surface M_ϕ and rotating the whole of $\mathbf{R}^3$ about r by $-\phi$ we see that the scenario is equivalent to a fixed surface M with a rotating view plane. This alternative description is often more convenient to work with mathematically. We refer to [Ryc, Section 3.3] and the related material for a more detailed discussion. The axis of rotation will then be taken to be the z-axis and the view direction given by a vector w on the unit sphere,

$$w = (-\cos\beta\cos\phi, -\cos\beta\sin\phi, -\sin\beta),$$

where β and ϕ are the latitude and longitude, respectively. The angle β is just the slant of the axis r in the 'rotating surface' set-up. The view direction will rotate about a circle of latitude for fixed β. We can assume the corresponding view plane (orthogonal to w) is always centred at the origin and choose the coordinate axes to be spanned by the unit vectors

$$\begin{aligned} u &= (-\sin\phi, \cos\phi, 0) \\ v &= (\sin\beta\cos\phi, \sin\beta\sin\phi, -\cos\beta) \end{aligned}$$

(so (u, v, w) forms a right-handed system). By abuse of notation we will denote the coordinates in the view plane by (u, v) as well. For fixed β we have a family of projection maps from $\mathbf{R}^3$ to the view plane parametrized by ϕ:

$$\begin{aligned} \mathbf{R}^3 \times \mathbf{R} &\longrightarrow \mathbf{R}^2 \\ (x, y, z, \phi) &\mapsto (u, v) \end{aligned}$$

where

$$\begin{aligned} u &= -x\sin\phi + y\cos\phi \\ v &= x\sin\beta\cos\phi + y\sin\beta\sin\phi - z\cos\beta. \end{aligned}$$

Figure 7.3: Fold and Cusp

M can be defined locally as the graph of a function $z = h(x, y)$ and (provided it is smooth) S may be parametrized as $t \mapsto (x(t), y(t), h(x(t), y(t)))$. The projections of S to the view plane are then given by the map

$$\begin{aligned} f : \mathbf{R} \times \mathbf{R} &\longrightarrow \mathbf{R}^2 \\ (t, \phi) &\mapsto (u, v) \end{aligned}$$

where

$$\begin{aligned} u &= -x(t) \sin \phi + y(t) \cos \phi \\ v &= x(t) \sin \beta \cos \phi + y(t) \sin \beta \sin \phi - h(x(t), y(t)) \cos \beta. \end{aligned}$$

The point now is that the envelope of the projections of S is given by the discriminant of f, $f(\Sigma)$, where Σ is the set of critical points of f; see [Ryc, Remark 3.2.4] and [BG5, P.84]. The phenomena of codimension ≤ 1 are described by the following singularities (using (x, y) as coordinates on $\mathbf{R}^2$ for convenience).

(x, y^2)	fold	stable
$(x, xy + y^3)$	cusp	stable
$(x, xy + y^4)$	swallowtail	$\mathcal{A}_e$-codim $= 1$
$(x, x^2y + y^3)$	lips	$\mathcal{A}_e$-codim $= 1$
$(x, x^2y - y^3)$	beaks	$\mathcal{A}_e$-codim $= 1$

The fold and cusp maps are in Figure 7.3. (In all diagrams representing map-germs $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$ we actually draw their discriminant, $f(\Sigma)$.)

When the projection map f above is a fold map (as a germ, to be precise) the corresponding point on the envelope is smooth, while when it is a cusp map we obtain isolated cusp singularities on the envelope. Rycroft gave several conditions for the occurrence of cusps on the envelope; [Ryc, Corollary 4.4.2, Proposition 4.4.5, Proposition 4.4.7]. Indeed, generically, we expect to find isolated cusps on the envelope; see [BG5, Section 7.13] (this is just Whitney's result referred

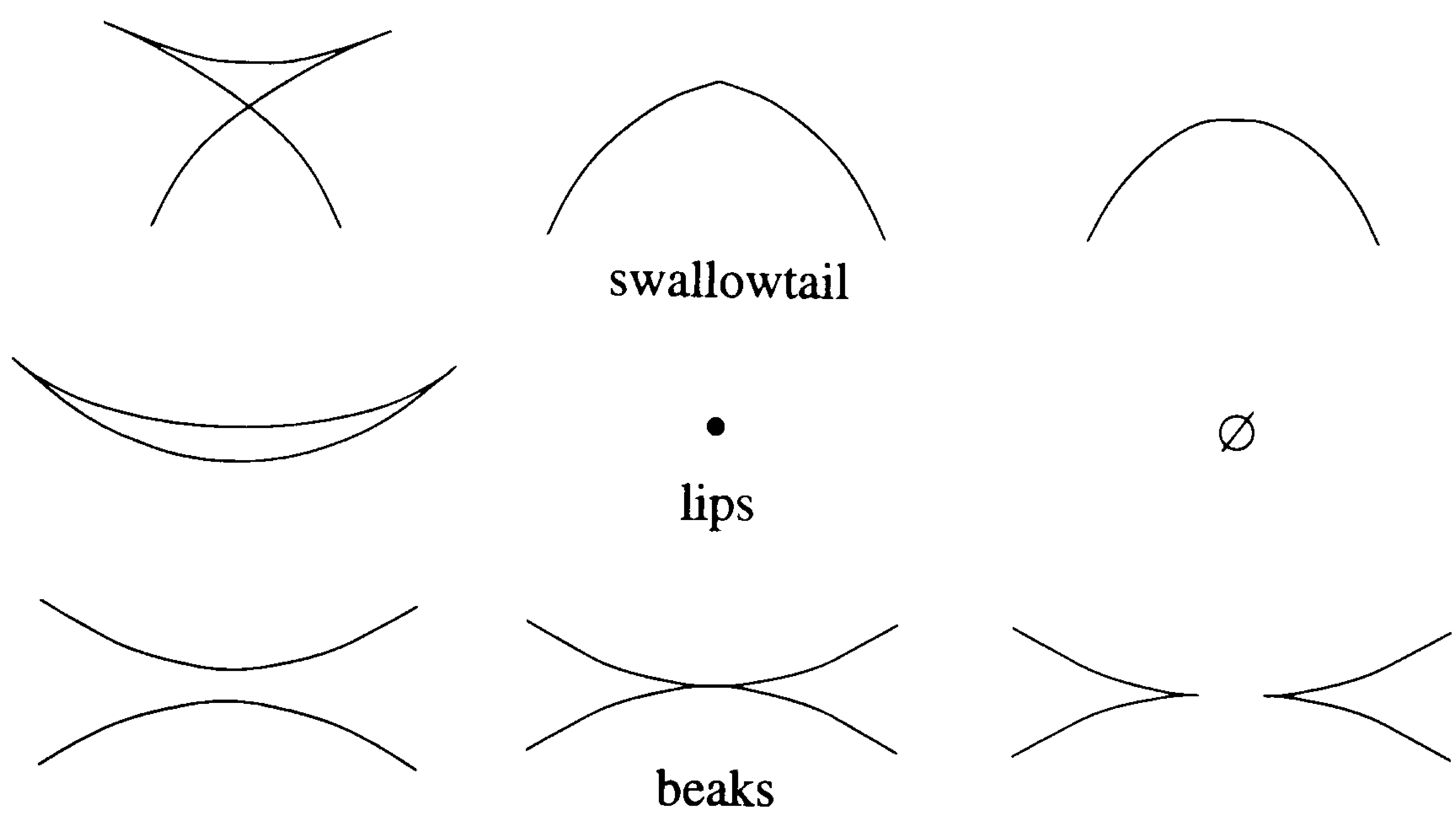


Figure 7.4: Codimension 1 Singularities

to at the start of this section). Such singularities were shown in our computer generated pictures of the envelopes in Figures 7.1 and 7.2.

We now turn our attention to the codimension 1 transitions. The maps

$$\begin{aligned}(x,y,u) &\mapsto (x, xy + y^4 + uy^2, u)\\ (x,y,u) &\mapsto (x, x^2y \pm y^3 + uy, u)\end{aligned}$$

are versal unfoldings of the swallowtail, lips and beaks maps and give the transitions shown in Figure 7.4.

Rycroft considered 1-parameter families of envelopes by varying the slant β and determined the possible transitions in the singularities. In particular, conditions for the envelope to have on-axis and off-axis swallowtail transitions were obtained (Propositions 4.7.2 and 4.7.8), together with on-axis lips/beaks transitions (Proposition 4.6.2); no off-axis lips/beaks transitions occur (Proposition 4.6.8). By on-axis/off-axis we mean the phenomena occur on/off the axis of symmetry of the envelope (the projection of the axis of rotation to the view plane). These results are proved by applying the recognition criteria of [Tar, Chapter 3], to the map-germ $f : (t, \phi) \longrightarrow (u, v)$ defined earlier. Rycroft also gave conditions for the map-germ

$$\begin{aligned}\mathbf{R}^2 \times \mathbf{R} &\longrightarrow \mathbf{R}^2 \times \mathbf{R}\\ (t,\phi,\beta) &\mapsto (u,v,\beta)\end{aligned}$$

to be a versal unfolding (in β) of the above codimension 1 singularities, at least generically (Propositions 4.6.4 and 4.7.5). Thus, by varying the slant β we can obtain computer pictures which exhibit the swallowtail, lips and beaks transitions. We conclude with such pictures (Figures 7.5 — 7.7). In addition, Figure 7.1(b) shows a lips transition, Figure 7.2(c) a beaks transition and Figure 7.2(b) several swallowtail transitions, including off-axis swallowtails. (Strictly speaking, these just depict the aftermath of such transitions.) The transitions are picked out clearly in the computer produced pictures of the envelopes, unfortunately we could not produce a reasonable PostScript version of these pictures. To make things a bit clearer we have oriented the figures so that the transitions always appear in the top left-hand corner of the image.

Computer investigations have led to many possibilities for future work. For example, we would expect the cusp singularities on the envelope to be ordinary, yet several pictures suggest they are of a rhamphoid nature. This is shown in the 'big smile' (or rather the aftermath of a lips transition) in Figure 7.1(b) where the cusps are extremely sharp and beaked. The animated sequence showing the projections of S suggests that the trajectories of the special points always pass through these cusps at the ends of the smiles. This will hopefully become clearer with future work.

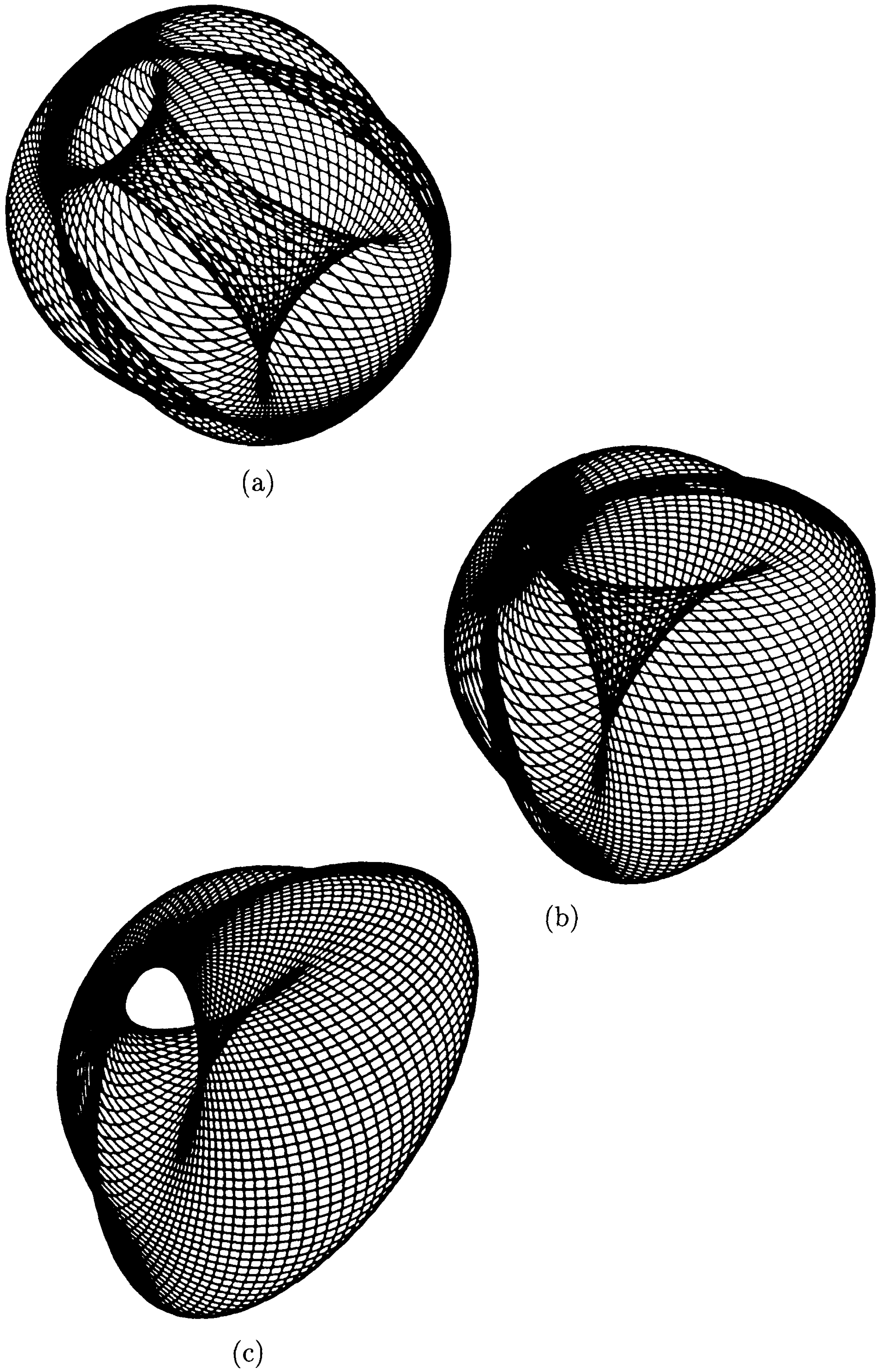
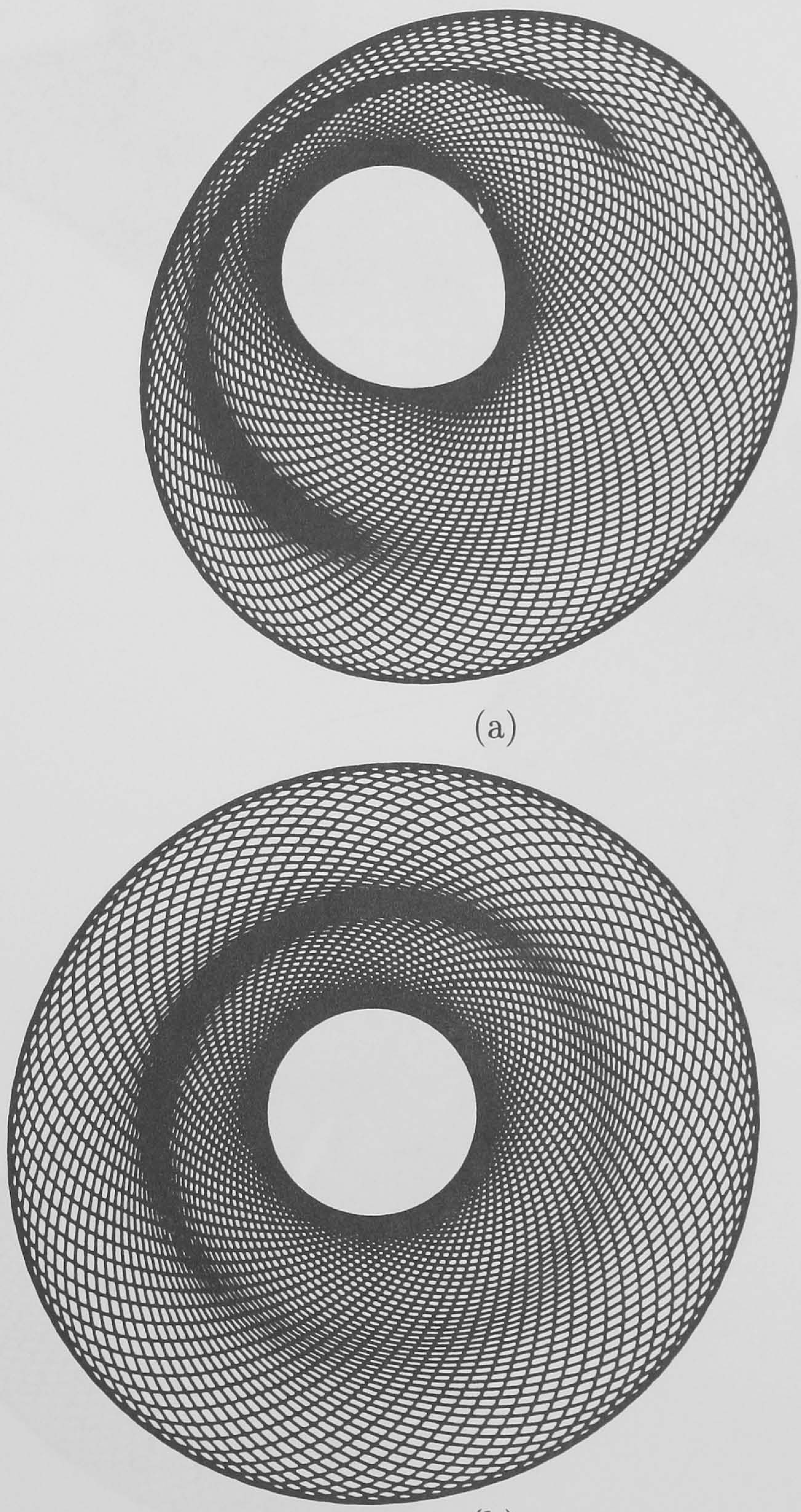


