ON THE GEOMETRY OF MECHANISMS

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Abstract

THIS THESIS presents a philosophy for studying the **kinematic geometry of mechanisms**. In particular, the aim is to bring together relevant ideas and theorems from modern algebraic geometry and to apply them to the special varieties which encapsulate the motion of mechansims.

The philosophy is to associate to each configuration of a mechanism a point in a higher-configuration space. The constraints on the motion of a mechanism may be expressed as polynomial equations in that **configuration space**, thus defining a **linkage variety**. The thesis describes the geometry of the linkage varieties for the planar and spherical four-bar mechanisms, the geared five-bar mechanism with gear ratio minus one and the Watt six-bar mechanisms.

The linkage varieties are real affine varieties. But it is natural to consider their complex projective closures. The geometry of these complex projective varieties are discussed in detail. The thesis computes the degree and genus of these varieties for these examples and, moreover, a complete list of their reductions into irreducible components is given in terms of the design parameters of the mechanism. The geometric genus is showed to be an invariant of the kinematic chain under inversion.

For the above examples the real affine linkage variety of the generic mechanism is an irreducible, compact and non-singular curve and therefore diffeomorphic to a disjoint union of circles. The thesis presents a general method for calculating the number of connected components by considering 'submechanisms' of the given mechanism. This philosophy is performed for the above mentioned examples and the number is determined in terms of the design parameters.

This Thesis is dedicated

to my wife Tine

and in loving memory

of my father.

Acknowledgements

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CONTENTS

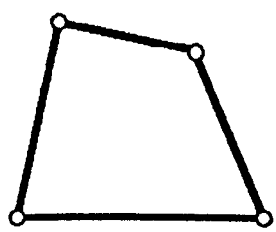
INTRODUCTION

THIS THESIS presents a philosophy for studying the kinematic geometry of mechanisms. In particular, the aim is to bring together relevant ideas and theorems from modern algebraic geometry and to apply them to the special varieties which encapsulate the motion of mechanisms. We begin with some basic definitions from the mechanisms literature.

A **kinematic chain** is a system of finitely many rigid bodies jointed together. The rigid bodies are called **bars** or **links.** A link, which is connected to two, three, four etc. other links, is called a **binary, ternary, quaternary** etc. link. If we fix some of the bars of the kinematic chain, we obtain a **mechanism** or **linkage.** The study of the relative motion of the moving links with respect to the fixed ones is called the kinematics of the mechanism. A mechanism, whose motion is wholly contained in the plane, is called a planar mechanism and mechanisms, which do not satisfy this property, are called spatial mechanisms. In general, the term 'mechanism' is used to describe any jointed system, whilst the term 'linkage' is used when the joints are hinges ('turning joints') or when the joints are 'simple'. Technically, a 'simple' joint is one of the lower order pairs. Note that for planar mechanisms the only 'simple' joints are hinges. The author hopes that the term 'hinge' is intuitively clear. The effect of hinging two rigid bodies is like that of pressing a drawing pin through two strips of strong card, so that the strips are free to turn about the pin. Indeed, the reader is urged to make models from card and drawing pins! Finally, the reader is warned that there is no consistent definition of the terms 'mechanism' and 'linkage' in the literature. A mechanism consisting of n bars is called an n-bar or n-link mechanism.

If the mechanism has one degree of freedom, then the locus of any point rigidly attached to one of the moving bars is a curve called the **coupler curve**. The tracing point is called the coupler point and the bar to which the **coupler point** is attached is called the **coupler bar**.

Throughout the thesis we shall need a diagramatic notation for planar mechanisms. We shall denote a hinge by an open circle ∘ . To describe any kinematic chain it is sufficient to denote which hinges are a fixed distance apart during the motion; we shall do this by joining such hinges by a line: ∘——∘. For example, the four-bar kinematic chain may be denoted in diagram form as



Let us restrict ourselves, for the time being, to linkages when the bars are jointed together only by hinges. Then it can be showed that planar linkages with one degree of freedom have an even number of bars greater than or equal to four. Thus, the simplest example is the four-bar which has just one kinematic chain. However, there are 2 possible six-bar kinematic chains (see Chapter 4 for details); 16 possible eight-bar kinematic chains

[Klein]; whilst there are an incredible 230 ten-bar [Davies&Crossley] and 6856 [Kiper&Schian + 1 case] twelve-bar kinematic chains.

The great explosion of interest in mechanisms came during the industrial revolution and the steam engine era. The most notable event was James Watt's (1736-1819) discovery in 1784 that a certain four-bar coupler curve could be used for an approximate straight-line motion to drive the piston rod on one of his steam engines. Approximate and exact straight-line motions proved to be the driving force for many of the investigations into coupler curves of mechanisms in the following years. Three other straight-line approximations by four-bar coupler curves were later discovered by R. Roberts (1789-1864), P. L. Chebyshev (1821-1894) and O. Evans (1755-1819). The production of exact straight-line motion by a planar linkage remained unsolved until 1864 when A. Peaucellier (1832-1913), an officer of the French Army Engineers, published his solution of the problem [Peaucellier]. The reader might be interested to know that Peaucellier's mechanism is used in the air conditioning machinery in the basement of the Houses of Parliament, London.

Before the publication of Reuleaux's (1829-1905) work [Reuleaux] in 1875, heralding a turning point in the study of mechanisms, the approach to mechanism design was purely trial and error. Reuleaux suggested that, instead of investigating individual mechanisms designed for specific purposes, one should study systems of finitely many rigid bodies and how they are connected. This led to the present concept of kinematic chain.

The first mathematical analyses of the coupler curves of mechanisms were made, as one might expect, on the four-bar linkage. These appeared in [Chebyshev] 1854 and in [S. Roberts] 1875, the same year as the publication of Reuleaux's classic work. Chebyshev gives an analytic description of how approximate straight-line motion may be attained by four-bar coupler curves and later in [Chebyshev 1899] he gives an analysis of the Watt linkage (i.e. the four-bar which Watt used for approximate straight-line motion).

On the other hand, S. Roberts (1827-1913) gives an analysis of the coupler curve of the four-bar linkage from an algebraic point of view and proves his Triple Generation Theorem (see Chapter 5): a result which is invaluable in coupler curve synthesis. It is of a historical interest to note that the equation of the four-bar coupler curve was well-known in the early 1800's, although it is unclear who first discovered it. Further results on the four-bar were achieved by [Johnson] and [Cayley] in 1876, [Bennett] in 1922 and [Morley] in 1923. By far the most intriguing result on mechanisms of this period is Kempe's Theorem [Kempe] which states: 'any plane curve of the n^{th} order may be generated by linkwork' i.e. any plane curve is part of the coupler curve of some linkage.

More recent work on the geared five-bar and six-bar mechanisms may be found in [Primrose et al] and [Freudenstein et al]. The general approach to the study of coupler curves taken by these authors and others depends entirely on being able to write down, quite explicitly, the defining equation of the coupler curve; the same approach which Cayley and others were using at the turn of the century in their studies of the four-bar and slider-crank linkages etc.. These authors determined the degree and genus of the coupler curves and the multiplicity of the circular points at infinity (assuming that the curve is irreducible) for the geared five-bar mechanisms and the six-bar linkages. They also gave an upper bound for the number of circuits of these curves; but, unfortunately, there are gaps in the proofs.

It is worth pointing out that, although the question of the irreducibility of the general coupler curve of a given mechanism is fundamental to the geometry, in all examples of engineering interest this remains an unproved hypothesis. The mechanisms literature has many of these well known "folklore" results for which no proofs exist. For example, the reductions of the planar four-bar have been known for over a century, but no proof has been published until recently [Gibson&Newstead].

It becomes apparent that despite the tremendous growth in techniques of mechanism synthesis (i.e. 'finding the best mechanism to do the job') in the last hundred years, little or no progress has been made in the study of coupler curves from an algebraic-geometric point of view.

A new general philosophy for the study of the motion of mechanisms, initiated in [Marsh;Gibson&Newstead], is to associate to each configuration of a mechanism a point in a higher-dimensional configuration space. The constraints on the motion of a mechanism may be expressed as polynomial equations in that configuration space, thus defining a **linkage variety.** A point on that variety corresponds to a configuration of the mechanism. It should be noted that the concept of a configuration space has been exploited in many other areas of science, but it seems to have been neglected in the field of Machine and Mechanism Theory.

Understanding the geometry of these linkage varieties is a fundamental and open problem of the subject. Coupler curves, or surfaces for mechanisms with two degrees of freedom etc., are obtained from these varieties by a birational linear projection.

For a natural study of these varieties we complexify and homogenise their defining equations thus giving a complex linkage variety in some complex projective space. We may then apply the machinery of algebraic geometry to obtain substantial information about the linkage varieties. In particular, this provides a natural approach to determining the degree and genus of the coupler curves. The first step is to determine the degree and genus of the linkage variety. Then we deduce the results for the coupler curves via the Projection Formula which relates the degree of a variety with that of its image under linear projection. Further, since the coupler projection is a birational map, the coupler curve has the same genus as the linkage variety.

For generic mechanisms of mobility one the real linkage variety is a non-singular curve. Thus, it is a compact one-dimensional real manifold and therefore diffeomorphic to a disjoint union of circles called the connected components. Determining the number of components is a central problem and is unlikely to be an easy matter so one seeks techniques which will at least apply to examples of engineering interest. Such a technique is provided in §1.4 and applied to the planar four-bar in §1.4, the spherical four-bar in Chapter 2, the geared five-bar in §3.3 and the Watt six-bar in §4.6.

The philosophy, which is emphasised throughout the thesis, is based on the following observation. Let M be a mechanism with linkage variety V and let M' be a submechanism of M with linkage variety V'. We mean by submechanism, that M' is obtained from M by removing a number of links. A configuration of M will determine a unique configuration of M' and this is realised, geometrically, as a natural linear projection of V onto V'. Indeed, one expects such a mapping to be finite in the technical sense of algebraic geometry on Zariski open subsets of V and V' i.e. between quasi-projective varieties. There is a fixed integer $d \geq 1$, such that for any generic configuration of M' there are d configurations of M. Thus, for the four-bar we may take any moving link and for the Watt I we may take an underlying four-bar. In both of these cases $d = 2$. For the real varieties V, V' the topology of V is related to d-fold coverings of V'. Thus, by choosing some or all possible submechanisms M', we hope to deduce information about the topology of V. Further, the projections between the complex varieties provide a technique for yielding information about the reductions of V from the lists of possible reductions of V'.

