THE CLASSIFICATION OF GERMS OF MAPS FROM SURFACES TO 3-SPACE, WITH APPLICATIONS TO THE DIFFERENTIAL GEOMETRY OF IMMERSIONS

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ABSTRACT

THE CLASSIFICATION OF GERMS OF MAPS FROM SURFACES TO 3-SPACE, WITH APPLICATIONS TO THE DIFFERENTIAL GEOMETRY OF IMMERSIONS.

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The original aim of the research project out of which this thesis grew was to study the differential geometry of immersions of surfaces in $\mathbb{R}^4$, by considering the $\mathcal{A}$ classes of germs of radial projections of the immersed surfaces into affine hyperplanes of $\mathbb{R}^4$, the immersions in question to be subjected to certain genericity requirements. As a preliminary step towards this programme, it was necessary to give an $\mathcal{A}$-invariant stratification of $J^k(2,3)$ in which all strata of codimension less than or equal to 6 should be $\mathcal{A}$-sufficient. This classification is presented in Chapter I, and includes the classification of some bi-germs.

In Chapter II, the exponential map $\exp_g : TM \longrightarrow \mathbb{R}^p$ associated with an immersion $g : M \longrightarrow \mathbb{R}^p$, is studied. The principal result is a transversality theorem for $\exp_g$, (Theorem II:1), from which it is possible to deduce that if $(2n,p)$ are nice dimensions (à la Mather), then generically $\exp_g$ is locally stable on TM-M. (Corollary II:2).

In Chapter III, the results of Chapters I and II are applied to the study of the differential geometry of generic immersions, as outlined in the first paragraph of this abstract.

In Chapter IV, some of the results of Chapter I are applied to give "normal forms" for the singularities of the tangent developable of a smooth space curve, via the introduction of the class of "map-germs with a cuspidal edge".

CONTENTS

INTRODUCTION

Let M be a smooth (C^{∞}) k-manifold, and $g:M \longrightarrow \mathbb{R}^n$ a smooth immersion. By studying the germs of radial projections of M from points q in $\mathbb{R}^n - M$ into affine hyperplanes, one may hope to obtain a description of the local differential geometry of the immersion g. Such a study has been made by C.T.C.Wall and his student J.M.Soares David, in the case where k = 1 and n = 3, and is reported by Wall in [28] , where a list is given of all $\mathcal{A}$-classes of germs (including multi-germs) of projections for "generic" immersions g. This list is used in a paper written by H.Morton and the author ([25]) in which it is proved that a generically embedded knot must have a quadrisecant.

Similar studies, replacing radial by orthogonal projection, have been made by Arnol'd [4] , by Gaffney and Ruas [10] , and by McCrory[23] , in the case where k = 2 and n = 3. In all of these, emphasis is placed on obtaining a list of the $\mathcal{A}$ classes to which germs of projection will belong, for a suitably large (residual) set of immersions $g:M \longrightarrow \mathbb{R}^3$, and in the last two the relation between the description so obtained, and the classical differential-geometric description, is also investigated.

In Chapter III of this thesis we study the germs of radial projections in the case k = 2 and n = 4, with the aims of determining, for a suitably defined residual set of immersions,

1) to which $\mathcal{A}$ classes the germs of projection will belong,

2) what is the relation between the $\mathcal{A}$ class of germs of projection

and the differential geometry of the immersion, and

3) what is the nature of the loci on M of the sets of points at which there are germs of projection belonging to the various possible $\mathcal{A}$ classes.

As a first step towards carrying out this programme, it is necessary to give an $\mathcal{A}$-invariant stratification of $J^k(M,\mathbb{R}^3)$, and this we do in Chapter I (Theorem I:2). All $\mathcal{A}$ orbits and modular strata of codimension less than or equal to 6 are given, along with normal forms for germs in these orbits and modular strata, their determinacy degrees, and their $\mathcal{A}$ tangent spaces. A list of all $\mathcal{A}$-simple germs of maps $(\mathbb{R}^2,0)\longrightarrow(\mathbb{R}^3,0)$ is also given (Theorem I:1). For determinacy estimates, and for the calculation of $\mathcal{A}$ tangent spaces, we rely on techniques developed by T.Gaffney ([8]) whose work in turn is based on that of J.Mather ([17]-[22]).

In the course of carrying out this classification, it was found that map-germs which are $\mathcal{K}$-equivalent to

$$(x,y)\longmapsto(x,y^2,0),$$

may all be put in pre-normal forms

$$(x,y)\longrightarrow(x,y^2,yp(x,y^2)),$$

and that the problem of $\mathcal{A}$ classification of such germs reduces to a much simpler problem of $\mathcal{K}$ classification. More precisely, if $\mathcal{K}^T$ is the subgroup of $\mathcal{K}$ which acts on the ring $\mathcal{E}_2^T$ of invariant function germs $p(x,y^2)$, then the $\mathcal{A}$ classification of map-germs of the form described above is equivalent to the $\mathcal{K}^T$ classification of the function germs $p(x,y^2)$ (Theorem I.5:8 and Corollary I.5:11). From this it is easy to deduce that the $\mathcal{A}$ classification problem reduces further, to that of the $\mathcal{K}^{\partial}$ classification of the function germs p(x,y), defined on the half-plane H^2, $\mathcal{K}^{\partial}$ being the contact group which acts naturally on the ring of such germs (Theorem I.5:16). A.du Plessis has informed us that these results may also be deduced from some results that he, T.Gaffney and L.Wilson have obtained ([11], [26]), but the methods of proof involved are quite different. In view of our Theorem I.5:16, we are able to use a classification that V.I.Arnol'd has given ([2]), although because of the rather sparse nature of his proofs, we have given proofs for the small part of his classification that we employ.

For any immersion $g:M \longrightarrow \mathbb{R}^n$, and for any point $(x,q) \in TM \subseteq M \times \mathbb{R}^n$, with $q \neq g(x)$, (here we refer to the usual embedding of TM in $M \times \mathbb{R}^n$), there is an isomorphism between the local algebra at (x,q) of the exponential map $\exp_g : TM \longrightarrow \mathbb{R}^n$, and the $\mathcal{A}$-invariant "singular algebra" (introduced in I.9) of the germ at x of the projection $p_g(q)$ (see II.1 for this notation). This is a consequence of our II.2:3 and II.2:7. This isomorphism is used in Chapter III to obtain information about the relation between the behaviour of the germs of projection and the behaviour of $\exp_g$, but its consequences are studied in more detail and in greater generality in Chapter II. In particular, it is used to prove a transversality theorem for $\exp_g$ (Theorem II:1) which has as a corollary that if (2n,p) are nice dimensions (in the sense of Mather, 21 , 22), then generically $\exp_g$ is locally stable away from the 0-section of TM.

The results in Chapter IV (on map-germs with a cuspidal edge) arose partly out of a desire to study the behaviour, at the 0-section of $T\mathbb{R}$, of the exponential map $\exp_g$ associated with a smooth immersion $g:\mathbb{R}\to\mathbb{R}^3$, and also because the "equivalence of equivalences" discussed in I.5 provided methods by which some of the germs of $\exp_g$, at the 0-section of $T\mathbb{R}$, may be classified up to $\mathcal{A}$ -equivalence. The usual methods of $\mathcal{A}$ classification (such as those employed in Chapter I, from I.6 onwards) are not applicable in this context, because map-germs $(\mathbb{R}^2,0)\longrightarrow(\mathbb{R}^3,0)$ with a non-isolated singularity are not finitely determined ([7]). By introducing the class of map germs with a cuspidal edge, we are able, in some cases, to circumvent this difficulty, and to provide a partial $\mathcal{A}$ classification. Using this classification, we are able to find normal forms for the germ of the tangent developable surface of a smooth space-curve, at points on the curve where the curvature does not vanish and the torsion vanishes to order k, with $0 \leqslant k \leqslant 4$, and at points where the curvature has a "non-degenerate" zero (Theorems IV.5:1 and IV.5:8). This extends work of J.Cleave ([6]) and of T.Gaffney and A.du Plessis ([9]). An analogous study, that of isolated line singularities, which involves the classification up to $\mathcal{R}$ equivalence of certain function germs of infinite $\mathcal{R}$ codimension, has been made by D.Siersma ([27]), but his work bears little relation to our own.

Finally, I would like to thank the people and institutions who have helped me during my stay in Liverpool. First, the University of Liverpool for providing me with financial support during the period of my studies, and my supervisor, Professor C.T.C.Wall, for suggesting the research project which led to this thesis, and for his extremely stimulating help and guidance. I would also like to thank my fellow student James Montaldi for a great many helpful discussions. Lastly, I would like to thank the National University of Colombia for granting me leave of absence in order to come to Liverpool.

CHAPTER I

CLASSIFICATION OF SINGULARITIES OF MAPS FROM SURFACES TO 3-SPACE

I.1 Introduction In this chapter we shall be concerned principally with the classification of smooth map-germs $(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ up to $\mathcal{A}$-equivalence, where $\mathcal{A}$ is the group $\mathrm{Diff}(\mathbb{R}^2, 0) \times \mathrm{Diff}(\mathbb{R}^3, 0)$. We shall also deal briefly with the classification of some multigerms.

We present two main results:

I:1 Theorem Simple map-germs $(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ belong to the $\mathcal{A}$-equivalence classes of the germs in the following list:

Germ		$\mathcal{A}$-codimension	Name
$f(x,y) = (x, y^2, xy)$		2	S_0
$f(x,y) = (x, y^2, y^3 + x^{k+1}y)$	$(k \geqslant 1)$	k+2	S_k
$f(x,y) = (x, y^2, x^2y + y^{2k+1})$	$(k \geqslant 2)$	k+2	B_k
$f(x,y) = (x, y^2, xy^3 + x^ky)$	$(k \geqslant 3)$	k+2	C_k
$f(x,y) = (x, y^2, x^3y + y^5)$		6	F_4
$f(x,y) = (x, xy + y^{3k-1}, y^3)$	$(k \geqslant 2)$	k+2	H_k

I:2 Theorem The following table gives an $\mathcal{A}$-invariant stratification $\mathcal{S}_0$ of $J^r(2,3)-\Sigma^2$ (where Σ^2 is the set of all jets of corank 2) for all r:

Stratum	$\mathcal{A}$-codim.	Name	Reference
$(x, y, 0)$	0	immersion	
(x, y^2, xy)	2	S_0	I.4
$(x, y^2, y^3 + x^{k+1}y)$ k= 1,2,3,4	k+2	S_k	I.5
$(x, y^2, x^2y + y^{2k+1})$ k= 2,3,4	k+2	B_k	I.5
$(x, y^2, xy^3 + x^ky)$ k= 3,4	k+2	C_k	I.5
$(x, y^2, x^3y + y^5)$	6	F_4	I.5
$(x, xy + y^{3k-1}, y^3)$ k= 2,3,4	k+2	H_k	I.6.1
$(x, xy + y^3, xy^2 + cy^4)$ $c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	5	P_3	I.6.2
$(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$	6		I.6.2
$(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$	6		I.6.2
$(x, xy + y^3, xy^2 + y^4 + y^6)$	6		I.6.2
$(x, xy + y^3, xy^2 + y^7)$	6		I.6.2
$(x, xy + y^3, y^4)$	6		I.6.4
$(x, xy + y^6 + by^7, xy^2 + y^4 + cy^6)$	6		I.6.3
$(x, y^3, x^2y + xy^2 + y^4)$	6		I.7
$(x, y^3 - x^2y, xy^2 + y^4)$	6		I.7
$\mathcal{A}^6(x, y^2, y^3)$	7		I.5
$\mathcal{A}^9(x, y^2, x^2y)$	7		I.5
$\mathcal{A}^5(x, y^2, x^3y)$	7		I.5
$\mathcal{A}^5(x, y^2, xy^3)$	7		I.5
$\mathcal{A}^4(x, y^2, 0)$	7		I.5
$\mathcal{A}^{11}(x, xy, y^3)$	7		I.6.1

Stratum	$\mathcal{A}$-codim.	Name	Reference
$\mathcal{A}^3(x,\ 0\ ,0)$	12		I.6
$\mathcal{A}^3(x,\ xy,\ 0)$	7		I.6
$\mathcal{A}^6(x,\ xy + y^3,\ xy^2 + \frac{1}{2}y^4 + y^6)$	7		I.6.2
$\mathcal{A}^6(x,\ xy + y^3,\ xy^2 + \frac{3}{2}y^4 + y^6)$	7		I.6.2
$\mathcal{A}^6(x,\ xy + y^3,\ xy^2 + \frac{1}{2}y^4)$	8		I.6.2
$\mathcal{A}^6(x,\ xy + y^3,\ xy^2 + \frac{3}{2}y^4)$	8		I.6.2
$\mathcal{A}^6(x,\ xy + y^3,\ xy^2 + y^4)$	7		I.6.2
$\mathcal{A}^7(x,\ xy + y^3,\ xy^2)$	7		I.6.2
$\mathcal{A}^4(x,\ xy + y^4,\ xy^2)$	7		I.6.3
$\mathcal{A}^4(x,\ xy,\ xy^2)$	8		I.6.3
$\mathcal{A}^6(x,\ xy,\ xy^2 + y^4 + y^6)$	7		I.6.3
$\mathcal{A}^6(x,\ xy,\ xy^2 + y^4)$	8		I.6.3
$\mathcal{A}^4(x,\ xy + y^3,\ xy^3)$	7		I.6.4
$\mathcal{A}^4(x,\ xy + y^3,\ 0)$	8		I.6.4
$\mathcal{A}^4(x,\ y^3,\ xy^2 + x^2y)$	7		I.7
$\mathcal{A}^4(x,\ y^3 - x^2y,\ xy^2)$	7		I.7
$\mathcal{A}^3(x,\ y^3,\ x^2y)$	7		I.7
$\mathcal{A}^3(x,\ y^3,\ 0)$	9		I.7
$\mathcal{A}^3(x,\ y^3,\ xy^2)$	7		I.7
$\mathcal{A}^3(x,\ y^3 + x^2y,\ 0)$	8		I.7
$\mathcal{A}^3(x,\ xy^2,\ x^2y)$	8		I.7
$\mathcal{A}^3(x,\ xy^2,\ 0)$	9		I.7
$\mathcal{A}^3(x,\ x^2y,\ 0)$	10		I.7

In the first part of this list, the strata are the $\mathcal{A}^r$ orbits of the germs given, or the union of $\mathcal{A}^r$ orbits where moduli figure. Each of these germs is $\mathcal{A}$-sufficient, in the sense that its degree of $\mathcal{A}$-determinacy is equal to the highest of the degrees of its component polynomials *.

In the second part of the list, the stratum in each case is the pre-image in $J^r(2,3)$ of the $\mathcal{A}^k$ orbit given, with respect to the natural projection $J^r(2,3) \longrightarrow J^k(2,3)$.

The fourth column of the table, marked "Reference", refers to the section of this chapter in which the stratum is studied.

A corresponding stratification of $J^r(2,3)$, for $r < 11$, may be obtained by projecting each of the given strata into $J^r(2,3)$. Of course, some of the strata may become identified in the process.

The bulk of the chapter is taken up by the classification of germs and jets which is tabulated in Theorem I:2. No further proof of this theorem is given. Since the classification was motivated by the desire to obtain a list of all simple germs of codimension less than or equal to 6, all unimodular germs of codimension less than or equal to 7, etc., (for reasons which will become clear in Chapter III), it is carried out accordingly. In particular, with a few exceptions, $\mathcal{A}^k$ orbits of codimension greater than 6 are taken as strata and not further subdivided. A proof of Theorem I:1 is given in section I: , although simple germs are shown to be so as they occur in the classification.

The results of the classification, some of which go slightly further than is needed to prove Theorem I:2, are shown diagramatically in Fig-

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* There is one exception to this: the germs (x,y) $(x, xy+y^6+by^7, xy^2+y^4+cy^6)$ are all 7-determined, in particular when b = 0, but not 6-determined.

ures 1,2 and 3. These show the stratifications of $J^r(2,3)$ for $2 \leqslant r \leqslant 11$. At each level (i.e. in each jet space) the strata are the $\mathcal{A}^r$ orbits of the jets shown. The lattice relation is simply one of projection — $\sigma_l < \rho_k$ (where σ_l is an l-jet and ρ_k is a k-jet, and $l > k$) means simply that $\pi_{l,k}(\sigma_l) = \rho_k$, (or, in one case, that these k-jets are $\mathcal{A}^k$-equivalent.). Those jets which are underlined are sufficient, and there the downward branch of the lattice ends, and similarly, those k-jets at which the downward branch ends, but which are not underlined, have $\mathcal{A}^k$-codimension greater than 6.

Figures 1, 2 and 3 also show certain infinite families of germs, for example $(x,y) \longrightarrow (x, y^2, y^3 + x^{k+1}y)$. These are indicated by the ending of the downward branch of the lattice in a dotted line, which means that the pattern of branching already established (and which should be clear from the diagram) continues ad infinitum.

I.2 <u>Notation</u> We identify $J^k(2,3)$ with the space of all triples of polynomials of degree less than k+1 , in two real variables, and with real coefficients. These are written in the form $(a(x,y), b(x,y), c(x,y))$, with $a(x,y) = \sum_{1 \leqslant i+j \leqslant k} a_{i,j} x^i y^j$, etc.

Since $J^k(2,3)$ is a vector space, we trivialise its tangent bundle and regard it as its own tangent space at each point (k-jet). In this connection we shall write its elements as column vectors,

$$\begin{bmatrix} a(x,y) \\ b(x,y) \\ c(x,y) \end{bmatrix} .$$

In order to specify linear subspaces of $J^k(2,3)$ we make use of the nat-


2
3
4
5
6
7
8
9
10
11
(x,y^2,0)
(x,y^2,y^3±x^2y)
(x,y^2,y^3)
(x,y^2,x^2y)
(x,y^2,0)
(x,y^2,y^3+x^3y)
(x,y^2,x^3y)
(x,y^2,xy^3±x^3y)
(x,y^2,xy^3)
(x,y^2,0)
(x,y^2,y^3±x^4y)
(x,y^2,x^2y±y^5)
(x,y^2,x^3y+y^5)
(x,y^2,x^3y)
(x,y^2,xy^3+x^4y)
(x,y^2,x^4y)
(x,y^2,x^4y±y^5+cx^2y^3)
c^2 ≠ 4 if coeff. of y^5 is +1
(x,y^2,x^4y+y^5±2x^2y^3)
(x,y^2,x^4y±x^2y^3)
(x,y^2,x^2y^3±y^5)
(x,y^2,x^2y^3)
(x,y^2,y^5)
(x,y^2,0)
(x,y^2,y^3+x^5y)
(x,y^2,x^3y+xy^5)
a=0
a≠0
(x,y^2,xy^3±x^5y)
(x,y^2,x^2y±y^7)
(x,y^2,x^3y+axy^5+by^7)
4a^3+27b^2 = 0
(x,y^2,x^3y+x^2y^3)
(x,y^2,x^3y)
(x,y^2,x^2y ±y^9)

FIGURE 1

2
$(x,xy,0)$
3
(x,xy,y^3)
$(x,xy+y^3,xy^2)$
(x,xy,xy^2)
$(x,xy+y^3,0)$
$(x,xy,0$
4
$(x,xy+y^3,xy^2)$
$(x,xy+y^3,xy^2+y^4)$
$(x,xy+y^3,xy^2+\frac{1}{2}y^4)$
$(x,xy+y^3,xy^2+cy^4)$
$c \neq 0,\frac{1}{2},1,\frac{3}{2}$
$(x,xy+y^3,xy^2+\frac{3}{2}y^4)$
$(x,xy+y^4,xy^2)$
$(x,xy,xy+y^4)$
(x,xy,xy^2)
$(x,xy+y^3,xy^3)$
$(x,xy+y^3,y^4)$
$(x,xy+y^3,0)$
5
$(x,xy+y^5,y^3)$
$(x,xy+y^3,xy^2+\frac{1}{2}y^4+y^5)$
$(x,xy+y^3,xy^2+\frac{1}{2}y^4)$
$(x,xy+y^3,xy^2+\frac{3}{2}y^4)$
$(x,xy+y^3,xy^2+\frac{3}{2}y^4+y^5)$
6
$(x,xy+y^3,xy^2+y^4)$
$(x,xy+y^3,xy^2+y^4\pm y^6)$
$(x,xy+y^3,xy^2+\frac{1}{2}y^4\pm y^6)$
$(x,xy+y^3,xy^2+\frac{3}{2}y^4\pm y^6)$
$(x,xy,xy^2+y^4+y^6)$
$(x,xy+y^6,xy^2+y^4+cy^6)$
$(x,xy,xy+y^4)$
7
$(x,xy+y^3,xy^2+y^7)$
$(x,xy+y^6+by^7,xy^2+y^4+cy^6)$
8
$(x,xy+y^8,y^3)$
9
10
$(x,xy+y^3,xy^2\pm y^{10})$
11
$(x,xy+y^{11},y^3)$

FIGURE 2


2
3
4
5
6
7
8
9
10
11
(x,y²,0)
(x,y²,y³±x²y)
(x,y²,y³)
(x,y²,x²y)
(x,y²,0)
(x,y²,y³+x³y)
(x,y²,x³y)
(x,y²,xy³±x³y)
(x,y²,xy³)
(x,y²,0)
(x,y²,y³±x⁴y)
(x,y²,x²y±y⁵)
(x,y²,x³y+y⁵)
(x,y²,x³y)
(x,y²,xy³+x⁴y)
(x,y²,x⁴y)
(x,y²,x⁴y±y⁵+cx²y³)
c² ≠ 4 if coeff. of y⁵ is +1
(x,y²,x⁴y+y⁵±2x²y³)
(x,y²,x⁴y±x²y³)
(x,y²,x²y³±y⁵)
(x,y²,x²y³)
(x,y²,y⁵)
(x,y²,0)
(x,y²,y³+x⁵y)
(x,y²,x³y+xy⁵)
a=0
a≠0
(x,y²,xy³±x⁵y)
(x,y²,x²y±y⁷)
(x,y²,x³y+axy⁵+by⁷)
4a³+27b² = 0
(x,y²,x³y+x²y³)
(x,y²,x³y)
(x,y²,x²y ±y⁹)

FIGURE 1

ural decomposition as the direct sum of three copies of $\mathfrak{m}_2/\mathfrak{m}_2^{k+1}$, and of the natural set of generators of $\mathfrak{m}_2/\mathfrak{m}_2^{k+1}$, namely $\{x^i y^j : 1 \leq i+j \leq k\}$. However, we shall generally omit the reference to $\mathfrak{m}_2^{k+1}$, since this will be obvious from the context. Thus, for example, by

$$\mathfrak{m}_2 - \{y, xy, y^3\} + \mathbb{R}\{xy + y^3\}$$

we would mean the linear subspace of $\mathfrak{m}_2$ generated by all $x^i y^j$ with $i + j \geq 1$, except for y, xy and y^3, and by $xy + y^3$. If we were doing calculations in $J^5(2,3)$, however, it would mean the projection into $\mathfrak{m}_2/\mathfrak{m}_2^6$ of this subspace.

By $\mathcal{E}_n$ we shall mean the algebra of germs at 0 of smooth functions of n variables, and by $\mathcal{E}_{n,p}$ the space of map- germs $(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$. The same symbols will also denote the tangent spaces to either of these spaces at any point.

In other respects, our notation is that of Mather ([17]-[22]).

I.3 <u>Techniques</u> Apart from using explicit coordinate changes to reduce k-jets to one of a number of normal forms, which becomes progressively more difficult as k increases, we shall use the infinitesimal technique due to Mather:

I.3:1 <u>Theorem</u> a) Let the Lie group G act smoothly on the Manifold M, and suppose that the submanifold $S \subseteq M$ has the following properties:

1) For all $x \in S$, $T_x Gx \supseteq T_x S$

2) The dimension of Gx is independent of the choice of $x \in S$

3) S is connected.

Then S is contained in a single G orbit.

b) Suppose $\pi : M_1 \longrightarrow M_2$ is a G-submersion and let $S = \pi^{-1}(x_o)$ for some $x_o \in M_2$. Then if (1) and (3) of part (a) hold, S is contained in a single G orbit.

This result may be found in [20] as Lemma 3.1 .

In order to go from the calculation of $\mathcal{A}^k$ tangent spaces to the calculation of $\mathcal{A}$ tangent spaces, we shall use

I.3:2 <u>Theorem</u> Let $f : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ be a smooth map-germ such that

$$tf(\Theta(n)) + f^*\mathcal{M}_p\Theta(f) \supseteq \mathcal{M}_n^1\Theta(f) , \qquad \text{let } n < p,$$

and let C be an $\mathcal{E}_p$ module (via f^*) such that

$$C \supseteq \mathcal{M}_n^k\Theta(f) \qquad (k \geqslant 1).$$

Then

(1) $\quad C = T\mathcal{A}f \quad$ if and only if

(2) $\quad C = T\mathcal{A}f + f^*\mathcal{M}_pC + \mathcal{M}_n^{k+1}\Theta(f).$

<u>Proof</u> That (1) $\Longrightarrow$ (2) is trivial. To see the converse, note that under the hypotheses of the theorem,

$$tf(\Theta(n)) + f^*\mathcal{M}_pC \supseteq \mathcal{M}_n^{k+1}\Theta(f),$$

for we have

$$\mathcal{M}_n^k\left\{tf(\Theta(n)) + f^*\mathcal{M}_p\Theta(f)\right\} \supseteq \mathcal{M}_n^k \cdot \mathcal{M}_n^1\Theta(f) = \mathcal{M}_n^{k+1}\Theta(f)$$

and so

$$tf(\mathcal{M}_n\Theta(n)) + f^*\mathcal{M}_p\mathcal{M}_n^k\Theta(f) \supseteq \mathcal{M}_n^{k+1}\Theta(f)$$

whence

$$tf(\mathcal{M}_n\Theta(n)) + f^*\mathcal{M}_pC \supseteq \mathcal{M}_n^{k+1}\Theta(f).$$

Thus, (2) implies

$$C = tf(\mathcal{M}_n\Theta(n)) + \omega f(\mathcal{M}_p\Theta(p)) + f^*\mathcal{M}_p C \quad .$$

A straightforward application of Nakayama's Lemma will complete the proof, if we can show that C is a finitely generated $\mathcal{E}_p$ module. But since

$$tf(\Theta(n)) + f^*\mathcal{M}_p\Theta(f) \supseteq \mathcal{M}_n^1\Theta(f)$$

and $n < p$, f is $\mathcal{E}$-finite ([30] page 494, Proposition 2.4), and so by the Preparation Theorem ([30] page 489), any finite $\mathcal{E}_n$ module is finitely generated over $\mathcal{E}_p$. In particular this holds for $\mathcal{M}_n^k\Theta(f)$. Since $C/\mathcal{M}_n^k\Theta(f)$ is a finite dimensional $\mathbb{R}$ vector space, it follows that C is a finite $\mathcal{E}_n$ module, and hence a finite $\mathcal{E}_p$ module∎

This theorem is due to Gaffney (verbal communication) and is related to the following result ([7] page 127) which is proved by a related method, and forms the basis of the determinacy estimates we make in this chapter.

I.3:3 <u>Theorem</u> Let $f : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ be a smooth map-germ and suppose that

$$tf(\Theta(n)) + \omega f(\Theta(p)) \supseteq \mathcal{M}_n^k\Theta(f)$$

and

$$tf(\Theta(n)) + f^*\mathcal{M}_p\Theta(f) \supseteq \mathcal{M}_n^1\Theta(f).$$

Then f is k + 1 determined for $\mathcal{A}$.

I.4 Classification

I.4:1 <u>Lemma</u> If $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ has rank 1 at the origin then there are coordinates in which f may be written

$$(x,y) \longrightarrow (x,\ b(x,y),\ c(x,y))$$

where $b, c \in \mathcal{M}_2^2$.

Proof Obvious ∎

Henceforth all germs and jets that we consider will be of this form.

I.4:2 Proposition (Classification of 2-jets) There are four $\mathcal{A}^2$ orbits in $J^2(2,3) \cap \Sigma^1$:

$$(x,y) \longrightarrow (x,\ y^2,\ xy)$$

$$(x,y) \longrightarrow (x,\ y^2,\ 0)$$

$$(x,y) \longrightarrow (x,\ xy,\ 0)$$

$$(x,y) \longrightarrow (x,\ 0,\ 0)$$

Proof Let $j^2f(0) = (x,\ b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2,\ c_{2,0}x^2 + c_{1,1}xy + c_{0,2}y^2)$.
Then the coordinate change in R^3 ("left coordinate change")

$$Y = \overline{Y} + b_{2,0}X^2 \qquad (X,Y,Z \text{ old coordinates, } \overline{X},\overline{Y},\overline{Z} \text{ new coordinates})$$

$$Z = \overline{Z} + c_{2,0}X^2$$

transforms $j^2f(0)$ into $(x,\ b_{1,1}xy + b_{0,2}y^2,\ c_{1,1}xy + c_{0,2}y^2)$.

Now if $\begin{vmatrix} b_{1,1} & c_{1,1} \\ b_{0,2} & c_{0,2} \end{vmatrix} \neq 0$, an appropriate (linear) left coordinate change transforms $j^2f(0)$ into $(x,\ y^2,\ xy)$.

If this determinant is zero, then after a left coordinate change, we can assume that $c_{1,1} = c_{0,2} = 0$. If now $b_{0,2} \neq 0$, we can complete the square in the second component; and the corresponding change in the y-variable then changes $j^2f(0)$ into $(x,\ y^2,\ 0)$.

If $b_{0,2} = 0$ then the 2-jet is equivalent either to $(x,\ 0,\ 0)$ or to $(x,\ xy,\ 0)$ ∎

I.4:3 Theorem The map-germ $f(x,y) = (x, y^2, xy)$ (the "cross-cap") is stable and 2-determined.

Proof It is a theorem of Whitney that f is stable ([33]) . It follows by Theorem I.3:3 that f is 3-determined, for clearly $f^*\mathcal{M}_3\Theta(f)$ contains $\mathcal{M}_2^2\Theta(f)$. Now by I.5:2 and I.5:3 (see below) if $j^3g(0)=j^2f(0)$, then g is equivalent to a germ of the form

$$(x,y) \quad (x, y^2, xy + c_{2,1}x^2y + c_{0,3}y^3 + O(4)).$$

The coordinate changes

$\overline{Z} = Z - c_{2,1}XZ$

$\overline{x} = x + c_{0,3}y^2$

$\overline{X} = X + c_{0,3}Y$

now transform g to a map whose 3-jet is equal to that of f. Since f is 3-determined, g must be equivalent to f∎

There now follow three sections, in which we classify, successively, map germs whose 2-jet is equivalent to $(x,y^2,0)$, $(x,xy,0)$ and $(x,0,0)$.

I.5 Classification of germs whose 2-jet is equivalent to $(x,y^2,0)$

I.5:1 Lemma a) The germ $f(x,y) = (x,y^2)$ has sufficient 2-jet.

b) $tf(\mathcal{M}_2^{k-1}\Theta(2)) + f(\mathcal{M}_2^{k-1}\Theta(2)) \supseteq \mathcal{M}_2^k\Theta(f)$, for $k \geqslant 1$.

Proof a) This is the Whitney fold, and is well known to be stable, so that $tf(\Theta(2)) + \omega f(\Theta(3)) \supseteq \mathcal{M}_2\Theta(f)$. Moreover, $tf(\Theta(2)) + f^*\mathcal{M}_3\Theta(f) \supseteq \mathcal{M}_2\Theta(f)$, so that by I.3:3 f is 2-determined.

b) As f is a $\mathcal{E}$-finite map-germ, $\mathcal{M}_2^k\Theta(f)$ is a finite $\mathcal{E}_2$ module via f^*, and so by Nakayama's Lemma we need only show

(1) $$tf(\mathcal{M}_2^{k-1}\Theta(2)) + \omega f(\mathcal{M}_2^{k-1}\Theta(2)) + f^*\mathcal{M}_2.\mathcal{M}_2^k\Theta(f) \supseteq \mathcal{M}_2^k\Theta(f) .$$

Now

$$f^*\mathcal{M}_2.\mathcal{M}_2^k\Theta(f) = \mathcal{M}_2^{k+1}\Theta(f) - \left\{\begin{bmatrix} 0 \\ y^{k+1}\end{bmatrix}, \begin{bmatrix} y^{k+1} \\ 0\end{bmatrix}\right\} ,$$

so we need only find the k-th order terms, plus $\begin{bmatrix} 0 \\ y^{k+1}\end{bmatrix}$ and $\begin{bmatrix} y^{k+1} \\ 0\end{bmatrix}$, in the left hand side of (1). And in fact

$$tf\begin{bmatrix} x^iy^j \\ 0\end{bmatrix} = \begin{bmatrix} x^iy^j \\ 0\end{bmatrix}, \quad \omega f\begin{bmatrix} 0 \\ x^k\end{bmatrix} = \begin{bmatrix} 0 \\ x^k\end{bmatrix} \text{ and } tf\begin{bmatrix} 0 \\ \frac{1}{2}x^iy^{j-1}\end{bmatrix} = \begin{bmatrix} 0 \\ x^iy^j\end{bmatrix}$$ ∎

I.5:2 <u>Corollary</u> Let $g(x,y) = (x + p(x,y), y^2 + q(x,y), c(x,y) + r(x,y))$, where $c \in \mathcal{M}_2^2$ and $p,q,r \in \mathcal{M}_2^k$ ($k \geqslant 3$). Then g is equivalent to a germ of the form

$$h(x,y) = (x, y^2, c(x,y) + \bar{r}(x,y))$$

where $\bar{r} \in \mathcal{M}_2^k$.

<u>Proof</u> Let $f_u : (R^2,0) \longrightarrow (R^2,0)$ be given by

$$f_u(x,y) = (x + up(x,y), y^2 + uq(x,y)).$$

By part (a) of the Lemma, f_u is equivalent to f_o for each $u \in \mathbb{R}$, and so by part (b),

$$tf_u(\mathcal{M}_2^{k-1}\Theta(2)) + \omega f_u(\mathcal{M}_2^{k-1}\Theta(2)) \supseteq \mathcal{M}_2^k\Theta(f_u)$$

for all $u \in R$. From this it follows by the Preparation Theorem and the usual Thom-Levine vector-field argument (see for example [30] page 488) that f_u is equivalent to f_o under diffeomorphisms in source and target whose k-2 jet is that of the identity. In other words, we have shown that f_u is $\mathcal{A}_{k-2}$ - equivalent to f_o. This holds in particular for u = 1. Now the diffeomorphism in the target may be regarded as a diffeomorphism in R^3 which leaves the last coordinate fixed. Writing the source diffeomorphism as

$x = \bar{x} + \alpha(\bar{x},\bar{y})$

$y = \bar{y} + \beta(\bar{x},\bar{y})$

where $\alpha, \beta \in \mathcal{M}_2^{k-1}$, we see that g is equivalent to

$$(\bar{x},\bar{y}) \quad (\bar{x}, \bar{y}^2, c(\bar{x} + \alpha(\bar{x},\bar{y}), \bar{y} + \beta(\bar{x},\bar{y})) + r(\bar{x} + \alpha(\bar{x},\bar{y}), \bar{y} + \beta(\bar{x},\bar{y}))$$

which in fact has the desired form ∎

I.5:3 Lemma Let $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ be given by

$$(x,y) \longmapsto (x, y^2, c(x,y))$$

with $c \in \mathcal{M}_2$. Then f is equivalent to

$$(x,y) \longmapsto (x, y^2, yp(x, y^2)),$$

where

$$yp(x,y^2) = \tfrac{1}{2}\{c(x,y) - c(x, -y)\}.$$

Proof We have

$$c(x,y) = \tfrac{1}{2}\{c(x,y) + c(x,-y)\} + \tfrac{1}{2}\{c(x,y) - c(x,-y)\} .$$

Since the first of these two summands is even in y, by a Theorem of Whitney ([32]) there exists a function germ $r : (\mathbb{R}^2,0) \longrightarrow \mathbb{R}$ such that

$$\tfrac{1}{2}\{c(x,y) + c(x,-y)\} = r(x,y^2).$$

The coordinate change $\bar{Z} = Z - r(x,Y)$ now transforms f into

$$(x,y) \longmapsto (x, y^2, \tfrac{1}{2} \ c(x,y) - c(x,-y) \ ).$$

Finally, since the last component of this germ is odd in y, it may be written in the form $yp(x,y^2)$, for some smooth function germ p, again by the theorem of Whitney ∎

I.5:4 Corollary Every map germ $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ whose $\mathcal{K}$-class is that of $(x,y) \longmapsto (x,y^2,0)$, is equivalent to a map-germ of the form $(x,y) \longmapsto (x, y^2, yp(x, y^2))$.

Proof This follows immediately from I.5:2 and I.5:3. ∎

Having acheived this reduction, it may be expected that the $\mathcal{A}$-classification of such map-germs will be related to the classification, under the action of an appropriate group, of the functions $p(x,y^2)$. In fact this is indeed the case, the group in question being $\mathcal{K}^T$, the subgroup of $\mathcal{K}$ which preserves the set of functions $\mathbb{R}^2 \longrightarrow \mathbb{R}$ invariant under $(x,y) \longmapsto (x,-y)$. The following sequence of definitions and lemmas goes towards proving this"equivalence of equivalences", and towards proving also "equivalence of versality" for unfoldings.

I.5:5 Lemma If $f(x,y) = (x, y^2, yp(x, y^2))$ then for any function $h \in \mathcal{E}_3$, there exist functions $r,s \in \mathcal{E}_2$ such that

$$h(f(x,y)) = r(x,y^2) + yp(x,y^2)s(x,y^2).$$

Proof By the Preparation Theorem, any function $h \in \mathcal{E}_3$ can be written

$$h(X,Y,Z) = h_1(X,Y,Z^2) + Zh_2(X,Y,Z^2)$$

for some functions h_1 and h_2. Thus,

$$h(f(x,y)) = h_1(x,y^2,y^2p^2(x,y^2))+yp(x,y^2)h_2(x,y^2,y^2p^2(x,y^2)).$$

Now put

$$r(x,w) = h_1(x,w,wp^2(x,w)) \quad \text{and} \quad s(x,w) = h_2(x,w,wp^2(x,w))$$

to obtain the desired equality. ∎

I.5:6 Definition i) $\mathcal{E}_2^T$ is the set of germs of functions $h : (\mathbb{R}^2,0) \longrightarrow \mathbb{R}$ such that $h \circ T = h$, where $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T(x,y) = (x,-y)$, and $\mathcal{M}_2^T = \mathcal{M}_2 \cap \mathcal{E}_2^T$.

ii) $\mathcal{R}^T$ is the subgroup of Diff $(\mathbb{R}^2,0)$ consisting of germs of diffeomorphisms φ such that $\varphi \circ T = T \circ \varphi$.

iii) $\mathcal{C}^T$ is the subgroup of $\mathcal{C}$ which acts naturally on $\mathcal{E}_2^T$, i.e. germs of diffeomorphisms $(\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$ of the form

$$(x,y,z) \longmapsto (x,y,h(x,y^2,z))$$

iv) $\mathcal{K}^T$ is the semi-direct product of $\mathcal{R}^T$ with $\mathcal{C}^T$.

v) For a germ $p(x,y^2) \in \mathcal{E}_2^T$,

$$T\mathcal{K}^T p(x,y^2) = \langle xp_x(x,y^2),\ y^2p_x(x,y^2),\ y^2p_y(x,y^2),\ p(x,y^2)\rangle \mathcal{E}_2^T .$$

vi) The $\mathcal{K}^T$ codimension of $p(x,y^2)$ is

$$\dim_R \frac{\mathcal{M}_2^T}{T\mathcal{K}^T p(x,y^2)}$$

vii) $T_e\mathcal{K}^T p(x,y^2) = \langle p_x(x,y^2),\ y^2p_y(x,y^2),\ p(x,y^2)\rangle \mathcal{E}_2^T.$

viii) The $\mathcal{K}_e^T$ codimension of $p(x,y^2)$ is

$$\dim_R \frac{\mathcal{E}_2^T}{T_e\mathcal{K}^T p(x,y^2)} .$$

I.5:7 <u>Proposition</u> If $f(x,y) = (x,\ y^2,\ yp(x,\ y^2))$, with $p(0,0) = 0$, then

$$\text{i) } T\mathcal{A}f = \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2^T \oplus y\{T\mathcal{K}^T p(x,y^2)\} \end{bmatrix}$$

$$\text{ii) } T_e\mathcal{A}f = \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{E}_2 \\ \mathcal{E}_2^T \oplus y\{T_e\mathcal{K}^T p(x,y^2)\} \end{bmatrix}$$

iii) $\operatorname{Codim}(\mathcal{A}_1)\, f = 2 + \operatorname{codim}(\mathcal{K}^T)\, p(x,y^2)$,

$\operatorname{Codim}(\mathcal{A}_e)\, f = \operatorname{codim}(\mathcal{K}_e^T)\, p(x,y^2)$.

<u>Proof</u> Let $\begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ 0 \end{bmatrix} \in \mathcal{M}_2\Theta(f)$.

Put $\quad g_i = g_{i,e} + g_{i,o} \quad$ (even and odd parts of g_i with respect to y).

Then

$$g_{i,e} = r_i(x,y^2) = f^*\{r_i(X,Y)\}$$

for some function r_i, so assume that $g_i = g_{i,o}$, i.e. $g_i \in y\,\mathcal{E}_2^T$.

If also $g_2 \in y\mathcal{M}_2$, then

$$tf\begin{bmatrix} g_1 \\ g_2/2y \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ yg_1p_x(x,y^2) + (g_2/2y)(p(x,y^2) + 2y^2p_y(x,y^2)) \end{bmatrix}$$

Since the bottom line is in $\mathcal{E}_2^T$, we can find it in $f^*\mathcal{M}_3$. Thus, we have

$$T\mathcal{A}f \supseteq \begin{bmatrix} \mathcal{M}_2^T + y\,\mathcal{E}_2^T \\ \mathcal{M}_2^T + y\mathcal{M}_2^T \\ \mathcal{M}_2^T \end{bmatrix} = \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2 - y \\ \mathcal{M}_2^T \end{bmatrix}$$

Now let g_3 be odd in y. Then $\begin{bmatrix} 0 \\ 0 \\ g_3 \end{bmatrix} \in T\mathcal{A}f$ if and only if

$$\begin{bmatrix} 0 \\ 0 \\ g_3 \end{bmatrix} = tf\begin{bmatrix} a \\ b \end{bmatrix} + \omega f\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$ where $a, b \in \mathcal{M}_2$ and $h_i \in \mathcal{M}_3$.

That is,

$$(*)\quad \begin{bmatrix} 0 \\ 0 \\ g_3(x,y) \end{bmatrix} = \begin{bmatrix} a(x,y) \\ 2yb(x,y) \\ ya(x,y)p_x(x,y^2) + b(x,y)(p(x,y^2)+2y^2p_y(x,y^2)) \end{bmatrix} + \begin{bmatrix} h_1(f(x,y)) \\ h_2(f(x,y)) \\ h_3(f(x,y)) \end{bmatrix}.$$

Clearly $a = -f^*h_1$ and $2yb = -f^*h_2$, so by Lemma I.5:5, write

$$a(x,y) = a_1(x,y^2) + yp(x,y^2)a_2(x,y^2)$$

$$2yb(x,y) = b_1(x,y^2) + yp(x,y^2)b_2(x,y^2)$$

$$f^*h_3(x,y) = r(x,y^2) + yp(x,y^2)\, s(x,y^2).$$

Note that since $a \in \mathcal{M}_2$, we have $a_1 \in \mathcal{M}_2^T$. Note also that the second of these three equalities implies that $b_1(x,y^2)$ is divisible by y^2, and so may be written as $y^2b_3(x,y^2)$. Equating the odd and even parts in both sides of $(*)$, we must have

$$g_3(x,y) = ya_1(x,y^2)p_x(x,y^2)+y^3b_3(x,y^2)p_y(x,y^2)+\tfrac{1}{2}yb_3(x,y^2)p(x,y^2)+yp(x,y^2)s(x,y^2)$$

That is,

$$g_3 \in y\left\{ \langle p_x(x,y^2)\rangle \mathcal{M}_2^T + \langle yp_y(x,y^2)\rangle\, y\mathcal{E}_2^T + \langle p(x,y^2)\rangle\, \mathcal{E}_2^T \right\}$$
$$= y\, T\mathcal{K}^T p(x,y^2).$$

Clearly we can reverse this construction, so that if $g \in T\mathcal{K}^T p(x,y^2)$,

then $\begin{bmatrix} 0 \\ 0 \\ yg(x,y) \end{bmatrix} \in T\mathcal{A}f$

This completes the proof of (i). Part (ii) is proved similarly, and part (iii) follows trivially from (i) and (ii)∎

I.5:8 <u>Theorem</u> Let $f_i(x,y) = (x, y^2, yp_i(x,y^2))$. Then if $p_1(x,y^2)$ is $\mathcal{K}^T$-equivalent to $p_2(x,y^2)$, f_1 is $\mathcal{A}$-equivalent to f_2.

<u>Proof</u> First suppose that $p_1(x,y^2)$ is $\mathcal{C}^T$-equivalent to $p_2(x,y^2)$. Then there exists a function $r(x,y^2)$ such that $p_1(x,y^2) = r(x,y^2)p_2(x,y^2)$, with $r(0,0) \neq 0$. Define $\Psi : (\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$ by

$$(X,Y,Z) \longmapsto (X,Y,r(X,Y)Z).$$

Clearly Ψ is a diffeomorphism at $0 \in \mathbb{R}^3$, and $\Psi \circ f_2 = f_1$.

Now consider the case where $p_1(x,y^2)$ is $\mathcal{R}^T$-equivalent to $p_2(x,y^2)$. Then there exists a diffeomorphism-germ $\varphi : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$, which may be written as

$$\varphi(x,y) = (r(x,y^2),\ ys(x,y^2)),$$

such that

$$p_1(r(x,y^2),\ y^2s^2(x,y^2)) = p_2(x,y^2).$$

Since φ is a diffeomorphism, $r_x(0,0) \neq 0 \neq s(0,0)$. We shall now show that $f_1 \circ \varphi$ is left-equivalent to f_2. We have

$$f_1(\varphi(x,y)) = (r(x,y^2),\ y^2s^2(x,y^2),\ ys(x,y^2)p_2(x,y^2)).$$

Consider the map-germ $q : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$

$$(x,y) \quad (r(x,y^2),\ y^2s^2(x,y^2)).$$

It is clear that

$$\frac{\mathcal{E}_2}{q^*(\mathcal{M}_2)\mathcal{E}_2} = \frac{\mathcal{E}_2}{\langle r(x,y^2),\ y^2s^2(x,y^2)\rangle} = \frac{\mathcal{E}_2}{\langle r(x,0),\ y^2\rangle}$$

$\cong \mathbb{R}\{1,y\}$, since $r_x(0,0) \neq 0 \neq s(0,0)$. By the Preparation Theorem, every function germ $k : (\mathbb{R}^2,0) \longrightarrow \mathbb{R}$ can be written in the form

$$k(x,y) = a(r(x,y^2),\ y^2s^2(x,y^2)) + yb(r(x,y^2),y^2s^2(x,y^2))$$

In particular, since $s(x,y^2)$ is even in y, we can write

$$s(x,y^2) = a(r(x,y^2),\ y^2s^2(x,y^2)).$$

Define $\Psi_1 : (\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$ by

$$(X,Y,Z,) \longmapsto (X, \frac{Y}{a^2(X,Y)}, \frac{Z}{a(X,Y)})$$

Then Ψ_1 is a diffeomorphism, since $a(0,0) \neq 0$, and moreover

$$\Psi_1 \circ f_1 \circ \varphi(x,y) = (r(x,y^2), y^2, yp_2(x,y^2)).$$

Now put $r_1(x,y) = r(x,y) - r(0,y)$. Clearly $r_1(x,y^2)$ is divisible by x, so write $r_1(x,y^2) = xr_2(x,y^2)$. Note that $r_2(0,0) \neq 0$.

Define $\Psi_2 : (\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$ by

$$(X,Y,Z) \longmapsto (X - r(0,Y), Y, Z).$$

Then Ψ_2 is a diffeomorphism, and

$$\Psi_2 \circ \Psi_1 \circ f_1 \circ \varphi(x,y) = (xr_2(x,y^2), y^2, yp_2(x,y^2)).$$

As in the construction of Ψ_1, $r_2(x,y^2)$ can be written in the form $b(xr_2(x,y^2),y^2)$, and defining a third diffeomorphism $\Psi_3 : (\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$ by

$$(X,Y,Z) \longmapsto (\frac{X}{b(X,Y)}, Y,Z)$$

we have $\quad \Psi_3 \circ \Psi_2 \circ \Psi_1 \circ f_1 \circ \varphi = f_2.$ ∎

In order to prove the converse, we need the following lemma, due to Gaffney and du Plessis ([9] page 10, Lemma 1.11) :

I.5:9 Lemma Let $f,g : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ be smooth map-germs, and suppose that there exists $h \in \mathcal{E}_n$ such that for $i = 1,\ldots,p$, $f_i - g_i \in \langle h \rangle$. (Here f_i, g_i are the component functions of f and g with respect to some coordinate system on $\mathbb{R}^p$). Then for any $k \in \mathcal{E}_p$, $f^*k - g^*k \in \langle h \rangle$ ∎

I.5:10 <u>Theorem</u> Let $f_i(x,y) = (x, y^2, yp_i(x,y^2))$, with $p_i(0,0)=0$, and suppose f_1 and f_2 are $\mathcal{A}$-equivalent. Then there exist diffeomorphisms $\tilde{\bar{\Phi}}$, $\tilde{\bar{\Psi}}$ of $(\mathbb{R}^2,0)$ and $(\mathbb{R}^3,0)$ respectively, of the form

$$\tilde{\bar{\Phi}}(x,y) = (r(x,y^2), ys(x,y^2))$$

$$\tilde{\bar{\Psi}}(X,Y,Z) = (r(X,Y), Ys^2(X,Y), Zt(X,Y))$$

such that $f_1 \circ \tilde{\bar{\Phi}} = \tilde{\bar{\Psi}} \circ f_2$.

<u>Proof</u> Suppose that $f_1 \circ \bar{\Phi} = \bar{\Psi} \circ f_2$ for some diffeomorphisms $\bar{\Phi}$ and $\bar{\Psi}$. Decomposing each component function of $\bar{\Phi}$ into odd and even parts (with respect to y) we can write

$$\bar{\Phi}(x,y) = (\varphi_{1,1}(x,y^2) + y\varphi_{1,2}(x,y^2),\ \varphi_{2,1}(x,y^2) + y\,\varphi_{2,2}(x,y^2)),$$

and then applying Lemma 1.5:5 and equating $f_1 \circ \bar{\Phi}$ and $\bar{\Psi} \circ f_2$ component by component, we have

(1) $\varphi_{1,1}(x,y^2) + y\,\varphi_{1,2}(x,y^2) = \psi_{1,1}(x,y^2) + yp_2(x,y^2)\psi_{1,2}(x,y^2)$

(2) $\{\varphi_{2,1}(x,y^2) + y\varphi_{2,2}(x,y^2)\}^2 = \psi_{2,1}(x,y^2) + yp_2(x,y^2)\psi_{2,2}(x,y^2)$

(3) $\{\varphi_{2,1}(x,y^2)+y\,\varphi_{2,2}(x,y^2)\}\,p_1(\varphi_{1,1}(x,y^2)+y\,\varphi_{1,2}(x,y^2),\ \varphi_{2,1}(x,y^2)+y\varphi_{2,2}(x,y^2))$

$$= \psi_{3,1}(x,y^2) + yp_2(x,y^2)\psi_{3,2}(x,y^2)$$

for some functions $\psi_{i,j}$.

Equating odd parts of (1) we see that $p_2(x,y^2)$ divides $\varphi_{1,2}(x,y^2)$ and hence that $\varphi_{1,2}(0,0) = 0$. Now since $\varphi(0,0) = (0,0)$, we have $\varphi_{i,1}(0,0)=0$ for $i = 1,2$, and hence $\frac{\partial}{\partial y}\varphi_{i,1}(x,y^2)$ is equal to 0 at $(x,y) = (0,0)$. It follows that $\det(d\bar{\Phi}(0,0)) = \varphi_{2,2}(0,0)\frac{\partial}{\partial x}\varphi_{1,1}(0,0)$, and so $\varphi_{2,2}(0,0) \neq 0 \neq \frac{\partial}{\partial x}\varphi_{1,1}(0,0)$. So we can define a diffeomorphism $\tilde{\bar{\Phi}} : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ by $\tilde{\bar{\Phi}}(x,y) = (\varphi_{1,1}(x,y^2), y\varphi_{2,2}(x,y^2))$.

Equating the odd parts of (2) we see that $p_2(x,y^2)$ divides

$\varphi_{2,1}(x,y^2)\varphi_{2,2}(x,y^2)$, and since $\varphi_{2,2}(0,0) \neq 0$, $p_2(x,y^2)$ must divide $\varphi_{2,1}(x,y^2)$.

It remains to be seen that we can construct a diffeomorphism $\tilde{\tilde{\Psi}}$ of $(\mathbb{R}^3,0)$, of the desired form, such that $f_1 \circ \tilde{\tilde{\Phi}} = \tilde{\tilde{\Psi}} \circ f_2$. There is no problem with the first two components of $\tilde{\tilde{\Psi}}$: simply set

$$\tilde{\tilde{\Psi}}_1(X,Y,Z) = \varphi_{1,1}(X,Y)$$
$$\tilde{\tilde{\Psi}}_2(X,Y,Z) = Y\{\varphi_{2,2}(X,Y)\}^2.$$

In order to define $\tilde{\tilde{\Psi}}_3$ in the desired way, we have to find $t(X,Y)$, with $t(0,0) \neq 0$, such that

$$(4) \quad y\varphi_{2,2}(x,y^2)p_1(\varphi_{1,1}(x,y^2),\{y\varphi_{2,2}(x,y^2)\}^2) = yt(x,y^2)p_2(x,y^2).$$

That is, we have to show that the expression on the left hand side of (4) is divisible by $p_2(x,y^2)$. Let a and b denote the left hand sides of (3) and (4) respectively, and let a_e and a_o, b_e and b_o, denote the even and odd parts (with respect to y) of a and b respectively. In fact $b_e = 0$. Now by equating the odd parts of (3), we find

$$a_o = yp_2(x,y^2)\psi_{3,2}(x,y^2).$$

Since both $\varphi_{1,2}(x,y^2)$ and $\varphi_{2,1}(x,y^2)$ are divisible by $p_2(x,y^2)$ it follows, by I.5:9, that a-b is divisible by $p_2(x,y^2)$. Hence

$$a_e + a_o - b_o = p_2(x,y^2)c = p_2(x,y^2)\ \{c_e + c_o\}$$

for some function c. Comparing odd parts of this equation, we have

$$a_o - b_o = c_o p_2(x,y^2)$$

and since a_o is divisible by $p_2(x,y^2)$, we conclude that $b = b_o$ is also divisible by $p_2(x,y^2)$. Clearly the quotient must be an odd function, and so there exists $t(X,Y)$ such that

$$b = yp_2(x,y^2)t(x,y^2).$$

In order to see that $t(0,0) \neq 0$, note that $b = \Phi^*(yp_1(x,y^2))$ and so the order of b is the same as the order of $yp_1(x,y^2)$. Now we claim that the orders of $yp_1(x,y^2)$ and $yp_2(x,y^2)$ are equal. For if the former were greater than the latter, then setting $k = \text{order}\,(yp_2(x,y^2))$ we have, from the fact that $\Psi \circ f_2 = f_1 \circ \Phi$, that

$$\Psi_3(x,y^2,\ yp_2(x,y^2)) = 0 \mod \mathcal{M}_2^{k+1}.$$

This implies that $\Psi_3 \in \mathcal{M}_3^2$, contradicting the fact that Ψ is a diffeomorphism. By symmettry, the two orders are equal, and so $t(0,0) \neq 0$. Thus, the map

$$\tilde{\Psi} : (\mathbb{R}^3,0) \longrightarrow (\mathbb{R}^3,0)$$

$$(X,Y,Z) \longmapsto (\varphi_{1,1}(X,Y), Y\{\varphi_{2,2}(X,Y)\}^2,\ Zt(X,Y))$$

is a diffeomorphism, and since

$$(5) \qquad \tilde{\tilde{\Psi}} \circ f_2 = f_1 \circ \tilde{\tilde{\Phi}},$$

the proof is complete ∎

I.5:11 <u>Corollary</u> Under the hypotheses of the preceeding theorem, $p_1(x,y^2)$ is $\mathcal{K}^T$-equivalent to $p_2(x,y^2)$.

<u>Proof</u> Equate the third components in (5) ∎

I.5:12 <u>Corollary</u> Let $p(x,y^2) \in \mathcal{M}_2$. Then $p(x,y^2)$ is k-dtermined for $\mathcal{K}^T$ if and only if the map germ $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ given by

$$(x,y) \longrightarrow (x,y^2,\ yp(x,y^2))$$

is k+1-determined for $\mathcal{A}$.

<u>Proof</u> Suppose $p(x,y^2)$ is k-determined for $\mathcal{K}^T$, and let

$$g(x,y) = (x + O(k+2),\ y^2 + O(k+2),\ yp(x,y^2) + O(k+2)).$$

By I.5:2, g is $\mathcal{A}$-equivalent to

$$(x,y) \longrightarrow (x,\ y^2,\ yp(x,y^2) + O(k+2)),$$

and by I.5:3 we may assume that the third component is

$$yp(x,y^2) + yh(x,y^2)$$

for some function $h(x,y^2) \in \mathcal{M}_2^{k+1}$. By hypothesis, $p(x,y^2) + h(x,y^2)$ is $\mathcal{K}^T$-equivalent to $p(x,y^2)$, and so by I.5:8 , g is $\mathcal{A}$-equivalent to f. The converse is an immediate consequence of I.5:11 ∎

I.5:13 Remark It is easy to see that pairs of diffeomorphisms like the $\tilde{\Phi}$ and $\tilde{\Psi}$ of I.5:10 form a subgroup of $\mathrm{Diff}(\mathbb{R}^2,0) \times \mathrm{Diff}(\mathbb{R}^3,0)$, which we shall call $\tilde{\mathcal{A}}$. If we denote by M the set of map-germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ of the form $(x,y) \longrightarrow (x,y^2,\ yp(x,y^2))$, then we have shown that for f, $g \in M$, $f \underset{\mathcal{A}}{\sim} g$ if and only if $\underset{\tilde{\mathcal{A}}}{\sim} g$.
One may check also that for $f \in M$, $T\mathcal{A}f = T\tilde{\mathcal{A}}f$, and that $\tilde{\mathcal{A}}M = M$.

I.5:14 Remark We have shown that the classification of map-germs in M with respe ct to $\mathcal{A}$, is the same as the classification of T-invariant functions under $\mathcal{K}^T$. Following Arnol'd, [3] , we now note that the classification of T-invariant germs $p(x,y^2)$ under $\mathcal{R}^T$ is the same as that of the function germs $p(X,Y)$ on $H^2 = \{(X,Y) \in \mathbb{R}^2 \mid Y \geq 0\}$, under the action of the group $\mathcal{R}^{\partial}$ of diffeomorphisms of $(H^2,0)$. A similar correspondence holds for $\mathcal{K}^T$ and $\mathcal{K}^{\partial}$:

I.5:15 Proposition Let $h : \mathbb{R}^2 \longrightarrow H^2$ be the fold map $(x,y) \longrightarrow (x,y^2)$, and let f and g be smooth germs $(H^2,0) \longrightarrow (\mathbb{R},0)$. Then

$$f \underset{\mathcal{K}^{\partial}}{\sim} g \iff f \circ h \underset{\mathcal{K}^T}{\sim} g \circ h.$$

Proof First suppose that $f \underset{\mathcal{C}^{\partial}}{\sim} g$, i.e. that there exists $r:(H^2,0) \longrightarrow \mathbb{R}$

such that $f(x,y) = r(x,y)g(x,y)$. Then $h^*f = (h^*r)(h^*g)$ and so h^*f and h^*g are $\mathcal{C}^T$-equivalent, since h^*r is T-invariant. The converse is equally trivial.

If now f and g are $\mathcal{R}^\partial$-equivalent, then there exists a diffeomorphism $\varphi : (H^2,0) \longrightarrow (H^2,0)$ such that $f \circ \varphi = g$. Since $\varphi(\partial H^2) = \partial H^2$, φ must have the form

$$(x,y) \longmapsto (\varphi_1(x,y),\ y\,\varphi_2(x,y))$$

with $\varphi_2(0,0) \neq 0 \neq \frac{\partial}{\partial x}\varphi_1(0,0)$. Thus,

$$f(\varphi_1(x,y),\ y\,\varphi_2(x,y)) = g(x,y) \qquad \text{for } y \geqslant 0,$$

and so

$$f(\varphi_1(x,y^2),\ y^2\varphi_2(x,y^2)) = g(x,y^2) \qquad \text{for all } y,$$

and hence

$$(h^*f)(\varphi_1(x,y^2),\ y\sqrt{\varphi_2(x,y^2)}) = (h^*g)(x,y).$$

The map germ defined by

$$(x,y) \longrightarrow (\varphi_1(x,y^2),\ y\sqrt{\varphi_2(x,y^2)})$$

is a member of $\mathcal{R}^T$, and thus we conclude that

$$f \circ h \underset{\mathcal{R}^T}{\sim} g \circ h.$$

A similar argument proves the converse ∎

Now putting the pieces together, we have

<u>I.5:16</u> <u>Theorem</u> Let $f_i(x,y) = (x,y^2,\ yp_i(x,y^2))$ $i = 1,2$ be smooth map-germs; then f_1 and f_2 are $\mathcal{A}$-equivalent if and only if the function germs p_i on $(H^2,0)$ are $\mathcal{K}^\partial$-equivalent.

<u>Proof</u> The theorem follows immediately from I.5:8, I.5:11 and I.5:15 ∎

I.5:17 <u>Remark</u> The close parallel between the action of $\mathcal{A}$ on M (see I.5:13) and the action of $\mathcal{K}^{\partial}$ on $C^{\infty}(H^2,0)$ (the space of smooth function germs on H^2 at 0) does not stop at classification, but carries over to versal unfoldings, which we examine in I.10 . Without going into any detail now, one may see this parallel quite simply displayed in the geometry of the germs in M: if $f \in M$, then the set of double points in the image of f is

$$\{(X,Y,Z) \in \mathbb{R}^3 \mid Z = 0,\ Y > 0,\ p(X,Y) = 0\}$$

and the closure of this set, which contains also the image under f of its singular points, is

$$\{(X,Y,Z) \in \mathbb{R}^3 \mid Z = 0,\ Y \geqslant 0,\ p(X,Y) = 0\}.$$

We now proceed to the classification itself. In [2] , Arnol'd gives a classification of critical points of functions in $C^{\infty}(H^2,0)$ with respect to the action of $\mathcal{R}^{\partial}$, and his lists may be adapted to give a classification under $\mathcal{K}^{\partial}$. The modifications required generally amount to no more than a reduction in the number of moduli, and in particular, the classification of those germs for which quasi-homogeneous normal forms exist is the same for $\mathcal{K}^{\partial}$ as for $\mathcal{R}^{\partial}$. Thus, by carrying out the appropriate modifications in Arnol'd's lists, one can obtain a far wider $\mathcal{A}$-classification of germs whose $\mathcal{K}$-class is that of $(x,y) \longrightarrow (x,y^2,0)$ than the one we give here.

The following table gives the classification of singular germs in $\mathcal{E}_2^T$ with respect to the action of $\mathcal{K}^T$, as far as we have carried it out, consistent with the aims stated in I.1. The proof follows, although we omit the calculation of tangent spaces to orbits, as they are very straightforward.