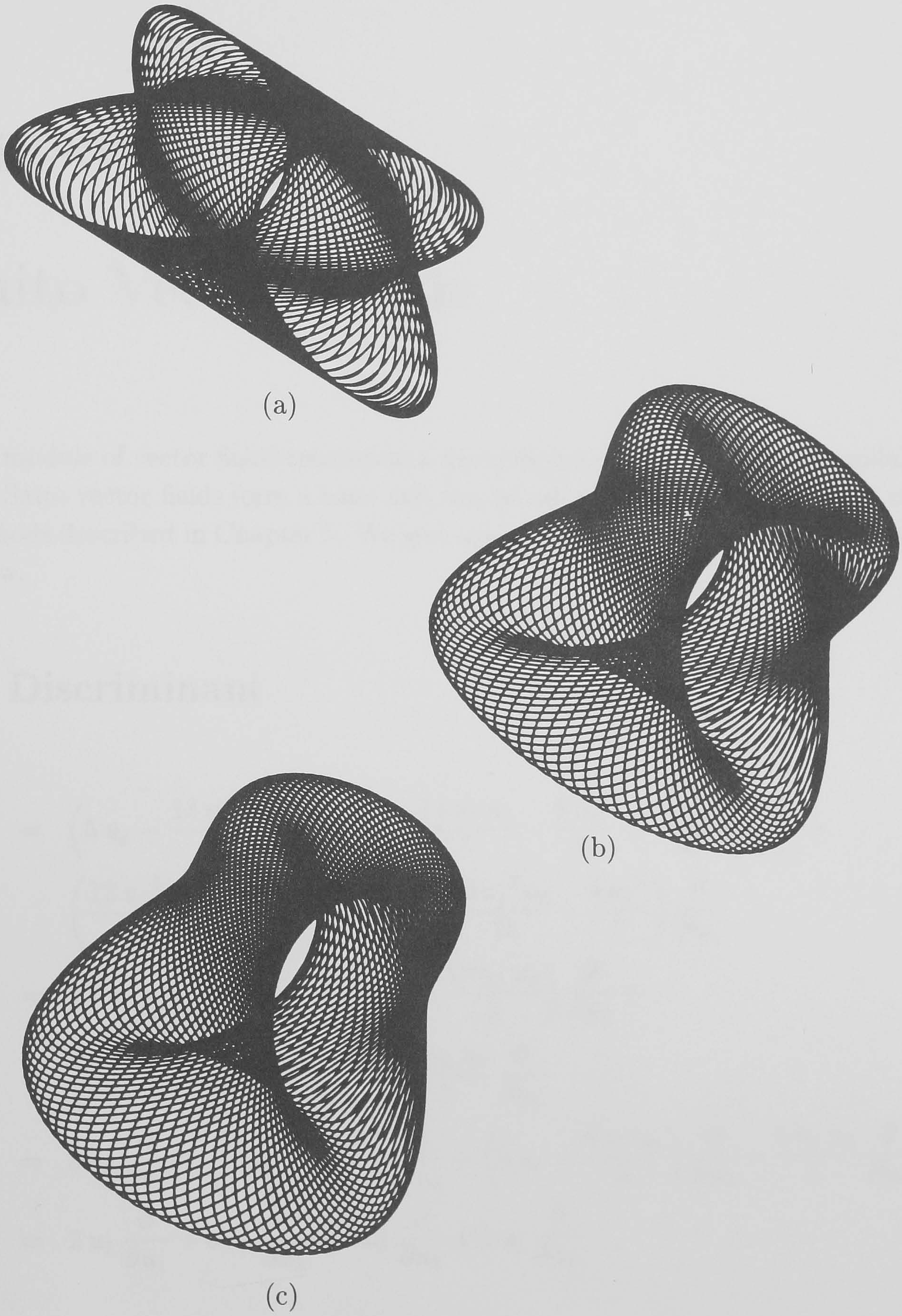
Figure 7.5: Beaks Transition



(a)

(b)

Figure 7.6: Lips Transition



(a) (b) (c)

Figure 7.7: Swallowtail Transition

Appendix A

Saito Vector Fields

The module of vector fields tangent to a discriminant variety is a free $\mathcal{O}_p$-module. The Saito vector fields form a basis and can be calculated by computer using the methods described in Chapter 5. We give some of the results of these calculations below.

A_4 Discriminant

$$\begin{aligned}
\theta_1 &= \left(5\,u_4 - \frac{13\,u_1\,u_2}{5}\right)\frac{\partial}{\partial u_1} + \left(-\frac{14\,u_1\,u_3}{5} - \frac{6\,u_2^{\,2}}{5} + \frac{18\,u_1^{\,3}}{25}\right)\frac{\partial}{\partial u_2} + \\
&\quad \left(\frac{12\,u_1^{\,2}u_2}{25} - \frac{11\,u_2\,u_3}{5}\right)\frac{\partial}{\partial u_3} + \left(\frac{6\,u_1^{\,2}u_3}{25} - \frac{4\,u_3^{\,2}}{5}\right)\frac{\partial}{\partial u_4} \\
\theta_2 &= \left(4\,u_3 - \frac{6\,u_1^{\,2}}{5}\right)\frac{\partial}{\partial u_1} + \left(5\,u_4 - \frac{13\,u_1\,u_2}{5}\right)\frac{\partial}{\partial u_2} + \\
&\quad \left(-\frac{2\,u_1\,u_3}{5} - \frac{6\,u_2^{\,2}}{5}\right)\frac{\partial}{\partial u_3} - \frac{3\,u_2\,u_3}{5}\frac{\partial}{\partial u_4} \\
\theta_3 &= 3\,u_2\frac{\partial}{\partial u_1} + \left(4\,u_3 - \frac{6\,u_1^{\,2}}{5}\right)\frac{\partial}{\partial u_2} + \left(5\,u_4 - \frac{4\,u_1\,u_2}{5}\right)\frac{\partial}{\partial u_3} - \frac{2\,u_1\,u_3}{5}\frac{\partial}{\partial u_4} \\
\theta_4 &= 2\,u_1\frac{\partial}{\partial u_1} + 3\,u_2\frac{\partial}{\partial u_2} + 4\,u_3\frac{\partial}{\partial u_3} + 5\,u_4\frac{\partial}{\partial u_4}
\end{aligned}$$

D_4 Discriminant

$$\theta_1 = \left(u_4 - \frac{5\,u_1\,u_3}{9} + \frac{4\,u_1{}^3}{27}\right)\frac{\partial}{\partial u_1} + \left(\frac{u_3\,u_2}{9} - \frac{u_1{}^2u_2}{27}\right)\frac{\partial}{\partial u_2} + \left(-\frac{u_2{}^2}{3} - \frac{2\,u_3{}^2}{9} + \frac{2\,u_1{}^2u_3}{27}\right)\frac{\partial}{\partial u_3} - \frac{u_1\,u_2{}^2}{36}\frac{\partial}{\partial u_4}$$

$$\theta_2 = -2\,u_2\,\frac{\partial}{\partial u_1} + u_4\,\frac{\partial}{\partial u_2} - \frac{3\,u_1\,u_2}{2}\,\frac{\partial}{\partial u_3} - u_3\,u_2\,\frac{\partial}{\partial u_4}$$

$$\theta_3 = \left(\frac{2\,u_3}{3} - \frac{2\,u_1{}^2}{9}\right)\frac{\partial}{\partial u_1} + \frac{u_1\,u_2}{18}\,\frac{\partial}{\partial u_2} + \left(u_4 - \frac{u_1\,u_3}{9}\right)\frac{\partial}{\partial u_3} - \frac{u_2{}^2}{3}\frac{\partial}{\partial u_4}$$

$$\theta_4 = \frac{u_1}{3}\,\frac{\partial}{\partial u_1} + \frac{2\,u_2}{3}\,\frac{\partial}{\partial u_2} + \frac{2\,u_3}{3}\,\frac{\partial}{\partial u_3} + u_4\,\frac{\partial}{\partial u_4}$$

D_5 Discriminant

$$\theta_1 = \left(u_5 - \frac{5\,u_1\,u_4}{8} - \frac{u_2{}^2}{4} + \frac{15\,u_1{}^2u_2}{32} - \frac{27\,u_1{}^4}{256}\right)\frac{\partial}{\partial u_1} + \left(-\frac{u_2\,u_4}{2} - \frac{5\,u_3{}^2}{16} + \frac{3\,u_1{}^2u_4}{64} + \frac{u_1\,u_2{}^2}{4} - \frac{9\,u_1{}^3u_2}{128}\right)\frac{\partial}{\partial u_2} + \left(\frac{3\,u_4\,u_3}{32} - \frac{u_1\,u_2\,u_3}{16} + \frac{9\,u_1{}^3u_3}{512}\right)\frac{\partial}{\partial u_3} + \left(\frac{u_1\,u_2\,u_4}{8} - \frac{3\,u_4{}^2}{16} - \frac{u_1\,u_3{}^2}{64} - \frac{9\,u_1{}^3u_4}{256}\right)\frac{\partial}{\partial u_4} + \left(\frac{3\,u_1{}^2u_3{}^2}{256} - \frac{u_2\,u_3{}^2}{32}\right)\frac{\partial}{\partial u_5}$$

$$\theta_2 = \left(\frac{3\,u_4}{4} - \frac{u_1\,u_2}{2} + \frac{9\,u_1{}^3}{64}\right)\frac{\partial}{\partial u_1} + \left(u_5 - \frac{u_1\,u_4}{16} - \frac{u_2{}^2}{4} + \frac{3\,u_1{}^2u_2}{32}\right)\frac{\partial}{\partial u_2} + \left(\frac{u_2\,u_3}{16} - \frac{3\,u_1{}^2u_3}{128}\right)\frac{\partial}{\partial u_3} + \left(-\frac{u_2\,u_4}{8} - \frac{5\,u_3{}^2}{16} + \frac{3\,u_1{}^2u_4}{64}\right)\frac{\partial}{\partial u_4} - \frac{u_1\,u_3{}^2}{64}\frac{\partial}{\partial u_5}$$

$$\theta_3 = -\frac{5\,u_3}{2}\frac{\partial}{\partial u_1} - 2\,u_1\,u_3\,\frac{\partial}{\partial u_2} + u_5\,\frac{\partial}{\partial u_3} - \frac{3\,u_2\,u_3}{2}\frac{\partial}{\partial u_4} - u_4\,u_3\,\frac{\partial}{\partial u_5}$$

$$\theta_4 = \left(\frac{u_2}{2} - \frac{3\,u_1{}^2}{16}\right)\frac{\partial}{\partial u_1} + \left(\frac{3\,u_4}{4} - \frac{u_1\,u_2}{8}\right)\frac{\partial}{\partial u_2} + \frac{u_1\,u_3}{32}\frac{\partial}{\partial u_3} +$$

$$\left(u_5 - \frac{u_1\, u_4}{16}\right)\frac{\partial}{\partial u_4} - \frac{5\, u_3^{\,2}}{16}\frac{\partial}{\partial u_5}$$

$$\theta_5 = \frac{u_1}{4}\frac{\partial}{\partial u_1} + \frac{u_2}{2}\frac{\partial}{\partial u_2} + \frac{5\, u_3}{8}\frac{\partial}{\partial u_3} + \frac{3\, u_4}{4}\frac{\partial}{\partial u_4} + u_5\frac{\partial}{\partial u_5}$$