In §4.6 we describe an important and very intuitive result: that the (residual) linkage variety of kinematically inverted linkages, under a quite general hypothesis, are (complex) birationally isomorphic and that their real (residual) linkage varieties are real isomorphic; this is explained in detail in §4.6. In particular, this shows that the linkage variety encapsulates the relative motion of the kinematic chain despite the fact that we derive its defining equations by fixing a bar. Moreover, it shows that the geometric genus of the (residual) linkage variety is an invariant of the kinematic chain.

In Chapter 1 we describe the geometry of the planar four-bar. The basic geometry of the linkage and Darboux varieties was first obtained by [Marsh;Gibson&Newstead] and this work is described in detail in the first section. In the following two sections we present two methods of finding the list of possible reductions of the four-bar linkage varieties, both different from the solution given in [Gibson&Newstead]. This provides an illustration of the techniques which we will use for the more complicated geared five-bar and Watt I six-bar mechanisms. In §1.4 we describe, more fully, the philosophy outlined above for determining the topology of linkage varieties. This philosophy is carried out for the planar four-bar; here the submechanisms are single links whose linkage varieties are circles. Thus, we have natural 2-fold coverings of circles and the topology is related to the number of critical points of these coverings. The result is that the generic four-bar mechanism has one or two components determined by a simple condition on the design parameters. This provides an alternative to the Morse Theoretic proof of the same result given in [Gibson&Newstead]. Moreover, the critical points relate to a concept familiar in the mechisms literature, namely, that of the 'limiting positions'. This leads to a natural way of classifying coupler curves based on the eight Hain types [Hain 1964].

In the next section we describe the geometry of the Segre quartic surface, a complete intersection of two quadric hypersurfaces in $P\mathbb{C}^4$. The geometry of this surface is important in the discussion of the four-bar coupler curves in §1.6 and of the coupler curves of the geared five-bar, when the coupler point is a hinge, in §3.4. In both of these cases the linkage varieties are the intersection of a net of quadric hypersurfaces in $P\mathbb{C}^4$. We recall that the coupler curves are obtained from the linkage varieties by linear projection. Then, in the cases at hand, the centre of projection is a line and there is a unique pencil of quadrics in the net containing that line. Provided that pencil is generic, a term which we make precise in the text, the intersection of every quadric in the pencil is a Segre quartic surface containing exactly sixteen lines one of which is the centre. Thus, the coupler projection is a projection from a line on the surface giving a birational map between the projective plane and the surface. We deduce, therefore, that the coupler projection is a generically 1-1 mapping from which we can deduce a number of properties of the coupler curves. These properties are discussed in §1.6.

Chapter 2 presents a non-planar mechanism namely the spherical four-bar. The author realised that the technique developed for determining the topology of the real linkage varieties need not be restricted to planar mechanisms. The basic geometry of the linkage variety was done in [Gibson&Selig] and we give a summary of the results which we shall need. In the spherical four-bar case there are natural maps of the linkage variety onto circles. Determining the critical points of these maps, yields the topology of the linkage variety. This result is entirely new.

In Chapter 3 we return to planar mechanisms with that of the geared five-bar mechanism with gear ratio -1. The linkage variety is showed to be isomorphic to an intersection of three quadrics in $P\mathbb{C}^4$. In §1.1 we describe the basic geometry of the linkage varieties. In particular, we show that it meets the hyperplane at infinity in two ordinary double points and four simple points. Hence, we may deduce that the linkage variety is a curve. Further, we discuss the condition for singularities to occur off the hyperplane at infinity. Section 3.2 is devoted to the reductions of the linkage variety. We give a complete list in terms of the design parameters. We show that the possible reductions are 8, 6+2, 4+2+2 and $4+2^2$. In particular, we show that the generic linkage variety is an irreducible curve of degree 8 and geometric genus 3. The list of reductions is an interesting new result.

In §3.3 we restrict our attention to the real linkage variety. In the generic case the real linkage variety is an irreducible non-singular curve thus diffeomorphic to a disjoint union of circles. By Harnack's Theorem this number is ≤ 4. We shall show how to determine this number in terms of the design parameters using the general philosophy which we present in §1.4. Finally, in §3.4 we describe the geometry of the coupler curves. When the coupler point is the hinge, we may apply the geometry of the Segre quartic surface as mentioned earlier. The result is that the generic coupler curve has degree 6, geometric genus 3 and ordinary double points at the circular points at infinity. This is the case described by [Freudenstein&Primrose]. For the general coupler point we show that the generic coupler curve has degree 8, geometric genus 3 and ordinary triple points at the circular points at infinity. Furthermore, we deduce the reductions of the coupler curves.

The Watt mechanisms present us with a number of problems which were not encountered in the previous examples. The main problem is that the linkage varieties are not set-theoretic complete intersections. In the previous examples we have relied on Bézout's Theorem to give us the degree of the variety so that we might deduce the degree of the <u>residual</u> linkage variety. Perhaps we should explain the term 'residual' here. We recall that the starting point is a set of polynomial constraints defining a real affine curve, which we then complexify and projectivise by introducing a complex variable w, so that we may apply the general theory of complex projective varieties. However, the projective set, which we produce, is not the smallest projective set containing the original variety. Indeed, we may introduce components in the hyperplane at infinity $w = 0$. We define the residual variety $\mathfrak{R}'$ to be the variety obtained from the linkage variety $\mathfrak{R}$ by removing any irreducible components lying in $w = 0$. The residual variety is the projectivisation in the technical sense which we require. In the first examples these 'extra' components were curves and therefore presented no problem since we could subtract the sum of their degrees from the degree of $\mathfrak{R}$ to obtain the degree of $\mathfrak{R}'$. However, in the Watt case these varieties are 2-planes and therefore we must argue very differently.

In §4.1 we set up the basic geometry of the Watt I and II linkage varieties and in §4.2 we describe the common Darboux variety of these mechanisms. We show that the Darboux variety is isomorphic to an intersection of two cubic surfaces in $P\mathbb{C}^3$. The variety consists of a line and a curve of degree 8 which we shall call the residual Darboux variety $\mathfrak{D}'$. In §4.3 we show that these two varieties are closely related thus giving the key tool for calculating the degrees of the Watt I and II linkage varieties. Moreover, we can deduce by careful reasoning the manner in which these varieties meet the hyperplane at infinity.

The reductions of the linkage varieties present a similar problem. However, we are able to deduce the reductions of the linkage variety from the reductions of the residual Darboux variety. Indeed, a component of $\mathfrak{D}'$ corresponds to a component of $\mathfrak{R}'$ of twice the degree. This list is obtained in terms of the design parameters in §4.4. In §4.5 we give a discussion of the Watt I coupler curves.

In Chapter 5 we are interested in the real geometry of the four-bar coupler curves. Here the emphasis lies on the natural classification of coupler curves hinted at in the above discussion of the critical points of the 2-fold coverings of circles in Chapter 1. We begin with the easier task of giving a classification of the <u>complex</u> coupler curves where there are fewer cases to consider. This analysis forms §5.2. In this chapter the real topology of the Segre quartic surface plays an important role. Therefore, we devote a whole section, §5.5, to its description. Sections 5.3, 5.4 and 5.6 prepare the way for a graphical analysis of the coupler curves.

In particular, we wish to determine exactly which types can occur by graphical methods. This is by no means easy. Indeed, we need sections 5.3-5.6 to give sufficient mathematical background before we can even begin this analysis. A catalogue of four-bar curves showing an example of each of the types found is presented in §5.7. This programme of work is still in its early stages, but it is clear that all the necessary techniques are available. This analysis ties up very nicely with the work of Müller [Müller].

The author has provided an appendix outlining all the results from algebraic geometry which will be used throughout the thesis. This gives some indication of how the theorems may be applied and references to where further discussion and proofs of the theorems may be found. In the text these theorems and sections in the appendix, to which the reader is referred, are prefixed with an 'A'.

CHAPTER 1. THE FOUR-BAR MECHANISM

Introduction

Despite the apparent simple structure of the four-bar mechanism, the reader might find himself surprised by the wide variety displayed by the (real) coupler curves. The simplest non-degenerate examples look very much like circles, while more complex ones may possess upto three real double points (which may be acnodes, crunodes, tacnodes or cusps) and have as many as eight real inflexions. Perhaps it is the simplicity, yet variety that makes the mechanism so popular and useful to the mechanical engineer.

One example with which the reader may be familiar, is the four-bar used in the design of the jib-crane and found on many building sites, scrapyards and docks. A typical example is showed in Fig 1.1. Here the structure of the crane is a quadrilateral OABC with bar OA fixed. The shape of the coupler curve is used to guide the gripper or pulley through a set path determined by the dimensions of the crane. The crane generally, has in addition to the four-bar motion, the freedom to revolve about its base.

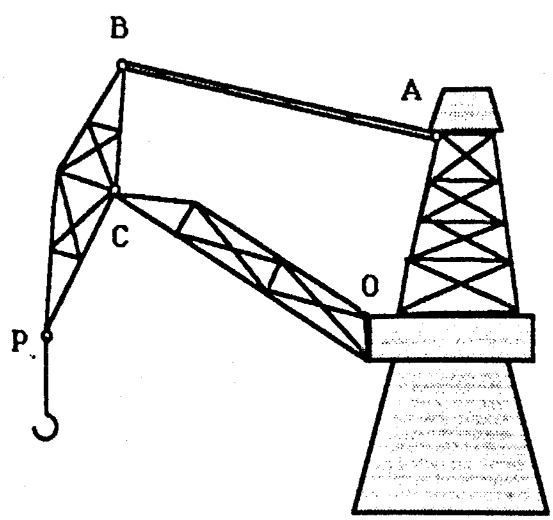


Fig. 1.1

On a much smaller scale four-bars, whose coupler curves approximate straight-line motion, are indispensible to the engineer. There are four famous classes of mechanisms which give straight-line approximations each named after their discoverer. These are the Chebyshev, Roberts, Evans and Watt mechanisms. Figure 1.2 shows how an Evans mechanism can be used in a measuring or recording apparatus. Finally, the reader is referred to Hain [Hain,1961] who gives examples of applications of four-bar coupler curves which have one, two or three cusps.