I.5:18 Table of $\mathcal{K}^T$ orbits

Name	Normal Form	$\mathcal{K}^T$-codim.	$\mathcal{K}^T$ tangent space	determinacy deg.
S_k	$y^2 + x^{k+1}$ $(k \geq 1)$	k	$m_2^T - \{x, x^2, \ldots, x^k\}$	k+1
T_k (B_k)	$x^2 + y^{2k}$ $(k \geq 2)$	k	$m_2^T - \{x, y^2, y^4, \ldots, y^{2k-2}\}$	2k
D_k (C_k)	$xy^2 + x^k$ $(k \geq 3)$	k	$m_2^T - \{x, y^2, x^2, x^3, \ldots, x^{k-1}\}$	k
E_6 (F_4)	$x^3 + y^4$	4	$m_2^T - \{x, x^2, y^2, xy^2\}$	4
$J_{10}(F_{1,0})$	$x^3 + \alpha xy^4 + \beta y^6$ $4\alpha^3 + 27\beta^2 \neq 0$	6	$m_2^T - \{x, x^2, x^3, y^2, xy^2, x^2y^2, xy^4, y^6\}$ $+ \mathbb{R}\{3x^2y^2 + xy^4,\ 3x^3 + xy^4,$ $2\alpha xy^4 + 3\beta y^6\}$	6
$X_9(K_{4,2})$	$x^4 + \alpha x^2y^2 \pm y^4$ $\alpha^2 \neq 4$ in positive case.	6	$m_2^T - \{x, x^2, x^3, x^4, y^2, xy^2, y^4, x^2y^2\}$ $+ \mathbb{R}\{2x^4 + \alpha x^2y^2,\ \alpha x^2y^2 + 2y^4\}$	4

Notes 1) The names given here (except for S_k and T_k) are those under which these germs, considered as members of $\mathcal{E}_2$, figure in Arnol'd's lists (see e.g. [1]). The names in brackets are those Arnol'd gives in [2] to the corresponding germs in $C^\infty(H^2,0)$, whose normal form is obtained in each case by dividing every exponent of y by 2. In the case of S_k, the corresponding germs in $C^\infty(H^2,0)$ are non-singular, and so do not figure in Arnol'd's list, but one should bear in mind that , unlike in the case of germs in $\mathcal{E}_n$, non-singular germs in $C^\infty(H^2,0)$ are not all equivalent, and in particular $f(x,y) = y$ is not even finitely determined for $\mathcal{K}^\partial$.

2) The family J_{10} $(F_{1,0})$ is in fact unimodal in the sense that two parameters are not necessary to list all the members: each member is equivalent to either $x^3 + \alpha xy^4 + y^6$ for some α, or to $x^3 \pm xy^4$. We have retained the two-parameter presentation in order to emphasise that we are dealing with a single family, and in order not to duplicate the expression for the $\mathcal{K}^T$ tangent space. In fact the normal form which Arnol'd gives, $x^3 + \alpha x^2y^2 + y^6$, is convenient in the complex case, but in the real case not every member can be brought to this normal form.

3) The $\mathcal{K}^T$-codimension in the table is that of each orbit; in the cases of J_{10} and X_9 the codimension of the stratum is 5.

4) There is some collapsing in the list, since for k even $y^2 + x^{k+1}$ (S_k^+) is equivalent to $y^2 - x^{k+1}$ (S_k^-), and D_k^+ is equivalent to D_k^-.

I.5:19 <u>Theorem</u> a) The above list includes normal forms for all $\mathcal{K}^T$- simple T-invariant germs, namely S_k, T_k, D_k and E_6.

b) The codimension, in $\mathfrak{M}_2^T$, of the complement of the germs listed, is 6.

c) Every singular germ $f \in \mathfrak{M}_2^T$ <u>either</u> is equivalent to one of the germs listed in I.5:18, <u>or</u> satisfies one of the following:

$j^\infty f \sim y^2$, and codim. f $= \infty$	$j^4 f \sim x^4 + x^2y^2$ and codim.$\mathcal{K}^{T4} f = 6$
$j^\infty f \sim x^2$, and codim. f $= \infty$	$j^4 f \sim y^4 + x^2y^2$ and codim.$\mathcal{K}^{T4} f = 6$
$j^\infty f \sim xy^2$, and codim. f $= \infty$	$j^4 f \sim x^4$ and codim.$\mathcal{K}^{T4} f = 7$
$j^6 f \sim x^3$ and codim.$\mathcal{K}^{T6} f = 7$	$j^4 f \sim y^4$ and codim.$\mathcal{K}^{T4} f = 7$
$j^6 f \sim x^3 + x^2y^2$ and codim.$\mathcal{K}^{T6} f = 6$	$j^4 f \sim x^2y^2$ and codim.$\mathcal{K}^{T4} f = 7$
$j^4 f \sim x^4 + 2x^2y^2 + y^4$ and codim.$\mathcal{K}^{T4} f = 6$	$j^4 f = 0$ and codim.$\mathcal{K}^{T4} f = 8$

Proof (of I.5:19 and of the assertions contained in I.5.:18):
For determinacy degrees we use the fact that if $f \in \mathcal{E}_2^T$ is k-determined for $\mathcal{R}$, then it is k-determined for $\mathcal{R}^T$ (see [29]page 2, Lemma 1.2).
We first construct a stratification of $\mathcal{M}_2^T$. Since any T-invariant function may be written as $p(x,y^2)$ for some $p \in \mathcal{E}_2$, its Taylor series will have the form

$$\sum_{i,j} a_{i,2j} x^i y^{2j},$$

and since we are principally interested in finitely determined germs, we will write $p(x,y^2)$ as a formal power series.

Case 1 $a_{1,0} \neq 0$. Then $p(x,y^2)$ is $\mathcal{K}^T$-equivalent to x. (Put $\bar{x} = p(x,y^2)$, $\bar{y} = y$).

Case 2 $a_{1,0} = 0$, $a_{2,0}$ and $a_{0,2}$ not both 0. If $a_{2,0}$ and $a_{0,2}$ are both non-zero, then $p(x,y^2)$ is a Morse function, equivalent to $x^2 + y^2$ and 2-determined. (S_1).
If $a_{2,0} = 0$, $a_{0,2} \neq 0$, then we can assume $a_{0,2} = 1$. By the Invariant Splitting Lemma ([31] page 42 ff) $p(x,y^2)$ is $\mathcal{R}^T$-equivalent to

$$y^2 + \eta(x)$$

for some $\eta \in \mathcal{M}_1^3$. The usual classification of functions of one variable now applies, so that $p(x,y^2)$ is $\mathcal{R}^T$-equivalent to

$$y^2 + x^{k+1}$$

for some $k \geq 2$, or has ∞-jet equivalent to y^2.
If $a_{0,2} = 0$, $a_{2,0} \neq 0$, similar considerations show that either $p(x,y^2)$ is $\mathcal{R}^T$-equivalent to $x^2 \pm y^{2k}$ (T_k) for some $k \geq 2$, or has ∞-jet equivalent to x^2.

Case 3 $a_{1,0} = a_{2,0} = a_{0,2} = 0$, $a_{1,2} \neq 0$. Assume $a_{1,2} = 1$

After multiplying $p(x,y^2)$ by $1-a_{2,2}x - a_{1,4}y^2$, we may assume that $a_{2,2} = a_{1,4} = 0$. Successive $\mathcal{K}^T$-equivalences of this kind will reduce $p(x,y^2)$, to any desired degree, to the form

$$xy^2 + \sum_{j\geq 2} \tilde{a}_{0,2j}\, y^{2j} + \sum_{i\geq 3} \tilde{a}_{i,0}\, x^i .$$

Put

$$x = \bar{x} - \sum_{j\geq 2} \tilde{a}_{0,2j}\, y^{2j} + \sum_{i\geq 3} \tilde{a}_{i,0}\, x^i$$

to reduce $p(x,y^2)$ to

$$\bar{x}y^2 + \sum_{i\geq 3} \tilde{a}_{i,0}(\bar{x} - \sum_{j\geq 2} \tilde{a}_{0,2j}\, y^{2j-2})^i.$$

Let k be the first integer such that $\tilde{a}_{k,0} \neq 0$. Then the last expression is equal to

$$\bar{x}y^2 + a_{k,0}\, \bar{x}^k + O(k+1)$$

which may be reduced, by an appropiate change of scale, to

$$\bar{x}y^2 + \bar{x}^k + O(k+1).$$

Since this germ is k-determined for $\mathcal{R}$, it is also for $\mathcal{R}^T$, and so $p(x,y^2)$ is $\mathcal{R}^T$-equivalent to $xy^2 + x^k$ (D_k). If, on the other hand, $\tilde{a}_{k,0} = 0$ for all k, then in fact $a_{k,0} = 0$ for all k, and so

$$p(x,y^2) = xy^2 + y^2(\sum a_{i,2j}\, x^i y^{2j}\ {}^{2})$$

which is formally equivalent to xy^2.

<u>Case 4</u> $a_{0,1} = a_{2,0} = a_{0,2} = a_{1,2} = 0$, $a_{3,0} \neq 0$. Assume $a_{3,0} = 1$. The coordinate change $x = \bar{x} - \frac{1}{3}a_{2,2}y^2$ removes any x^2y^2 term from the 4-jet of $p(x,y^2)$, leaving $x^3 + a_{0,4}y^4$. If $a_{0,4} \neq 0$, $p(x,y^2)$ is equivalent to $x^3 + y^4 + O(5)$ and we have an E_6, which is 4-determined for $\mathcal{R}$

and hence also for $\mathcal{R}^T$.

Now assume $a_{0,4} = 0$. One then shows, by using straightforward $\mathcal{K}^T$ equivalences, that the 6-jet of $p(x,y^2)$ is equivalent to

$$x^3 + \alpha xy^4 + \beta y^6$$

for some $\alpha, \beta \in \mathbb{R}$. This is 6-determined for $\mathcal{R}$ if $4\alpha^3 + 27\beta^2 \neq 0$, and can be placed in one of the two normal forms

$$x^3 + \alpha xy^4 + y^6 \quad \text{or} \quad x^3 \pm xy^4.$$

One can check by direct substitution that α is indeed a modulus — for different values of α, the germs are inequivalent.

The remaining 6-jet orbits are those of

$$x^3 - \frac{3}{\sqrt[3]{4}}\, xy^4 + y^6 \ \ (\text{codim.}6) \text{ and } \quad x^3 \quad (\text{codim.}7).$$

The former has the more convenient normal form $x^3 + x^2y^2$ since it is a cubic with a repeated root.

<u>Case 5</u> $a_{1,0} = a_{0,2} = a_{2,0} = a_{1,2} = a_{3,0} = 0$. Then the 4-jet of $p(x,y^2)$ is

$$a_{4,0}x^4 + a_{2,2}x^2y^2 + a_{0,4}y^4.$$

If $a_{4,0}$ and $a_{0,4}$ are both non-zero, appropiate changes of scale reduce this to

$$x^4 + \alpha x^2y^2 + y^4,$$

for some $\alpha \in \mathbb{R}$. In the case where the coefficient of y^4 is 1, this jet is sufficient for $\mathcal{R}$ iff $\alpha^2 \neq 4$, and when the coefficient of y^4 is -1, it is sufficient for all values of α.

It is easy to see that we have here a unimodular family: if we apply

any $\mathcal{K}^T$-equivalence, then it is only the linear part of the diffeomorphism which acts on the 4-jet, and one checks that if the coefficients of x^4 and y^4 are unchanged, then α is also unchanged.
The remaining 4-jet orbits are those of $x^4 \pm 2x^2y^2 + y^4$, $x^4 \pm x^2y^2$, $x^4 \pm y^4$, x^4, x^2y^2, y^4 and 0, which have codimension 6, 6, 6, 7, 7, 7 and 8 respectively. This completes the stratification that we require. It is shown in diagramatic form on page 34.

Parts (b) and (c) of the theorem follow directly from this stratification, and the assertions in I.5:18 have also been dealt with.

Proof of (a) It is clear that if $p(x,y^2)$ is not equivalent to one of the S_k, T_k, D_k or E_6 then it is not simple, for it must adjoin J_{10}, in the case that the 4-jet of $p(x,y^2)$ is equivalent to x^3, or X_9, in the case that the 3-jet is zero, or it must adjoin a countably infinite number of distinct orbits, in the cases where the ∞-jet is equivalent to y^2, to x^2 or to xy^2.
Conversely, the simplicity of S_k, T_k, D_k and E_6 is more or less immediate from the stratification. For S_k, we have $p(x,y^2) = y^2 \pm x^{k+1}$, and small perturbations will remove neither the y^2 term nor the x^{k+1} term. Thus, the only orbits in which small perturbations will lie are those of $y^2 \pm x^{i+1}$ for $i \leqslant k$. The argument for T_k is similar. For D_k, we have $p(x,y^2) = xy^2 \pm x^k$, and small perturbations may either introduce an x^i term for some $i \leqslant k$, in which case D_i results, or introduce quadratic terms, in which case S_i or T_i $(i \leqslant k)$ will result. For E_6, small perturbations will result in S_i, T_i or D_i, with $i \leqslant 6$ ∎

Stratification of $\mathfrak{m}_2^T$

Jets with double underlining are $\mathcal{K}^T$-sufficient

The stratification constructed in the preceeding proof is slightly finer than is needed for the proof of Theorem I:2; for that purpose we obtain, from the proof of I.5:19, the following stratification:

I.5:20 Corollary The following is a $\mathcal{K}^T$-invariant stratification of $\mathcal{M}_2^T$.

Stratum		$\mathcal{K}^T$-codim.
x		0
$y^2 + x^{k+1}$	$1 \leqslant k \leqslant 4$	k
$x^2 + y^{2k}$	$2 \leqslant k \leqslant 4$	k
$xy^2 + x^k$	$3 \leqslant k \leqslant 4$	k
$x^3 + y^4$		4
${\mathcal{K}^T}^4 x^3$		5
${\mathcal{K}^T}^3 0$		5
${\mathcal{K}^T}^5 y^2$		5
${\mathcal{K}^T}^9 x^2$		5
${\mathcal{K}^T}^4 xy^2$		5

Note In other words, every germ $\in \mathcal{M}_2^T$ is $\mathcal{K}^T$-equivalent to one of the germs listed in the first four places in the table, or its k-jet lies in one of the ${\mathcal{K}^T}^k$-orbits listed in the remaining places.

I.5:21 Remark The part of the stratification of Theorem I:2 which lies within the $\mathcal{K}$-orbit of the map germ $(x,y) \longrightarrow (x,y^2,0)$ can easily be obtained from this stratification by applying I.5:7, I.5:8, I.5:11 and I.5:12.

I.6 Classification of germs whose 2-jet is equivalent to (x,xy,0)

I.6:1 Proposition In $J^3(2,3)$ there are five $\mathcal{A}^3$-orbits of jets whose 2-jet is equivalent to (x,xy,0):

Orbit	$\mathcal{A}^3$-codim.
(x, xy, y^3)	4
$(x, xy + y^3, xy^2)$	5
(x, xy, xy^2)	6
$(x, xy + y^3, 0)$	6
$(x, xy, 0)$	7

Proof Any 3-jet over (x,xy,0) is equivalent to one of the form

$$(x,\ xy + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3,\ c_{2,1}x^2y + c_{1,2}xy^2 + c_{0,3}y^3)\ .$$

If $c_{0,3} \neq 0$, complete the cube in the third component by adding appropriate multiples of X^3 and XY to Z (i.e. by a left coordinate change) and then change the y coordinate to get a 3-jet of the form

$$(x,\ xy + b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3,\ y^3).$$

After an obvious left coordinate change we may take $b_{2,1}$ to be 0. Put $\bar{y} = y(1 + b_{1,2}y^2)$ and then remove the y^3 term from the second component by the obvious left coordinate change to obtain

$$(x,\ xy,\ y^3).$$

If $c_{0,3} = 0$, the 3-jet is obviously left-equivalent to

$$(x,\ xy + b_{1,2}\ xy^2 + b_{0,3}\ y^3,\ c_{1,2}\ xy^2)$$

and then, if $c_{1,2} \neq 0$, it is equivalent to

$$(x,\ xy + b_{0,3}y^3,\ xy^2).$$

This is equivalent to the second of the 3-jets in the table, if $b_{0,3} \neq 0$, or to the third, if $b_{0,3} = 0$.

<u>If $c_{0,3} = c_{1,2} = 0$,</u> the 3-jet is left-equivalent to

$$(x,\ xy + b_{1,2}xy^2 + b_{0,3}y^3,\ 0).$$

Putting $\overline{y} = y(1 + b_{1,2}y)$, this becomes equivalent to the fourth of the 3-jets in the table, if $b_{0,3} \neq 0$, or to the fifth otherwise.

It is straightforward to calculate the $\mathcal{A}^3$-tangent spaces to each of the orbits listed, and to count the codimension from them ∎

The remainder of I.6 is divided into four subsections, numbered from I.6.1 to I.6.4. These deal respectively with the classification of germs whose 3-jet is equivalent to one of the first four listed in I.6:1. Since the $\mathcal{A}^3$-codimension of the fifth is 7, the preimage in $J^k(2,3)$ of the $\mathcal{A}^3$-orbit of this jet forms one of the strata in our stratification of $J^k(2,3)$ (Theorem I:2).

I.6.1 <u>Classification of germs whose 3-jet is equivalent to $(x,\ xy,\ y^3)$</u>

Here we are able to acheive a complete classification of finitely determined map-germs.

I.6.1:1 <u>Lemma</u> All 4-jets whose 3-jet is equivalent to (x,xy,y^3) belong to a single $\mathcal{A}^4$ orbit

<u>Proof</u> Apply Theorem I.3:1 to the set $S = \pi_{4,3}^{-1}((x,xy,y^3)) \subseteq J^4(2,3)$, taking G as $\mathcal{A}^4$, which acts on $J^3(2,3)$ by taking mod $\mathcal{M}_2^4$, and π as $\pi_{4,3}: J^4(2,3) \longrightarrow J^3(2,3)$. The hypotheses of the Theorem are easily seen to hold; in particular, $T_\sigma S = \bigoplus_1^3 \mathcal{M}_2^4$, for any $\sigma \in S$, and it is straightforward to calculate that

$$T_\sigma \mathcal{A}^4 \sigma \supseteq T_\sigma S \quad \blacksquare$$

I.6.1:2 <u>Theorem</u> a) If $j^3 f(0)$ is equivalent to (x,xy,y^3) and f is of finite $\mathcal{A}$-codimension then for some $l \geq 2$, f is equivalent to

$$(x,y) \longrightarrow (x,\ xy + y^{3l-1},\ y^3).$$

b) If $f(x,y) = (x,\ xy + y^{3l-1},\ y^3)$ $(l \geq 2)$ then

$$T\mathcal{A}f = C = \begin{bmatrix} \mathcal{M}_2 - \{y, y^4, y^7, \ldots, y^{3l-5}\} \\ \mathcal{M}_2 - \{y, y^2, y^5, \ldots, y^{3l-4}\} \\ \mathcal{M}_2 - \{y, y^2\} \end{bmatrix} + \mathbb{R}\left\{ \begin{bmatrix} y \\ y^2 \\ 0 \end{bmatrix} \begin{bmatrix} y^4 \\ y^5 \\ 0 \end{bmatrix} \begin{bmatrix} y^7 \\ y^8 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} y^{3l-5} \\ y^{3l-4} \\ 0 \end{bmatrix} \right\}$$

f has $\mathcal{A}$-codimension $l+2$, f is $3l-1$ determined (for $\mathcal{A}$), and f is simple.

<u>Proof</u> By induction on k. Suppose that σ is a k-jet of the form

$$(x + a(x,y),\ xy + b(x,y),\ y^3 + c(x,y)),$$

where a, b and c are homogeneous polynomials of degree k. By the coordinate change $\bar{x} = f^*X$, we may assume that a = 0. Now it is straightforward to check that the following hold for any f such that $j^k f(0) = \sigma$:

(1) $$f^*\mathcal{M}_3 + \mathcal{M}_2^{3l+1} \supseteq \mathcal{M}_2^{3l} - \{xy^{3l-1}\} \qquad (l \geq 2)$$

$$(2) \qquad f^*\mathfrak{m}_3 + \mathfrak{m}_2^{3l+2} \supseteq \mathfrak{m}_2^{3l+1} - \{y^{3l+1}\} \qquad (l \geq 1)$$

$$(3) \qquad f^*\mathfrak{m}_3 + \mathfrak{m}_2^{3l+3} \supseteq \mathfrak{m}_2^{3l+2} - \{y^{3l+2}\} \qquad (l \geq 1)$$

This means that by applying left coordinate changes we can reduce σ to

$$(x,\ xy + bxy^{k-1} + b'y^k,\ y^3 + cxy^{k-1} + c'y^k)$$

where b,b',c and c' are now constants.

By Lemma I.6.1:1 we may now take k to be greater than 4. In what follows we make a succession of coordinate changes, denoting at each stage the variables which we are replacing by x,y (in the source) and by X,Y,Z (in the target), and denoting the new variables by $\bar{x},\bar{y}$ and by $\bar{X},\bar{Y},\bar{Z}$.

<u>If k = 3l</u> we can assume that b' = c' = 0, by (1).

Coordinate Change	resulting k-jet
	$(x,\ xy + bxy^{3l-1},\ y^3 + cxy^{3l-1})$
$x = \bar{x} - b\bar{x}\bar{y}^{3l-2}$	
	$(x - bxy^{3l-2},\ xy,\ y^3 + cxy^{3l-1})$
$\bar{X} = X + bYZ^{l-1}$	
	$(x,\ xy,\ y^3 + cxy^{3l-1})$
$y = \bar{y} - \frac{1}{3}cxy^{3l-2}$	
	$(x,\ xy - \frac{1}{3}cx^2y^{3l-3},\ y^3)$
$\bar{Y} = Y + \frac{1}{3}cX^2Z^{l-1}$	
	$(x,\ xy,\ y^3)$

If $k = 3l+1$ we can assume $b = c = 0$, by (2).

Coordinate Change	Resulting k-jet
	$(x,\ xy + b'y^{3l+1},\ y^3 + c'y^{3l+1})$
$x = \bar{x} - b'y^{3l}$	
	$(x - b'y^{3l},\ xy,\ y^3 + c'y^{3l+1})$
$\bar{X} = X + b'Z^l$	
	$(x,\ xy,\ y^3 + c'y^{3l+1})$
$y = \bar{y} - \frac{1}{3}c'\bar{y}^{3l-1}$	
	$(x,\ xy - \frac{1}{3}c'xy^{3l-1},\ y^3)$
$x = \bar{x} + \frac{1}{3}c'\overline{xy}^{3l-2}$	
	$(x + \frac{1}{3}c'xy^{3l-2},\ xy,\ y^3)$
$\bar{X} = X - \frac{1}{3}c'YZ^{l-1}$	
	$(x,\ xy,\ y^3)$

If $k = 3l+2$ we can assume $b = c = 0$, by (3).

Coordinate Change	Resulting k-jet
	$(x,\ xy + b'y^{3l+2},\ y^3 + c'y^{3l+2})$
$y = \bar{y} - c'\bar{y}^{3l}$	
	$(x,\ xy - \frac{1}{3}c'xy^{3l} + b'y^{3l+2},\ y^3)$
$\bar{Y} = Y + \frac{1}{3}c'XZ^l$	
	$(x,\ xy + b'y^{3l+2},\ y^3)$

Now if $b' \neq 0$,

$y = \bar{y}/|b'|^{\frac{1}{3l+1}}$, $\bar{Y} = |b'|^{\frac{1}{3l+1}}Y$,

$\bar{Z} = |b'|^{\frac{3}{3l+1}}Z$

$(x,\ xy + y^{3l+2},\ y^3)$.

Finally, $(x,\ xy + y^{3l+2},\ y^3)$ and $(x,\ xy - y^{3l+2},\ y^3)$ are equivalent under changes of sign in the x, X and Y coordinates. We replace the $3l+2$ here by $3l-1$ in the statement of the theorem, since this is the $\mathcal{A}_e$-codimension of the germ f of part (b).

In order to complete the proof of the theorem, it will be enough to prove part (b), since we have shown that for any f with $j^3f(0)$ equivalent to (x,xy,y^3), either $j^{3l+2}(0)$ is equivalent to $(x,xy + y^{3l+2},y^3)$ for some l, or for all finite k, $j^kf(0)$ is equivalent to (x,xy,y^3).

<u>Proof of part (b)</u> Let $j^{3l-1}g(0) = (x,xy + y^{3l-1},y^3)$. Then

$$g^*\mathcal{M}_3\Theta(g) \supseteq \mathcal{M}_2^3\Theta(g),$$

for clearly the inclusion holds mod. $\mathcal{M}_2^4\Theta(g)$, and we can apply Nakayama's Lemma. Hence, g is a finite map-germ, and C is a finite $\mathcal{E}_3$ module via g^* (see the argument at the end of the proof of I.3:2). It is, of course, easy to see that C is indeed closed under the action of $\mathcal{E}_3$ via g^*. Now, since

$$C \supseteq \mathcal{M}_2^{3l-3}\Theta(g)$$

we may conclude from I.3:2 that $C = T\mathcal{A}g$, if we can prove

(4) $$C = T\mathcal{A}g + g^*\mathcal{M}_3.C + \mathcal{M}_2^{3l}\Theta(g)\ .$$

We have

$$g^*\mathcal{M}_3.C + \mathcal{M}_2^{3l}\ \Theta(g) \supseteq g^*\{X,Z\}.C + \mathcal{M}_2^{3l}\Theta(g) = \{x,y^3\}.C + \mathcal{M}_2^{3l}\Theta(g)$$

$$\supseteq \left[C \cap \mathcal{M}_2^2\Theta(g)\right] - \left\{ \begin{bmatrix}xy\\0\\0\end{bmatrix} \begin{bmatrix}y^2\\0\\0\end{bmatrix} \begin{bmatrix}y^3\\0\\0\end{bmatrix} \begin{bmatrix}0\\xy\\0\end{bmatrix} \begin{bmatrix}0\\xy^2\\0\end{bmatrix} \begin{bmatrix}0\\y^3\\0\end{bmatrix} \begin{bmatrix}0\\y^4\\0\end{bmatrix} \begin{bmatrix}0\\0\\xy\end{bmatrix} \begin{bmatrix}0\\0\\xy^2\end{bmatrix} \begin{bmatrix}0\\0\\y^3\end{bmatrix} \begin{bmatrix}0\\0\\y^4\end{bmatrix} \begin{bmatrix}0\\0\\y^5\end{bmatrix} \begin{bmatrix}y^{3l-2}\\0\\0\end{bmatrix} \begin{bmatrix}0\\y^{3l-1}\\0\end{bmatrix} \right.$$

Then, working mod $\mathfrak{m}_2^{3l}\Theta(g)$, we have

$$tg\begin{bmatrix}0\\y\end{bmatrix} - \omega g\begin{bmatrix}0\\Y\\3Z\end{bmatrix} = \begin{bmatrix}0\\(3l-1)y^{3l-1}\\0\end{bmatrix}, \quad tg\begin{bmatrix}y^{3l-2}\\0\end{bmatrix} = \begin{bmatrix}y^{3l-2}\\y^{3l-1}\\0\end{bmatrix}$$

and so the right hand side of (4) contains $\mathfrak{m}_2^{3l-2}\Theta(g)$.

Now working mod $\mathfrak{m}_2^{3l-2}\Theta(g)$, we have

$xy = g^*Y$,

$$\begin{bmatrix}0\\xy^2\\0\end{bmatrix} = tg\begin{bmatrix}xy\\0\end{bmatrix} - \begin{bmatrix}xy\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\3y^4\end{bmatrix} = tg\begin{bmatrix}0\\y^2\end{bmatrix} - \begin{bmatrix}0\\xy^2\\0\end{bmatrix},$$

$$\begin{bmatrix}y\\y^2\\0\end{bmatrix} = tg\begin{bmatrix}y\\0\end{bmatrix},$$

$$\begin{bmatrix}y^2\\0\\0\end{bmatrix} = tg\begin{bmatrix}y^2\\0\end{bmatrix} - \omega g\begin{bmatrix}0\\Z\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\0\\xy^2\end{bmatrix} = tg\begin{bmatrix}0\\x\end{bmatrix} - \omega g\begin{bmatrix}0\\X^2\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\y^4\\0\end{bmatrix} = tg\begin{bmatrix}y^3\\0\end{bmatrix} - \omega g\begin{bmatrix}Z\\0\\0\end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 3y^5 \end{bmatrix} = tg \begin{bmatrix} 0 \\ y^3 \end{bmatrix} - \omega g \begin{bmatrix} 0 \\ XZ \\ 0 \end{bmatrix},$$

and since $y^3 = g^*Z$, $x = g^*X$, this completes the proof that (4) holds, and hence that $T\mathcal{A}g = C$.

Now, by Theorem I.3:3, g is $3l$-determined, but by an application of I.3:1, we see that the $3l$-jets of all such g are equivalent, so that in fact g is $(3l-1)$-determined.

<u>Proof of simplicity</u> By inspection of the $\mathcal{A}$-tangent space to f, we see that an $\mathcal{A}$-versal unfolding of f is given by

$$(x,y,u_1,\ldots,u_{l+2}) \longrightarrow (x, xy+u_1y+u_2y^2+u_3y^5+\ldots u_ly^{3l-4}+y^{3l-1}, u_{l+1}y+u_{l+2}y^2+y^3, u)$$

$$= (f_u(x,y), u).$$

If u_1 or u_{l+1} is non-zero, the f_u is an immersion.

Assume $u_1 = u_{l+1} = 0$. If $u_{l+2} \neq 0$, the 2-jet of f_u is $(x, xy + u_2y^2, u_{l+2}y^2)$, and so f_u is a cross-cap, equivalent to $(x,y) \longrightarrow (x, y^2, xy)$.

Assume $u_1 = u_{l+1} = u_{l+2} = 0$. If $u_2 \neq 0$, then the coordinate changes

$$\bar{y} = y + \frac{x}{2u_2}, \qquad \bar{Y} = \frac{Y}{u_2} + \frac{X^2}{4u_2^2}$$

transform the 3-jet of f_u into

$$(x, \bar{y}^2, \bar{y}^3 - \frac{3x\bar{y}^2}{2u_2} + \frac{3x^2\bar{y}}{4u_2^2} - \frac{x^3}{8u_2^3})$$

which is equivalent to

$$(x, y^2, y^3 + x^2y).$$

Since this is the 3-jet of S_1 (see I.5:18) and is sufficient (I.5:19), f_u is equivalent to S_1.

Assume $u_1 = u_2 = u_{l+1} = u_{l+2} = 0$. Then

$$f_u(x,y) = (x,\ xy + u_3y^5 + u_4y^8 + \dots + u_ly^{3l-4} + y^{3l-1},\ y^3).$$

By the above, this is equivalent to

$$(x,y) \longrightarrow (x,\ xy + y^{3i-1},\ y^3)$$

where i is the first sub-index for which $u_i \neq 0$, or to f_0 if u=0. Thus, only a finite number of different $\mathcal{A}$-orbits are met by an $\mathcal{A}$-versal unfolding of f, and so it follows that f is simple ∎

I.6.2 <u>Classification of germs whose 3-jet is equivalent to $(x,xy+y^3,xy^2)$</u>

We now embark on a rather tedious series of calculations. Unlike in the preceeding subsection, the explicit coordinate changes needed to bring jets to the appropriate normal form are rather complicated, so that it is in fact easier to make the corresponding infinitesimal calculations and apply Theorem I.3.1.

I.6.2:1 <u>Lemma</u> Every 4-jet whose 3-jet is equivalent to $(x,xy + y^3,\ xy^2)$, is equivalent to one of

$$\sigma_c = (x,\ xy + y^3,\ xy^2 + cy^4).$$

<u>Proof</u> Let σ be a 4-jet of the form $(x,\ xy + y^3 + b(x,y),\ xy^2 + c(x,y))$, where $b,c \in \mathcal{M}_2^4$. Since, mod $\mathcal{M}_2^5$, we have

$X^4 = x^4,\ \ X^2Y = x^3y,\ \ Y^2 = x^2y^2,$

we may assume, after appropriate left coordinate changes, that σ is

$$(x,\ xy + y^3 + b_{1,3}xy^3 + b_{0,4}y^4,\ xy^2 + c_{1,3}xy^3 + c_{0,4}y^4).$$

Putting $\bar{y} = y(1+c_{1,3}y)^{\frac{1}{2}}$ removes the xy^3 term from the third component, but introduces an xy^2 term in the second. This, however, may be removed by a left coordinate change, bringing σ to the form

$$(x,\ xy + y^3 + b_{1,3}xy^3 + b_{0,4}y^4,\ xy^2 + c_{0,4}y^4)$$

Putting $\bar{y} = y(1+b_{1,3}y^2)$ transforms this 4-jet into

$$(x,\ xy + y^3 + b_{0,4}y^4,\ xy^2 + c_{0,4}y^4). \qquad (1)$$

One then calculates that for any 4-jet of this form,

$$tf\begin{bmatrix} y^3 \\ 0 \end{bmatrix} - \omega f\begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -xy \\ y^4 \\ 0 \end{bmatrix},$$

$$tf\begin{bmatrix} -2xy \\ y^2 \end{bmatrix} + \omega f\begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} -2xy \\ (3+c)y^4 \\ 0 \end{bmatrix} \quad \text{and}$$

$$tf\begin{bmatrix} (2-4c)xy - 4b(xy^2 + cy^4) \\ x \end{bmatrix} + \omega f\begin{bmatrix} 4bZ \\ (4c-5)Z - X^2 \\ -cXY \end{bmatrix} = \begin{bmatrix} (2-4c)xy \\ (4c-5)y^4 \\ 0 \end{bmatrix}$$

(where $c = c_{0,4}$) so that

$$\begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix} \in T\mathcal{A}^4\sigma .$$

From there it is relatively straightforward to calculate $T\mathcal{A}^4\sigma$ — it is in fact equal, mod. $\mathcal{M}_2^5$, to the space C of I.6.2:3, except that the fourth supplementary generator of C is replaced by

$$\begin{bmatrix} 0 \\ 0 \\ xy + y^3 + b_{0,4}y^4 \end{bmatrix}.$$

Hence the dimension of $T\mathcal{A}^4\sigma$ is independent of $b_{0,4}$, and a straightforward application of Theorem I.3:1 (a) with $G = \mathcal{A}^4$ and $S = S_c = \{$4-jets of the form (1) with $c_{0,4}=c\}$ completes the proof∎

I.6.2:2 <u>Lemma</u> Every 6-jet whose 3-jet is equivalent to $(x, xy + y^3, xy^2)$ is equivalent to a 6-jet of the form

$$(x, xy+y^3+b_{0,5}y^5+b_{0,6}y^6, xy^2+cy^4+c_{1,4}xy^4+c_{0,5}y^5+c_{1,5}xy^5+c_{0,6}y^6).$$

<u>Proof</u> Let σ be such a 6-jet. By the preceeding lemma, and by obvious left coordinate changes, the 5-jet of σ is equivalent to

$$(x, xy + y^3, + b_{1,4}xy^4 + b_{0,5}y^5, xy^2 + cy^4 + c_{1,4}xy^4 + c_{0,5}y^5).$$

By writing $\bar{y} = y + b_{1,4}y^4$ we can remove the xy^4 term from the second component. A similar proceedure at 6-jet level reducues σ to the desired form∎

Now for convenience write $b_{0,5} = b$, $c_{1,4} = d$, $c_{0,5} = e$, and $c_{0,6} = g$.

I.6.2:3 <u>Proposition</u> Let σ be a 6-jet

$$(x, xy + y^3 + by^5 + b_{0,6}y^6, xy^2 + cy^4 + dxy^4 + ey^5 + c_{1,5}xy^5 + gy^6). \qquad (1)$$

If $c \neq \frac{1}{2}, 1, \frac{3}{4}$,

or if $c = 1$ and $g-d-b-e^2 \neq 0$,

or if $c = \frac{1}{2}$ or $\frac{3}{2}$ and $e \neq 0$,

then

$$T\mathcal{A}^6\sigma = C = \begin{bmatrix} \mathfrak{m}_2 - \{y, y^2\} \\ \mathfrak{m}_2 - \{y, xy, y^2, y^3\} \\ \mathfrak{m}_2 - \{y, xy, y^2, xy^2, y^3, y^4\} \end{bmatrix} + \mathbb{R}\left\{ \begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix} \begin{bmatrix} 00 \\ xy+y^3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy+y^3 \end{bmatrix} \begin{bmatrix} 0 \\ xy \\ xy^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^2+cy^4 \end{bmatrix} \right\}$$

and in the other cases, $T\mathcal{A}^6\sigma \subsetneq C$.

<u>Proof</u> First, with one exception, it is easy to see that the generators of $T\mathcal{A}^6\sigma$ (i.e. tf of the natural generators of $\overset{2}{\underset{1}{\oplus}}\mathfrak{m}_2$ and ωf of the natural generators of $\overset{3}{\underset{1}{\oplus}}\mathfrak{m}_3$; see I.2) are contained in C; note that the supplementary generators of C are, respectively,

$$tf\begin{bmatrix} y \\ 0 \end{bmatrix},\ tf\begin{bmatrix} y^2 \\ 0 \end{bmatrix},\ \omega f\begin{bmatrix} 0 \\ Y \\ 0 \end{bmatrix},\ \omega f\begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix},\ tf\begin{bmatrix} x \\ 0 \end{bmatrix} - \omega f\begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix} \text{ and } \omega f\begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix}$$

all mod. $\mathfrak{m}_2^5$. The exception is $tf\begin{vmatrix} 0 \\ y \end{vmatrix}$, but this is equal, again mod. $\mathfrak{m}_2^5$, to

$$\omega f\begin{bmatrix} 2X \\ 3Y \\ 4Z \end{bmatrix} - tf\begin{bmatrix} 2x \\ 0 \end{bmatrix},$$ and so is indeed contained in C. Hence, $T\mathcal{A}^6\sigma \subseteq C$.

To see that $T\mathcal{A}^6\sigma \supseteq C$, note first that by Nakayama's Lemma this is equivalent to $T\mathcal{A}^6\sigma + \sigma^*\mathfrak{m}_3.C \supseteq C$, where σ^* is the obvious morphism $\mathcal{E}_3 \to \mathcal{E}_2$ obtained by considering σ as a polynomial germ. Nakayama's Lemma is certainly applicable here since we are dealing with finite dimensional real vector spaces, and thus certainly with finite modules.

Since $\sigma^*\mathcal{M}_3.C \supseteq x.C$, it contains all those monomial generators of C (i.e. those listed in the statement of the proposition) which are divisible by x, except for

$$(2) \qquad \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} xy^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ x^2y \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy^2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy^3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x^2y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x^2y^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^4 \end{bmatrix}.$$

Moreover, all terms of the form $\begin{bmatrix} a(x,y) \\ 0 \\ 0 \end{bmatrix}$, where $a \in \mathcal{M}_2^6$, belong to $T\mathcal{A}^6\sigma$

since $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = t\sigma \begin{bmatrix} a \\ 0 \end{bmatrix}$, where $t\sigma$ is the morphism $\bigoplus_1^2 \mathcal{M}_2 \to \bigoplus_1^3 \mathcal{M}_2$ obtained by regarding σ as a polynomial germ.

Using $\omega\sigma \begin{bmatrix} 0 \\ y^2 \\ 0 \end{bmatrix}$ we obtain $\begin{bmatrix} 0 \\ y^6 \\ 0 \end{bmatrix}$ in $T\mathcal{A}^6\sigma$.

Of the generators of C we now need to find those of group (2), and

$$(3) \qquad \begin{bmatrix} y^3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^5 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^6 \end{bmatrix}$$

for once these are found, then by the remark in the first paragraph of the proof, the supplementary generators of C will then belong to $T\mathcal{A}^6\sigma$ as a consequence. By listing the generators of $T\mathcal{A}^6\sigma$ in which the terms of groups (2) and (3) figure, it becomes clear that $T\mathcal{A}^6\sigma \supseteq C$ if and only if the matrix (4) of coefficients (shown on next page) has maximal rank. The rows have been numbered in order to facilitate the succeeding calculations, and, as can be seen, the first eleven rows are already in s tep form. Rows ⑫ and ⑬ turn out to be dependent, modulo the first eleven rows, and after operating, with rows ① to ⑪, on rows ⑫ and ⑭ to ⑰, we obtain, in the last four columns and the last five rows, matrix (5), whose corank is thus equal to that of (4).

<u>Matrix (5)</u>	$\begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} y^4 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ y^6 \end{bmatrix}$
⑫ - c① - ④ + c⑥ - (c-1)⑩ - (c-2)⑪	0	0	$2c-c^2$	1-c
⑭ - 2① - 3③ + ⑥ - ⑩	0	0	-3	4c-4
⑮ - 5e① - 4c⑤ + 5e⑥ - 2⑧ -(5-4c)⑨	$4c^2-5c$	2-4c	e(5-9c)	-4ce
⑯ - 4c② - 2⑤ + 4c⑦ + ⑨	3-3c	4c-2	-e	4e
⑰ -2d① -e② -2b③ +2d⑥ +e⑦ -2d⑩ -2d⑪	-e	e	-2(b+cd)	2(g-b-d)

Now, the first two rows of this matrix are independent if and only if

$$(c - 1)(2c - 3)(2c - 1) \neq 0$$

and in this case, matrix (5) has the same corank as

$$\begin{bmatrix} 4c^2 - 5c & 2 - 4c \\ 3 - 3c & 4c - 2 \\ -e & e \end{bmatrix}.$$

However, the determinant of the first two rows of this matrix is equal to $2(2c - 1)^2(2c - 3)$ and so if $(c - 1)(2c - 3)(2c - 1) \neq 0$, we conclude

	xy^4	y^5	o	x^2y^2	xy^3	o	c	x^2y	o	[illegible]	[illegible]	c	c	o	y^6	
$tf\begin{bmatrix}xy^2\\0\end{bmatrix}$	1	0	0	0	0	1	0	0	0	0	1	0	[illegible]	0	0	(1)
$tf\begin{bmatrix}y^3\\0\end{bmatrix}$	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	(2)
$tf\begin{bmatrix}y^4\\0\end{bmatrix}$	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	(3)
$tf\begin{bmatrix}x^2\\0\end{bmatrix} - wf\begin{bmatrix}x^2\\0\\0\end{bmatrix}$	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	(4)
$tf\begin{bmatrix}xy\\0\end{bmatrix}$	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	(5)
$wf\begin{bmatrix}0\\xy\\0\end{bmatrix}$	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	(6)
$wf\begin{bmatrix}y\\0\\0\end{bmatrix}$	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	(7)
$wf\begin{bmatrix}0\\0\\xy\end{bmatrix} - tf\begin{bmatrix}x\\0\end{bmatrix}$	0	0	0	0	0	0	0	1	-1	0	0	0	[illegible]	0	0	(8)
$wf\begin{bmatrix}0\\z\\0\end{bmatrix} - tf\begin{bmatrix}xy^4\\0\end{bmatrix}$	0	0	0	0	0	0	0	0	1	0	0	0	0	-e	-e	(9)
$wf\begin{bmatrix}x^2\\2xy\\y^2\end{bmatrix} - tf\begin{bmatrix}x^2+2xy^2\\0\end{bmatrix}$	0	0	0	0	0	0	0	0	0	1	-2	0	0	0	11	(10)
$wf\begin{bmatrix}z\\0\\0\end{bmatrix}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	(11)
$wf\begin{bmatrix}0\\0\\xz\end{bmatrix}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	(12)
$tf\begin{bmatrix}0\\xy\end{bmatrix}$	4c	0	0	2	6	3	0	0	0	1	0	0	0	0	0	(13)
$tf\begin{bmatrix}0\\y\end{bmatrix}$	2	0	3	0	0	1	0	0	0	0	0	0	0	0	[illegible]	(14)
$tf\begin{bmatrix}0\\x\end{bmatrix} - wf\begin{bmatrix}0\\x^2\\0\end{bmatrix}$	5e	0	0	0	4c	0	0	2	3	0	0	0	0	0	0	(15)
$tf\begin{bmatrix}0\\y^2\end{bmatrix}$	0	4c	0	0	2	0	0	0	1	0	0	0	0	0	5e	(16)
$tf\begin{bmatrix}2x\\y\end{bmatrix} - wf\begin{bmatrix}2x\\3y\\4z\end{bmatrix}$	7d	e	2b	0	0	0	0	0	0	0	0	0	0	0	7c	(17)

that matrix (4) has maximal rank, proving that in this case $T\mathcal{A}^6\sigma = C$.

If c = 1 matrix (5) has maximal rank iff $g - b - d - e^2 \neq 0$.

If $c = \frac{3}{2}$ matrix (5) has maximal rank iff $e \neq 0$

If $c = \frac{1}{2}$ matrix (5) has maximal rank iff $e \neq 0$ ∎

I.6.2:4 Corollary The 6-jets σ of form (1) (of I.6.2:3) are classified (for $\mathcal{A}$) as follows:

	recognition principle	normal form	name	$\mathcal{A}^6$ codim.
i	$c \neq \frac{1}{2}, 1, \frac{3}{2}$	$(x, xy + y^3, xy^2 + cy^4)$		6
ii	$c = \frac{1}{2}, e \neq 0$	$(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$	$(\frac{1}{2},1)$	6
iii	$c = \frac{3}{2}, e \neq 0$	$(x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^5)$	$(\frac{1}{2},1)$	6
iv	$c = 1, g-b-d \neq e^2$	$(x, xy + y^3, xy^2 + y^4 \pm y^6)$	$(1,+1)$	6
v	$c=\frac{1}{2}, e=0, b+2d \neq 3g$	$(x, xy + y^3, xy^2 + \frac{1}{2}y^4 \pm y^6)$	$(\frac{1}{2},+1)$	7
vi	$c=\frac{3}{2}, e=0, 5b+6d \neq 3g$	$(x, xy + y^3, xy^2 + \frac{3}{2}y^4 \pm y^6)$	$(\frac{1}{2},+1)$	7
vii	$c=\frac{1}{2}, e=0, b+2d= 3g$	$(x, xy + y^3, xy^2 + \frac{1}{2}y^4)$	$(\frac{1}{2},0)$	8
viii	$c=\frac{3}{2}, e=0, 5b+6d= 3g$	$(x, xy + y^3, xy^2 + \frac{3}{2}y^4)$	$(\frac{1}{2},0)$	8
ix	$c=1, gg-b-d = e^2$	$(x, xy + y^3, xy^2 + y^4)$	$(1,0)$	7

Proof (i) If $c \neq \frac{1}{2}, 1$ or $\frac{3}{2}$, let $S_c = \{$6-jets of the form (1), for fixed $c\}$. Then apply Theorem I.3:1 (a) with $S = S_c$ and $G = \mathcal{A}^6$.

(ii) If $c = \frac{1}{2}$, let $S(\frac{1}{2},1) = \{$6-jets of the form (1), with $c=\frac{1}{2}$, $e \neq 0\}$

and apply Theorem I.3:1(a) to conclude that each component of $S(\frac{1}{2},1)$ is contained in a single $\mathcal{A}^6$ orbit. Normal forms $\sigma(\frac{1}{2}, \pm 1)$ may then be chosen, one for each component of $S(\frac{1}{2},1)$, but are clearly equivalent.

iii) As in (ii).

v) In the case where $c = \frac{1}{2}$ and $e = 0$, then by modifiying the calculations made in the proof of I.6.2:3 and applying I.3:1(a) with $S = \{\sigma \text{ of form (1) with } c = \frac{1}{2},\ e = 0,\ b + 2d \neq 3g\}$, one shows that each component of S lies in a single $\mathcal{A}^6$ orbit. The condition $b + 2d \neq 3g$ is necessary and sufficient (in this case) for matrix (4) to drop rank by exactly one. Normal forms $\rho(\frac{1}{2},\pm 1)$ may be chosen, one for each component of S.

vii) This is proved by a similar method.

vi), viii). As in (v), (vii).

iv), ix). If $c = 1$, then, as can be seen by inspection, the condition that matrix (5) drops rank by exactly 1 is $g - b - d \neq e^2$. It is clear that $T\mathcal{A}^5\sigma = C \pmod{\mathcal{M}_2^6}$, so by I.3:1 there is only one orbit at 5-jet level. We may thus take $b = d = e = 0$. The proof of (iv) then proceeds as in (v), and (ix) follows by similar methods∎

I.6.2:5 <u>Theorem</u> Each of the map germs

(i) $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + cy^4)$ $\qquad c \neq 0, \frac{1}{2}, 1, \frac{1}{2}$

ii) $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + cy^4 + y^5)$ $\qquad c = \frac{1}{2}, \frac{3}{2}$

iii) $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + y^4 \pm y^6)$

iv) $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + y^7)$

has $\mathcal{A}$ tangent space equal to

$$C = \begin{bmatrix} \mathcal{M}_2 - \{y,y^2\} \\ \mathcal{M}_2 - \{y,xy,y^2,y^3\} \\ \mathcal{M}_2 - \{y,xy,y^2,xy^2,y^3,y^4\} \end{bmatrix} + \mathbb{R}\left\{ \begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix} \begin{bmatrix} 0 \\ xy+y^3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy+y^3 \end{bmatrix} \begin{bmatrix} 0 \\ xy \\ xy^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^2+cy^4 \end{bmatrix} \right\}$$

and $\mathcal{A}$-codimension 6. Their degrees of determinacy are, respectively, 4, 5, 6 and 7.

Note In case (iv), since c = 0, the $\mathcal{A}$-tangent space has a simpler presentation.

Proof In each case,

$$T\mathcal{A}f + \mathcal{M}_2^7\Theta(f) = C$$

by I.6.2:3. By Theorem I.3.:2,

$$T\mathcal{A}f = C \quad \text{if and only if} \quad T\mathcal{A}f + f^*\mathcal{M}_3.C + \mathcal{M}_2^8\Theta(f) = C,$$

since it is clear that $f^*\mathcal{M}_3\Theta(f) \supseteq \mathcal{M}_2^3\Theta(f)$ and $C \supseteq \mathcal{M}_2^5\Theta(f)$. Hence, in order to prove that $T\mathcal{A}f = C$ it is enough to prove that

$$T\mathcal{A}f + f^*\mathcal{M}_3.C + \mathcal{M}_2^8\Theta(f) \supseteq \mathcal{M}_2^7\Theta(f).$$

It is straightforward to show that this holds, in each of the cases. Note that the case c = 0, which was not exceptional previously, becomes so at this level. This may be seen from the fact that the germ

$$(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2),$$

is not finitely determined, since, considered as a map-germ $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, it has a complex line of points of triple self-intersection in its image, namely $\{0\} \times \mathbb{C} \times \{0\}$ (See [30], Proposition 1.7 and Theorem 2.1).

For determinacy estimates, note first that all of the germs listed are

8-determined, by Theorem I.3:3. However, the fact that $T\mathcal{A}f = C$, depends only on $j^4f(0)$ in case (i), on $j^5f(0)$ in case (ii), on $j^6f(0)$ in case (iii) and on $j^7f(0)$ in case (iv), so that a straightforward application of I.3:1(a) shows that these germs are respectively, 4, 5, 6 and 7-detremined. That these estimates are exact follows from I.6.2:4 in cases (ii) and (iii) and from the fact, already noted, that the map-germ $(x,y) \longrightarrow (x, xy + y^3, xy^2)$ is not finitely determined, in cases (i) and (iv) ∎

I.6.2:6 Remark We have already noted that one can verify that the value $c = 0$, in the unimodal family $(x,y) \longrightarrow (x, xy + y^3, xy^2 + cy^4)$, is indeed exceptional, by applying Proposition 1.8 and Theorem 2.1 of [30]. One can check similarly that for each of the other exceptional values $c = \frac{1}{2}, 1, \frac{3}{2}$, the corresponding germ is not finitely determined, by calculating that in the first case there is a line of tangential self-intersection (the image of points $(-y^2,y)$ and $(-y^2,-y)$) in $\mathbb{R}^3$, in the second case there is a line of triple self intersection (the image of points $(-y^2,0)$, $(-y^2,y)$ and $(-y^2,-y)$) in $\mathbb{R}^3$, and in the third case the image of f is the "swallowtail surface" which has a cuspidal edge. In each of these three cases the set of unstable points in the image is 1-dimensional, and so by the results cited above, the three germs are not finitely $\mathcal{A}$-determined.

To finish this subsection, we prove the existence of another infinite family of germs:

I.6.2:7 Theorem Let $j^6f(0)$ be equivalent to $(x, xy + y^3, xy^2)$. Then either for some l, f is equivalent to $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + y^{3l+1})$, or for all k, $j^k f(0)$ is equivalent to $(x,\ xy + y^3,\ xy^2)$ and f has infinite $\mathcal{A}$-codimension.

b) The germ $f(x,y) = (x,\ xy + y^3,\ xy^2 + y^{3l+1})$ $(l \geq 2)$ has $\mathcal{A}$ tangent space

$$C = \begin{bmatrix} \mathcal{M}_2 - \{y, y^2, y^5, y^8, \ldots, y^{3l-4}\} \\ \mathcal{M}_2 - \{y, y^2\} \\ \mathcal{M}_2 - \{y, xy, y^2, y^3, y^4, y^7, y^{10}, \ldots, y^{3l-2}\} \end{bmatrix} + \mathbb{R}\left\{ \begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \\ y^4 \end{bmatrix} \begin{bmatrix} y^5 \\ 0 \\ y^7 \end{bmatrix} \cdots \begin{bmatrix} y^{3l-4} \\ 0 \\ y^{3l-2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy+y^3 \end{bmatrix} \right\}$$

has $\mathcal{A}$-codimension $l+4$ and is $3l+1$ determined.

<u>Proof</u> a) By induction on k: we suppose $j^{k-1}f(0) = (x, xy+y^3, xy^2)$ and consider seperately each of the three cases $k=3l-1$, $k=3l$ and $k=3l+1$.

i) If $j^{3l-1}f(0) = (x,\ xy + y^3 + b(x,y),\ xy^2 + c(x,y))$ where $b,c \in \mathcal{M}_2^{3l-1}$, then $T\mathcal{A}^{3l-1}f = C \pmod{\mathcal{M}_2^{3l}\Theta(f)}$. The calculations involved in showing this are very similar to those involved in the proof of (b), and so we omit them here. In particular, $T\mathcal{A}^{3l-1}f \supseteq \mathcal{M}_2^{3l-1}\Theta(f)$. Applying Theorem I.3:1(b), taking as x_0 the $3l-2$-jet $(x,\ xy + y^3,\ xy^2)$, as π the projection $J^{3l-1}(2,3) \longrightarrow J^{3l-2}(2,3)$, and as G the group $\mathcal{A}^{3l-1}$, we conclude that all such jets lie in a single $\mathcal{A}^{3l-1}$ orbit.

ii) If $j^{3l}f(0) = (x,\ xy + y^3 + b(x,y),\ xy^2 + c(x,y))$, with $b,c \in \mathcal{M}_2^{3l}$, then $T\mathcal{A}^{3l}f$ is again equal to $C \pmod{\mathcal{M}_2^{3l+1}\Theta(f)}$, and the proof proceeds as in (i).

iii) If $j^{3l+1}f(0) = (x,\ xy + y^3 + b(x,y),\ xy^2 + c(x,y))$, with $b,c \in \mathcal{M}_2^{3l+1}$, then there are two cases to consider: first, when $c_{0,3l+1} = 0$, and second when $c_{0,3l+1} \neq 0$. In the first case,

$$T\mathcal{A}^{3l+1}f = \begin{bmatrix} \mathcal{M}_2 - \{y, y^2, y^5, \ldots, y^{3l-1}\} \\ \mathcal{M}_2 - \{y, y^2\} \\ \mathcal{M}_2 - \{y, y^2, xy, y^3, y^4, \ldots, y^{3l+1}\} \end{bmatrix} + \mathbb{R}\left\{\begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \\ y^4 \end{bmatrix} \cdots \begin{bmatrix} y^{3l-1} \\ 0 \\ y^{3l+1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy + y^3 + b' y^{3l+1} \end{bmatrix}\right\},$$

where $b' = b_{0,3l+1}$, and applying Theorem I.3:1(a) with $G = \mathcal{A}^{3l+1}$ and $S = \{3l+1\text{-jets } (x,\ xy+y^3+b(x,y), xy^2+c(x,y)) \text{ with } b,c \in \mathcal{M}_2^{3l+1},\ c_{0,3l+1} = 0\}$ we conclude that S is contained in a single $\mathcal{A}^{3l+1}$ orbit, that of $(x,\ xy + y^3,\ xy^2)$.

We consider the second case in (b).

b) Suppose that $j^{3l+1}f(0) = (x,\ xy + y^3 + b(x,y),\ xy^2 + c(x,y))$ with $b,c \in \mathcal{M}_2^{3l+1}$ and $c_{0,3l+1} \neq 0$. Then $T\mathcal{A}f = C$. To show this, it is enough, by Theorem I.3:2, to show that

$$T\mathcal{A}f + f^*\mathcal{M}_3.C + \mathcal{M}_2^{3l+2}\Theta(f) = C,$$

for $f^*\mathcal{M}_3\Theta(f) \supseteq \mathcal{M}_2^3\Theta(f)$ and $C \supseteq \mathcal{M}_2^{3l-1}\Theta(f)$.

Working first in the top line, we find that $f^*\mathcal{M}_3.C$ contains all monomial generators of C (i.e. those listed in the statement of the theorem) divisible by x except for

$$\begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} xy^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} xy^5 \\ 0 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} xy^{3l-4} \\ 0 \\ 0 \end{bmatrix}.$$

Then we have (working mod $\mathcal{M}_2^{3l+2}$)

$$\begin{bmatrix} y^{3i} \\ 0 \\ 0 \end{bmatrix} = f^*Y \begin{bmatrix} y^{3i-3} \\ 0 \\ 0 \end{bmatrix} - f^*X \begin{bmatrix} y^{3i-2} \\ 0 \\ 0 \end{bmatrix} \in f^*\mathcal{M}_3.C \qquad (i \geq 2)$$

$$\begin{bmatrix} xy^{3i+2} \\ 0 \\ 0 \end{bmatrix} = f^*Z \begin{bmatrix} y^{3i} \\ 0 \\ 0 \end{bmatrix} \in f^*\mathcal{M}_3.C \qquad (i \geq 1)$$

$$\begin{bmatrix} y^{3i+1} \\ 0 \\ 0 \end{bmatrix} = f^*Y \begin{bmatrix} y^{3i-2} \\ 0 \\ 0 \end{bmatrix} - f^*Z \begin{bmatrix} y^{3i-3} \\ 0 \\ 0 \end{bmatrix} \in f^*\mathcal{M}_3.C \qquad (i \geq 2) \quad \text{and}$$

$$\begin{bmatrix} xy^2 \\ 0 \\ 0 \end{bmatrix} = \omega f \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix} \bmod \begin{bmatrix} \mathcal{M}_2^{3l+1} \\ 0 \\ 0 \end{bmatrix} \quad \text{but since} \begin{bmatrix} \mathcal{M}_2^{3l+1} \\ 0 \\ 0 \end{bmatrix} \subseteq T\mathcal{A}f + \mathcal{M}_2^{3l+2}\Theta(f),$$

$$\begin{vmatrix} xy^2 \\ 0 \\ 0 \end{vmatrix} \in T\mathcal{A}f + \mathcal{M}_2^{3l+2}\Theta(f).$$

Thus, of the monomial generators of C in the top line, we only need

$$\begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y^{3l-1} \\ 0 \\ 0 \end{bmatrix}.$$

Similar operations provide us with all of the monomial generators of C in the second and third lines, except for

$$\begin{bmatrix} 0 \\ xy \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x^2y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^{3l+1} \end{bmatrix}$$

Then, working successively,

$$tf\begin{bmatrix} xy^2 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ 0 \\ xy^4 \end{bmatrix}$$

$$tf\begin{bmatrix} 0 \\ x \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ 0 \\ x^2y \end{bmatrix}$$

$$\omega f\begin{bmatrix} 0 \\ 0 \\ xy \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ 0 \\ xy^3 \end{bmatrix}$$

$$tf\begin{bmatrix} xy \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$$

$$\omega f\begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} y^3 \\ 0 \\ 0 \end{bmatrix}$$

$$tf\begin{bmatrix} 0 \\ y^2 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ y^4 \\ 0 \end{bmatrix}$$

$$tf\begin{bmatrix} y^3 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ 0 \\ y^5 \end{bmatrix}$$

$$\omega f\begin{bmatrix} 0 \\ 0 \\ y^2 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 0 \\ 0 \\ y^6 \end{bmatrix}$$

$$tf\begin{bmatrix}0\\y^3\end{bmatrix} \quad \text{gives} \quad \begin{bmatrix}0\\y^5\\0\end{bmatrix} \quad \text{and}$$

$$tf\begin{bmatrix}y^4\\0\end{bmatrix} \quad \text{gives} \quad \begin{bmatrix}y^4\\0\\0\end{bmatrix}.$$

Then $tf\begin{bmatrix}x\\0\end{bmatrix}$, $tf\begin{bmatrix}0\\y\end{bmatrix}$, $\omega f\begin{bmatrix}0\\Y\\0\end{bmatrix}$ and $\omega f\begin{bmatrix}0\\0\\Z\end{bmatrix}$ together give

$$\begin{bmatrix}0\\xy\\0\end{bmatrix}\begin{bmatrix}0\\y^3\\0\end{bmatrix}\begin{bmatrix}0\\0\\xy^2\end{bmatrix}\begin{bmatrix}0\\0\\y^{3l+1}\end{bmatrix} \quad \text{and} \quad tf\begin{bmatrix}y^{3l-1}\\0\end{bmatrix} \quad \text{gives} \quad \begin{bmatrix}y^{3l-1}\\0\\0\end{bmatrix}.$$

Finally, $\begin{bmatrix}y\\y^2\\y^3\end{bmatrix} = tf\begin{bmatrix}y\\0\end{bmatrix}$ and $\begin{bmatrix}y^{3i-1}\\0\\y^{3i+1}\end{bmatrix} = tf\begin{bmatrix}y^{3i-1}\\0\end{bmatrix} - \begin{bmatrix}0\\y^{3i}\\0\end{bmatrix}$.

This completes the proof that $T\mathcal{A}f = C$. It then follows that all such f are $3l+1$ determined, for by I.3:2 they are all $3l+2$ determined, while by I.3:1 their $3l+2$-jets are all equivalent. In fact by I.3:1(a) their $3l+1$-jets all lie in at most two $\mathcal{A}^{3l+1}$ orbits, those of

$$(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 \pm y^{3l+1}),$$

and these two are clearly equivalent under a change of sign in the y and Y coordinates. Note that $3l+1$ is the exact degree of determinacy, since $(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2)$ is not finitely determined∎

I.6.3 Classification of Germs whose 3-jet is equivalent to (x, xy, xy^2)

I.6.3:1 Proposition The 4-jets whose 3-jet is equivalent to (x, xy, xy^2) lie in one of the following $\mathcal{A}^4$ orbits:

		codim.
i)	$(x, xy, xy^2 + y^4)$	6
ii)	$(x, xy + y^4, xy^2)$	7
iii)	(x, xy, xy^2)	8

Proof Any such 4-jet can be transformed by obvious coordinate changes to a 4-jet of the form

$$\sigma = (x, xy + b_{0,4}y^4, xy^2 + c_{1,3}xy^3 + c_{0,4}y^4).$$

If $c_{0,4} \neq 0$ then

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathfrak{m}_2 - \{y, y^2, y^3\} \\ \mathfrak{m}_2 - \{y, y^2, y^3\} \\ \mathfrak{m}_2 - \{y, y^2, y^3\} \end{bmatrix} + \mathbb{R}\begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix}\begin{bmatrix} y^2 \\ y^3 \\ 0 \end{bmatrix}$$

and there is only one orbit (by I.3:2(a)), that of (i).

If $c_{0,4} = 0$, $b_{0,4} \neq 0$,

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathfrak{m}_2 - \{y, y^2\} \\ \mathfrak{m}_2 - \{y, y^2, y^3\} \\ \mathfrak{m}_2 - \{y, y^2, xy, y^3, y^4\} \end{bmatrix} + \mathbb{R}\begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix}\begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ xy + b_{0,4}y^4 \end{bmatrix}$$

and there is only one orbit, that of (ii).

If $c_{0,4} = b_{0,4} = 0$

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathfrak{m}_2 - \{y, y^2, y^3\} \\ \mathfrak{m}_2 - \{y, y^2, y^3, y^4\} \\ \mathfrak{m}_2 - \{y, y^2, xy, y^3, y^4\} \end{bmatrix} + \mathbb{R}\begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix}\begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix}\begin{bmatrix} y^3 \\ y^4 \\ 0 \end{bmatrix}$$

and again there is only one orbit, that of (iii)∎

I.6.3:2 <u>Proposition</u> The 5-jets whose 4-jet is equivalent to (x,xy,xy^2+y^4) all lie in a single $\mathcal{A}^5$ orbit.