E_7 Discriminant

$$\theta_1 = \left(u_7 - \frac{10240\, u_1^{\,6} u_2}{81} - \frac{40\, u_1^{\,2} u_6}{9} - u_2\, u_5 - \frac{28\, u_3\, u_4}{27} + \frac{31\, u_1\, u_3^{\,2}}{27} - \right.$$
$$u_2^{\,3} - \frac{20\, u_1\, u_2\, u_4}{3} + \frac{65536\, u_1^{\,9}}{729} + \frac{160\, u_1^{\,3} u_5}{27} + \frac{136\, u_1^{\,2} u_2\, u_3}{9} +$$
$$\left.\frac{1792\, u_1^{\,4} u_4}{81} + \frac{352\, u_1^{\,3} u_2^{\,2}}{9} - \frac{2560\, u_1^{\,5} u_3}{81}\right)\frac{\partial}{\partial u_1} +$$
$$\left(-\frac{32\, u_3\, u_6}{27} - \frac{4096\, u_1^{\,6} u_4}{243} - \frac{4352\, u_1^{\,5} u_2^{\,2}}{27} - u_4\, u_5 - 3\, u_2^{\,2} u_4 + \right.$$
$$\frac{7\, u_1\, u_3\, u_5}{3} + \frac{11\, u_2\, u_3^{\,2}}{9} - \frac{28\, u_4^{\,2} u_1}{9} - \frac{640\, u_1^{\,4} u_2\, u_3}{9} - \frac{22\, u_1\, u_2\, u_6}{3} +$$
$$\frac{32768\, u_1^{\,8} u_2}{243} + \frac{20480\, u_1^{\,7} u_3}{729} + \frac{43\, u_2^{\,2} u_1\, u_3}{3} + \frac{128\, u_1^{\,4} u_6}{81} +$$
$$\frac{32\, u_1^{\,2} u_2\, u_5}{3} + \frac{800\, u_1^{\,2} u_4\, u_3}{81} - \frac{1888\, u_1^{\,3} u_3^{\,2}}{243} + 32\, u_2^{\,3} u_1^{\,2} +$$
$$\left.\frac{1216\, u_1^{\,3} u_2\, u_4}{27} - \frac{256\, u_1^{\,5} u_5}{81}\right)\frac{\partial}{\partial u_2} +$$
$$\left(\frac{256\, u_1^{\,5} u_6}{81} - \frac{1664\, u_1^{\,4} u_3^{\,2}}{243} - \frac{8192\, u_1^{\,7} u_4}{243} + \frac{16384\, u_1^{\,8} u_3}{729} - \right.$$
$$\frac{23\, u_2\, u_3\, u_4}{9} + \frac{19\, u_1\, u_2\, u_3^{\,2}}{9} - 2\, u_1\, u_5\, u_4 + \frac{1024\, u_1^{\,4} u_2\, u_4}{27} +$$
$$\frac{19\, u_4\, u_6}{9} - \frac{8\, u_1^{\,2} u_2\, u_6}{3} - \frac{64\, u_1^{\,2} u_4^{\,2}}{9} + \frac{1216\, u_1^{\,3} u_3\, u_4}{81} +$$
$$\frac{20\, u_2^{\,2} u_1^{\,2} u_3}{3} - \frac{49\, u_1\, u_3\, u_6}{27} + \frac{4\, u_3^{\,3}}{81} - 6\, u_2^{\,2} u_1\, u_4 +$$
$$\left.\frac{4\, u_1^{\,2} u_5\, u_3}{3} - \frac{256\, u_1^{\,5} u_2\, u_3}{9}\right)\frac{\partial}{\partial u_3} +$$
$$\left(u_1\, u_5^{\,2} - u_5\, u_6 + u_2\, u_3\, u_5 - \frac{512\, u_1^{\,4} u_2\, u_5}{27} + \frac{32768\, u_1^{\,8} u_4}{729} + \right.$$
$$\frac{4096\, u_1^{\,7} u_5}{243} - 3\, u_2^{\,2} u_6 + \frac{4096\, u_1^{\,7} u_2^{\,2}}{81} + 3\, u_2^{\,3} u_3 + \frac{832\, u_1^{\,3} u_4^{\,2}}{81} +$$

$$9\,u_2^4 u_1 + \frac{20\,u_1\,u_3^3}{81} - \frac{256\,u_1^5 u_3^2}{243} - \frac{1024\,u_1^6 u_6}{243} - \frac{512\,u_1^4 u_2^3}{9} -$$

$$\frac{8\,u_4^2 u_2}{3} - \frac{7\,u_3^2 u_4}{81} - \frac{5120\,u_1^5 u_2\,u_4}{81} + \frac{1024\,u_1^6 u_3\,u_2}{243} - \frac{80\,u_1\,u_4\,u_6}{27} +$$

$$\frac{56\,u_1^2 u_4\,u_5}{9} + \frac{28\,u_1^2 u_3\,u_6}{27} - \frac{128\,u_1^3 u_3\,u_5}{27} - \frac{4\,u_1^2 u_3^2 u_2}{27} +$$

$$\frac{128\,u_1^3 u_2\,u_6}{27} + \frac{88\,u_2^2 u_1^2 u_4}{3} + 6\,u_2^2 u_1\,u_5 - \frac{3328\,u_1^4 u_3\,u_4}{243} -$$

$$\left.\frac{512\,u_1^3 u_2^2 u_3}{27} + \frac{176\,u_1\,u_2\,u_3\,u_4}{27}\right) \frac{\partial}{\partial u_4} +$$

$$\left(\frac{4096\,u_1^7 u_2\,u_3}{243} + u_1\,u_5\,u_2\,u_3 - \frac{512\,u_1^3 u_2\,u_3^2}{81} - u_1\,u_5\,u_6 -\right.$$

$$\frac{16\,u_2\,u_3\,u_6}{9} + \frac{2\,u_5\,u_3^2}{27} - \frac{10\,u_4^2 u_3}{27} + \frac{1024\,u_1^6 u_3^2}{729} - \frac{28\,u_1^2 u_3^3}{81} + \frac{7\,u_6^2}{9} -$$

$$\frac{512\,u_1^4 u_2^2 u_3}{27} + \frac{62\,u_1\,u_3^2 u_4}{81} - 3\,u_2^2 u_1\,u_6 - \frac{32\,u_1^2 u_4\,u_6}{9} +$$

$$\frac{416\,u_1^3 u_3\,u_6}{81} + 3\,u_2^3 u_1\,u_3 - \frac{512\,u_1^5 u_4\,u_3}{243} + u_2^2 u_3^2 + \frac{16\,u_1^2 u_2\,u_4\,u_3}{3} +$$

$$\left.\frac{512\,u_1^4 u_2\,u_6}{27} - \frac{4096\,u_1^7 u_6}{243}\right) \frac{\partial}{\partial u_5} +$$

$$\left(\frac{448\,u_1^3 u_4\,u_6}{81} + \frac{47\,u_1\,u_3^2 u_5}{81} + u_2^2 u_1\,u_3^2 - \frac{17\,u_3^2 u_6}{81} + \frac{16384\,u_1^8 u_6}{729} -\right.$$

$$\frac{400\,u_1^3 u_3^3}{243} - \frac{29\,u_1\,u_6^2}{27} + \frac{8\,u_2\,u_3^3}{27} + \frac{4096\,u_1^7 u_3^2}{729} - \frac{1024\,u_1^5 u_4^2}{243} +$$

$$\frac{4\,u_1^2 u_5\,u_6}{3} + \frac{40\,u_1^2 u_3^2 u_4}{81} - \frac{512\,u_1^4 u_2\,u_3^2}{81} + 2\,u_2^2 u_3\,u_4 +$$

$$\frac{124\,u_1\,u_3\,u_4^2}{81} - \frac{256\,u_1^5 u_2\,u_6}{9} + \frac{2048\,u_1^6 u_3\,u_4}{729} + \frac{20\,u_2^2 u_1^2 u_6}{3} -$$

$$\frac{u_5\,u_3\,u_4}{27} + \frac{8192\,u_1^7 u_2\,u_4}{243} - \frac{1024\,u_1^4 u_2^2 u_4}{27} + \frac{32\,u_1^2 u_2\,u_4^2}{3} -$$

$$\frac{256\,u_1^5 u_3\,u_5}{243} - \frac{1664\,u_1^4 u_3\,u_6}{243} - \frac{20\,u_4^3}{27} + \frac{19\,u_1\,u_2\,u_3\,u_6}{9} +$$

$$2\,u_1\,u_5\,u_2\,u_4 - \frac{1024\,u_1^3 u_2\,u_3\,u_4}{81} + \frac{8\,u_1^2 u_2\,u_3\,u_5}{9} + 6\,u_2^3 u_1\,u_4 -$$

$$\left.\frac{23\,u_2\,u_4\,u_6}{9}\right) \frac{\partial}{\partial u_6} +$$

$$\left(-\frac{28\,u_1^2 u_3^2 u_6}{81} - \frac{512\,u_1^4 u_2^2 u_6}{27} - \frac{512\,u_1^5 u_4\,u_6}{243} - \frac{5\,u_5\,u_3\,u_6}{27} -\right.$$

$$\frac{10\,u_4^2 u_6}{27} + \frac{1024\,u_1^6 u_3\,u_6}{729} - \frac{8\,u_2\,u_6^2}{9} + \frac{8\,u_2\,u_3^2 u_5}{27} +$$

$$
\frac{4096\,u_1{}^7 u_3\,u_5}{729} + \frac{4096\,u_1{}^7 u_2\,u_6}{243} + u_2{}^2 u_3\,u_6 + \frac{u_1\,u_5{}^2 u_3}{3} -
\frac{512\,u_1{}^4 u_2\,u_3\,u_5}{81} + \frac{62\,u_1\,u_3\,u_4\,u_6}{81} + u_1\,u_5\,u_2\,u_6 + 3\,u_2{}^3 u_1\,u_6 +
\frac{16\,u_1{}^3 u_6{}^2}{81} - \frac{400\,u_1{}^3 u_3{}^2 u_5}{243} + u_2{}^2 u_1\,u_3\,u_5 + \frac{16\,u_1{}^2 u_2\,u_4\,u_6}{3} +
\left.\frac{32\,u_1{}^2 u_4\,u_3\,u_5}{27} - \frac{512\,u_1{}^3 u_2\,u_3\,u_6}{81}\right)\frac{\partial}{\partial u_7}
$$

$$
\begin{aligned}
\theta_2 \;=\; & \left(-\frac{32\,u_1{}^2 u_4}{9} - 3\,u_2{}^2 u_1 + \frac{400\,u_1{}^3 u_3}{81} + \frac{7\,u_6}{9} - u_1\,u_5 - \frac{8\,u_2\,u_3}{9} + \frac{512\,u_1{}^4 u_2}{27} - \frac{4096\,u_1{}^7}{243}\right)\frac{\partial}{\partial u_1} + \\
& \left(\frac{280\,u_1{}^2 u_2\,u_3}{27} - \frac{8\,u_1{}^2 u_6}{27} + \frac{16\,u_1{}^3 u_5}{27} - u_2{}^3 - \frac{20\,u_1\,u_2\,u_4}{3} + u_7 - u_2\,u_5 + \frac{256\,u_1{}^4 u_4}{81} - \frac{28\,u_3\,u_4}{27} + \frac{208\,u_1{}^3 u_2{}^2}{9} - \frac{1280\,u_1{}^5 u_3}{243} + \frac{31\,u_1\,u_3{}^2}{27} - \frac{2048\,u_1{}^6 u_2}{81}\right)\frac{\partial}{\partial u_2} + \\
& \left(-\frac{62\,u_1\,u_3\,u_4}{27} - \frac{u_2{}^2 u_3}{3} + \frac{28\,u_1{}^2 u_3{}^2}{27} - \frac{16\,u_1{}^3 u_6}{27} + \frac{u_2\,u_6}{3} - \frac{2\,u_5\,u_3}{9} + \frac{112\,u_1{}^3 u_2\,u_3}{27} + \frac{10\,u_4{}^2}{9} - \frac{16\,u_1{}^2 u_2\,u_4}{3} + \frac{512\,u_1{}^5 u_4}{81} - \frac{1024\,u_1{}^6 u_3}{243}\right)\frac{\partial}{\partial u_3} + \\
& \left(-\frac{2\,u_1\,u_2\,u_6}{3} - \frac{44\,u_4{}^2 u_1}{27} - 3\,u_2{}^2 u_4 + \frac{25\,u_2{}^2 u_1\,u_3}{9} + \frac{56\,u_1{}^2 u_4\,u_3}{27} + \frac{8\,u_1{}^2 u_2\,u_5}{3} + \frac{u_2\,u_3{}^2}{27} + \frac{64\,u_1{}^4 u_6}{81} - \frac{4\,u_3\,u_6}{27} - u_4\,u_5 + 8\,u_2{}^3 u_1{}^2 + \frac{16\,u_1{}^3 u_3{}^2}{81} + \frac{256\,u_1{}^3 u_2\,u_4}{27} - \frac{64\,u_1{}^4 u_2\,u_3}{81} - \frac{256\,u_1{}^5 u_5}{81} + \frac{19\,u_1\,u_3\,u_5}{27} - \frac{2048\,u_1{}^6 u_4}{243} - \frac{256\,u_1{}^5 u_2{}^2}{27}\right)\frac{\partial}{\partial u_4} + \\
& \left(\frac{25\,u_1\,u_2\,u_3{}^2}{27} - \frac{7\,u_1\,u_3\,u_6}{9} - \frac{7\,u_2\,u_3\,u_4}{9} + \frac{4\,u_3{}^3}{81} + \frac{8\,u_2{}^2 u_1{}^2 u_3}{3} + \frac{32\,u_1{}^3 u_3\,u_4}{81} + \frac{5\,u_4\,u_6}{9} - \frac{64\,u_1{}^4 u_3{}^2}{243} - \frac{8\,u_1{}^2 u_2\,u_6}{3} + \frac{256\,u_1{}^5 u_6}{81} - \frac{256\,u_1{}^5 u_2\,u_3}{81}\right)\frac{\partial}{\partial u_5}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{8\,u_1\,u_4\,u_6}{9}-\frac{u_2\,u_3\,u_5}{9}-\frac{u_2{}^2u_6}{3}-\frac{14\,u_4{}^2u_2}{9}+\frac{50\,u_1\,u_2\,u_3\,u_4}{27}+\right.\\
&\frac{28\,u_1{}^2u_3\,u_6}{27}+\frac{20\,u_1\,u_3{}^3}{81}-\frac{7\,u_3{}^2u_4}{81}-\frac{2\,u_5\,u_6}{9}+\frac{8\,u_1{}^2u_3{}^2u_2}{9}+\\
&\frac{16\,u_2{}^2u_1{}^2u_4}{3}+\frac{64\,u_1{}^3u_4{}^2}{81}+\frac{16\,u_1{}^3u_3\,u_5}{81}+\frac{112\,u_1{}^3u_2\,u_6}{27}-\\
&\left.\frac{128\,u_1{}^4u_3\,u_4}{243}-\frac{1024\,u_1{}^6u_6}{243}-\frac{256\,u_1{}^5u_3{}^2}{243}-\frac{512\,u_1{}^5u_2\,u_4}{81}\right)\frac{\partial}{\partial u_6}+\\
&\left(\frac{4\,u_3{}^2u_6}{81}+\frac{20\,u_1\,u_3{}^2u_5}{81}+\frac{25\,u_1\,u_2\,u_3\,u_6}{27}-\frac{256\,u_1{}^5u_2\,u_6}{81}+\right.\\
&\frac{8\,u_2{}^2u_1{}^2u_6}{3}-\frac{u_1\,u_6{}^2}{27}-\frac{5\,u_5\,u_3\,u_4}{27}-\frac{256\,u_1{}^5u_3\,u_5}{243}-\frac{64\,u_1{}^4u_3\,u_6}{243}+\\
&\left.\frac{8\,u_1{}^2u_2\,u_3\,u_5}{9}+\frac{32\,u_1{}^3u_4\,u_6}{81}-\frac{7\,u_2\,u_4\,u_6}{9}\right)\frac{\partial}{\partial u_7}
\end{aligned}
$$