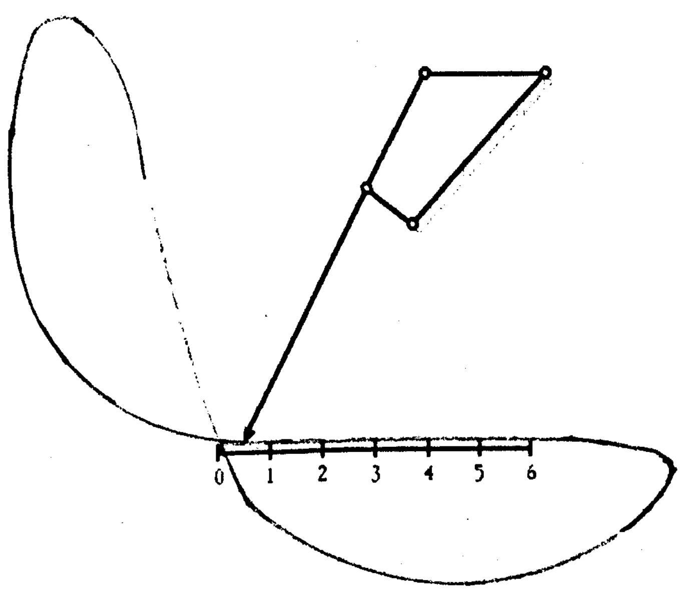


Fig. 1.2

This first chapter serves two purposes. Firstly, it is a study of the geometry of the planar four-bar, illustrating the general strategy and introducing much of the terminology used in later chapters. Many of the results of this section are well known, but the emphasis is on developing a sufficiently general framework and methodology, so that similar results can be derived for <u>any</u> planar mechanism. Secondly, the four-bar provides a testing ground on which we can try out new techniques.

In §1.1 we recall from [Marsh;Gibson&Newstead] the basic geometry of the linkage curve $\mathfrak{R}$ and the Darboux curve $\mathfrak{D}$. In §1.2 and §1.3 we introduce two further approaches to determining the reductions of $\mathfrak{R}$ to the one given in [Gibson&Newstead]. These illustrate the methods which we shall use for the more complicated Geared five-bar mechanism in Chapter 3 and the Watt mechanism in Chapter 4. In §1.4 we describe a new technique for determining the topology of the linkage variety which provides the basis for calculating the topology of the linkage variety of the spherical four-bar, the geared five-bar and the Watt six-bar in the following chapters. In §1.6 we view the coupler curves as a projection from a line in 4-space of the residual curve $\mathfrak{R}'$ (as in [Marsh;Gibson&Newstead]) and we recall from [Gibson&Newstead] how we can explain the geometry of the projection in terms of the geometry of the Segre quartic surface. Thus, as a necessary preliminary we dedicate §1.5 to a description of this surface and some of its properties.

§1.1 Geometry of the Linkage and Darboux Curves.

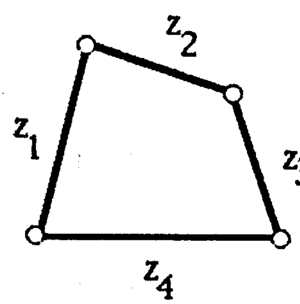


Fig. 1.3

Let the sides of the quadrilateral have positive lengths d_1, d_2, d_3, d_4 and directions given by the unit complex numbers z_1, z_2, z_3, z_4 as showed in Fig. 1.3. We fix the fourth link by setting $z_4 = -1$. The constraints of motion are $|z_j| = 1$ for $j = 1, 2, 3$ and the single complex relation

$$d_1z_1 + d_2z_2 + d_3z_3 = d_4$$

expressing the closure of the quadrilateral. Set $z_j = x_j + iy_j$ for $j = 1, 2, 3$ with x_j, y_j real numbers. Then the constraints of motion may be expressed by the real equations

$$\left.\begin{array}{l} d_1x_1 + d_2x_2 + d_3x_3 = d_4 \\ d_1y_1 + d_2y_2 + d_3y_3 = 0 \\ x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = 1 \end{array}\right\}$$

Thus, we obtain five equations in six unknowns defining an algebraic variety in $\mathbb{R}^6$. To study real affine varieties it is natural to consider them as "real parts" (i.e. the set of real points) of a complex projective variety. Therefore, we shall complexify and projectivise the equations by allowing the variables $x_1, y_1, x_2, y_2, x_3, y_3$, to be complex and introducing a complex homogenising co-ordinate w to give the following set of equations

$$\left.\begin{aligned} d_1x_1 + d_2x_2 + d_3x_3 &= d_4w \\ d_1y_1 + d_2y_2 + d_3y_3 &= 0 \\ x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 &= w^2 \end{aligned}\right\} \quad (1.1)$$

These equations define a projective variety $\mathfrak{R}$ in the configuration space $P\mathbb{C}^6$ which we refer to as the **linkage variety**. For convenience, we shall call the hyperplane $w = 0$ the **hyperplane at infinity**.

It is showed in [Marsh;Gibson&Newstead] that there is a necessary and sufficient condition for equations (1.1) to have at least one real solution. If we re-label d_1, d_2, d_3, d_4 in increasing order of magnitude as e_1, e_2, e_3, e_4, then that condition may be expressed as $e_4 \leq e_1 + e_2 + e_3$. Whenever this condition is satisfied, we shall say that we are in the **constructible** case. Henceforth, we shall assume that we are in the constructible case, so that equations (1.1) do possess a real solution.

By setting $w = 0$ in equations (1.1) one obtains the intersection of $\mathfrak{R}$ with the hyperplane at infinity. We find that $y_k = \pm \mathbf{i} x_k$ $(k = 1, 2, 3)$ thus giving eight possibilities. The two sign combinations +++ and --- give two complex conjugate lines $L, \overline{L}$. These are given by the equations

$$L: \begin{cases} d_1x_1 + d_2x_2 + d_3x_3 = 0 \\ y_k = \mathbf{i}x_k \quad (k = 1, 2, 3) \end{cases} \qquad \overline{L}: \begin{cases} d_1x_1 + d_2x_2 + d_3x_3 = 0 \\ y_k = -\mathbf{i}x_k \quad (k = 1, 2, 3) \end{cases}$$

The remaining six sign combinations give three pairs of complex conjugate points, namely P_1 , P_2 , P_3 lying on L, and $\overline{P}_1$, $\overline{P}_2$,

$\bar{P}_3$ lying on $\bar{L}$. Their co-ordinates are :

$-++: P_1 = (0,0,d_3,id_3,-d_2,-id_2,0)$ $+--: \bar{P}_1 = (0,0,d_3,-id_3,-d_2,id_2,0)$

$+-+: P_2 = (d_3,id_3,0,0,-d_1,-id_1,0)$ $-+-: \bar{P}_2 = (d_3,-id_3,0,0,-d_1,id_1,0)$

$++-: P_3 = (d_2,id_2,-d_1,-id_1,0,0,0)$ $--+: \bar{P}_3 = (d_2,-id_2,-d_1,id_1,0,0,0)$

To determine the singularities of $\mathfrak{R}$, we note that, if P is a point where the Jacobian matrix $\mathfrak{J}$ of equations (1.1) has non-maximal rank, then *either* P is a singular point *or* a component of $\mathfrak{R}$ with dimension ≥2 passes through P. The matrix is

$$\mathfrak{J} = \begin{pmatrix} d_1 & 0 & d_2 & 0 & d_3 & 0 & -d_4 \\ 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 \\ 2x_1 & 2y_1 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 2x_2 & 2y_2 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 2x_3 & 2y_3 & -2w \end{pmatrix}$$

When $w = 0$, the singularities are easily seen to be the points P_1, P_2, P_3, $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$. Thus, we may deduce that $\mathfrak{R}$ is a curve. Since, any component of dimension ≥2 would necessarily meet $w = 0$ in L or $\bar{L}$ and a simple contradiction follows by observing that $\mathfrak{J}$ would have to have non-maximal rank at every point on the line: this is clearly impossible.

When $w \neq 0$, singularities occur if and only if $y_1 = y_2 = y_3 = 0$. Substituting $y_1 = y_2 = y_3 = 0$ into equations (1.1), gives the condition

$$\pm d_1 \pm d_2 \pm d_3 \pm d_4 = 0 \qquad (1.2)$$

We shall refer to condition (1.2) as the **Grashof equality**. Writing d_1, d_2, d_3, d_4 in increasing order of magnitude as e_1, e_2, e_3, e_4, (1.2) yields four distinct possibilities (using the notation of [Gibson & Newstead]):

(I) $e_1 + e_4 = e_2 + e_3$ but none of the cases below.

(I') $e_1 + e_2 + e_3 = e_4$

(II) $e_1 = e_2 \neq e_3 = e_4$

(III) $e_1 = e_2 = e_3 = e_4$.

The four cases are called the circumscriptible, the collapse, the kite/parallelogram, and the rhombus, respectively. Whenever condition (1.2) does not hold, we say that we are in the **generic** case: thus, in the generic case $\mathfrak{R}$ has no singular points with $w \neq 0$. The numbering of cases indicates the number of singular points of $\mathfrak{R}$ with $w \neq 0$: the singular points have the form $(\pm 1,0,\pm 1,0,\pm 1,0,1)$, so that in cases (I) and (I') there is just one such singular point and in cases (II) and (III) there are two and three singular points respectively.

To obtain the degree of $\mathfrak{R}$ one observes that equations (1.1) express it as a set theoretic complete intersection of five hypersurfaces in $P\mathbb{C}^6$: two hyperplanes and three quadric hypersurfaces. Thus, Bézout's Theorem (A3) yields that $\mathfrak{R}$ has degree 8. Therefore the **residual curve** $\mathfrak{R}'$, obtained from $\mathfrak{R}$ by deleting the two lines at infinity, has degree 6. Moreover, the points $P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3$ are singular on $\mathfrak{R}$, but not on the lines at infinity: thus they must lie on $\mathfrak{R}'$. Hence, $\mathfrak{R}'$ intersects the hyperplane at infinity in precisely six points with intersection multiplicity one. Therefore $\mathcal{R}'$ meets the hyperplane transversally at each of these points. Since the intersection multiplicity is always larger than the multiplicity at a point (see §A2 for details) it follows that all six points are ordinary double points of $\mathcal{R}$ and simple points of $\mathcal{R}'$.