<u>Proof</u> Straightforward calculations show that for any such 5-jet σ, $T\mathcal{A}^5\sigma \supseteq \bigoplus_1^3 \mathcal{M}_2^5$. Apply Theorem I.3:2(b)∎

I.6.3:3 <u>Proposition</u> The 6-jets whose 5-jet is equivalent to (x,xy,xy^2+y^4) lie in the orbits of

		codim.
i)	$(x,\ xy + y^6,\ xy^2 + y^4 + cy^6)$	7
ii)	$(x,\ xy,\ xy^2 + y^4 + y^6)$	7
iii)	$(x,\ xy,\ xy^2 + y^4)$	8

<u>Note</u> In the first case, the codimension of the stratum is 6.

<u>Proof</u> Calculations show that for the 6-jet $\sigma = (x,xy+b(x,y),xy^2+y^4+c(x,y))$,

$$T\mathcal{A}^6\sigma = \begin{bmatrix} \mathcal{M}_2 - \{y,y^2,y^3,xy^2,y^4,y^5\} \\ \mathcal{M}_2 - \{y,xy,y^2,y^3,y^4,y^5,y^6,xy^3\} \\ \mathcal{M}_2 - \{y,xy,y^2,y^3,y^4,y^6,xy^2\} \end{bmatrix} + \mathbb{R}\left\{ \begin{bmatrix} 0 \\ xy+b_{0,6}y^6 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy+b_{0,6}y^6 \end{bmatrix}, \right.$$

$$\begin{bmatrix} xy^2+y^4+c_{0,6}y^6 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ y^4+c_{0,6}y^6 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^2+y^4+c_{0,6}y^6 \end{bmatrix} \begin{bmatrix} 0 \\ xy \\ xy^2 \end{bmatrix} \begin{bmatrix} y \\ y^2+b_{1,5}y^6 \\ y^3+c_{1,5}y^6 \end{bmatrix} \begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix} \begin{bmatrix} y^3 \\ y^4 \\ y^5 \end{bmatrix}$$

$$\left. \begin{bmatrix} y^4 \\ y^5 \\ y^6 \end{bmatrix} \begin{bmatrix} y^5 \\ y^6 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy+6b_{0,6}y^6 \\ 2xy^2+4y^4+6c_{0,6}y^6 \end{bmatrix} \begin{bmatrix} xy^2 \\ xy^3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy^3 \\ 4y^6 \end{bmatrix} \right\} = C.$$

The supplementary generators in this expression are, respectively,

$$\omega f\begin{bmatrix}0\\Y\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\0\\Y\end{bmatrix},\ \omega f\begin{bmatrix}Z\\0\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\Z\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\0\\Z\end{bmatrix},\ tf\begin{bmatrix}x\\0\end{bmatrix} - \omega f\begin{bmatrix}X\\0\\0\end{bmatrix},$$

$$tf\begin{bmatrix}y\\0\end{bmatrix},\ \ldots,\ tf\begin{bmatrix}y^5\\0\end{bmatrix},\ tf\begin{bmatrix}0\\y\end{bmatrix},\ tf\begin{bmatrix}xy^2\\0\end{bmatrix},\ \text{and } tf\begin{bmatrix}0\\y^3\end{bmatrix},$$ modulo the

principal part of C. They are not necessarily independent: in fact

$$tf\begin{bmatrix}2x\\y\end{bmatrix} - f\begin{bmatrix}2X\\3Y\\4Z\end{bmatrix} = \begin{bmatrix}0\\3b_{0,6}y^6\\2c_{0,6}y^6\end{bmatrix}$$ (modulo the principal part of C)

and if $b_{0,6} = c_{0,6} = 0$, the codimension increases by 1. Since

$C \supseteq \begin{bmatrix}\mathfrak{m}_2^6\\ \mathfrak{m}_2^6 - \{y^6\}\\ \mathfrak{m}_2^6 - \{y^6\}\end{bmatrix}$ whatever the values of $b_{0,6}$ and $c_{0,6}$, by Theorem I.3:1

we conclude that σ is equivalent to $(x,\ xy + b_{0,6}y^6,\ xy^2 + y^4 + c_{0,6}y^6)$. These 6-jets form a linear subspace S of $J^6(2,3)$ which is foliated by the integral curves of the vector field χ equal to

$$\begin{bmatrix}0\\3b_{0,6}y^6\\2c_{0,6}y^6\end{bmatrix}$$

each of which is contained in a single $\mathcal{A}^6$-orbit, the orbits being locally distinct, in the sense that the values of a first integral of χ (defined locally on $\acute{S}$) are $\mathcal{A}^6$ invariants. It is clear that each integral curve of $\mathcal{X}$ contains one of the 6-jets in the statement of the proposition, the last one corresponding to the degenerate, 1-point, curve ∎

I.6.3:4 Remark We take orbits (ii) and (iii) as strata in the stratification of Theorem I:2. Although in fact the orbits of (i) and (ii) together make up a smooth unimodal stratum of $J^6(2,3)$, the treatment of 7-jets over (ii) is different from that of 7-jets over (i), and we prefer to omit it.

I.6.3:5 Theorem a) The germs

$$(x,y) \longrightarrow (x,\ xy + y^6 + b_{0,7}y^7,\ xy^2 + y^4 + c_{0,6}y^6)$$

form a bimodal family (with respect to $\mathcal{A}$) each member of which has $\mathcal{A}$-codimension 8 and is 7-determined. The $\mathcal{A}$ tangent space is

$$C = \begin{bmatrix} \mathfrak{m}_2 - \{y, xy, y^2, y^3, xy^2, y^4, y^5, y^6\} \\ \mathfrak{m}_2 - \{y, y^2, xy, xy^2, y^3, xy^3, y^4, y^5, y^6, y^7\} \\ \mathfrak{m}_2 - \{y, y^2, xy, xy^2, y^3, y^4, y^5, y^6\} \end{bmatrix} + \mathbb{R} \begin{bmatrix} xy+y^6 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy+y^6+b_{0,7}y^7 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ xy+y^6 \end{bmatrix} \begin{bmatrix} xy^2+y^4+c_{0,6}y^6 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy^2+y^4+c_{0,6}y^6 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy+y^4+c_{0,6}y^6 \end{bmatrix} \begin{bmatrix} 0 \\ xy \\ xy^2 \end{bmatrix} \begin{bmatrix} xy \\ xy^2 \\ 0 \end{bmatrix} \begin{bmatrix} xy^2 \\ xy^3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ y^3 \\ y^4 \end{bmatrix} \begin{bmatrix} y^3 \\ y^4 \\ y^5 \end{bmatrix} \begin{bmatrix} y^4 \\ y^5 \\ y^6 \end{bmatrix} \begin{bmatrix} y^5 \\ y^6 \\ 0 \end{bmatrix} \begin{bmatrix} y^6 \\ y^7 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ xy+6y^6+7b_{0,7}y^7 \\ 2xy^2+4y^4+6c_{0,6}y^6 \end{bmatrix} \begin{bmatrix} 0 \\ xy^2+6y^7 \\ 2xy^3+4y^5 \end{bmatrix} \begin{bmatrix} 0 \\ xy^3 \\ 4y^6 \end{bmatrix}$$

b) Every germ whose 6-jet is equivalent to $(x,\ xy + y^6,\ xy^2 + y^4 + c_{0,6}y^6)$

belongs to one of the orbits of (a).

Proof a) Since $C \supseteq \mathfrak{M}_2^8\Theta(f)$ and $f^*\mathfrak{M}_3\Theta(f) \supseteq \mathfrak{M}_2^4\Theta(f)$ for any such germ f, we need only prove

$$C = T\mathcal{A}f + f^*\mathfrak{M}_3.C + \mathfrak{M}_2^{12}\Theta(f).$$

Straightforward checking shows that the inclusion from left to right holds; note that the supplementary generators of C are, respectively,

$$\omega f\begin{bmatrix}Y\\0\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\Y\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\0\\Y\end{bmatrix},\ \omega f\begin{bmatrix}Z\\0\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\Z\\0\end{bmatrix},\ \omega f\begin{bmatrix}0\\0\\Z\end{bmatrix},\ tf\begin{bmatrix}x\\0\end{bmatrix} - \omega f\begin{bmatrix}X\\0\\0\end{bmatrix}$$

$$tf\begin{bmatrix}xy\\0\end{bmatrix},\ tf\begin{bmatrix}xy^2\\0\end{bmatrix},\ tf\begin{bmatrix}y\\0\end{bmatrix},\ldots,tf\begin{bmatrix}y^6\\0\end{bmatrix},\ tf\begin{bmatrix}0\\y\end{bmatrix},\ tf\begin{bmatrix}0\\y^2\end{bmatrix},\ tf\begin{bmatrix}0\\y^3\end{bmatrix},$$

modulo the principal part of C.

To show the opposite inclusion, note first that using $f^*\mathfrak{M}_3.C$ we obtain all of the monomial generators of C except for

$$\begin{bmatrix}0\\y^{11}\\0\end{bmatrix}\begin{bmatrix}y^{10}\\0\\0\end{bmatrix}\begin{bmatrix}0\\y^{10}\\0\end{bmatrix}\begin{bmatrix}0\\0\\y^{10}\end{bmatrix}\begin{bmatrix}y^9\\0\\0\end{bmatrix}\begin{bmatrix}0\\y^9\\0\end{bmatrix}\begin{bmatrix}0\\0\\y^9\end{bmatrix}\begin{bmatrix}y^8\\0\\0\end{bmatrix}\begin{bmatrix}0\\y^8\\0\end{bmatrix}\begin{bmatrix}0\\0\\y^8\end{bmatrix}\begin{bmatrix}y^7\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\y^7\end{bmatrix}$$

$$\begin{bmatrix}0\\xy^7\\0\end{bmatrix}\begin{bmatrix}xy^6\\0\\0\end{bmatrix}\begin{bmatrix}0\\xy^6\\0\end{bmatrix}\begin{bmatrix}0\\0\\xy^6\end{bmatrix}\begin{bmatrix}xy^5\\0\\0\end{bmatrix}\begin{bmatrix}0\\xy^5\\0\end{bmatrix}\begin{bmatrix}0\\0\\xy^5\end{bmatrix}\begin{bmatrix}xy^4\\0\\0\end{bmatrix}\begin{bmatrix}0\\xy^4\\0\end{bmatrix}\begin{bmatrix}0\\0\\xy^4\end{bmatrix}$$

$$\begin{bmatrix}xy^3\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\xy^3\end{bmatrix}\begin{bmatrix}x^2y^2\\0\\0\end{bmatrix}\begin{bmatrix}0\\x^2y^3\\0\end{bmatrix}\begin{bmatrix}0\\x^2y^2\\0\end{bmatrix}\begin{bmatrix}0\\0\\x^2y^2\end{bmatrix}\begin{bmatrix}x^2y\\0\\0\end{bmatrix}\begin{bmatrix}0\\x^2y\\0\end{bmatrix}\begin{bmatrix}0\\0\\x^2y\end{bmatrix}\begin{bmatrix}x\\0\\0\end{bmatrix}\begin{bmatrix}0\\x\\0\end{bmatrix}\begin{bmatrix}0\\0\\x\end{bmatrix}$$

The last three of these are obviously in $T\mathcal{A}f$. We now work inductively, i.e. at each stage calculations are made modulo the generators of C already obtained.

From $\omega f \begin{bmatrix} Y^2 \\ 0 \\ 0 \end{bmatrix}$ and $\omega f \begin{bmatrix} 0 \\ 0 \\ Y^2 \end{bmatrix}$ we obtain $\begin{bmatrix} x^2y^2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ x^2y^2 \end{bmatrix}$;

$tf \begin{bmatrix} x^2y^2 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} 00 \\ x^2y^3 \\ 0 \end{bmatrix}$;

taking together $tf \begin{bmatrix} x^2y \\ 0 \end{bmatrix}$, $tf \begin{bmatrix} xy^6 \\ 0 \end{bmatrix}$, $\omega f \begin{bmatrix} XY \\ 0 \\ 0 \end{bmatrix}$ and $\omega f \begin{bmatrix} 0 \\ Y^2 \\ 0 \end{bmatrix}$ we obtain

$\begin{bmatrix} x^2y \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x^2y^2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} xy^6 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ xy^7 \\ 0 \end{bmatrix}$;

$\omega f \begin{bmatrix} XZ \\ 0 \\ 0 \end{bmatrix}$ then gives $\begin{bmatrix} xy^4 \\ 0 \\ 0 \end{bmatrix}$;

$tf \begin{bmatrix} x^2 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} 0 \\ x^2y \\ 0 \end{bmatrix}$, then from $\omega f \begin{bmatrix} 0 \\ XY \\ 0 \end{bmatrix}$ we get $\begin{bmatrix} 0 \\ xy^6 \\ 0 \end{bmatrix}$, from $tf \begin{bmatrix} xy^5 \\ 0 \end{bmatrix}$

we get $\begin{bmatrix} xy^5 \\ 0 \\ 0 \end{bmatrix}$, from $\omega f \begin{bmatrix} YZ \\ 0 \\ 0 \end{bmatrix}$ we get $\begin{bmatrix} y^{10} \\ 0 \\ 0 \end{bmatrix}$, from $tf \begin{bmatrix} y^{10} \\ 0 \end{bmatrix}$ we get $\begin{bmatrix} 0 \\ y^{11} \\ 0 \end{bmatrix}$, and from $\omega f \begin{bmatrix} Z^2 \\ 0 \\ 0 \end{bmatrix}$ comes $\begin{bmatrix} y^8 \\ 0 \\ 0 \end{bmatrix}$;

from tf $\begin{bmatrix} 0 \\ 0 \\ y^7 \end{bmatrix}$ comes $\begin{bmatrix} 0 \\ 0 \\ y^{10} \end{bmatrix}$, and tf $\begin{bmatrix} y^8 \\ 0 \end{bmatrix}$ then gives $\begin{bmatrix} 0 \\ y^9 \\ 0 \end{bmatrix}$;

from tf $\begin{bmatrix} 0 \\ y^6 \end{bmatrix}$ we get $\begin{bmatrix} 0 \\ 0 \\ y^9 \end{bmatrix}$;

$f \begin{bmatrix} 0 \\ XZ \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} 0 \\ xy^4 \\ 0 \end{bmatrix}$;

taking together tf $\begin{bmatrix} xy^4 \\ 0 \end{bmatrix}$, $\omega f \begin{bmatrix} 0 \\ 0 \\ Z^2 \end{bmatrix}$, $\omega f \begin{bmatrix} 0 \\ YZ \\ 0 \end{bmatrix}$ and tf $\begin{bmatrix} 0 \\ y^5 \end{bmatrix}$ we get

$\begin{bmatrix} 0 \\ xy^5 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ xy^6 \end{bmatrix} \begin{bmatrix} 0 \\ y^{10} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ y^8 \end{bmatrix}$;

From $\omega f \begin{bmatrix} 0 \\ 0 \\ XY \end{bmatrix}$ we get $\begin{bmatrix} 0 \\ 0 \\ x^2y \end{bmatrix}$;

From tf $\begin{bmatrix} y^9 \\ 0 \end{bmatrix}$ comes $\begin{bmatrix} y^9 \\ 0 \\ 0 \end{bmatrix}$

from $\omega f \begin{pmatrix} 0 \\ Z^2 \\ 0 \end{pmatrix}$ we get $\begin{pmatrix} 0 \\ y^8 \\ 0 \end{pmatrix}$ and then from tf $\begin{pmatrix} y^7 \\ 0 \end{pmatrix}$ comes $\begin{pmatrix} y^7 \\ 0 \\ 0 \end{pmatrix}$;

from f $\begin{pmatrix} 0 \\ 0 \\ YZ \end{pmatrix}$ we get $\begin{pmatrix} 0 \\ 0 \\ xy^5 \end{pmatrix}$, and tf $\begin{pmatrix} 0 \\ x \end{pmatrix}$ now gives $\begin{pmatrix} 0 \\ 0 \\ x^2y^3 \end{pmatrix}$;

from tf $\begin{pmatrix} xy^3 \\ 0 \end{pmatrix}$ we now get $\begin{pmatrix} xy^3 \\ 0 \\ 0 \end{pmatrix}$;

from $\omega f \begin{pmatrix} 0 \\ 0 \\ XZ \end{pmatrix}$ we get $\begin{pmatrix} 0 \\ 0 \\ xy^4 \end{pmatrix}$,

and finally tf $\begin{pmatrix} 0 \\ y^4 \end{pmatrix}$ gives $\begin{pmatrix} 0 \\ 0 \\ y^7 \end{pmatrix}$.

This completes the proof that $T\mathcal{A}f = C$. Now note that these calculations also work for any germ whose 7-jet is the same as that of the statement of the theorem, and this, together with Theorem I.3:1(a), proves the 7-determinacy of all such germs. Inspection of the $\mathcal{A}$ tangent spaces shows that this is the exact determinacy degree.

b) In fact for the 7-jet

$$= (x + a(x,y),\ xy + y^6 + b(x,y),\ xy^2 + y^4 + c_{0,6}y^6 + c(x,y))$$

where $a,b,c \in \mathfrak{m}_2^7$, it is easy to see, by modifying the proof of (a) a and taking everything mod $\mathfrak{m}_2^8\Theta(f)$, that

$$T\mathcal{A}^7\sigma \supseteq \begin{bmatrix} \mathcal{M}_2^7 \\ \mathcal{M}_2^7 - \{y^7\} \\ \mathcal{M}_2^7 \end{bmatrix}$$

and that the dimensions of all these $\mathcal{A}^7$ tangent spaces are the same. It follows that σ is equivalent to the 7-jet

$$(x,\ xy + y^6 + b_{0,7}y^7,\ xy^2 + y^4 + c_{0,6}y^6).$$

This completes the proof of (b)■

I.6.4 Classification of germs whose 3-jet is equivalent to $(x,\ xy + y^3,\ 0)$

I.6.4:1 Proposition The 4-jets whose 3-jet is equivalent to $(x,xy+y^3,0)$ belong to the $\mathcal{A}^4$ orbits of the following 4-jets:

i)	$(x,\ xy + y^3,\ y^4)$	codim. 6
ii)	$(x,\ xy + y^3,\ xy^3)$	" 7
iii)	$(x,\ xy + y^3,\ 0)$	" 8

Proof Since the map germ $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$

$$(x,y) \longrightarrow (x,\ xy + y^3)$$

is 3-determined, any germ $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ whose 3-jet is $(x,xy+y^3,0)$ is equivalent to a germ of the form

$$(x,y) \longrightarrow (x,\ xy + y^3,\ c(x,y))$$

where $c \in \mathcal{M}_2^4$. After obvious left coordinate changes, we may suppose that its 4-jet is $(x,\ xy + y^3,\ c_{1,3}xy^3 + c_{0,4}y^4)$.

<u>If $c_{0,4} \neq 0$</u> then

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2 - \{y,\ y^2\} \\ \mathcal{M}_2 - \{y,\ xy,\ y^2,\ xy^2,\ y^3\} \end{bmatrix} + \mathbb{R}\left\{\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}\right\}$$

and so all such jets belong to the $\mathcal{A}^4$ orbit of (i), by I.3:1.

<u>If $c_{0,4} = 0 \neq c_{1,3}$</u>

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2 - \{y,\ y^2\} \\ \mathcal{M}_2 - \{y,\ y^2,\ xy,\ y^3,\ xy^2,\ y^4\} \end{bmatrix} + \mathbb{R}\left\{\begin{pmatrix} y \\ y^2 \\ c_{1,3}y^4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ xy\ +y^3 \end{pmatrix}\right\}$$

and all such jets belong to the orbit of (ii).

<u>If $c_{0,4} = c_{1,3} = 0$</u>

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2 - \{y,\ y^2\} \\ \mathcal{M}_2 - y,\ xy,\ y^2,\ xy^2,\ y^3,\ x^2y,\ xy^3,\ y^4\} \end{bmatrix} + \mathbb{R}\left\{\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x^2y+xy^3 \end{pmatrix}\right\} \blacksquare$$

I.6.4:2 <u>Theorem</u> The germ $(x,y) \longrightarrow (x,\ xy + y^3,\ y^4)$ has $\mathcal{A}$ tangent space

$$C = \begin{bmatrix} \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2 - \{y,\ y^2\} \\ \mathcal{M}_2 - \{y,\ xy,\ y^2,\ xy^2,\ y^3\} \end{bmatrix} + \mathbb{R}\left\{\begin{pmatrix} y \\ y^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ xy+y^3 \end{pmatrix}\right\}$$

and is 4-determined

<u>Proof</u> Since $C \supseteq \mathcal{M}_2^4\,\Theta(f)$ and $f^*\mathcal{M}_3\Theta(f) \supseteq \mathcal{M}_2^3\,\Theta(f)$, we need only prove

$$C = T\mathcal{A}f + f^{*}\mathcal{M}_{3}.C + \mathcal{M}_{2}^{7}\Theta(f).$$

It is straightforward to check that this holds for any germ whose 4-jet is equal to that of f. From this the 4-determinacy of f follows immediately∎

I.7 Classification of germs whose 2-jet is equivalent to (x, 0, 0)

I.7:1 Proposition The 3-jets whose 2-jet is equivalent to (x, 0, 0) belong to the $\mathcal{A}^3$ orbits of the following 3-jets:

	3-jet	$\mathcal{A}^3$ codimension
i)	$(x,\ y^3,\ x^2y + xy^2)$	6
ii)	$(x,\ y^3 - x^2y,\ xy^2)$	6
iii)	$(x,\ y^3,\ x^2y)$	7
iv)	$(x,\ y^3,\ xy^2)$	7
v)	$(x,\ y^3,\ 0)$	9
vi)	$(x,\ y^3 + x^2y,\ 0)$	8
vii)	$(x,\ xy^2,\ x^2y)$	8
viii)	$(x,\ xy^2,\ 0)$	9
ix)	$(x,\ x^2y,\ 0)$	10
x)	$(x,\ 0,\ 0)$	12

Proof Any 3-jet whose 2-jet is equivalent to (x, 0, 0) is obviously equivalent to $(x,\ b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3,\ c_{2,1}x^2y + c_{1,2}xy^2 + c_{0,3}y^3)$, for some value of the coefficients. If either $b_{0,3}$ or $c_{0,3}$ is non-zero, then this jet is left-equivalent to one of the form $(x,\ y^3 + b_{2,1}x^2y + b_{1,2}xy^2, + c_{2,1}x^2y + c_{1,2}xy^2)$. Now we can complete the cube in the the second component, by making the coordinate change

$$\overline{Y} = Y + \alpha X^3 + \beta Z$$

for appropriate values of α and β, if and only if

$$3(c_{2,1})^2 \geqslant 4c_{1,2}(b_{1,2}c_{2,1} - b_{2,1}c_{1,2}).$$

If this holds, the 3-jet is equivalent to one of the form $(x,\ y^3,\ c_{2,1}x^2y + c_{1,2}xy^2)$. By changes of scale in the coordinates, if necessary, we can reduce this to one of (i), (iii), (iv) or (v).

If $3(c_{2,1})^2 < 4c_{1,2}(b_{1,2}c_{2,1} - b_{2,1}c_{1,2})$ then in particular $c_{1,2} \neq 0$. The coordinate change

$$y = \overline{y} - \frac{c_{2,1}}{2c_{1,2}}\, x,$$

followed by certain obvious left coordinate changes, then transforms the 3-jet to

$$\left(x,\ y^3 + \frac{-4c_{1,2}(b_{1,2}c_{2,1} - b_{2,1}c_{1,2}) + 3(c_{2,1})^2}{4(c_{1,2})^2}\, x^2y,\ xy^2\right)$$

and this is clearly equivalent to $(x,\ y^3 - x^2y,\ xy^2)$.

If $b_{0,3} = c_{0,3} = 0$, then left coordinate changes, followed if necessary by a change of scale in the coordinates, will transform the 3-jet into one of the normal forms (vii) — (x).

The codimensions of the $\mathcal{A}^3$ orbits are easily found by calculating the $\mathcal{A}^3$ tangent spaces.

In order to distinguish between the orbits, note first that (vi) and (vii) are not even $\mathcal{K}^3$ equivalent; the same goes for (v) and (viii). For the equi-codimensional pairs (i), (ii) and (iii), (iv), see Note I.7:6. Other than these four pairs, all of the orbits listed are distinguished by their codimension∎

I.7:2 Proposition All 4-jets whose 3-jet is equivalent to (x, y^3, x^2y+xy^2) lie in the $\mathcal{A}^4$ orbits of

i) $(x, y^3, x^2y + xy^2 + y^4)$ codimension 6

ii) $(x, y^3, x^2y + xy^2)$ " 7

Proof After obvious coordinate changes, any such 4-jet may be assumed to be of the form $(x, y^3 + b_{2,2}x^2y^2 + b_{0,4}y^4, x^2y + xy^2 + c_{2,2}x^2y^2 + c_{0,4}y^4)$

If $c_{0,4} \neq 0$ then

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2 - \{y, y^2, xy\} \\ \mathfrak{m}_2 - \{y, y^2, xy\} \end{bmatrix}$$

and so all such 4-jets lie in the orbit of (i).

If $c_{0,4} = 0$ then

$$T\mathcal{A}^4\sigma = \begin{bmatrix} \mathfrak{m}_2 - \{y, y^2\} \\ \mathfrak{m}_2 - \{y, y^2, xy\} \\ \mathfrak{m}_2 - \{y, y^2, xy, y^3, y^4\} \end{bmatrix} + \mathbb{R}\begin{bmatrix} y \\ 0 \\ y^3 \end{bmatrix} \begin{bmatrix} y^2 \\ 0 \\ y^4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y^3 + b_{0,4}y^4 \end{bmatrix}$$

and all such 4-jets belong to the orbit of (ii)■

I.7:3 Theorem The germ $f:(x,y)\longrightarrow(x,\ y^3,\ x^2y + xy^2 + y^4)$ is 4-determined and has $\mathcal{A}$ tangent space

$$C = \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2 - \{y,\ xy,\ y^2\} \\ \mathcal{M}_2 - \{y,\ xy,\ y^2\} \end{bmatrix}$$

Proof Let g have the same 4-jet as f. Then we claim that $T\mathcal{A}g = C$. The theorem is a consequence of this claim. To prove it, note that for any such g, $g^*\mathcal{M}_3\Theta(g) \supseteq \mathcal{M}_2^3\Theta(g)$ and so by I.3:2,

$T\mathcal{A}g = C$ if and only if $C = T\mathcal{A}g + g^*\mathcal{M}_3\Theta(g) + \mathcal{M}_2^6\Theta(g)$.

It is straightforward to verify that the last equality holds■

I.7:4 Proposition All 4-jets whose 3-jet is equivalent to $(x,\ y^3 - x^2y,\ xy^2)$ lie in the $\mathcal{A}^4$ orbits of

i) $(x,\ y^3 - x^2y,\ xy^3 + y^4)$ codimension 6

ii) $(x,\ y^3 - x^2y,\ xy^3)$ " 7

Proof The proof is almost exactly the same as that of I.7:2. In particular, the two cases $c_{0,4} \neq 0$ and $c_{0,4} = 0$ give rise to the same $\mathcal{A}^4$ tangent spaces,as in I.7:2■

I.7:5 Theorem The germ $f(x,y) = (x,\ y^3 - x^2y,\ xy^2 + y^4)$ has the same tangent space as the germ of Theorem I.7:3, and is 4-determined.

Proof As in I.7:3■

I.7:6 Note Inequivalence of the two germs

$$f(x,y) = (x,\ y^3,\ x^2y + xy^2 + y^4)$$

and

$$g(x,y) = (x,\ y^3 - x^2y,\ xy^2 + y^4)$$

may be proved by calculating their singular algebras (see I.9 below), as the isomorphism class of these algebras is an $\mathcal{K}$-invariant. It is easy to calculate that the two algebras are, respectively,

$$\frac{\mathcal{E}_2}{\langle y^2,\ x^2+2xy\rangle} \simeq \frac{\mathcal{E}_2}{\langle y^2,\ x^2\rangle} \quad \text{and} \quad \frac{\mathcal{E}_2}{\langle 3y^2-x^2,\ 2xy+4y^3\rangle} \simeq \frac{\mathcal{E}_2}{\langle x^2-y^2,\ xy\rangle}$$

which are not isomorphic (as real algebras).

In fact, for any two map germs whose 3-jets are the same as those of f and g, respectively, the corresponding singular algebras are isomorphic to those of f and g, and hence are not isomorphic to one another. This serves to distinguish between the $\mathcal{A}^3$ orbits of (i) and (ii) in I.7:1.

Suppose now that $\bar{f}$ and $\bar{g}$ have 3-jets $(x,\ y^3,\ x^2y)$ and $(x,\ y^3,\ xy^2)$. Then one calculates that the singular algebra of $\bar{f}$ has real dimension 4, while that of $\bar{g}$ has dimension greater than 4. This proves that the two 3-jets (iii) and (iv) of I.7:1 are not equivalent.

I.8 Proof of Theorem I:2

Let $f\colon (\mathbb{R}^2,0)\longrightarrow(\mathbb{R}^3,0)$ be a smooth map-germ. If f is non-singular at $0\in\mathbb{R}^2$ then it is equivalent to $(x,y)\longrightarrow(x,y,0)$.

Let f have a Σ^1 singularity at $0\in\mathbb{R}^2$. Then by I.4:2, $j^2f(0)$ is equivalent to (x,y^2,xy), $(x,y^2,0)$, $(x,xy,0)$ or $(x,0,0)$. In the first case, f is 2-determined and stable, and hence simple. Now suppose $j^2f(0)\sim(x,y^2,0)$. Then any small perturbation of f will either be an immersion or will have 2-jet equivalent to $(x,y^2,0)$ or to (x,y^2,xy). This is because $\mathcal{A}^2(x,y^2,xy)\bigcup\mathcal{A}^2(x,y^2,0)$ is open in $J^2(2,3)\bigcap\{\Sigma^1\bigcup\Sigma^2\}$, as can be seen in the proof of I.4:2. By I.5:4, we may suppose that f is of the form $f(x,y) = (x,\ y^2,\ yp(x,y^2))$, and moreover, any small perturbation of f will be equivalent, if it is neither an immersion nor a cross-cap, to a germ of the form $(x,\ y^2,\ yq(x,y^2))$, where q is a small perturbation of p. This follows from the fact that any (germ of a) one parameter deformation of f is induced from the $\mathcal{A}$-versal deformation of f which may be calculated from the expression for $T\mathcal{A}f$ given in I.5:7. (We are supposing, of course, that f is finitely determined — otherwise it is certainly not simple.) Now by I.5:8 and I.5:11 it follows that f is simple if and only if the germ $p(x,y^2)$ is simple for $\mathcal{K}^T$, and from the list given in I.5:19, we obtain the following list of $\mathcal{A}$-simple map-germs $(\mathbb{R}^2,0)\longrightarrow(\mathbb{R}^3,0)$.

$f(x,y) = (x,\ y^2,\ y^3 + x^{k+1}y)$	$(k\geq1)$	S_k
$f(x,y) = (x,\ y^2,\ x^2y + y^{2k+1})$	$(k\geq2)$	B_k
$f(x,y) = (x,\ y^2,\ xy^3 + x^ky)$	$(k\geq3)$	C_k
$f(x,y) = (x,\ y^2,\ x^3y + y^5)$		F_4

We have given them the same names as the function germs p(x,y) receive in Arnol'd's list ([2]), except for S_k, since there the function p(x,y) is non-singular and does not figure in his list.

Next, consider map-germs whose 2-jet is equivalent to (x,xy,0). If $j^3f(0)$ is equivalent to (x, xy, y^3) then either f is equivalent to $(x, xy + y^{3k-1}, y^3)$ for some $k \geqslant 2$, in which case it is simple, or it is not finitely determined and therefore not simple (Theorem I.6.1:2). If $j^3f(0)$ is not equivalent to (x, xy, y^3) then we claim f is not simple. For in this case $j^3f(0)$ is equivalent to $(x, xy + y^3, xy^2)$, to (x,xy,xy^2), to $(x, xy + y^3, 0)$ or to (x,xy,0). From I.6.2 we see that there are no simple germs with the first of these 3-jets, and since arbitrarily small perturbations of the remaining 3-jets will lie in the orbit of this one, there can be no simple germs with any of these 3-jets either.

Next, consider germs whose 2-jet is equivalent to (x,0,0). None of these can be simple either, since arbitrarily small perturbations of any 3-jet having this 2-jet will lie in the orbit of $(x, xy+y^3, xy^2)$.

Finally, no germ having a Σ^2 singularity at $0 \in R^2$ can be simple. This is because the set of 3-jets of such   is of codimension 6 in $J^3(2,3)$ and hence of dimension 21, while the tangent space to the $\mathcal{A}^3$ orbit of such a 3-jet is generated by only 19 elements∎

I.9 The Singular Algebra of a Smooth Map-Germ

If $f:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^p,0)$ is a smooth map-germ, then so is $df:(\mathbb{R}^n,0)\longrightarrow L(n,p)$ (where $L(n,p)$ is the space of linear maps $\mathbb{R}^n\longrightarrow\mathbb{R}^p$), and so graph(df) is a smooth submanifold-germ of $\mathbb{R}^n\times L(n,p)$. Since Σ^k, the set of linear maps $\mathbb{R}^n\longrightarrow\mathbb{R}^p$ of corank k, is a smooth submanifold of $L(n,p)$, we can investigate the algebra of contact of graph(df) with $\mathbb{R}^n\times\Sigma^k$ at $(0, df(0))$. If $df(0)\notin\Sigma^k$ then this algebra is 0 by definition, but if $df(0)\in\Sigma^k$ then we obtain what turns out to be an interesting $\mathcal{A}$-invariant of the germ f, which we call the singular algebra of f at 0 and denote by $Q\Sigma_f(0)$. Recall (see e.g. [12] pages 170-173) that if $df(0)\in\Sigma^k$ then

$$Q\Sigma_f(0) = \frac{\mathcal{E}_n}{df^*(I(\Sigma^k,df(0))\mathcal{E}_n}$$

where $I(\Sigma^k,df(0))$ is the ideal of germs at $df(0)$ of functions on $L(n,p)$ which vanish on Σ^k.

I.9:1 Example Let $f:(\mathbb{R}^2,0)\longrightarrow(\mathbb{R}^3,0)$ be given by $f(x,y) = (x,p(x,y),q(x,y))$ and suppose that $\frac{\partial p}{\partial y}(0) = \frac{\partial q}{\partial y}(0) = 0$, so that $df(0)\in\Sigma^1$. Now

$$df(0) = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

and so $I(\Sigma^1,df(0))$ is generated by the two functions ρ_1 and ρ_2,

$$\rho_1(a_{i,j}) = a_{1,1}a_{2,2} - a_{2,1}a_{1,2} \quad , \quad \rho_2(a_{i,j}) = a_{1,1}a_{3,2} - a_{3,1}a_{1,2}$$

where the $a_{i,j}$ are the usual coordinates on $L(n,p)$. It follows that

$$Q\Sigma_f(0) = \frac{\mathcal{E}_2}{\langle\rho_1\circ df,\rho_2\circ df\rangle} = \frac{\mathcal{E}_2}{\langle\frac{\partial p}{\partial y},\frac{\partial q}{\partial y}\rangle}$$

For example, if $f(x,y) = (x, y^2, y^3 + x^{k+1}y)$ (S_k) then

$$Q\Sigma_f(0) = \frac{\mathcal{E}_2}{\langle y, x^{k+1}+3y^2\rangle} \cong \frac{\mathcal{E}_1}{\langle x^{k+1}\rangle} \cong \mathbb{R}\{1, x, x^2, \ldots, x^k\} .$$

I.9:2 <u>Theorem</u> If $f,g : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ are $\mathcal{A}$-equivalent then

$$Q\Sigma_f(0) \cong Q\Sigma_g(0).$$

<u>Proof</u> Take $df(0)$ and $dg(0)$ to be in Σ^k, and suppose that $f = \psi \circ g \circ \varphi$, where $\varphi \in \mathrm{Diff}(\mathbb{R}^n,0)$ and $\psi \in \mathrm{Diff}(\mathbb{R}^p,0)$. Define

$$\Theta : (\mathbb{R}^n \times L(n,p), (0,df(0))) \longrightarrow (\mathbb{R}^n \times L(n,p), (0,dg(0)))$$

$$(x, A) \longmapsto (\varphi^{-1}(x), d\psi(g(x)).A.d\varphi(\varphi^{-1}(x))).$$

Then Θ is a diffeomorphism, $\Theta(\mathrm{graph}(df)) = \mathrm{graph}(dg)$ (as germs of manifolds), and $\Theta(\mathbb{R}^n \times \Sigma^k) = \mathbb{R}^n \times \Sigma^k$. That is, graph(df) and graph(dg) have equivalent contact with $\mathbb{R}^n \times \Sigma^k$. It follows by the usual argument (see e.g. [12] page 173) that $Q\Sigma_f(0) \cong Q\Sigma_g(0)$. ∎

Instead of considering the algebra $Q\Sigma_f(0)$, we can also define a map-germ associated with f, whose local algebra is $Q\Sigma_f(0)$, as follows. Let $\rho : (L(n,p), df(0)) \longrightarrow (\mathbb{R}^c,0)$ be a submersion such that $\Sigma^k = \rho^{-1}(0)$, and set $\tilde{df} = \rho \circ df$. Then $Q\Sigma_f(0)$ is (isomorphic to) the local algebra $Q_{\tilde{df}}(0)$, and so from I.9:2 we deduce

I.9:3 <u>Corollary</u> If f and g are $\mathcal{A}$-equivalent, then $\tilde{df}$ and $\tilde{dg}$ are $\mathcal{K}$-equivalent.

<u>Proof</u> This follows immediately from the fact that their local algebras are isomorphic ∎

I.10 The Classification of some Multigerms

We consider "bi-germs" $f:(\mathbb{R}^2,S)\longrightarrow(\mathbb{R}^3,0)$, where S is a set consisting of two points, and give the beginning of a classification (with respect to $\mathcal{A}$).

Notation Recall from e.g. [19] that for a multi-germ $f:(\mathbb{R}^n,S)\longrightarrow(\mathbb{R}^p,f(S))$, where S is a finite point set, the space $\Theta(f)_S$ of germs at S of vector fields along f is just the direct product of the $\Theta(f_i)$, where f_i is the germ of f at $x_i \in S$. We shall denote elements of $\Theta(f)_S$ by matrices, the i^{th} column of which corresponds to a menber of $\Theta(f_i)$. The same goes for $\Theta(n)_S$ and for $\Theta(p)_{f(s)}$, which are after all equal, by definition, to $\Theta(1_{\mathbb{R}^n})_S$ and $\Theta(1_{\mathbb{R}^p})_{f(S)}$ respectively. Then

$$tf:\Theta(n)_S\longrightarrow\Theta(f)_S$$

is just the product of the maps tf_i. Since we shall be dealing only with cases where f(S) is a single point, the map

$$\omega f:\Theta(p)_{f(S)}\longrightarrow\Theta(f)_S$$

presents no special notational problems, and is perhaps best described with an example.(I.10:1).

For our coordinate notation we shall assume that each $x_i \in S$ lies at the centre of a coordinate patch in $\mathbb{R}^n$, and has coordinate 0 in the corresponding coordinate system. This should cause no confusion, since any expression involving the coordinates arround x_i will always appear in the i^{th} column of a matrix.

I.10:1 Example
$$\begin{cases}(x,y)\longrightarrow(a(x,y),\ b(x,y),\ c(x,y))\\ (x,y)\longrightarrow(\bar{a}(x,y),\ \bar{b}(x,y),\ \bar{c}(x,y))\end{cases}$$

denotes a bi-germ. For this bi-germ we have

$$tf\begin{bmatrix} h_1 & \bar{h}_1 \\ h_2 & \bar{h}_2 \end{bmatrix} + \omega f\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} h_1a_x+h_2a_y & \bar{h}_1\bar{a}_x+\bar{h}_2\bar{a}_y \\ h_1b_x+h_2b_y & \bar{h}_1\bar{b}_x+\bar{h}_2\bar{b}_y \\ h_1c_x+h_2c_y & \bar{h}_1\bar{c}_x+\bar{h}_2\bar{c}_y \end{bmatrix} + \begin{bmatrix} u(a,b,c) & u(\bar{a},\bar{b},\bar{c}) \\ v(a,b,c) & v(\bar{a},\bar{b},\bar{c}) \\ w(a,b,c) & w(\bar{a},\bar{b},\bar{c}) \end{bmatrix}$$

Bi-germs of immersions

Given two immersed planes meeting at $0 \in \mathbb{R}^3$, it is possible to choose coordinates in $\mathbb{R}^3$ and at each of the two source points with respect to which the bi-germ of the immersion is written

$$(1) \quad \begin{cases} (x,y) \longrightarrow (x,\ y,\ \varphi(x,y)) \\ (x,y) \longrightarrow (x,\ y,\ 0) \end{cases}$$

We shall refer to φ as the separation function. We now have the elementary

I.10:2 Theorem a) Bi-germs of immersions, expressed in the form (1), are classified for $\mathcal{A}$ by the $\mathcal{K}$-class of the separation function.

b) For a bi-germ f of the form (1),

$$T_e\mathcal{A}f = \begin{bmatrix} \mathcal{E}_2 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_2 \\ T_e\mathcal{K}\varphi & T_e\mathcal{K}\varphi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Delta(\mathcal{E}_2 \times \mathcal{E}_2) \end{bmatrix}$$

where $\Delta(\mathcal{E}_2 \times \mathcal{E}_2)$ is the diagonal in $\mathcal{E}_2 \times \mathcal{E}_2$.

Proof a) This is precisely the definition of $\mathcal{K}$ — see for example [12] pages 170-172.

b) Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be projection onto the XY-plane. The equality

$$\begin{bmatrix} g_1 & \bar{g}_1 \\ g_2 & \bar{g}_2 \\ 0 & 0 \end{bmatrix} = tf \begin{bmatrix} 0 & \bar{g}_1 - g_1 \\ 0 & \bar{g}_2 - g_2 \end{bmatrix} + \omega f \begin{bmatrix} \pi^* g_1 \\ \pi^* g_2 \\ 0 \end{bmatrix}$$

for any $g_1, g_2, \bar{g}_1, \bar{g}_2$, shows that

$$T_e \mathcal{A} f \supseteq \begin{bmatrix} \mathcal{E}_2 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_2 \\ 0 & 0 \end{bmatrix} .$$

We may therefore ignore the first two rows of matrices in $\Theta(f)_S$ and $\Theta(3)$ in the calculations which follow.

Now let $g_3 \in T_e \mathcal{K} \varphi$, so that g_3 can be written $g_3 = h_1 \varphi_x + h_2 \varphi_y + c \varphi$ for some functions $h_1, h_2 \in \mathcal{E}_2$, $c \in \mathcal{M}_2$. Then (looking only at the bottom row of the matrices in $\Theta(f)_S$)

$$\begin{bmatrix} 0 & g_3 \end{bmatrix} = \omega f \ (\pi^* g_3 - Z\pi^* c) - tf \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} g_3 & 0 \end{bmatrix} = \omega f \ (Z\pi^* c) + tf \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix}$$

This shows that (on the bottom row) $T_e \mathcal{A} f \supseteq \begin{bmatrix} T_e \mathcal{K} \varphi & T_e \mathcal{K} \varphi \end{bmatrix}$.

Now for any $g \in \mathcal{E}_2$, we have $\begin{bmatrix} g & g \end{bmatrix} = \omega f(\pi^* g)$, and this completes the proof of the inclusion from left to right in the statement of (b). Conversely, if $\begin{bmatrix} g_3 & \bar{g}_3 \end{bmatrix} \in T_e \mathcal{A} f$, then we must have

$$g_3 = w(x, y, \varphi) + h_1 \varphi_x + h_2 \varphi_y$$

$$\bar{g}_3 = w(x, y, 0)$$

for some $w \in \mathcal{E}_3$ and $h_1, h_2 \in \mathcal{E}_2$. It follows that $[g_3 - \bar{g}_3] \in T_e \mathcal{K}\varphi$. This completes the proof.

I.10:3 Remark Given a bigerm of immersions, it is not necessary to reduce it to the form (1) in order to find out what the $\mathcal{K}$-class of the separation function is. In fact it is easily seen that it is enough to choose a direction transverse to both immersed surfaces and measure the distance between the two surfaces in this direction, as a function of the co rdinates on one of the two surfaces.

In view of I.10:2 we shall refer to $\mathcal{A}$-classes of bi-germs of immersions by the $\mathcal{K}$-class of the separation function. Thus,

$$\begin{cases} (x,y) \longrightarrow (x,\ y,\ x^2 + y^2) \\ (x,y) \longrightarrow (x,\ y,\ -x^2) \end{cases}$$

will be referred to as an $[A_1^+]$, or as $[x^2+y^2]$.

The interest of this theorem lies in the way that it helps in the study of the $\mathcal{A}_e$-versal unfoldings of some of the germs classified earlier, in particular those whose 2-jet is equivalent to $(x, y^2, 0)$. For these map-germs, the only singular (i.e. unstable) multi-germs present in their $\mathcal{A}_e$-versal unfoldings are bi-germs of immersions. Theorem I.10:2 allows us to extend the isomorphism of classifications (Theorem I.5:16), to obtain isomorphisms of versal unfoldings. In this context the sheaf-theoretic approach is perhaps the most appropriate, so we make the following definitions: for any unfolding

$$F: (\mathbb{R}^2 \times \mathbb{R}^a, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}^a, 0)$$
$$(x,y,u) \longrightarrow (f_u(x,y), u)$$

and each $u \in (\mathbb{R}^a, 0)$, define a sheaf $\mathcal{S}_{\mathcal{A}}(f_u)$ as follows: first, let $\theta(f_u)$ be the sheaf of germs of vector fields along f_u, and let $\Theta(\mathbb{R}^k)$ be the sheaf of germs along $1_{\mathbb{R}^k}$. Define $\mathcal{S}_{\mathcal{R}}(f_u)$ by the exact sequence

$$0 \longrightarrow \Theta(\mathbb{R}^2) \xrightarrow{tf_u} \theta(f_u) \longrightarrow \mathcal{S}_{\mathcal{R}}(f_u) \longrightarrow 0 ,$$

let $(f_u)_* \mathcal{S}_{\mathcal{R}}(f_u)$ be the push-forward of $\mathcal{S}_{\mathcal{R}}(f_u)$ by f, and define $\mathcal{S}_{\mathcal{A}}(f_u)$ by

$$0 \longrightarrow \Theta(\mathbb{R}^3) \longrightarrow (f_u)_* \mathcal{S}_{\mathcal{R}}(f_u) \longrightarrow \mathcal{S}_{\mathcal{A}}(f_u) \longrightarrow 0.$$

Note that if $f = f_o$ is $\mathcal{K}$-finite, then for all $u \in (\mathbb{R}^a, 0)$, f_u is finite-to-one ([30] page 493) and so the stalk over $y \in \mathbb{R}^3$ of $(f_u)_* \mathcal{S}_{\mathcal{R}}(f_u)$ is a direct sum of the stalks of $\mathcal{S}_{\mathcal{R}}(f_u)$ over the finite set $f_u^{-1}(y)$; moreover, the stalk of $\mathcal{S}_{\mathcal{A}}(f_u)$ over y may be identified with

$$\frac{\theta_S(f_u)}{T_e\mathcal{A}(f_u)_S}$$

where $S = f_u^{-1}(y)$ (see [30] pages 492 and 493 for more details of this construction, or [7] where it is described at some length).

Now let
$$P:(H^2 \times \mathbb{R}^a, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^a, 0)$$

$$(x,y,u) \longrightarrow (p_u(x,y),u)$$

be any unfolding and for each u define a sheaf $\mathcal{S}_{\mathcal{K}^\partial}(p_u)$ over H^2 by

$$\mathcal{S}_{\mathcal{K}^\partial}(p_u) = \frac{\Theta(p_u)}{tp_u(\Theta(H^2)) + p_u^*\mathfrak{m}_1\Theta(p_u)}$$

If $f(x,y) = (x, y^2, yp(x,y^2))$ then by I.5:7 and the Versality Theorem

([30] Theorem 3.3) there exists an $\mathcal{A}_e$-versal unfolding of f of the form

$$F:(x,y,u) \longmapsto (x,\ y^2,\ yp(x,y^2),\ u)$$

and clearly one can regard the map-germ

$$P:(x,y,u) \longmapsto (p(x,y),\ u)$$

defined on $(H^2 \times \mathbb{R}^a, 0)$ as an unfolding of the germ $p:(H^2,0) \longrightarrow \mathbb{R}$. An $\mathcal{R}^\partial_e$-versality theorem has been proved by Arnol'd, ([2] page), namely that the unfolding P is $\mathcal{R}_e$-versal if and only if

$$T_e\mathcal{R}^\partial p + \mathbb{R}\{\dot{P}_1,\dots,\dot{P}_a\} = C^\infty(H^2,0)$$

and one can go on from this to prove the corresponding $\mathcal{K}^\partial_e$-versality theorem, but for the purposes of brevity we shall adopt the following definition: we shall say that the unfolding P is $\mathcal{K}^\partial_e$-versal if and only if the corresponding infinitesimal condition holds, namely

$$T_e\mathcal{K}^\partial p + \mathbb{R}\{\dot{P}_1,\dots,\dot{P}_a\} = C^\infty(H^2,0)$$

where as usual

$$\dot{P}_i = \frac{\partial p_u}{\partial u_i}(x,y,0).$$

Then we have

I.10:4 <u>Lemma</u> Let $f(x,y) = (x,\ y^2,\ yp(x,y^2))$ and let $F:(\mathbb{R}^2 \times \mathbb{R}^a,0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}^a,0)$ be an unfolding of f of the form

$$(x,y,u) \longrightarrow (x,\ y^2,\ yp_u(x,y^2),\ u).$$

Let $P:(H^2 \times \mathbb{R}^a,0) \longrightarrow (\mathbb{R} \times \mathbb{R}^a,0)$ be the unfolding of p defined by

$$(x,y,u) \longmapsto (p_u(x,y),\ u).$$

Then F is an $\mathcal{A}_e$-versal unfolding of f if and only if P is a $\mathcal{K}_e^{\partial}$-versal unfolding of p.

Proof Immediate from I.5:7 and the Versality Theorem for $\mathcal{A}$ ∎

We now show that for unfoldings F and P as in the lemma, for all $u \in (\mathbb{R}^a,0)$ the sheaves $\mathcal{S}_{\mathcal{A}}(f_u)$ and $\mathcal{S}_{\mathcal{K}^\partial}(p_u)$ are essentially the same. Of course the former is a sheaf over $\mathbb{R}^3$ while the latter is a sheaf over H^2, and so in order to compare them we make use of the projection π introduced in the proof of I.10:2, and consider, instead of $\mathcal{S}_{\mathcal{A}}(f_u)$, its push-forward $\pi_*(\mathcal{S}_{\mathcal{A}}(f_u))$. Now for an unfolding F as in Lemma I.10:4, and for any u,

$$\operatorname{supp}(\mathcal{S}_{\mathcal{A}}(f_u)) \subseteq \{(X,Y,Z) \in \mathbb{R}^3 : Z = 0,\ Y \geq 0\}$$

(we shall see why in the proof of the next result), and so π is one-to-one on $\operatorname{supp}(\mathcal{S}_{\mathcal{A}}(f_u))$ and, also, $\operatorname{supp}(\pi_*\mathcal{S}_{\mathcal{A}}(f_u)) \subseteq H^2$. In view of this, regard $\pi_*\mathcal{S}_{\mathcal{A}}(f_u)$ as a sheaf on H^2; then we have

I.10:5 Theorem The sheaves $\pi_*\mathcal{S}_{\mathcal{A}}(f_u)$ and $\mathcal{S}_{\mathcal{K}^\partial}(p_u)$ are isomorphic in the weak sense that for each $y \in H^2$, the respective stalks over y have the same dimension as real vector spaces.

Proof We consider two cases, where y is on the boundary of H^2, and where y is in the interior.

Let $(X_0,0) \in H^2$. Then $\mathcal{S}_{\mathcal{K}^\partial}(p_u)_{(X_0,0)}$, the stalk of $\mathcal{S}_{\mathcal{K}^\partial}(p_u)$ over $(X_0,0)$, is

$$\frac{C^\infty(H^2,(X_0,0))}{\left\langle \dfrac{\partial p_u}{\partial X},\ Y\dfrac{\partial p_u}{\partial Y},\ p_u \right\rangle}$$

$$\cong \frac{\mathcal{E}_2^T}{\left\langle \frac{\partial p_u}{\partial X}(X+X_0,\ Y^2),\ Y^2\frac{\partial p_u}{\partial Y}(X+X_0,\ Y^2),\ p_u(X+X_0,\ Y^2)\right\rangle}$$

which by I.5:7 is isomorphic to

$$\frac{\Theta(f_u|(X_0,0))}{T_e\mathcal{A}(f_u|(X_0,0))}$$

where $(f_u|(X_0,0))$ is the germ of f_u at the point $(X_0,0)$ in $\mathbb{R}^2$. But this is of course just

$$\mathcal{S}_{\mathcal{A}}(f_u)(X_0,0,0)$$

and thus equal to

$$\pi_*\mathcal{S}_{\mathcal{A}}(f_u)(X_0,0)\ .$$

Now let $(X_0,Y_0)\in H^2$ with $Y_0>0$. Then $\mathcal{S}_{\mathcal{K}}\partial(p_u)_{(X_0,Y_0)}$ is equal to

$$\frac{C^\infty(H^2,\ (X_0,Y_0))}{\left\langle \frac{\partial p_u}{\partial X}\ ,\ Y\frac{\partial p_u}{\partial Y}\ ,\ p_u\right\rangle}$$

which is equal to

$$\frac{\mathcal{E}_2}{\left\langle \frac{\partial p_u}{\partial X}(X+X_0,Y+Y_0),\ \frac{\partial p_u}{\partial Y}(X+X_0,Y+Y_0),\ p_u(X+X_0,Y+Y_0)\right\rangle} \tag{2}$$

since the function Y is a unit in the ring $C^\infty(H^2,\ (X_0,Y_0))$ when $Y_0>0$. Now this space is different from 0 if and only if

$$p_u(X_0,Y_0) = \frac{\partial p_u}{\partial X}(X_0,Y_0) = \frac{\partial p_u}{\partial Y}(X_0,Y_0) = 0 \quad ,$$

but that is precisely the condition that f_u have a singular bi-germ at $S = \{(X_0,\sqrt{Y_0}),\ (X_0,-\sqrt{Y_0})\}$ (f_u has a singular bi-germ at S if and only if $f_u(X_0,\sqrt{Y_0}) = f_u(X_0,-\sqrt{Y_0})$ and the two tangent planes to the image coincide). In this case we have

$$\pi_* \mathcal{S}_{\mathcal{A}}(f_u)(X_0,Y_0) = \mathcal{S}_{\mathcal{A}}(f_u)(X_0,Y_0,0)$$

$$= \frac{\theta_S(f_u)}{T_e \mathcal{A} f_u}$$

(3) $$= \frac{\mathcal{E}_2}{T_e \mathcal{K} \varphi}$$

where φ is a separation function, by I.10:2. We now complete the proof by showing that we may take, as φ, the function $(x,y) \longmapsto p_u(x+X_0,\ y+Y_0)$. To see this, we reduce the bigerm of f_u at S to the form (1) defined before I.10:2. Put

$$x = X_0 + \bar{x} \qquad X = X_0 + \bar{X}$$
$$y = Y_0 + \bar{y} \qquad Y = Y_0 + \bar{Y}$$
$$Z = \bar{Z} \quad .$$

Then in the new coordinates $(\bar{x},\bar{y})$ and $(\bar{X},\bar{Y},\bar{Z})$, the germ of f_u at $(X_0,\sqrt{Y_0})$ is

$$(\bar{x},\bar{y}) \longmapsto (\bar{x},\ \bar{y}^2 + 2\sqrt{Y_0}\,\bar{y},\ (\bar{y} + \sqrt{Y_0})p_u(\bar{x} + X_0,\ (\bar{y} + \sqrt{Y_0})^2).$$

Now put $\tilde{y} = \bar{y}^2 + 2\sqrt{Y_0}\,\bar{y}$, so that $\bar{y} = -\sqrt{Y_0} + \sqrt{Y_0 + \tilde{y}}$. Then the germ of f_u at $(X_0,\ Y_0)$ becomes

$$(\bar{x},\tilde{y}) \longmapsto (\bar{x},\ \tilde{y},\ \sqrt{Y_0 + \tilde{y}}\ \ p_u(\bar{x} + X_0,\ \tilde{y} + Y_0)).$$

By symmettry, the germ of f_u at $(X_0,\ -\sqrt{Y_0})$ is equivalent to

Since the germ of f at x_2 is a cross-cap, a linear autormphism of the XY-plane in the target (leaving the Z coordinate unchanged) reduces the 2-jet of (4) to

$$\begin{cases}(x,\ y,\ 0)\\(y^2,\ xy,\ x).\end{cases}$$

By applying the Splitting Lemma ([15] page) to the first component of the germ of f at x_2, we can reduce f to

$$\begin{cases}(x,y)\longrightarrow(x,\ y,\ 0)\\(x,y)\longrightarrow(y^2+r(x),\ xy+s(x,y),\ x)\end{cases}$$

where $s\in\mathcal{M}_2^3$ and $r\in\mathcal{M}_1^3$. Now clearly

$$f_2{}^*\mathcal{M}_3+\mathcal{M}_2^{2k+1}=\mathcal{M}_2-\{y,\ y^3,\ \dots,\ y^{2k-1}\}$$

for any $k<\infty$, and so, for arbitrarily high $l<\infty$, the l-jet of f is equivalent to

$$\begin{cases}(x,\ y,\ 0)\\(y^2,\ xy+\sum_{3\leq 2i+1\leq l}a_iy^{2i+1},\ x)\end{cases}$$

under left coordinate changes.

I.10:6 <u>Theorem</u> i) The bi-germ

$$\begin{cases}(x,y)\longrightarrow(x,\ y,\ 0)\\(x,y)\longrightarrow(y^2,\ xy+y^{2k+1},\ x)\end{cases}$$

is 2k+1-determined, and has $\mathcal{A}_e$ tangent space

$$C=\begin{bmatrix}\mathcal{E}_2 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_2-\{y,y^3,\dots,y^{2k-1}\} \\ \mathcal{M}_2-\{x,x^2,\dots x^{k-1}\} & \mathcal{M}_2-\{y^2,y^4,\dots,y^{2k-2}\}\end{bmatrix}+\mathbb{R}\left\{\begin{bmatrix}0&0\\0&0\\x^i&y^{2i}\end{bmatrix}\begin{bmatrix}0&0\\0&y^{2i+1}\\0&y^{2i}\end{bmatrix}\right\}_{0\le i\le k-1}$$

ii) Every finitely determined germ of an immersion and a cross-cap meeting transversely, is equivalent to one of the germs defined in (i).

<u>Proof</u> i) Since

i) Since $tf\begin{bmatrix}h_1 & 0\,0\\ h_2 & 0\end{bmatrix}=\begin{bmatrix}h_1&0\\h_2&0\\0&0\end{bmatrix}$, we have $\begin{bmatrix}\mathcal{E}_2&0\\\mathcal{E}_2&0\\0&0\end{bmatrix}\subseteq T_e\mathcal{A}f.$

Next,

$$tf\begin{bmatrix}-p(x,y^2)&0\\-q(x,y^2)&0\end{bmatrix}+\omega f\begin{bmatrix}p(Z,X)\\q(Z,X)\\0\end{bmatrix}=\begin{bmatrix}0&p(x,y^2)\\0&q(x,y^2)\\0&0\end{bmatrix}\text{ shows }\begin{bmatrix}0&\mathcal{E}_2^T\\0&\mathcal{E}_2^T\\0&0\end{bmatrix}\subseteq T_e\mathcal{A}f,$$

where, as in I.5, $\mathcal{E}_2^T$ is the ring of functions invariant under $(x,y)\longrightarrow(x,-y)$.

Then

$$tf\begin{bmatrix}0&0\\0&p(x,y^2)\end{bmatrix}\equiv\begin{bmatrix}0&2yp(x,y^2)\\0&0\\0&0\end{bmatrix}\bmod\begin{bmatrix}0&0\\0&\mathcal{E}_2^T\\0&0\end{bmatrix}\text{ shows }\begin{bmatrix}0&\mathcal{E}_2\\0&0\\0&0\end{bmatrix}\subseteq T_e\mathcal{A}f.$$

By comparing

$$tf\begin{bmatrix}0&0\\0&yp(x,y^2)\end{bmatrix}\quad\text{and}\quad\omega f\begin{bmatrix}0\;\;0\\Yp(Z,X)\\0\end{bmatrix}$$

we see now that

$$T_e\mathcal{A}f\supseteq\begin{bmatrix}0&0\\0&\mathcal{E}_2-\{y,y^3,\dots,y^{2k-1}\}\\0&0\end{bmatrix},$$

and then $tf\begin{bmatrix}0 & a(x,y)\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & ya(x,y)\\0 & a(x,y)\end{bmatrix}$ shows that

$$T_e\mathcal{A}f \supseteq \begin{vmatrix}0 & 0\\0 & 0\\0 & \mathcal{M}_2 - \{y^2,\ldots,y^{2k-2}\}\end{vmatrix}.$$

Next, $\omega f\begin{bmatrix}0\\0\\a(X,Y)\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\\a(x,y) & a(y^2,\ xy+y^{2k+1})\end{bmatrix}$ shows that $\begin{bmatrix}0 & 0\\0 & 0\\a(x,y) & 0\end{bmatrix} \in T_e\mathcal{A}f$

for all $a \in \mathcal{E}_2$ such that $a(y^2,\ xy+y^{2k+1}) \in \mathcal{M}_2 - \{y^2, y^4,\ldots,y^{2k-2}\}$; that is, for $a \in \mathcal{M}_2 - \{x,x^2,\ldots,x^{k-1}\}$.

Finally, to conclude that $T_e\mathcal{A}f \supseteq C$, note that

$$\omega f\begin{bmatrix}0\\0\\x^i\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\\0 & y^{2i}\end{bmatrix} \quad \text{and } tf\begin{bmatrix}0 & y^{2i}\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & y^{2i+1}\\0 & y^{2i}\end{bmatrix}.$$

The proof that $T_e\mathcal{A}f \subseteq C$ is a trivial matter of checking generators, and we omit it.

To see that f is 2k+1-determined, note first that $T_e\mathcal{A}f \supseteq \mathcal{M}_2^{2k}\Theta_S(f)$, and, as an easy calculation shows, $f^*\mathcal{M}_3\Theta_S(f) \supseteq \mathcal{M}_2^2\Theta_S(f)$. It follows from Gaffney's formula (Theorem I.3:3), whose proof works equally well in the multi-jet case, that f is 2k+2-determined. However, by the remarks preceeding the statement of the theorem, it is clear that if $j^{2k+1}f(S) = j^{2k+1}g(S)$, then $j^{2k+2}g(S)$ is equivalent to $j^{2k+2}f(S)$. From this we conclude that f is 2k+2-determined.

ii) This now follows immediately from (i) and from the remarks preceeding the statement of the theorem.

I.10:7 Definition Let $f:(\mathbb{R}^2,\{x_1,x_2\})\longrightarrow(\mathbb{R}^3,0)$ be a bigerm such that the germ of f at x_1 is an immersion and the germ of f at x_2 is a cross-cap. Write $(f|x_i)$ for the germ of f at x_i. Then $(f|x_2)$ has a curve of double-points which, in the source, is a smooth curve D passing through x_2. Let I(D) be the ideal in $C^\infty(\mathbb{R}^2, x_2)$ of germs of functions vanishing on D, and let $I(f(\mathbb{R}^2,x_i))$ be the ideal in $\mathcal{E}_3$ of germs of functions vanishing on $f(\mathbb{R}^2,x_i)$. Then the algebra of contact between the curve of double points of $(f|x_2)$, and $f(\mathbb{R}^2,x_1)$, is defined to be

$$\frac{C^\infty(\mathbb{R}^2,x_2)}{[I(D) + f^*(I(f(\mathbb{R}^2,x_1)))]\, C^\infty(\mathbb{R}^2,x_2)}\, .$$

I.10:8 Corollary Finitely-determined germs of an immersion and a cross-cap meeting transversely, are classified, for $\mathcal{A}$, by the isomorphism class of the algebra of contact between the curve of double points of the cross-cap and the immersed plane.