$$
\begin{aligned}
\theta_3 \;=\; &\left(\frac{28\,u_1{}^2u_3}{27}-\frac{2\,u_5}{9}-\frac{1024\,u_1{}^6}{243}+\frac{112\,u_1{}^3u_2}{27}-\frac{22\,u_1\,u_4}{27}-\frac{u_2{}^2}{3}\right)\frac{\partial}{\partial u_1}+\\
&\left(\frac{4\,u_1{}^2u_5}{27}-\frac{512\,u_1{}^5u_2}{81}+\frac{44\,u_2{}^2u_1{}^2}{9}+\frac{4\,u_3{}^2}{81}+\frac{64\,u_1{}^3u_4}{81}-\right.\\
&\left.\frac{320\,u_1{}^4u_3}{243}+\frac{19\,u_1\,u_2\,u_3}{9}-\frac{29\,u_1\,u_6}{27}-\frac{7\,u_2\,u_4}{9}\right)\frac{\partial}{\partial u_2}+\\
&\left(\frac{17\,u_1\,u_3{}^2}{81}-\frac{10\,u_1\,u_2\,u_4}{9}-\frac{256\,u_1{}^5u_3}{243}+\frac{128\,u_1{}^4u_4}{81}+\frac{8\,u_1{}^2u_2\,u_3}{9}-\right.\\
&\left.\frac{4\,u_1{}^2u_6}{27}+u_7-\frac{13\,u_3\,u_4}{27}\right)\frac{\partial}{\partial u_3}+\\
&\left(-\frac{2\,u_5\,u_3}{9}-\frac{10\,u_4{}^2}{27}-\frac{64\,u_1{}^4u_2{}^2}{27}-\frac{512\,u_1{}^5u_4}{243}+\frac{5\,u_2{}^3u_1}{3}+\right.\\
&\frac{34\,u_1\,u_3\,u_4}{81}+\frac{16\,u_1{}^3u_6}{81}+\frac{56\,u_1{}^2u_2\,u_4}{27}+\frac{4\,u_1{}^2u_3{}^2}{81}+\frac{5\,u_2{}^2u_3}{9}-\\
&\left.\frac{64\,u_1{}^4u_5}{81}-\frac{16\,u_1{}^3u_2\,u_3}{81}+\frac{5\,u_1\,u_2\,u_5}{9}-\frac{8\,u_2\,u_6}{9}\right)\frac{\partial}{\partial u_4}+\\
&\left(-\frac{5\,u_1\,u_2\,u_6}{9}-\frac{64\,u_1{}^4u_2\,u_3}{81}+\frac{64\,u_1{}^4u_6}{81}+\frac{5\,u_2{}^2u_1\,u_3}{9}+\right.\\
&\left.\frac{8\,u_1{}^2u_4\,u_3}{81}+\frac{5\,u_2\,u_3{}^2}{27}-\frac{16\,u_1{}^3u_3{}^2}{243}-\frac{11\,u_3\,u_6}{27}\right)\frac{\partial}{\partial u_5}+\\
&\left(-\frac{19\,u_4\,u_6}{27}-\frac{256\,u_1{}^5u_6}{243}-\frac{128\,u_1{}^4u_2\,u_4}{81}-\frac{64\,u_1{}^4u_3{}^2}{243}+\frac{4\,u_3{}^3}{81}+\right.\\
&\frac{17\,u_1\,u_3\,u_6}{81}+\frac{5\,u_1\,u_2\,u_3{}^2}{27}+\frac{10\,u_2{}^2u_1\,u_4}{9}+\frac{4\,u_1{}^2u_5\,u_3}{81}+\frac{16\,u_1{}^2u_4{}^2}{81}+
\end{aligned}
$$

$$\frac{10\,u_2\,u_3\,u_4}{27}+\frac{8\,u_1{}^2u_2\,u_6}{9}-\frac{2\,u_5{}^2}{9}-\frac{32\,u_1{}^3u_3\,u_4}{243}\Bigg)\frac{\partial}{\partial u_6}+$$

$$\Bigg(\frac{5\,u_1\,u_5\,u_2\,u_3}{27}-\frac{64\,u_1{}^4u_3\,u_5}{243}+\frac{5\,u_2\,u_3\,u_6}{27}+\frac{4\,u_5\,u_3{}^2}{81}-\frac{7\,u_6{}^2}{27}+$$

$$\frac{5\,u_2{}^2u_1\,u_6}{9}+\frac{8\,u_1{}^2u_4\,u_6}{81}-\frac{16\,u_1{}^3u_3\,u_6}{243}-\frac{64\,u_1{}^4u_2\,u_6}{81}\Bigg)\frac{\partial}{\partial u_7}$$

$$\theta_4 \;=\; \Bigg(\frac{5\,u_4}{9}-\frac{20\,u_1\,u_3}{27}-\frac{8\,u_1{}^2u_2}{3}+\frac{256\,u_1{}^5}{81}\Bigg)\frac{\partial}{\partial u_1}+$$

$$\Bigg(-u_1\,u_5-\frac{8\,u_2\,u_3}{9}+\frac{7\,u_6}{9}-3\,u_2{}^2u_1-\frac{16\,u_1{}^2u_4}{27}+\frac{80\,u_1{}^3u_3}{81}+$$

$$\frac{128\,u_1{}^4u_2}{27}\Bigg)\frac{\partial}{\partial u_2}+$$

$$\Bigg(\frac{2\,u_2\,u_4}{3}+\frac{u_1\,u_6}{9}-\frac{4\,u_3{}^2}{27}-\frac{5\,u_1\,u_2\,u_3}{9}+\frac{64\,u_1{}^4u_3}{81}-$$

$$\frac{32\,u_1{}^3u_4}{27}\Bigg)\frac{\partial}{\partial u_3}+$$

$$\Bigg(-u_2\,u_5-\frac{8\,u_3\,u_4}{27}+u_7-\frac{4\,u_1{}^2u_6}{27}-\frac{4\,u_1\,u_2\,u_4}{3}-u_2{}^3-\frac{u_1\,u_3{}^2}{27}+$$

$$\frac{4\,u_1{}^2u_2\,u_3}{27}+\frac{16\,u_1{}^3u_5}{27}+\frac{128\,u_1{}^4u_4}{81}+\frac{16\,u_1{}^3u_2{}^2}{9}\Bigg)\frac{\partial}{\partial u_4}+$$

$$\Bigg(\frac{u_2\,u_6}{3}-\frac{2\,u_5\,u_3}{9}-\frac{u_2{}^2u_3}{3}-\frac{2\,u_1\,u_3\,u_4}{27}+\frac{4\,u_1{}^2u_3{}^2}{81}-\frac{16\,u_1{}^3u_6}{27}+$$

$$\frac{16\,u_1{}^3u_2\,u_3}{27}\Bigg)\frac{\partial}{\partial u_5}+$$

$$\Bigg(-\frac{5\,u_1\,u_2\,u_6}{9}-\frac{4\,u_3\,u_6}{27}-\frac{4\,u_4\,u_5}{9}-\frac{u_2\,u_3{}^2}{9}-\frac{2\,u_2{}^2u_4}{3}-\frac{u_1\,u_3\,u_5}{27}-$$

$$\frac{4\,u_4{}^2u_1}{27}+\frac{8\,u_1{}^2u_4\,u_3}{81}+\frac{64\,u_1{}^4u_6}{81}+\frac{32\,u_1{}^3u_2\,u_4}{27}+\frac{16\,u_1{}^3u_3{}^2}{81}\Bigg)\frac{\partial}{\partial u_6}+$$

$$\Bigg(\frac{4\,u_1{}^2u_3\,u_6}{81}-\frac{2\,u_1\,u_4\,u_6}{27}+\frac{16\,u_1{}^3u_3\,u_5}{81}+\frac{16\,u_1{}^3u_2\,u_6}{27}-\frac{u_2{}^2u_6}{3}-$$

$$\frac{2\,u_5\,u_6}{9}-\frac{u_2\,u_3\,u_5}{9}\Bigg)\frac{\partial}{\partial u_7}$$

$$\theta_5 \;=\; \Bigg(-\frac{5\,u_1\,u_2}{9}-\frac{4\,u_3}{27}+\frac{64\,u_1{}^4}{81}\Bigg)\frac{\partial}{\partial u_1}+$$

$$\Bigg(\frac{20\,u_1{}^2u_3}{81}-\frac{22\,u_1\,u_4}{27}-\frac{u_2{}^2}{3}-\frac{2\,u_5}{9}+\frac{32\,u_1{}^3u_2}{27}\Bigg)\frac{\partial}{\partial u_2}+$$

$$\left(\frac{7\,u_6}{9}-\frac{u_2\,u_3}{9}+\frac{16\,u_1{}^3u_3}{81}-\frac{8\,u_1{}^2u_4}{27}\right)\frac{\partial}{\partial u_3}+$$
$$\left(-\frac{u_1\,u_6}{27}-\frac{7\,u_2\,u_4}{9}+\frac{u_1\,u_2\,u_3}{27}+\frac{4\,u_1{}^2u_5}{27}-\frac{4\,u_3{}^2}{27}+\frac{4\,u_2{}^2u_1{}^2}{9}+\right.$$
$$\left.\frac{32\,u_1{}^3u_4}{81}\right)\frac{\partial}{\partial u_4}+$$
$$\left(u_7-\frac{4\,u_1{}^2u_6}{27}+\frac{u_1\,u_3{}^2}{81}-\frac{5\,u_3\,u_4}{27}+\frac{4\,u_1{}^2u_2\,u_3}{27}\right)\frac{\partial}{\partial u_5}+$$
$$\left(-\frac{10\,u_5\,u_3}{27}-\frac{10\,u_4{}^2}{27}-\frac{u_2\,u_6}{9}+\frac{2\,u_1\,u_3\,u_4}{81}+\frac{16\,u_1{}^3u_6}{81}+\frac{4\,u_1{}^2u_3{}^2}{81}+\right.$$
$$\left.\frac{8\,u_1{}^2u_2\,u_4}{27}\right)\frac{\partial}{\partial u_6}+$$
$$\left(\frac{4\,u_1{}^2u_2\,u_6}{27}+\frac{u_1\,u_3\,u_6}{81}+\frac{4\,u_1{}^2u_5\,u_3}{81}-\frac{5\,u_4\,u_6}{27}-\frac{2\,u_5{}^2}{9}\right)\frac{\partial}{\partial u_7}$$

$$\theta_6 = \left(\frac{u_2}{3}-\frac{16\,u_1{}^3}{27}\right)\frac{\partial}{\partial u_1}+\left(\frac{5\,u_4}{9}-\frac{8\,u_1{}^2u_2}{9}-\frac{20\,u_1\,u_3}{27}\right)\frac{\partial}{\partial u_2}+$$
$$\left(\frac{2\,u_5}{3}-\frac{4\,u_1{}^2u_3}{27}+\frac{2\,u_1\,u_4}{9}\right)\frac{\partial}{\partial u_3}+$$
$$\left(\frac{7\,u_6}{9}-\frac{8\,u_1{}^2u_4}{27}-\frac{u_2{}^2u_1}{3}-\frac{u_1\,u_5}{9}-\frac{4\,u_2\,u_3}{9}\right)\frac{\partial}{\partial u_4}+$$
$$\left(-\frac{u_1\,u_2\,u_3}{9}-\frac{4\,u_3{}^2}{27}+\frac{u_1\,u_6}{9}\right)\frac{\partial}{\partial u_5}+$$
$$\left(u_7-\frac{4\,u_1{}^2u_6}{27}-\frac{u_1\,u_3{}^2}{27}-\frac{2\,u_1\,u_2\,u_4}{9}-\frac{8\,u_3\,u_4}{27}\right)\frac{\partial}{\partial u_6}+$$
$$\left(-\frac{u_1\,u_2\,u_6}{9}-\frac{u_1\,u_3\,u_5}{27}-\frac{4\,u_3\,u_6}{27}\right)\frac{\partial}{\partial u_7}$$

$$\theta_7 = \frac{u_1}{9}\frac{\partial}{\partial u_1}+\frac{u_2}{3}\frac{\partial}{\partial u_2}+\frac{4\,u_3}{9}\frac{\partial}{\partial u_3}+\frac{5\,u_4}{9}\frac{\partial}{\partial u_4}+\frac{2\,u_5}{3}\frac{\partial}{\partial u_5}+$$
$$\frac{7\,u_6}{9}\frac{\partial}{\partial u_6}+u_7\frac{\partial}{\partial u_7}$$

Appendix B

Transversal Code

A fully documented listing of the Transversal code is given in this appendix. Section B.1 contains the code for the main routine jetcalc; Section B.2 the code for the user functions; Section B.3 contains examples of liealg routines; and Section B.4 the code for the subroutines. We do not give the code for all the liealg routines, nor the 'set-up' routines, nor the functions which deal with the weighted case (the main function for use with weighted filtrations, wtcalc, is similar to jetcalc only more complicated).