We may deduce further information about singular points with $w \neq 0$ (i.e. when the Grashof equality holds) by making a local co-ordinate calculation. Make $\mathcal{R}$ affine by setting $w = 1$ in equations (1.1) and suppose that $\mathcal{R}$ has a singular point of the form $(\varepsilon_1,0,\varepsilon_2,0,\varepsilon_3,0,1)$ where $\varepsilon_j = \pm 1$. Applying the affine transformation $x_j \rightarrow x_j + \varepsilon_j$ and leaving y_j fixed ($j = 1, 2, 3$), we may assume that the singular point is at the origin. Using equations (1.1) and the Implicit Function Theorem, we may smoothly eliminate variables x_3, y_3, x_1, x_2 to obtain a curve in the (y_1,y_2)-plane with a singular point at the origin with equation

$$(d_1y_1 + d_2y_2)^2 \pm d_1d_3y_1^2 \pm d_2d_3y_2^2 + O(3) = 0$$

Details of this calculation are given in [Marsh]. The discriminant of the quadratic part is $\pm 4d_1d_2d_3d_4$ which is always non-zero. We conclude therefore that the singular points are always ordinary double points.

In §1.2 and §1.3 we describe two approaches to the problem of determining the possible reductions of $\mathcal{R}'$ into its components (i.e. its irreducible subvarieties). The list in [Gibson & Newstead] was obtained by arguments involving the

Genus Formula (Theorem A8). This approach suffers two drawbacks. Firstly, it gives little illumination to the geometry of $\mathfrak{R}'$ and secondly, it does not provide a suitable method of working out the reductions for more complicated mechanisms.

In preparation for the first approach we introduce a second variety. The reader may recall that we began with the single constraint of motion

$$d_1 z_1 + d_2 z_2 + d_3 z_3 = d_4$$

To express it as two real relations one adds the complex conjugate condition

$$d_1 \bar{z}_1 + d_2 \bar{z}_2 + d_3 \bar{z}_3 = d_4$$

However, the complex numbers z_1, z_2, z_3 are unit length so we may express the second equation as

$$\frac{d_1}{z_1} + \frac{d_2}{z_2} + \frac{d_3}{z_3} = d_4$$

Homogenising the two equations by introducing a new variable z_4, we obtain

$$\left.\begin{aligned} d_1 z_1 + d_2 z_2 + d_3 z_3 &= d_4 z_4 \\ \frac{d_1}{z_1} + \frac{d_2}{z_2} + \frac{d_3}{z_3} &= \frac{d_4}{z_4} \end{aligned}\right\} \qquad (1.3)$$

Regarding z_1, z_2, z_3, z_4 as co-ordinates in $P\mathbb{C}^3$, then equations (1.3) define the intersection of a hyperplane and a cubic surface, giving a plane cubic curve $\mathfrak{D}$. It was Darboux [Darboux] who in

1879 first studied this curve, connecting the motion of quadrilaterals with the above cubic. Henceforth, we shall refer to this cubic as the **Darboux cubic**. It is easy to show that the cubic is singular if and only if the Grashof equation holds (see [Marsh] for details). In particular, in the generic case the cubic is non-singular and therefore irreducible. In cases (I) and (I') the cubic is nodal, in case (II) the cubic is a conic and chord and in case (III) the cubic is a triangle. Since equations (1.3) have real coefficients, we may ask which real types of cubics occur. We find that we can make two further distinctions. Firstly, we may show by looking more carefully at the local co-ordinate calculation described above, that cases (I) and (I') give crunodal and acnodal cubics respectively. Secondly, when we are in the generic case, we can distinguish between the two types of real non-singular cubics. One can determine the type, for instance, by calculating the number of real tangents to the cubic through the point (0,0,0,1): for in the one component case there are two such tangents and in the two component case there are four. The details of this may be found in [Marsh]. The result is that when $e_1+e_4<e_2+e_3$ we have two components and when $e_1+e_4>e_2+e_3$ we have just one component.

§1.2 The Reductions of the Linkage Variety - Approach 1

We may now proceed to describe the first approach to the problem of determining the reductions of $\mathfrak{R}'$. A similar approach will be used in Chapter 5 for the Watt six-bar mechanism. Consider the linear projection $\pi: P\mathbb{C}^6 \rightarrow P\mathbb{C}^3$ given by

$$(x_1, y_1, x_2, y_2, x_3, y_3, w) \rightarrow (x_1+iy_1, x_2+iy_2, x_3+iy_3, -w)$$

Then the restriction map $\varphi = \pi|_{\mathfrak{R}'}$ of π to the residual linkage variety maps $\mathfrak{R}'$ into the Darboux variety $\mathfrak{D}$. The centre of projection is given by $x_1+iy_1 = x_2+iy_2 = x_3+iy_3 = w = 0$ i.e. the line at infinity L. We shall use the Projection Formula (Theorem A11) to determine the degrees of the components of $\mathfrak{R}'$, given the reduction of $\mathfrak{D}$. The first step is to show that the map φ is finite (see §A7 for a formal definition) i.e. that no component of $\mathfrak{R}'$ maps to a point on $\mathfrak{D}$. To do this it is sufficient to show that any point P on $\mathfrak{D}$ has only finitely many pre-images on $\mathfrak{R}'$.

Let $P = (z_1,z_2,z_3,z_4)$ be any point on $\mathfrak{D}$. If $z_4 = 0$, then its pre-images satisfy $w = 0$ and is therefore a subset of the three points on $\bar{L}$: a finite number. Now suppose that $z_j = 0$ for some $j = 1,2,3$ then P is one of the points $Q_1 = (-d_4,0,0,d_1)$, $Q_2 = (0,-d_4,0,d_2)$ or $Q_3 = (0,0,-d_4,d_3)$. However, these points have no pre-image on $\mathfrak{R}'$: they are the points added to $\varphi(\mathfrak{R}')$ to make the image Zariski closed. (The reader should note that the image of a projection is Zariski closed, whenever the map is regular. However, in this situation the centre of projection meets $\mathfrak{R}'$ at the points P_1, P_2 P_3 and hence fails to be regular at these points. Thus we take the closure of the image by adding finitely many points.) We may now suppose that $z_j \neq 0$ for $j = 1,2,3,4$. Write $z_j = x_j+iy_j$ and $z_4 = -w$. Then we may eliminate variables y_1, y_2, y_3, w by writing them as polynomials in x_1, x_2, x_3, z_1, z_2, z_3, z_4. Equations (1.1) become

$$\left.\begin{aligned} d_1x_1 + d_2x_2 + d_3x_3 &= -d_4z_4 \\ i(d_1x_1 + d_2x_2 + d_3x_3) - i(d_1z_1 + d_2z_2 + d_3z_3) &= 0 \\ z_1(z_1 - 2x_1) = z_2(z_2 - 2x_2) = z_3(z_3 - 2x_3) &= z_4^2 \end{aligned}\right\}$$

Finally, we may use the last three equations to write x_1, x_2, x_3 as rational maps in the variables z_1, z_2, z_3, z_4. Therefore, we can write all the variables $x_1, x_2, x_3, y_1, y_2, y_3, w$ as rational maps in z_1, z_2, z_3, z_4. Thus we have showed that each point of $\mathcal{D}$ has a unique pre-image on $\mathcal{R}'$. Thus φ has degree one implying that $\mathcal{R}'$ and $\mathcal{D}$ are birationally equivalent. This result was proved for a more general case in [Gibson&Newstead].

Eliminate z_3 using the linear equation of (1.3) to obtain $\mathcal{D}$ as a cubic in the (z_1,z_2,z_4)-plane. Then $\mathcal{D}$ meets $z_4=0$ in the three points $P'_1 = (0,1,0)$, $P'_2 = (1,0,0)$ and $P'_3 = (d_2,-d_1,0)$. These points P'_j have, as their pre-image on $\mathcal{R}'$, the points $\overline{P}_j$ lying on $\overline{L}$. It is easily seen that the points $Q'_1 = (d_3,0,-d_1)$, $Q'_2 = (0,d_2,-d_3)$ and $Q'_3 = (0,0,1)$ have no pre-image on $\mathcal{R}'$. Therefore these points are the closure points of the image $\varphi(\mathcal{R}')$. By the general theory of projections (Theorem A13) they are the images of an osculating n-plane to $\mathcal{R}'$ at points lying in the centre of projection L. But the tangents to $\mathcal{R}'$ at points on L do not lie in the hyperplane $w=0$ and it follows that the points Q'_j are the images of these tangent lines; indeed, the tangent to P_j maps to Q'_j. Note that no line component of $\mathcal{D}$ is one of the lines $z_j=0$. Thus any line is necessarily one of the lines through the pairs of points $P'_jQ'_j$ whose pre-image on $\mathcal{R}'$ is a component of $\mathcal{R}'$ passing through $P_j\overline{P}_j$ and no other points of $\mathcal{R}'$ meeting $w=0$. It follows that the pre-image is a real conic. The reductions are now easily determined. When $\mathcal{D}$ is three lines $\mathcal{R}'$ is three real conics, when $\mathcal{D}$ is a line and conic $\mathcal{R}'$ is a real conic and real quartic, and finally, when $\mathcal{D}$ is an irreducible cubic then $\mathcal{R}'$ is an irreducible sextic.

Thus, in the generic case, $\mathcal{R}'$ is an irreducible sextic of genus one: the genus of a non-singular planar cubic. In the circumscriptible case, $\mathcal{R}'$ is irreducible with one ordinary double point and has the genus of a nodal cubic, i.e. zero. When we are in the kite/parallelogram case, the Darboux cubic reduces to a conic and chord, thus $\mathcal{R}'$ reduces to a conic and quartic (both having genus zero) intersecting in two points. Finally, in the rhombus case we deduce that $\mathcal{R}'$ reduces to three conics, so that each meets the other two transversally in one point.

It should be pointed out that in [Gibson&Newstead] the construction of the Darboux variety is put in a more general setting. The authors show that for any set of constraints of the form

$$f_j(z_1,...,z_r) = 0\ ,\quad |z_k|^2 = 1 \qquad \text{for } 1 \le j \le s, \text{ and } 1 \le k \le r$$

where $z_1,...,z_r$ are complex numbers and $f_1,...,f_s$ are complex polynomials in those variables, we may associate two varieties whose geometries are related. The first variety is obtained by setting $z_k = x_k + iy_k$ for $1 \le k \le r$, and $f_k = u_k + iv_k$ for $1 \le k \le r$, where u_k and v_k are real polynomials in the real variables x_k,

y_k. Then we may rewrite the constraints as $u_k = v_k = 0$ and $x_k^2 + y_k^2 = 1$, defining a variety in $\mathbb{R}^{2r}$. Complexifying and homogenising these equations by introducing the complex homogenising variable w yields a variety in $P\mathbb{C}^{2r}$ given by the homogeneous equations

$$u'_k(x_1,y_1,\ldots,x_r,y_r,w) = 0 \quad : \quad v'_k(x_1,y_1,\ldots,x_r,y_r,w) = 0$$
$$\text{and} \qquad x_k^2 + y_k^2 = w^2 \qquad \text{for } 1 \le k \le s.$$

This variety is the generalised form of the linkage variety which we shall denote by $\mathfrak{R}$.