Proof If f is such a bi-germ then by I.10:7, it is equivalent to

$$\begin{cases}(x,y)\longrightarrow(x,\ y,\ 0)\\ (x,y)\longrightarrow(y^2,\ xy + y^{2k+1},\ x)\end{cases}$$

for some $k\geq 1$. A brief calculation shows that in this case D is the curve $x = -y^{2k}$, and the algebra of contact is

$$\frac{\mathcal{E}_2}{\langle x+y^{2k};\ f^*Z\rangle} = \frac{\mathcal{E}_2}{\langle x+y^{2k},\ x\rangle} \cong \mathbb{R}\{1,y,y^2,\dots,y^{2k-1}\}\, .$$

For different k, these algebras are non-isomorphic, and the corollary now follows from the fact that the isomorphism class of the algebra is an $\mathcal{A}$-invariant ∎

I.11 Adjacencies

The $\mathcal{A}$ class L of singularities is adjacent to the $\mathcal{A}$ class K ($L \longrightarrow K$) if every map-germ $f \in L$ can be deformed to a map-germ in K by an arbitrarily small perturbation. This is the same as saying that K specialises to L. Thus, the $\mathcal{A}$ classes to which L is adjacent are those which appear in the $\mathcal{A}$-versal unfolding of any germ in L, and may be determined by studying the $\mathcal{A}$-versal unfolding of such a germ. Here we omit the generally straightforward calculations involved in the study of these unfoldings, and simply show the adjacencies, for some of the $\mathcal{A}$ classes in our classification. In fact adjacencies, except for those involving bi-germs, have already been calculated for the simple singularities, in the proofs of I.5:19 and I.6.1:2.

We refer to $\mathcal{A}$ classes by the symbols given in I:2 and I.10.

Diagram 1

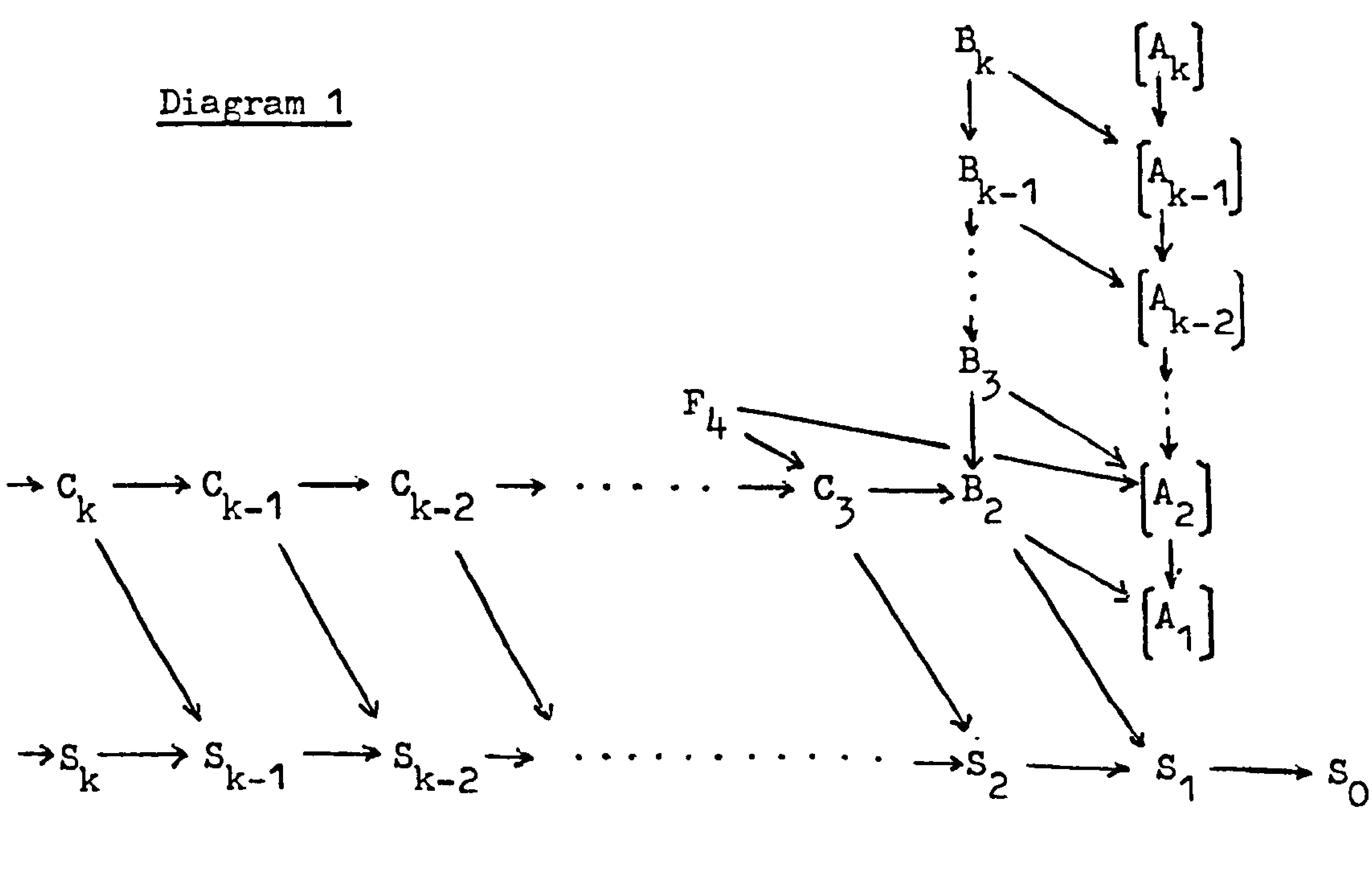
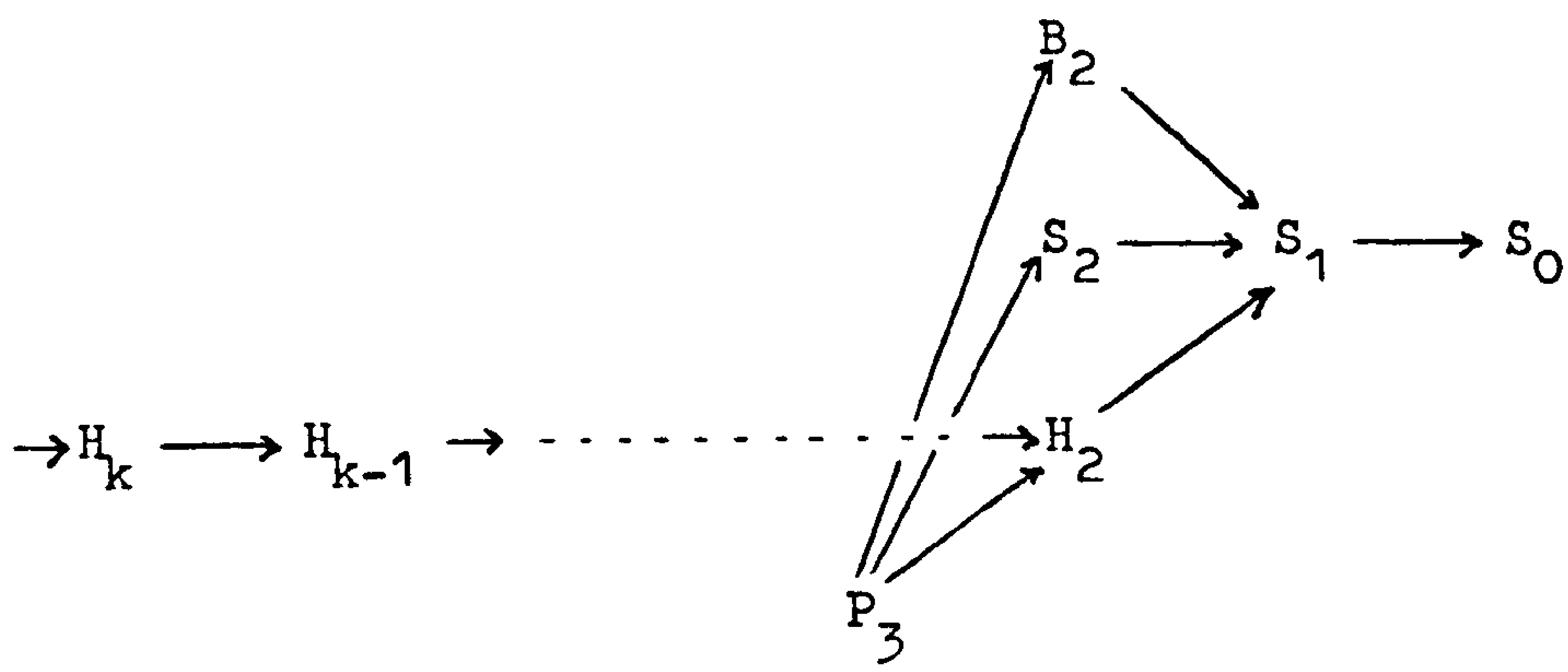


Diagram 2



I.11:1 Remarks i) The situation regarding multi-germs in the unfoldings of germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ with 2-jet equivalent to $(x,xy,0)$ is considerably more complicated than in the case of germs with 2-jet equivalent to $(x,y^2,0)$, and is not shown here. In fact we know that if $f:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ is in the same $\mathcal{K}$ class as $(x,y) \longrightarrow (x,y^k,0)$, then there will be arbitrarily small deformations of f presenting k-tuple points in the image. Moreover, such k-tuple points will also specialise to more singular multi-germs, such as multi-germs involving cross-caps as well as immersed planes. This occurs in the $\mathcal{A}$-versal deformation of H_2

$$f_{a,b}:(x,y) \longrightarrow (x,\ xy + y^5 + ay^2,\ y^3 + by).$$

When $4b^3 + 27a^2 = 0$, then $f_{a,b}$ has an S_0 (cross-cap) singularity at

$$\left(\frac{-5b^2}{9} + 2a\sqrt{\frac{-b}{3}}\ ,\ -\sqrt{\frac{-b}{3}} \right)$$

whose image in $\mathbb{R}^3$ is the same as that of

$$\left(\frac{-5b^2}{9} + 2a\sqrt{\frac{-b}{3}}\ ,\ 2\sqrt{\frac{-b}{3}} \right).$$

It seems likely that in the $\mathcal{A}$-versal deformation of H_k $(k \geq 2)$, there will be bi-germs consisting of a cross-cap and an immersed plane, $\mathcal{A}$-equivalent to

$$\begin{cases} (x,y) \longmapsto (x, y, 0) \\ (x,y) \longmapsto (y^2, xy + y^{2k-1}, x), \end{cases}$$

but we have not carried out the calculations involved in verifying this.

ii) In our adjacency diagrams we have not distinguished between the $\mathcal{A}$ classes S_k^+,

$$(x,y) \longmapsto (x, y^2, y^3 + x^{k+1}y),$$

and S_k^-,

$$(x,y) \longmapsto (x, y^2, y^3 - x^{k+1}y),$$

which are inequivalent (as map-germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$) when k is odd. Similarly, we have not distinguished between B_k^+ and B_k^-, $[A_k^+]$ and $[A_k^-]$, etc. In fact interesting phenomena present themselves when these signs are taken into consideration. For example, B_2^+ is adjacent to $[A_1^+]$ but not to $[A_1^-]$, while for B_2^- the reverse is true.

CHAPTER II

THE EXPONENTIAL MAP OF THE TANGENT BUNDLE

In this chapter we study the exponential map $\exp_g$ from the tangent bundle TM of a smooth manifold M into $\mathbb{R}^p$, associated with a smooth immersion $g:M \longrightarrow \mathbb{R}^p$. This map is given, in trivial local coordinates on TM, by

$$\exp_g(x,v) = g(x) + dg(x)(v)$$

where $v \in T_xM$ and $dg(x)$ is the differential of g, from T_xM to $T_{g(x)}\mathbb{R}^p = \mathbb{R}^p$. We show that the behaviour of $\exp_g$ may be studied by looking at families of central (radial) projections p_g of M into hyperplanes H of $\mathbb{R}^p$, and in particular that every contact-invariant submanifold W of $J^k(TM-M, \mathbb{R}^p)$ corresponds in a well-defined way to a submanifold W' of $J^{k+1}(M, H)$, such that

$$j^{k+1}p_g(q)(x) \in W' \quad \text{if and only if} \quad j^k\exp_g(x,q) \in W$$

and

$$j^{k+1}p_g \pitchfork W' \quad \text{if and only if} \quad j^k\exp_g(x,q) \pitchfork W.$$

(Theorem II.3:5).

This allows us to prove

II:1 <u>Theorem</u> Let $W \subseteq {}_rJ^k(TM-M, \mathbb{R}^p)$ be a contact-invariant manifold; then $\{g \in \mathrm{Imm}(M, \mathbb{R}^p) : {}_rj^k\exp_g|_{TM-M} \pitchfork W\}$ is residual.

From this we deduce

II:2 <u>Corollary</u> If $(2n,p)$ are nice dimensions and if $\dim M = n$, then $\{g \in \mathrm{Imm}(M, \mathbb{R}^p) : \exp_g$ is locally stable on $TM-M\}$ is residual in $\mathrm{Imm}(M, \mathbb{R}^p)$.

By "$\exp_g$ is locally stable on TM-M" we mean that every multi-germ of $\exp_g$ on TM-M is stable.

The converse to II:2 does not hold, and we give an example of dimensions (n,p) for which $(2n,p)$ are not nice dimensions but $\exp_g: TM-M \longrightarrow \mathbb{R}^p$ is generically locally stable. We also give examples of certain reasonably large ranges of dimensions for which local stability of $\exp_g$ on TM-M is not generic.

II.1 Preliminaries

Given an immersion $g:M \longrightarrow \mathbb{R}^p$, we embed TM in $M \times \mathbb{R}^p$, as

$$\{(x,q) \in M \times \mathbb{R}^p : q - g(x) = dg(x)(v) \text{ for some } v \in T_xM\}.$$

The map $\exp_g$ is then simply projection onto the second factor.

The starting point for the proof of the results described above is the observation that TM is locally the singularity manifold of the family of central projections of M from points of $\mathbb{R}^p$ into a fixed hyperplane. More precisely, let $x_0 \in M$, let $q_0 \neq g(x_0)$ be a point in $\mathbb{R}^p$, and choose an affine hyperplane H, with associated height function h (i.e. such that $H = h^{-1}(0)$) such that

(1) $\quad h(q_0) \neq h(g(x_0))$.

Then for $(y,q) \in \mathbb{R}^p \times \mathbb{R}^p$ near $(g(x_0),q_0)$, the projection of y from q into H is defined by

$$(y,q) \longmapsto p(q,y) = p(q)(y) = \frac{h(q)y - h(y)q}{h(q) - h(y)}$$

and we define a family of projections

$$P_g : (M,x_0) \times (\mathbb{R}^p,q_0) \longrightarrow H \times \mathbb{R}^p$$

by

$$(x,q) \longmapsto (p_g(q)(x),\ q) = (p(q)(g(x))\ ,\ q)$$

Note that if (H', h') also satisfies (1), then P_g and P'_g are left-equivalent families:

$$\begin{array}{ccc} (M,x_0)\times(\mathbb{R}^p,q_0) & \xrightarrow{P_g} & (H,\ p_g(q_0)(x_0))\times(\mathbb{R}^p,q_0) \\ & \searrow^{P'_g} & \downarrow \bar{\Psi} \\ & & (H',p'_g(q_0)(x_0))\times(\mathbb{R}^p,q_0) \end{array}$$

where $\bar{\Psi}(y,q) = (p'(q)(y),\ q)$.

Now, for such a family P_g, define the singularity manifold S_{P_g} by

$$S_{P_g} = \{(x,q) \in M\times\mathbb{R}^p : p_g(q) \text{ is not an immersion at } x\}.$$

Clearly $p_g(q)$ is not an immersion at x precisely when the kernel of $dp(q)$ intersects $T_{g(x)}g(M)$, but since this kernel is spanned by the vector $q-g(x)$, we conclude

II.1:1 <u>Proposition</u> $S_{P_g} = TM \cap \text{domain}(P_g)$ and moreover,

$$\exp_g : TM \cap \text{domain}(P_g) \longrightarrow \mathbb{R}^p \quad \text{and} \quad \pi : S_{P_g} \longrightarrow \mathbb{R}^p$$

are the same map. Here π is just projection onto the second factor ∎

Note that $\text{domain}(P_g)$ is an open subset of $M\times\mathbb{R}^p$.

This identity will allow us to study $\exp_g : TM - M \longrightarrow \mathbb{R}^p$ from the viewpoint of singularity theory. Incidentally, it justifies the use of the word "manifold" in the term "singularity manifold".

II.2 For any immersion $g: M \longrightarrow \mathbb{R}^p$, we have identified $TM - M \subseteq M \times \mathbb{R}^p$ as being, locally, the singularity manifold of the family P_g of radial projections of M into a hyperplane. It will be convenient to identify it further as the 0-set of a submersion $(M, x_0) \times (\mathbb{R}^p, q_0) \longrightarrow \mathbb{R}^{p-n}$. Before proceeding, note that if g is an immersion, all the singularities of P_g (and of p_g) are of type Σ^1.

II.2:1 Definition Let

$$F: \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}^p \qquad (n \leq p)$$

$$(x,u) \longmapsto (f(x,u),u) = (f_u(x),u)$$

be a smooth unfolding , and suppose that F (and hence f_0) has a Σ^1 singularity at x = u = 0. Let $\rho : (L(n,p), df_0(0)) \longrightarrow (\mathbb{R}^{p-n+1}, 0)$ be the germ of a submersion such that locally $\Sigma^1(n,p) = \rho^{-1}(0)$, and define map-germs $\tilde{d}f : (\mathbb{R}^n \times \mathbb{R}^a, 0) \longrightarrow (\mathbb{R}^{p-n+1}, 0)$ and $\bar{d}F : (\mathbb{R}^n \times \mathbb{R}^a, 0) \longrightarrow (\mathbb{R}^{p-n+1} \times \mathbb{R}^a, 0)$ by

$$\tilde{d}f(x,u) = \rho(df_u(x))$$

and

$$\bar{d}F(x,u) = (\tilde{d}f(x,u), u).$$

II.2:2 Remark 1) If $\rho' : (L(n,p), df_0(0)) \longrightarrow (\mathbb{R}^{p-n+1}, 0)$ is another submersion such that locally $\Sigma^1(n,p) = \rho'^{-1}(0)$, then ρ and ρ' are $\mathcal{C}$-equivalent, so that the ideals

$$\rho'^* \mathfrak{m}_{p-n+1} C^\infty(L(n,p), df_0(0)) \text{ and } \rho^* \mathfrak{m}_{p-n+1} C^\infty(L(n,p), df_0(0))$$

of $C^\infty(L(n,p),\ df_0(0))$ are equal. From this it follows that

$$(\rho\circ d_x f)^*\mathcal{M}_{p-n+1}\mathcal{E}_{n+a} = (\rho'\circ d_x f)^*\mathcal{M}_{p-n+1}\mathcal{E}_{n+a}$$

(where $d_x f$ is the differential of f with respect to x), and so the two versions of $\tilde{d}f$ are $\mathcal{C}$-equivalent. The same goes for $\bar{d}F$. We shall be interested only in the $\mathcal{K}$-class of $\tilde{d}f$ and $\bar{d}F$, and so the ambiguity in the definition will not trouble us.

2) The local algebra of $\tilde{d}f$ is just the algebra of contact of $d_x f$ with $\Sigma^1(n,p)$, as studied in I.9.

Using this definition, we may restate II.1:1 as

II.2:3 <u>Proposition</u> $TM = \tilde{d}p_g^{-1}(0) = \bar{d}P_g^{-1}(\{0\}\times\mathbb{R}^p)$ and

$\exp_g$: $TM \cap \text{domain}(P_g) \longrightarrow \mathbb{R}^p$ is just $\pi : \bar{d}P_g^{-1}(\{0\}\times\mathbb{R}^p) \longrightarrow \mathbb{R}^p$ ∎

We are therefore led to study projections $\pi : F^{-1}(\{0\}\times\mathbb{R}^a) \longrightarrow \mathbb{R}^a$, where $F : \mathbb{R}^n\times\mathbb{R}^a \longrightarrow \mathbb{R}^p\times\mathbb{R}^a$ is an unfolding. As a starting point we have the theorem of Martinet ([16] page 27, though see also [30] page 502 for this formulation)

II.2:4 <u>Theorem</u> Let $F:(\mathbb{R}^n\times\mathbb{R}^a,0)\longrightarrow(\mathbb{R}^p\times\mathbb{R}^a,0)$

$$(x,u)\longmapsto(f(x,u),\ u) = (f_u(x),u)$$

be a regular unfolding of f_0 (i.e. $F \pitchfork \{0\}\times\mathbb{R}^a$), and let $V_F = F^{-1}(\{0\}\times\mathbb{R}^a)$. Then the projection $\pi:(V_F,0)\longrightarrow(\mathbb{R}^a,0)$ is an $\mathcal{A}$-stable map-germ if and only if F is a $\mathcal{K}_e$-versal unfolding of f_0 ∎

It follows immediately from this theorem and from II.2:3 that the germ of $\exp_g$ at (x_0,q_0) is $\mathcal{A}$-stable if and only if $\overline{d}p_g$ is a $\mathcal{K}_e$-versal unfolding of the germ of $\tilde{d}p_g(q_0)$ at x_0. However, in order to prove Theorem II:1, we need a more general result. First we give a characterisation of $\mathcal{K}_e$-versality in terms of transversality:

II.2:5 <u>Proposition</u> Let $F:(\mathbb{R}^n\times\mathbb{R}^a,0)\longrightarrow(\mathbb{R}^p\times\mathbb{R}^a,0)$

$$(x,u)\longmapsto(f(x,u),u) = (f_u(x),u)$$

be a regular unfolding of f_0. Then F is $\mathcal{K}_e$-versal if and only if the restriction to V_F of the map

$$j^k_x f : (\mathbb{R}^n\times\mathbb{R}^p,0)\longrightarrow J^k(\mathbb{R}^n,\mathbb{R}^p)$$
$$(x,u)\longmapsto j^k f_u(x)$$

is transverse to the contact class containing $j^k f_0(0)$, where $k\geqslant a+1$.

<u>Proof</u> F is a $\mathcal{K}_e$-versal unfolding of f_0 if and only if

(1) $$tf_0(\mathcal{M}_n\Theta(n)) + f_0^*\mathcal{M}_p\Theta(f_0) + \mathbb{R}\left\{\frac{\partial f_0}{\partial x_i}, \frac{\partial f_0}{\partial u_j}\Big|_{u=0}\right\}_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant a}} + \mathcal{M}_n^{a+2}\Theta(f_0)$$
$$= \Theta(f_0).$$

(See [15], Sec.2, p.125) Note that (1) implies that F is regular. Now let $\{\alpha_{i,j}\}_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n+a}}$ and $\{\beta_{l,j}\}_{\substack{1\leqslant l\leqslant a\\ 1\leqslant j\leqslant n+a}}$ be real numbers such that if

$$\partial_j = \sum_i \alpha_{i,j}\frac{\partial}{\partial x_i} + \sum_l \beta_{l,j}\frac{\partial}{\partial u_j}$$

then $\partial_1, \dots, \partial_{n+a-p}$ is a basis for $T_{(0,0)}V_F$ and

$\partial_1, \dots, \partial_{n+a}$ is a basis for $T_{(0,0)}\mathbb{R}^n\times\mathbb{R}^a$.

Then $\partial_{n+a-p+1}f(0), \ldots, \partial_{n+a}f(0)$ is a basis for T_0R^p, and so (1) is equivalent to

$$(2) \quad tf_0(\mathcal{M}_n\theta(n)) + f_0{}^*\mathcal{M}_p\theta(f_0) + \mathbb{R}\left\{\partial_1 f|_{u=0}, \ldots, \partial_{n+a-p}f|_{u=0}\right\}$$

$$+\mathcal{M}_n^{a+2}\theta(f_0) = \mathcal{M}_n\theta(f_0).$$

But (2) is simply the statement that $(j_x^{a+1}f)|_{V_F}$ is transverse to the $\mathcal{K}^{a+1}$-orbit (contact class) of $j^{a+1}f_0(0)$ ∎

Since we know ([20] page 227) that $\pi:(V_F, 0)\longrightarrow(\mathbb{R}^a,0)$ is stable if and only if

$$j^{a+1}(\pi|_{V_F}) \pitchfork \mathcal{K}^{a+1}(j^{a+1}(\pi|_{V_F})(0,0))$$

in $J^{a+1}(V_F, R^a)$, we can restate Martinet's Theorem (II.2:4) as

II.2:6 Theorem With the hypotheses of II.2:4, we have

$$(j_x^{a+1}f)|_{V_F} \pitchfork \mathcal{K}^{a+1}(j^{a+1}f_0(0))$$

if and only if

$$j^{a+1}(\pi|_{V_F}) \pitchfork \mathcal{K}^{a+1}(j^{a+1}(\pi|_{V_F})(0,0))$$ ∎

As a final preparatory step for the generalisation of II.2:4, we have

II;2:7 Lemma Let $F:(\mathbb{R}^n\times\mathbb{R}^a,0)\longrightarrow(\mathbb{R}^p\times\mathbb{R}^a,0)$

$$(x,u)\longmapsto(f(x,u),u) = (f_u(x), u)$$

be a regular unfolding. Then for $(x,u) \in V_F$, the local algebra $Q_{(\pi|_{V_F})}(x,u)$ is isomorphic to the local algebra $Q_{f_u}(x)$.

<u>Proof</u> Assume, for simplicity of notation, and without loss of generality, that $(x,u) = (0,0)$. We have

$$Q_{(\pi|V_F)}(0,0) = \frac{C^{\infty}(V_F,(0,0))}{(\pi|_{V_F})^* \mathfrak{m}_a C^{\infty}(V_F,(0,0))}$$

$$\cong \frac{\mathcal{E}_{n+a}\big/ f^*\mathfrak{m}_p \mathcal{E}_{n+a}}{\pi^*\mathfrak{m}_a\left[\mathcal{E}_{n+a}\big/ f^*\mathfrak{m}_p \mathcal{E}_{n+a}\right]}$$

$$\cong \frac{\mathcal{E}_{n+a}}{\left[\pi^*\mathfrak{m}_a + f^*\mathfrak{m}_p\right]\mathcal{E}_{n+a}}$$

$$\cong \frac{\mathcal{E}_n}{f_0{}^*\mathfrak{m}_p \mathcal{E}_n} \qquad = Q_{f_0}(0) \blacksquare$$

II.2:6 and II.2:7 suggest a generalisation of Martinet's Theorem, (our Theorem II·2:12) which in fact provides the principal step in the proof of Theorem II:1. Before stating it we need some preparatory definitions and lemmas. First, we establish a correspondence between smooth $\mathcal{K}$-invariant submanifolds of different jet-bundles. If $W \subseteq J^k(N,P)$ is a smooth, $\mathcal{K}$-invariant submanifold, then W is a union of contact classes W_λ corresponding to isomorphism types of local algebras Q_λ , $\lambda \in \Lambda$. Let s be any integer greater than min. $\{-n, -p\}$ and let $\tilde{N}$, $\tilde{P}$ be smooth manifolds such that dim.$\tilde{N}$ = n+s, dim.$\tilde{P}$ = p+s, where n and p are the dimensions of N and P respectively. Define $\tilde{W} \subseteq J^k(\tilde{N}, \tilde{P})$ to be the union of contact classes in $J^k(\tilde{N}, \tilde{P})$ corresponding to the same set of local algebra isomorphism types Q_λ , $\lambda \in \Lambda$.(Of course if $s < 0$ then some of these contact classes may be empty.)

II.2:8 Lemma $\tilde{W}$, as defined in the preceeding paragraph, is a smoothly embedded submanifold of $J^k(\tilde{N}, \tilde{P})$, whose codimension is the same as that of W in $J^k(N, P)$, provided that it is non-empty.

Proof Since W and $\tilde{W}$ are locally trivial fibre bundles over $N \times P$ and $\tilde{N} \times \tilde{P}$ respectively, it will be enough to consider the fibre of W and of $\tilde{W}$, which we shall continue to refer to as W and $\tilde{W}$, and so we work in $J^k(n,p)$ and $J^k(n+s,p+s)$. There are two cases:

i) $s \leq 0$. If $s = 0$, there is nothing to prove, so assume $s < 0$. Write $b = -s$, and define an immersion $J^k(n-b,p-b) \xrightarrow{i} J^k(n,p)$ as follows: if $z = j^k f(0)$ for some map-germ $f:(\mathbb{R}^{n-b},0) \longrightarrow (\mathbb{R}^{p-b},0)$, then define $F:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ by $F(x_1,\ldots, x_n) = (f(x_1,\ldots, x_{n-b}), x_{n-b+1},\ldots, x_n)$ and let $i(z) = j^k F(0)$. Clearly the local algebras of f and F are isomorphic, so we have

$$i(\tilde{W}) = W \cap i(J^k(n-b,p-b)).$$

It is only necessary now to show that $W \overline{\pitchfork}\, i(J^k(n-b,p-b))$. In fact this is straightforward; indeed, if W_λ is any contact class contained in W then $W_\lambda \overline{\pitchfork}\, i(J^k(n-b,p-b))$. To see this, identify the tangent space to $J^k(n,p)$ with

$$\mathcal{m}_n \mathcal{E}_{n,p} \big/ \mathcal{m}_n^{k+1} \mathcal{E}_{n,p}$$

and that of $J^k(n-b,p-b)$ with

$$\mathcal{m}_{n-b} \mathcal{E}_{n-b,p-b} \big/ \mathcal{m}_{n-b}^{k+1} \mathcal{E}_{n-b,p-b} \quad .$$

If we write elements of either of these as column vectors having, respectively, p and p-b entries, and, in order to simplify notation, abandon reference to $\mathfrak{m}_n^{k+1}\mathcal{E}_{n,p}$ and $\mathfrak{m}_{n-b}^{k+1}\mathcal{E}_{n-b,p-b}$, then the tangent space to $i(J^k(n-b,p-b))$ at any point becomes

$$\begin{bmatrix} \mathfrak{m}_{n-b} \\ \vdots \\ \mathfrak{m}_{n-b} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \uparrow \\ p-b \\ \downarrow \\ \uparrow \\ b \\ \downarrow \end{matrix}$$

Now if W_λ does not meet $i(J^k(n-b,p-b))$, transversality holds vacuously, so assume the contrary and let $z \in W_\lambda \cap i(J^k(n-b.p-b))$. Then z is the k-jet of a map germ of the form $F:(x_1,\ldots,x_n) \longrightarrow (f(x_1,\ldots,x_{n-b}), x_{n-b+1},\ldots,x_n)$.

Then

$$T_zW = \mathfrak{m}_n\left\{\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ 0 \end{bmatrix},\ldots,\begin{bmatrix} \frac{\partial f}{\partial x_{n-b}} \\ 0 \end{bmatrix}, e_{p-b+1},\ldots,e_p\right\}$$

$$+ \left[f^*\mathfrak{m}_{p-b} + \langle x_{n-b+1},\ldots,x_n\rangle\right]\mathcal{E}_{n,p} \quad .$$

where e_i is the i^{th} vector in the natural basis of $\mathbb{R}^p$. Thus,

$$T_zW \supseteq \begin{bmatrix} 0 \\ \mathfrak{m}_n\,\mathcal{E}_{n,b} \end{bmatrix} + \begin{bmatrix} \mathfrak{m}_b\,\mathcal{E}_{n,p-b} \\ \mathfrak{m}_b\,\mathcal{E}_{n,b} \end{bmatrix}$$

where $\mathfrak{m}_b = \langle x_{n-b+1},\ldots,x_n\rangle$. From this, and from the formula for $T_z i(J^k(n-b,p-b))$, the transversality of W and $i(J^k(n-b,p-b))$ follows.

ii) <u>$s>0$</u> Define a similar immersion $i:J^k(n,p) \longrightarrow J^k(n+s,p+s)$. As before, we have $i(W) = \tilde{W} \cap i(J^k(n,p))$. Now, $\tilde{W}$ is the union of the orbits under the contact group of elements in $i(W)$, and in order to show that it is a smoothly immersed submanifold of $J^k(n+s,p+s)$it is enough to show that the map

$$\Gamma : \mathcal{X}^k \times i(W) \longrightarrow J^k(n+s,p+s),$$

determined by the action of $\mathcal{X}^k$, has constant rank. Let $z \in W_\lambda$; then the rank of Γ at $(H,i(z)) \in \mathcal{X}^k \times i(W)$ is

$$\dim T_{H(i(z))}\widetilde{W}_\lambda + \dim T_{H(i(z))}H(i(W)) - \dim T_{H(i(z))}\widetilde{W}_\lambda \cap T_{H(i(z))}H(i(W))$$

$$= \dim \widetilde{W}_\lambda + \dim W - \dim T_{i(z)}\widetilde{W}_\lambda \cap T_{i(z)}i(W).$$

However, since $i(W_\lambda) = \widetilde{W}_\lambda \cap i(W) = \widetilde{W}_\lambda \cap i(J^k(n,p))$, and the second of these intersections is transverse, we have

$$T_{i(z)}i(W\ ) \subseteq T_{i(z)}\widetilde{W}_\lambda \cap T_{i(z)}i(W)$$

$$T_{i(z)}\widetilde{W}_\lambda \cap T_{i(z)}i(J^k(n,p)) = T_{i(z)}(\widetilde{W}_\lambda \cap i(J^k(n,p)))$$

$$= T_{i(z)}i(W_\lambda).$$

So in particular

$$T_{i(z)}\widetilde{W}_\lambda \cap T_{i(z)}i(W) = T_{i(z)}\widetilde{W}_\lambda \cap T_{i(z)}i(J^k(n,p)),$$

and the rank of Γ at $(H,i(z))$ is equal to

$$\dim W + \dim \widetilde{W}_\lambda - \dim \widetilde{W}_\lambda \cap i(J^k(n,p))$$

$$= \dim W + \operatorname{codim} i(J^k(n,p)).$$

This proves that W is a smoothly immersed submanifold of $J^k(n+s,p+s)$ whose codimension is equal to that of W in $J^k(n,p)$. That W is embedded now follows from the fact that each $\mathcal{X}^k$ orbit $\widetilde{W}_\lambda$ is embedded ([21] page 305) that for differing λ and μ , $\widetilde{W}_\lambda$ and $\widetilde{W}_\mu$ are disjoint, and from the fact that the adjacency relations among the $\widetilde{W}_\lambda$ are the same as those among the W_λ ∎

II.2:9 Theorem Let $F:(\mathbb{R}^n \times \mathbb{R}^a, 0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^a, 0)$

$$(x,u) \longrightarrow (f(x,u),u) = (f_u(x),u)$$

be a regular unfolding.

i) If $W \subseteq J^k(V_F, \mathbb{R}^a)$ is a smooth, $\mathcal{K}$-invariant submanifold, let $\tilde{W} \subseteq J^k(\mathbb{R}^n, \mathbb{R}^p)$ be defined as in II.2:8. Then

$$j^k(\pi|_{V_F}) \overline{\pitchfork} W \quad \text{if and only if} \quad (j^k_x f)|_{V_F} \overline{\pitchfork} \tilde{W}.$$

ii) If $W \subseteq J^k(\mathbb{R}^n, \mathbb{R}^p)$ is a smooth, $\mathcal{K}$-invariant submanifold, let $\tilde{W} \subseteq J^k(V_F, \mathbb{R}^a)$ be defined as in II.2:8. Then

$$j^k(\pi|_{V_F}) \overline{\pitchfork} \tilde{W} \quad \text{if and only if} \quad (j^k_x f)|_{V_F} \overline{\pitchfork} W.$$

Remark Since for $(x,u) \in V_F$, the local algebra of $\pi|_{V_F}$ at (x,u) is isomorphic to the local algebra of f_u at x, these two statements mean the same thing in practice. However they are not identical inasmuch as if we start with $W \subseteq J^k(\mathbb{R}^n, \mathbb{R}^p)$, define $\tilde{W} \subseteq J^k(V_F, \mathbb{R}^a)$ as in II.2:8 and then use $\tilde{W}$ to define "$\tilde{\tilde{W}}$" $\subseteq J^k(\mathbb{R}^n, \mathbb{R}^p)$, then $\tilde{\tilde{W}}$ is not necessarily equal to W, although it is contained in it, since if $a < p$, some $\tilde{W}_\lambda$ may be empty.

Proof of II.2:9 i) Let $\bar{F}:(\mathbb{R}^n \times \mathbb{R}^a \times \mathbb{R}^b, 0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^a \times \mathbb{R}^b, 0)$

$$(x,u,v) \longrightarrow (\bar{f}(x,u,v),u,v) = (\bar{f}_{u,v}(x),u,v),$$

where $\bar{f}(x,u,0) = f(x,u)$, be a $\mathcal{K}^k_e$-versal unfolding of f_0, in the sense that

$$T_e \mathcal{K} f_0 + \mathbb{R}\{\dot{\bar{F}}_1; \ldots, \dot{\bar{F}}_{a+b}\} + \mathfrak{m}_n^{k+1}\theta(f_0) = \theta(f_0).$$

Such a $\mathcal{K}_e^k$- versal unfolding will always exist, even if f_0 is not

$\mathcal{K}$-finite, since we can always replace f_0 by a $\mathcal{K}$-finite map-germ having the same k-jet as f_0 ([30], Theorem 5.1). Now let $\overline{W} \subseteq J^k(V_{\overline{F}}, \mathbb{R}^a \times \mathbb{R}^b)$ be defined, like $\widetilde{W}$, as the union of contact classes corresponding to the same local algebra isomorphism-types as W.

If $j^k_x f(0,0) \notin \widetilde{W}$, then $j^k \pi(0,0) \notin W$ and transversality holds vacuously, so assume $j^k_x f(0,0) \in \widetilde{W}$. Since $\overline{F}$ is $\mathcal{K}_e$-versal, by II.2:5 we have

$$(j^k_x \overline{f})|_{V_{\overline{F}}} \,\overline{\pitchfork}\, \mathcal{K}^k(j^k f_0(0)),$$

from the fact that $\widetilde{W}$ is $\mathcal{K}$-invariant we have $\widetilde{W} \supseteq \mathcal{K}^k(j^k f_0(0,0))$, and so

$$(j^k_x \overline{f})|_{V_{\overline{F}}} \,\overline{\pitchfork}\, \widetilde{W}.$$

Moreover, since $j^k(\overline{\pi}|_{V_{\overline{F}}})$ is the k-jet of a stable map-germ (by II.2:4), a similar argument shows that

$$j^k(\overline{\pi}|_{V_{\overline{F}}}) \,\overline{\pitchfork}\, \overline{W}.$$

Note that by II.2:7,

$$(j^k \overline{\pi}|_{V_{\overline{F}}})^{-1}(\overline{W}) = (j^k_x \overline{f})^{-1}(\widetilde{W}) \cap V_{\overline{F}} \quad .$$

Now suppose that $(j^k_x f)|_{V_F} \,\overline{\pitchfork}\, \widetilde{W}$. Choose a transversal X to $(j^k_x f)^{-1}(\widetilde{W}) \cap V_F$ in V_F, with dim X = codim $\widetilde{W}$. Then we claim that for no vector $\hat{x} \in T_{(0,0)}X$, $\hat{x} \neq 0$, is

$$dj^k(\pi|_{V_F})(0,0)(\hat{x}) \in T_w W$$

where $w = j^k(\pi|_{V_F})(0,0)$. For if this were the case, then since $j^k(\overline{\pi}|_{V_{\overline{F}}}) \,\overline{\pitchfork}\, \overline{W}$, there exists a smooth curve $\gamma(t)$ in $j^k(\overline{\pi}|_{V_{\overline{F}}})^{-1}(\overline{W})$ with $\gamma'(0) = \hat{x}$. By II.2:7, $j^k_x \overline{f}(\gamma(t)) \in \widetilde{W}$ for all t, and so we have

$dj_x^k\bar{f}(0,0,0)(\hat{x}) \in T_{\tilde{w}}\tilde{W}$, where $\tilde{w} = j^k f_0(0)$. Since $(j_x^k\bar{f})|V_F = (j_x^k f)|V_F$, this implies that $dj_x^k f(0,0)(\hat{x}) \in T_{\tilde{w}}\tilde{W}$. However, this last is absurd, by the definition of X. We conclude, from the fact that codim $\tilde{W}$ is equal to codim W, that

$$j^k(\pi|V_F) \bar{\pitchfork} W,$$

and so we conclude that

$$(j_x^k f)|V_F \bar{\pitchfork} \tilde{W} \implies j^k(\pi|V_F) \bar{\pitchfork} W.$$

The proof of the reverse implication is practically identical.

ii) If $j_x^k f(0,0) \notin W$ then transversality holds vacuously. If $j_x^k f(0,0) \in W$, then there is nothing to distinguish this case from (i), and so the same proof applies∎

The Multi-germ Case

Multi-germ versions of II.2:4 - II.2:9 hold, and in fact all, with the exception of the multi-germ version of II.2:4 (Martinet's Theorem), follow directly from the corresponding results for "mono-germs". We have preferred to state and prove the mono-germ versions first, as they are both notationally and conceptually simpler.

First we give the multi-germ version of II.2:8. In order to define the desired correspondence between $\mathcal{K}$-invariant submanifolds of ${}_rJ^k(\tilde{N},\tilde{P})$ and ${}_rJ^k(N,P)$, where dim $\tilde{P}$ - dim $\tilde{N}$ = dim P - dim N, note that since the contact group action on ${}_rJ^k(n,p)$ is just the r-fold product of the contact group action on $J^k(n,p)$, it follows that any $\mathcal{K}$-invariant submanifold of ${}_rJ^k(n,p)$ is of the form $W_1\times\ldots\times W_r$, with each W_i a contact invariant submanifold of $J^k(n,p)$. Define the corresponding submanifold $\tilde{W}$ of ${}_rJ^k(n+s,p+s)$ as $\tilde{W}_1\times\ldots\times\tilde{W}_r$, where each $\tilde{W}_i$ is defined as before. This done, we have

II.2:10 Lemma $\tilde{W}$, as defined in the preceeding paragraph, is a smoothly embedded submanifold of ${}_rJ^k(N,P)$.

Proof Obvious∎

If $F: (\mathbb{R}^n\times\mathbb{R}^a,\ S\times\{0\})\longrightarrow(\mathbb{R}^p\times\mathbb{R}^a,\ f_0(S)\times\{0\})$ is a regular unfolding of the multi-germ $f_0: (\mathbb{R}^n,S)\longrightarrow(\mathbb{R}^p,f(S))$, where $S\subseteq\mathbb{R}^n$ is a set consisting of r points ($r<\infty$), and $W\subseteq{}_rJ^k(\mathbb{R}^n,\mathbb{R}^p)$ is a contact-invariant submanifold, then defining $\tilde{W}\subseteq{}_rJ^k(V_F,\mathbb{R}^a)$ as above, it follows immediately from II.2:7 that for $S' = \{(x_1,u_1),\ \ldots,\ (x_r,u_r)\}\subseteq V_F$,

$${}_rj^k_xf(S')\in W \iff {}_rj^k(\pi|_{V_F})(S')\in\tilde{W}.$$

Here $V_F = F^{-1}\ (f_0(S)\times\mathbb{R}^a)$, and by a regular unfolding we mean one that is transverse to $f_0(S)\times\mathbb{R}^a$.

The corresponding result when we start with W in ${}_rJ^k(V_F,\mathbb{R}^a)$ and define $\tilde{W}\subseteq{}_rJ^k(\mathbb{R}^n,\mathbb{R}^p)$ as above, is also an immediate consequence of II.2:7.

In order to prove the multi-jet version of II.2:9 we need first a multi-jet version of Martinet's Theorem. And in fact, since Martinet's proof of his theorem works equally well in the multi-jet version, we shall state the result without further proof

II.:11 Theorem Let $F : (\mathbb{R}^n\times\mathbb{R}^a,S\times\{0\})\longrightarrow(\mathbb{R}^p\times\mathbb{R}^a,\ f_0(S)\times\{0\})$ be a regular unfolding of the multi-germ f_0. Then F is $\mathcal{K}_e$-versal if and only if the multi-germ $\pi:(V_F, S\times\{0\})\longrightarrow(\mathbb{R}^a,0)$ is stable∎

From this we may deduce the multi-jet version of II.2:9

II.2:12 Theorem Let F be as in II.2:11.

i) If $W\subseteq {}_rJ^k(V_F,\mathbb{R}^a)$ is a smooth, $\mathcal{K}$-invariant submanifold, let $\tilde{W}\subseteq {}_rJ^k(\mathbb{R}^n,\mathbb{R}^p)$ be defined as in II.2:10. Then

$$ {}_rj^k(\pi|_{V_F}) \,\bar{\pitchfork}\, W \Longleftrightarrow ({}_rj^k_x f)|_{V_F} \,\bar{\pitchfork}\, \tilde{W}. $$

ii) If $W\subseteq {}_rJ^k(\mathbb{R}^n,\mathbb{R}^p)$ is a smooth, $\mathcal{K}$-invariant submanifold, let $\tilde{W}\subseteq {}_rJ^k(V_F,\mathbb{R}^a)$ be defined as in II.2:10. Then

$$ {}_rj^k(\pi|_{V_F}) \,\bar{\pitchfork}\, \tilde{W} \Longleftrightarrow ({}_rj^k_x f)|_{V_F} \,\bar{\pitchfork}\, W. $$

Proof As in II.2:9∎

This theorem will be our principal singularity-theoretic tool in the proof of II:1. We now return to the geometrical context of the exponential map.

II.3

The results of the preceeding section, in as much as they apply to the study of $\exp_g:TM\text{-}M\longrightarrow\mathbb{R}^p$, may be summarised as follows:

II.3:1 Proposition With the notation of II.1 and II.2, let $W\subseteq {}_rJ^k(TM,\mathbb{R}^p)$ be a smooth, $\mathcal{K}$-invariant manifold, and let $\tilde{W}\subseteq {}_rJ^k(M,\mathbb{R}^{p-n})$ be the corresponding $\mathcal{K}$-invariant submanifold, as in II.2:10. Then

$${}_rj^k(\exp_g|_{(\text{domain } P_g)\cap TM}) \pitchfork W \iff ({}_rj^k_x\tilde{d}p_g)|_{((\text{domain } P_g)\cap TM)^{(r)}} \pitchfork \tilde{W}.$$

Similarly, if $W \subseteq {}_rJ^k(M, \mathbb{R}^{p-n})$ is a $\mathcal{K}$-invariant submanifold, and we define $\tilde{W} \subseteq {}_rJ^k(TM,\mathbb{R}^p)$ as in II.2:10, then

$${}_rj^k(\exp_g|_{(\text{domain } P_g)\cap TM}) \pitchfork \tilde{W} \iff ({}_rj^k_x\tilde{d}p_g)\ _{((\text{domain } P_g)\cap TM)^{(r)}} \pitchfork W.$$

Proof The statement of this proposition is just a translation of II.2:12 into the context of $\exp_g$. For $\overline{d}P_g$ is a regular unfolding of $\tilde{d}p_g$, (domain P_g)$\cap$ TM is just $V_{\overline{d}P_g}$, and $\exp_g$ is just $\pi : V_{\overline{d}P_g} \longrightarrow \mathbb{R}^p$ ∎

Leaving aside for the moment the problem of going from a local to a global result, which is due to the fact that domain P_g is not all of $M \times \mathbb{R}^p$ and does not, in general, contain all of TM-M, we see from II.3:1 that in order to prove that for a residual set of $g \in \text{Imm}(M,\mathbb{R}^p)$, ${}_rj^k(\exp_g|_{TM-M}) \pitchfork W$, we have to prove the corresponding statement about ${}_rj^k_x\tilde{d}p_g$. And in fact this is not difficult: it is an essentially easy consequence of the following transversality theorem, due to Soares David ([28] page 742), (although we give here a slightly sharper version).

II.3:2 Theorem Let M be a smooth manifold, let H be an affine hyperplane in $\mathbb{R}^p$ ($p >$ dim M) with associated height function h, and for an immersion $g:M \longrightarrow \mathbb{R}^p$ let P_g be the family of radial projections of M into H, as in II.1. Let

$${}_rG_g(H) = \{(x_1, \ldots, x_r, q) \in M^{(r)} \times \mathbb{R}^p : h(g(x_i)) \neq h(q),\ i = 1,\ldots,r;\ h(q) \neq 0\}.$$

Then for any smooth manifold $W \subseteq {}_rJ^k(M,H)$, the set

$$\{g \in C^{\infty}(M,\mathbb{R}^p) : {}_rj^k p_g|_{{}_rG_g(H)} \pitchfork W\}$$

is residual in $C^{\infty}(M,\mathbb{R}^p)$.

Proof We follow closely the proof of Soares David's Theorem that Wall gives in [28] , and indeed the only difference between our version and his is that we remove the restriction that the maps $g \in C^{\infty}(M,\mathbb{R}^p)$ that one considers should map M into one component of $\mathbb{R}^p$-H while the points q of projection should lie in the other.

As in Wall's proof, consider first the case where r = 1, and try to find a sufficiently large space A of perturbations of g, such that by varying g in A we obtain a submersion

$$A \times M \times \mathbb{R}^p \xrightarrow{j_x^k p_g} J^k(M,H).$$

The difficulty is, of course, that $p_g(x,q)$ is not defined for all (x,q), and moreover that the set ${}_1G_g(H)$ of (x,q) for which p_g is defined varies with g. To obviate this, set

$${}_1G(H) = \{(g,x,q) \in A \times M \times \mathbb{R}^p : (x,q) \in {}_1G_g(H)\}.$$

Clearly ${}_1G_g(H) = {}_1G(H) \cap \{g\} \times M \times \mathbb{R}^p$, and also that ${}_1G(H)$ is open. Hence, by choosing A so that one obtains a submersion

$${}_1G(H) \xrightarrow{j^k p_g} J^k(M,H),$$

one can deduce that for a set of g in A whose complement is of measure 0,

$$j_x^k p_g|_{{}_1G_g(H)} \pitchfork W.$$

We define A as follows: let $i:M \longrightarrow \mathbb{R}^N$ be a k-th order non-degenerate embedding (see [28] pages 721 and 722 for definitions and for proof of the existence of such embeddings) and let

$$B = \mathrm{Aff}(\mathbb{R}^N, H),$$

i.e. the space of affine maps from $\mathbb{R}^N$ to H. Now define A by

$$A = \{ g + \varphi \cdot i \ : \ \varphi \in B \}.$$

Then one checks, exactly as in Wall's proof, that one does indeed obtain a submersion

$${}_1G(H) \xrightarrow{j^k p_g} J^k(M,H).$$

From this it follows, by Sard's Theorem, that for a set of g in A whose complement is of measure 0, $j^k p_g \,\overline{\pitchfork}\, W$.

The rest of the proof is exactly as in [28]. The multi-jet version is proved similarly, replacing ${}_1G(H)$ by ${}_rG(H)$■

We now show that the transversality of $({}_rj^k_x \tilde{d}p_g)|(TM-M)^{(r)}$ to a given $\mathcal{K}$-invariant submanifold of ${}_rJ^k(M,\mathbb{R}^{p-n})$ is equivalent to the transversality of

$${}_rj^k_x p_g \ : \ G_g(H) \longrightarrow {}_rJ^k(M,H)$$

to a certain submanifold of ${}_rJ^k(M,H)$. We make use of the following lemma in elementary differential topology:

II.3:3 <u>Lemma</u> Let $f:N \longrightarrow P$ be a smooth map, N and P smooth manifolds, and suppose that $f \overline{\pitchfork} Y$, where Y is a smooth submanifold of P. If Z is another smooth submanifold of P such that $Z \overline{\pitchfork} Y$, then

$$f|_{f^{-1}(Y)} : f^{-1}(Y) \longrightarrow P \text{ is transverse to } Z \Longleftrightarrow f \overline{\pitchfork} Y \cap Z.$$

Proof First it is clear that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

For $x \in f^{-1}(Y \cap Z)$ and $y = f(x)$, we have

$$(1) \qquad f \overline{\pitchfork} Y \cap Z \text{ at } x \Longleftrightarrow df(x)(T_xN) + T_yY \cap T_yZ = T_yP$$

and

$$(2) \qquad f|_{f^{-1}(Y)} \overline{\pitchfork} Z \text{ at } x \Longleftrightarrow df(x)(T_xN) \cap T_yY + T_yZ = T_yP.$$

Intersect both sides of (1) with T_yY to obtain

$$(1') \qquad df(x)(T_xN) \cap T_yY + T_yY \cap T_yZ = T_yY.$$

If we now add $df(x)(T_xN)$ to both sides of (1'), we obtain (1), so we have (1)$\Longleftrightarrow$(1'). Intersecting both sides of (2) with T_yY also gives (1'), and then adding T_yZ to both sides of (1') gives (2), so that (2) also is equivalent to (1')∎

From this result we see that, with the notation and definitions of II.2:12,

$$({}_rj^k_xf)|_{V_F} \overline{\pitchfork} W \subseteq {}_rJ^k(R^n,R^p) \Longleftrightarrow {}_rj^k_xf \overline{\pitchfork} W \cap \beta^{-1}(f(x_1),\ldots, f(x_r))$$

where $\beta : {}_rJ^k(R^n,R^p) \longrightarrow (R^p)^r$ is just the target projection of the jet bundle. Note that since W is a locally trivial fibre bundle over $(R^n)^{(r)} \times (R^p)^r$, it follows that $W \overline{\pitchfork} \beta^{-1}(f(x_1),\ldots, f(x_r))$; it is also clear that $V_F = ({}_rj^k_xf)^{-1}(\beta^{-1}(f(x_1), \ldots, f(x_r)))$.

II.3:4 Lemma Let $W \subseteq {}_rJ^k(M, \mathbb{R}^{p-n})$ be a smooth, $\mathcal{K}$-invariant submanifold. Then there exists a smooth submanifold W' of ${}_rJ^{k+1}(M,H)$ such that

i) ${}_rj^{k+1}p_g((x_1, q_1), \ldots, (x_r, q_r)) \in W'$

if and only if

$${}_rj^k_x \,\tilde{d}p_g((x_1, q_1), \ldots, (x_r, q_r)) \in W,$$

for $(x_1, q_1), \ldots, (x_r, q_r) \in TM \cap \text{domain}(P_g)$.

ii) ${}_rj^{k+1}_x p_g \pitchfork W' \Longleftrightarrow ({}_rj^k_x \,\tilde{d}p_g)\big|(TM \cap \text{domain}(P_g))^{(r)} \pitchfork W.$

Proof We define a submersion

$${}_r\tilde{d}_k : {}_rJ^{k+1}(M,H) - (\Sigma^2 \cup \ \ldots \cup \Sigma^n) \longrightarrow {}_rJ^k(M,\mathbb{R}^{p-n})$$

such that

$${}_r\tilde{d}_k \circ ({}_rj^{k+1}p_g) = {}_rj^k \tilde{d}p_g,$$

and then set

$$W' = ({}_r\tilde{d}_k)^{-1}(W \cap \beta^{-1}(0, \ldots, 0).$$

To begin with, we define ${}_1\tilde{d}_k$, which we shall denote simply by $\tilde{d}_k$. In fact it is not possible to make a global definition, so we work locally, but as we shall see, this causes no problems.

Suppose then that $z_0 = j^{k+1}f(x_0) \in J^{k+1}(M,H)$ and that rank $df(x_0) \geq n-1$. Choose a submersion $\rho : U \longrightarrow \mathbb{R}^{p-n}$, where U is a neighbourhood of $df(x_0)$ in $L(n,p-1) = L(R^n,H)$, with U contained in $\Sigma^0 \cup \Sigma^1$, such that

$$\Sigma^1 = \rho^{-1}(0).$$

Then set $\tilde{d}_k(z) = j^k(\tilde{d}\ f)(x_0)$. Evidently $z \in \Sigma^1 \Longleftrightarrow \tilde{d}_k(z) \in \beta^{-1}(0)$, where β is the target projection of $J^k(M,\mathbb{R}^{p-n})$. Clearly there is ambiguity in the definition of $\tilde{d}_k$, just as there was in the definition of $\tilde{d}$, but for exactly the same reason as in that case, it will not matter: if ρ and ρ' are two submersions such that $\Sigma^1 = \rho^{-1}(0) = \rho'^{-1}(0)$, then $\rho \cdot df$ and $\rho' \cdot df$ are $\mathcal{C}$-equivalent, so that in particular $\tilde{d}_k(z)$ and $\tilde{d}_k'(z)$ lie in the same $\mathcal{K}$-orbit in $J^k(M,\mathbb{R}^{p-n})$. Since we also have $\tilde{d}_k(z) \in \beta^{-1}(0)$ if and only if $\tilde{d}_k'(z) \in \beta^{-1}(0)$, and our purpose in introducing $\tilde{d}_k$ is to pull back $W \cap \beta^{-1}(0)$, where W is a $\mathcal{K}$-invariant submanifold of $J^k(M,\mathbb{R}^{p-n})$, the ambiguity in the definition of d_k will not be an obstacle.

Now, it is clear that $\tilde{d}_k$, as thus defined using ρ, is well defined on a neighbourhood U_1 of z in $J^k(M,H)$.

Claim $\tilde{d}_k$ is a submersion.

Proof of claim After an appropriate choice of local coordinates on M, centred at x_0, and of local coordinates on H, we can assume that the jet z_0 is equal to $j^{k+1}f(x_0)$, where f is of the form

$$f(x_1,\ldots,x_n) = (x_1,\ldots x_{n-1},f_n(x),\ldots,\ f_{p-1}(x))$$

$$= (x_1,\ldots x_{n-1},\tilde{f}(x))$$

Then we may take $\tilde{d}f(x)$ to be equal to

$$\left(\frac{\partial f_n}{\partial x_n}(x),\ \ldots,\ \frac{\partial f_{p-1}}{\partial x_n}(x)\right),$$

and trivialising $J^k(M,\mathbb{R}^{p-n})$ locally as $\mathbb{R}^n \times \mathbb{R}^{p-n} \times J^k(n,p-n)$, we have

$$\tilde{d}_k(z) = (x, d\tilde{f}(x)(e_n), d^2\tilde{f}(x)(e_n), \ldots, d^{k+1}\tilde{f}(x)(e_n))$$

$$\in \mathbb{R}^n \times \mathbb{R}^{p-n} \times L(n,p-n) \times L_S^2(n,p-n) \times \ldots \times L_S^k(n,p-n),$$

where $L_S^i(n,p-n)$ is the space of symmetric i-linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$.

It is easy to see that in these coordinates $\tilde{d}_k$ is a submersion. However, whether or not $\tilde{d}_k$ is a submersion does not depend on the choice of coordinates or on the choice of ρ in the definition of $\tilde{d}_k$, since as was shown in I.9, if f is $\mathcal{A}$-equivalent to g then $\tilde{d}f$ is $\mathcal{K}$-equivalent to $\tilde{d}g$, so that a change of coordinates on M and H is compensated by a $\mathcal{K}^k$-equivalence (a diffeomorphism) on $J^k(M, \mathbb{R}^{p-n})$. This proves the claim.

Thus, we can define W' locally by setting

$$W' \cap U_1 = \tilde{d}_k^{-1}(W \cap \beta^{-1}(0)).$$

By adopting a similar proceedure on a neighbourhood of every k+1-jet z in $J^{k+1}(M,H) - (\Sigma^2 \cup \ldots \cup \Sigma^n)$, we obtain a well-defined submanifold W' of $J^{k+1}(M,H)$, since if U_1 and U_2 are intersecting neighbourhoods of z_1 and z_2 on which d_k is defined by using ρ and ρ', then by the construction, the definitions of W' coincide on $U_1 \cap U_2$.

By the construction, and by Lemma II.3:3, W' has the properties listed in the statement of this lemma.

Now, turning to the multi-jet case, if $W \subseteq {}_rJ^k(M,\mathbb{R}^{p-n})$ is a smooth $\mathcal{K}$-invariant submanifold, then W is of the form

$$(W_1 \times \ldots \times W_r) \cap \alpha^{-1}(M^{(r)})$$

where each W_i is a $\mathcal{K}$-invariant submanifold of $J^k(M,\mathbb{R}^{p-n})$ and

$$\alpha : {}_rJ^k(M,\mathbb{R}^{p-n})^r \longrightarrow M^r$$

is the source projection.

Define W' as

$$W_1' \times \ldots \times W_r' \cap \alpha^{-1}(M^{(r)})$$

where each W_i' is defined as in the preceeding paragraphs. Again it is clear that W' has the desired properties■

We shall refer to the submanifold W' as defined in the proof of this lemma, as ${}_r\tilde{d}_k^{-1}(W \cap \beta^{-1}(0))$.

II.3:5 <u>Theorem</u> Let $H \subseteq \mathbb{R}^p$ be a hyperplane, let $g:M \longrightarrow \mathbb{R}^p$ be an immersion, let ${}_rG_g(H)$ be defined as in II.3:2, and let $W \subseteq {}_rJ^k(TM,\mathbb{R}^p)$ be a $\mathcal{K}$-invariant submanifold. If $\tilde{W} \subseteq {}_rJ^k(M,\mathbb{R}^{p-n})$ is defined, starting from W, as in II.3:1, and

$${}_r\tilde{d}_k^{-1}(\tilde{W} \cap \beta^{-1}(0)) \subseteq {}_rJ^{k+1}(M,H)$$

is defined as in II.3:4, then

$${}_rj^k\exp_g \big| TM^{(r)} \cap {}_rG_g(H) \quad \text{is transverse to } W$$

if and only if

$${}_rj_x^{k+1}p_g : {}_rG_g(H) \rightarrow {}_rJ^{k+1}(M,H) \text{ is transverse to } {}_r\tilde{d}_k^{-1}(\tilde{W} \cap \beta^{-1}(0)).$$

<u>Proof</u> First,

$${}_rj_x^{k+1}p_g \,\overline{\pitchfork}\, {}_r\tilde{d}_k^{-1}(\tilde{W} \cap \beta^{-1}(0))$$

if and only if

$$ {}_r\tilde{d}_k \circ {}_r j_x^{k+1} p_g \;\overline{\pitchfork}\; \tilde{W} \cap \beta^{-1}(0), $$

by II.3:4. But by the definition of ${}_r\tilde{d}_k$, this is equivalent to

$$ {}_r j_x^k \tilde{d} p_g \;\overline{\pitchfork}\; \tilde{W} \cap \beta^{-1}(0), $$

and by II.3:1, this is equivalent to

$$ ({}_r j_x^k \tilde{d} p_g)|_{TM^{(r)} \cap\, {}_r G_g(H)} $$

being transverse to W. By II.2:12, this last holds if and only if

$$ {}_r j^k \exp_g : TM^{(r)} \cap {}_r G_g(H) \longrightarrow {}_r J^k(TM, \mathbb{R}^p) $$

is transverse to W∎

<u>Proof of Theorem II:1</u>

Choose a set $H_1, \ldots, H_s$ of hyperplanes in $\mathbb{R}^p$ such that for any immersion $g:M \longrightarrow \mathbb{R}^p$ the corresponding sets ${}_r G_g(H_i)$ cover

$$ X = \{(x_1, \ldots, x_r, q) \in M^{(r)} \times \mathbb{R}^p : g(x_i) \neq q \text{ for } i = 1, \ldots, r\}. $$

This may be done by taking any set of p hyperplanes in general position (so that for no (x,q) with $g(x) \neq q$ is $h_i(g(x)) = h_i(q)$ for all i) and then adding to it a further p hyperplanes parallel to but not coincident with the hyperplanes in the first set (so that we can always find H_i with $h_i(g(x)) \neq h_i(q)$ and $h_i(q) \neq 0$).

With this choice of hyperplanes, we then have 2p "maps"

$$ {}_r\tilde{d}_k \; : {}_r J^{k+1}(M, H_i) \longrightarrow {}_r J^k(M, \mathbb{R}^{p-n}) $$

and thus obtain in each bundle ${}_rJ^{k+1}(M,H_i)$ a submanifold

$$ {}_r\tilde{d}_k^{-1}(\tilde{W} \cap \beta^{-1}(0)) $$

defined as in the preceeding Theorem.

Let

$$ S_i = \{g \in C^\infty(M,\mathbb{R}^p) : {}_rj_x^{k+1}p_g : {}_rG_g(H_i) \longrightarrow {}_rJ^{k+1}(M,H_i) \text{ is } \pitchfork \text{ to } {}_r\tilde{d}_k^{-1}(\tilde{W} \cap \beta^{-1}(0)). $$

Then for each i, S_i is residual in $C^\infty(M,\mathbb{R}^p)$, by II.3:2, and so if $S = S_1 \cap \ldots \cap S_{2p}$ then S is also residual.