B.1 Jetcalc

```
jetcalc := proc(f,k)
   local i,j,num_vectors,deg,rank,count,poly,row,col,row2,scalar,temp,
         least_deg,tgtcopy,pre_tgtspace,pre_num_vectors,poly_table,num_polys,
         index_ref,deg_ref,left_deg_lim,left_index_ref,left_deg_ref,
         num_nilp,nilp_terms,nilp_pt_ref,nilp_wt_ref,weight;

# define tables

   tgtcopy := table(); pre_tgtspace := table(); poly_table := table();
   index_ref := table(); deg_ref := table();
   left_index_ref := table(); left_deg_ref := table();
   nilp_terms := table(); nilp_pt_ref := table(); nilp_wt_ref := table();

# define global variable tables (used to store results)

   gl_index_ref := table(); gl_deg_ref := table();
   tgtspace := table();
   compbasis := table();
```

```
   checklist := table();
   coeffarray_ref := table();

# basic global variable checks (always a good idea !!!)

   if not assigned(liealg) then
      ERROR(`global variable 'liealg' unassigned`);
   fi;
   if not type(compltrans,boolean) then
      ERROR(`global variable 'compltrans' must be of type 'boolean'`);
   fi;
   if equiv<>'A' and equiv<>'K' and equiv<>'R' and equiv<>'L'
         and equiv<>'C' then
      ERROR(`global variable 'equiv' must be set as A,K,R,L or C`);
   fi;
   if not type(source_power,nonnegint) then
      ERROR(`global variable 'source_power' must be a non -ve integer`);
   fi;
   if not type(target_power,nonnegint) then
      ERROR(`global variable 'target_power' must be a non -ve integer`);
   fi;
   if liealg=stdjacobian and not type(source_dim,posint) then
      ERROR(`global variable 'source_dim' must be a +ve integer`);
   fi;
   if nilp=true or nilp='true_order' then
      if not type(R_nilp,list) then
         ERROR(`global variable 'R_nilp' must be of type 'list'`);
      fi;
      if not type(L_nilp,list) then
         ERROR(`global variable 'L_nilp' must be of type 'list'`);
      fi;
      if nilp='true_order' then
         if not type(nilp_source_wt,list) then
            ERROR(`global variable 'nilp_source_wt' must be of type 'list'`);
         fi;
         if not type(nilp_target_wt,list) then
            ERROR(`global variable 'nilp_target_wt' must be of type 'list'`);
         fi;
      fi;
   else
      if nilp<>false then ERROR
   (`global variable 'nilp' must be set as 'true', 'false', or 'true_order'`);
      fi;
   fi;

# basic parameter checks

   if not type(f,list) then
      ERROR(`parameter 'f' must be of type 'list'`);
   fi;
   if not type(k,nonnegint) then
```

```
      ERROR(`parameter 'k' must be a non -ve integer`);
   fi;

# set up initial variables
# check nilpotent global variables assigned in 'liealg'

   print(`defined map:`);
   print(f);
   target_dim := nops(f);
   print(`working in `.k.`-jet space with `.equiv.`-equivalence`);
      # get the variables stored in 'liealg'
   liealg(f,target_dim,tgtspace);
   print(`defined coordinates:`);
   print(coords);
   num_coords := nops(coords);
   num_vectors := nops(convert(tgtspace,list));
   num_polys := 0;   # used in L and C tgtspace routines
      # create and print the nilpotent terms
   num_nilp := 1;
   if nilp=true or nilp='true_order' then
      for i from 1 to nops(R_nilp) do
         if not type(R_nilp[i],list) or nops(R_nilp[i])<>2 then
          ERROR(``R_nilp[`.i.`]' must be of type 'list' with 2 entries`);
         fi;
         if R_nilp[i][2]>num_vectors then
          ERROR(``R_nilp[`.i.`][2]' > number of vectors in Lie algebra`);
         fi;
         nilp_terms[num_nilp] := scalar_multn( R_nilp[i][1],
                                 tgtspace[R_nilp[i][2]],target_dim);
         num_nilp := num_nilp+1;
      od;
      for i from 1 to nops(L_nilp) do
         if not type(L_nilp[i],list) or nops(L_nilp[i])<>2 then
          ERROR(``L_nilp[`.i.`]' must be of type 'list' with 2 entries`);
         fi;
         if L_nilp[i][1]>target_dim then
            ERROR(``L_nilp[`.i.`][1]' > dimension of target`);
         fi;
         if L_nilp[i][2]>target_dim then
            ERROR(``L_nilp[`.i.`][2]' > dimension of target`);
         fi;
         nilp_terms[num_nilp] := canonical_vector(L_nilp[i][2],
                                 f[L_nilp[i][1]],target_dim);
         num_nilp := num_nilp+1;
      od;
   fi;
      # store the number of terms in 'nilp_terms';
   num_nilp := num_nilp-1;

# Check map-germ 'f' has target 0, Ie. f(0)=0 or equivalently all poly.
# components of f are of degree > 1 (needed for "left tgt. space"
```

```
# procedure to work). Also checks for monomials with negative indices.

   if ldegree_vector(f,k,target_dim) <= 0 then
      ERROR ('f has non-zero target or monomials with negative indices');
   fi;

# Create index and degree reference tables for the source coordinates.
# Store these for global use too.
# Store the jet space degree (k) as a global variable.

   get_ref_tables(k,num_coords,index_ref,deg_ref);
   gl_index_ref := copy(index_ref); gl_deg_ref := copy(deg_ref);
   jetspace_deg := k;

# Use right tangent space (given by 'liealg') in the equivalence ???

   if equiv=R or equiv=A or equiv=K then

#   ***   RIGHT TANGENT SPACE ROUTINE    ***

# Calculate a real spanning set for the (image of the) E(n)-module
#                     (m(n)^source_power) . < LR.f >
# in the jet space J^k(n,p), where LR is the Lie algebra of the 'source
# coordinate change group' and LR.f is obtained from the procedure liealg.

   print('***   calculating right tangent space   ***');
   tgtcopy := copy(tgtspace);
   count := 1;
      # Multiply tgtcopy[i] by the monomials in m(n)^source_power.
      # That is multiply by the monomials of degree source_power and higher
      # until a zero k-jet is reached.
      # The set of all such vectors is stored in tgtspace and gives
      # all the possible non-zero k-jets which result from tgtcopy[i].
   for i from 1 to num_vectors do
         # Check m(n)^source_power*tgtcopy[i] gives non zero k-jets.
         # (Case when tgtcopy[i] is the zero vector, 'ldegree_vector'
         # RETURNS k+1 and the jet will be ignored as required.)
      least_deg := ldegree_vector(tgtcopy[i],k,target_dim);
      if source_power+least_deg<=k then
         for deg from deg_ref[source_power] to
                  deg_ref[k-least_deg+1]-1 do
               # multiply tgtcopy[i] by monomial corresponding to deg
               # and store in the tangent space table
            tgtspace[count] := scalar_multn(get_monomial(index_ref[deg]),
                                          tgtcopy[i],target_dim);

            count := count+1;
         od;
      fi;
   od;
      # store the number of vectors in 'tgtspace'
   num_vectors := count-1;
```

```
#   ***   END OF RIGHT TANGENT SPACE ROUNTINE   ***

   else
        # no right tangent space ... ignore vectors stored in tgtspace
     num_vectors := 0;
   fi;

# Use left tangent space in the equivalence ???

   if equiv=L or equiv=A then

#   ***   LEFT TANGENT SPACE ROUTINE   ***

# Calculate a real spanning set for the (image of the) E(p)-module
#                      (f^*)(m(p)^target_power) . { e(i) }
# in the jet space J^k(n,p).

   print('***   calculating left tangent space   ***');
      # Get maximum degree for left ref tables, Ie. the maximum degree by
      # which the components of 'f' may be raised giving non-zero k-jets.
      # Do not get 'division by zero' below, Ie. 'ldegree_vector' <> 0,
      # as 'f' does not contain constant terms (cf. check at start of
      # procedure). And the case f=0 gives a RETURNED least degree of
      # k+1; thus left_deg_lim=0 giving just the vectors with constant
      # (unit) components in the left tangent space.
   ldegree_vector(f,k,target_dim);
   left_deg_lim := iquo(k,");

      # Get all possible jets formed by the 'appropriate combinations' of
      # the coord. functions of 'f' (using monomials of degree target_power
      # to left_deg_lim, with the indeterminates substituted with the
      # components of f).
      # Store all these polys in the table poly_table.

      # first check that the left tangent space gives non zero jets
   if target_power<=left_deg_lim then
      get_ref_tables(left_deg_lim,target_dim,
                     left_index_ref,left_deg_ref);
      count := 1;
      for deg from left_deg_ref[target_power] to
                   left_deg_ref[left_deg_lim+1]-1 do
           # Get the polynomial f[1]^left_index_ref[deg][1]* ...
           # ... *f[target_dim]^left_index_ref[deg][target_dim].
           # NB: require f[i]^0 = 1 for any value of f[i], in particular
           #     if f[i]=0, for all the possible powers of the f[i].
         poly := 1;
         for i from 1 to target_dim do
           if left_index_ref[deg][i]=0 then
              1;
           else
```

```
               f[i]^left_index_ref[deg][i];
            fi;
            poly := poly * ";
         od;
            # Check poly is a non-zero k-jet.
            # NB: for 'ldegree' to return the total least degree the list
            #     of indeterminates must in fact be of data type 'set'.
         if poly<>0 and ldegree(expand(poly),convert(coords,set))<=k then
            poly_table[count] := expand(poly);
            count := count+1;
         fi;
      od;
         # store the number of polys in poly_table
      num_polys := count-1;
   fi;

#   ***   END OF LEFT TANGENT SPACE ROUTINE   ***

   fi;

# Use the C tangent space in the equivalence ???

   if equiv=C or equiv=K then

#   ***   C ( LEFT CONTACT ) TANGENT SPACE ROUTINE   ***

# Calculate a real spanning set for the (image of the) E(n)-module
#                (m(n)^target_power) . (f^*)(m(p)) . E(n,p)
# in the jet space J^k(n,p).

   print(`***   calculating C tangent space   ***`);
   count := 1;
      # Multiply f[i] by the monomials in m(n)^target_power.
      # That is multiply by the monomials of degree target_power and higher
      # until a zero k-jet is reached.
      # Store all these polys in the table poly_table.
   for i from 1 to target_dim do
         # check (m(n)^target_power)*f[i] gives non zero k-jets
      least_deg := ldegree(expand(f[i]),convert(coords,set));
      if target_power+least_deg<=k and f[i]<>0 then
         for deg from deg_ref[target_power] to
                      deg_ref[k-least_deg+1]-1 do
               # Multiply f[i] by monomial corresponding to deg and store.
            poly_table[count] := expand(get_monomial(index_ref[deg])*f[i]);
            count := count+1;
         od;
      fi;
   od;
      # store the number of polys in poly_table
   num_polys := count-1;
```

```
#   ***   END OF C TANGENT SPACE ROUTINE   ***

   fi;

#   ***   INCLUDE THE NILPOTENT TERMS IN THE TANGENT SPACE   ***

      # These are just added to the table tgtspace (since there are only a
      # few of them, so no need to incorporate and pre-tangent routine).
   count := num_vectors+1;
   for i from 1 to num_nilp do
         # use 'copy' here as 'nilp_terms[i]' is an indexed table
      tgtspace[count] := copy(nilp_terms[i]);
      count := count+1;
   od;
      # store the number of vectors in 'tgtspace'
   num_vectors := count-1;

#   ***   END OF 'NILPOTENT' ROUTINE   ***

#   ***   REDUCTION ROUTINE TO PRODUCE BASIS FOR TANGENT SPACE   ***

   # Basic procedure: take k-jets and extract coefficients of the jets to form
   # a matrix (coeffarray) and reduce to echelon form using Gaussian
   # elimination. However, matrix very sparse (and large) and it is more
   # efficient to have a reference table which allows us to obtain each
   # coefficient (ie. each 'would be' entry (i,j) of coeffarray) from the table
   # of polynomial vectors, tgtspace, as required.

   # The right tangent space (currently stored in the table tgtspace (with the
   # the nilpotent vectors)) and the left tangent space (to be calculated from
   # the table of polynomials poly_table) are dealt with separately so that we
   # can exploit the symmetry present in the left tangent space (cf. the
   # pre-tangent space routine below).

   print(`***   performing Gaussian elimination   ***`);

      # Firstly set up the 'coeffarray' reference table.
      # The rth column of 'coeffarray' gives the coefficient of a monomial of a
      # component of V (V being some tangent vector), where
      # coeffarray_ref[r][1] indicates the component of V,
      # coeffarray_ref[r][2] gives the monomial via a pointer to its index.

      # Use the ordering induced by the nilpotent weights ?
   if nilp='true_order' then
      print(`using ordering induced by the nilpotent weights`);
         # firstly create the nilpotent reference tables by ordering the
         # degree k monomials by their weight
      count := 0;
      for weight from k to k*num_coords do
         nilp_wt_ref[weight] := count;
         for deg from deg_ref[k] to deg_ref[k+1]-1 do
```

```
         if get_wt(index_ref[deg],num_coords,nilp_source_wt)=weight then
            nilp_pt_ref[count] := deg;
            count := count+1;
         fi;
      od;
   od;
      # store pointer to end of table
   nilp_wt_ref[k*num_coords+1] := count;

      # Create 'coeffarray' reference table for the monomial vectors up to
      # and including degree k-1.
   count := 1;
   for deg from 0 to deg_ref[k]-1 do
      for j from 1 to target_dim do
         coeffarray_ref[count][1] := j;
         coeffarray_ref[count][2] := deg;
         count := count+1;
      od;
   od;

      # Create 'coeffarray' reference table for the monomial vectors of
      # degree k, now using the nilpotent ordering.
      # Ie. order via the (vector) weights k thru' k*num_coords+target_dim-1.
   for weight from k to k*num_coords+target_dim-1 do
      for j from 1 to target_dim do
         weight+nilp_target_wt[j];
         if k<=" and "<=k*num_coords then
            for i from nilp_wt_ref["] to nilp_wt_ref["+1]-1 do
               coeffarray_ref[count][1] := j;
               coeffarray_ref[count][2] := nilp_pt_ref[i];
               count := count+1;
            od;
         fi;
      od;
   od;

else
      # Do not use nilpotent weights, use default ordering.
   print(`using default ordering`);
      # Create 'coeffarray' reference table for all the monomial vectors
      # of degree up to and including k.
   for deg from 0 to deg_ref[k+1]-1 do
      for j from 1 to target_dim do
         coeffarray_ref[deg*target_dim+j][1] := j;
         coeffarray_ref[deg*target_dim+j][2] := deg;
      od;
   od;
fi;

   # Now reduce to echelon form using Gaussian elimination.
   # If equiv=R then only the right tangent space need be reduced.
```

```
    # Otherwise, a routine is firstly used to reduce the left tangent space
    # (the pre_tgtspace) which has a lot of symmetry, followed by a full
    # reduction to create a basis for the whole tangent space.
    # ('Indexed' Gaussian elimination is used via the matrix coeffarray_ref).

if equiv=R then

      # Calculate tangent space: just need to reduce "matrix" tgtspace.

      # Check the (right) tangent space is non-empty; if empty
      # then add a zero vector so that the rest of the
      # prodedure may continue with the calculations.
   if num_vectors=0 then
      num_vectors := 1;
      tgtspace[1] := array(sparse,1..target_dim);
   fi;

   print(`calculating tangent space`);
   print(`matrix dimensions: `.num_vectors,deg_ref[k+1]*target_dim);
      # Gaussian elimination bit.
      # NB: pivotal elements are left as they are found, they must NOT be
      #     scaled to produce a 1 for the pivotal element.
      # NB: number of columns = deg_ref[k+1]*target_dim.
      #     Number of vectors in the table tgtspace was stored as the
      #     variable num_vectors earlier.
   row := 1;
   for col from 1 to deg_ref[k+1]*target_dim while
                row <= num_vectors do
         # find pivot
      for i from row to num_vectors while
                 coeff_table(tgtspace,i,col) = 0 do   od;
         # Give preference to numeric pivots, but otherwise
         # choose most efficient pivot from a symbolic point of view
         # (that is the one of least length as a Maple expression).
      for j from i+1 to num_vectors do
         coeff_table(tgtspace,j,col);
         if "<>0 and
            not type(coeff_table(tgtspace,i,col),numeric) and
            ( type(",numeric) or
            length(") < length(coeff_table(tgtspace,i,col)) )
            then
            i := j;
         fi;
      od;
         # If a pivot has been found then perform "row reduction" and move
         # to next row, otherwise (zero column) stay on the same row.
      if i <= num_vectors then
            # swap rows first ?
         if i <> row then
            temp := copy(tgtspace[i]);
            tgtspace[i] := copy(tgtspace[row]);
```

```
            tgtspace[row] := copy(temp);
         fi;
         for i from row+1 to num_vectors do
               # reduce row i ?
            if coeff_table(tgtspace,i,col) <> 0 then
               scalar := normal(coeff_table(tgtspace,i,col)/
                                coeff_table(tgtspace,row,col));
               for j from 1 to target_dim do
                  tgtspace[i][j] := normal(tgtspace[i][j]-
                                     scalar*tgtspace[row][j]);
               od;
            fi;
         od;
         row := row+1;
      fi;
   od;
      # Finally store the rank as the global variable basis_dim.
   basis_dim := row-1;
      # And store the number of columns as a global variable.
   tgtstore_lim := deg_ref[k+1]*target_dim;

else

      # Calculate pre_tangent space ('left bit').

      # Check the table poly_table is non-empty; if empty
      # then add a zero term so that the rest of the
      # procedure may continue with the calculations.
   if num_polys=0 then
      num_polys := 1;
      poly_table[1] := 0;
   fi;

   print(`calculating pre-tangent space`);
   print(`matrix dimensions: `.num_polys,deg_ref[k+1]);
      # First reduce the table poly_table using Gaussian elimination.
      # NB: pivotal elements are left as they are found, they must NOT be
      #     scaled to produce a 1 for the pivotal element.
   row := 1;
   for col from 0 to deg_ref[k+1]-1 while row <= num_polys do
         # find pivot
      for i from row to num_polys while
                 normal(get_coeff(poly_table[i],index_ref[col])) = 0 do
      od;
         # Give preference to numeric pivots, but otherwise
         # choose most efficient pivot from a symbolic point of view
         # (that is the one of least length as a Maple expression).
      for j from i+1 to num_polys do
         normal(get_coeff(poly_table[j],index_ref[col]));
         if "<>0 and
            not type(get_coeff(poly_table[i],index_ref[col]),numeric) and
```

```
         ( type(",numeric) or
         length(") < length(get_coeff(poly_table[i],index_ref[col])) )
         then
            # NB: above condition is not as time-inefficient to check as
            #     first seems as "<>0 will fail for most entries and the
            #     rest of the expression is avoided (Maple uses the
            #     McCarthy evaluation rules for logical operators).
         i := j;
      fi;
   od;
      # If a pivot has been found then perform "row reduction" and move
      # to next row, otherwise (zero column) stay on the same row.
   if i <= num_polys then
         # swap rows first ?
      if i <> row then
         temp := copy(poly_table[i]);
         poly_table[i] := copy(poly_table[row]);
         poly_table[row] := copy(temp);
      fi;
      for i from row+1 to num_polys do
            # reduce row i ?
         if normal(get_coeff(poly_table[i],index_ref[col])) <> 0 then
            scalar := normal(get_coeff(poly_table[i],index_ref[col])/
                             get_coeff(poly_table[row],index_ref[col]));
               poly_table[i] := normal(poly_table[i]-
                                       scalar*poly_table[row]);
         fi;
      od;
      row := row+1;
   fi;
od;
   # store rank
num_polys := row-1;

   # Store (in the table pre_tgtspace) target_dim number of copies of
   # poly_table to form (echelon) matrix for whole of left tangent space.
count := 1;
   # The order below is CRUCIAL: do i loop before j loop.
for i from 1 to num_polys do
   for j from 1 to target_dim do
      pre_tgtspace[count] := canonical_vector(j,poly_table[i],target_dim);
      count := count+1;
   od;
od;
   # Store number of vectors (note that pre_num_vectors IS >= 1
   # because we ensured num_polys >= 1 at the start of this bit.
pre_num_vectors := count-1;

   # If using the ordering induced by the nilpotent weights, then
   # re-order the last block of the matrix (represented by) pre_tgtspace
   # (the block of terms of degree k) and then reduce this block.
```

```
if nilp=true_order then
      # Find the start of the degree k terms, these start at
      # column deg_ref[k]*target_dim+1.
   row := 1;
   col := 1;
   while row<=pre_num_vectors and col<deg_ref[k]*target_dim+1 do
      if coeff_table(pre_tgtspace,row,col)=0 then
         col := col+1;
      else
            # hit a pivotal element so ...
         row := row+1;
         col := col+1;
      fi;
   od;
      # Reduce this last block of degree k terms using Gaussian
      # elimination. Use value of row just calculated.
      # NB: pivotal elements are left as they are found, they must NOT be
      #     scaled to produce a 1 for the pivotal element.
   for col from deg_ref[k]*target_dim+1 to deg_ref[k+1]*target_dim while
                row <= pre_num_vectors do
         # find pivot
      for i from row to pre_num_vectors while
                 coeff_table(pre_tgtspace,i,col) = 0 do   od;
         # Give preference to numeric pivots, but otherwise
         # choose most efficient pivot from a symbolic point of view
         # (that is the one of least length as a Maple expression).
      for j from i+1 to pre_num_vectors do
         coeff_table(pre_tgtspace,j,col);
         if "<>0 and
            not type(coeff_table(pre_tgtspace,i,col),numeric) and
            ( type(",numeric) or
            length(") < length(coeff_table(pre_tgtspace,i,col)) )
            then
            i := j;
         fi;
      od;
         # If a pivot has been found then perform "row reduction" and move
         # to next row, otherwise (zero column) stay on the same row.
      if i <= pre_num_vectors then
            # swap rows first ?
         if i <> row then
            temp := copy(pre_tgtspace[i]);
            pre_tgtspace[i] := copy(pre_tgtspace[row]);
            pre_tgtspace[row] := copy(temp);
         fi;
         for i from row+1 to pre_num_vectors do
               # reduce row i ?
            if coeff_table(pre_tgtspace,i,col) <> 0 then
               scalar := normal(coeff_table(pre_tgtspace,i,col)/
                                coeff_table(pre_tgtspace,row,col));
               for j from 1 to target_dim do
```

```
                pre_tgtspace[i][j] := normal(pre_tgtspace[i][j]-
                                      scalar*pre_tgtspace[row][j]);
             od;
          fi;
       od;
       row := row+1;
    fi;
  od;   # NB: rank already stored as pre_num_vectors and will not
        #     have changed due to this reordering.
fi;   # otherwise nilp <> true_order and everything already OK.

  # Now perform full Gaussian elimination to create table tgtspace.
  # Note that there is no need to reduce pre_tgtspace as this is
  # already in echelon form. Just adjoin the right tangent space
  # (currently stored as tgtspace) and then apply Gaussian elimination
  # using the previously calculated pivotal elements of pre_tgtspace as
  # pivots, where possible, and avoiding interchanging the rows (except
  # where necessary in tgtcopy).
  # This method is very efficient.

  # Check the (right) tangent space is non-empty; if empty
  # then add a zero vector so that the rest of the
  # prodedure may continue with the calculations.
if num_vectors=0 then
  num_vectors := 1;
  tgtspace[1] := array(sparse,1..target_dim);
fi;

print(`calculating tangent space`);
print(`matrix dimensions: `.num_vectors,deg_ref[k+1]*target_dim);
tgtcopy := copy(tgtspace);
  # Use variables row and col for pre_tgtspace "matrix",
  # row2 and col for tgtcopy "matrix"
  # (this is now used to store the right tangent space),
  # and variable count for each element ("row") in the final (echelon)
  # "matrix" (THIS will now be stored as the global table tgtspace).
  # NB: number of columns = deg_ref[k+1]*target_dim.
  #     Number of vectors in the table tgtcopy was stored as the
  #     variable num_vectors earlier.
row := 1;
row2 := 1;
count := 1;
tgtspace := table();
  # Gaussian elimination
  # NB: pivotal elements are left as they are found, they must NOT be
  #     scaled to produce a 1 for the pivotal element.
  # Note that below we must take col up to deg_ref[k+1]*target_dim and
  # not have a termination condition dependent on row (or row2) as in
  # previous cases. This ensures that ALL the appropriate vectors from
  # pre_tgtspace and tgtcopy are included in the final table tgtspace.
  # When row or row2 become too large the appropriate section of code
```

```
   # below is ignored accordingly.
for col from 1 to deg_ref[k+1]*target_dim do
      # Use current element in pre_tgtspace as a pivot to reduce tgtcopy?
      # NB: following statement OK because if row>pre_num_vectors then
      #     second half of 'and' statement ignored (McCarthy rules).
   if row<=pre_num_vectors and coeff_table(pre_tgtspace,row,col)<>0 then
         # store current row from pre_tgtspace in tgtspace
      tgtspace[count] := copy(pre_tgtspace[row]);
      count := count+1;
         # perform elimination on tgtcopy
      for i from row2 to num_vectors do
         if coeff_table(tgtcopy,i,col)<>0 then
            scalar := normal(coeff_table(tgtcopy,i,col)/
                             coeff_table(pre_tgtspace,row,col));
            for j from 1 to target_dim do
               tgtcopy[i][j] := normal(tgtcopy[i][j]-
                                  scalar*pre_tgtspace[row][j]);
            od;
         fi;
      od;
      row := row+1;
   else
         # Whole of the current column in pre_tgtspace must be zero so
         # find appropriate pivot from tgtcopy.
         # Now only need to reduce column in tgtcopy.
      for i from row2 to num_vectors while
                coeff_table(tgtcopy,i,col) = 0 do   od;
         # Give preference to numeric pivots, but otherwise
         # choose most efficient pivot from a symbolic point of view
         # (that is the one of least length as a Maple expression).
      for j from i+1 to num_vectors do
         coeff_table(tgtcopy,j,col);
         if "<>0 and
            not type(coeff_table(tgtcopy,i,col),numeric) and
            ( type(",numeric) or
            length(") < length(coeff_table(tgtcopy,i,col)) )
            then
            i := j;
         fi;
      od;
         # If a pivot has been found then store current row from tgtcopy
         # in tgtspace, perform "row reduction" and move to next row of
         # tgtcopy; otherwise (zero column) stay on the same row.
      if i <= num_vectors then
            # swap rows first ?
         if i <> row2 then
            temp := copy(tgtcopy[i]);
            tgtcopy[i] := copy(tgtcopy[row2]);
            tgtcopy[row2] := copy(temp);
         fi;
            # store current row from tgtcopy in tgtspace
```

```
            tgtspace[count] := copy(tgtcopy[row2]);
            count := count+1;
               # perform elimination
            for i from row2+1 to num_vectors do
                  # reduce row i ?
               if coeff_table(tgtcopy,i,col) <> 0 then
                  scalar := normal(coeff_table(tgtcopy,i,col)/
                                 coeff_table(tgtcopy,row2,col));
                  for j from 1 to target_dim do
                     tgtcopy[i][j] := normal(tgtcopy[i][j]-
                                        scalar*tgtcopy[row2][j]);
                  od;
               fi;
            od;
            row2 := row2+1;
         fi;
      fi;   # if row<=pre_num_vectors then ... else ... fi;
   od;      # col loop

      # Finally store the rank (number of vectors in tgtspace) as the
      # global variable basis_dim.
   basis_dim := count-1;
      # And store the number of columns as a global variable.
   tgtstore_lim := deg_ref[k+1]*target_dim;
fi;

#   ***   END OF REDUCTION ROUTINE   ***

#   ***   ROUTINE TO CALCULATE BASIS FOR COMPLEMENTARY SPACE   ***

   # Calculates and stores (as global variables) the basis, the codimension,
   # and a checklist of all non-numeric pivotal elements in the echelon matrix
   # (represented by) tgtspace.

   i := 1;   # the row
   j := 1;   # the column
      # set up global count variables
   basis_count := 1;
   clist_count := 1;

      # Search through the "matrix" tgtspace looking for pivotal elements.
   while i<=basis_dim and j<=deg_ref[k+1]*target_dim do
         # Check for zero entry, this corresponds to a vector in the basis.
      if coeff_table(tgtspace,i,j)=0 then
         canonical_vector(coeffarray_ref[j][1],get_monomial
                        (index_ref[coeffarray_ref[j][2]]),target_dim);
         compbasis[basis_count][1] := ";
            # check for 'degree(monomial) = k'
         compbasis[basis_count][2] := false;
         if deg_ref[k]<=coeffarray_ref[j][2] and
               coeffarray_ref[j][2]<deg_ref[k+1] then
```

```
            compbasis[basis_count][2] := true;
         fi;
         basis_count := basis_count+1;
         j := j+1;
      else
            # Hit a pivotal element; see if numeric and store in checklist
            # along with corresponding monomial vector (for which this element
            # is a coefficient).
         coeff_table(tgtspace,i,j);
         if not type(",numeric) then
            checklist[clist_count][1] := ";
            checklist[clist_count][2] := canonical_vector(coeffarray_ref[j][1],
             get_monomial(index_ref[coeffarray_ref[j][2]]),target_dim);
            clist_count := clist_count+1;
         fi;
         i := i+1;
         j := j+1;
      fi;
   od;
      # Check for i > basis_dim, for then the vectors corresponding
      # to the remaining columns (Ie. from j to deg_ref[k+1]*target_dim) are
      # to be included in the complementary basis.
   if i>basis_dim then
      while  j<=deg_ref[k+1]*target_dim do
         canonical_vector(coeffarray_ref[j][1],get_monomial
                          (index_ref[coeffarray_ref[j][2]]),target_dim);
         compbasis[basis_count][1] := ";
            # check for 'degree(monomial) = k'
         compbasis[basis_count][2] := false;
         if deg_ref[k]<=coeffarray_ref[j][2] and
              coeffarray_ref[j][2]<deg_ref[k+1] then
            compbasis[basis_count][2] := true;
         fi;
         basis_count := basis_count+1;
         j := j+1;
      od;
   fi;

      # store the codimension
   codim := deg_ref[k+1]*target_dim-basis_dim;

#   ***   END OF COMPLEMENTARY SPACE ROUTINE   ***

# Print warning if checklist non-empty and RETURN NULL.

   if clist_count>1 then
      print(`WARNING: global variable 'checklist' is non-empty !!!`);
   fi;
   print();
   print(`Ready.`);
   NULL;
```

```
end:
```