The second variety that we may derive from the set of constraints is obtained by conjugating the polynomials $f_k(z_1,\ldots,z_r) = 0$ to give the polynomials $\overline{f}_k(\overline{z}_1,\ldots,\overline{z}_r)$. Then, since the complex numbers are unit length, $\overline{z}_k = \frac{1}{z_k}$ and we may substitute for $\overline{z}_k$ and clear the denominators of the polynomials $\overline{f}_k$ to give the set of polynomials $F_k(z_1,\ldots,z_r)$. Thus the set of equations $f_k = F_k = 0$ for $1 \le k \le s$ defines a variety in $\mathbb{C}^r$. We may now homogenise these equations by introducing the homogenising parameter w to give a variety in $P\mathbb{C}^r$ given by the homogeneous equations

$$f'_k(z_1,\ldots,z_r,w) = 0 \quad : \quad F'_k(z_1,\ldots,z_r,w) = 0 \quad \text{for } 1 \le k \le s$$

This variety is the generalised form of the Darboux variety which we shall denote by $\mathfrak{D}$.

The main result of [Gibson&Newstead] is that, if we remove from $\mathfrak{R}$ all components lying in the hyperplane $w = 0$ to give a residual variety $\mathcal{R}'$ and if we remove from $\mathcal{D}$ all components lying in any one of the hyperplanes $z_k = 0$ $(1 \le k \le r)$ leaving a residual variety $\mathcal{D}'$, then the varieties $\mathcal{R}'$ and $\mathcal{D}'$ are birationally isomorphic. We shall give a slightly different proof to that given in [Gibson&Newstead]. Consider the projection $\pi: (x_1,y_1,\dots,x_r,y_r,w) \mapsto (x_1+iy_1,\dots,x_r+iy_r,w)$. First we will show that the image of $\mathcal{R}'$ is $\mathcal{D}'$ and that the projection is generically one to one; thus by Lemma A3, π is birational. Note that it is sufficient to consider the Zariski open subsets of $\mathcal{R}'$ of points with $w \neq 0$. Let $P = (x_1,y_1,\dots,x_r,y_r,1)$ be a point on $\mathcal{R}'$ with $w \neq 0$. Since $w \neq 0$ we may write $z_k = x_k + iy_k = 1/(x_k - iy_k)$ for all k. Hence,

$$f'_k(x_1+iy_1,\dots,x_r+iy_r,1) = f_k(x_1+iy_1,\dots,x_r+iy_r) = u_k(x_1,y_1,\dots,x_r,y_r) + iv_k(x_1,y_1,\dots,x_r,y_r) = u'_k(x_1,y_1,\dots,x_r,y_r,1) + iv'_k(x_1,y_1,\dots,x_r,y_r,1)$$

and since $u'_k(P) = v'_k(P) = 0$ we have $f'_k(\pi(P)) = 0$. Further,

$$F'_k(x_1+iy_1,\dots,x_r+iy_r,1) = \overline{f_k}(1/(x_1+iy_1),\dots,1/(x_r+iy_r)) = \overline{f_k}(x_1-iy_1,\dots,x_r-iy_r).$$

Conjugating the right hand equation yields $F'(\pi(P)) = 0$, since

$$f_k(\overline{x}_1+i\overline{y}_1,\dots,\overline{x}_r+i\overline{y}_r) = u_k(\overline{x}_1,\overline{y}_1,\dots,\overline{x}_r,\overline{y}_r) - iv_k(\overline{x}_1,\overline{y}_1,\dots,\overline{x}_r,\overline{y}_r)$$

vanishes, whenever $u_k(x_1,y_1,\dots,x_r,y_r) = v_k(x_1,y_1,\dots,x_r,y_r) = 0$. Thus $\pi(P)$ lies on $\mathcal{D}'$. To show that the restriction of $\pi|\mathcal{R}'$ is generically one to one let $P = (z_1,\dots,z_r)$ be any point of $\mathcal{D}'$ with $z_1 \neq 0,\dots,z_r \neq 0$ and suppose that $(x_1,y_1,\dots,x_r,y_r,1)$ and

$(x'_1,y'_1,\ldots,x'_r,y'_r,1)$ are two distinct points on $\mathcal{R}'$ which map to P. Then $x_j+iy_j=x'_j+iy'_j$, and $(x_j+iy_j)(x_j-iy_j)=x_k^2+y_k^2=x'_k{}^2+y'_k{}^2=(x'_j+iy'_j)(x'_j-iy'_j)$. Thus $(x_j+iy_j)[(x_j-iy_j)-(x'_j-iy'_j)]=0$. Hence, either $(x_j+iy_j)=0$ contradicting the fact that $z_k \neq 0$ or $x_j-iy_j=x'_j-iy'_j$. In the latter case the above condition together with the condition $x_j+iy_j=x'_j+iy'_j$ yields $x_k=x'_k$, $y_k=y'_k$. Thus, P has a unique pre-image. Combining this with the fact that π is rational, yields that π is a birational map between $\mathcal{R}'$ and $\mathcal{D}'$.

§1.3 The Reductions of the Linkage Variety - Approach 2

In this section we introduce a new technique which we need to determine the reductions of the four-bar is this section and to determine the reductions of the geared five-bar in Chapter 3. An extension of this technique will then be used to determine the topology of the real linkage varieties of the generic four-bar in the next section and the generic spherical four-bar, the generic geared five-bar and the generic Watt six-bar in Chapters 2, 3 and 4 respectively. We now describe the philosophy of this technique.

Suppose we have a mechanism M with linkage variety V, and a submechanism M' with linkage variety V' : by this we mean that M' is obtained from M by removing a number of links. Any configuration of M will determine a unique configuration of M' (i.e. for any point P on V we may associate a point P' on V' obtained from P by projecting onto some of the co-ordinates), yielding a natural projection $\pi : V \rightarrow V'$. One expects π to be a finite mapping. Indeed, there should be a fixed integer

$d \geq 1$ such that for almost all (i.e. generic) configurations of M' there are just d possible corresponding configurations of M. For instance, when M is a four-bar we could take M' to be a single link and when M is the Watt I mechanism we may take M' to be the underlying planar four-bar mechanism: in both of these cases $d = 2$. Of course, for a given mechanism M there are a number of possible choices of M' and by considering some (or all) of these choices one reasonably hopes to obtain positive information about the geometry of V. The particular interest of this point of view is that in a number of engineering examples one need only consider 2-fold coverings of varieties V' whose geometry we know sufficiently well to deduce properties of V.

For the example at hand we will take V' to be one of the moving links of the four-bar. Consider then, the projection $\pi_j: \mathcal{R}' \to C_j$ from the residual linkage variety to the circle C_j whose equation is $x_j^2 + y_j^2 = w^2$ given by $(x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (x_j, y_j, w)$ for $j = 1,2,3$. Write equations (1.1) as

$$\left.\begin{aligned} & d_j x_j + d_k x_k + d_\ell x_\ell - d_4 w = 0 \\ & d_j y_j + d_k y_k + d_\ell y_\ell = 0 \\ & x_j^2 + y_j^2 = x_k^2 + y_k^2 = x_\ell^2 + y_\ell^2 = w^2 \end{aligned}\right\} \quad (1.4)$$

We shall now show

(i) that in the generic and circumscriptible case π_j maps $\mathcal{R}'$ onto C_j with degree 2,

(ii) that in the kite/parallelogram case $\mathcal{R}'$ contains a conic component, which is mapped by π_j to a point, whilst the remainder of the curve is mapped onto C_j with degree 2 and

(iii) that in the rhombus case $\mathcal{R}'$ is the union of three conics: one of them being mapped by π_j to a point and the other two mapped with degree 1 to C_j.

Consider a fixed point $P = (x_j, y_j, w)$ on C_j. We can find its pre-image by eliminating x_ℓ, y_ℓ from equations (1.4) to give

$$(d_j x_j + d_k x_k - d_4 w)^2 + (d_j y_j + d_k y_k)^2 = d_\ell^2 w^2$$
$$-2d_4 w(d_j x_j + d_k x_k) + 2d_j d_k(y_j y_k + x_j x_k) + (d_4^2 + d_j^2 + d_k^2 - d_\ell^2)w^2 = 0$$

These equations define a conic and line in the (x_k, y_k) - plane, so one expects two solutions in general implying that P has two pre-images and exceptionally, when the line is tangent to the conic, there is only one solution and P has just one pre-image. We cannot exclude the possibility that the conic reduces and that the line is contained in the conic: implying that the point has infinitely many pre-images (a component of $\mathcal{R}'$). This yields the conditions $d_j x_j = d_4 w$, $y_j = 0$ and $((d_4^2 + d_j^2 + d_k^2 - d_\ell^2)w - 2d_4 d_j x_j) = 0$. The first two equations imply that the point must be $(-d_4, 0, d_j)$. Substituting the co-ordinates of the point into the third equation gives the condition $d_j = d_4$, $d_k = d_\ell$ on the design parameters. Further, we may deduce that the pre-image is given by the equations

$$d_j x_j + d_k x_k + d_\ell x_\ell - d_4 w = 0$$
$$d_j y_j + d_k y_k + d_\ell y_\ell = 0$$
$$d_k^2(x_k^2 + y_k^2) = d_\ell^2 w^2$$
$$-2d_4(d_4 w + d_k x_k) + 2d_k d_4 x_k + (d_4^2 + d_j^2 + d_k^2 - d_\ell^2)w = 0,$$

defining a conic component of $\mathcal{R}'$. Thus in the generic and circumscriptible cases, all of the projections π_j are finite and have degree two. In the kite/parallelogram case exactly one condition of the form $d_j = d_4$, $d_k = d_l$ holds simultaneously, so that one of the projections π_j maps a conic component to a point. In the rhombus case all three conditions of the form $d_j = d_4$, $d_k = d_l$ hold simultaneously, so that all three projections π_j, $j = 1,2,3$ map a conic to a point.