It follows by Theorem II.3:5 that if $g \in \mathrm{Imm}(M,\mathbb{R}^p) \cap S$, then at all points

$$ ((x_1,q), \ldots, (x_r,q)) $$

in $(TM-M)^{(r)}$, ${}_rj^k\exp_g \pitchfork W$, and from this it is an easy deduction that ${}_rj^k\exp_g : (TM-M)^{(r)} \longrightarrow {}_rJ^k(TM,\mathbb{R}^p)$ is transverse to W∎

Proof of Corollary II.2

The proof is almost exactly the one Mather gives for his Theorem 8.1 in [21] pp.326-327. The only difference is that we are not looking for globally stable maps $TM-M \longrightarrow \mathbb{R}^p$, but locally stable ones. We do not attempt to find globally stable ones, because $\exp_g : TM-M \longrightarrow \mathbb{R}^p$ is not proper. Now the local stability of $\exp_g$ at $(x_1,q_1),\ldots,(x_r,q_r)$ is equivalent to the transversality of ${}_rj^k\exp_g$ at $(x_1,q_1),\ldots,(x_r,q_r)$ to the contact class in ${}_rJ^k(TM,\mathbb{R}^p)$ containing ${}_rj^k\exp_g((x_1,q_1),\ldots,(x_r,q_r))$ for $k \geq p+1$ (see [20] page 229). Thus, to prove that

$$ \{g \in \mathrm{Imm}(M,\mathbb{R}^p) : \exp_g \text{ is locally stable on } TM-M\} $$

is residual, we must show that

$$\{g \in \mathrm{Imm}(M,R^p) : {}_r j^{p+1}\exp_g|_{(TM-M)(r)} \text{ is transverse to every contact class in } {}_r J^{p+1}(TM,R^p)\}$$

is residual. But when $2n < (2n,p)$, this follows from our Theorem II.1 exactly as Mather's Theorem 8.1 follows from his Corollary 3.4∎

II.4 Outside the Nice Dimensions

By following our Theorem II.1 with Mather's argument, we have been able to prove that when $(2n,p)$ are nice dimensions then local stability on TM-M is a generic property of the exponential map $\exp_g$. However, the considerably more sophisticated arguments which Mather used in [21] to prove that local stability is not generic in $C^\infty(N,P)$ when $(\dim N, \dim P)$ are not nice dimensions, do not apply in our case. The reason for this is that they depend upon the possibility of constructing maps $f:N \longrightarrow P$ such that $j^k f(N)$ meets a given manifold in $J^k(N,P)$, and since we are interested not so much in $j^k g$ (where $g \in \mathrm{Imm}(M,\mathbb{R}^p)$) as in the behaviour of $j^k \exp_g$, this construction is not available to us. That is, given a submanifold $W \subseteq J^k(TM-M,\mathbb{R}^p)$, of codimension $\leqslant 2n$, there is in general no way of constructing an immersion $g:M \longrightarrow \mathbb{R}^p$ such that $j^k\exp_g$ meets W transversely.

II.4:1 Example Consider immersions $g:M \longrightarrow \mathbb{R}^6$, where $\dim M = 4$, and let $W = \Sigma^4(4,6) \subseteq J^1(M,R^6)$. Then codim $W = 8$, and so if g were such that $j^1\exp_g : TM-M \longrightarrow \mathbb{R}^6$ met W transversely, it would follow that the set

$(j^1\exp_g)^{-1}(W)$ consisted of isolated points. However, writing $\exp_g$, in terms of trivial local coordinates on TM, as

$$\exp_g(x,v) = g(x) + dg(x)(v),$$

we have, for $(\hat{x},\hat{v}) \in T_{(x,v)}TM$,

$$\begin{aligned} d\exp_g(x,v)(\hat{x},\hat{v}) &= dg(x)(\hat{x}) + d^2g(x)(v,\hat{x}) + dg(x)(\hat{v}) \\ &= dg(x)(\hat{x}+\hat{v}) + d^2g(x)(v,\hat{x}) \end{aligned}$$

and so

$$d\exp_g(x,\lambda v)(\hat{x},\lambda\hat{v} + (\lambda-1)\hat{x}) = \lambda\, d\exp_g(x,v)(\hat{x},\hat{v}).$$

Thus, rank $d\exp_g(x,\lambda v)$ = rank $d\exp_g(x,v)$ for $\lambda \neq 0$, and so if $(x,v) \in (j^1\exp_g)^{-1}(W)$ then all points $(x,\lambda v)$ are also in $(j^1\exp_g)^{-1}(W)$. Therefore it is impossible for $j^1\exp_g$ to meet W transversely.

The same argument applies to all cases where codim $\Sigma^k(2n,p) = 2n$.

Now by applying Theorem II:1, we may conclude that for a residual set of immersions $g:M \longrightarrow \mathbb{R}^6$, the rank of $d\exp_g$ is greater than or equal to 5 at all points in TM-M. And in fact we can go further:

II.4:2 <u>Proposition</u> For a residual set of immersions $g:M \longrightarrow \mathbb{R}^6$, (where dim M = 4), $\exp_g$ is locally stable on TM-M.

<u>Proof</u> Recall (cf. [21]) that the genericity of stability in the nice dimensions is proved by showing that for a residual set of f in $C^\infty(N,P)$, $j^pf(N)$ does not meet $\Pi^p(N,P)$, where $\Pi^k(N,P)$ is defined, for any k, to be the sub-bundle of $J^k(N,P)$ whose fibre, $\Pi^k(n,p)$, is the smallest clo-

sed, algebraic, $\mathcal{K}^k$-invariant subset of $J^k n,p)$ whose complement contains only a finite number of $\mathcal{K}^k$ orbits. In [22], Mather calculates the codimension of $\Pi^k(n,p)$ by decomposing it as the union of $\Pi^k_r(n,p)$, where $\Pi^k_r(n,p)$ is the set of k-jets in $\Pi^k(n,p)$ whose kernel rank is r, and then calculating individually the codimension $\sigma^k_r(n,p)$ of each $\Pi^k_r(n,p)$.

In our case, (n = 8, p = 6) we need only consider $\Pi^k_2(8,6)$, $\Pi^k_3(8,6)$, and $\Pi^k_4(8,6)$, since the kernel rank of $\exp_g$ on TM-M is at most 4. However, $\Pi^k_2(8,6)$ is empty, since all k-jets of submersions belong to a single $\mathcal{K}^k$ orbit, $\Pi^k_3(8,6) \geq 9$ ([22] page 250), and as we have just seen, for a generic immersion $g\colon M \longrightarrow \mathbb{R}^6$, the rank of $d\exp_g$ on TM-M is always greater than 4.

It follows that for a residual set of immersions $g\colon M \longrightarrow \mathbb{R}^6$,

$$j^6 \exp_g(TM-M) \cap \Pi^6(TM, R^6) = \phi$$

and the proof then concludes in the usual way∎

Note that locally stable maps are not residual in $C^\infty(TM-M, \mathbb{R}^6)$, since (8,6) are not nice dimensions.

In order to provide examples of dimensions (n,p) for which local stability of $\exp_g$ on TM-M is not generic, we must prove an existence theorem to take the place of the construction that Mather gives on the last page of [21]. Here we limit ourselves to proving such a theorem for the 1-jets of exponential maps, which allows us to give the examples shown in the diagram on page 139.

Trivialise TM over a neighbourhood of $x_0 \in M$, as $\mathbb{R}^n \times \mathbb{R}^n$. Then

$$\exp_g(x,v) = g(x) + dg(x)(v)$$

By choosing local coordinates on M appropriately, we can write

$$g(x) = (x,\ h(x))$$

where $h:\mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$, and with respect to these coordinates, $d\exp_g(x,v)$ has matrix

$$\begin{bmatrix} I_n & I_n \\ dh(x) + d^2h(x)(v) & dh(x) \end{bmatrix}.$$

Define $\bar{d}^2h : TM \longrightarrow L(n,p-n)$ to be the map $(x,v) \longrightarrow d^2h(x)(v)$.

II.4:3 <u>Lemma</u> $d\exp_g(x,v) \in \Sigma^k(2n,p) \iff \bar{d}^2h(x,v) \in \Sigma^k(n,p-n)$

and

$$d\exp_g \pitchfork \Sigma^k(2n,p) \iff \bar{d}^2h \pitchfork \Sigma^k(n,p-n).$$

<u>Proof</u> Define $B \in Gl(2n)$, and $\gamma:\mathbb{R}^n \longrightarrow Gl(p)$, by

$$B = \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \qquad \gamma(x) = \begin{bmatrix} I_n & 0 \\ -dh(x) & I_{p-n} \end{bmatrix}$$

Then the map $L(2n,p) \longrightarrow L(2n,p)$ given by $A \longmapsto \gamma(x).A.B$ defines a diffeomorphism of $L(2n,p)$ which leaves $\Sigma^k(2n,p)$ setwise invariant. Thus, $d\exp_g$ is transverse to $\Sigma^k(2n,p)$ if and only if the map

$$r:(x,v) \longmapsto \gamma(x).d\exp_g(x,v).B$$

is transverse to $\Sigma^k(2n,p)$. Now let $\eta:L(n,p-n) \longrightarrow L(2n,p)$ be defined by

$$\eta(D) = \begin{bmatrix} 0 & I_n \\ D & 0 \end{bmatrix}$$

Then $\eta \overline{\pitchfork} \Sigma^k(2n,p)$ for all k, and $\eta^{-1}(\Sigma^k(2n,p)) = \Sigma^k(n,p-n)$. Moreover, $r = \eta \circ \bar{d}^2 h$, and so

$$\mathrm{dexp}_g \overline{\pitchfork} \Sigma^k(2n,p) \Longleftrightarrow r \overline{\pitchfork} \Sigma^k(2n,p) \Longleftrightarrow \bar{d}^2 h \overline{\pitchfork} \Sigma^k(n,p-n).$$

The first affirmation of the lemma is obvious from these calculations ∎

We now aim to show, by using this lemma, that if $2n \geqslant \mathrm{codim}\, \Sigma^k(2n,p)-1$, then it is possible to construct immersions $g: M \longrightarrow \mathbb{R}^p$ (where dim $M = n$) such that $j^1 \exp_g : TM-M \longrightarrow J^1(TM,\mathbb{R}^p)$ meets $\Sigma^k(2n,p)$ transversely. By the lemma, it will be enough to show that we can construct maps $h: \mathbb{R}^n \longmapsto \mathbb{R}^{p-n}$ such that $\bar{d}^2 h$ meets $\Sigma^k(n,p-n)$ transversely.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n \times (\mathbb{R}^n - \{0\}) & \xrightarrow{\quad d^2h \times 1 \quad} & L^2_S(\mathbb{R}^n,\mathbb{R}^{p-n}) \times (\mathbb{R}^n - \{0\}) \\ & \searrow^{\bar{d}^2 h} & \swarrow_{ev} \\ & & L(n,p-n) \end{array}$$

where $L^2_S(\mathbb{R}^n,\mathbb{R}^{p-n})$ is the space of symmettric bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$ and ev is defined by $ev(\varphi,v) = \varphi(v)$. Clearly ev is a submersion, and it follows that $\bar{d}^2 h \overline{\pitchfork} \Sigma^k(n,p-n) \Longleftrightarrow (d^2h \times 1) \overline{\pitchfork} ev^{-1} \Sigma^k(n,p-n)$.

Now $(d^2h \times 1) \overline{\pitchfork} ev^{-1} \Sigma^k$ if and only if $d^2h \overline{\pitchfork} \pi|_{ev^{-1}\Sigma^k}$, where π is the projection of $L^2_S(\mathbb{R}^n,\mathbb{R}^{p-n}) \times (\mathbb{R}^n - \{0\})$ onto the first factor, and so the problem of constructing $h: \mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$ such that $\bar{d}^2 h$ meets $\Sigma^k(n,p-n)$

transversely, becomes that of constructing h such that $d^2h \pitchfork \pi|_{ev^{-1}\Sigma^k}$ and meets $\pi(ev^{-1}\Sigma^k)$.

II.4:4 Lemma $\pi(ev^{-1}\Sigma^k)$ contains a smooth immersed manifold of codimension $\max\{k(p-2n+k) - n + 1, 0\}$.

Proof Let

$R_{k,i} = \{(A,v) \in ev^{-1}\Sigma^k$: kernel rank $(\pi|_{ev^{-1}\Sigma^k})$ at (A,v) equals $i\}$.

Then $R_{k,0}$ is empty for all k, for if $(A,v) \in ev^{-1}\Sigma^k$ then $(A,\lambda v) \in ev^{-1}\Sigma^k$ for all real $\lambda \neq 0$, so that

$$(0,v) \in T_{(A,v)}ev^{-1}\Sigma^k .$$

Since $d\pi(A,v)(0,v) = 0$, this gives $d(\pi|_{ev^{-1}\Sigma^k})(A,v)$ a non-trivial kernel.

It is clear that

$$\dim \ker d(\pi|_{ev^{-1}\Sigma^k}) \geqslant n - \operatorname{codim}\Sigma^k(n,p-n)$$

with equality when $\pi|_{ev^{-1}\Sigma^k}$ is a submersion, and so by the preceeding paragraph,

$$\dim \ker d(\pi|_{ev^{-1}\Sigma^k}) \geqslant \max\{1, n - \operatorname{codim}\Sigma^k(n,p-n)\} .$$

Claim Let $i = \max\{1, n - \operatorname{codim}\Sigma^k(n,p-n)\}$. Then $R_{k,i}$ is non-empty and open in $ev^{-1}\Sigma^k$.

To see this, we construct a pair $(A,v_0) \in ev^{-1}\Sigma^k$ such that at (A,v_0), $\ker d(\pi|_{ev^{-1}\Sigma^k})$ has the requisite dimension. Regard A as being a (p-n)-tuple of symmetric $n \times n$ matrices, $A^1, \ldots, A^{p-n}$. Then we have

$$ev(A,v) = ev((A^1, \ldots, A^{p-n}), v) = \begin{bmatrix} v^t A^1 \\ \vdots \\ v^t A^{p-n} \end{bmatrix}$$

where v^t is the transpose of v.

For a fixed k, define A as follows: first, let

$$A^i = \overbrace{\begin{bmatrix} 0 & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}}^{\longleftarrow\ i\ \longrightarrow} \updownarrow i \qquad \text{for } 1 \leqslant i \leqslant n-k$$

and let all the remaining A^i have first row and column identically zero.

<u>Case 1</u> $n-1 \leqslant k(p-2n+k)$. Let $r \in \mathbb{N}$ be such that $rk < n-1 \leqslant (r+1)k$. Let

$$C^{n-k+j} = \overbrace{\begin{bmatrix} 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}}^{\longleftarrow\ n-k\ \longrightarrow} \begin{matrix} \updownarrow (j-1)k+1 \\ \updownarrow jk+1 \end{matrix}$$

and let $A^{n-k+j} = -(C^{n-k+j})^t + C^{n-k+j}$ for $1 \leqslant j \leqslant r$.

Let

$$A^{n-k+r+1} = \begin{bmatrix} 0 & . & . & 0 & 0 & . & . & & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & 0 & 1 & 0 & . & & 0 \\ 0 & . & . & 0 & 0 & 1 & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & 0 & 0 & . & . & 0 & 1 \end{bmatrix}$$

(the first $rk+1$ columns and the first $rk+1$ rows are marked: $\leftarrow rk+1 \rightarrow$ across the top, $\uparrow rk+1 \downarrow$ at the right)

and let the remaining A^i be arbitrary, except that they have first row and column identically zero.

<u>Case 2</u> $n-1 > k(p-2n+k)$. Define $A^{n-k+1}, \ldots, A^{p-n}$ as in Case 1.

In each case, let $v_0 = e_1 = (1, 0, \ldots, 0)^t$.

Then in Case 1, $(A,v_0) \in R_{k,1}$, and in Case 2, $(A,v_0) \in R_{k,n-k(p-2n+k)}$. To see this, regard $L(n,p-n)$ as its own tangent space at each point, and note that $T_{ev(A,v_0)}\Sigma^k(n,p-n)$ consists of all matrices of the form

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{matrix} \updownarrow n-k \\ \updownarrow p-2n+k \end{matrix}$$

$$\leftarrow n-k \rightarrow \quad \leftarrow k \rightarrow$$

with $M_{2,2} = 0$. Write $M_{2,2}(A,v)$ for the $M_{2,2}$ component of $ev(A,v)$. Then in Case 1, we have to show that for no vector $v \neq e_1$ is $M_{2,2}(A,v)$ equal to 0. This amounts to showing that $M_{2,2}(A,e_2), \ldots, M_{2,2}(A,e_n)$ are independent. But, for i such that $(j-1)k+1 < i \leq jk+1$, the first $j-1$ rows of $M_{2,2}(A,e_i)$ are all-equal to 0; so provided we can show that each set of k matrices $M_{2,2}(A,e_{jk+2}), \ldots, M_{2,2}(A,e_{(j+1)k+1})$ is independent, for $j = 0, 1, \ldots, r$, and similarly that $M_{2,2}(A,e_{(r+1)k+2}), \ldots, M_{2,2}(A,e_n)$ is an independent set, the claim (for Case 1) is proved.

However, for $(j-1)k+1 < i \leqslant jk+1$, we see that the j^{th} row of $M_{2,2}(A,e_i)$ (which is just the i^{th} row of A^{n-k+j}) has a 1 in the $(i-(j-1)k-1)^{th}$ place, followed only by zeros. Thus, $M_{2,2}(A,e_{(j-1)k+2}),\ldots,M_{2,2}(A,e_{jk+1})$ are independent.

In Case 2, the same construction shows that the composition of

$$\{0\}\times T_{e_1}\mathbb{R}^n \xrightarrow{dev(A,e_1)} T_{ev(A,e_1)}L(n,p-n) \longrightarrow \frac{T_{ev(A,e_1)}L(n,p-n)}{T_{ev(A,e_1)}\Sigma^k(n,p-n)}$$

is surjective.

This completes the proof that $R_{k,\max\{1,\ n-\operatorname{codim}\Sigma^k(n,p-n)\}}$ is not empty. That it is open in $ev^{-1}\Sigma^k(n,p-n)$ is straightforward: in Case 1, $(A,v) \in R_{k,1}$ if $ev(A,v) \in \Sigma^k(n,p-n)$ and the map

$$\frac{T_v\mathbb{R}^n}{Sp\{v\}} \longrightarrow \frac{T_{ev(A,v)}L(n,p-n)}{T_{ev(A,v)}\Sigma^k(n,p-n)}$$

induced by $dev(A,v)$, is injective, while in Case 2, $(A,v) \in R_{k,\ n-k(p-2n+k)}$ if $ev(A,v) \in \Sigma^k(n,p-n)$ and this same map is surjective. Both of these are evidently open conditions on $ev^{-1}\Sigma^k$.

This completes the proof of the claim.

Now, in Case 1 the corank of $\pi|_{R_{k,1}}$ is everywhere equal to 1, and so

$\pi(R_{k,1})$ is a smoothly immersed manifold of codimension $k(p-2n+k)-n+1$ in $L^2_S(\mathbb{R}^n,\mathbb{R}^{p-n})$, while in Case 2, $\pi|_{R_{k,n-k(p-2n+k)}}$ is a submersion, and so its image is open in $L^2_S(R^n,R^{p-n})$ ∎

II.4:5 Theorem Let $2n \geqslant k(p-2n+k) - 1$, and let $p > n$. Then there exists an immersion $g:M \longrightarrow \mathbb{R}^p$ (where dim M = n) such that $j^1\exp_g$ meets $\Sigma^k(2n,p) \subseteq J^1(TM-M,R^p)$ transversely.

Proof Choose any immersion $f_0:M \longrightarrow \mathbb{R}^p$, and take a coordinate chart φ on a neighbourhood U of $x_0 \in M$ so that $f_0 \cdot \varphi^{-1}$ has the form

$$x \longmapsto (x,\ h_0(x))$$

where $h_0:\mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$ and h_0 and all its first derivatives vanish at 0. Since $2n > k(p-2n+k)$, we have $n \geqslant k(p-2n+k) - n + 1$, and so by Theorem 6.1 there exists a map $h_1:\mathbb{R}^n \longrightarrow \mathbb{R}^{p-n}$ such that d^2h_1 meets $\pi(R_{k,\ \max\{1,\ n-k(p-2n+k)\}})$ transversely at 0. It is clear that we may take $h_1(0) = 0$ and $dh_1(0) = 0$.

Now if $g_1:R^n \longrightarrow R^{p-n}$ is defined by $g_1(x) = (x,\ h_1(x))$ then it follows from Lemma II.4:3 that $j^1\exp_{g_1}$ meets $\Sigma^k(n,p-n) \subseteq J^1(TR^n-R^n,\mathbb{R}^p)$ transversely at some point $(0,v) \in T\mathbb{R}^n-\mathbb{R}^n$.

Let $\sigma:M \longrightarrow \mathbb{R}$ be a C^∞ mapping with compact support in U, identically equal to 1 in a neighbourhood of x_0. Define $g:M \longrightarrow \mathbb{R}^p$ by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in M-U \\ \sigma(x)g_1(\varphi(x)) + (1-\sigma(x))g_0(\varphi(x)) & \text{if } x \in U \end{cases}$$

Then g is a smooth immersion $M \longrightarrow \mathbb{R}^p$, and $j^1\exp_g$ meets $\Sigma^k(2n,p)$ transversely■

We are now in a position to give examples of dimensions (n,p) for which $\{g \in Imm(M,R^p) : exp_g$ is locally stable on $TM-M\}$ is not dense. We use the calculations in [22] , but at this stage it is perhaps worthwhile to point out in slightly more detail the difference between our case and Mather's. In [21] , he shows that if there is a smooth stratum W_c contained in $J^k(N,P)$ consisting of $\mathcal{K}$ orbits, with codim $W_c = c \leqslant n$, but such that each $\mathcal{K}$ orbit contained in W_c has codimension greater than c, then it is possible to construct a proper map $f:N \longrightarrow P$ such that $j^k f$ meets W_c transversely but is not transverse to any of the $\mathcal{K}$ orbits contained in W_c, so that f is not stable. This alone is not enough to prove that stable maps are not dense in $C^{\infty}_{pr}(N,P)$; and a crucial step in Mather's argument consists in showing that it is possible to construct f such that the phenomenon "$j^k f$ meets W_c transversely but is not transverse to any of the $\mathcal{K}$ orbits contained in W_c" is "stabley" or "transversely" present. From this it follows that sufficiently small perturbations of f will exhibit the same phenomenon, and will therefore be unstable also. Because, as we have remarked earlier, exp_g is subject to the constraint of being an exponential map and we are free only in our construction of g, it is not clear, from Mather's work alone, that it is always possible to construct g so that exp_g will exhibit the phenomenon described above. However, there is one case, or set of cases, in which to show that local stability of exp_g is not a generic property of immersions g, it is enough merely to construct g so that $j^k exp_g$ meets W_c transversely. These are of course the cases in which each $\mathcal{K}$ orbit contained in W_c has codimension greater than 2n. It is to such cases that we now turn our attention.

Since we have at our disposal only the rather weak Theorem II.4:5, we look at the cases in which $\Sigma^k(2n,p) \subseteq \Pi^k(2n,p)$, and we aim to find dimensions (n,p) for which there exists k satisfying

i) $k(p-2n+k) < 2n$

ii) the codimension of each of the $\mathcal{K}^2$ orbits in $(\pi_{2,1})^{-1}(\Sigma^k(2n,p))$ (where $\pi_{2,1}: J^2(2n,p) \to J^1(2n,p)$ is the usual projection) is greater than 2n.

Recall from [22] pp.214-219 that the 2-jets of map-germs $(\mathbb{R}^m,0) \to (\mathbb{R}^q,0)$ having corank k at the origin are classified up to $\mathcal{K}^2$-equivalence by the orbit under the action of the projective linear group PGl(k) of $\operatorname{Ker}\varphi \in Gr(s, \frac{1}{2}k(k+1))$, where φ is the linearisation of the second intrinsic derivative D^2f (here we depart from Mather's notation) as in

$$\begin{array}{ccc} \operatorname{Ker} df \times \operatorname{Ker} df & & \\ \downarrow & \searrow^{D^2f} & \\ S^2(\operatorname{Ker} df) & \xrightarrow[\varphi]{} & \operatorname{Coker} df, \end{array}$$

and $s = \dim \operatorname{Ker} \varphi$. Two such 2-jets exhibiting different values of s are not $\mathcal{K}^2$-equivalent.

If $\dim PGl(k) < \dim Gr(s, \frac{1}{2}k(k+1))$ then the set of all such 2-jets is contained in $\Pi^2_k(m,q)$. If we denote by $\Sigma^k(s)$ the set of all 2-jets whose kernel has rank k and whose second intrinsic derivative, regarded as a linear map $S^2\mathbb{R}^k \to \operatorname{Coker} df = \mathbb{R}^{q-m+k}$, has an s-dimensional kernel, then by the above, $\Sigma^k(s) \subseteq \Pi^2_k(m,q)$ if $s(\frac{1}{2}k(k+1)-s) < k^2-1$.

Moreover, $\Sigma^k(s)$ is a smooth manifold of codimension $s(q-m+k-\frac{1}{2}k(k+1)+s)$ in Σ^k, and hence of codimension $s(q-m-\frac{1}{2}k(k-1)+s) + k(q-m+k)$ in $J^2(m,q)$, provided, of course, that

$$\min\left\{\tfrac{1}{2}k(k+1)-q+m-k,\ 0\right\} \leqslant s \leqslant \tfrac{1}{2}k(k+1),$$

(for otherwise it is empty).

The codimension of each of the $\mathcal{K}^2$ orbits contained in $\Sigma^k(s)$ is at least

$$\operatorname{codim}\Sigma^k(s) + \dim Gr(s,\tfrac{1}{2}k(k+1)) - \dim PGl(k).$$

Since we have no existence theorem for immersions $g:M\longrightarrow\mathbb{R}^p$ such that $\exp_g$ has a given 2-jet, we restrict our attention to those values of k and s for which $\Sigma^k(s)$ is open and dense in Σ^k, that is, for which the linearisation $\varphi :S^2\mathbb{R}^k\longrightarrow\mathbb{R}^{q-m+k}$ of D^2f is either an epimorphism or an injection.

From the calculations on pp.218-224 of [22] , we see that the only cases that interest us are

i) $0 \leqslant p-2n \leqslant \frac{1}{2}k(k-1) - 2$ and $k \geqslant 4$ (Mather's Case Ic, p.218 op. cit.)

ii) $2n-p \geqslant 2$ and $k \geqslant 2n-p+2$

iii) $2n-p = 1$ and $k \geqslant 4$

} (Mather's Case IIa, p.218 op. cit.)

In all these cases, we have $p-2n+k \leqslant \frac{1}{2}k(k+1)$, so that the value of s for which $\Sigma^k(s)$ is open in Σ^k is $\frac{1}{2}k(k+1)-p+2n-k$.

Thus, we seek those values of n,p and k within these ranges such that

$$\operatorname{codim}\Sigma^k(2n,p) < 2n < \operatorname{codim}\Sigma^k(2n,p) + \dim Gr(\tfrac{1}{2}k(k+1)-p+2n-k,\ \tfrac{1}{2}k(k+1)) - \dim PGl(k)$$

that is,

$$k(p-2n+k) < 2n < k(p-2n+k) + (p-2n+k)(\tfrac{1}{2}k(k+1)-p+2n-k) - k^2 + 1,$$

which simplifies to

$$k(p-2n+k) < 2n < \tfrac{1}{2}k(k-1)(p-2n+k) - (p-2n)^2 + 1. \qquad \text{(iv)}$$

From the preceeding analysis, and from II.4:5, we conclude

II.4:6 <u>Theorem</u> For any pair (n,p) with $0 < n < p$, if there exists $k > 0$ such that n,p and k satisfy one of the conditions (i), (ii) or (iii) on the previous page and satisfy also (iv), then if $\dim M = n$, the set

$$\{g \in \mathrm{Imm}(M,\mathbb{R}^p) : \exp_g \text{ is locally stable on } TM-M\}$$

is not dense in $\mathrm{Imm}(M,\mathbb{R}^p)$ ∎

<u>Calculation of Examples</u>

In order not to devote too much space to the solution of diophantine inequalities, we do not attempt to give a complete list of dimensions to which this theorem applies.

Put $2n - p = r$. Then (iv) becomes

$$k(k-r) < 2n < \tfrac{1}{2}k(k-1)(k-r) - r^2 + 1.$$

<u>If $r = 0$</u> then setting $k = 4$ gives $16 < 2n < 25$

$k = 5$ gives $25 < 2n < 51$

$k = 6$ gives $36 < 2n < 91$

. . . .

and so II.4:6 applies to all values of $n \geq 9$.

If $r = -1$ then setting $k = 4$ gives $20 < 2n < 30$

$k = 5$ gives $30 < 2n < 60$

$k = 6$ gives $42 < 2n < 115$

. . . .

and so II.4:6 applies to all values of $n \geqslant 11$, except for $n = 15$.

If $r = -2$ then setting $k = 4$ gives $24 \quad 2n \quad 33$

$k = 5$ gives $35 \quad 2n \quad 67$

$k = 6$ gives $48 \quad 2n \quad 117$

. . . .

and so II.4:6 applies to all values of $n \geqslant 13$, except for $n = 17$.

If $r = -3$ then setting $k = 4$ gives $28 < 2n < 34$

$k = 5$ gives $40 < 2n < 72$

$k = 6$ gives $54 < 2n < 127$

. . . .

and so II.4:6 applies to $n = 15, 16$ and $n \geqslant 21$.

If $r = 1$ then setting $k = 4$ gives $12 < 2n < 18$

$k = 5$ gives $20 < 2n < 40$

$k = 6$ gives $30 < 2n < 75$

and so II.4:6 applies to $n = 7, 8$ and to $n \geqslant 11$.

If r = 2 then setting $k = 4$ gives $8 < 2n < 9$

$k = 5$ gives $15 < 2n < 27$

$k = 6$ gives $24 < 2n < 57$

. . . .

and so II.4:6 applies to $n \geqslant 8$.

If r = 3 then setting $k = 5$ gives $10 < 2n < 12$

$k = 6$ gives $18 < 2n < 37$

$k = 7$ gives $28 < 2n < 76$

. . . .

and so II.4:6 applies to $n \geqslant 10$.

By fixing r and substituting successive values of k into (iv), we obtain a continuous set of $n \in \mathbb{N}$ once the value of the right hand side of (iv) for k is greater than the value of the left hand side for k+1. That is, once

$$\tfrac{1}{2}k(k-1)(k-r) - r^2 + 1 > (k+1)(k+1-r)$$

providing the additional condition (i), (ii) or (iii) is met. This inequality is more conveniently expressed as

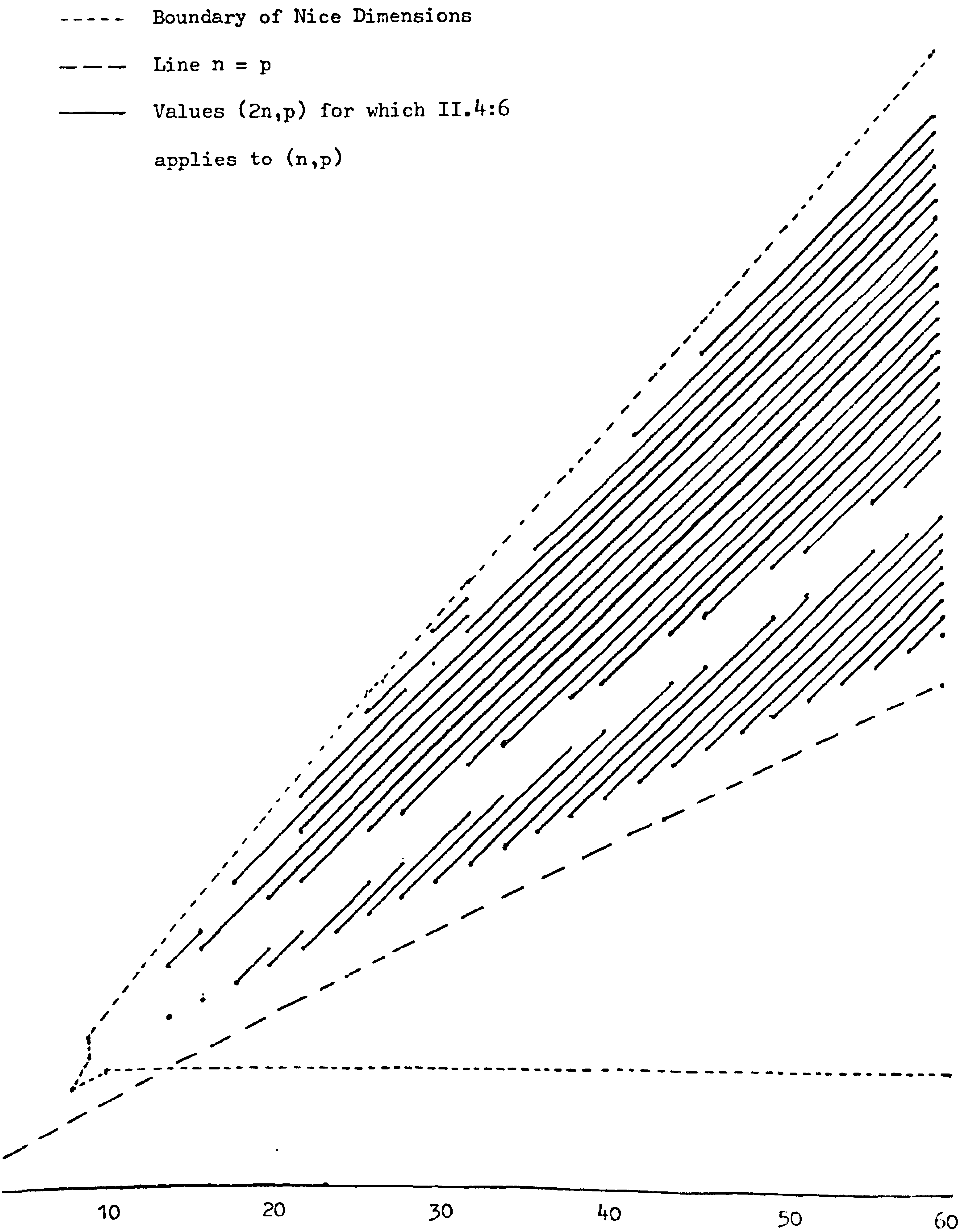
$$k^3 - (r+3)k^2 + (3r-4)k + 2r - 2r^2 > 0.$$

It is satisfied, along with condition (i), if $r \leqslant -4$ and $k = 1-r$ (although there are of course other sets of values which will satisfy both the inequality and condition (i)). Putting $k = 1-r$ in the left hand side of (iv), we find that II.4:6 applies to all dimension-pairs $(n,2n-r)$ if $2n > 2r^2 - 3r + 1$ and $-4 \geqslant r$.

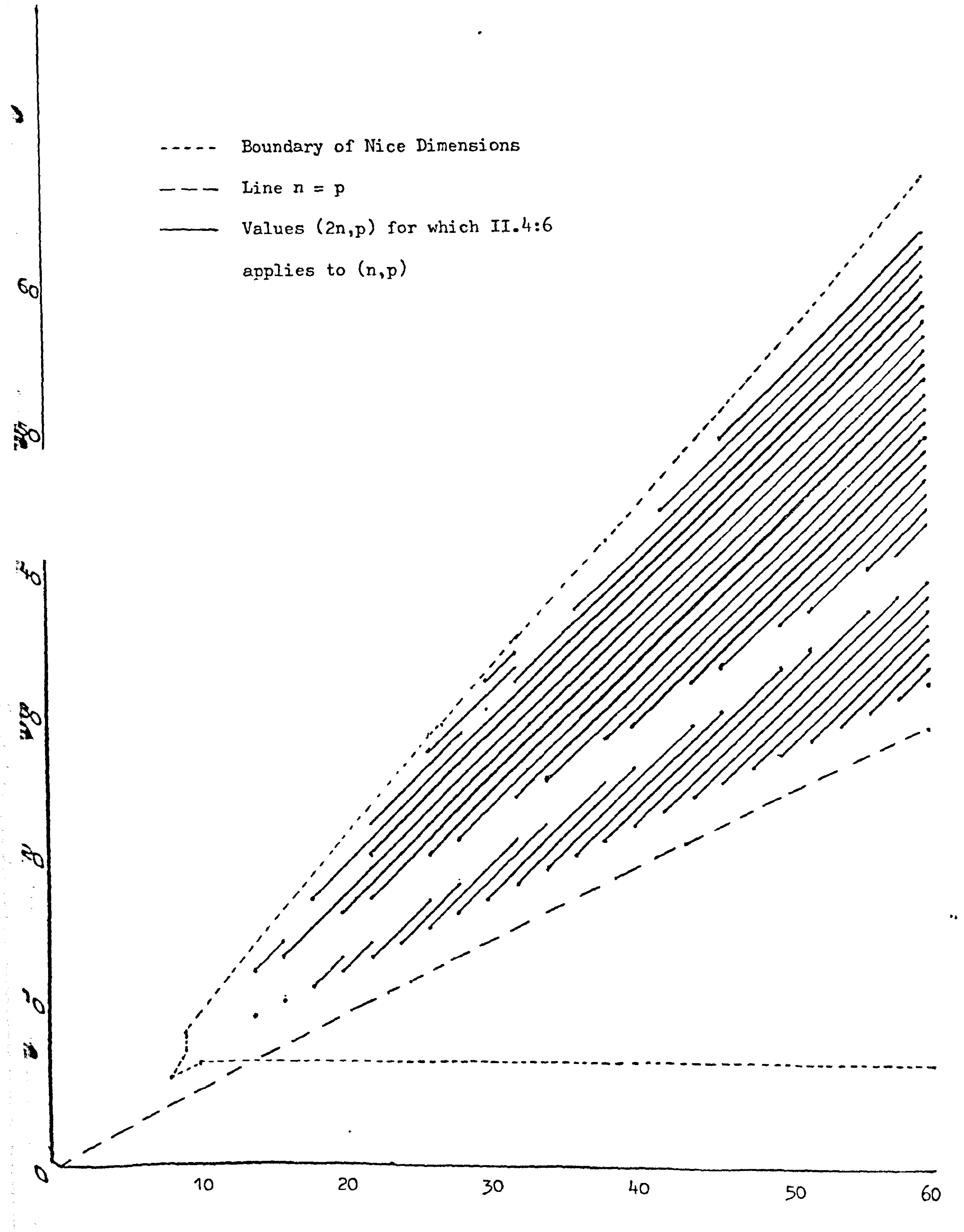
The inequality is satisfied, along with condition (ii), if $r \geqslant 4$ and $k = r+3$. Putting $k = r+3$ into the left hand side of (iv), we find that II.4:6 applies to all dimension-pairs $(n,2n-r)$ if $2n > 3r+9$ and $4 \leqslant r < n$.

By taking $k = r+2$ for $r \geqslant 4$, one finds that II.4:6 also applies to all dimension-pairs $(n,2n-r)$ for which $2r+4 < 2n < 3r+3$.

The results of these calculations are displayed on the next page. It will be appreciated that a fairly large part of Mather's "bad dimensions" is covered by those pairs $(2n,p)$ for which II.4:6 applies to (n,p).



Boundary of Nice Dimensions
Line n = p
Values (2n,p) for which II.4:6 applies to (n,p)
10
20
30
40
50
60



----- Boundary of Nice Dimensions
— — — Line n = p
——— Values (2n,p) for which II.4:6 applies to (n,p)
60
50
40
30
20
10
0
10
20
30
40
50
60

CHAPTER III

APPLICATIONS TO THE GEOMETRY OF SURFACES IN $\mathbb{R}^4$

In this chapter we shall study radial projections into hyperplanes of surfaces M (of dimension 2) embedded in $\mathbb{R}^4$. In particular we shall be interested in the $\mathcal{A}$-classification of the germs of these projections. We shall try to answer the following questions: for a "generic" embedding $M \longrightarrow \mathbb{R}^4$,

1) which $\mathcal{A}$-classes do the germs of projection belong to?

2) what correlation is there between the $\mathcal{A}$-class of the germs of the projections $p_g(q)$ at $x_0 \in M$, and the differential geometry of the embedding g of the surface at x_0?

3) For a given $\mathcal{A}$-class S, what is the locus of points $x \in M$ for which for some $q \in \mathbb{R}^4$, the germ at x of the projection $p_g(q)$ belongs to S?

Our answers to these questions are by no means complete, as will become apparent, and at the end of the chapter we raise several questions which might be susceptible to further investigation.

To start with, we must give a precise meaning to the term "generic". Our philosophy will be that we are interested in proving theorems about the members of residual subsets of $\mathrm{Imm}(M,\mathbb{R}^4)$, and since the countable intersection of residual subsets is still residual, we shall not hesitate to introduce further conditions on the immersions for which our theorems hold, as and when they become necessary.

III.1:1 Definition E_1 is the set of all immersions $g:M \longrightarrow \mathbb{R}^4$ such that for any hyperplane H in $\mathbb{R}^4$, $j_x^k p_g : {}_1G_g(H) \longrightarrow J^k(M,H)$ is transverse to the stratification $\mathcal{S}$ induced in $J^k(M,H)$ by the stratification $\mathcal{S}_0$ of Theorem I:2.

III.1:2 Proposition E_1 is residual in $\mathrm{Imm}(M,\mathbb{R}^4)$.

Proof That for any one hyperplane H, the set $E_1(H)$ of immersions g such that $j_x^k p_g : {}_1G_g(H) \longrightarrow J^k(M,H)$ is transverse to $\mathcal{S}$, is an immediate consequence of our transversality theorem II.3:2, bearing in mind that $\mathcal{S}$ consists of only a finite number of strata. That E_1 is also residual, although it is the intersection of all $E_1(H)$, follows from the fact that it contains the intersection of a finite number of them. To see this, let $H_i = \{(X_1,X_2,X_3,X_4) \in \mathbb{R}^4 : X_i = 0 \quad \text{for } i = 1,\dots,4\}$ and let $H_i = \{(X_1,X_2,X_3,X_4) \in \mathbb{R}^4 : X_{i-4} = 1 \quad \text{for } i = 5,\dots,8\}$. Then

$$E_1 = \bigcap_{i=1}^{8} E_1(H_i).$$

For if H is any hyperplane, and $(x,q) \in {}_1G_g(H)$, then for some i, $(x,q) \in {}_1G_g(H_i)$, and the germs at x of the projections $p_g(q)$ into H and into H_i are left-equivalent (see II.1). Since $\mathcal{S}$ is an $\mathcal{A}$-invariant stratification, if the projection $p_g(q)$ into H_i is transverse to $\mathcal{S}$ at x then so is the projection $p_g(q)$ into H.

From this it follows that E_1 contains the intersection of the $E_1(H_i)$, $i = 1,\dots,8$ ∎

As a first consequence of our definition, we have

III.1:3 <u>Proposition</u> If $g \in E_1$, then each germ of $\exp_g$ at a point in TM-M is stable.

<u>Proof</u> It is convenient to continue to use the embedding of TM into $M \times \mathbb{R}^4$ defined by g, as discussed in II.1.

Let $(x_0, q_0) \in$ TM-M, and choose a hyperplane H such that $(x_0, q_0) \in {}_1G_g(H)$. Since by hypothesis $j^k_x p_g$ is transverse to the stratum of $\mathcal{S}$ containing $j^k p_g(q_0)(x_0)$, it is also transverse to any manifold containing this stratum. Now it is easy to check, for any stratum X of codimension $\leqslant 6$, and for $z \in X$, that

$$\tilde{d}_k^{-1}(\mathcal{K}^k \tilde{d}_k(z)) \supseteq X.$$

For the strata which consist of only one $\mathcal{A}$-orbit, this just the statement that the $\mathcal{K}$-class of $\tilde{d}f$ is an $\mathcal{A}$-invariant of f; for the modular strata in $\mathcal{S}$, one checks by direct calculation that the $\mathcal{K}^k$-class of $\tilde{d}_k z$ does not vary as z varies within each stratum.

Hence, we can conclude that

$$j^{k+1}_x p_g \,\overline{\pitchfork}\, \tilde{d}_k^{-1}(\mathcal{K}^k \tilde{d}_k(j^{k+1} p_g(q_0)(x_0))),$$

from which it follows, by II.3:5, that

$$j^k \exp_g \overline{\pitchfork}\, \mathcal{K}^k j^k \exp_g(x_0, q_0)$$

at (x_0, q_0). By taking k = 5, we conclude that the germ of $\exp_g$ at (x_0, q_0) is stable∎

In what follows, we shall be talking only about immersions $g \in E_1$, and thus will assume that all (mono-germ) singularities of $\exp_g$, away from the 0-section, are stable. From this it follows, on the grounds of $\mathcal{A}$ - codimension alone, that the only singularities that $\exp_g$ can present are the Morin singularities corresponding to the Boardman strata $\Sigma^{1,0}$, $\Sigma^{1,1,0}$, $\Sigma^{1,1,1,0}$, and $\Sigma^{1,1,1,1,0}$, and the two codimension 4 Σ^2 singularities. However, a simple argument, in essence that of II.4:1, shows that for $g \in E_1$, $\exp_g$ has no Σ^2 singularities:

III.1:4 <u>Proposition</u> For $g \in E_1$, $\exp_g$ presents only the Morin singularities listed in the preceeding paragraph.

<u>Proof</u> That for a residual set of immersions g, $\exp_g$ has no Σ^2 points, is straightforward. For by II:1, there is a residual set of g such that $j^1\exp_g \pitchfork \Sigma^2$, and for such g the codimension in TM-M of the set of Σ^2 points of $\exp_g$ (if it is not empty) must be 4. It follows that this set must be empty, for, as was shown in II.4:1, the rank of $d\exp_g$ is constant along 1-dimensional vector subspaces of the fibres of TM.

In fact it is precisely this same reasoning that shows that for $g \in E_1$, $\exp_g$ has no Σ^2 points. The link is the fact that the local algebra of $\exp_g$ at (x,q) is isomorphic to the local algebra of $\tilde{d}p_g(q)$ at x (II.2:3 and II.2:7). The only two strata in $\mathcal{S}$ of jets $z = j^kf(0)$ such that $\tilde{d}f$ has a Σ^2 singularity at 0, are both of codimension 6. Hence, for $g \in E_1$ the set of points (x,q) such that $p_g(q)(x)$ lies in either of these strata is of codimension 6 in $M \times R^4$, if it is not empty, and thus consists of isolated points. But this is impossible, since the rank of $\tilde{d}p_g(q)$ at x remains constant as q moves along the line joining g(x) and q ∎

As a prelude to further description of the geometry of embeddings $g \in E_1$, we present a table of the strata of $\mathcal{S}$ which may be met by $j^k_x p_g$ for g in a residual set E_4 of immersions (see Theorem III.3:1, below, for the definition of E_4 and proof that these are the only strata met).

Singularity of $\exp_g$	$\mathcal{A}$-codim	Stratum of $\mathcal{S}$	Name	$\mathcal{A}$-codim	d
submersion	0	(x,y^2,xy)	S_0	2	0
$\Sigma^{1,0}$ (fold)	1	(x,y^2,y^3+x^2y)	S_1	3	0
		(x,y^2,x^2y+y^5)	B_2	4	1
		$(x,xy+y^5,y^3)$	H_2	4	1
		(x,y^2,x^2y+y^7)	B_3	5	2
		$(x,xy+y^8,y^3)$	H_3	5	2
		(x,y^2,x^2y+y^9)	B_4	6	3
		$(x,xy+y^{11},y^3)$	H_4	6	3
$\Sigma^{1,1,0}$ (cusp)	2	(x,y^2,y^3+x^3y)	S_2	4	0
		(x,y^2,xy^3+x^3y)	C_3	5	1
		$(x,xy+y^3,xy^2+cy^4)$ *	P_3	5	1
		(x,y^2,x^3y+y^5)	F_4	6	2
$\Sigma^{1,1,1,0}$	3	(x,y^2,y^3+x^4y)	S_3	5	0
		(x,y^2,xy^3+x^4y)	C_4	6	1
$\Sigma^{1,1,1,1,0}$	4	(x,y^2,y^3+x^5y)	S_4	6	0

The number d in the last column is defined on the next page.

* $c \neq 0, \frac{1}{3}, 1, \frac{1}{2}$.

The number d is defined as follows:

For X a stratum of $\mathcal{S}$ and $g \in \mathrm{Imm}(M,\mathbb{R}^4)$, let

$$X(g) = \{(x,q) \in M \times \mathbb{R}^4 : j^k p_g(q)(x) \in X\} ,$$

and let π_1 and π_2 be the projections of $M \times \mathbb{R}^4$ onto M and $\mathbb{R}^4$ respectively.

Suppose that the $\mathcal{K}$-class of $\tilde{d}_k z$, for $z \in X$, (which is an invariant of X), is that of $(x,y) \longmapsto (x,y^r)$. Then by II.3:4,

$$\pi_2(X(g)) \in \exp_g(\Sigma^{1_r}\exp_g).$$

Moreover, for $g \in E_1$, every mono-germ of $\exp_g$ on TM-M is stable, by III.1:3, and so is a smooth immersion when restricted to its equisingularity manifold $\Sigma^{1_r}\exp_g$. Hence the image of this equisingularity manifold is an immersed manifold in $\mathbb{R}^4$, of codimension r.

We define d to be the codimension of $\pi_2(X(g))$ in $\exp_g(\Sigma^{1_r}\exp_g)$. That d is then well-defined follows from

III.1:5 <u>Proposition</u> For each $X \in \mathcal{S}$, and for $g \in E_1$, $\pi_2|_{X(g)}$ is an immersion.

<u>Proof</u> For the strata of $\mathcal{S}$ which are $\mathcal{A}$-orbits, this is a consequence of

III.1:6 <u>Lemma</u> Let $F:(\mathbb{R}^n \times \mathbb{R}^a,0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^a,0)$

$$F(x,u) = (f(x,u),\ u) = (f_u(x),\ u)$$

be an $\mathcal{A}_e$-versal unfolding of f_0, and let

$$S_F = \{(x,u) \in (\mathbb{R}^n \times \mathbb{R}^a, 0) : \text{the germ of } f_u \text{ at } x \text{ is } \mathcal{A}\text{-equivalent to the germ of } f_0 \text{ at } 0\}.$$

Then S_F is the germ of a smooth manifold, and if $\pi_1 : \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}^a$ are the usual projections, then $\pi_2|S_F$ is of constant rank, and is an immersion if f_0 is not stable.

Proof Since F is $\mathcal{A}_e$-versal, it is equivalent as an unfolding to a constant unfolding $\overline{G}$ of a miniversal unfolding G of f_0. That is, there is a commutative diagram

$$\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{a-c}, 0) & \xrightarrow{\ \overline{G}\ } & (\mathbb{R}^p \times \mathbb{R}^c \times \mathbb{R}^{a-c}, 0) \\
\overline{\Phi} \downarrow & & \downarrow \overline{\Psi} \\
(\mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{a-c}, 0) & \xrightarrow{\ h^*F\ } & (\mathbb{R}^p \times \mathbb{R}^c \times \mathbb{R}^{a-c}, 0)
\end{array}$$

where $\overline{G}(x,v,w) = (g(x,v),v,w)$, $h : (\mathbb{R}^a, 0) \longrightarrow (\mathbb{R}^a, 0)$ is the germ of a diffeomorphism, $h^*F(x,v,w) = (f(x,h(v,w)), v, w)$, $\overline{\Phi}$ and $\overline{\Psi}$ are a-parameter unfoldings of the identity of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, and c is the $\mathcal{A}_e$-codimension of f_0.

If f_0 is not stable, then

$$S_{\overline{G}} = \{0\} \times \{0\} \times \mathbb{R}^{a-c},$$

for a non-stable singularity is necessarily of isolated singularity type ([30] , page 492), so

$$S_{h^*F} = \overline{\Phi}(S_{\overline{G}}) = \{(\phi_{(0,w)}(0), 0, w) : w \in (\mathbb{R}^{a-c}, 0)\}$$

and

$$S_F = \{(\varphi_{(0,w)}, h(0,w)) : w \quad (R^{a-c},0)\} .$$

Thus, $\pi_1 : (S_F, 0) \longrightarrow (\mathbb{R}^n,0)$ is $\mathcal{A}$-equivalent to the map-germ

$$(\mathbb{R}^{a-c},0) \longrightarrow (\mathbb{R}^n,0)$$

$$w \longmapsto \varphi_{(0,w)}(0)$$

and $\pi_2 : (S_F,0) \longrightarrow (\mathbb{R}^n,0)$ is $\mathcal{A}$-equivalent to the map-germ

$$(\mathbb{R}^{a-c},0) \longrightarrow (\mathbb{R}^a,0)$$

$$w \longmapsto h(0,w).$$

Hence, $\pi_2|_{S_F}$ is an immersion.

If f_0 is stable, then $c = 0$ and $\pi_2 : (S_F,0) \longrightarrow (\mathbb{R}^a,0)$ is a submersion∎

Proof of III.1:5 (continued)

If the stratum X is an $\mathcal{A}$ orbit, and if $j^k_x p_g \pitchfork X$, then P_g is an $\mathcal{A}_e$-versal unfolding of the germ whose $\mathcal{A}$-orbit X is, and $X = S_{P_g}$.

For those strata of $\mathcal{S}$ which are not $\mathcal{A}$ orbits but instead are uni- or bi-modular, a slight modification of the preceeding argument suffices to prove the result: in a miniversal unfolding, the set

$$\{(x,u) \in \mathbb{R}^2 \times \mathbb{R}^c : \text{the germ of } f_u \text{ at } x \text{ belongs to the modular family}\},$$

instead of being a single point, is a smooth submanifold which projects immersively into $\mathbb{R}^c$∎

III.1:7 Calculation of the value of d

From the preceeding remarks, it is clear that for a stratum X of $\mathcal{S}$ whose codimension is c and which corresponds to Σ^{1_r} singularities of $\exp_g$, we have

$$d = c - r - 2.$$

III.1:8 Remark As can be seen in the proof of III.1:5, Singularity Theory alone tells us nothing about $\pi_1 : X(g) \longrightarrow M$ ($= \pi_1 : S_{P_g} \longrightarrow M$); geometrical factors intervene to determine the form of this map, and we shall see that even for $g \in E_1$, for certain strata $X \in \mathcal{S}$, $\pi_1|_{X(g)}$ is not in general of constant rank.

III.2 Calculations

Since we are interested in the $\mathcal{A}$ class of the germs of projections $p_g(q)$, in seeking "normal forms" for the germs of immersions $g : M \longrightarrow \mathbb{R}^4$, in order to facilitate succeeding calculations, we cannot allow ourselves arbitrary coordinate changes in $\mathbb{R}^4$, and in fact we will restrict ourselves to using only isometric coordinate changes, corresponding to isometries of $\mathbb{R}^4$. However, by using these, as well as arbitrary smooth coordinate changes in M, it is clear that we may bring the germ at $x_0 \in M$ of any immersion $g : M \longrightarrow \mathbb{R}^4$ to the form

$$g(x,y) = (x,\ y,\ b(x,y),\ c(x,y)), \quad \text{with } b,c \in \mathcal{M}_2^{\,2}.$$

and in all our calculations we will suppose the germ of g at $x_0 = 0$ to be of this form , which we shall refer to as Monge form. We shall write the Taylor series of b and c as

$$\sum_{i,j} b_{i,j} x^i y^j \quad \text{and} \quad \sum_{i,j} c_{i,j} x^i y^j$$

<u>The Curvature Ellipse</u> An extremely useful second order invariant of the embedding $g:M \longrightarrow \mathbb{R}^4$ is the curvature ellipse, studied in some detail in [14] . It is defined as follows:

let $S^1(T_xM)$ be the unit circle in T_xM (with respect to the metric induced by g), and parametrise it by $(\cos\theta , \sin\theta)$. Let N_x be the orthogonal complement in $\mathbb{R}^4$ to $T_{g(x)}\, g(M)$, and let $\pi_x : \mathbb{R}^4 \longrightarrow N_x$ be orthogonal projection. Then the curvature ellipse (at x) is defined to be the image of $S^1(T_xM)$ under the map

$$\eta(\theta) = \pi_x \, (d^2g(x)((\cos\theta , \sin\theta), (\cos\theta , \sin\theta)).$$

For $g:M \longrightarrow \mathbb{R}^4$ in Monge form , the curvature ellipse at 0 is thus the image of the map

$$\theta \longmapsto 2(b_{2,0}\cos^2\theta + b_{1,1}\cos\theta\sin\theta + b_{0,2}\sin^2\theta,\ c_{2,0}\cos^2\theta + c_{1,1}\cos\theta\sin\theta + c_{0,2}\sin^2\theta).$$

The map $\pi_x \cdot d^2g$ is of course the second fundamental form.

We shall say that a point $x \in M$ is <u>hyperbolic</u>, <u>elliptic</u> or <u>parabolic</u> according to whether the point $0 \in N_xM$ lies inside, outside or on the curvature ellipse.

We shall refer to the directions θ in T_xM for which $\frac{\partial\eta}{\partial\theta}$ is parallel to $\eta(\theta)$ as <u>asymptotic directions</u>. At a hyperbolic point, then providing the ellipse is not a radial line segment, there are two distinct asymptotic directions, while at parabolic points, again providing that the ellipse is not a radial line segment, there is only one asymptotic direction. When the ellipse is a radial line segment then every tangent direction is asymptotic.

The set of points $x \in M$ such thet the curvature ellipse passes through $0 \in N_x M$ is the <u>parabolic curve</u> .

It may also be characterised by the contact between the surface $g(M)$ and its tangent plane $T_{g(x)}g(M)$ at the point x, but we shall not go into that here.

It turns out that the configuration of the ellipse and $0 \in N_x M$ is of considerable importance in determining the $\mathcal{A}$-classes of the germs of projection $p_g(q)$ at x when the line joining q and $g(x)$ is tangent to $g(M)$ at $g(x)$.(Projections in non-tangent directions are of course non-singular). In particular, projections in tangent directions which are not asymptotic are all cross-caps (III.2: 2). Apart from the hyperbolic, parabolic and elliptic configurations, two other configurations are important in determining the $\mathcal{A}$-classes of projections in tangent directions: when the ellipse is a radial line segment (see III.2:8) and when it is a radial line segment with one end at the origin. In each of these cases, all tangent directions at x are asymptotic, while in the second case $\exp_g$ has a Σ^2 singularity in the fibre of TM over x (III.2:2).

III.2:1 <u>Lemma</u> If $g(x,y) = (x,y,b(x,y),c(x,y))$, $q = (0,v,0,0)$ $(v \neq 0)$ and $H = (0,1,0,0)$, then

i) If $b_{0,2} \neq 0$, $j^3 p_g(q)(0)$ is equivalent to

$$\Big(x,\ y^2+0(3),\ (b_{0,2}c_{1,1}-c_{0,2}b_{1,1})xy+\Big[c_{2,1}b_{0,2}-b_{2,1}c_{0,2}-\frac{(c_{1,2}b_{0,2}-b_{1,2}c_{0,2})b_{1,1}}{b_{0,2}}$$

$$+\frac{3b_{1,1}^2}{4b_{0,2}^2}(c_{0,3}b_{0,2}-b_{0,3}c_{0,2})+\frac{c_{0,2}b_{2,0}-b_{0,2}c_{2,0}}{v}\Big]x^2y+(c_{0,2}b_{0,2}-b_{0,3}c_{0,2})y^3\Big).$$

ii) If $b_{0,2} = c_{0,2} = 0$, and $b_{1,1} \neq 0$, then $j^3p_g(q)(0)$ is equivalent to

$$(x,\ b_{1,1}xy+b_{1,2}xy^2+b_{0,3}y^3, (c_{1,2}b_{1,1}-b_{1,2}c_{1,1})xy^2 + (c_{0,3}b_{1,1}-b_{0,3}c_{1,1})y^3).$$

Proof We have

$$p_g(q)(x,y) = \left(\frac{x}{1-\frac{y}{v}}, \frac{b(x,y)}{1-\frac{y}{v}}, \frac{c(x,y)}{1-\frac{y}{v}} \right)$$

and putting $\tilde{x} = \dfrac{x}{1-\frac{y}{v}}$, $j^3p_g(q)(0)$ is equal to

$$(\tilde{x},\ b_{2,0}\tilde{x}^2+b_{1,1}\tilde{x}y+b_{0,2}y^2+(b_{2,1}-\frac{b_{2,0}}{v})\tilde{x}^2y+b_{1,2}\tilde{x}y^2 + (b_{0,3}+\frac{b_{0,2}}{v})y^3,$$

$$c_{2,0}\tilde{x}^2+c_{1,1}\tilde{x}y+c_{0,2}y^2+(c_{2,1}-\frac{c_{2,0}}{v})\tilde{x}^2y+c_{1,2}\tilde{x}y^2 + (c_{0,3}+\frac{c_{0,2}}{v})y^3).$$

i) Remove powers of $\tilde{x}$ from the second and third components by the obvious left coordinate changes, and then put $\overline{Z} = b_{0,2}Z - c_{0,2}Y$ to obtain, in the third componnnt,

$$(c_{1,1}b_{0,2}-b_{1,1}c_{0,2})\tilde{x}y+(c_{2,1}b_{0,2}-b_{2,1}c_{0,2}+\frac{c_{0,2}b_{2,0}-b_{0,2}c_{2,0}}{v})\tilde{x}^2y +$$

$$+ (c_{1,2}b_{0,2} - b_{1,2}c_{0,2})\tilde{x}y^2 + (c_{0,3}b_{0,2} - b_{0,3}c_{0,2})y^3.$$

Now put $\tilde{y} = y + \dfrac{b_{1,1}}{2b_{0,2}}\tilde{x}$ and eliminate all $\tilde{x}^2$, $\tilde{x}^3$ and $\tilde{x}\tilde{y}^2$ terms from the second and third components by the obvious left coordinate changes to bring $j^3p_g(q)(0)$ to the desired form.

ii) The coordinate change $\overline{Z} = b_{1,1}Z - c_{1,1}Y$, followed by obvious left coordinate changes to remove the $\tilde{x}^2$, $\tilde{x}^3$ and $\tilde{x}^2y$ terms from the second and third components, bring $j^3p_g(q)(0)$ to the desired form∎

III.2:2 Proposition

i) If the direction of projection is tangential but not asymptotic at $x \in M$ then the germ at x of $p_g(q)$ is a cross-cap.

ii) With the hypotheses of the preceeding lemma, if O is a hyperbolic point and the direction of projection is asymptotic, then either $b_{0,2}$ or $c_{0,2}$ is different from 0, and so either $j^3p_g(q)(0)$ is equivalent to

$$\Big(x,\ y^2,\ \Big[c_{2,1}b_{0,2} - b_{2,1}c_{0,2} + b_{1,2}c_{1,1} - c_{1,2}b_{1,1} + \frac{3b_{1,1}^2}{4b_{0,2}^2}(c_{0,3}b_{0,2} - b_{0,3}c_{0,2})$$
$$+ \frac{c_{0,2}b_{2,0} - b_{0,2}c_{2,0}}{v}\Big]x^2y + (c_{0,3}b_{0,2} - b_{0,3}c_{0,2})y^3\ \Big),$$

or it becomes so after b and c are interchanged. If the coefficients of both x^2y and y^3 in the third component of this jet are non-zero, then the jet is sufficient and the germ of $p_g(q)$ at 0 is $\mathcal{A}$-equivalent to

$$(x,y) \longmapsto (x,\ y^2,\ y^3 + x^2y). \qquad (S_1).$$

If the coefficient of y^3 is non-zero and $g \in E_1$, then for some k, $1 \leqslant k \leqslant 4$, the germ of $p_g(q)$ at O is equivalent to

$$(x,y) \longmapsto (x,\ y^2,\ y^3 + x^{k+1}y) \qquad (S_k),$$

and moreover, $\exp_g$ has a Σ^{1_k} singularity at (0,q). In this case the $\mathcal{A}$-class of $\exp_g$ at (0,q) determines the $\mathcal{A}$-class of $p_g(q)$ at 0, and vice-versa.

iii) If $x \in M$ is a hyperbolic point, then the germs at x of all projections in tangent directions are equivalent to germs of the form

$$(x,y) \longmapsto (x,\ y^2,\ yp(x,y^2)).$$

iv) For g as in the preceeding lemma, if $b_{0,2} = c_{0,2} = b_{1,1} = c_{1,1} = 0$, then $\exp_g$ has a Σ^2 singularity at $(0,q)$, and it follows by III:4 that if $g \in E_1$, then this possibility can be excluded. Thus, after interchanging b and c if necessary, either case (i) or case (ii) of the preceeding lemma applies to g.