B.2 User Functions

```
# The parameters passed to this procedure are all of type list and specify
# vectors in J^k(n,p). If the space spanned by these vectors is in the
# previously calculated tangent space (given by coeffarray) then true is
# RETURNED, otherwise false. (Specifically, true is RETURNED when this set
# of vectors and the basis given by coeffarray form a dependent set of
# vectors, and false when they form an independent set.)
# A set of vectors by which the tangent space basis must be extended to give
# a basis for the whole space is calculated and stored as the global variable
# ext_tangent (a matrix whose rows give the coordinates of the basis elements).
# So rank(ext_tangent)=(number of parameters) <=> independent.
# In some cases this may contain non_numeric elements (even if the parameters
# passed to the procedure do not) and the rank may drop for certain values.
# In such cases ext_tangent is output to allow the user to determine the
# degenerate cases. It is global and may therefore be inspected later too.

# NB: the appropriate reference tables (whether for standard jet-spaces or
#     weighted jet-spaces) will have already been created by jetcalc or wtcalc
#     as appropriate, and stored as the common global variables gl_index_ref
#     and gl_deg_ref. These tables must be used below and allow the routine
#     to work for both cases without the need to re-calculate the reference
#     tables depending on the case in question.

intangent := proc()
   local i,j,jj,k,t,v,rank;

      # check parameter type
   if nargs<1 then
      ERROR(`parameters must be of type 'list'`);
   fi;
   for jj from 1 to nargs do
      if not type(args[jj],list) then
         ERROR(`parameters must be of type 'list'`);
      fi;
   od;

      # Get coefficient matrix (v) for the vector parameters.
   v := array(1..nargs,1..tgtstore_lim);
   for jj from 1 to nargs do
      for j from 1 to tgtstore_lim do
         expand(args[jj][coeffarray_ref[j][1]]);
         v[jj,j] := get_coeff(",gl_index_ref[coeffarray_ref[j][2]]);
      od;
   od;
```

```
   # row reduce each vector (row of v) using the matrix coeffarray
   # NB: no need to use full Gaussian elimination as coeffarray is
   #     already in echelon form
for jj from 1 to nargs do
   i := 1;
   j := 1;
   while i<=basis_dim and j<=tgtstore_lim do
      if coeff_table(tgtspace,i,j)=0 then
         j := j+1;
      else
         if v[jj,j]<>0 then
               # Perform Gaussian elimination. We do not need to use
               # fraction-free Gaussian elimination: instead a scaled copy
               # of the row containing the pivotal element is subtracted
               # from the current row, jj, (cf. use of t below). However,
               # it is important NOT to scale the row itself (as in standard
               # Gaussian elimination) in order to make the pivotal element
               # equal to 1.
               # NB: this is done more efficiently "manually" than using
               #     the 'linalg' library for row manipulations
            t := normal(v[jj,j]/coeff_table(tgtspace,i,j));
            for k from j+1 to tgtstore_lim do
               v[jj,k] := normal(v[jj,k]-t*coeff_table(tgtspace,i,k));
            od;
               # and finally ...
            v[jj,j] := 0;
         fi;
         i := i+1;
         j := j+1;
      fi;
   od;
od;

   # Now reduce matrix v to echelon form and obtain its rank.
   # (NB: Maple V procedure 'rank' will not use standard Gaussian
   # elimination, so call the 'gausselim' procedure explicitly.)
   # Store row reduced v as the global variable ext_tangent.
   # Then if rank(v) = nargs, the tangent space (coeffarray) and vectors
   # passed as parameters form an independent set
ext_tangent := gausselim(v,'rank');
   # print warning if matrices not numeric
if not type(ext_tangent,'matrix'(numeric)) then
   print
    (`WARNING: matrix contains non-numeric elements, check ext_tangent:`);
   print(`tangent space basis to be extended by (the rows of)`);
   print(ext_tangent);
fi;
if clist_count>1 then
   print
    (`WARNING: original matrix contains non-numeric elements, check checklist !!!`)
```

```
   fi;
   if rank = nargs then
         # RETURN false
      false;
   else
         # rank < nargs; RETURN true
      true;
   fi;

end:


# prints the table 'compbasis' output from the 'jetcalc' procedure

pcomp := proc()
   local i,print_flag;

   print_flag := false;
   for i from 1 to basis_count-1 do
         # print vector?
      if compltrans=false or compbasis[i][2]=true then
         print(convert(compbasis[i][1],list));
         print_flag := true;
      fi;
   od;
   if print_flag=false then
      print(`***   TRANSVERSAL EMPTY   ***`);
   fi;
      # RETURN NULL
   NULL;

end:


# prints the table 'det_store' output from the 'Adetermined' procedure

pdetterms := proc()
   local i;

   for i from 1 to det_count-1 do
      print(convert(det_store[i],list));
   od;

end:


# Prints the table 'checklist' output from the 'jetcalc' or 'wtcalc' procedures.
# Each element in 'checklist' represents a monomial vector and the precise
# format is to output the number of the element in the table checklist followed
# by the coefficient and monomial term of the vector. Thus the actual vector is
# given by 'coefficient' * 'monomial term'. The number of the element is output
```

```
# so that the vector can be accessed by the user; the coefficient and monomial
# term of the r-th entry in checklist being given by checklist[r][1] and
# checklist[r][2] respectivey. For instance, to factor the coefficient of the
# r-th vector one would use (in Maple)
# factor(checklist[r][1]);

# Parameters may be passed to the procedure; they must take the form of a
# flag with value 'P' or 'F'. If 'P' (PAUSE) is passed then the elements of the
# checklist are printed out in turn with the procedure waiting for the user
# to type C;[RETURN] (C followed by a semi-colon followed by the [RETURN] key)
# (CONTINUE) before proceeding with the next element. Alternatively the
# procedure may be terminated at this point by typing E;[RETURN] instead of
# C;[RETURN] (for EXIT).
# If 'F' (FACTOR) is passed then the elements in checklist which are the
# coefficients (of the appropriate vector) are factored before being output.

plist := proc()
   local i,pause_flag,factor_flag,input_char;

      # check parameter
   pause_flag := false;
   factor_flag := false;
   for i from 1 to nargs do
      if args[i]='P' or args[i]='F' then
         if args[i]='P' then
            pause_flag := true;
         else
            factor_flag := true;
         fi;
      else
         ERROR(`parameter must be the flag 'P' or 'F' or empty`);
      fi;
   od;

      # print checklist
   for i from 1 to clist_count-1 do
      if factor_flag then
         factor(checklist[i][1]);
      else
         checklist[i][1];
      fi;
      print(`#`.i,'',checklist[i][2]);
      if pause_flag then
         input_char := NULL;
         while input_char<>'E' and input_char<>'C' do
            input_char := readstat
               (`type E;[RETURN] to exit, or C;[RETURN] to continue ... `);
         od;
         if input_char='E' then
            RETURN(NULL);
         fi;
```

```
      fi;
   od;
   if clist_count=1 then
      print(`***   CHECKLIST EMPTY   ***`);
   fi;
      # RETURN NULL
   NULL;

end:
```

```
# Prints the set of homogeneous monomial vectors of degree k1. If a second
# (optional) parameter, k2, is given then all vectors of degree k1 to degree
# k2 are output.

# The order in which these monomial vectors are printed is that used by jetcalc
# when creating the array of coefficients, coeffarray, with the lower order
# terms being output first. If global variable nilp is set to false then the
# default order is used; however if nilp is set to 'true-order' then the order
# induced by the nilpotent filtration is used.
# (NB: the global lists nilp_source_wt and nilp_target_wt must be defined and
# are used to determine this order.)
# In such a case the homogeneous monomial vectors of degree k are partitioned
# into their appropriate M(r,s) jet-level, this level being output too.

# NB: coordinates are specified by the global variable coords, which must
# therefore be defined beforehand. Note that liealg procedures will assign
# values to coords when appropriately called (usually by jetcalc).

pmons := proc(k1)
   local i,j,k,k2,count,level,weight,wt,index_ref,deg_ref,
         nilp_pt_ref,nilp_wt_ref;

      # set up initial variables
   if not type(coords,list) then
      ERROR(`coordinates undefined (variable coords not of type list)`);
   fi;
   print(`defined coordinates:`);
   print(coords);
   num_coords := nops(coords);
      # initialise ordering variables
   if nilp='true_order' then
      if not type(nilp_source_wt,list) or not type(nilp_target_wt,list) then
         ERROR(`variables nilp_source_wt and nilp_target_wt not of type list`);
      fi;
      target_dim := nops(nilp_target_wt);
      if nops(nilp_source_wt)<>num_coords then
         ERROR(`source dimension and list nilp_source_wt incompatible`);
      fi;
      print(`using ordering induced by the nilpotent weights`);
      nilp_pt_ref := table(); nilp_wt_ref := table();
```

```
else
   if nilp=true or nilp=false then
      print(`using default ordering`);
   else
      ERROR
(`global variable 'nilp' must be set as 'true', 'false', or 'true_order'`)
   fi;
fi;

   # get degree (k) limits
if nargs=2 then
   k2 := args[2]
else
   k2 := k1;
fi;

   # create reference tables
index_ref := table();
deg_ref := table();
get_ref_tables(k2,num_coords,index_ref,deg_ref);

   # go through all required degrees (k1 to k2) and output vectors
for k from k1 to k2 do
   print(`monomial vectors of degree `.k);
   if nilp='true_order' then
         # use nilpotent filtration and corresponding jet-space
      level := 1;
         # run through all weights for the degree k monomials
      for weight from k to k*num_coords+target_dim-1 do
            # firstly create the nilpotent reference tables by ordering the
            # degree k monomials by their weight
         count := 0;
         for wt from k to k*source_dim do
            nilp_wt_ref[wt] := count;
            for deg from deg_ref[k] to deg_ref[k+1]-1 do
               if get_wt(index_ref[deg],num_coords,nilp_source_wt)=
                     wt then
                  nilp_pt_ref[count] := deg;
                  count := count+1;
               fi;
            od;
         od;
            # store pointer to end of table
         nilp_wt_ref[k*source_dim+1] := count;
            # now print the terms at this level
         print(`level: (`.k.`,`.level.`)`);
         for j from 1 to target_dim do
            weight+nilp_target_wt[j];
            if k<=" and "<=k*source_dim then
               for i from nilp_wt_ref["] to nilp_wt_ref["+1]-1 do
                  print(convert( canonical_vector(j,get_monomial
```

```
                    (index_ref[nilp_pt_ref[i]]),target_dim), list));
               od;
            fi;
         od;
         level := level+1;
      od;
   else
         # use standard ordering and jet-spaces
      for i from deg_ref[k] to deg_ref[k+1]-1 do
         for j from 1 to target_dim do
            print(convert( canonical_vector(j,get_monomial
             (index_ref[i]),target_dim), list));
         od;
      od;
   fi;
   print(`------------`);
od;
   # RETURN NULL
NULL;

end:


# Prints a canonical basis for the tangent space as output by jetcalc (in
# the matrix coeffarray). Vectors are represented as a sum, the constituent
# monomial vectors and their coefficients being output for each vector.
# If a (monomial vector) parameter is passed only tangent vectors containing
# it as a term are output ( NB: parameter must be of type list )

# NB: the appropriate reference tables (whether for standard jet-spaces or
#     weighted jet-spaces) will have already been created by jetcalc or wtcalc
#     as appropriate, and stored as the common global variables gl_index_ref
#     and gl_deg_ref. These tables must be used below and allow the routine
#     to work for both cases without the need to re-calculate the reference
#     tables depending on the case in question.

ptangent := proc()
   local i,j,ii,mon_term,print_flag;

      # check correct number of parameters
   if nargs<>0 and nargs<>1 then
      ERROR(`wrong number of parameters passed`);
   fi;
      # check parameter of type list
   if nargs = 1 and not type(args[1],list) then
      ERROR(`parameter must be of type 'list'`);
   fi;

   print(`***   basis for tangent space   ***`);
   print(`vectors output as monomial terms and corresponding coefficients`);
   for i from 1 to basis_dim do
```

```
    print_flag := true;
    if nargs = 1 then
          # search for the monomial passed as a parameter (args[1]) as
          # a term in the tangent vector and print vector iff it is present
       print_flag := false;
       j := 1;
       while print_flag=false and j<=tgtstore_lim do
             # first check coefficent of term corresponding to j is zero
          if coeff_table(tgtspace,i,j)<>0 then
             print_flag := true;
                # now see if monomial term corresponding to j is
                # equal to parameter passed to procedure
             mon_term := canonical_vector(coeffarray_ref[j][1],get_monomial
                         (gl_index_ref[coeffarray_ref[j][2]]),target_dim);
             for ii from 1 to nops(args[1]) do
                if mon_term[ii]<>args[1][ii] then
                   print_flag := false;
                   break;
                fi;
             od;
          fi;
          j := j+1;
       od;
    fi;
    if print_flag = true then
          # print vector
       print(`vector`.i);
       for j from 1 to tgtstore_lim do
          if coeff_table(tgtspace,i,j)<>0 then
             print(coeff_table(tgtspace,i,j), convert(canonical_vector
                  (coeffarray_ref[j][1],get_monomial(gl_index_ref
                  [coeffarray_ref[j][2]]),target_dim), list));
          fi;
       od;
    fi;
  od;
     # RETURN NULL
  NULL;

end:


# prints-out the global variables

pvars := proc()

  print(`liealg = `.liealg);
  print(`equiv = `.equiv);
  print(`compltrans = `.compltrans);
  print(`source_dim = `.source_dim);
  print(`source_power = `.source_power);
```

```
   print('target_power = '.target_power);
   print('nilp = '.nilp);
   print('R_nilp:');
   print(R_nilp);
   print('L_nilp:');
   print(L_nilp);
      # RETURN NULL
   NULL;

end:


# Given a germ f, test whether it is k-A-determined using the determinacy
# criterion 'm(n)^(k+1).E(n,p) contained in LG.f + m(n)^(2k+2).E(n,p)' (G a
# unipotent subgroup of A) implies f is k-A-determined.
# This is done by calculating successive complete transversals (from k+1 up to
# 2k+1) and checking that they are empty. If the determinacy criterion fails
# then output the first non-empty transversal.

# If an optional third parameter r is passed then (provided it lies between
# k+1 and 2k+1) the complete transversals are only checked up to degree r.
# This provides an implementation of the 'extended determinacy criterion':
# 'm(n)^(k+1).E(n,p) contained in LG.f + m(n)^(k+1).f^(*)(m(p)).E(n,p) +
# m(n)^(2k+2).E(n,p)' implies f is k-determined.

Aclassify := proc(f,k)
   local i,r,kk;

      # set the limit degree r
   r := 2*k+1;
   if nargs=3 then
      if type(args[3],posint) and args[3]>=k+1 and args[3]<2*k+1 then
         r := args[3];
      else
         ERROR('third parameter must be a postive integer r; k+1 <= r < 2k+1');
      fi;
   fi;

      # check tranversals from degree k+1 to degree r
   print('***   checking transversals up to degree '.r.'   ***');
   for kk from k+1 to r do
      jetcalc(f,kk);
         # check transversal is empty
      for i from 1 to basis_count do
         if compbasis[i][2]=true then
               # NB: the degree 'kk' will have been stored globally as
               # 'jetspace_deg' should the user need to refer to it
            print(f);
            print('the '.kk.' tranversal is non-empty:');
            pcomp();
            print('given determinacy condition failed');
```

```
            RETURN(NULL);
         fi;
      od;
      print(`the `.kk.` transversal is empty`);
      print(`--------------------`);
   od;

      #  all tranversals empty: germ is k-A-determined
   print(f);
   print(`germ is `.k.`-A-determined`);

      # RETURN NULL
   NULL;

end:


# Uses criterion 'm(n)^(k+1).E(n,p) contained in LG.f + m(n)^(2k+2).E(n,p)
# implies f is k-A-determined' (G a unipotent subgroup of A) to test for k
# determinacy of the map-germ f.
# If determinacy fails the offending terms are stored in the global
# variable det_store and then output using the function pdetterms.

# If an optional third parameter r is passed then (provided it lies between
# k+1 and 2k+1) the jet-space used is of degree r instead of degree 2k+1.
# This provides an implementation of the `extended determinacy criterion':
# 'm(n)^(k+1).E(n,p) contained in LG.f + m(n)^(k+1).f^(*)(m(p)).E(n,p) +
# m(n)^(2k+2).E(n,p)' implies f is k-determined.


# NB: the index and degree reference tables have already been calculated by
#     jetcalc and stored as the globals gl_index_ref and gl_deg_ref.

Adetermined := proc(f,k)
   local j,count,u,r;

      # set the limit degree r
   r := 2*k+1;
   if nargs=3 then
      if type(args[3],posint) and args[3]>=k+1 and args[3]<2*k+1 then
         r := args[3];
      else
         ERROR(`third parameter must be a postive integer r; k+1 <= r < 2k+1`);
      fi;
   fi;

      # calculate tangent space LG.f
   jetcalc(f,r);

      # check determinacy condition
   print(`*** checking determinacy condition ***`);
```

```
      # set up global table for storing any vectors which fail determinacy test
   det_store := table();
      # see if the monomial vectors m(n)^(k+1).E(n,p) are in the tangent space
   print(`number of vectors to check`,gl_deg_ref[r+1]*target_dim-
         gl_deg_ref[k+1]*target_dim);
   count := 1;
   for j from gl_deg_ref[k+1]*target_dim+1 to gl_deg_ref[r+1]*target_dim do
         # more user info ???
      if 1 < printlevel then
         lprint(`adetermined: checking vector`,j-gl_deg_ref[k+1]*target_dim);
      fi;
      u := canonical_vector(coeffarray_ref[j][1],get_monomial
                       (gl_index_ref[coeffarray_ref[j][2]]),target_dim);
      if not intangent(convert(u,list)) then
         det_store[count] := copy(u);
         count := count+1;
      fi;
   od;
      # store count as a global
   det_count := count;

      # RETURN the result
   print(f);
   if det_count=1 then
      print(`germ is `.k.`-A-determined`);
     else
      print(`given determinacy condition failed, due to missing vectors:`);
      pdetterms();
   fi;

      # now RETURN NULL
   NULL;

end:
```