We are now in a position to list the reductions. First we note that the only components, which are mapped to a point under a π_j, are the conics described above. We may deduce therefore that there are no line components of $\mathfrak{R}'$: for a line would have to map to either a line component of C_j or to a point, giving a clear contradiction. This also rules out the possibility of $\mathfrak{R}'$ having a quintic component, for $\mathfrak{R}'$ would also have a line component. Second, we should note that any component of $\mathfrak{R}'$ of degree two is one of the conics C_j, described above, thus occuring only when we are in the rhombus or kite/parallelogram cases. This fact follows from the observation that the centre of projection π_j is a 3-space meeting $\mathfrak{R}'$ in the points $P_j, \overline{P}_j$. Applying the Projection Formula, we find that any conic, which passes through P_j but not through $\overline{P}_j$, would be mapped by π_j to a line giving a contradiction; while any conic, which passes through both P_j and $\overline{P}_j$ (for some j), is mapped by π_j to a point and is therefore one of the conics C_j.

The final step to the list of possible reductions is to eliminate the possibility of $\mathfrak{R}'$ reducing to two cubics. In this case both cubics must be mapped onto C_j with degree one: for otherwise one of the cubics would be mapped to a point giving a contradiction.

From the general theory of finite mappings (see §A7) we know that for any map of degree d onto a non-singular curve the number of pre-images of a given point is always $\leq d$. The required contradiction now follows from the observation that one of the two cubics passes through at least two of the points P_1, P_2, P_3, and that all three points are mapped to the point $I = (1,i,0)$ by π_j, implying that I has >1 pre-images. The above arguments yield the following reductions:

(i) in the rhombus case the three conics C_1, C_2, C_3

(ii) in the kite/parallelogram case a conic C_j (for some j) and an irreducible quartic and

(iii) in the circumscriptible and generic cases an irreducible sextic.

§1.4 The Topology of the Real Linkage Variety

In this section we restrict our attention to the real geometry of the linkage variety. The most important feature of the real linkage variety is its underlying topology and determining this for specific examples appears to be a central problem of the subject. For generic mechanisms of **mobility** one (i.e. mechanisms whose linkage varieties are curves) the real linkage variety is a compact non-singular curve (conjecture!) and the topology is completely specified by the number of connected components. Determining this number in terms of the design parameters is not likely to be an easy matter, so one seeks techniques which will at least apply to examples of engineering interest.

We recall from the previous section that for any mechanism M with linkage variety V, we have submechanisms M' with linkage variety V' and a natural projection $\pi : V \to V'$ of degree d. The crucial extension of the outlined philosophy lies in the fact that, if we are dealing with generic mechanisms M, M' of mobility one, then both of the real linkage varieties are non-singular compact curves (so diffeomorphic to a finite union of circles) and the topology of V is related to d-fold coverings of circles (each a topological component of V') whose branching can be described in terms of the design parameters. For a suitably chosen M' for which one knows something of the topology of V', one hopes to obtain positive information about the topology of V.

A motion of M is to be thought of as a connected component of V (necessarily diffeomorphic to a circle) which maps under π into a connected component of V' (likewise diffeomorphic to a circle). We can then distinguish a **crank**, when the image under π is the whole circle, from a **rocker**, when the image under π is just a proper closed subarc of the circle. This appears to be a useful generalisation of the concept long familiar to engineers, when M' is chosen to be a single link of a planar mechanism M and one is simply distinguishing the case when M' rotates full circle during the motion from that when M' rocks backwards and forwards. Thus given a motion of M, the question, whether a given submechanism M' is moving as a crank or a rocker, is intimately related to the branching of the projection $\pi : V \to V'$. Indeed, in the case of 2-fold coverings it is decided by the absence or presence of branching. Either way, one can reasonably hope, for a given example, to answer this important question in terms of the design parameters.

We begin by recalling some facts on finite mappings. The key objects in our discussion are surjective morphisms $\pi : V \to V'$ between irreducible complex projective curves V, V'. In examples π is usually a projection from a projective subspace and V' is the image of V under π. Such mappings satisfy the technical condition of "finiteness" (or at least on Zariski open subsets of V and V'). The nomenclature derives from the following basic result. There exists an integer $d \geq 1$ (called the **degree** of the mapping) such that every point on V' has $\leq d$ pre-images in V and for all but finitely many points on V' (called **branch points**) there are precisely d pre-images in V. It is a basic fact that for non-singular varieties V and V' the number B of branch points (counting multiplicities) is related to the genera g, g' of V, V' by the Hurwitz formula (A10)

$$2(g-1) = 2d(g'-1) + B \qquad (1.5)$$

In the situations under discussion one should view (1.5) as a device for computing g in terms of g' and B. Generally we shall know g' and we can compute B. Another basic fact, which we shall use in the sequel, is that a critical value of π (i.e. the image under π of a point in V where the differential has rank zero) is necessarily a branch point. The reader is referred to §A7 for details.

It is, however, the real geometry which interests us, when we have a morphism $\pi : V \to V'$ between irreducible non-singular real projective curves V, V' for which the complexification has the properties described above. More particularly, we are concerned with the case when the morphism has degree 2. Note that in this case the pre-image of a critical value necessarily comprises a single point. This situation seems to arise naturally when elucidating the geometry of a number of mechanisms and has the virtue that one can give the following qualitative description of the real mapping π.

Let us consider one fixed component X' of V' whose pre-image under π must comprise finitely many components $X_1,\ldots,X_n$ of V. Any X_k maps under π, either onto an arc A_k of X', or onto X' itself. We assert that there are three essentially distinct qualitative pictures which we shall state in the form of a Proposition.

Proposition 1.1

Let $\pi : V \to V'$ be a map of degree two between two real curves. Then there are three possibilities

(I) There is just one component X_1 mapped immersively onto X' as a double cover (Fig. 1.4a)

(II) There are just two components X_1, X_2 each mapped diffeomorphically onto X' (Fig. 1.4b)

(III) There are n components $X_1,\ldots,X_n$ mapping onto disjoint arcs $A_1,\ldots,A_n$ of X', with exactly $2n$ critical values, namely the end-points of $A_1,\ldots,A_n$ (Fig. 1.4c).

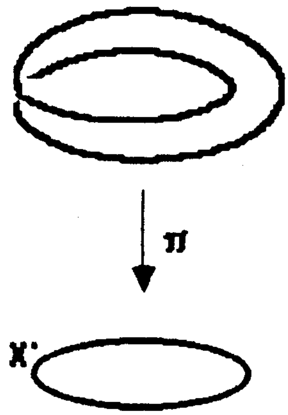


Fig. 1.4a

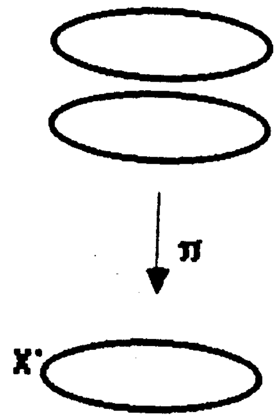


Fig. 1.4b

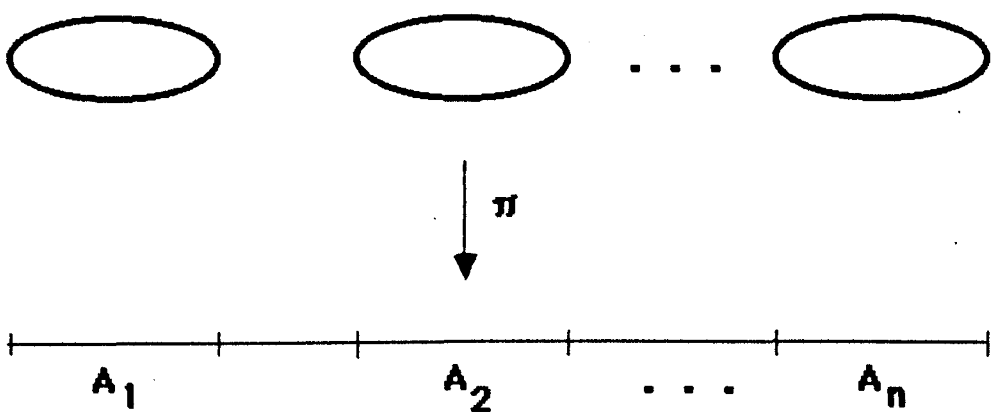


Fig. 1.4c

Proof: Suppose first that at least one component X_i maps onto an arc A_i of X'. By the Inverse Function Theorem the two end-points of A_i must be critical points each having a pre-image of a single point. Thus X_i is split into two arcs with common end-points each necessarily mapping onto A_i. In particular every interior point of A_i has exactly two distinct pre-images in X_i so cannot be a critical value. Since no point on X' has ≥ 3 pre-images in V we see that $X_1, \ldots, X_n$ must all map to arcs

$A_1,\ldots,A_n$ which are necessarily disjoint and we have the situation described under III. It remains to consider what happens when every component X_i maps onto X'. In this situation it is evident that $n \leq 2$. When there are two components X_1, X_2 each must map injectively and hence homeomorphically onto X' : clearly, there can be no branching, so in fact these mappings are diffeomorphisms. That is case II above. We are left with the case when there is just one component X_1. Choose a point on X' with two distinct pre-images on X_1 : thus X_1 is split into two arcs with these points as common end-points. One possibility is that these arcs map to arcs of X', splitting X' in the same way : but then we have at least one critical value on X' with two distinct pre-images, a contradiction. The only remaining possibility is when both arcs map onto X'. In that case every point on X' has exactly two pre-images on X_1, so there is no branching and X_1 is mapped immersively to X' as a double cover. That completes the proof.

■

Thus, in the case of mappings of degree 2 we can distinguish a <u>double crank</u>, when one is in case I, from two <u>single cranks</u>, when one is in case II. Note, however, that the absence of branching does not tell us which case we are dealing with; in a given example one has to look for some special feature which will distinguish these cases. By contrast, in the case of rockers the number of real branch points completely determines the number of real components in the pre-image.

For the remainder of this section we shall <u>assume that none</u>

of the Grashof equations $e_1 \pm e_2 \pm e_3 \pm e_4 = 0$ are satisfied, so that $\mathcal{R}'$ is a non-singular irreducible complex curve.

For the planar four-bar mechanism M the philosophy of the introduction is realised by taking M' to be any one of the moving links. The configuration space for the j^{th} link is the complex projective plane $P\mathbb{C}^2$ with co-ordinates x_j, y_j, w, and the linkage variety is the conic defined by $x_j^2 + y_j^2 = w^2$, a non-singular irreducible complex curve C_j. The natural projections $P\mathbb{C}^6 \rightarrow P\mathbb{C}^2$ given by $(x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (x_j, y_j, w)$ then restrict to the finite mappings $\pi_j : \mathcal{R}' \rightarrow C_j$. Note that the centre of π_j is the 3-space defined by $x_j = 0, y_j = 0, w = 0$. The intersection of the centre with the 4-space defined by the linear equations in (1.1) is precisely the line joining $P_j, \overline{P}_j$. Note also that the complex map $\pi_j: \mathcal{R}' \rightarrow C_j$ is indeed surjective. We showed in §1.3 that the degree d_j of π_j is 2 for $j = 1,2,3$.