Proof

i) It is apparent from the lemma that the germ of $p_g(q)$ at 0 is a cross-cap if and only if $b_{1,1}c_{0,2} - c_{1,1}b_{0,2} \neq 0$. Now the curvature ellipse at 0 is the image of the unit circle in T_0M under the map

$$\eta : (x,y) \longrightarrow (b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2,\ c_{2,0}x^2 + c_{1,1}xy + c_{0,2}y^2),$$

and so the direction (0,1) (which is the direction determined by the point q of projection) is asymptotic if and only if the differential at (0,1) of η, applied to the unit tangent vector in $T_{(0,1)}S^1$, gives a vector parallel to $\eta(0,1)$. That is, if $(b_{1,1}, c_{1,1})$ is parallel to $(b_{0,2}, c_{0,2})$.

ii) Assume that the direction of projection is asymptotic, so that $b_{1,1}c_{0,2} - c_{1,1}b_{0,2} = 0$. Then the second-order term disappears from the third component of the 3-jet in the statement of III.2:1(i). It follows by I.5:2 that the 3-jet of $p_g(q)$ at 0 is equivalent to the 3-jet in the statment of this proposition. The remaining assertions follow by I.5:12 and I.5:18 (on the $\mathcal{A}$-classification and degree of determinacy of map-germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$) and by the isomorphism between the local algbras of $\exp_g$ at $(0,q)$ and of $\tilde{d}p_g(q)$ at x (II.2:3 and II.2:7).

ii) This follows, by I.5:3 and I.5:4, from the fact that if x is a hyperbolic point then the 2-jet at x of the projection in any tangential direction is equivalent either to (x,y^2, xy) or $(x,y^2,0)$.

iv)If $b_{0,2} = c_{0,2} = b_{1,1} = c_{1,1} = 0$ then $j^2p_g(q)(0) = (x, 0, 0)$ and so $\tilde{d}p_g(q)$ has a Σ^2 singularity at 0.

III.2:3 Remark One may check by straightforward calculation (which we omit here) that the coefficient of y^3 in the third component of the 3-jet of III.2:2(ii) vanishes if and only if the torsion of the curve $V \cap g(M)$ vanishes at 0, where V is any hyperplane meeting g(M) transversely at 0 and containing the vector $\overrightarrow{0q}$. Here we consider $V \cap g(M)$ as a 3-space curve, contained as it is in V. If (as in this case) $\overrightarrow{0q}$ is an asymptotic direction, then the vanishing of the torsion of $V \cap g(M)$ is independent of the choice of hyperplane V, but if $\overrightarrow{0q}$ is not an asymptotic direction then this independence is lost. However, it does make sense to speak of twisting and non-twisting asymptotic directions, the first when the torsion of $V \cap g(M)$ at x is non-zero, the second when it is zero. Thus, by III.2:(ii), projections in twisting asymptotic directions are equivalent to S_k for some k.

One might expect, on naive grounds of codimension, that generically, hyperbolic points at which one or other asymptotic direction is non-twisting form a curve, and this is indeed the case, although the condition on $g \in Imm(M,R^4)$ is not just $g \in E_1$.

III.2:4 Lemma Let A_0 be the set of 3-jets of the form $(x,y,b(x,y),c(x,y))$ in $J^3(2,4)$, such that b, $c \in \mathcal{M}_2^{\,2}$ and also the following conditions hold:

i) $b_{0,2}c_{1,1} - c_{0,2}b_{1,1} = b_{0,2}c_{0,3} - c_{0,2}b_{0,3} = 0$

ii) $(b_{0,2}, c_{0,2}) \neq (0,0)$

iii) $(b_{2,0}c_{0,2} - c_{2,0}b_{0,2}, b_{0,2}c_{1,2} - c_{0,2}b_{1,2}) \neq (0,0)$.

Let

$$A = \{(x,y,b(\alpha x+\beta y, -\beta x+\alpha y),c(\alpha x+\beta y, -\beta x+\alpha y) : (x,y,b,c)\in A_0, \alpha^2+\beta^2=1\},$$

let B_0 be the set of 3-jets in $J^3(2,4)$ whose orbit under the action of $\mathrm{Diff}(\mathbb{R}^2,0)\times \mathrm{Isom}(\mathbb{R}^4,0)$ meets A, and let B be the sub-bundle of $J^3(M,\mathbb{R}^4)$ (over $M\times\mathbb{R}^4$) whose fibre is B_0.

Then B is a smoothly immersed submanifold of $J^3(M,R^4)$, of codimension 1.

<u>Proof</u> First, an explanation of conditions (i)-(iii) in the statement. (i) means that if $j^3g(0)\in A_0$ then (0,1) is a non-twisting asymptotic direction, (ii) means that 0 is a hyperbolic point (and also guarantees that A_0 is smooth), and (iii) is introduced to guarantee that the map Γ $S^1\times A_0 \longrightarrow J^3(2,4)$ whose image is A,has constant rank.

Let D be the set of 3-jets of the form $(x,y,b(x,y),c(x,y))$ with $b,c\in \mathcal{M}_2^{\,2}$, and let $\rho :D\longrightarrow \mathbb{R}^2$ be defined by

$$\rho(x,y,b(x,y),c(x,y)) = (b_{1,1}c_{0,2} - c_{1,1}b_{0,2}, c_{0,3}b_{0,2} - b_{0,3}c_{0,2}).$$

Then A_0 is the set of 3-jets in $\rho^{-1}(0,0)$ such that $(b_{0,2}, c_{0,2}) \neq (0,0)$, and since at such points ρ is a submersion, A_0 is a smooth submanifold of D of codimension 2.

Now define a map $\Gamma :S^1\times A_0\longrightarrow D$ by

$$\Gamma((x,y,b(x,y),c(x,y)),(\alpha,\beta)) = (x,y,b(\alpha x+\beta y,-\beta x+\alpha y),c(\alpha x+\beta y,-\beta x+\alpha y)).$$

Γ is of constant rank if and only if it is of constant rank at all points of $\{(1,0)\}\times A_0$, and replacing (α, β) by $(\cos\Theta, \sin\Theta)$, it is of constant (maximal) rank at $(0,z)$ if

$$d\rho(z)\left(\frac{\partial}{\partial\Theta}\Gamma(\Theta, z)\Big|_{\Theta=0}\right) \neq 0.$$

Straightforward calculation shows that

$$d\rho(z)\left(\frac{\partial}{\partial\Theta}\Gamma(\Theta, z)\Big|_{\Theta=0}\right) = \begin{bmatrix} 2(b_{2,0}c_{0,2} - c_{2,0}b_{0,2}) \\ b_{0,2}c_{1,2} - c_{0,2}b_{1,2} \end{bmatrix}$$

if $z \in A_0$, and thus condition (iii) guarantees that Γ is of constant rank. A is thus a smoothly immersed submanifold of D, of codimension 1 (self-intersection takes place at jets which have more than one non-twisting asymptotic direction).

For any jet $z \in A$, the orbit of z under the action of $\mathrm{Diff}(\mathbb{R}^2,0)\times \mathrm{Isom}(\mathbb{R}^4,0)$ is transverse to A in $J^3(2,4)$, and from this and the fact that A is an immersed submanifold, one deduces that B_0 is an immersed submanifold of $J^3(2,4)$ whose codimension is the same as that of A in D, i.e. 1. The conclusion of the lemma then follows∎

III.2:5 <u>Remark</u> Condition (iii) of the statement of the lemma has the following geometrical interpretation: first, if $z \in A_0$, and if also $b_{2,0}c_{0,2} - c_{2,0}b_{0,2} = 0$, then

$$\begin{bmatrix} b_{2,0} & b_{1,1} & b_{0,2} \\ c_{2,0} & c_{1,1} & c_{0,2} \end{bmatrix}$$

has rank 1 (since the last column is non-zero), and so the curvature ellipse is a radial line segment; second, one calculates that if

$V_\theta = Sp\{(-\sin\theta, \cos\theta, 0, 0), (0,0,1,0), (0,0,0,1)\} \subseteq \mathbb{R}^4$, then the derivative with respect to θ of the torsions at 0 of $V_\theta \cap M$, taken at $\theta = 0$, is a non-zero multiple of $(b_{0,2}c_{1,2} - c_{0,2}b_{1,2})$.

It follows from the lemma that if $g \in Imm(M,\mathbb{R}^4)$ and $j^3g \bar{\pitchfork} B$, and if moreover j^3g does not meet B'-B, where B' is defined exactly like B except that condition (iii) of the statement of the lemma is omitted, then the set of points in the hyperbolic region of M at which there is a non-twisting asymptotic direction, is an immersed submanifold of codimension 1 i.e. an immersed curve.

By inspection of the 3-jet in the statement of III.2:2(ii), one sees that unless $c_{0,2}b_{2,0} - b_{0,2}c_{2,0} = 0$, there is at most one value of v for which the coefficient of x^2y in the third component vanishes. Now this condition, taken together with the condition $c_{0,2}b_{1,1} - b_{0,2}c_{1,1} = 0$ which was assumed in order to obtain this 3-jet, is equivalent, providing $(b_{0,2}, c_{0,2}) \neq (0,0)$, to

$$\text{rank}\begin{bmatrix} b_{2,0} & b_{1,1} & b_{0,2} \\ c_{2,0} & c_{1,1} & c_{0,2} \end{bmatrix} = 1,$$

and thus to the curvature ellipse being a radial line segment. Thus, we have

III.2:6 <u>Proposition</u> Let x be a point in the hyperbolic region of M (with respect to the immersion $g:M \longrightarrow \mathbb{R}^4$), and suppose that the curvature ellipse at x is not a radial line segment. Suppose that L is a twisting asymptotic tangent line at x. Then for all points q on L, except perhaps for one, the germ at x of the projection $p_g(q)$ is equivalent to S_1. If now L is a non-twisting asymptotic tangent line at x, then for all points q on L, except perhaps for one, the germ at x of the projection $p_g(q)$ is equivalent to B_k for some $k \geqslant 2$.

Proof By means of an isometry on $\mathbb{R}^4$ and an appropriate choice of coordinates in M arround x, bring the germ of the immersion g at x to Monge form. After a rotation in the tangent plane of M at x, we may suppose that the asymptotic tangent line L is the line through O in $\mathbb{R}^4$, parallel to the vector (0,1,0,0). The result now follows from III.2:2(ii)∎

In order to control the behaviour of the projections $p_g(q)$ in tangent directions at points x in M where the curvature ellipse is a radial line segment, we make the following sequence of definitions, leading up to the definition of a submanifold in the jet bundle $J^3(M,\mathbb{R}^4)$ to which we shall insist that j^3g be transverse.

First, as in the proof of III.2:4, let

$$D = \{(x,y,b(x,y),c(x,y)) \in J^3(2,4) \ : \ b,c \in \mathcal{M}_2^{\,2}\}$$

and let

$$A_1 = \left\{ z \in D \ : \ \text{rank} \begin{bmatrix} b_{2,0} & b_{1,1} & b_{0,2} \\ c_{2,0} & c_{1,1} & c_{0,2} \end{bmatrix} = 1 \right\}.$$

Then A_1 is a smooth submanifold of D of codimension 2, and if $j^3g(0) \in A_1$ then every tangent direction at 0 is asymptotic. Now let

$$A_1^{\,h} = \{z \in A_1 \ : \ b_{1,1}^{\,2} - 4b_{2,0}b_{0,2} < 0, \ c_{1,1}^{\,2} - 4c_{2,0}c_{0,2} < 0\} ,$$

so that if $j^3g(0) \in A_1^{\,h}$ then 0 is a hyperbolic point. Note that $A_1 - A_1^{\,h}$ has non-empty interior in A_1.

If $z \in A_1^h$, then $(b_{0,2}, c_{0,2}) \neq (0,0) \neq (b_{2,0}, c_{2,0})$.

As in the proof of Lemma III.2:4, let $\Gamma : S^1 \times D \longrightarrow D$ be defined by

$$\Gamma((\alpha,\beta),(x,y,b(x,y),c(x,y))) = (x,y,b(\alpha x+\beta y, -\beta x+\alpha y),c(\alpha x+\beta y, -\beta x+\alpha y))$$

and let the coefficients of $x^i y^j$ in the third and fourth components of $((\alpha,\beta),(x,y,b,c))$ be written as $\tilde{b}_{i,j}(\Theta)$, $\tilde{c}_{i,j}(\Theta)$ respectively.

Now it follows from III.2:2(ii) that if $g \in \mathrm{Imm}(M,\mathbb{R}^4)$ is (locally) in Monge form and $j^3 g \in A_1^h$, and if $q = (-v\sin\Theta, v\cos\Theta, 0, 0)$, then $j^3 p_g(q)(0)$ is equivalent to a 3-jet

$$(x, y^2, R_{2,1}(\tilde{b}_{i,j}(\Theta), \tilde{c}_{i,j}(\Theta))x^2 y + R_{0,3}(\tilde{b}_{i,j}(\Theta), \tilde{c}_{i,j}(\Theta))y^3)$$

where, if $\tilde{b}_{0,2}(\Theta) \neq 0$, $R_{2,1}$ is equal to

$$4\tilde{b}_{0,2}^2(\tilde{c}_{2,1}\tilde{b}_{0,2}-\tilde{b}_{2,1}\tilde{c}_{0,2}+\tilde{b}_{1,2}\tilde{c}_{1,1}-\tilde{c}_{1,2}\tilde{b}_{1,1}) + 3\tilde{b}_{1,1}^2(\tilde{c}_{0,3}\tilde{b}_{0,2}-\tilde{b}_{0,3}\tilde{c}_{0,2})$$

and $R_{0,3}$ is equal to

$$4\tilde{b}_{0,2}^2(\tilde{c}_{0,3}\tilde{b}_{0,2} - \tilde{b}_{0,3}\tilde{c}_{0,2}),$$

and if $\tilde{c}_{0,2}(\Theta) \neq 0$, $R_{2,1}$ and $R_{0,3}$ are equal to the same functions except that the $\tilde{b}_{i,j}$ and $\tilde{c}_{i,j}$ are interchanged. If both $\tilde{b}_{0,2} \neq 0$ and $\tilde{c}_{0,2} \neq 0$, the two 3-jets are of course equivalent.

By taking the discriminants of the four homogeneous polynomials in $(\cos\Theta, \sin\Theta)$ thus obtained, we obtain four polynomial functions of the original coefficients $b_{i,j}$, $c_{i,j}$, which we shall write as $f_1,\ldots,f_4$. It is easy to check that none of these is identically zero. Now let

$$A_1^{h'} = \{z \in A_1^h : f_1(z)f_2(z)f_3(z)f_4(z) = 0, \text{ or } R_{2,1}(z) \text{ and } R_{0,3}(z) \text{ have a common root}\},$$

let $B_1^{h'}$ be the set of 3-jets in $J^3(2,4)$ whose orbit under the action of $\mathrm{Diff}(\mathbb{R}^2,0)\times\mathrm{Isom}(\mathbb{R}^4,0)$ meets $A_1^{h'}$, and let $B^{h'}$ be the sub-bundle of $J^3(M,\mathbb{R}^4)$ whose fibre is $B_1^{h'}$. Clearly $B_1^{h'}$ is algebraic. Then we have

III.2:7 Lemma $B^{h'}$ is an algebraic subset of $J^3(M,\mathbb{R}^4)$ whose codimension is equal to 3.

Proof By this we mean that ~~we mean that~~ the intersection of $B^{h'}$ with each fibre of $J^3(M,\mathbb{R}^4)$ over $M\times\mathbb{R}^4$ is algebraic, and thus $B^{h'}$ has a canonical stratification, induced from the canonical algebraic stratification of $B_1^{h'}$, whose open stratum has the same codimension in $J^3(M,\mathbb{R}^4)$ as that of the open stratum of $B_1^{h'}$ in $J^3(2,4)$. It is immediate from the definition of $B_1^{h'}$ that this codimension is 3 ∎

III.2:8 Proposition Let $g\in\mathrm{Imm}(M,\mathbb{R}^4)$, let B^h be the sub-bundle of $J^3(M,\mathbb{R}^4)$ consisting of those 3-jets whose orbit under the action of $\mathrm{Diff}(M)\times\mathrm{Isom}(\mathbb{R}^4)$ meets A^h, and suppose that $j^3g \pitchfork B^h$ and $j^3g \pitchfork B^{h'}$. Then

i) There is a set of isolated points in the hyperbolic region of M at which the curvature ellipse is a radial line segment

and

ii) If x is one of these points, then at x there is a finite number of tangent lines L_i, such that if $q\in L_i$ then $j^3p_g(q)(x)$ is equivalent to (x,y^2,y^3), there is a finite number of tangent lines L_i' such that if $q\in L_i'$ then $j^3p_g(q)(x)$ is equivalent to (x,y^2,x^2y), and if q lies in $T_{g(x)}g(M)$ but does not lie on any of the L_i or L_i' then the germ at x of $p_g(q)$ is equivalent to

$$(x,y)\longrightarrow(x,\ y^2,\ y^3 \pm x^2y).$$

Proof First, it is clear that B^h is a smooth submanifold of $J^3(M,\mathbb{R}^4)$, of codimension 2, so that the first affirmation of the proposition follows. Now $j^3g \overline{\pitchfork} B^{h'}$ means that it does not meet it, so that the homogeneous polynomials $R_{2,1}(b_{i,j}, c_{i,j})$ and $R_{0,3}(b_{i,j}, c_{i,j})$ defined by bringing the germ of g at x to Monge form by means of an appropriate choice of coordinates on M at x, and an isometry of $\mathbb{R}^4$, have only simple roots, none of which is common to both of them. Thus, in the tangent directions for which $R_{2,1}$ vanishes (which are finite in number), $R_{0,3}$ does not vanish, and vice-versa. Projections in the tangent directions for which neither $R_{2,1}$ nor $R_{0,3}$ vanish have 3-jets at x equivalent to $(x, y^2, y^3 \pm x^2y)$, and since this 3-jet is sufficient, the last statement follows∎

By the Thom Transversality Theorem, the conclusion of this proposition applies to a residual set of $g \in \mathrm{Imm}(M,\mathbb{R}^4)$.

We now turn our attention to the behaviour of the projections $p_g(q)$ at parabolic points of M. There are two cases to consider: first, when the curvature ellipse is not a radial line segment, so that there is only one asymptotic direction, and second, where the ellipse is a radial line segment, in which case every tangent direction is asymptotic. Before considering these two cases, however, we prove

III.2:9 Proposition There is a residual set E_2 of $g \in \mathrm{Imm}(M,\mathbb{R}^4)$ for which the parabolic set is a smoothly immersed curve with transverse self-intersection at points where the curvature ellipse is a radial line segment.

Proof Choose a moving frame e_1, e_2, e_3, e_4 defined on an open set U in

M such that for all x in U, $e_1(x)$ and $e_2(x)$ are a basis for $T_{g(x)}g(M)$ and $e_3(x)$ and $e_4(x)$ are a basis for N_xM. Then the point $x \in U$ is parabolic if there exists a unit vector $u \in T_xM$ such that

$$e_3(x)\cdot d^2g(x)(u,u) = e_4(x)\cdot d^2g(x)(u,u) = 0 \quad .$$

Trivialise TM over U as $\mathbb{R}^2 \times \mathbb{R}^2$. Then the parabolic set is the projection into the first factor of the set P defined by

$$\{(x,u) \in \mathbb{R}^2 \times S^1 \ : \ e_3(x)\cdot d^2g(x)(u,u) = e_4(x)\cdot d^2g(x)(u,u) = 0\} \ .$$

We want to show that generically this projection is non-singular, but first we must show that P itself is, generically, a smooth submanifold of $\mathbb{R}^2 \times S^1$. In order to do this we look at the differential of the map ρ used to define P. We find that $d\rho(x,u)$ has matrix

$$\begin{bmatrix} de_3(x)\cdot d^2g(x)(u,u) + e_3(x)\cdot d^3g(x)(u,u) & 2e_3(x)\cdot d^2g(x)(u) \\ de_4(x)\cdot d^2g(x)(u,u) + e_4(x)\cdot d^3g(x)(u,u) & 2e_4(x)\cdot d^2g(x)(u) \end{bmatrix}$$

Now suppose that g is in Monge form , that x = 0, that u = (0,1) and that $(x,u) \in P$. Then $b_{0,2} = c_{0,2} = 0$, and this matrix becomes

$$(1) \qquad \begin{bmatrix} 2b_{1,2} & 6b_{0,3} & 2b_{1,1} & 0 \\ 2c_{1,2} & 6c_{0,3} & 2c_{1,1} & 0 \end{bmatrix} .$$

Now the codimension in D of the algebraic set Q_0 defined by

$$\{(x,y,b(x,y),c(x,y)) \ : \ b_{0,2} = c_{0,2} = 0, \text{ and matrix (1) has rank} < 2\}$$

is 4, and so the image Q of the map $\Gamma : S^1 \times Q \longrightarrow D$ defined as in III.2:4 is of codimension 3 in D. Now proceeding as in III.2:4 to define a sub-bundle Q_1 of $J^3(M,\mathbb{R}^4)$ by taking as its fibre the set of 3-jets in $J^3(2,4)$ whose orbit under the action of $\mathrm{Diff}(\mathbb{R}^2,0)\times \mathrm{Isom}(\mathbb{R}^4,0)$ meets Q, we see that the codim of Q_1 in $J^3(M,\mathbb{R}^4)$ is also 3. It follows, since Q is algebraic, that for a residual set of $g \in \mathrm{Imm}(M,\mathbb{R}^4)$, j^3g does not meet Q_1. But that is to say that generically the map ρ defined on the previous page is a submersion at all points of P, so that generically P is a smooth curve.

Now in order to show that generically the projection $\pi : P \longrightarrow \mathbb{R}^2$ is non-singular, we look once again at the matrix (1). It is clear that $\pi|_P$ has non-trivial kernel if and only if matrix (1), applied to the vector ((0,0),(1,0)), is equal to 0 ((1,0) is of course the unit tangent vector to S^1 at u), and this happens if and only if $b_{1,1} = c_{1,1} = 0$. However, since we are assuming that $b_{0,2} = c_{0,2} = 0$, this means that $\exp_g$ has rank 2 at all points (0,q) in TM, where $q = (0,q_2)$, $q_2 \in \mathbb{R}-\{0\}$. Now by III.1:4, if $g \in E_1$ then $\exp_g$ has no Σ^2 singularities on TM-M, and so we conclude that if $g \in E_1$ and also P is a smooth curve, then the parabolic set in M is a smoothly immersed curve.

Now, nothing in the previous analysis excludes the possibility that $\pi : P \longrightarrow M$ should have multiple points i.e. self-intersections of the parabolic curve on M. In fact, if the curvature ellipse is not a line segment at x then there is only one value of $u \in S^1$ such that $\rho(x,u) = 0$, and so x is a simple point of the parabolic curve, but if the curvature ellipse is a line segment (and we assume that it does not degenerate to a point) there are two distinct values of u, unless also the ellipse is a line segment whose centre is at 0. This last case can be excluded for a residual set of g on the grounds of codimension - in the case we are considering, with $b_{0,2} = c_{0,2} = 0$, it is equivalent to having also $b_{2,0} = c_{2,0} = 0$.

Assuming, then, that $b_{0,2} = c_{0,2} = 0$, and that the ellipse is a line segment whose centre is not at 0, we find that as well as $(0,(0,1))$ we have

$$(0, \frac{(-b_{1,1}, b_{2,0})}{\|(-b_{1,1}, b_{2,0})\|}) \in P.$$

Now, it is easy to check that the condition that the tangent vectors to the two branches of $\pi(P)$ at 0 should be parallel, defines an algebraic set in D whose codimension is 3, and then by the usual proceedure one concludes that for a residual set of $g \in \mathrm{Imm}(M,\mathbb{R}^4)$, all self-intersections of the parabolic curve are transverse. This completes the proof∎

III.2:10 Remark For future reference, we note that if, in the notation of the preceding proof,

$$e_3(0)\cdot d^2g(0)((0,1),(0,1)) = e_4(0)\cdot d^2g(0)((0,1),(0,1)) = 0$$

and g is in Monge form , then the tangent line to the corresponding branch of the parabolic curve at 0,is generated by the vector

$$\frac{1}{(b_{1,2}c_{0,3} - c_{1,2}b_{0,3})}(3(b_{0,3}c_{1,1} - c_{0,3}b_{1,1}), (b_{1,1}c_{1,2} - c_{1,1}b_{1,2}))$$

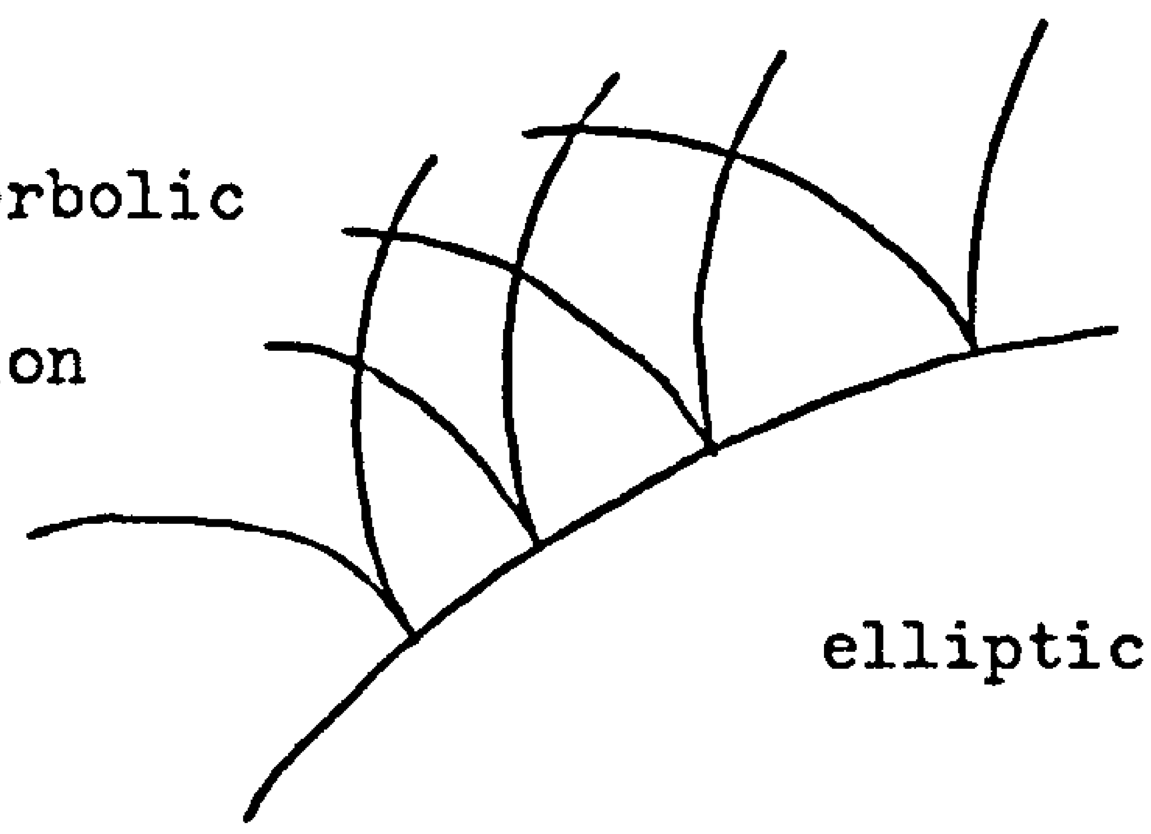
if the determinant in the denominator is not 0, and by the vector

$$(-3c_{0,3}, c_{1,2}) \quad (\text{or } (-3b_{0,3}, b_{1,2}) \text{ if this is } 0)$$

if the determinant is 0. The vanishing of this determinant corresponds to the vanishing of the component in T_uS^1 of the unit tangent vector to P at (0,u).

III.2:11 Remark We would like to stress at this point that although generically the parabolic curve is 1-dimensional and the set of points at which the curvature ellipse is a radial line segment is 0-dimensional, it is not the case that generically these two do not meet. As noted on page 158, that the ellipse should pass through the origin is a codimension 0 condition on the set of 2-jets of germs of immersions for which the ellipse is a radial line segment, and since this (algebraic) set has a smooth stratum of codimension 2, it is possible to construct immersions g for which j^2g is transverse to the part of this stratum corresponding to the ellipse passing through the origin.

That the parabolic curve can present transverse self-intersection raises interesting problems about the configuration of the asymptotic curves in the neighbourhood of such points, the asymptotic curves being the integral curves of the multi-valued line field defined on the union of the elliptic and parabolic regions by the asymptotic directions. Lak Dara, in a paper on the local form of solutions to multiform differential equations in the plane, ([13]), conjectures local normal forms which would be applicable in this context except at points at which the curvature ellipse is a radial line segment, and Banchoff, Gaffney and McCrory have made use of these conjectures in [5] to draw pictures of the (generically) possible configurations of asymptotic curves and parabolic locus for surfaces immersed in $\mathbb{R}^3$. Lak Dara does prove one result ([13], page 122) which may be used to show that at simple points of the parabolic locus on a surface immersed in $\mathbb{R}^4$ where also the asymptotic direction is not tangent to the parabolic curve, the configuration of the asymptotic curves is diffeomorphic to the one drawn below, in which each curve has a first-order cusp where it meets the parabolic locus.



However, proofs of his other conjectures (which relate in our case to the configuration of asymptotic curves at points on the parabolic locus where the (unique) asymptotic direction is tangent to the parabolic locus) have not been forthcoming.

Before leaving this area, we will make a conjecture of our own, about the generic configuration of asymptotic curves in the neighbourhood of a point on M where the parabolic curve meets itself transversely. First, a lemma:

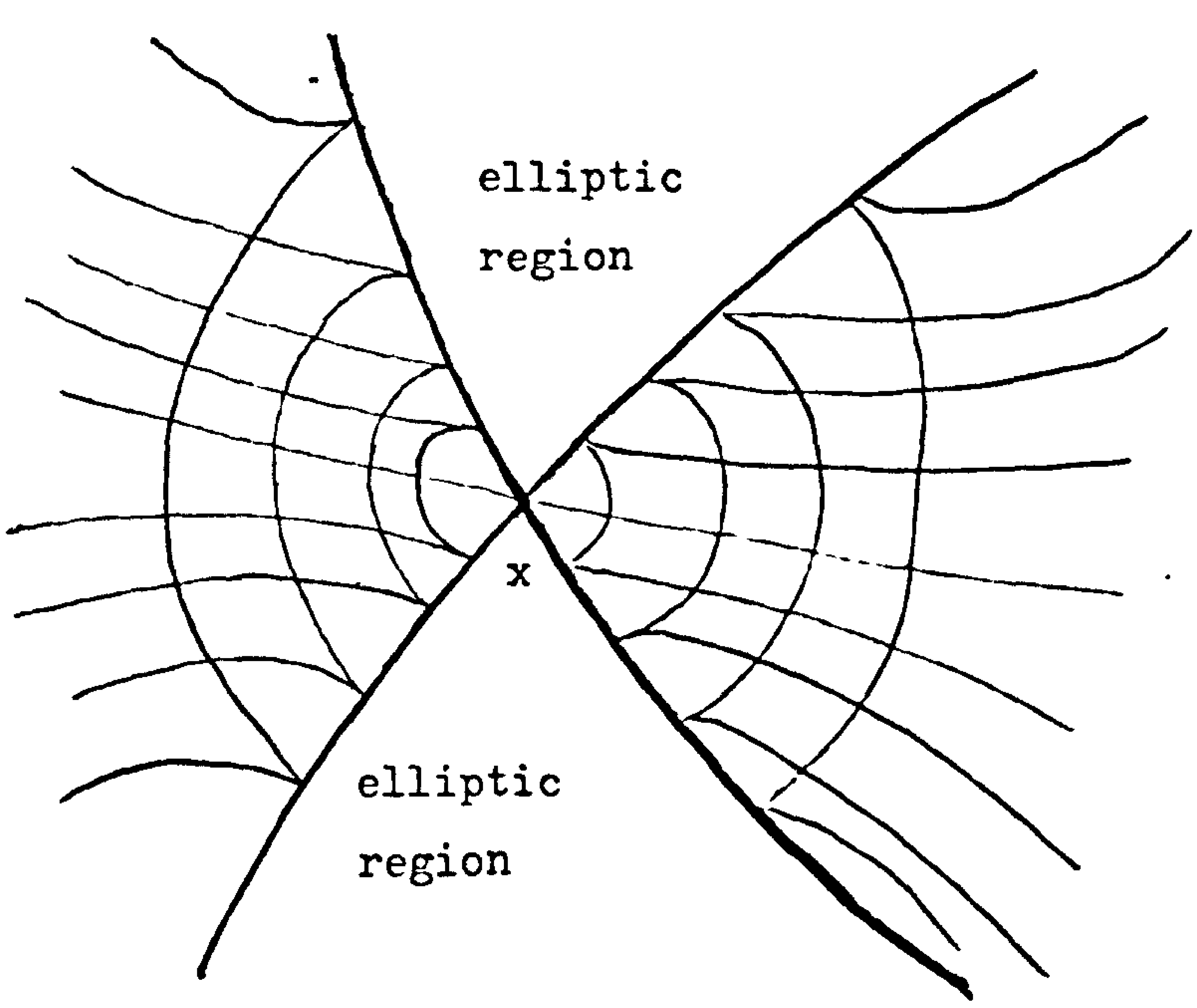
III.2:12 <u>Lemma</u> There is a residual set E_3 of $g \in \mathrm{Imm}(M,\mathbb{R}^4)$, contained in E_2, such that at each point $x \in M$ at which the parabolic locus crosses itself, neither of the two root-directions of $e_3(x)\cdot d^2g(x) = e_4(x)\cdot d^2g(x) = 0$ is tangent to either branch of the parabolic locus.

<u>Proof</u> Assume, as in the proof of III.2:9, that g is in Monge form, that x = 0, and that the curvature ellipse is a line segment passing through 0. Then after a rotation of the normal plane N_0M we may assume that $c_{2,0} = c_{1,1} = c_{0,2} = 0$, and after a rotation of the tangent plane we may assume that $b_{0,2} = 0$. Then if $g \in E_2$, both $b_{1,1}$ and $b_{2,0}$ are diff-

erent from O. One calculates that the two root directions of the second fundamental form are $(0,1)$ and $(-b_{1,1}, b_{2,0})$, and that the tangent directions to the two corresponding branches of the parabolic locus are determined, respectively, by the vectors $(-3c_{0,3}, c_{1,2})$ and $(2b_{1,1}^2c_{2,1}-4b_{1,1}b_{2,0}c_{1,2}+6b_{2,0}^2c_{0,3}, -(2b_{2,0}^2c_{1,2}-4b_{1,1}b_{2,0}c_{2,1}+6b_{1,1}^2c_{0,3}))$.

It is easy to see from this that the condition that either of the two root-directions should be tangent to either of the branches of the parabolic locus, raises by 1 the codimension of the set of 3-jets in question, and since this set is already of codimension 2, an application of the Thom Transversality Theorem concludes the proof∎

III.2:13 <u>Conjecture</u> Generically, the configuration of asymptotic curves in the neighbourhood of a point $x \in M$ where the parabolic locus crosses itself transversely, is equivalent, under a homeomorphism of (M,x) which is C^∞ except at x, to the configuration drawn below.



The parabolic locus is represented by the thick lines, the asymptotic curves by the thin lines.

Returning now to the study of germs of projection, we have

III.2:14 Proposition Let $g \in E_2$. If x is a parabolic point of M at which the curvature ellipse is not a radial line segment, then

i) If the (unique) asymptotic direction is not tangent to the parabolic curve at x, then for all points $q \neq g(x)$ on the asymptotic line at x, the germ of $p_g(q)$ at x is equivalent (if finitely determined) to

$$(x,y) \quad (x,\ xy + y^{3k-1},\ y^3) \qquad (H_k)$$

for some $k \geqslant 2$.

ii) If the asymptotic direction is tangent to the parabolic curve, and if, moreover, $g \in E_1$, then for all points $q \neq g(x)$ on the asymptotic line at x, $j^4 p_g(q)(0)$ is equivalent to

$$(x,\ xy + y^3,\ xy^2 + cy^4)$$

for some value of c which does not vary with q.

Proof Assume that g is in Monge form and that x = 0. Assume also, after a rotation in T_0M, that the unique asymptotic direction is spanned by (0,1), so that $b_{0,2} = c_{0,2} = 0$. Then at least one of $b_{1,1}$, $c_{1,1}$ is different form 0, since the curvature ellipse is not a radial line segment, and hence after a rotation in the normal plane we may assume that $b_{1,1}$ is different from 0. Then by III.2:1(ii), $j^3 p_g(q)(0)$ is equivalent to

$$(x, b_{1,1}xy+b_{1,2}xy^2+b_{0,3}y^3,\ (c_{1,2}b_{1,1}-b_{1,2}c_{1,1})xy^2+(c_{0,3}b_{1,1}-b_{0,3}c_{1,1})y^3)\ .$$

Now if the coefficient of y^3 in the third component is non-zero, $j^3 p_g(q)(0)$ is equivalent to (x,xy,y^3) (see the proof of I.6:1) and so it follows by

I.6.1:2 that the germ at 0 of $p_g(q)$ is equivalent to H_k for some $k \geq 2$. However, from III.2:10 we see that this coefficient vanishes precisely when the asymptotic direction is tangent to the parabolic curve at x. This completes the proof of (i).

Now assume that the asymptotic direction is tangent to the parabolic curve at x, and that $g \in E_1$. Then by I.6:1, $j^3p_g(q)(0)$ is equivalent to one of $(x, xy+y^3, xy^2)$, (x, xy, xy^2), $(x, xy+y^3, 0)$, and $(x, xy, 0)$. We now show that only the first of these is possible. The last can be eliminated immediately, since the codimension of its $\mathcal{A}^3$ orbit, to which $j^3_x p_g$ is transverse, is 7. The codimension of the $\mathcal{A}^3$ orbits of each of the two remaining 3-jets is 6, and thus since $j^3_x p_g$ is transverse to these orbits, their preimage in $M \times \mathbb{R}^4$, if not empty, consists of a collection of isolated points. However, as can be seen from the expression for $j^3p_g(q)(0)$, its $\mathcal{A}^3$ class does not vary as q moves along the asymptotic tangent line, and thus for no point q can $j^3p_g(q)(0)$ be equivalent to either (x,xy,xy^2) or $(x,xy+y^3,0)$.

It now follows, by I.6.2:1, that $j^4p_g(q)(0)$ is equivalent to

$$(x,\ xy + y^3,\ xy^2 + cy^4)$$

for some $c \in \mathbb{R}$. To show that the value of c does not vary as q moves along the asymptotic tangent line, we carry out a straightforward calculation, which is slightly facilitated by assuming, after a rotation in N_0M, that $c_{1,1} = 0$. Our assumption that the asymptotic tangent direction is tangent to the parabolic curve at 0 now becomes simply $c_{0,3} = 0$, and by what we have shown about $j^3p_g(q)(0)$, we must have $b_{0,3} \neq 0 \neq c_{1,2}$. As usual $j^4p_g(q)(0)$ is equivalent to

$$(x, \sum_{2\leqslant i+j\leqslant 4} b_{i,j}x^i y^j(1-\frac{y}{v})^{i-1}, \sum_{2\leqslant i+j\leqslant 4} c_{i,j}x^i y^j(1-\frac{y}{v})^{i-1})$$

and since $b_{1,1} \neq 0$ it is clear that we can remove all x^4, x^3y and x^2y^2 terms from the third and fourth components of the 4-jet by means of left coordinate changes. Similarly we may remove all x^2, x^3 and x^4 terms from the second and third components by obvious left coordinate changes. These we shall carry out tacitly at each stage in the ensuing calculation. Thus, $j^4p_g(q)(0)$ is equivalent to

$$(x, b_{1,1}xy + (b_{2,1} - \frac{b_{2,0}}{v})x^2y + b_{1,2}xy^2 + b_{1,3}xy^3 + (b_{0,4} + \frac{b_{0,3}}{v})y^4,$$

$$(c_{2,1} - \frac{c_{2,0}}{v})x^2y + c_{1,2}xy^2 + c_{1,3}xy^3 + c_{0,4}y^4).$$

Now put $\overline{Y} = Y - (\frac{b_{2,1}}{b_{1,1}} - \frac{b_{2,0}}{vb_{1,1}})XY$, $\overline{Z} = Z - (\frac{c_{2,1}}{b_{1,1}} - \frac{c_{2,0}}{vb_{1,1}})XY$ to obtain

$$(x, b_{1,1}xy + b_{1,2}xy^2 + b_{0,3}y^3 + (b_{1,3} - \frac{b_{0,3}}{b_{1,1}}(b_{2,1} - \frac{b_{2,0}}{v}))xy^3 + (b_{0,4} + \frac{b_{0,3}}{v})y^4,$$

$$c_{1,2}xy^2 + (c_{1,3} - \frac{b_{0,3}}{b_{1,1}}(c_{2,1} - \frac{c_{2,0}}{v}))xy^3 + c_{0,4}y^4),$$

and put $\overline{y} = y + \frac{1}{c_{1,2}}(c_{1,3} - \frac{b_{0,3}}{b_{1,1}}(c_{2,1} - \frac{c_{2,0}}{v}))y^2$ to obtain

$$(x, b_{1,1}xy + \tilde{b}_{1,2}xy^2 + \tilde{b}_{1,3}xy^3 + \tilde{b}_{0,4}y^4, c_{1,2}xy^2 + c_{0,4}y^4),$$

where the $\tilde{b}_{i,j}$ are functions of the original coefficients $b_{i,j}$, $c_{i,j}$ and of v. The coordinate change

$$\overline{Y} = Y - \frac{\tilde{b}_{1,2}}{c_{1,2}} Z$$

transforms this to

$$(x,\ b_{1,1}xy + b_{0,3}y^3 + \hat{b}_{1,3}xy^3 + \hat{b}_{0,4}y^4,\ c_{1,2}xy^2 + c_{0,4}y^4)$$

where once again $\hat{b}_{1,3}$ and $\hat{b}_{0,4}$ are functions of the original coefficients and of v, and now the coordinate change

$$\bar{y} = y + \frac{b_{1,3}}{b_{1,1}} y^3$$

transforms it to

$$(x,\ b_{1,1}xy + b_{0,3}y^3 + b_{0,4}y^4,\ c_{1,2}xy^2 + c_{0,4}y^4).$$

Now by the calculations in the proof of I.6.2:1, this is equivalent to

$$(x,\ b_{1,1}xy + b_{0,3}y^4,\ c_{1,2}xy^2 + c_{0,4}y^4),$$

and finally the coordinate changes

$$\underline{x} = \frac{b_{1,1}}{b_{0,3}} x,\ \bar{X} = \frac{b_{1,1}}{b_{0,3}} X,\ \bar{Y} = \frac{11}{b_{0,3}} Y \text{ and } \bar{Z} = \frac{b_{1,1}}{c_{1,2}b_{0,3}} Z$$

transform this to

$$(x,\ xy + y^3,\ xy^2 + \frac{c_{0,4}b_{1,1}}{c_{1,2}b_{0,3}} y^4).$$

Evidently, the coefficient of y^4 in the last component of this 4-jet does not vary with v∎

III.2:15 Remark We do not attempt here to find any relation between the value of this coefficient and the "traditional" differential geometric invariants of the immersion g at 0.

III.2:16 Proposition There is a residual set of $g \in \mathrm{Imm}(M, R^4)$, contained in $E_1 \cap E_2$, for which the set of points on the parabolic locus at which the curvature ellipse is not a radial line segment and the asymptotic direction is tangent to the parabolic curve, is of dimension 0, and such that at each of these points the germs of projection in the asymptotic direction are equivalent to

$$(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + cy^4), \qquad (P_3)$$

the values of c (one for each such point on the parabolic locus) all being different from 0, $\frac{1}{2}$, 1, and $\frac{3}{2}$.

Proof That generically the set of points on the parabolic curve at which the ellipse is not a radial line segment and the asymptotic direction is tangent to the parabolic curve, is of dimension 0, is intuitively obvious from the calculations made in III.2:10, and may be proved by methods similar to those employed in III.2:4. We omit the proof here, as it is clear how it proceeds.

For an immersion g having this property, there will only be a countable number of these points, and thus a countable number of values of the coefficient c in the normal form for the 4-jet of the projections in asymptotic directions. Thus, for a residual set of immersions g, none of these values will be equal to 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and hence for such immersions the 4-jets of projections in asymptotic directions are all sufficient, by I.6.2:5 ∎

The last case that we shall consider is that of the projections at points on the parabolic curve at which the ellipse is a radial line segment.

III.2:1 Proposition If the immersion g belongs to $E_1 \cap E_3$, then at all points x on the parabolic locus at which the curvature ellipse is a radial line segment, the germs at x of projections in the root directions of the second fundamental form are equivalent to

$$(x,y)\longmapsto(x,\ xy + y^{3k-1},\ y^3)$$

for values of k between 2 and 4. Projections in other tangent directions are, of course, equivalent to germs of the form

$$(x,y)\longmapsto(x,\ y^2,\ yp(x,y^2)).$$

Proof By the definition of E_3 (see III.2:12), if $g \in E_3$ then at each point x at which the curvature ellipse is a line segment containing the origin, neither of the two root directions of the second fundamental form is tangent to either of the two branches of the parabolic curve. The first affirmation now follows by the arguments used in the proof of III.2:14 ◻

III.3 Conclusions

We are now able to give at least partial answers to the questions we asked at the beginning of this chapter. First,

III.3:1 Theorem There is a residual set E_4 of immersions for which all singular germs of projection lie in the $\mathcal{A}$ -classes of the germs shown in the table on page 144.

Proof Take as E_4 the intersection with E_3 of the set defined in III.2:15. On the grounds of codimension alone, it is clear that if $g \in E_4$ then those strata of $\mathcal{S}$ which will be met by $j^k_x p_g$ all correspond to strata of $\mathcal{S}_0$ which figure in the first part of the list given on page 2 (Theorem I:2). By III.1:4, if $g \in E_1$ (which contains E_4), then $j^4_x p_g$ does not meet the last two of these; and the proof is completed by noting that, by III.2:14, III.2:16 and III.2:17, if $g \in E_4$ then $j_x^7 p_g$ does not meet any of the remaining strata which figure on page 2 but not on page 144 ∎

We shall not attempt here to prove any theorem concerning the existence of immersions $g : M \longrightarrow \mathbb{R}^4$ for which $j_x^k p_g$ meets any or all of the strata shown on page 144; however, it seems in particular that classical methods (such as those employed in [34]) may be useful in counting the number of points on the parabolic locus at which the asymptotic direction is tangent to the parabolic curve.

For the second question, we have provided a number of partial (and highly incomplete) answers in III.1 and III.2, of which perhaps the most satisfactory is the relation between the singularities of the exponential map $\exp_g : TM \longrightarrow \mathbb{R}^4$ and the singular algebra of the germs of projection, discussed in III.1. We shall not state any further results here. However, it seems worthwhile to ask two further questions at this point.

First, since the key to understanding the index k in the case of germs of projection equivalent to S_k or C_k, is in fact the index r in the Boardman symbol Σ^{1^r} of $\exp_g$, one may ask what, in the geometry of the immersion g, is the key to understanding the index k in the case of germs of projection which belong to the other infinite families B_k or H_k?

Second, after noting that for a generic immersion g, the germs of projection in almost all tangent directions at all points on M are equivalent to germs of the form

$$(x,y) \longrightarrow (x,\ y^2,\ yp(x,y^2)),$$

and moreover that the $\mathcal{A}$ classification of such germs is equivalent to the $\mathcal{K}^T$ classification of the function germs $p(x,y^2)$ and to the $\mathcal{K}^\partial$ classification of the function germs $p(x,y)$ (see I.5), one may ask if any other relation may be found between these functions and the geometry of the immersion g.

As regards the third of the questions asked at the beginning of the chapter, we shall not say more here, except to point out two things: first, as was mentioned in III.1:8, even for a sufficient stratum $X \in \mathcal{S}$, it is not the case that generically $\pi_1:X(g) \longrightarrow M$ is of constant rank. This is shown for example by taking as X the $\mathcal{A}$-orbit of S_2. As can be seen from III.2:6, if M_h is the hyperbolic region of M then at most points in M_h, $\pi_1:X(g) \longrightarrow M$ is a double cover, but at those isolated points $x \in M_h$ where the curvature ellipse is a radial line segment, $\pi_1^{-1}(x) \cap X(g)$ may contain one or more lines (III.2:8).

Second, it seems that much may be learned from the adjacency relations for the strata of $\mathcal{S}_0$ (see I.11). To take just one example, we have

III.3:2 <u>Proposition</u> If $g \in E_4$, then the closure in M of the set of points in M_h at which there is a non-twisting asymptotic direction, can only meet the parabolic curve at points at which either the unique asymptotic asymptotic direction is tangent to the parabolic curve, or at points where the curvature ellipse is a radial line segment.

<u>Proof</u> If x is a parabolic point at which the ellipse is not a radial line segment, and at which the unique asymptotic direction is not tangent to the parabolic curve, then the germs at x of projections in tangent dir-

ections are equivalent either to the cross-cap S_0 or to H_k for some $k \geq 2$. Since for $g \in E_4$ the germs of projections in non-twisting asymptotic directions at hyperbolic points are either equivalent to B_2 or unfold to B_2, and since B_2 does not specialise to H_k or to S_0, it follows that x cannot lie in the closure of the set of points in M_h at which there is a non-twisting asymptotic direction.

On the other hand, if x is a parabolic point at which the ellipse is not a radial line segment, but at which the unique asymptotic direction is tangent to the parabolic curve, then since $g \in E_4$, the germs of projections in the asymptotic direction are equivalent to P_3. Now P_3 unfolds to B_2, and moreover at no point on the parabolic curve near x is any germ of projection equivalent to B_2, and so we are forced to conclude that there must be hyperbolic points arbitrarily near x at which there is a germ of projection equivalent to B_2, and hence a non-twisting asymptotic direction.

As regards points on the parabolic curve at which the curvature ellipse is a radial line segment, not enough has been said to determine whether or not any germ of projection at these points is either equivalent to B_2 or unfolds to B_2, and so we leave open the question of whether such points can lie in the closure of the set of points in M_h at which there is a non-twisting asymptotic direction ∎

CHAPTER IV

MAP-GERMS WITH A CUSPIDAL EDGE

IV.1 Introduction Map germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ "with a cuspidal edge" occur naturally in differential geometry, the simplest example being the tangent developable (the exponential map of the tangent bundle) of a smooth curve in 3-space. Such map-germs are not finitely $\mathcal{A}$ -determined because ([30] Theorem 2.1 and Proposition 1.7) of their non-isolated singularity, and so a knowledge of the local differential topology of the tangent developable of a space-curve is not immediately obtainable by the standard methods of singularity theory. Several authors have, however, studied the problem from the point of view of singularity theory, making use of a variety of techniques to avoid this difficulty: see [6] , [9] , [24]. Here we show that a large class of map-germs $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ with a cuspidal edge may be easily classified using some of the results of Chapter I, and we also lay some of the groundwork for a sytematic study, from the point of view of singularity theory, of these map-germs.

IV.2

IV.2:1 Definition a) Let $f:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ be a smooth map-germ; it is a map-germ with a cuspidal edge if there is a curve (i.e. a homeomorph of the open interval $]0,1[\subseteq \mathbb{R}$), containing $0 \in \mathbb{R}^2$, at each point of which df has rank 1. We shall denote by CE the space of all such map-germs, and for any $f \in CE$, we shall denote by S_f the set of points at which df has rank 1.

b) If $f \in CE$, we shall say that <u>f is k-determined in CE</u> if every other member of CE, whose k-jet is that of f, is $\mathcal{A}$-equivalent to f.

IV.2:2 <u>Proposition</u> The germ $f:(x,y) \longrightarrow (x,y^2,y^3)$ is 3-determined in CE.

<u>Proof</u> If $j^3g(0) = (x,y^2,y^3)$ then by I.5:2 and I.5:3, g is $\mathcal{A}$-equivalent to a germ of the form

$$\tilde{g}:(x,y) \longrightarrow (x,y^2,yp(x,y^2)).$$

Now

$$d\tilde{g}(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 2y \\ yp_x(x,y^2) & p(x,y^2) + 2y^2p_y(x,y^2) \end{bmatrix}$$

and since this is non-singular when $y \neq 0$, the curve of singular points of g must lie in the x-axis. Hence $p(x,0) = 0$ for all $x \in (\mathbb{R},0)$, and so we can write $p(x,y^2) = y^2\bar{p}(x,y^2)$, and

$$\tilde{g}(x,y) = (x,\ y^2,\ y^3\bar{p}(x,y^2)).$$

Since $j^3\tilde{g}$ is equivalent to (x,y^2,y^3), we must have $\bar{p}(0,0) \neq 0$, so

$$(X,Y,Z) \longrightarrow (X,\ Y,\ \frac{Z}{\bar{p}(X,Y)})$$

defines a diffeomorphism $\bar{\Psi}$ of $(\mathbb{R}^3,0)$ such that $\bar{\Psi}\circ\tilde{g}(x,y) = (x,y^2,y^3)$ ∎

IV.2:3 <u>Remark</u> ~~Since~~ It is easy to check that if $0 = \gamma(0)$ is a point of non-zero torsion and curvature of the smooth space curve γ, then the exponential map $\exp_\gamma : T\mathbb{R} \longrightarrow \mathbb{R}^3$, given (with respect to trivial coordinates on $T\mathbb{R}$) by $\exp_\gamma(t,u) = \gamma(t) + u\gamma'(t)$, has 3-jet at $(0,0)$ equivalent to (x,y^2,y^3). Since $d\exp_\gamma$ has rank 1 at all points $(t,0)$, the germ of $\exp_\gamma$ at $(0,0)$ belongs to CE, and it follows by IV.2:2 that it is equivalent to $(x,y) \longrightarrow (x,y^2,y^3)$.

IV.2:4 <u>Lemma</u> Let $f \in CE$ and suppose that $j^2f(0,0)$ is equivalent to $(x,y^2,0)$. Then f is equivalent to a germ of the form

$$(x,y) \longrightarrow (x,y^2,y^3p(x,y^2)).$$

<u>Proof</u> As in IV.2:2∎

IV.2:5 <u>Theorem</u> Let $f_i(x,y) = (x,\ y^2,\ y^3p_i(x,y^2))$,

$$g_i(x,y) = (x,\ y^2,\ yp_i(x,y^2))$$

for $i = 1,2$. Then f_1 is $\mathcal{A}$-equivalent to f_2 if and only if g_1 is $\mathcal{A}$-equivalent to g_2.

<u>Proof</u> Let $g_1 = \Psi \circ g_2 \circ \varphi$. We may suppose, by I.5:10, that Ψ is of t the form

$$(X,Y,Z) \longrightarrow (r(X,Y),\ Ys(X,Y),\ Zt(X,Y))$$

with $s(0,0) \neq 0 \neq t(0,0)$.and $r_X(0,0) \neq 0$.
Defining $\Lambda : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $\Lambda(X,Y,Z) = (X,Y,YZ)$, we see that $f_i = \Lambda \circ g_i$, and in order to conclude that f_1 is $\mathcal{A}$-equivalent to f_2 we need only complete the diagram

$$\begin{array}{ccccc} (\mathbb{R}^2,0) & \xrightarrow{g_1} & (\mathbb{R}^3,0) & \xrightarrow{\Lambda} & (\mathbb{R}^3,0) \\ \varphi \downarrow & & \uparrow \Psi & & \vdots\uparrow \tilde{\Psi} \\ (\mathbb{R}^2,0) & \xrightarrow{g_2} & (\mathbb{R}^3,0) & \xrightarrow{\Lambda} & (\mathbb{R}^3,0) \end{array}$$

with another diffeomorphism $\tilde{\tilde{\Psi}}$. In fact, we can define $\tilde{\tilde{\Psi}}$ directly, by

$$\tilde{\tilde{\Psi}}(X,Y,Z) = (r(X,Y), Ys(X,Y), Zt(X,Y)s(X,Y)),$$

and it is easily checked that $\tilde{\tilde{\Psi}}$ is a diffeomorphism and does indeed complete the diagram.

Now suppose that $f_1 = \Psi \cdot f_2 \circ \varphi$. Again, we may take Ψ to be of the form

$$\Psi(X,Y,Z) = (r(X,Y), Ys(X,Y), Zt(X,Y)),$$

with $s(0,0) \neq 0 \neq t(0,0)$ and $r_X(0,0) \neq 0$, and if we now define $\tilde{\tilde{\Psi}}$ by

$$\tilde{\tilde{\Psi}}(X,Y,Z) = (r(X,Y), Ys(X,Y), Z\frac{t(X,Y)}{s(X,Y)})$$

then $\tilde{\tilde{\Psi}}$ is a diffeomorphism and $\tilde{\tilde{\Psi}} \circ g_2 \circ \varphi = g_1$ ∎

IV.2:6 <u>Corollary</u> Let p_i: $(H^2,0) \longrightarrow (\mathbb{R},0)$, $i = 1,2$, and define q_i: $(H^2,0) \longrightarrow (\mathbb{R},0)$ by $q_i(x,y) = yp_i(x,y)$. Then p_1 and p_2 are $\mathcal{K}^\partial$-equivalent if and only if q_1 and q_2 are.

<u>Proof</u> Immediate from I.5:16 and IV.2:5, although it can also be proved directly (and easily) by a method similar to that of IV.2:5 ∎

IV.2:7 <u>Corollary</u> The germs f_1 and f_2 of IV.2:5 are $\mathcal{A}$-equivalent if and only if p_1 and p_2 are $\mathcal{K}^\partial$-equivalent.

<u>Proof</u> Immediate from IV.2:5 and I.5:16.

IV.2:8 <u>Corollary</u> Let $f(x,y) = (x, y^2, y^3p(x,y^2))$

$$g(x,y) = (x, y^2, yp(x,y^2)).$$

Then the following are equivalent:

i) f is k+2-determined in CE.

ii) g is k-determined for $\mathcal{A}$.

iii) $p(x,y^2)$ is k-1-determined for $\mathcal{K}^T$.

<u>Proof</u> (ii)$\Longleftrightarrow$(iii) by I.5:12, and (i)$\Longleftrightarrow$(iii) follows easily from I.5:15 and IV.2:7 by the same argument as that used in the proof of I.5:12 ∎

IV.3

From the results of the preceeding section we see that a classification of those map-germs in CE whose 2-jet at 0 is equivalent to $(x,y^2,0)$ may be obtained from the classification that we give in I.5:18 and I.5:19. Before proceeding to exploit these results, we shall prove some more general results about CE.

As a first step, we calculate some tangent spaces.

IV.3:1 <u>Definition</u> i) $T(CE)_f = \left\{ \frac{df_t}{dt}\Big|_{t=0} : f_t \in CE \text{ for } t \in (\mathbb{R},0),\ f_0 = f \right\}$

ii) $\mathcal{CE}$ is the sub-sheaf of $\mathcal{E}^\infty(\mathbb{R}^2,\mathbb{R}^3)$ whose stalk over $x \in \mathbb{R}^2$ is CE_x,

$= \left\{ f:(\mathbb{R}^2,x) \longrightarrow \mathbb{R}^3 : S_f \text{ contains a homeomorph of }]0,1[, \text{ containing } x \right\}$.

iii) $T_e(CE)_f = \left\{ \frac{df_t}{dt}\Big|_{t=0} : f_t \in \mathcal{CE} \text{ for all } t \in (\mathbb{R},0) \right\}$

iv) $D_x = \left\{ g:(\mathbb{R}^2,x) \longrightarrow (\mathbb{R}^2,0) : g^{-1}(0) \text{ contains a homeomorph of }]0,1[\right\}$.

v) $\mathcal{D}$ is the sub-sheaf of $\mathcal{E}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ whose stalk over $x \in \mathbb{R}^2$ is D_x.

vi) For $g \in D_0$, $T(D)_g = \left\{ \frac{dg_t}{dt}\Big|_{t=0} : g_t \in D_0, \ g_0 = g \right\}$.

vii) For $g \in D_0$, $T_e(D)_g = \left\{ \frac{dg_t}{dt}\Big|_{t=0} : g_t \in \mathcal{D}, \ g_0 = g \right\}$.

viii) Let $b:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ be given by $b(x,y) = (x,0)$.

We introduce D_0 and $\mathcal{D}$ because of the obvious

IV.3:2 <u>Proposition</u> $CE = \tilde{d}^{-1}(D_0)$

$$j^{k+1}(CE) = \tilde{d}_k^{-1}(j^k(D_0))$$

where $j^{k+1}(CE) = \{ j^{k+1}f(0) : f \in CE \}$, $j^k(D_0) = \{ j^k g(0) : g \in D_0 \}$, and $\tilde{d}$ and $\tilde{d}_k$ are as defined in II.2:1 and II.3:4.

IV.3:3 <u>Remark</u> Recall that

i) If f is $\mathcal{A}$-equivalent to g, then $\tilde{d}f$ is $\mathcal{K}$-equivalent to $\tilde{d}g$.

ii) $\tilde{d}_k$ is a submersion.

(proofs in I.9:2 and II.3:4).

In view of IV.3:2, we can learn a lot about CE by studying D_0. Perhaps the first thing to note is that D_0 is not everywhere smooth. In general D_0 has a singularity at $g \in D_0$ "similar" to that of $g^{-1}(0)$ at 0; this is related to the fact that $g^{-1}(0) = \check{g}^{-1}(\mathcal{D})$, where $\check{g}$ is g considered as a section of $\mathcal{E}^\infty(\mathbb{R}^2,\mathbb{R}^2)$. We shall not make these rather vague statements precise, except with

IV.3:4 <u>Lemma</u> i) $\mathcal{K}b$ is open in D_0, in the sense that any smooth 1-parameter deformation in D_0 of a germ $\mathcal{K}$-equivalent to b, is also $\mathcal{K}$-equivalent to b for small values of the parameter.

ii) Let $g \in D_0$, and suppose that $g^{-1}(0)$ contains two non-singular curves, meeting transversely at 0. Then $T(D_0)_g$ is not a vector space.

<u>Proof</u> i) This follows easily from the fact that for any $g \in D_0$, $g \in \mathcal{K}b$ if and only if $j^1g(0) \neq 0$. The "only if" of this equivalence is obvious; to see "if", note that if $j^1g(0) \neq 0$ then after a change of coordinates in the source, g has the form

$$(x,y) \longmapsto (x, g_2(x,y)).$$

Then $g \in D_0$ if and only if x divides g_2, in which case g is $\mathcal{K}$-equivalent to b.

ii) We may take the two non-singular curves referred to in the hypothesis as coordinate axes in a new system of coordinates on $\mathbb{R}^2$, and thus may suppose that

$$g(x,y) = (xyh_1(x,y),\ xyh_2(x,y)).$$

Then

$$g'_t(x,y) = (tx + xyh_1(x,y),\ xyh_2(x,y))$$

and

$$g''_t(x,y) = (xyh_1(x,y),\ xyh_2(x,y) + ty)$$

are smooth 1-parameter deformations of g in D_0, so that

$$\frac{dg'_t}{dt}\Big|_{t=0} = \begin{bmatrix} x \\ 0 \end{bmatrix} \qquad \text{and} \qquad \frac{dg''_t}{dt}\Big|_{t=0} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

belong to $T(D_0)_g$. However, we claim that $\begin{bmatrix} x \\ y \end{bmatrix} \notin T(D_0)_g$. For if it were a member, then there would exist a 1-parameter deformation g'''_t of g, necessarily of the form

$$g'''_t(x,y) = (tx + h_1(x,y)xy + t^2k_1(x,y,t),\ ty + h_2(x,y)xy + t^2k_2(x,y,t)),$$

with $g'''_t \in D_0$ for all $t \in (\mathbb{R},0)$. This, however, is impossible, since for sufficiently small $t \neq 0$, g'''_t is the germ of a diffeomorphism ∎

The corresponding results in $J^k(2,2)$, where we have a well defined differentiable structure to speak of, are contained in

IV.3:5 Corollary Let $g \in D_0$. Then

i) If g is $\mathcal{K}$-equivalent to b, then $j^k(D_0)$ is a smooth manifold in a neighbourhood of $j^k g(0)$.

ii) If $g^{-1}(0)$ contains two non-singular curves meeting transversely at 0, then $j^k(D_0)$ is not a smooth manifold in any neighbourhood of $j^k g(0)$.