B.3 Liealg Routines

```
# RIGHT-EQUIVALENCE SET-UP PROCEDURE

# procedure to define standard Jacobian Lie algebra ( pseudo tgt.space )
# dimension of source manifold given by the global variable 'source_dim'

stdjacobian := proc(f,target_dim,tgtspace)
   local i,j,coords_temp;

      # DEFINE COORDINATES (global variable, data type 'list')
      # denote coordinates as x1,x2,...xn, where n = source_dim
```

```
    # first check all source coordinates are formal indeterminates,
    # Ie. are unassigned as Maple expressions
  for i from 1 to source_dim do
    if assigned(`x`.i) then
      ERROR(`not all source coordinates are unassigned Maple names`);
    fi;
  od;
  coords_temp := array(1..source_dim);
  for i from 1 to source_dim do
    coords_temp[i] := `x`.i;
  od;
    # now convert coords_temp to type list to form coords
  coords := convert(coords_temp,list);

    # DEFINE LIE ALGEBRA GENERATING SET (data type 'table')
  tgtspace := table();
  for i from 1 to source_dim do
    for j from 1 to target_dim do
      tgtspace[i][j] := diff(f[j],coords[i]);
    od;
  od;

    # RETURN NULL
  NULL;

end:


# RIGHT-EQUIVALENCE SET-UP PROCEDURE

# procedure to define Lie algebra tgt. to the cusp discriminant

cusp := proc(f,target_dim,tgtspace)

    # DEFINE COORDINATES (global variable, data type 'list')
  if assigned(`u1`) or assigned(`u2`) then
    ERROR(`not all source coordinates are unassigned Maple names`);
  fi;
  coords := [u1,u2];

    # DEFINE LIE ALGEBRA GENERATING SET (data type 'table')
  tgtspace := table();
  tgtspace[1] := [9*u2*diff(f[1],u1) - 2*u1^2*diff(f[1],u2)];
  tgtspace[2] := [2*u1*diff(f[1],u1) + 3*u2*diff(f[1],u2)];

    # DEFINE WEIGHTS (global variables, data type 'list')
  source_wt := [2,3];
  target_wt := [0];

    # DEFINE NILPOTENT VECTORS (global variables, data type 'list')
  R_nilp := [ [1,1] ];
```

```
   L_nilp := [];

      # RETURN NULL
   NULL;

end:
```