We are now in a position to apply the above description to the real mappings $\pi_j : \mathcal{R}' \rightarrow C_j$. We need to determine the number of real branch points, i.e. the number of real critical points. Critical points occur when the tangent line to $\mathcal{R}'$ meets the centre of π_j, i.e. when these projective subspaces fail to span a 5-space. In computational terms that means that the 5 x 4 matrix, obtained from the Jacobian matrix of the equations (1.1) by deleting the columns corresponding to the co-ordinates x_j, y_j and w, should have rank < 4. It is a minor exercise to check that this happens precisely when the vectors (x_i, y_i) and (x_k, y_k) are linearly dependent, where i, j, k is a permutation of 1, 2, 3. In the real case the physical interpretation of this condition is that the two moving links i and k are parallel. Using the equations (1.1) we see that our condition is equivalent to $x_k = \varepsilon x_i$ and $y_k = \varepsilon y_i$, where $\varepsilon = \pm 1$. Substituting for x_k, y_k in (1.1), we see that for each choice of sign ε there are exactly two (complex) critical values corresponding to the finite intersections of the real circles with equations

$$x_i^2 + y_i^2 = 1$$
$$(d_i x_i - d_4)^2 + d_i^2 y_i^2 = (d_j + \varepsilon d_k)^2 .$$

Note here that the circles can only be tangent when one of the Grashof conditions is satisfied. Therefore we obtain four distinct complex critical values. At this point it is worth remarking that, since C_j has genus zero, the Hurwitz Formula (A10) tells us that the residual curve $\mathcal{R}'$ has genus 1 (i.e. is elliptic) confirming a fact proved by a different method in [Marsh;Gibson&Newstead].

In the real case it is an elementary matter to decide in terms of the link lengths, whether the circles intersect in no or two real points. Expanding the second equation and using the first equation to substitute for $x_i^2 + y_i^2$, we find that

$$x_i = \frac{(d_j + \varepsilon d_k)^2 - d_i^2 - d_4^2}{-2 d_i d_4}$$

and it is now clear that we have a real solution if and only if $-1 \leq x_i \leq 1$. Thus for $\varepsilon = +1$ we yield two real solutions if and only if $d_j + d_k \leq d_i + d_4$ and for $\varepsilon = -1$ we yield two real solutions if and only if *either* $d_j + d_i \geq d_k + d_4$ and $d_j + d_4 \geq d_i + d_k$ *or* $d_j + d_i \leq d_k + d_4$ and $d_j + d_4 \leq d_i + d_k$. Combining these two cases of $\varepsilon = \pm 1$, we may have

0, 2 or 4 real critical points. To summarise the results set $E = e_1 - e_2 - e_3 + e_4$, where we continue to write d_1, d_2, d_3, d_4 in increasing order of magnitude as e_1, e_2, e_3, e_4. Then one finds that

π_j has four real critical points	$\Leftrightarrow$	if $E < 0$ and the shortest link length is d_i or d_k
π_j has two real critical points	$\Leftrightarrow$	$E > 0$
π_j has no real critical points	$\Leftrightarrow$	$E < 0$ and the shortest link length is d_j or d_4.

On this basis it is an easy matter to determine the real topology of $\mathcal{R}'$ and answer the crank/rocker question. If $E > 0$, we see that all three projections have exactly two critical points, so that we can deduce from Proposition 1.1: firstly that $\mathcal{R}'$ has just one connected component and secondly that all three moving links are rockers. Suppose now that $E < 0$. Note first that only one link can have the shortest length: indeed, if $e_1 + e_4 < e_2 + e_3$ and $e_1 = e_2$, then $e_4 < e_3$, giving a contradiction. If the shortest link length is d_i, then the two projections π_j for $j \neq i$ have four critical points and π_i has no critical points We then deduce from §1.1: firstly that $\mathcal{R}'$ has two connected components, secondly that the link of shortest length moves as a single crank for each component and thirdly that the other two moving links are rockers for each component. It remains to discuss the case when the fixed link has the shortest length. In that case all three projections have no critical points; certainly then all three moving links are cranks, but we do not know whether $\mathcal{R}'$ has one or two connected components. (See the closing comments of §1.1.) In fact we claim that, if π_j has no critical points, then necessarily $\mathcal{R}'$ has two topological components: in other words the theoretical possibility of a double crank cannot arise. The key observation is that the curve $\mathcal{R}'$ possesses a natural involution, namely, that given by reversing the signs of y_1, y_2, y_3. Physically, one is just reflecting the mechanism in the line determined by the fixed link. Now consider the smooth function $x_i y_k - x_k y_i$ on $\mathcal{R}'$ with i, j, k as above a permutation of 1, 2, 3. The effect of the involution is to reverse the sign of this function, so that it assumes both positive and negative values on $\mathcal{R}'$. If $\mathcal{R}'$ has just one connected component, the function would vanish somewhere on $\mathcal{R}'$ and hence π_j would have a critical point, a contradiction, establishing our claim. Thus, when $E < 0$ and the fixed link has the shortest length, all three moving links are single cranks. That completes our analysis for the planar four-bar. Note in particular that we have confirmed the result in [Gibson&Newstead], namely that $\mathcal{R}'$ has one/two connected components according as $E < 0 / E > 0$.

The above is easily related to the established engineering literature. The branch points of the projections π_j for $j = 1,3$ are precisely the "limiting positions" described in [Hain, 1964]. Moreover, one recovers the eight basic types of planar four-bar isolated in [Hain, 1964]. When $E < 0$ we need a further distinction. Recall that in that case all three links move as rockers and that the condition for π_j to have a critical point is that the vectors z_i, z_k are equal, in which case we have an <u>inward</u> rocker, or opposite, in which case we have an <u>outward</u> rocker. Notice that once this distinction is made for two of the moving links, it is automatically made for the third. Thus choosing, for example, links 1 and 3 we can distinguish four cases, namely inward/outward, outward/outward, outward/inward and inward/inward denoted by Hain [Hain 1964] as R_{io}, R_{oo}, R_{oi}, R_{ii}, respectively. And it is an easy matter to verify the somewhat classical result that these cases correspond precisely to whether d_1, d_2, d_3 or d_4 is the longest link length. When $E>0$ Hain distinguishes four types denoted by CR_1, DR, CR_2, DL depending on whether d_1, d_2, d_3 or d_4 is the shortest. The notation may be explained in the following way.

The letter C refers to a crank and the letter R stands for a rocker. Hence, the first three types simply mean that the motion of bars one and three are crank/rocker, double rocker and rocker/crank. Whilst when d_4 is the shortest, bars one and three crank and such a mechanism is called a drag link.

§1.5 The Segre Quartic Surface

In §1.6 we will show how for any coupler point one can associate a pencil of quadrics in $P\mathbb{C}^4$. The base variety, that is the intersection of all the quadrics in the pencil, is in general a quartic surface, whose geometry seems to be crucial to the study of four-bar (and indeed geared five-bar) coupler curves. Such surfaces may be given as the intersection of any two quadrics in the pencil. The purpose of this section is to outline the results achieved by Segre on the 'generic' intersection of two quadrics in $P\mathbb{C}^4$ and named in his honour - the **Segre quartic surface**. These results are expressed as Theorem 1.1 and closely follow the sketch of this result given in [Jessop 1916].

We shall begin by making the term 'generic' more precise. Consider the pencil of quadrics $\lambda Q_1 + \mu Q_2$ in $P\mathbb{C}^4$ generated by two quadrics Q_1 and Q_2. Let Q be any quadric in the pencil. Then Q is singular if and only if its Jacobian matrix, a 5×5 matrix with coefficients involving λ and μ, has non-maximal rank. This is equivalent to saying that the matrix has zero determinant: thus the condition is the vanishing of a homogeneous binary quintic polynomial in λ and μ - the **discriminant** of the pencil. Thus in general, the discriminant will have <u>five</u> solutions $(\lambda_1,\mu_1),\ldots,(\lambda_5,\mu_5)$ each one corresponding to a singular quadric in the pencil. More precisely, each of the singular quadrics is a point cone. Whenever the pencil has five distinct cones, we will say that the pencil is **generic**. Exceptionally, the polynomial may have less than five solutions. In this case we have the concept of multiplicity of a solution: we simply write the polynomial as a product of linear factors $(a_1\lambda + b_1\mu)^{\alpha_1}\cdot\ldots\cdot(a_r\lambda + b_r\mu)^{\alpha_r}$ over $\mathbb{C}$ and say that the root $(b_s,-a_s)$ has multiplicity α_s. In this case the singular quadrics corresponding to roots with multiplicity α_s remain cones, but the dimension of the vertex (i.e. the singular set of the quadric) may be positive depending on the form of the pencil in question. More exceptionally, the polynomial may be identically zero. In this case every member of the pencil is a cone and we refer to the pencil as **singular**.

We may now prove the main result

Theorem 1.1: Let Q_1 and Q_2 be two quadrics in $P\mathbb{C}^4$ and suppose that the pencil generated by the two quadrics is generic. Then

(1) the intersection of Q_1 and Q_2 is a quartic surface $\mathcal{S}$ containing exactly sixteen lines

(2) the configuration of lines is such that any one of them meets five other lines and

(3) projecting from any one of the lines, defines a birational map between the surface and the projective plane. The five lines meeting the line of projection map to points - which we call base points. The other lines map to a line passing through two base points and conversely, any line through two base points is the image of a line on the surface. Finally, the line of projection corresponds to the unique conic passing through the five base points. Points on this conic are precisely those points for which the projection has *either* no pre-image *or* a line of pre-images on the surface.