Proof i) This follows directly from (i) of IV.3:4 and from the fact that $\mathcal{K}^k$ orbits in jet spaces are smoothly embedded manifolds.([21] page 303).

ii) As in the proof of IV.3:4(ii), the tangent space to $j^k(D_0)$ at $j^k g(0)$ contains, mod $\mathfrak{m}_2^{k+1}$, $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$ but not $\begin{bmatrix} x \\ y \end{bmatrix}$ ∎

Translating these results back to CE, we have

IV.3:6 Lemma Let $f \in CE$.

i) If $j^2f(0)$ is equivalent to $(x,y^2,0)$ or $(x,xy,0)$ then $j^{k+1}(CE)$ is smooth in a neighbourhood of $j^{k+1}f(0)$, and

$$T_z j^{k+1}(CE) = (\tilde{d}_k)_*^{-1}(T\mathcal{K}^k \tilde{d}f)$$

where $z = j^{k+1}f(0)$ and $(d_k)_*$ is the differential of d_k.

ii) If $j^2f(0)$ is not equivalent to $(x,y^2,0)$ or to $(x,xy,0)$ then it is equivalent to $(x,0,0)$ and in general $j^{k+1}(CE)$ is not smooth in any neighbourhood of $j^{k+1}f(0)$.

Proof If $f \in CE$, then $j^2f(0)$ is equivalent to $(x,y^2,0)$, to $(x,xy,0)$ or to $(x,0,0)$, since by I.4:3 it cannot be equivalent to (x,y^2,xy). Only in the first two cases is $\tilde{d}f$ equivalent to b. In either of these two cases $\mathcal{K}df = \mathcal{K}b$ and is open in D_0, and correspondingly $\mathcal{K}^k j^k\tilde{d}f(0)$ is open in $j^k(D_0)$, so that

$$T_z j^{k+1}(CE) = (d_k)_*^{-1}(T_{z'}(D_0)) = (d_k)_*^{-1}(T\mathcal{K}^k df)$$

where $z' = j^k df(0) = d_k(j^{k+1}f(0))$ ∎

Our next result is a natural extrapolation of IV.3:6(i).

IV.3:7 Definition Let $\widetilde{CE} = \{f \in CE : j^2f(0)$ is equivalent to $(x,y^2,0)$ or $(x,xy,0)\}$.

IV.3:8 Theorem Let $f \in \widetilde{CE}$ and suppose that $f(x,y) = (x, p(x,y), q(x,y))$. Then

i) $$T(CE)_f = d_*^{-1}(T\mathcal{K}df)$$

ii) $$T_e(CE)_f = d_*^{-1}(T_e\mathcal{K}df).$$

Note d_* is the partial differential operator $\Theta(f) \longrightarrow \Theta(\tilde{d}f)$ defined by

$$\left.\frac{df_t}{dt}\right|_{t=0} \longmapsto \left.\frac{d(\tilde{d}f_t)}{dt}\right|_{t=0} .$$

For f as in the statement of the theorem, and $\tilde{d}$ defined by using

$\rho: L(2,3) \longrightarrow R^2$ given by $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \mapsto \left(\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \right)$,

one calculates that $\tilde{d}_*$ has matrix

$$\begin{bmatrix} \frac{\partial p}{\partial y}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial y} & \frac{\partial}{\partial y} & 0 \\ \frac{\partial q}{\partial y}\frac{\partial}{\partial x} - \frac{\partial q}{\partial x}\frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial y} \end{bmatrix} .$$

<u>Proof of IV.3:8</u> (i) Let $M \subseteq \mathcal{E}_{2,3}$ consist of those map-germs of the form

$$(x,y) \longmapsto (x,\ r(x,y),\ s(x,y)).$$

Then $\mathcal{E}_{2,3} - \Sigma^2 = \mathcal{A}\cdot M$, and $CE = \mathcal{A}\cdot(M \cap CE)$.

<u>Claim</u>: $T(CE)_f \cap T_f M = \tilde{d}_*^{-1}(T\mathcal{K}\,df) \cap T_f M.$ $(*)$

<u>Proof of claim</u> The inclusion from left to right is obvious from the definitions. To prove that the opposite inclusion holds, let

$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \in T\mathcal{K}\,df$ and choose a 1-parameter family (a_t, b_t) of germs $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$

such that $\frac{d}{dt}(a_t,b_t)|_{t=0} = \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$, $(a_t,b_t) \in \mathcal{K}\,df$, and $(a_0,b_0) = df$.

Then for arbitrary $r, s \in \mathcal{M}_1$, the map-germ defined by

$$f_t(x,y) = \left(x,\ \int a_t\,dy + tr(x),\ \int b_t\,dy + ts(x)\right)$$

belongs to CE for each value of t, satisfies $f_0 = f$, and we have

$$\tilde{d}_*\left(\frac{df_t}{dt}\Big|_{t=0}\right) = \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$$

where $\tilde{d}_*$ is defined as in the note before this proof.

This shows that $\tilde{d}_*(T(CE)_f \cap T_f M) \supseteq T\mathcal{X}\tilde{d}f$, and moreover, since

$\ker \tilde{d}_* \cap T_f M = \left\{ \begin{bmatrix} 0 \\ r(x) \\ s(x) \end{bmatrix} : r,s \in \mathcal{M}_1 \right\}$, we have in fact proved the claim.

Now let $\hat{f}$ be any element of $\tilde{d}_*^{-1}(T\mathcal{X}\tilde{d}f)$. Take any 1-parameter deformation f_t of f such that $\frac{df_t}{dt}\Big|_{t=0} = \hat{f}$, (for example $f_t = f + t\hat{f}$) and "project it" into CE as follows: first, choose a smooth 1-parameter family of $\mathcal{A}$-equivalences (φ_t, ψ_t) such that $\psi_0 = 1_{\mathbb{R}^3}$, $\varphi_0 = 1_{\mathbb{R}^2}$, and $\psi_t \circ f_t \circ \varphi_t = h_t \in M$ for all t. Now we claim that

$$\frac{dh_t}{dt}\Big|_{t=0} \in \tilde{d}_*^{-1}(T\mathcal{X}\tilde{d}f) \cap T_f M.$$

This holds because $\frac{dh_t}{dt}\Big|_{t=0} = \frac{df_t}{dt}\,_{t=0} + tf(\xi) + \omega f(\eta)$,

where ξ and η are the vector fields on $(\mathbb{R}^2,0)$ and $(\mathbb{R}^3,0)$ defined by

$\xi = \frac{d\varphi_t}{dt}\Big|_{t=0}$ and $\eta = \frac{d\psi_t}{dt}\Big|_{t=0}$,

so that

$$\tilde{d}_*\left(\frac{dh_t}{dt}\Big|_{t=0}\right) = \tilde{d}_*\hat{f} + \tilde{d}_*(tf(\xi) + \omega f(\eta)).$$

Since

$$\tilde{d}_*(tf(\xi) + \omega f(\eta)) = \tilde{d}_*\left(\frac{d\psi_t \circ f \circ \varphi_t}{dt}\Big|_{t=0}\right) = \frac{d\, \tilde{d}(\psi_t \circ f \circ \varphi_t)}{dt}\Big|_{t=0}$$

and $\tilde{d}(\psi_t \circ f \circ \varphi_t) \in \mathcal{K}\,\tilde{d}f$ for all t, certainly $\tilde{d}_*(tf(\xi) + \omega f(\eta)) \in T\mathcal{X}\tilde{d}f$, and this is enough to prove the claim.

Now, by (*) there exists a smooth 1-parameter family g_t such that $g_t \in \widetilde{CE}$ for all t, $g_0 = f$ and $\frac{dg_t}{dt}\Big|_{t=0} = \frac{dh_t}{dt}\Big|_{t=0}$. Set $k_t = \Psi_t^{-1} \cdot g_t \cdot \varphi_t^{-1}$.

It is easily checked that $k_t \in \widetilde{CE}$ for all t, that $k_0 = f$, and that

$$\frac{dk_t}{dt}\Big|_{t=0} = \hat{f}.$$

This completes the proof of (i).

The proof of (ii) proceeds along similar lines, the only added complication being in the choice of the 1-parameter deformation of $\tilde{d}f$ in the proof of the first claim. This we now sketch.

Given $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$ $T_e\mathcal{K}\tilde{d}f$, we have to find a 1-parameter family (a_t, b_t) such that $(a_0, b_0) = \tilde{d}f$, $\frac{d}{dt}(a_t, b_t)\Big|_{t=0} = \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$ and such that for each value of t, the germ of (a_t, b_t), now a germ $(\mathbb{R}^2, x_t) \longrightarrow (\mathbb{R}^2, 0)$ where $x_0 = 0$, should have local algebra isomorphic to that of $\tilde{d}f$. To do this, let

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = u_1 \frac{\partial \tilde{d}f}{\partial x} + u_2 \frac{\partial \tilde{d}f}{\partial y} + \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} \begin{bmatrix} \rho_1 \circ df \\ \rho_2 \circ df \end{bmatrix}$$

where $u_1, u_2 \in \mathcal{E}_2$, $v_{i,j} \in \mathcal{E}_2$, and ρ is the map $L(2,3) \longrightarrow \mathbb{R}^2$ used to define $\tilde{d}$ in the note after the statement of the theorem. Such functions u_i, $v_{i,j}$ exist by the definition of $T_e\mathcal{K}df$. Define the germ of a diffeomorphism $H : (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, 0)$ by setting

$$H(x,y,X,Y,t) = (x + tu_1(x,y), y + tu_2(x,y), X + \sum_j v_{1,j}\rho_j \circ df, Y + \sum_j v_{2,j}\rho_j \circ df, t)$$

and let (a_t, b_t) be the map germ $(\mathbb{R}^2, (tu_1(0,0), (tu_2(0,0)) \longrightarrow (\mathbb{R}^2, 0)$ whose graph is $H_t(\text{graph } \tilde{d}f)$, where H_t is defined by $H = (H_t, 1_{\mathbb{R}})$. Then one checks that (a_t, b_t) has the desired properties.

The rest of the proof proceeds as in (i)■

IV.3:9 Definition For $f \in \widetilde{CE}$, set

$$\mathrm{cod}_{CE}(f,\mathcal{A}) = \dim_{\mathbb{R}} \frac{T(CE)_f}{T\mathcal{A}f}$$

and

$$\mathrm{cod}_{CE}(f,\mathcal{A}_e) = \dim_{\mathbb{R}} \frac{T_e(CE)_f}{T_e\mathcal{A}f}$$

IV.3:10 Theorem Let $f(x,y) = (x, y^2, y^3p(x,y^2))$ (so that $f \in \widetilde{CE}$). Then

a) i) $$T(CE)_f = \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2 - \{y\} \\ \mathcal{M}_2^T + y^3\mathcal{E}_2^T \end{bmatrix}$$

ii) $$T_e(CE)_f = \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{E}_2 \\ \mathcal{E}_2^T + y^3\mathcal{E}_2^T \end{bmatrix}$$

iii) $$T\mathcal{A}f = \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2 - y \\ \mathcal{M}_2^T + y^3T\mathcal{K}^Tp(x,y^2) \end{bmatrix}$$

iv) $$T_e\mathcal{A}f = \begin{bmatrix} \mathcal{E}_2 \\ \mathcal{E}_2 \\ \mathcal{E}_2^T + y^3T_e\mathcal{K}^Tp(x,y^2) \end{bmatrix}$$

b) $\mathrm{cod}_{CE}(f,\mathcal{A}) = \mathrm{cod}(g,\mathcal{A}) - 1$ and $\mathrm{cod}_{CE}(f, \mathcal{A}_e) = \mathrm{cod}(g, \mathcal{A}_e)$, where

$$g(x,y) = (x,\ y^2,\ yp(x,y^2)).$$

Proof a)i) By Theorem IV.3:8, $T(CE)_f = \tilde{d}_*^{-1}(T\mathcal{K}df)$. We have

$$\tilde{d}f(x,y) = (2y,\ 3y^2p(x,y^2) + 2y^4p_y(x,y^2)),$$

and so

$$T\mathcal{K}\tilde{d}f = \begin{bmatrix} \mathcal{M}_2 \\ y\mathcal{E}_2 \end{bmatrix}.$$

We find that

$$\tilde{d}_* \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} 2ya_x + b_y \\ c_y + a_x(3y^2p(x,y^2) + 2y^4p_y(x,y^2)) - a_yy^3p_x(x,y^2) \end{bmatrix}$$

and so $\tilde{d}_* \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} \in T\mathcal{K}\tilde{d}f$ if and only if $\hat{b}_y \in \mathcal{M}_2$ and $\hat{c}_y \in y\mathcal{E}_2$.

It is clear that the condition on $\hat{b}$ is just $\hat{b} \in \mathcal{M}_2 - \{y\}$.
If we write

$$\hat{c}(x,y) = r(x,y^2) + ys(x,y^2)$$

then

$$\hat{c}_y(x,y) = 2yr_y(x,y^2) + s(x,y^2) + 2y^2s_y(x,y^2)$$

and in order that $\hat{c}_y$ be divisible by y, $s(x,y^2)$ must be divisible by y and hence by y^2. We can thus write

$$\hat{c}(x,y) = r(x,y^2) + y^3s'(x,y^2)$$

for some $s' \in \mathcal{E}_2$. This completes the proof of (a)(i). The proof of (a)(ii) is similar.

a)iii) By I.5:7,

$$T\mathcal{A}f = \begin{bmatrix} \mathfrak{m}_2 \\ \mathfrak{m}_2 - y \\ \mathfrak{m}_2^T + yT\mathcal{K}^T(y^2p(x,y^2)) \end{bmatrix}$$

and so it remains only to show that

$$T\mathcal{K}^T(y^2p(x,y^2)) = y^2T\mathcal{K}^Tp(x,y^2).$$

This is straightforward, if slightly unexpected. The proof of (a)(iv) is similar.

b) This now follows from (a) and I.5:7∎

IV.3:11 <u>Definition</u> i) Let $f \in \dot{C}E$, and let

$$F:(\mathbb{R}^2 \times \mathbb{R}^a,0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}^a,0)$$
$$(x,y,u) \longmapsto (f_u(x,y),\ u)$$

unfold f, with $f_0 = f$. Then F is a <u>CE-unfolding</u> if for all u, $f_u \in CE$, and F is a <u>$(CE)_e$-unfolding</u> if there exists a homeomorphism

$$\Gamma : (\mathbb{R} \times \mathbb{R}^a,0) \longrightarrow (\mathbb{R}^2 \times \mathbb{R}^a,\ 0)$$
$$(t,u) \longmapsto (\gamma(t,u),u) = (\gamma_u(t),u)$$

such that for all t, the map $u \longmapsto \gamma(t,u)$ is smooth, and $\mathrm{rank}(df_u(\gamma_u(t)))=1$ for all (t,u).

ii) A CE-unfolding ($(CE)_e$-unfolding) is <u>$\mathcal{A}$-versal</u> ($\mathcal{A}_e$-versal) if it is $\mathcal{A}$-versal ($\mathcal{A}_e$-versal) within the category of

CE-unfoldings ($(CE)_e$-unfoldings).

IV.3:12 <u>Lemma</u> Let $f(x,y) = (x, y^2, y^3p(x,y^2))$ and suppose that

$$T_e\mathcal{A}f + \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ y^3p_i(x,y^2) \end{bmatrix}_{i=1,\ldots,k} = T_e(CE)_f .$$

Then the $(CE)_e$-unfolding of f,

$$F(x,y,u_1,\ldots,u_k) = (x, y^2, y^3p(x,y^2) + \sum_{i=1}^{k} u_ip_i(x,y^2), u_1,\ldots,u_k),$$

is $\mathcal{A}_e$-versal.

<u>Proof</u> Let

$$\begin{aligned} H:(\mathbb{R}^2\times \mathbb{R}^1,0) &\longrightarrow (\mathbb{R}^3\times \mathbb{R}^1,0) \\ (x,y,v) &\longmapsto (h(x,y,v),v) = (h_v(x,y),v) \end{aligned}$$

be any $(CE)_e$-unfolding of f. Since the germ $r:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ given by $r(x,y) = (x,y^2)$ is $\mathcal{A}$-stable, the unfolding H_1 of r defined by

$$H_1(x,y,v) = (h_1(x,y,v),h_2(x,y,v),v)$$

(where h_1 and h_2 are the first two component functions of h) is trivial, and there exist diffeomorphisms Φ and Ψ_1, which are 1-parameter unfoldings of the identity on $\mathbb{R}^2$, such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^2\times \mathbb{R}^1,0) & \xrightarrow{H_1} & (\mathbb{R}^2\times \mathbb{R}^1,0) \\ \Phi \downarrow & & \downarrow \Psi_1 \\ (\mathbb{R}^2\times \mathbb{R}^1,0) & \xrightarrow[r\times 1_{\mathbb{R}^1}]{} & (\mathbb{R}^2\times \mathbb{R}^1,0) \end{array}$$

Writing $\Psi_1(x,y,v) = (\psi_1(x,y,v),v)$, define a diffeomorphism

$$\Psi : (\mathbb{R}^3 \times \mathbb{R}^1, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}^1, 0)$$

by $$\Psi(X,Y,Z,v) = (\psi_1(X,Y,v),Z,v).$$

Then $\Psi \circ H \circ \Phi^{-1}$ is an unfolding of f of the form

(1) $$Q : (x,y,v) \longmapsto (x,y^2,q_v(x,y),v).$$

Since H is a $(CE)_e$-unfolding of f, $\Psi \circ H \circ \Phi^{-1}$ is also. It is clear by inspection of (1) that the map Γ associated with this unfolding (see IV.3:11) must take the form

$$(t,v) \longmapsto (t,0,v),$$

which is to say that $\frac{\partial q_v}{\partial y}(x,0) = 0$ for all (small) x and v. By first removing from q(x,y,v) its linear and quadratic part in x, and its quadratic part in y, by the smooth 1-parameter coordinate change in $\mathbb{R}^3$

$$\overline{Z} = Z - \frac{\partial q}{\partial x}(0,0,v)X - \frac{\partial^2 q}{\partial x^2}(0,0,v)X^2 - \frac{\partial^2 q}{\partial y^2}(0,0,v)Y$$

and then applying parametrised versions of I.5:3 and IV.2:4, we see that Q, and therefore also H, is equivalent to an unfolding F' of f, of the form

$$F'(x,y,v) = (x,\ y^2,\ y^3 p'_v(x,y^2),v).$$

Now define unfoldings $G:(\mathbb{R}^2 \times \mathbb{R}^k,0) \to (\mathbb{R}^3 \times \mathbb{R}^k,0)$ and $G':(\mathbb{R}^2 \times \mathbb{R}^1,0) \to (\mathbb{R}^3 \times \mathbb{R}^1,0)$ by

$$G(x,y,u) = (x,\ y^2,\ yp(x,y^2) + \sum_{i=1}^{k} u_i y p_i(x,y^2),\ u)$$

$$G'(x,y,v) = (x,\ y^2,\ yp'_v(x,y^2),\ v).$$

It follows from the hypothesis of the Theorem, from IV.3:10(ii), from I.5.7(ii) and from the Versality Theorem ([30], page 499) that G is an $\mathcal{A}_e$-versal unfolding of the finitely-determined map-germ g defined by

$$g(x,y) = (x,\ y^2,\ yp(x,y^2))$$

and hence G' is induced from G. That is, there exists a map-germ $h:(\mathbb{R}^1,0)\longrightarrow(\mathbb{R}^k,0)$, and 1-parameter unfoldings $\bar{\Phi}'$ and $\bar{\Psi}'$ of the identity on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, such that $\bar{\Psi}'\circ h^*G = G'\circ\bar{\Phi}'$. Now applying a parametrised version of IV.2:5 to the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^2\times\mathbb{R}^1,0) & \xrightarrow{\ h^*G\ } & (\mathbb{R}^3\times\mathbb{R}^1,0) & \xrightarrow{\ \Lambda\times 1\ } & (\mathbb{R}^3\times\mathbb{R}^1,0)\\
\downarrow{\scriptstyle\bar{\Phi}'} & & \downarrow{\scriptstyle\bar{\Psi}'} & & \vdots\downarrow\\
(\mathbb{R}^2\times\mathbb{R}^1,0) & \xrightarrow[\ G'\ ]{} & (\mathbb{R}^3\times\mathbb{R}^1,0) & \xrightarrow[\ \Lambda\times 1\ ]{} & (\mathbb{R}^3\times\mathbb{R}^1,0)
\end{array}$$

where Λ is as in the proof of IV.2:5, we conclude that $(\Lambda\times 1)\circ G'$ is induced from $(\Lambda\times 1)\circ G$, that is, Q is induced from F. Since H is equivalent to Q, it follows that H is induced from F ∎

IV.3:13 <u>Theorem</u> Let $f\in CE$, with $j^2f(0)$ equivalent to $(x,y^2,0)$. Then for a $(CE)_e$-unfolding F of f, the following are equivalent

i) F is an $\mathcal{A}_e$-versal $(CE)_e$-unfolding of f.

ii) $T_e\mathcal{A}f + \mathbb{R}\{\dot{F}_i\} = T_e(CE)_f$.

<u>Proof</u> (ii)$\Longrightarrow$(i). By applying equivalences of unfoldings similar to those used in the proof of the preceeding lemma, we may assume that

$$F(x,y,u) = (x,y^2,y^3p_u(x,y^2),u).$$

By the usual rules of transformation of $\mathcal{A}_e$-tangent spaces, (ii) then transforms to

$$\text{ii)'} \qquad T_e\mathcal{A}f + \mathbb{R}\left\{\begin{bmatrix} 0 \\ 0 \\ \left.\dfrac{\partial p_u}{\partial u_i}\right|_{u=0} \end{bmatrix}\right\} = T_e(CE)_f .$$

After a coordinate change in $\mathbb{R}^k$, we may assume that

$$F(x,y,u) = (x,y^2,y^3p(x,y^2) + \sum_{i=1}^{k} u_iy^3p_i(x,y^2),\ u)$$

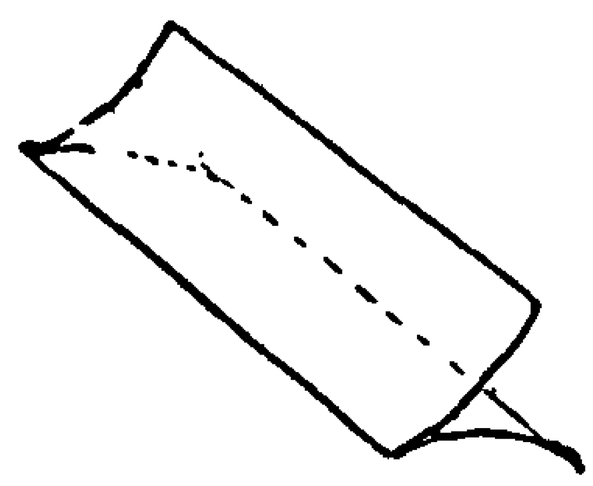
and (ii)' then transforms to the hypothesis of the lemma.

(i)$\Longrightarrow$(ii). This is more or less immediate from the definitions. Let f_t be any smooth 1-parameter deformation of f as in the definition of $T_e(CE)_f$ (IV.3:1). Then the map $G(x,y,t) = (f_t(x,y),t)$ is a $(CE)_e$-unfolding of f, and as such is induced from the versal $(CE)_e$-unfolding F. Thus there is a map-germ $h:(\mathbb{R},0)\longrightarrow(\mathbb{R}^k,0)$ and there are 1-parameter unfoldings Φ and Ψ of the identity on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively such that $\Psi\circ h^*F = G\circ\Phi$. The result follows by differentiating this equality with respect to t and setting t = 0∎

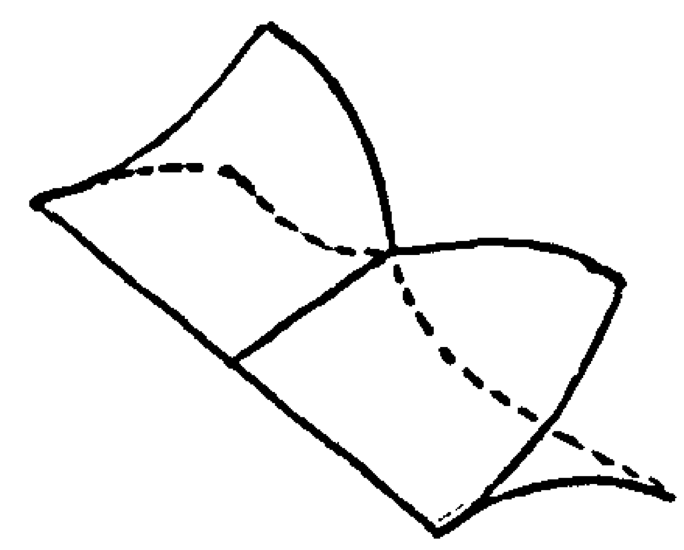
IV.3:14 <u>Definition</u> Let $f\in CE$. Then f is CE-stable if every $(CE)_e$-unfolding of f is trivial.

IV.3:15 <u>Theorem</u> The map-germs $(x,y)\longmapsto(x,y^2,y^3)$ ("cuspidal immersion") and $(x,y)\longmapsto(x,y^2,xy^3)$ ("cuspidal cross-cap") are CE-stable.

Proof Apply IV.3:10, and make an easy calculation to deduce "infinitesimal CE-stability". Then apply IV.3:13 to deduce the result∎



cuspidal immersion



cuspidal cross-cap

IV.4

So far we have only studied in any detail germs in CE whose 2-jet at 0 is equivalent to $(x,y^2,0)$, and the results we have obtained are due more to the "equivalence of equivalences" I.5:8 and I.5:11 than to any deep understanding of CE. A general "infinitesimal CE-stability implies CE-stability" result is at present beyond our reach, and with it general results on CE-determinacy and classification.

It seems worth pointing out at this stage that it is likely that for germs in CE but not in $\widetilde{CE}$ (i.e. germs $f \in CE$ such that S_f is not a non-singular curve germ), no such thing as an $\mathcal{A}_e$-versal CE_e-unfolding exists, at least as such unfoldings are defined in IV.3:11. This is because CE is not in general smooth at such points. What seems more probable is that the concept of CE_e-unfolding will have to be widened to include unfoldings whose base (i.e. parameter space) is the germ of an algebraic variety rather than the germ of a smooth manifold.

We now look at examples of germs in CE whose 2-jet is equivalent to $(x,xy,0)$. The first example, in the sense that the $\mathcal{A}^4$ orbit of its 4-jet at 0 has the lowest codimension, is

$$f(x,y) = (x,\ xy + y^3,\ xy^2 + \tfrac{3}{2}y^4)$$

whose image is the swallowtail surface. In fact f is $\mathcal{A}$-equivalent to the projection into the parameter space $\mathbb{R}^3$ of the bifurcation set of an $\mathcal{R}_e$-versal unfolding of an A_4 singularity.

Since the map-germ defined by the first two component functions of f is the Whitney cusp, which is 3-determined for $\mathcal{A}$, it is clear that any germ g whose 4-jet is that of f, is equivalent to

$$(x,y) \longrightarrow (x,\ xy + y^3,\ xy^2 + \tfrac{3}{2}y^4 + c(x,y)),$$

for some $c \in \mathcal{M}_2^4$, and from the fact that $g \in CE$ one deduces that $x+3y^2$ must divide c_y. Some rather tedious calculations, which we shall not go into here, show that after coordinate changes in $(\mathbb{R}^2,0)$ and $(\mathbb{R}^3,0)$ we can replace the germ $c \in \mathcal{M}_2^4$ by a germ $c' \in \mathcal{M}_2^k$, for any $k < \infty$, but attempts to prove that f is actually 4-determined in CE have not yet been successful. However, the following argument, suggested by Andrew du Plessis (personal communication), shows that f is 4-determined in CE^ω, where CE^ω is the space of analytic map-germs in CE. First,

IV.4:1 <u>Lemma</u> Let $g:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ be a map-germ such that $j^4g(0)$ is equivalent to

$$(x,\ y^3 + xp(x,y),\ y^4 + xq(x,y))$$

where $p,q \in \mathcal{E}_2$. Then g is $\mathcal{A}$-equivalent to the map-germ h defined by

$$(x,y) \longrightarrow (x,\ y^3 + a(x)y,\ y^4 + b(x)y^2 + c(x)y)$$

for some a, b, $c \in \mathcal{M}_1$.

<u>Proof</u> First, let $r:(\mathbb{R},0) \longrightarrow (\mathbb{R}^2,0)$ be defined by $r(y) = (y^3, y^4)$. Then one calculates, using I.3:2, that

$$T_e\mathcal{A}r = \begin{bmatrix} \mathcal{E}_1 - \{y\} \\ \mathcal{E}_1 - \{y, y^2\} \end{bmatrix}$$

since clearly $r^*\mathcal{M}_2\mathcal{E}_1 \supseteq \mathcal{M}_1^3$. It follows, by I.3:3, that r is 6-determined, but in fact an easy argument based on I.3:1(a), shows that it is actually 4-determined.

From this we deduce that the map-germ $(\mathbb{R},0)\longrightarrow(\mathbb{R}^2,0)$ obtained from g by setting x equal to 0 and omitting the first component, is $\mathcal{A}$-equivalent to r, and hence that g itself is equivalent to

$$g':(x,y)\longrightarrow(x,\ y^3 + xp'(x,y),\ y^4 + xq'(x,y))$$

for some p', $q' \in \mathcal{E}_2$. Now this germ is a 1-parameter unfolding of r, and as such must be induced from an $\mathcal{A}_e$-versal unfolding. From the expression for $T_e\mathcal{A}r$, and by the versality theorem for $\mathcal{A}$, we see that

$$R:(y,\ u_1,\ u_2,\ u_3)\longrightarrow(y^3 + u_1y,\ y^4 + u_2y^2 + u_3y)$$

is an $\mathcal{A}_e$-versal unfolding of r, and so there exists a map-germ $h:(\mathbb{R},0)\rightarrow(\mathbb{R}^3,0)$, which we write as $x\longmapsto(a(x),\ b(x),\ c(x))$, such that g' is equivalent, as an unfolding of r, and a fortiori as a map-germ, to h^*R. But that is to say that g is equivalent to

$$(x,y)\longrightarrow(x,\ y^3 + a(x)y,\ y^4 + b(x)y^2 + c(x)y)\ \blacksquare$$

IV.4:2 <u>Proposition</u> The map-germ $f(x,y) = (x,\ y^3 + xy,\ xy^2 + \frac{3}{2}y^4)$ is 4-determined in CE^{ω}.

<u>Proof</u> If g is an analytic map-germ such that $j^4g(0) = j^4f(0)$, then by the lemma, g is equivalent to

$$h:(x,y) \longrightarrow (x,\ y^3 + a(x)y,\ y^4 + b(x)y^2 + c(x)y)$$

for some $a,b,c \in \mathcal{M}_1$. However, since all the results on $\mathcal{A}_e$-versality hold in the analytic category, we may take a, b and c to be analytic.

We now aim to show that after coordinate change in the x and X variables we may take a(x) to be equal to x. To see this, we compare $\tilde{d}g^*\mathcal{M}_2\mathcal{E}_2$ and $\tilde{d}h^*\mathcal{M}_2\mathcal{E}_2$, which are, respectively,

$$\langle 3y^2 + x + O(4),\ 6y^3 + 2xy + O(4)\rangle$$

and

$$\langle 3y^2 + a(x),\ 4y^3 + 2b(x)y + c(x)\rangle .$$

Since h and g are $\mathcal{A}$-equivalent, these two ideals must be equivalent, in the sense that $\tilde{d}g^*\mathcal{M}_2\mathcal{E}_2 = \varphi^*(\tilde{d}h^*\mathcal{M}_2\mathcal{E}_2)$ for some analytic diffeomorphism φ (I.9:2). Thus, since the first has a non-singular generator, the second must have also. It follows that either a or c is non-singular. If c is non-singular, then, since $3y^2 + a(x)$ must vanish along a curve contained in the 0-locus of $4y^3 + 2b(x)y + c(x)$ (for $h \in CE$), it follows that the latter must actually divide the former. But this is absurd, since for each value of x, the 0-locus of the latter contains 3 points (in $\mathbb{C}^2$) whereas the 0-locus of the former contains only 2. Hence, c cannot be non-singular, and therefore a must be. Thus, a is $\mathcal{R}$-equivalent to $1_{\mathbb{R}}$, and applying the same coordinate change to the X coordinate in $\mathbb{R}^3$, we see that h is $\mathcal{A}$-equivalent to the map- germ h' defined by

$$h'(x,y) = (x,\ y^3 + xy,\ y^4 + b(x)y^2 + c(x)y).$$

But now $3y^2 + x$ must divide $4y^3 + 2b(x)y + c(x)$, and writing

$$4y^3 + 2b(x)y + c(x) = \tfrac{4}{3}y(3y^2 + x) + (2b(x) - \tfrac{4}{3}x)y + c(x)$$

we see that $(2b(x) - \frac{4}{3}x)y + c(x)$ must vanish identically on $3y^2+x = 0$, i.e.

$$(2b(-3y^2) + 4y^2)y + c(-3y^2) = 0.$$

for all y. Separating odd and even powers of y, we see that

$$2b(-3y^2) + 4y^2 = 0 \qquad \text{and} \qquad c(-3y^2) = 0$$

for all y, from which we deduce (since b and c are analytic) that

$$b(x) = \tfrac{2}{3}x \qquad \text{and} \qquad c(x) = 0$$

for all x. From this it is immediate that h (and hence also g) is equivalent to f∎

IV.4:3 <u>Corollary</u> The map-germ f of the proposition is CE^{ω}-stable.

<u>Proof</u> Let $F:(\mathbb{R}^2 \times \mathbb{R},0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R},0)$ be any 1-parameter $(CE)_e^{\omega}$ unfolding of f. By the stability of the Whitney cusp, we may suppose that F is of the form

$$F(x,y,t) = (x,\ xy + y^3,\ xy^2 + \tfrac{3}{2}y^4 + c_t(x,y),\ t)$$

where $c_0 = 0$. Since F is a $(CE)_e$ unfolding, $x + 3y^2$ must divide $\frac{\partial c_t}{\partial y}(x,y)$, and so we can write

$$\frac{\partial c_t}{\partial y}(x,y) = \sum_{i,j} c_{i,j}(t)x^i y^j(x + 3y^2)$$

Hence

$$c_t(x,y) = \sum_{i,j} c_{i,j}(t)\left(\frac{x^{i+1}y^{j+1}}{j+1} + \frac{3x^i y^{j+3}}{j+3} \right)$$

The analytic 1-parameter coordinate change in $\mathbb{R}^3$ given by

$$\overline{Z} = \frac{1}{1+\frac{1}{2}c_{0,1}}\left(Z - c_{0,0}Y - c_{1,0}XY - c_{2,0}X^2Y - \tfrac{1}{2}c_{1,1}Y^2 \right)$$

then transforms F into

$$(x,y,t) \longrightarrow (x,\ xy + y^3,\ xy^2 + \tfrac{1}{2}y^4 + \tilde{c}_{0,2}(t)xy^3 + \overline{c}_t(x,y))$$

where $\tilde{c}_{0,2} = \frac{1}{1+\frac{1}{2}c_{0,1}}\, c_{0,2}$ and $\overline{c}_t \in \mathcal{M}_2^5$.

Now if we write (the transformed) F as $(x,y,t) \longmapsto (f_t(x,y),t)$, we find

$$tf_t\begin{bmatrix} 3(xy + y^3) \\ y^2 - 2c_{0,2}xy \end{bmatrix} + \omega f_t\begin{bmatrix} -3Y \\ -4Z + 2c_{0,2}XY \\ 2c_{0,2}Y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5xy^3 \end{bmatrix} \qquad (1).$$

Letting φ_t and ψ_t be the analytic 1-parameter families of analytic diffeomorphisms of $(\mathbb{R}^2,0)$ and $(\mathbb{R}^3,0)$ whose differentials with respect to t are, respectively,

$$\frac{c_{0,2}}{5}\xi \qquad \text{and} \qquad \frac{c_{0,2}}{5}\eta$$

where ξ and η are the vector fields on $(\mathbb{R}^2,0)$ and $(\mathbb{R}^3,0)$ which figure in (1), we find that

$$j^4(\psi_t \circ f_t \circ \varphi_t^{-1})(0) = j^4 f(0)$$

and then it follows from the 4-determinacy of f in CE that the unfolding F is trivial∎

IV.4:4 Remark It follows from the proof we give for the preceeding corollary that if f is 4-determined in CE then it is CE-stable, since the existence of the coordinate changes that we give in the proof does not depend on the analyticity of the unfolding F, although their analyticity does.

IV.5 Applications to the study of the geometry of the tangent developable of a smooth space-curve

IV.5:1 Theorem Let $\gamma:(\mathbb{R},0)\longrightarrow(\mathbb{R}^3,0)$ be the germ of a smooth curve with nowhere vanishing velocity and curvature. Let τ (the torsion of the curve) vanish at the origin to order k, $0\leqslant k\leqslant 4$. Then the germ of the exponential map

$$\exp_\gamma:(T\mathbb{R},0)\longrightarrow(\mathbb{R}^3,0)$$

$$(t,u)\longmapsto\gamma(t)+u\gamma'(t)$$

is $\mathcal{A}$-equivalent to

$$(x,y)\longrightarrow(x,\ y^2,\ y^3) \qquad \text{if } k=0 \quad (\text{i.e. if } \tau(0)\neq 0)$$

$$(x,y)\longrightarrow(x,\ y^2,\ xy^3) \qquad \text{if } k=1$$

$$(x,y)\longrightarrow(x,\ y^2,\ y^5+x^2y^3) \qquad \text{if } k=2$$

$$(x,y)\longrightarrow(x,\ y^2,\ xy^5+x^2y^3) \qquad \text{if } k=3$$

$$(x,y)\longrightarrow(x,\ y^2,\ y^7+2\sqrt{\tfrac{21}{5}}\,x^2y^5+x^4y^3) \qquad \text{if } k=4.$$

Proof The proof is very straightforward, and depends principally on the fact that for each value of k, the germ shown is k+3-determined in CE, by I.5:18 and IV.2:8. It remains only to show that in each case the k+3-jet of exp is $\mathcal{A}^k$-equivalent to the k+3-jet of the germ shown.

First, by taking as basis vectors in R^3 a Serret-Frenet set $\mathbf{t}(0)$, $\mathbf{n}(0)$ and $\mathbf{b}(0)$, redefining the t coordinate on R, and making, if necessary, a change of scale in R^3, we can write

$$\gamma(t) = (t,\ t^2 + b(t),\ c(t))$$

where b, $c \in \mathcal{M}_1^{\,3}$. Since on $\mathbb{R}^3$ only linear coordinate changes have been used in bringing γ to this form, the $\mathcal{A}$-class of exp $:(T\mathbb{R},0) \longrightarrow (\mathbb{R}^3,0)$ remains unchanged. We shall write the Taylor series of b and c as

$$\sum_{i\geqslant 3} b_i t^i \qquad \text{and} \qquad \sum_{i\geqslant 3} c_i t^i\ .$$

Now it is easy to check that γ vanishes to order k at 0 if and only if

$$c_3 = \cdots = c_{k+2} = 0,\ c_{k+3} \neq 0.$$

It follows that if γ vanishes to order k at 0, we have

$$j^{k+3}\exp\ (0,0) = (t + u,\ t^2 + 2ut + \sum_{i=3}^{k+3} b_i(t^i + iut^{i-1}),\ c_{k+3}(t^{k+3} + (k+3)ut^{k+2})).$$

By putting $t = v - u$ and dividing the Z coordinate by c_{k+3}, the k+3-jet becomes

$$(v,\ v^2 - u^2 + O(3),\ (v - u)^{k+2}(v + (k+2)u)).$$

By putting $\bar{Y} = -Y + X^2$ we remove the v^2 from the second component, and then an application of I.5:2 shows that this k+3-jet is equivalent to

$$(v,\ u^2,\ (v - u)^{k+2}(v + (k+2)u)).$$

After removing all even powers of u from the third component by using left coordinate changes (see I.5:3), and then making a change of scale in the coordinates if necessary, one obtains the k+3 jet of the corresponding germ listed in the statement of the theorem. In the last case, once the coefficients of y^7 and x^4y^3 have been reduced to 1, then the coefficient of x^2y^5 is an $\mathcal{A}$-invariant (see case 5 in the proof of I.5:19), and so cannot be replace by ±1■

The tangent developable surface is simply the image of $\exp_\gamma$, and so from the theorem we can obtain information about the topology of the germ at O of this surface. For example,

IV.5:2 Corollary Under the hypotheses of the theorem, the tangent developable surface has one curve of self-intersection beginning at O if k = 1 or 3, and none if k = 0, 2 or 4.

Proof Straightforward calculation■

By the use of IV.3:13 (the Versality Theorem) we can also gain a certain ammount of information about the behaviour of the tangent developable surface as the curve γ is deformed, since deformations of γ induce $(CE)_e$-unfoldings of $\exp_\gamma$.

Iv.5:3 Example Consider the curve

$$\gamma_\lambda(t) = (t,\ t^2 + b_\lambda(t),\ \ t^3 + c_5t^5 + c_\lambda(t))$$

where $c_5>0$, $b_\lambda \in \mathcal{M}_1{}^3$ and $c_\lambda \in \mathcal{M}_1{}^6$ for all λ. When $\lambda = 0$, τ vanishes at 0 to second order, but for small $\lambda < 0$, τ has two distinct first-order zeros in a small neighbourhood of O. Now, it is easy to check that the $(CE)_e$-unfolding of $\exp_{\gamma_0}$ given by

$$E: (t,u,\lambda) \longrightarrow (\ \gamma_\lambda(t) + u\gamma_\lambda'(t), \lambda)$$

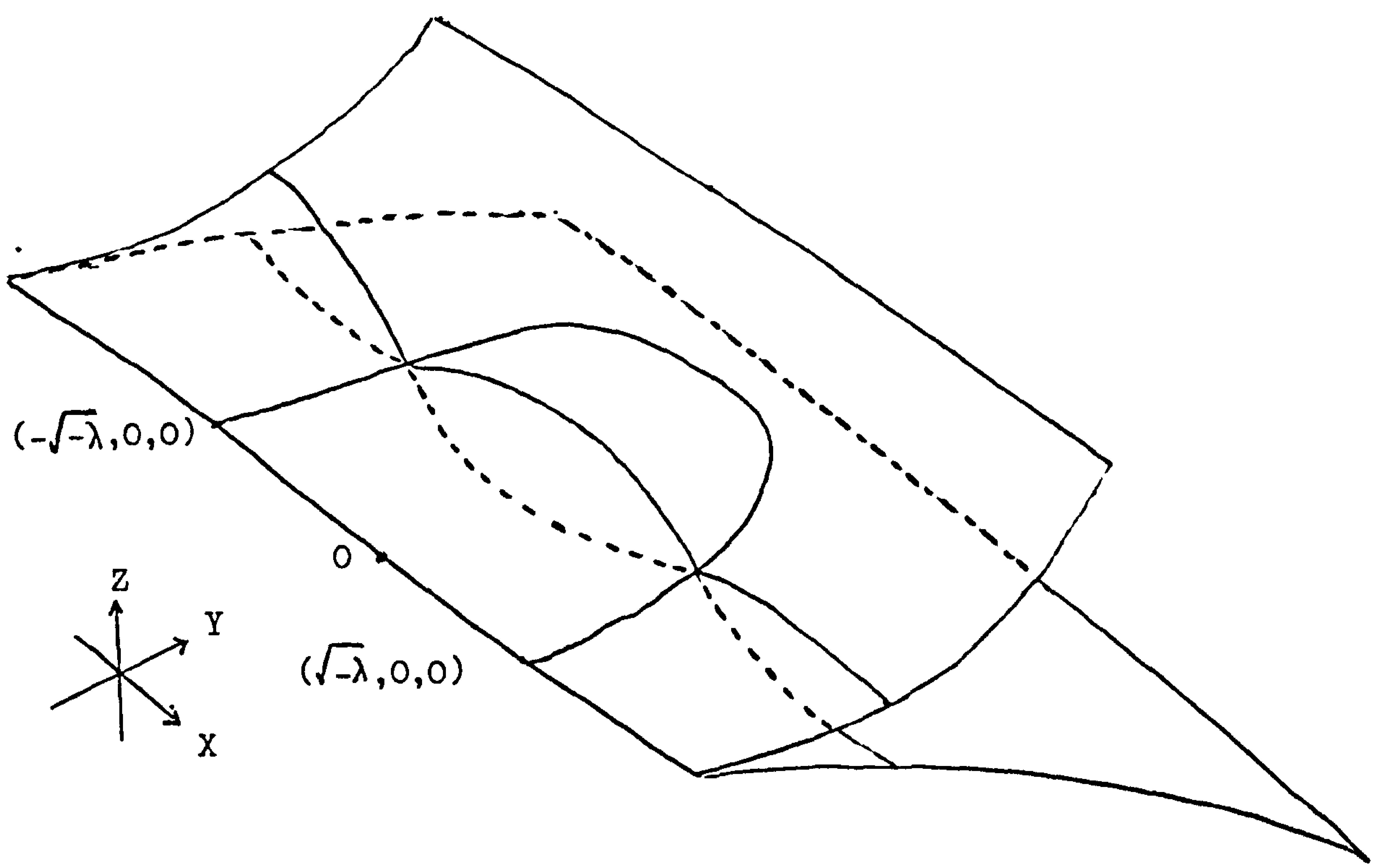
is $\mathcal{A}_e$-versal, and from this and the fact that $F:(\mathbb{R}^2 \times \mathbb{R},0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R},0)$

defined by

$$F(x,y,\lambda) = (x,\ y^2,\ y^5 + x^2y^3 + \lambda y^3,\ \lambda)$$

is an $\mathcal{A}_e$-versal $(CE)_e$-unfolding of $(x,y) \longmapsto (x,\ y^2,\ y^5 + x^2y^3)$, we deduce that E and F are equivalent unfoldings.

By examining the geometry of F, one deduces that for sufficiently small $\lambda < 0$, the image of $\exp_{\gamma_\lambda}$ is diffeomorphic to the singular surface sketched below. Note the cuspidal edge along the X-axis, and the two points $(\pm\sqrt{-\lambda},\ 0,\ 0)$ at each of which the surface is a cuspidal cross-cap. Joining these two points is a curve of self-intersection of the surface. As $\lambda \longrightarrow 0-$, this curve contracts to a point.



IV.5:4 <u>Remark</u> This, and the cases where k = 0 or 1, (when $\exp_\gamma$ is stable), are the only cases where the singularity of $\exp_\gamma$ can be $\mathcal{A}_e$-versally unfolded in CE by varying the curve γ. It is easy to see that for $k > 2$, an $\mathcal{A}_e$-versal $(CE)_e$-unfolding cannot take place: the CE germ

$$(x,y) \longmapsto (x,\ y^2,\ x^2y^3 + y^7)$$

occurs in the $\mathcal{A}_e$-versal $(CE)_e$-unfolding of

$$(x,y) \longmapsto (x,\ y^2,\ xy^5 + x^3y^3)$$

(case k = 3 of IV.5:1) and hence in the $\mathcal{A}_e$-versal $(CE)_e$-unfolding of every singularity of $\exp_\gamma$ presented at a point on the curve where $\kappa \neq 0$ and τ vanishes to order > 3; but it does not occur as a singularity of $\exp_\gamma$ itself, since the only germ in CE of the form

$$(x,y) \longmapsto (x,\ y^2,\ y^3p(x,y^2))$$

with $p(x,y^2)$ of order 2, to occur as a singularity of $\exp_\gamma$, is

$$(x,y) \longmapsto (x,\ y^2,\ y^5 + x^2y^3).$$

IV.5:5 Remark From IV.5:1 one sees that the topology of the tangent developable of a space-curve in the neighbourhood of a point at which $\kappa \neq 0$, is determined by the order of vanishing of the torsion τ, when this order is no greater than 4. Now when τ vanishes to order $k > 4$, the $\mathcal{A}$-class of $\exp_\gamma$ is no longer determined only by this order- briefly, k+3-jets of the form

$$(x,\ y^2,\ y^3p(x,y^2)),$$

where $p(x,y^2)$ is a homogeneous polynomial of degree k, are not sufficient in CE if $k > 4$. However, one can prove the following result:

Theorem Let $p(x,y^2)$ be a homogeneous polynomial of degree k, with no repeated real root. Then the map-germ $(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,0)$ defined by

$$(x,y) \longmapsto (x,\ y^2,\ yp(x,y^2))$$

is $(k+1)$-C^0-$\mathcal{A}$-determined.

From this it is possible to deduce, by proving a C^0 variant of IV.2:5, that

$$(x,y) \longrightarrow (x,\ y^2,\ y^3p(x,y^2))$$

is $(k+3)$-C^0-$\mathcal{A}$-determined in CE.

As a final application of some of the forgoing ideas to the study of the geometry of the tangent developable of a space curve, we compute the $\mathcal{A}$-class of the germ of $\exp_\gamma:(T\mathbb{R},0)\longrightarrow(\mathbb{R}^3,0)$ when 0 is a "non-degenerate" point of zero curvature of the smooth curve γ.

IV.5:6 <u>Definition</u> Let $\gamma:(\mathbb{R},0)\longrightarrow(\mathbb{R}^3,0)$ be the germ of a regular smooth curve with zero curvature at $t = 0$. We shall say that 0 is a non-degenerate point of zero curvature if $\gamma'(0)$, $\gamma'''(0)$ and $\gamma''''(0)$ are linearly independent vectors.

IV.5:7 <u>Lemma</u> Let γ have a non-degenerate point of zero curvature at $t=0$. Then

i) By an appropriate choice of t coordinate, and after a linear coordinate change in $\mathbb{R}^3$, $\gamma(t)$ may be written

$$\gamma(t) = (t,\ t^3 + b(t),\ t^4 + c(t))$$

where b and $c \in \mathcal{M}_1^5$.

ii) 0 is an isolated point of zero curvature.

iii) If (t,u) are locally trivial coordinates on $T\mathbb{R}$ at 0, then for γ as in (i),

$$\tilde{d}(\exp_\gamma)^* \mathcal{M}_2 . C^\infty(\mathbb{R},0) = \langle ut \rangle .$$

iv) The function τ (the torsion of the curve), which is not defined at points of zero curvature, has a C^∞ extension on a neighbourhood of 0 in $\mathbb{R}$, whose value at $t = 0$ is non-zero.

Proof i) Let γ be parametrised by $v \in \mathbb{R}$. Choose a basis for $\mathbb{R}^3$ consisting of the vectors

$$\frac{\gamma'(0)}{\|\gamma'(0)\|} , \frac{\gamma'''(0)}{\|\gamma'''(0)\|} \text{ and } \frac{\gamma''''(0)}{\|\gamma''''(0)\|}$$

and define a coordinate system on $\mathbb{R}^3$ by taking the dual to this basis. Choose a t coordinate on $\mathbb{R}$ by setting

$$t = \gamma(v) \cdot \frac{\gamma'(0)}{\|\gamma'(0)\|} .$$

Then in this coordinate system, γ has the desired form.

ii) This is obvious from (i).

iii) Using $\tilde{d}$ defined as in IV.3:8, we have

$$\tilde{d}(\exp_\gamma)^* \mathcal{M}_2 . C^\infty(\mathbb{R},0) = \langle u(6t + b''(t)), u(12t^2 + c''(t)) \rangle$$

and since $b'', c'' \in \mathcal{M}_1^3$, this ideal is equal to $\langle ut \rangle$.

iv) At points where the curvature is non-zero,

$$\tau(t) = \frac{[\gamma'(t) \times \gamma''(t)] \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2} .$$

Since this is invariant under a parameter change in $\mathbb{R}$, and is subject only to multiplication by a non-zero constant under a linear isomorphism of $\mathbb{R}^3$, we can use the expression for $\gamma(t)$ as in (i). We find that

$$\tau(t) = \frac{72t^2 + O(3)}{36t^2 + O(3)}$$

and thus after dividing numerator and denominator by t^2, the same expression may be used to define τ at O∎

IV.5:8 <u>Theorem</u> Let γ have a non-degenerate point of zero curvature at $t = 0$. Then the germ of $\exp_\gamma : (T\mathbb{R},0) \longrightarrow (\mathbb{R}^3,0)$ is $\mathcal{A}$-equivalent to

$$(x,y) \longmapsto (x,\ y^3 - x^2y,\ y^4 - \tfrac{2}{3}x^2y^2).$$

<u>Proof</u> Let $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear isomorphism used in IV.5:7(i) to bring γ to the form given. Since $\exp_{T\circ\gamma} = T\circ\exp_\gamma$, we may assume that γ is as in IV.5:7(i). Then we have

$$j^4\exp_\gamma(0) = (u + t,\ t^3 + 3ut^2,\ t^4 + 4ut^3).$$

After the coordinate change $u = v - t$, this becomes

$$(v,\ 3vt^2 - 2t^3,\ 4vt^3 - 3t^4)$$

From this it is immediate, by IV.4:1, that the germ of $\exp_\gamma$ at $0 \in T\mathbb{R}$ is $\mathcal{A}$-equivalent to

$$h(v,t) = (v,\ t^3 + a(v)t,\ t^4 + b(v)t^2 + c(v)t)$$

for some functions $a,b,c \in \mathcal{M}_1$.

As in the proof of IV.4:2, in order to decide what the functions a, b and c are, we consider the ideal

$$dh^*\mathcal{M}_2 \cdot \mathcal{E}_2 = \langle 3y^2 + a(x),\ 4y^3 + 2b(x)y + c(x) \rangle .$$

Since the zero-set of this ideal is contained in that of $3y^2 + a(x)$, and must, by IV.5:7(iii), contain two non-singular curves meeting transversely at $0 \in \mathbb{R}^2$, we must have $a'(0) = 0$, $a''(0) < 0$. A change in x and X coordinates then transforms h to

$$(x,y) \longmapsto (x,\ y^3 - x^2y,\ y^4 + b(x)y^2 + c(x)y).$$

Since $4y^3 + 2b(x)y + c(x)$ vanishes on the zero set of $3y^2 - x^2$, which is a non-degenerate polynomial, it must be divisible by it, and putting

$$4y^3 + 2b(x)y + c(x) = \tfrac{4}{3}y(3y^2 - x^2) + (2b(x) + \tfrac{4}{3}x^2)y + c(x)$$

we see that the remainder $(2b(x) + \frac{4}{3}x^2)y + c(x)$ must vanish on $3y^2 - x^2 = 0$ which is equal to $(x - \sqrt{3}\,y)(x + \sqrt{3}\,y) = 0$. Substituting successively $x = \sqrt{3}\,y$ and $x = -\sqrt{3}\,y$ in the expression for the remainder, and setting it equal to 0, we obtain

$$(2b(x) - \tfrac{4}{3}x^2)x + 3\,c(x) = 0$$

$$(2b(x) - \tfrac{4}{3}x^2)x - 3\,c(x) = 0$$

for allx. From this it is immediate that

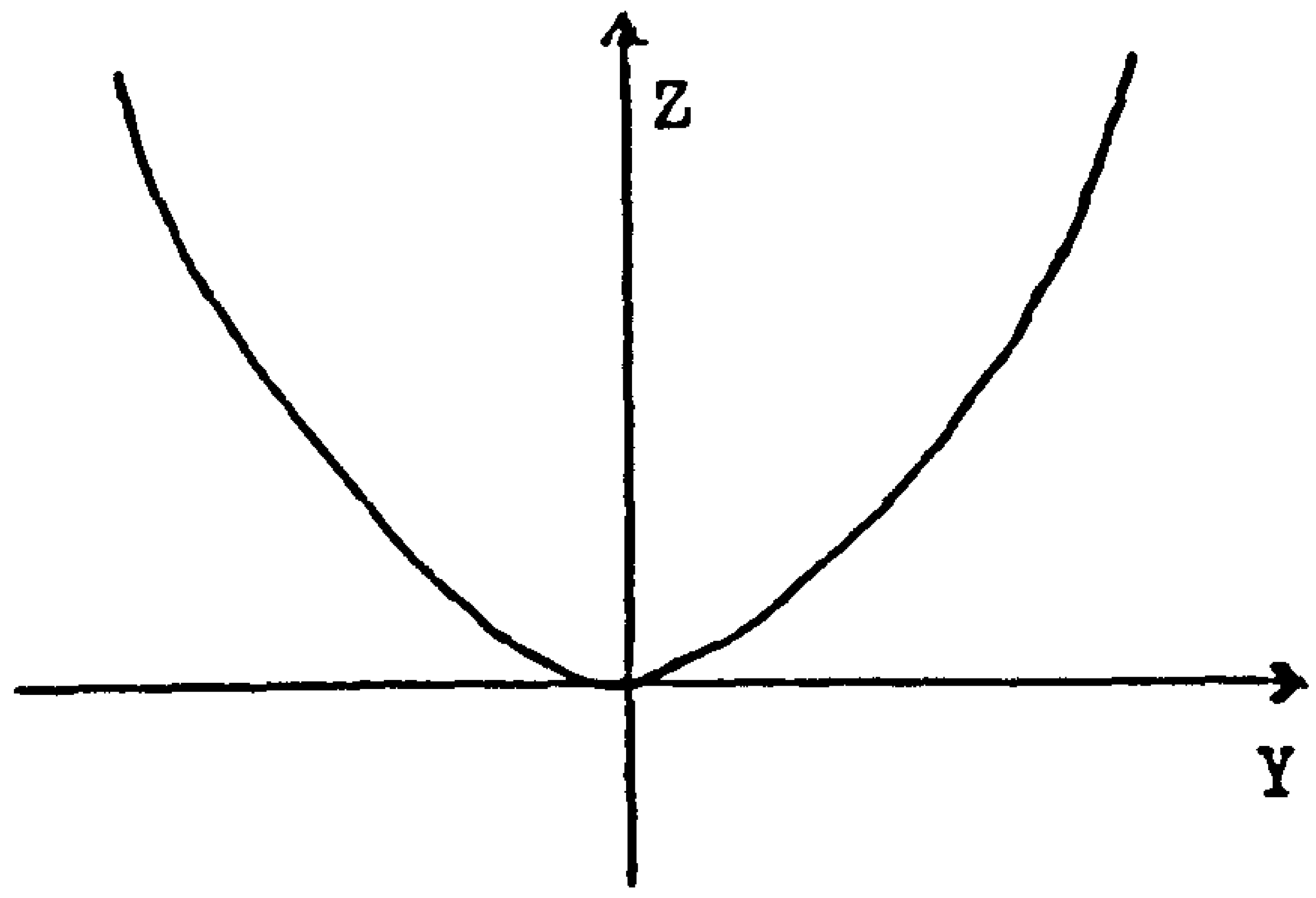
$$b(x) = -\tfrac{2}{3}x^2 \qquad \text{and} \qquad c(x) = 0$$

for all x. This concludes the proof∎

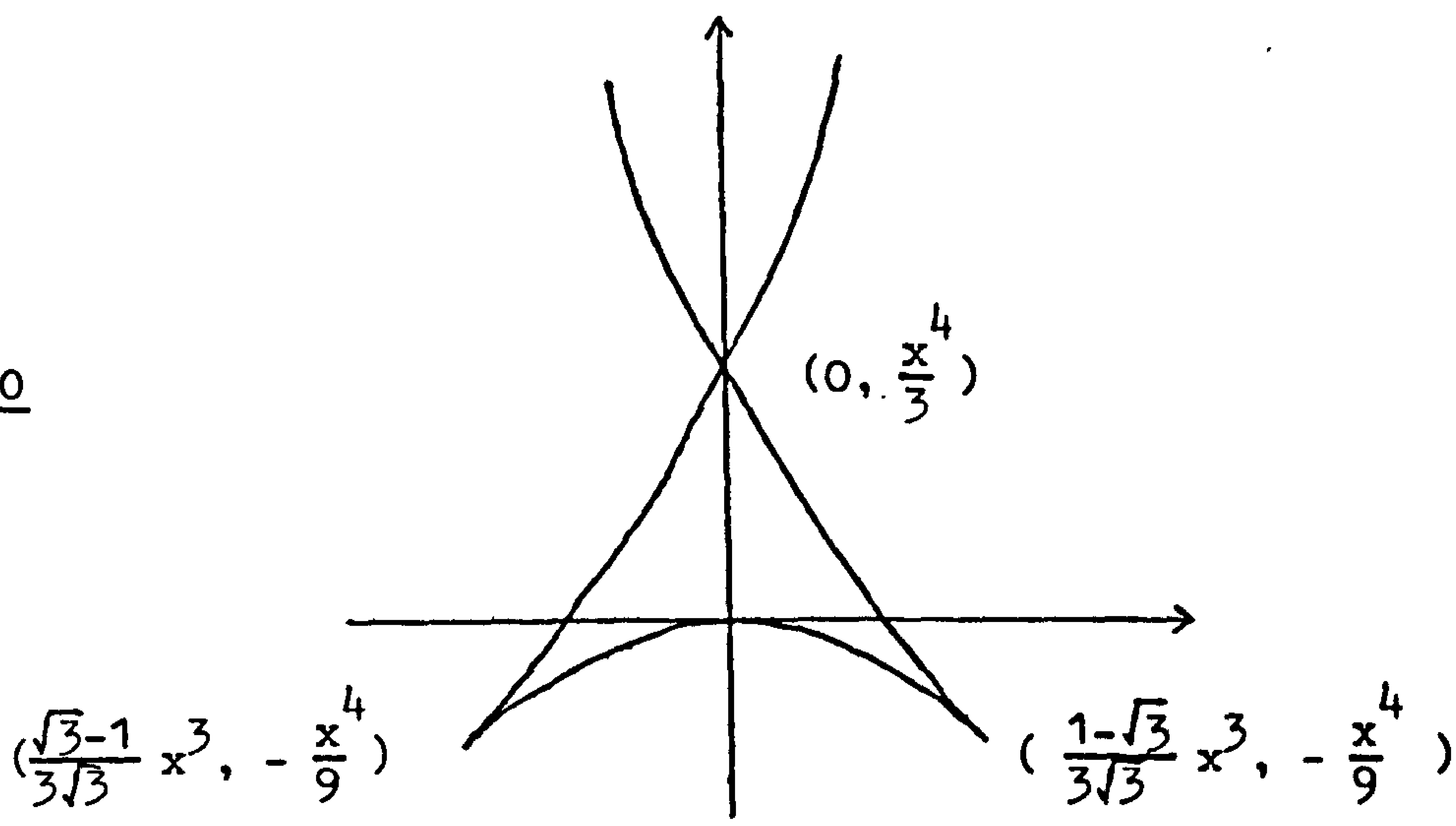
IV.5:9 <u>Geometry of the map-germ $(x,y) \longmapsto (x,\ y^3 - x^2y,\ y^4 - \frac{2}{3}x^2y^2)$.</u>

In order to obtain a picture of this map, it is perhaps best to regard it as a 1-parameter unfolding of $y \longmapsto (y^3, y^4)$, obtaining for each value of x a deformation of this curve, which is simply the intersection of the image of f with the plane $X = x$.

x = 0



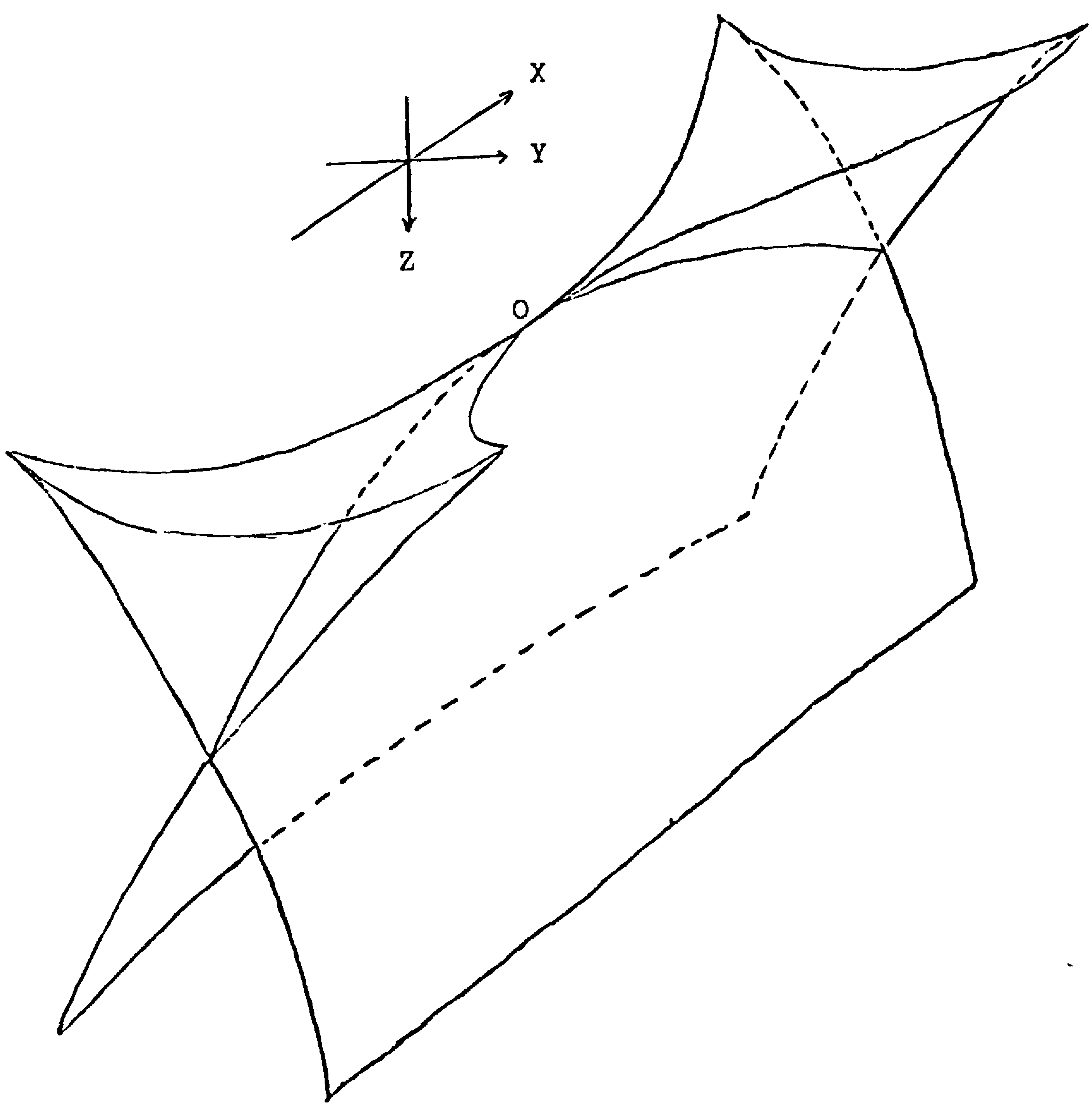
x ≠ 0



Piecing all these together, we obtain the surface (the "double swallow-tail") sketched on the next page. Note the two cuspidal edges, which meet inflexionally at 0, and the curve of self-intersection.