B.4 Subroutines

```
# RETURNS the vector ( data type 'sparse array' ) of dimension 'n'
# whose ith component contains 'poly', all other entries being zero

canonical_vector := proc(i,poly,n)
   local vector;

   vector := array(sparse,1..n);
   vector[i] := poly;
      # RETURN 'vector'
   op(vector);

end:


# returns the coeff of a monomial ( specified via index )
# in the given ( multivariate ) polynomial

get_coeff := proc(poly,index)
   local p,i;

   p := expand(poly);
   for i from 1 to num_coords do
      coeff(p,coords[i],index[i]);
      p := ";
   od;

end:


# returns the monomial with indices given by the table 'index'
# the indeterminates are specified by global variable 'coords'

get_monomial := proc(index)
   local i,monomial;

   monomial := 1;
   for i from 1 to num_coords do
      monomial := monomial*coords[i]^index[i];
   od;
```

```
end:


# set up index and degree reference tables
# index_ref() : a table of indices corresponding to our monomials
# deg_ref[r] : where degree r indices begin in index_ref table
# ( indices go from deg 0 to deg k in 'num_indets' indeterminates )

get_ref_tables := proc(k,num_indets,index_ref,deg_ref)
   local count,deg,i,index;

   index := table();
   count := 0;
   for deg from 0 to k do
      deg_ref[deg] := count;
         # set up first index of degree 'deg', namely [0,...,0,deg]
      for i from 1 to num_indets-1 do
         index[i] := 0;
      od;
      index[num_indets] := deg;
      index_ref[count] := copy(index);
         # increase index list by one starting with 'num_indets-1' index
         # 'increment' procedure returns false when no longer possible
      while increment(index,deg,num_indets) do
         count := count+1;
         index_ref[count] := copy(index);
      od;
         # move onto indices of the next degree
      count := count+1;
   od;
      # store pointer to end of table
   deg_ref[k+1]:=count;
      # RETURN NULL
   NULL;

end:


# tries to increase the set of indices stored in table parameter 'index'
# by one, starting from the n-1 st. index ( nth index is predetermined
# by 'homogenity' requirement ), subject to the total index degree
# being <= deg. If this holds then TRUE is returned, otherwise FALSE

increment := proc(index,deg,n)
   local i,j;

      # first try to increase n-1 st. index
   i  := n-1;
   while i>0 do
      index[i] := index[i]+1;
```

```
         # if the sum of the indices exceeds the degree then try
         # to increase the next index along to the left instead
      if get_deg(index,n-1)>deg then
         for j from i to n-1 do
            index[j] := 0;
         od;
         i := i-1;
      else
            # increment worked ok so leave the while loop
         i := -1;
      fi;
   od;
      # i=0 means all indices of given degree obtained: return 'false'
   if i=0 then
      RETURN(false);
   fi;
      # otherwise set the nth component of the list appropriately
      # to give indices of the required degree and return 'true'
   index[n] := deg-get_deg(index,n-1);
   true;

end:


# calculates the degree of the monomial with indices given by the table
# a, in m indeterminates

get_deg := proc(a,m)
   local i,sum;

   sum := 0;
   for i from 1 to m do
      sum := sum+a[i];
   od;
   sum;

end:


# RETURNS the minimum degree of all the non-zero components of the
# polynomial vector 'p' ( or 'k+1' if no components of degree <= k )
# NB: MAPLE defines the degree of the zero polynomial to be 0, but we are
#     not interested in such cases and therefore ignore zero components.
#     Thus if 'p' is identically zero degree 'k+1' is RETURNED

ldegree_vector := proc(p,k,n)
   local i,least;

   least := k+1;
   for i from 1 to n do
      if p[i]<>0 then
```

```
         # NB: for ldegree to return the total least degree the list
         #     of variables must in fact be of data type 'set'
      ldegree(expand(p[i]),convert(coords,set));
      if "<least then
         least := ";
      fi;
   fi;
od;

   # RETURN 'least'
least;

end:


# multiply vector 'v' ( of dim 'n' ) by scalar 's'

scalar_multn := proc(s,v,n)
   local i,v_copy;

   v_copy := table();
   v_copy := copy(v);
   for i from 1 to n do
      v_copy[i] := s*v_copy[i];
   od;
      # return v_copy
   op(v_copy);

end:


# Gets a particular monomial coefficient from a table of polynomial vectors.

# M specifies a table of tangent vectors (polynomial vectors).
# The global refence tables gl_index_ref and coeffarray_ref defined in jetcalc
# and wtcalc are required.
# This function then obtains the component of the ith vector in M indicated
# by the value of coeffarray_ref[j][1]. This component is a polynomial; the
# coefficient of the monomial with index given by the value of
# gl_index_ref[coeffarray_ref[j][2]] is then RETURNED.

coeff_table := proc(M,i,j)

   M[i][coeffarray_ref[j][1]];
   normal(get_coeff(",gl_index_ref[coeffarray_ref[j][2]]));
      # this coefficient is RETURNED

end:
```

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