IV.5:10 Remark From IV.5:7 (iv) it follows that if γ_λ is a smooth 1-parameter deformation of a curve γ_0 which has a non-degenerate point of zero curvature at $t = 0$, then for small values of λ the curve γ_λ has non-vanishing torsion in a neighbourhood of $t = 0$. Thus, if γ_λ is a generic 1-parameter deformation of γ_0, in the sense that for $\lambda \neq 0$ each γ_λ has non-vanishing curvature in a neighbourhood of $t = 0$, then the tangent developable of each curve γ_λ is a cuspidal immersion, equivalent at each point on the curve to $(x,y) \longmapsto (x,y^2,y^3)$. This might appear to contradict the fact that the tangent developable of γ_0 has transverse self-intersection, which must persist under deformations of the curve, but in fact what happens is that this curve of self-intersection moves away from the curve γ_λ as λ moves away from 0, and thus does not appear in the germ of $\exp_{\gamma_\lambda}$ at $(t,0)$, for $\lambda \neq 0$.

The Double Swallowtail



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CLOSED CURVES WITH NO QUADRISECANTS

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§1. INTRODUCTION

IN HIS PAPER [1], Wall discusses the idea of a generically embedded curve in R^3, related to the local appearance of the radially projected curve as viewed from different points in R^3.

Wall shows that a generic curve never meets a straight line in five points. There may, however, be finitely many straight lines, quadrisecants, which meet the curve in four points. The local view of the curve from distant points on a quadrisecant will have four branches crossing transversely.

In this paper we ask how simple the embedding of a closed curve must be if the curve has no quadrisecants. Under a slight extension of Wall's term generic to ensure a similar regularity of appearance of the curve when viewed from other points of the curve itself, we prove:

THEOREM. *A generically embedded closed curve with no quadrisecants is unknotted.*

We expect that in the non-generic case a knotted closed curve will have a genuine quadrisecant. (If we widened the definition of quadrisecant to allow the intersections of a line with the curve to be counted with multiplicity, including, e.g. a tangent line which meets the curve in two further points, then a limiting argument would produce at least a "fake" quadrisecant for a knotted closed curve.)

Although the theorem deals with differentiable embeddings, it can be proved by a similar argument that a knotted *PL* curve in general position must have a quadrisecant, other than one of its own straight edges. This result will probably hold for any knotted curve, whether differentiable or not.

It seems possible to estimate the number of quadrisecants for a generic knot in terms of its minimum crossing number. Experiments with wire in the simplest case agree with the following conjecture.

CONJECTURE. *A generic knot with crossing number n has at least* $(1/2)n(n-1)$ *quadrisecants.*

§2. GENERICALLY EMBEDDED CURVES

Wall's analysis of a generic curve shows that the projection of the curve from a point of R^3 only has simple crossings as its singularities except when the point lies on certain two, one or zero-dimensional subsets of R^3. The singularities shown in Fig. 1, tacnode, simple cusp and triple point can be seen from certain two-dimensional subsets, while anything more complicated, e.g. a curve and an inflectional tangent, are seen only from finitely many lower dimensional subsets, if at all. See [1] for the complete list.

We shall ask that our curve be generic in the sense of Wall, and that it meet these subsets "transversely". By this we mean that for each point p on the curve, a

†The second author wishes to thank the University of Liverpool for financial support during the period of this research.

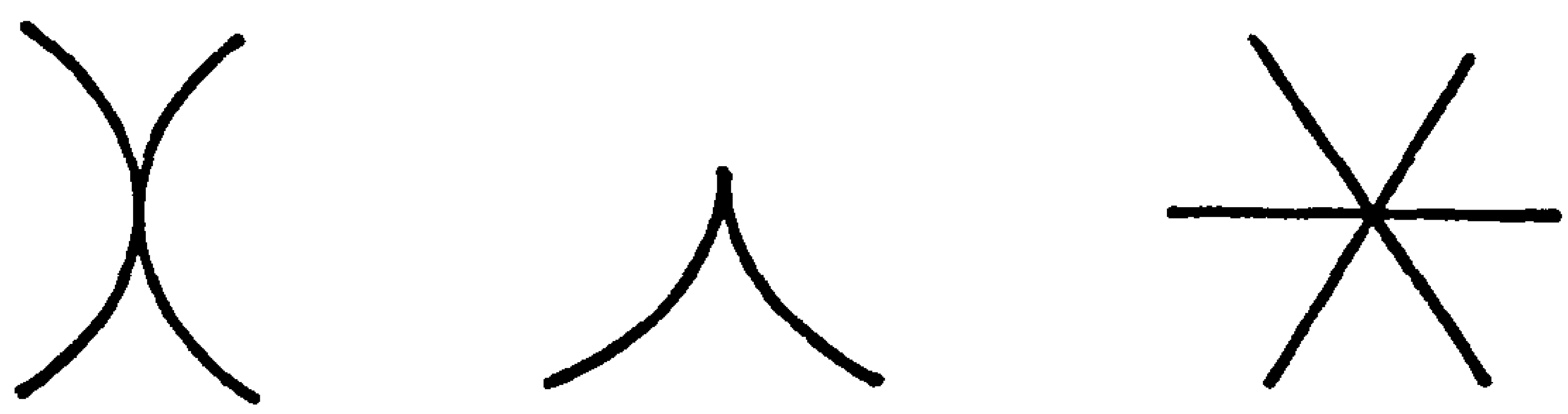

Fig. 1

sufficiently small interval round p meets the subsets defined by the complement of the interval transversely, and in particular meets only the two-dimensional subsets. The result is to restrict the type of singularities seen when projecting from a point p on the curve to simple crossings or, from finitely many points, those in Fig. 1, with one extra possibility. This arises, since the image is no longer a closed curve, but has two ends corresponding to the tangent at p. Where the tangent meets the curve again, at q say, one of these ends will lie on another branch of the projected curve, as in Fig. 2. This can happen for finitely many q, when the picture viewed from q has a simple cusp at p.

In §5 we rephrase this definition of genericity in the language of multijet transversality, and show that generic embeddings form a residual subset of all embeddings. In the following sections we shall prove the theorem using the properties of generic embeddings.

§3. THE FAMILY OF PROJECTIONS OF A CURVE

Let us assume that our curve is generically embedded in the sense of the previous section. From each point p on the curve project the curve radially to a sphere S^2 centre p. The image of the curve, $D_p \subset S^2$, is a path joining opposite ends of the diameter which is tangent to the curve at p. For all but finitely many critical values of p the path D_p is immersed with a finite number of simple crossings. In such cases we can view D_p as a knot diagram for the original knotted curve by distinguishing an over and an undercrossing at each simple crossing. If these crossings are separated accordingly, and the end points of D_p joined by the diameter we recover a curve in $\mathbb{R}^3$ isotopic to the original.

Because of the transversality requirement in our definition we can picture how D_p changes as p passes a critical value. There are four possible singularity types that can appear in D_p at a critical value of p, shown in Figs. 1 and 2. We can assume that only one occurs at any critical value. The changes in D_p in the neighbourhood of the singular point as p passes the critical value are shown in Fig. 3 with over and undercrossings indicated. The rest of D_p changes simply by isotopy in S^2. For reference we shall distinguish the critical values as types 1, 2, 3 or 4 as in Fig. 3. Notice that a critical value p of type 4 lies on a quadrisecant of the curve through p and the apparent triple point.

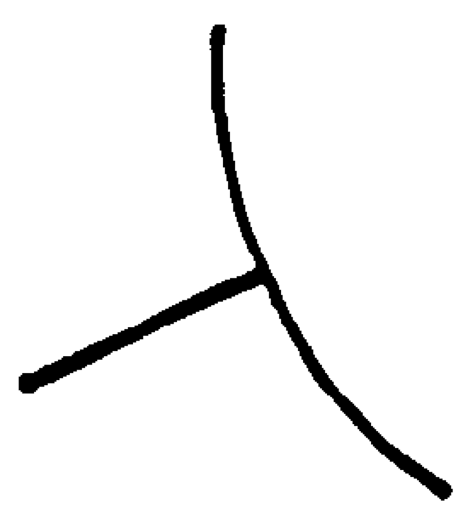

Fig. 2.

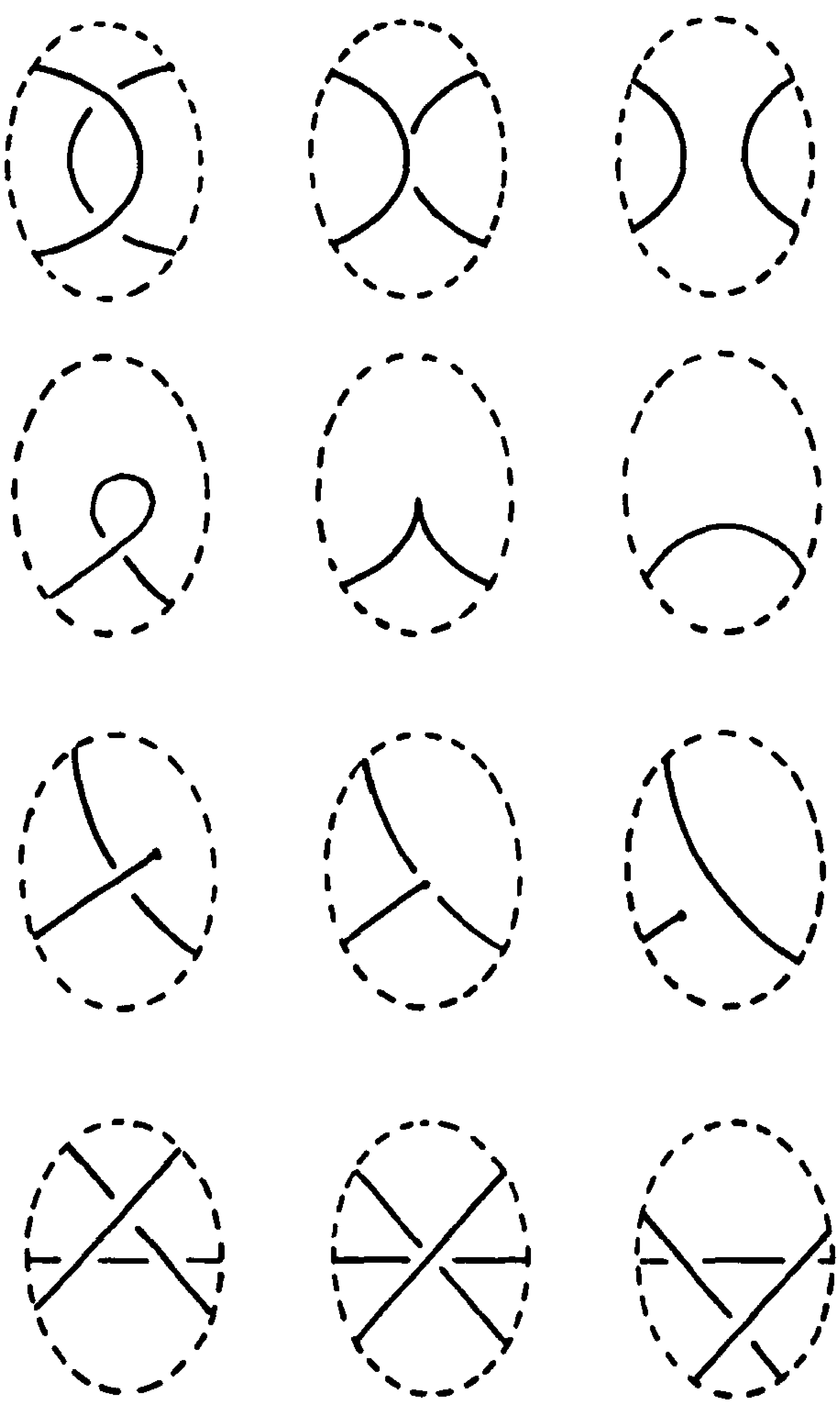

Fig. 3.

THEOREM. *A generically embedded closed curve with no quadrisecants must be unknotted.*

Starting from the curve let us consider the family of diagrams D_p constructed by projection for each p on the curve as a subset $D \subset S^2 \times S^1$, with $D \cap S^2 \times \{p\} \cong D_p$ for each p. We shall look at the singular subset $X \subset D$ consisting of the family of singular points of D_p for all p, in connection with the natural projection $\pi: X \to S^1$. Away from critical values of p, $\pi^{-1}\{p\}$ is the finite set of crossing points of D_p and π is a covering. The nature of π above critical values of each type is indicated schematically in Fig. 4, where the inverse image of an interval round the critical value is shown.

Let us now assume that the curve has no quadrisecants. Then there are no critical values of type 4, and X is a 1-manifold, with boundary points above critical values of types 2 and 3. If we choose an orientation for our curve then each critical value p can be termed a "birth" or a "death" depending on whether the number of crossings increase or decrease as p is passed. A point of X in $\pi^{-1}\{p\}$ can be viewed as a trisecant meeting the curve in three distinct points, except at a boundary point of X where two of these coincide and yield a tangent line. One of the extreme points of the three on the trisecant line is the point p. Write $\mu: X \to S^1$ for the map which selects the middle point of the three on each trisecant line. So $\mu(x) = \pi(x)$ only at a boundary point $x \in X$.

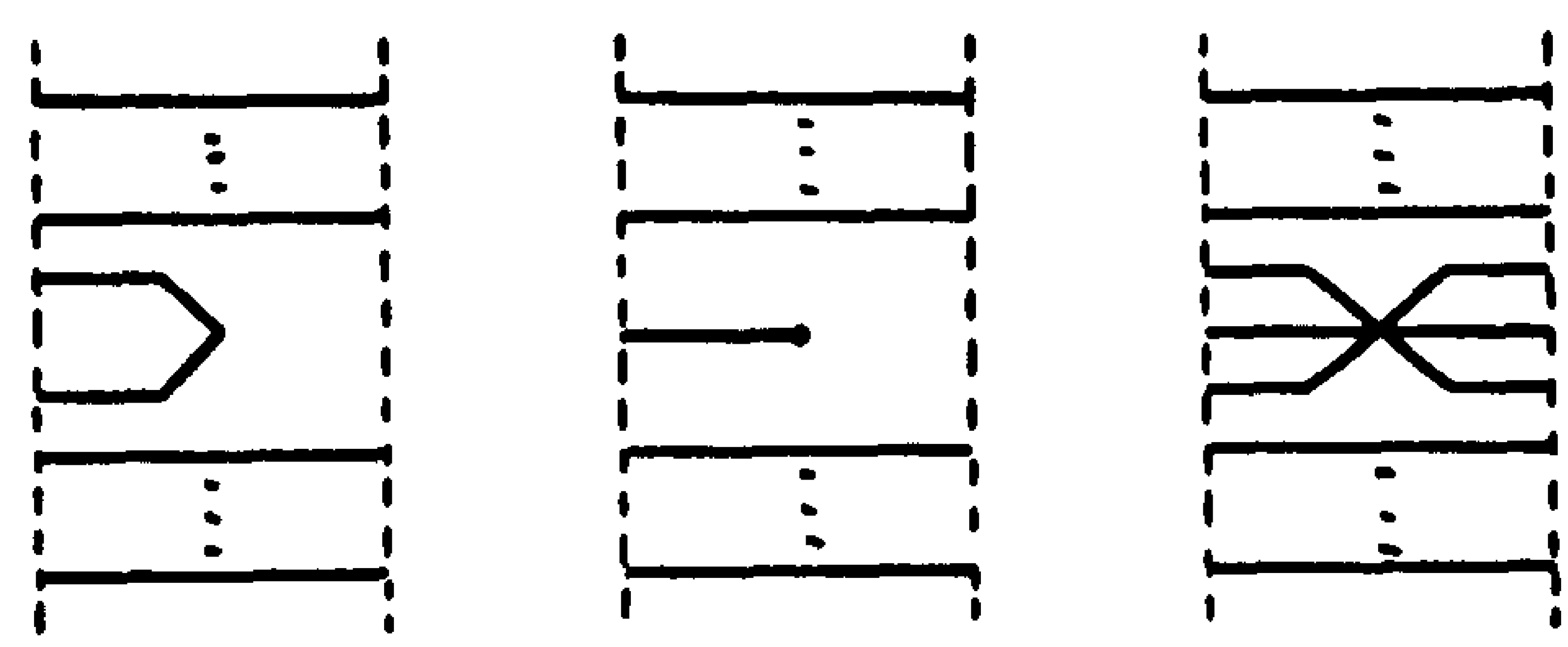

Fig. 4.

LEMMA 1. *The map π is inessential on every component of X.*

Proof. Suppose not. Then let $Y \subset X$ be a component, without a boundary, on which π has degree $k \neq 0$. Since $\pi(y) \neq \mu(y)$ for any $y \in Y$ it follows that μ also has degree k on Y, and hence that μ: $Y \to S^1$ is onto. Thus every point of the curve is the middle point of some trisecant. This is impossible, since every height function on $\mathbb{R}^3$ has extreme points on the curve which are not middle points.

§4. MODIFYING FAMILIES OF DIAGRAMS

In this section we shall deal with certain subsets of $S^2 \times S^1$, including, but not confined to, the family D of projections of a curve as described in §3.

Definition. Call the subset $D \subset S^2 \times S^1$ a *restricted family of diagrams* if (i) the sets $D_p = D \cap S^2 \times \{p\}$ consist, for all but finitely many p, of an arc immersed in S^2 with simple crossings, at which an over and undercrossing are distinguished, and (ii) on passing a critical value of p, D_p changes as for types 1, 2 or 3 in Fig. 3 near one singular point, and otherwise it changes smoothly with p, preserving over and undercrossings.

The curves in $\mathbb{R}^3$ formed by separating over and undercrossings of $D_p \subset S^2$, and joining the end points by a diameter of S^2 will yield isotopic knots for all choices of p. Call this isotopy class the *knot class* of D. If, for example, D has been constructed by projections of an embedded smooth curve then this curve belongs to the knot class of D.

The singular set $X \subset D$ of a restricted family of diagrams is a 1-manifold, possibly with boundary. Call components of X *essential* or *inessential* according to their behaviour under the projection π: $X \to S^1$.

LEMMA 2. *For any restricted family of diagrams D, with singular set X, there is another family D^1 with the same knot class, whose singular set has the same number of essential components as X, but has no inessential components.*

Proof of Theorem. Every component of the singular set X in the restricted family of diagrams D constructed from the given curve by projection is inessential, by Lemma 1. Hence, by Lemma 2, there exists a family of diagrams D^1 with the same knot-class, having no singular set. This knot-class can be represented, using any of the diagrams D_p^1, by an *embedded* arc in S^2 and a diameter. Such a closed curve is unknotted.

Proof of Lemma 2. If π: $X \to S^1$ has no critical values then π is a covering and there is nothing to prove. Otherwise, having chosen an orientation for S^1, there must occur among the critical values a birth followed immediately by a death. (If say, a birth occurs as we pass each critical value, then the number of points in $\pi^{-1}\{p\}$ would increase as p makes a circuit of S^1.) We shall alter D over an interval containing only these two critical values to give a new restricted family of diagrams $\bar{D}$, with at most two critical values in this interval. Clearly, since D and $\bar{D}$ agree over part of S^1 they will have the same knot class.

The new singular set $\bar{X} \subset \bar{D}$ will be more regular in relation to the projection, with essential components straightened out slightly, and possibly an inessential component of X omitted. See Fig. 5 for the modifications to the singular set.

More precisely, there is a homeomorphism h from $\bar{X}$ to a subset $Y \subset X$, which commutes up to homotopy with projection to S^1, or $\bar{X} = \emptyset = Y$, where $Y = X$ or $X - Y$ is an inessential component of X. Moreover $\bar{X}$ has fewer points lying above critical values in S^1.

Since any inessential component of X either disappears in $\bar{X}$, or continues, via h^{-1}, to form an inessential component of $\bar{X}$, we shall reach the required D^1 whose singular set has no inessential components after a finite number of such alterations. In fact we could continue until the essential components, if any, cover S^1 regularly.

The alteration from D to $\bar{D}$ must now be prescribed for a consecutive birth and death of any of the three types. Let us suppose that the birth and death occur at critical values p and q in the interval $[a, b] \subset S^1$. Select an intermediate value c so that $a < p < c < q < b$ in the orientation order. The new family $\bar{D}$ over $[a, b]$ will be constructed with the same initial and final diagrams, $\bar{D}_a = D_a$, $\bar{D}_b = D_b$, but a new intermediate diagram $\bar{D}_c$, related to the initial and final diagrams either by passing through a single birth of types 1, 2 or 3, or simply by an isotopy. This passage from $\bar{D}_c$ to $\bar{D}_a$ and $\bar{D}_b$ by way of the simple birth or isotopy then provides the way to complete $\bar{D}$ over the intervals $[a, c]$ and $[c, b]$.

The diagrams in Fig. 5 show X and $\bar{X}$ above $[a, b]$ with π viewed as projection, in the various cases depending on the type of birth and death, and whether any singular point is involved in both. The map h can readily be produced in each case.

It remains to show that a suitable candidate for the new intermediate diagram $\bar{D}_c$ can always be found. Let us start with the original diagram $D_c \subset S^2$. The diagrams D_a

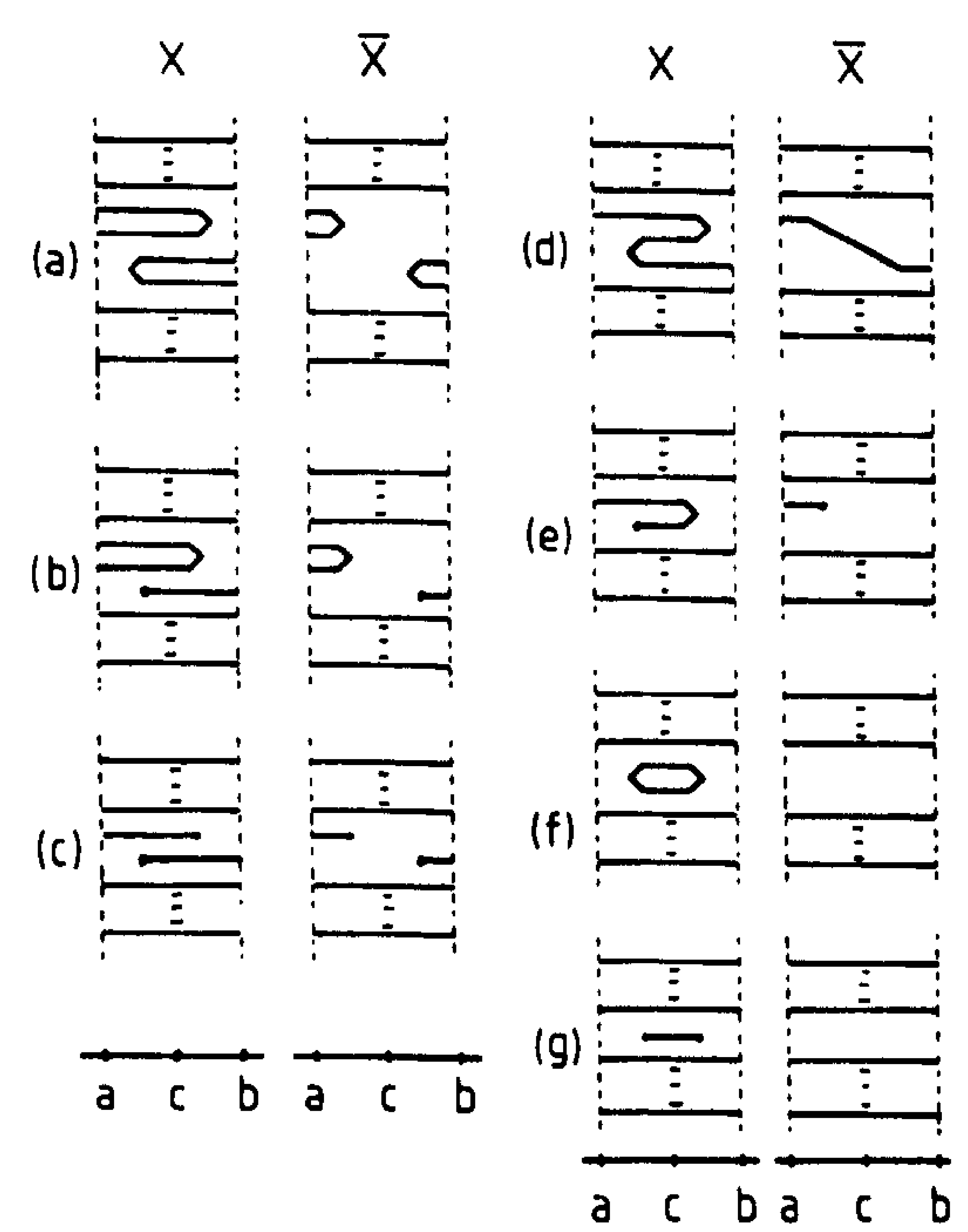

Fig. 5.

and D_b are both reached from this by the simple death of one or two singular points. For a type 1 death of two singular points $x, y \in D_c$ there will be two arcs joining x to y in D_c which bound a disc $A \subset S^2$ meeting D_c only in these arcs. The change to D_a takes place (up to isotopy on S^2) only in a neighbourhood of A.

There is a similar disc for a type 2 death bounded by a loop joining the singular point to itself, and for a type 3 death there is an arc in D_c from the singular point to an end-point of D_c with all the action again taking place in a close neighbourhood.

Write A for the disc (or arc in type 3) determined in this way in S^2 by the death which yields D_a from D_c, and write B for the disc or arc yielding D_b from D_c. These discs or arcs only contain the singular point(s) of D_c which are to die.

Then D_a and D_b only differ, up to isotopy in S^2, in a neighbourhood of $A \cup B$.

If the singular points which die in passing from D_c to D_a are different from those which die between D_c and D_b then A and B are disjoint, for neither contains any part of D_c in its interior, and they have no boundary in common unless they have some common singular point. Choose disjoint neighbourhoods of A and B, and alter D_c to $\bar{D}_c$ by killing the singular points in each. Then D_a and D_b are both reached from $\bar{D}_c$ by a simple birth, in the neighbourhood of B and A respectively, so in $\bar{D}$ the death in B is followed by the birth in A. The corresponding alterations to the singular set are those in 5(a), (b) or (c).

In the remaining cases A and B will either meet at one singular point only, or, since D_c must be a single immersed arc, if they have more boundary in common they will coincide.

If $A = B$, then D_a and D_b will be isotopic, and D_c can be replaced by D_a with the isotopy connecting D_a and D_b completing $\bar{D}$. The singular set will change as in 5(f) or (g).

We are left with the cases where A and B meet in a single point. The possible configurations for D_c in a neighbourhood of $A \subset B$ are shown with appropriate over and undercrossings in Figs. 6–8. The corresponding diagrams for the singular set are given by 5(d), (e) and (g) respectively. The different cases in Figs. 7 and 8 cover the possibilities of type 2 or type 3 singularities. The configurations 8(ii) and (iii) can be excluded since D_c is an immersed arc. In the remaining cases $\bar{D}_c$ is given by altering D_c in the neighbourhood as shown. Outside the neighbourhood D_a and D_b agree with D_c up to isotopy, and within it they are also shown on the diagrams. It can be seen that in each case D_a and D_b are reached from $\bar{D}_c$ either by isotopy or by a simple birth, enabling $\bar{D}$ to be completed as claimed.

Remark. The first author has been able to show that a link with two components in general position has at least k^2 quadrisecants which meet the components alternately, where k is the linking number. The proof uses the technique of representing trisecants on copies of $S^1 \times S^1$ by taking the middle point and one end point of each trisecant. A

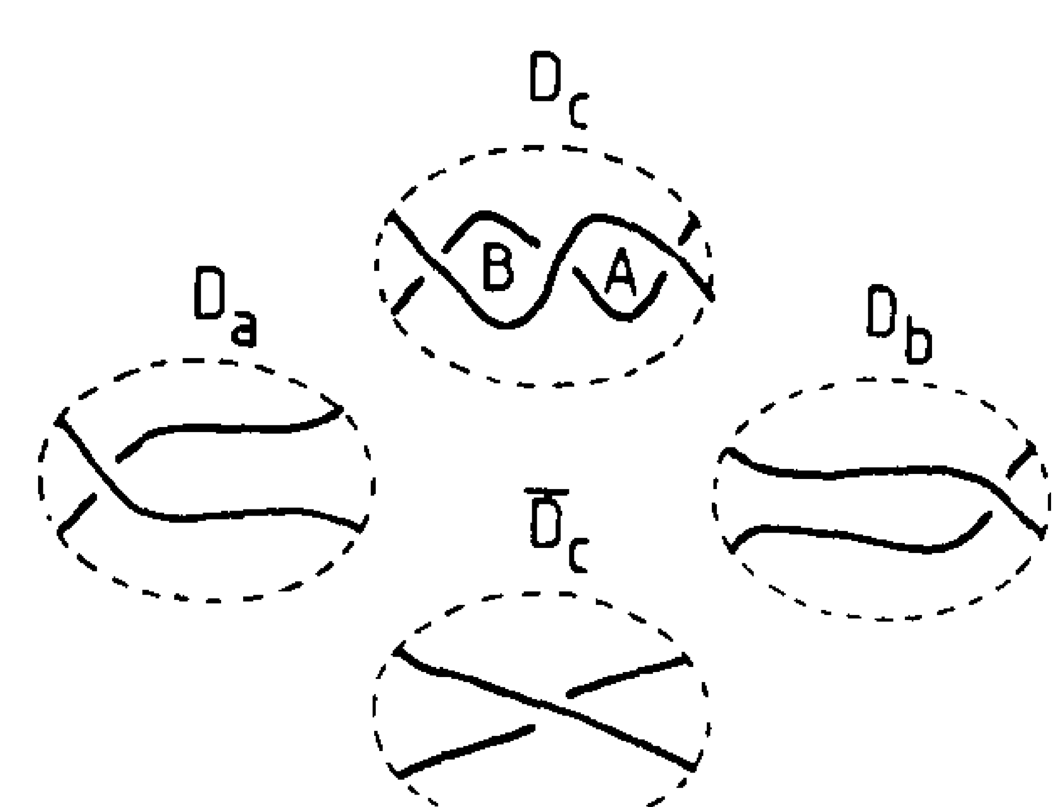

Fig. 6.

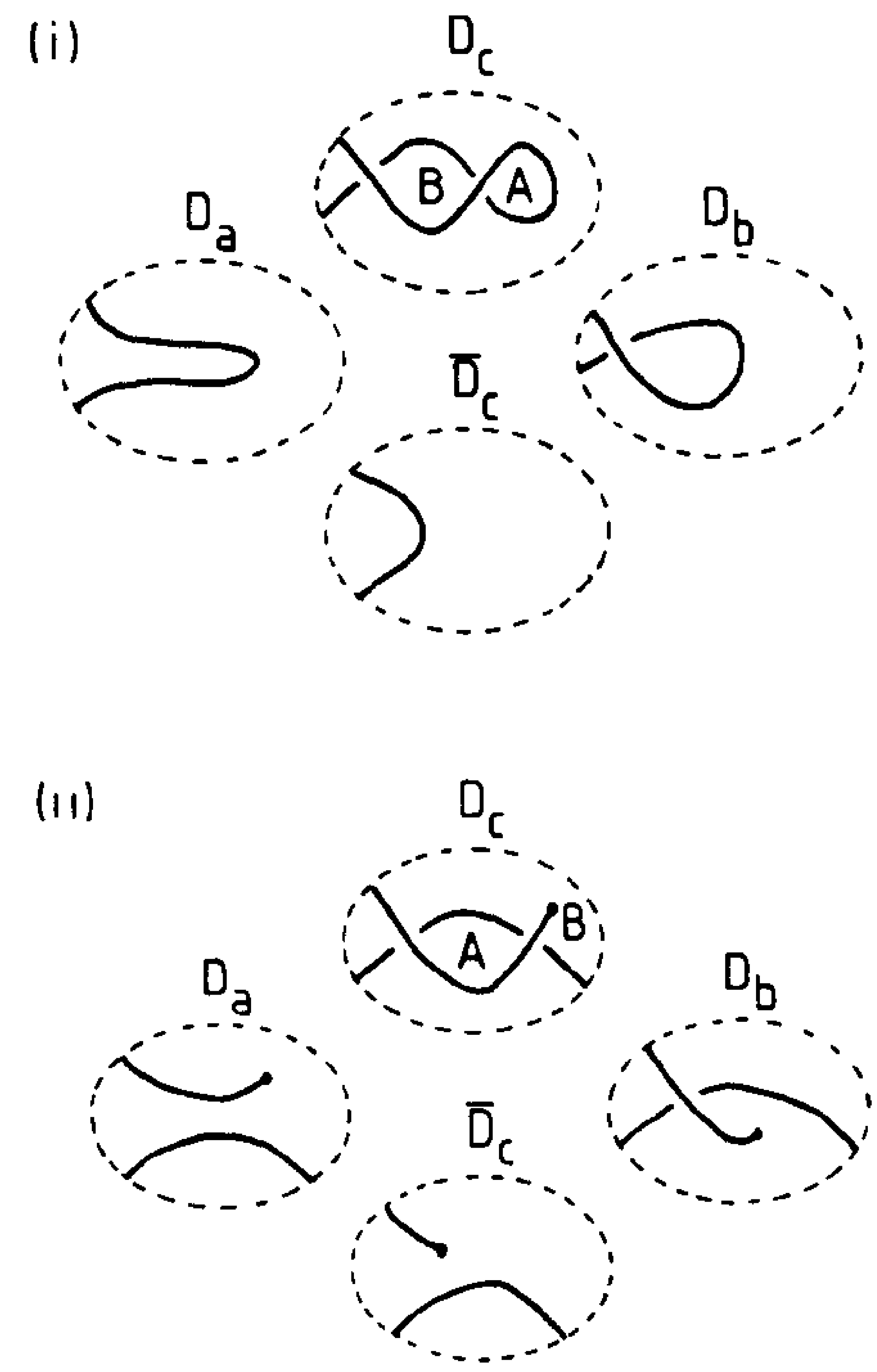

Fig. 7.

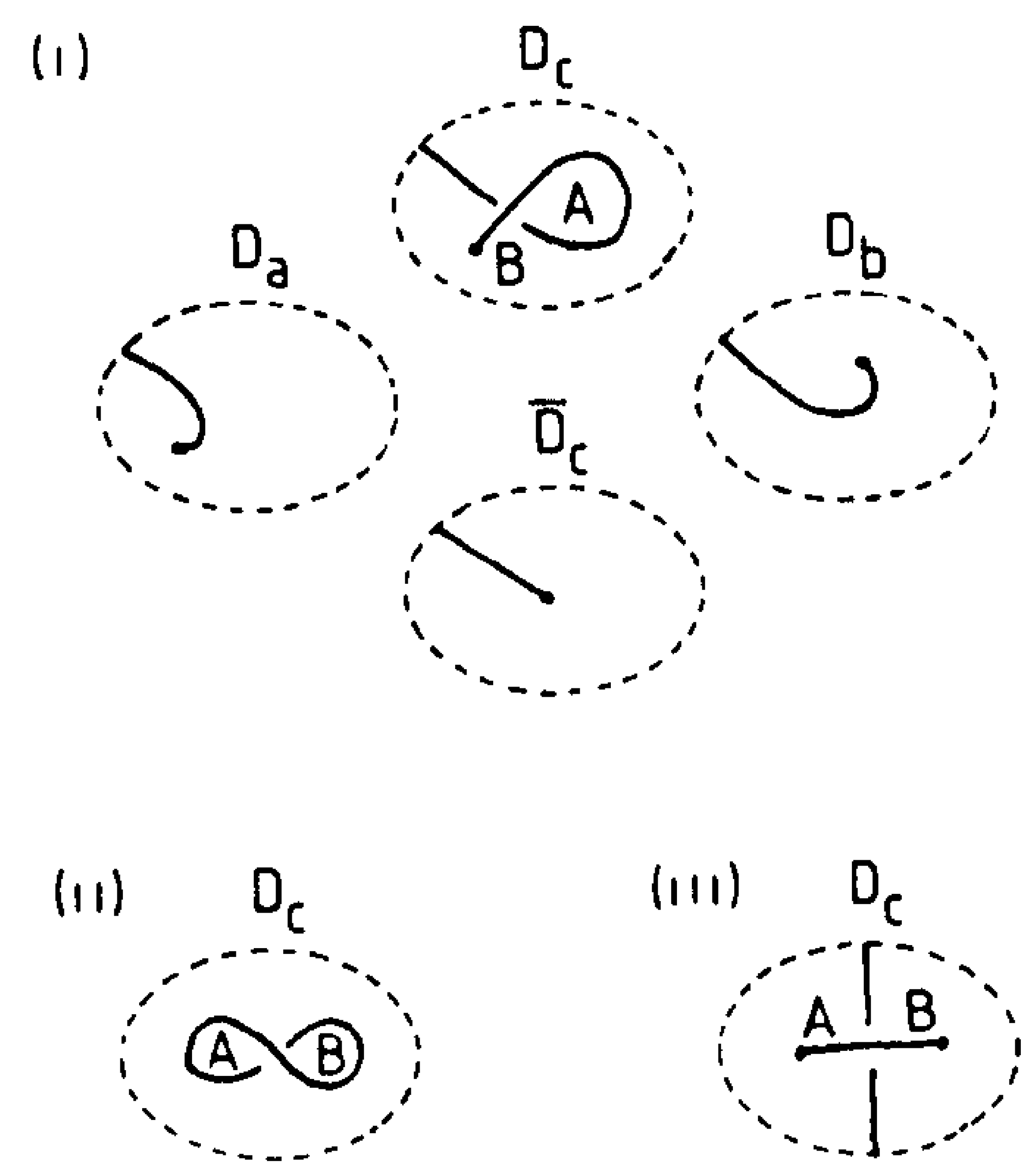

Fig. 8.

combination of this technique and the methods used in this section shows that a two-component link without quadrisecants must be trivial.

§5. GENERIC EMBEDDINGS

It remains to show that for a sufficiently large class of embeddings of S^1 in $\mathbb{R}^3$, the family of diagrams obtains by projecting from points on the curve is indeed a restricted family of diagrams as defined in §4.

LEMMA 3. *For a residual set of embeddings of S^1 in $\mathbb{R}^3$, the image of the curve after projection from any one of its points contains, apart from simple crossings, only the singularities shown in Fig. 1, and as the point of projection moves along the curve, these singularities are unfolded as shown in Fig. 3.*

Proof. In [1] Wall defines a residual set (which we shall call C) of embeddings of S^1 in $\mathbb{R}^3$ for which only the singularities in a list which he gives there are visible under projection from points in $\mathbb{R}^3$ off the curve. Moreover, each of these singularities, with the exception of the quadruple crossing, is versally unfolded by the family of mappings obtained by letting the point of projection vary in three-space. By versality we always mean versality with respect to the group $\mathscr{A}$ of local diffeomorphisms at source and target. If a singularity S is visible from a point on the curve then from points in space on the line joining this point to the perceived singularity, an "augmented" singularity $\bar{S}$ is seen, consisting of S plus an additional branch of the curve. Thus for a curve in C we need only look at the singularities in Wall's list involving two or more branches of the curve, and such that on removing one of the branches, a singularity other than a simple crossing remains. These are shown in Fig. 9. Removal of the starred branch in (i) or (iii) leaves the simple cusp or triple point, as shown in Fig. 1. Now the situation is slightly complicated in the case of (ii) and (iv). The difference between them is that the two branches of the curve which have non-transverse contact have inflexional (3rd order) contact in (iv), while in (ii) they have only tangential (2nd order) contact; but the degree of contact may be altered by moving the point of projection along the line of sight, and so what will appear as 2nd order contact between two branches of the curve from all except one of the points on the line of sight, will appear as higher order contact from this one remaining point. Thus, we cannot use Wall's list alone to exclude the possibility that, for a curve in C, two branches of the curve will appear to have 3rd order contact when viewed from a point on a 3rd branch. However, a straightforward if tedious calculation shows that in the multijet space ${}_3J^2(S^1, \mathbb{R}^3)$ the set of multi-jets exhibiting this circumstance is algebraic of codimension four, so that (by Mather's multijet transversality theory, [1], p. 741) for a residual set of curves it will not occur. Intersecting this set with C, we obtain a residual set $\bar{C}$ of curves satisfying the first statement of the lemma.

A perceived singularity that is versally unfolded by the family of projections, is versally unfolded as the eye moves along a *curve* in space if and only if this curve meets the corresponding equisingularity manifold transversely. (This is a simple consequence of the finite $\mathscr{A}$-determinacy of all of the singularities in question.) Thus for example, a first order cusp is versally unfolded if the path that the eye follows in space crosses the tangent developable of the embedded curve transversely. It is easy to see, in the case of the cusp, tacnode and trisecant, that if the branch of the curve from which we are viewing fails to cross the equisingularity manifold transversely,

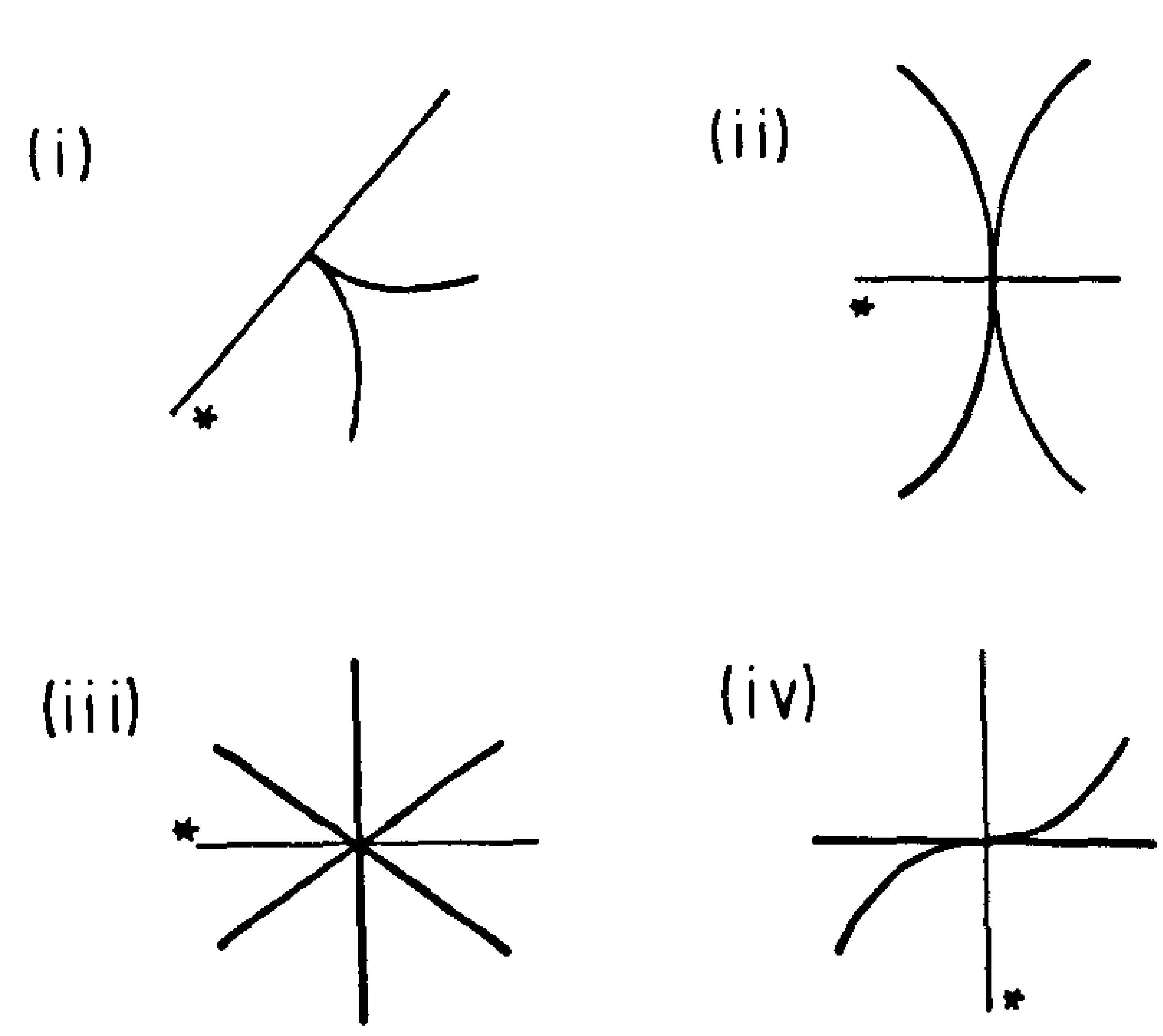

Fig. 9.

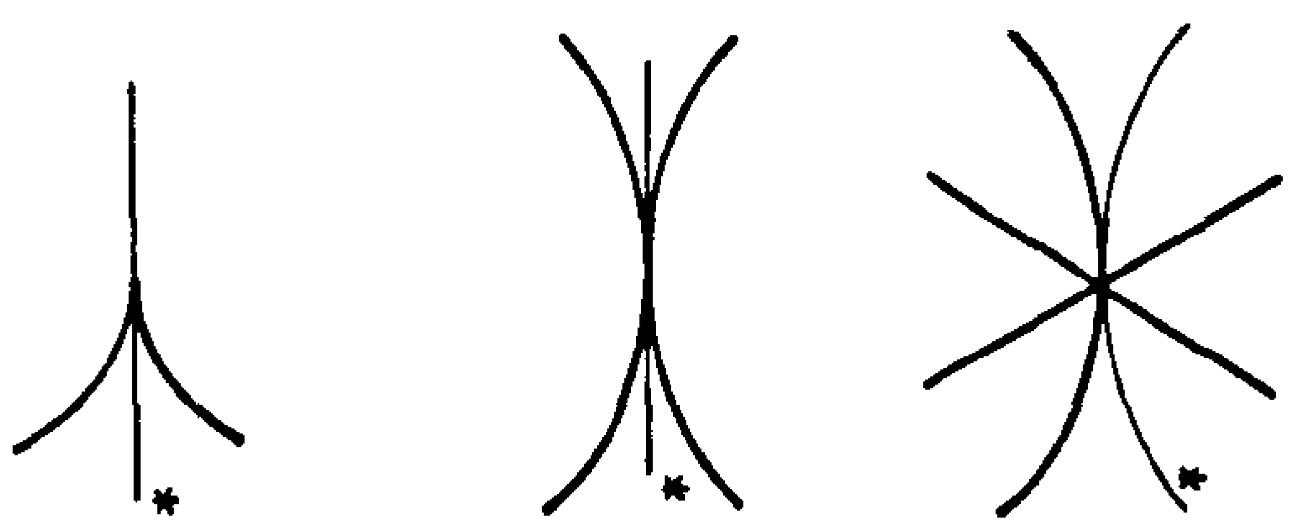

Fig. 10.

then the augmented singularities seen are those shown in Fig. 10. None of these figures is in Wall's list for the curves of C. Hence the original singularities, when seen from a point on the curve, must be versally unfolded as the eye moves along the curve. The drawings in Fig. 3 are simply representations of these versal unfoldings as one-parameter families of curve germs.

COROLLARY. *There is a residual set of curves,* $\tilde{C}$, *such that, for every curve in* $\tilde{C}$, *the family of diagrams obtained by projecting from the points on the curve is a restricted family of diagrams.*

Proof. After Lemma 3 all that we need do is examine the singularity of Fig. 2. But in fact there is nothing new here—it is just the same as the cusp, in as much as in each case the augmented singularity is that of Fig. 9(i).

In the case of the singularity shown in Fig. 2, the starred branch is being viewed from the point on the other branch which appears here as the vertex of the cusp, while a cusp appears in D_p when p is on the starred branch. In any case, the same argument as that used in the proof of the lemma shows that for a curve in $\bar{C}$, if the singularity of Fig. 2 appears then it is versally unfolded as shown in Fig. 3.

Finally, the assumption that for each $p \in S^1$, at most one of the singularities of Figs. 1 and 2 appears in D_p, is in fact the assumption that the corresponding equisingularity manifolds do not meet on the curve. Although this could be proved by an extension of Soares's theorem (see [1], p. 742), in our case it is sufficient, since we are only interested in three specific and algebraically definable singularities, to assure the reader that in the corresponding multi-jet spaces ${}_kJ^i(S^1, \mathbb{R}^3)$, the codimension of the phenomenon in question is in each case greater than k, so that for a subset $\tilde{C} \subset \bar{C}$, still residual, it does not occur.

Acknowledgement—The authors would like to thank Jean-Pierre Otal for drawing their attention to the paper "Eine elementargeometrische Eigenschaft von Verschlingungen und Knoten". *Math. Annalen* **108** (1933), 629–672, E. Pannwitz, where similar results are obtained.

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On the tangent developable of a space curve

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Summary

In this paper we investigate the local form of the tangent developable of a curve in 3-space, and obtain results relating this to the order of vanishing of the torsion. In particular,

THEOREM 1. *If at the point $\gamma(t)$ on the curve γ, the torsion vanishes to order k ($0 \leqslant k \leqslant 4$), then the germ of the tangent developable at that point has one curve of self-intersection if k is odd, and none if k is even.*

The proof uses the techniques of singularity theory, especially the notion of finite determinacy and also the idea of 'blowing up' a variety along a singular set, in order to obtain a less singular variety.

A similar result, for the cases $k = 0, 1$ (that is, for generic curves) was obtained by different methods by J. Cleave (1), and also by T. Gaffney and A. du Plessis (7).

1. *Definitions*

Throughout, $\gamma: (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ will be the germ of a regular C^∞ curve with non-vanishing curvature. The tangent developable is the surface generated by the lines tangent to γ, and so has a natural parametrization

$$\phi: (t, w) \to \gamma(t) + w\gamma'(t). \tag{1}$$

Given a map germ $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the group $\mathscr{A} = \text{Diff}(\mathbb{R}^n, 0) \times \text{Diff}(\mathbb{R}^p, 0)$ acts on it in the obvious way. We say that a map germ f is *k-determined* if any other map-germ g, such that $j^k(g) = j^k(f)$, is equivalent to f under the action of $\mathscr{A}$. A k-jet σ is *sufficient* if any two map-germs f and g such that $j^k(f) = j^k(g) = \sigma$, are equivalent under $\mathscr{A}$. By abuse of language, a polynomial germ f of degree d will be called sufficient if its d-jet is sufficient.

2. *Blowing up the tangent developable*

Since we are going to use finite determinacy, we are interested only in the Taylor series of the curve γ. After an appropriate choice of coordinate in $\mathbb{R}$, and a non-singular linear transformation of $\mathbb{R}^3$, (which will transform the tangent developable of the original curve into that of the transformed curve), we may assume that

$$\begin{aligned}\gamma(t) &= (t, t^2 + b_3t^3 + \ldots, c_3t^3 + \ldots) \\ &= (t, b(t), c(t))\end{aligned}$$

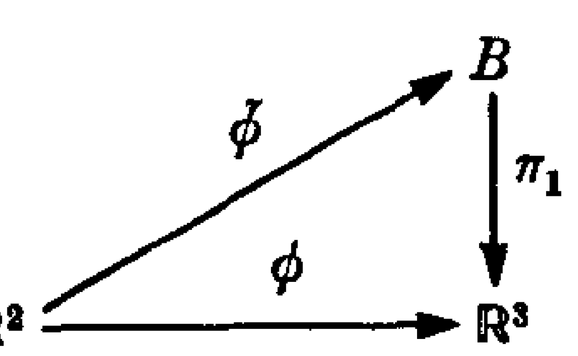
in a neighbourhood of $t = 0$.

As can be seen from (1), ϕ is singular when $w = 0$, and so the tangent developable has a cuspidal edge along the curve. Since map-germs $\mathbb{R}^2 \to \mathbb{R}^3$ with non-isolated singularities cannot be finitely determined (2), we cannot apply this notion directly to ϕ. We therefore define the blow-up

$$\begin{aligned} B &= \{((x,y,z),(u:v)) \in \mathbb{R}^3 \times \mathbb{P}^1 : u(z-c(x)) = v(y-b(x))\} \\ &= \{((x,y,z),(u:v)) \in \mathbb{R}^3 \times \mathbb{P}^1 : (u,v) \text{ parallel to } (y-b(x), z-c(x))\}. \end{aligned}$$

Let π_1 and π_2 be the natural projections from B onto $\mathbb{R}^3$ and $\mathbb{P}^1$ respectively. It is easy to check that B is a manifold, and that $\pi_1 : B \to \mathbb{R}^3$ is a diffeomorphism on $B - \pi_1^{-1}(\gamma)$.

We now aim to lift ϕ by a map $\tilde{\phi} : (\mathbb{R}^2, 0) \to B$, to obtain the commutative diagram



Since π_1 is a diffeomorphism away from $\pi_1^{-1}(\gamma)$, $\tilde{\phi}$ is automatically defined off the t-axis in $\mathbb{R}^2$, by

$$\begin{aligned} \tilde{\phi}(t,w) = ((t+w, b(t)+wb'(t), c(t)+wc'(t)), (b(t)+wb'(t)-b(t+w) : c(t) \\ + wc'(t) - c(t+w))). \end{aligned}$$

The Taylor series of $\pi_2 \circ \tilde{\phi}$ has the form

$$(-w^2(1+b_3(3t+w)+\ldots) : -w^2(c_3(3t+w)+\ldots)) \tag{2}$$

and so for small $w \neq 0$ and small t,

$$\pi_2 \circ \tilde{\phi}(t,w) \in U_1 = \{(u:v) \in \mathbb{P}^1 : u \neq 0\}.$$

By taking local coordinates $(x, y, v/u)$ on $B \cap (\mathbb{R}^3 \times U_1)$ we can write $\tilde{\phi}$ as a map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$,

$$\left(t+w, b(t)+wb'(t), \frac{c(t)+wc'(t)-c(t+w)}{b(t)+wb'(t)-b(t+w)}\right) \tag{3}$$

and looking at (2) we see that after cancellation of w^2, (3) may be used to definite $\tilde{\phi}$ along the t-axis as well, i.e. in an open neighbourhood of $(0,0) \in \mathbb{R}^2$.

3. *The germ of* $\tilde{\phi} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$

(*a*) Assume $c_3 \neq 0$. Then the 1-jet of $\tilde{\phi}$ is

$$(t,w) \to (t+w, 0, c_3(3t+w))$$

and so $\tilde{\phi}$ is an immersion.

(*b*) Assume $c_3 = 0, c_4 \neq 0$. The 2-jet of $\tilde{\phi}$ is

$$(t,w) \to (t+w, t^2+2tw, 6c_4 t^2 + 4c_4 tw + c_4 w^2).$$

Putting $t = s - w$, this becomes

$$(s,w) \to (s, s^2 - w^2, c_4(3w^2 - 8sw + 6s^2)),$$

which is equivalent under coordinate changes in $\mathbb{R}^3$ (briefly, 'left-equivalent') to

$$(s, w) \to (s, w^2, sw).$$

This is the 2-jet of a 'cross-cap', and is sufficient (for determinacy estimates see section 5).

(*c*) Assume $c_3 = c_4 = 0, c_5 \neq 0$. The 3-jet of $\tilde{\phi}$ is now, after the variable change $t = s - w$,

$$(s, w) \to (s, s^2 - w^2 + b_3(s^3 - 3sw^2 + 2w^3), c_5(10s^3 - 20s^2w + 15sw^2 - 4w^3)).$$

This is left-equivalent as a 3-jet to

$$(s, w) \to (s, w^2 - 2b_3 w^3, w^3 + 5s^2 w),$$

and putting $\bar{w} = w(1 - 2b_3 w)^{\frac{1}{2}}$, the 3-jet becomes

$$(s, \bar{w}) \to (s, \bar{w}^2, \bar{w}^3 + 5s^2\bar{w}),$$

which can be reduced to

$$(s, w) \to (s, w^2, w^3 + s^2 w)$$

by a change of scale. This is a sufficient jet.

(*d*) Assume $c_3 = c_4 = c_5 = 0, c_6 \neq 0$. Similar coordinate changes show that the 4-jet of $\tilde{\phi}$ is equivalent to

$$(s, \bar{w}) \to (s, \bar{w}^2, \bar{w}^3 s + s^3 \bar{w}),$$

which is again a sufficient jet.

(*e*) If $c_3 = \ldots = c_6 = 0, c_7 \neq 0$, then $\tilde{\phi}$ has 5-jet equivalent to

$$(s, \bar{w}) \to (s, \bar{w}^2, \bar{w}^5 + 2\sqrt{\tfrac{21}{5}}\, s^2 \bar{w}^3 + s^4 \bar{w}).$$

which is again sufficient. In fact

$$(s, w) \to (s, w^2, w^5 + \lambda s^2 w^3 + s^4 w),$$

is a unimodular family of germs with modulus λ, and different members (i.e. with different values of λ) are inequivalent. Thus we cannot bring the 5-jet of $\tilde{\phi}$ to a simpler form not involving a coefficient like $2\sqrt{\frac{21}{5}}$.

4.

The fact that in each of the cases considered in 3, the jet given is sufficient, tells us that from it we may gain complete information about the germ of the image of $\tilde{\phi}$. In fact, it is easy to see that π_2 gives a homeomorphism of $\tilde{\phi}(R^2, 0) \subset B$ on to $\phi(R^2, 0) \subset \mathbb{R}^3$, and so by looking at the appropriate jet of the curve γ (which of course determines the jet of $\tilde{\phi}$) we can gain complete information about the topology of $\phi(\mathbb{R}^2, 0)$. Straightforward calculations show that the images of the jets of (*a*)–(*e*) (considered as polynomial maps $\mathbb{R}^2 \to \mathbb{R}^3$) have respectively 0, 1, 0, 1, 0 curves of self-intersection, and since cases (*a*)–(*e*) correspond to the torsion of γ at $t = 0$ being respectively non-zero, zero to first order, zero to second order, etc., all that needs to be done to complete the proof of Theorem 1 is to show that the relevant jets are sufficient.

5. Determinacy Estimates

The normal forms for the jets found in 3 figure in a classification of some map-germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ obtained by the author (6). Here we sketch a proof of their sufficiency. We rely heavily on results obtained by Gaffney (3), and use his notation.

(*a*) Let f be an immersion. Then it is well known that its 1-jet is sufficient.

(*b*) Let f be given by $(x, y) \to (x, y^2, xy)$. Then as is well known (see e.g. (4), page 213, although it is easy to check), f is stable, i.e. $tf(\theta(2)) + \omega f(\theta(3)) = \theta(f)$. Since $f^* \mathfrak{m}_3 \theta(f) \supset \mathfrak{m}_2^2 \theta(f)$, it follows from the theorem of (3) that f is 3-determined. However, given a 3-jet of the form

$$(x, y) \to (x + p_1(x, y), y^2 + p_2(x, y), xy + p_3(x, y)),$$

where the p_i are homogeneous cubics, then making the coordinate change

$$\bar{x} = x + p_1(x, y)$$

we can assume that $p_1 = 0$. Then using left coordinate changes, we can remove the x^3, xy^2 and x^2y terms from p_2, so we are left with a 3-jet of the form

$$(x, y) \to (x, y^2 + ay^3, xy + p_3(x, y)).$$

Now put

$$\bar{y} = y(1 + ay)^{\frac{1}{2}}$$

to get as 3-jet

$$(x, y) \to (x, \bar{y}^2, x\bar{y} + \bar{p}_3(x, \bar{y})).$$

Remove the x^3, $x\bar{y}^2$ and $x^2\bar{y}$ terms from p_3 as before, to get a 3-jet of the form

$$(x, \bar{y}) \to (x, \bar{y}^2, x\bar{y} + b\bar{y}^3),$$

then put $\bar{x} = x + b\bar{y}^2$ to get as 3-jet

$$(\bar{x}, \bar{y}) \to (\bar{x} - b\bar{y}^2, \bar{y}^2, \bar{x}\bar{y}).$$

Finally, remove the $b\bar{y}^2$ from the first component by the obvious (linear) coordinate change in $\mathbb{R}^3$. Hence, the 2-jet

$$(x, y) \to (x, y^2, xy)$$

is sufficient.

(*c*) Let $f(x, y) = (x, y^2, y^3 + x^2y)$. Then one calculates, using the techniques introduced in (3), that

$$tf(\theta(2)) + \omega f(\theta(3)) \supset \mathfrak{m}_2^3 \theta(f)$$

$$tf(\theta(2)) + f^* \mathfrak{m}_3 \theta(f) \supset \mathfrak{m}_2^3 (f)$$

so that by the Theorem of (3), f is 5-determined. Manipulations similar to those of (*b*) then show that any 5-jet, whose 3-jet is equal to that of f, is equivalent as a 5-jet to the 5-jet of f. So f is sufficient.

(*d*) The remaining cases are dealt with in a similar way.

6. Remarks

1. It will be noted that in the calculations made in 3, none of the coefficients of the Taylor series of $b(t)$ appear (except for b_2, which is equal to 1 by assumption). This is because in making the calculations we have only had to use the constant term (again

equal to 1) of the power-series inverse of $(-1/w^2)\{b(t)+wb'(t)-b(t+w)\}$, since the sufficiency of the jets concerned means that higher terms are irrelevant. However, for $k > 5$, no k-jet of the form $(x, y) \to (x, y^2, p(x, y))$, where p is a homogeneous polynomial of degree k, is sufficient, and so an analysis of further cases (τ vanishing to order greater than 4) would have involved looking at the coefficients of $b(t)$. It is possible, however, that such k-jets may be topologically (as opposed to $\mathscr{A}$) sufficient.

2. By looking at the versal deformation (for definitions see (5)) of such germs as $(x, y) \to (x, y^2, y^3 + x^2y)$ (case (c) of 3), we can get a picture of what happens to the tangent developable of a curve as we deform it by varying some parameter. For example, for $\lambda = 0$ the curve

$$\gamma_\lambda(t) = (t, t^2 + \ldots, \lambda t^3 + c_5 t^5 + \ldots) \quad (c_5 > 0) \tag{4}$$

has torsion vanishing to second order at $t = 0$, but for $\lambda < 0$ the torsion has two (distinct) first order zeros in a small neighbourhood of $t = 0$. The versal deformation of $(x, y) \to (x, y^2, y^3 + x^2y)$ is

$$(x, y, \lambda) \to (x, y^2, y^3 + x^2y + \lambda y)$$

For $\lambda < 0$, there two 'cross-caps' or 'pinch-points' (the type of singularity of 3(b)), at $(x, y) = (\pm\sqrt{-\lambda}, 0)$, and their images in $\mathbb{R}^3$ are linked by a curve of self-intersection of the image of $(\mathbb{R}^2, 0)$. As $\lambda \to 0-$, this curve of self intersection contracts to a point. Since varying λ in (4) induces a versal deformation of ϕ, exactly the same must happen in the tangent developable of γ_λ as $\lambda \to 0-$.

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