Proof: (1) The fact that the intersection of two quadrics in $P\mathbb{C}^4$ is a quartic surface follows from Bézout's Theorem (A3). The first step to show that the surface has sixteen lines is to make a complex projective change of co-ordinates putting the quadrics into their Weierstrass normal form (see [Jessop 1903]). This leads us to a number of different (complex) types denoted by their **Segre symbol**, the principal ones of which are

[11111]: $Q_1 = \sum_{i=1}^{5} x_i^2$ and $Q_2 = \sum_{i=1}^{5} a_i x_i^2$

[1112] : $Q_1 = x_1^2 + x_2^2 + x_3^2 + 2x_4x_5$ and
$Q_2 = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_4x_5 + x_4^2$

[122] : $Q_1 = x_1^2 + 2x_2x_3 + 2x_4x_5$ and
$Q_2 = a_1x_1^2 + 2a_2x_2x_3 + 2a_3x_4x_5 + x_2^2 + x_4^2$

[113] : $Q_1 = x_1^2 + x_2^2 + x_4^2 + 2x_3x_5$ and
$Q_2 = a_1x_1^2 + a_2x_2^2 + a_3(x_4^2 + 2x_3x_5) + 2x_3x_4$

[23] : $Q_1 = 2x_1x_2 + 2x_3x_5 + x_4^2$ and
$Q_2 = 2a_1x_1x_2 + x_1^2 + a_2(x_4^2 + 2x_3x_5) + 2x_3x_4$

[14] : $Q_1 = x_1^2 + 2x_2x_5 + 2x_3x_4$ and
$Q_2 = a_1x_1^2 + 2a_2(x_2x_5 + x_3x_4) + 2x_2x_4 + x_3^2$

[5] : $Q_1 = 2x_1x_5 + 2x_2x_4 + x_3^2$ and
$Q_2 = a_1(2x_1x_5 + 2x_2x_4 + x_3^2) + 2x_1x_4 + 2x_2x_3$

Note that in the real case any non-singular pencil can only be reduced to the form

$$Q_1 = \sum_{i=1}^{r} a_i x_i^2 + \sum_{i=1}^{s} 2u_i v_i \quad \text{and} \quad Q_2 = \sum_{i=1}^{r} b_i x_i^2 + \sum_{i=1}^{s} \{\beta_i(x_i^2 - v_i^2) + 2\alpha_i u_i v_i\}$$

We may assume that $a_i = \pm 1$ (see [Muth]). But in the real forms we must distinguish those forms which differ only by the sign of an a_i. We shall not need the real type here.

In particular, for the generic pencil [11111] we may reduce the pair of matrices to diagonal form. In this manner the form of the cones is easily obtained. Suppose that the quadrics are $Q_1 = x_1^2 + \dots + x_5^2$ and $Q_2 = a_1x_1^2 + \dots + a_5x_5^2$ then the pencil $\lambda Q_1 + \mu Q_2$ has five cones of the form

$$(a_2-a_1)x_2^2 + (a_3-a_1)x_3^2 + (a_4-a_1)x_4^2 + (a_5-a_1)x_5^2 = 0 \text{ etc.,}$$

Thus, the cones are point cones over a non-singular 2-dimensional quadric. A general 2-dimensional quadric contains two families of generating lines. Any two lines in the same family do not meet, whilst a line from one family meets any line from the other family in a point. Thus each cone in the pencil has two families of generating planes. Likewise, any two planes in the same family meet only in the vertex of the cone, whilst any plane from one family meets a plane from the other in a line.

Claim: Any generating plane $\mathfrak{T}$ of one of the five cones $\mathfrak{C}$ meets the quartic surface $\mathcal{S}$ in a conic.

Proof of claim: Since $\mathcal{S}$ is the intersection of all the quadrics in the pencil it is the intersection of any two quadrics in the pencil i.e. we can generate the pencil by any two quadrics contained in it. Let the pencil be generated by $\mathfrak{C}$ and any other quadric Q'. But $\mathfrak{T}$ lies in $\mathfrak{C}$, thus the intersection of $\mathfrak{T}$ and Q' lies in $\mathcal{S}$ i.e. a conic.

Conversely, any conic c on $\mathcal{S}$ lies on a generating plane of a cone in the pencil. For, consider any point on the plane containing c, but not lying on c. Then there is a unique quadric $\mathfrak{C}$ in the pencil containing the point and the conic; but any quadric meeting a plane in a conic and a point must contain the plane. The claim follows since any 2-dimensional quadric, which contains a plane, is necessarily a cone.

Then let the pencil be generated by a cone and one other quadric, say $\Sigma_{i=2}^{5}(a_i - a_1)x_i^2 = 0$ and $\Sigma_{i=1}^{5}x_i^2 = 0$. By a projective change of co-ordinates we may write these equations as $x_2x_3 = x_4x_5$ and $x_1^2 + f(x_2,x_3,x_4,x_5) = 0$, where f is a quadratic polynomial. Then the generators of the cone have either the form $x_2 = \alpha x_4$, $x_5 = \alpha x_3$ or the form $x_3 = \alpha x_4$, $x_5 = \alpha x_2$. Any such plane meets the second quadric in a conic lying on $\mathcal{S}$. The condition for the conic to reduce to two lines is a quartic polynomial one in α. Note that this quartic cannot be identically zero. For then every generating plane meets $\mathcal{S}$ in two lines; in particular, every point on the surface is contained in a line on the surface. We can then derive a contradiction in the following manner. It is clear from the Weierstrass normal form that there is a point P of the form $(p_1,p_2,0,p_4,0)$ on $\mathcal{S}$. Let $\mathcal{L}_p$ be any line through P. Then $\mathcal{L}_p$ is contained in the tangent plane to $\mathcal{S}$ at P. The tangent plane has the form $\lambda x_1 + \mu x_2 + \nu x_4 = 0$ and $\lambda a_1x_1 + \mu a_2x_2 + \nu a_4x_4 = 0$ and therefore contains the line of points of the form $(0, 0, p_3, 0, p_5)$. Thus $\mathcal{L}_p$ must meet this line in some point $(0,0,p,0,q)$. But it is clear from the normal form that no such point can lie on $\mathcal{S}$. Hence $\mathcal{L}_p$ cannot lie on $\mathcal{S}$ and we have a contradiction. It follows, therefore, that there are four generating planes in each family, which meet the quadric in two lines, implying that there are in all <u>at most sixteen lines</u> lying on the quartic surface.

We must now show that these sixteen lines are distinct.

From the above analysis it is clear that there exists at least one line on $\mathcal{S}$. Let us assume that the pair of generating quadrics are in Weierstrass normal form and consider the natural automorphisms of $\mathcal{S}$, taking lines to lines, defined by $x_j \to \varepsilon_j x_j$ for $j = 1,..5$, where $\varepsilon_j = \pm 1$. Note that the choice of signs $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$ and $(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3,-\varepsilon_4,-\varepsilon_5)$ give identical automorphisms leaving just sixteen distinct automorphisms for the generic pencil. There are two possibilities for fixed points of the automorphisms: *either* points defined by the vanishing of four of the variables x_j *or* lines defined by the vanishing of any three variables x_j. However, in the latter case no such line can ever lie on the surface. Thus given one line on the surface we may obtain for each automorphism one new line lying on the surface; implying that there are <u>at least sixteen lines</u> in all. The result now follows.

(2) All sixteen lines lie on the five cones of the pencil. Therefore, the plane through the vertex of a cone and any one of the lines $\mathcal{L}$ is a generating plane; thus meeting $\mathcal{S}$ in one more line which meets $\mathcal{L}$ in a point. Thus <u>five</u> of the sixteen lines on $\mathcal{S}$ meet $\mathcal{L}$.

(3) Let Q_1 and Q_2 be the generators of the pencil and let $\mathcal{L}$ be one of the sixteen lines on $\mathcal{S}$. Further, suppose that $\mathcal{K}$ is any plane disjoint from $\mathcal{L}$. Make a projective change of co-ordinates taking the line $\mathcal{L}$ onto the line $x_3 = x_4 = x_5 = 0$; we may assume that $\mathcal{K}$ has co-ordinates x_3, x_4, x_5. Then we may write the quadrics in the form

$$x_1\ell_1 + x_2\ell_2 + f = 0, \qquad x_1m_1 + x_2m_2 + g = 0,$$

where ℓ_1, m_1, ℓ_2, m_2 are homogeneous of degree 1 and f, g are homogeneous of degree 2 in x_3, x_4, x_5. Let P be any point on $\mathcal{K}$ and denote by $\mathcal{F}$ the plane through P and $\mathcal{L}$. The equation of $\mathcal{K}$ may be obtained explicitly by substituting for the x_3, x_4, x_5 co-ordinates of P in ℓ_1, m_1, ℓ_2, m_2. Thus $\mathcal{F}$ meets Q_i in $\mathcal{L}$ and one other line $\mathcal{L}_i$ (i=1,2). These two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ meet in a unique point, lying on the surface $\mathcal{S}$, provided $\ell_1 \cdot m_2 - \ell_2 \cdot m_1 \neq 0$: thus defining a 1-1 correspondence between $\mathcal{S}$ and $\mathcal{K}$. Therefore, the correspondence only fails to be 1-1 when $\ell_1 \cdot m_2 - \ell_2 \cdot m_1 = 0$; this condition defines a conic on $\mathcal{K}$. Points on this conic have no pre-images in general, whilst exceptionally, when the two lines coincide, they have a line of pre-images. Thus these points are the images of the five lines meeting $\mathcal{L}$. We will call these five distinguished points on the conic the **base points**. Note that the conic is necessarily irreducible; for otherwise three of the base points would lie on a line implying that three of the lines on $\mathcal{S}$ lie on a 3-space, contrary to the configuration described above.

Suppose that $\mathcal{L}'$ is one of the ten lines of $\mathcal{S}$ which are skew to $\mathcal{L}$. Then the hyperplane $\mathcal{H}$ spanned by $\mathcal{L}$ and $\mathcal{L}'$ meets $\mathcal{S}$ in $\mathcal{L}$, and in two transversals of $\mathcal{L}$ and $\mathcal{L}'$. Thus, $\mathcal{H}$ meets $\mathcal{K}$ in a line passing through <u>two</u> of the five base points which by definition is the image of $\mathcal{L}'$.

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§1.6 The Four-bar Coupler Curves.

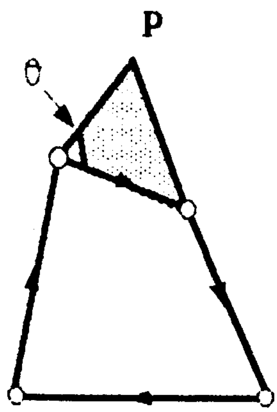


Fig. 1.5

In this section we consider the two parameter family of coupler curves, which are the loci of a point $\mathfrak{P}$ rigidly attached to the coupler bar (that is, the bar opposite the fixed bar), as showed in Fig 1.5. Thus $\mathfrak{P} = d_1 z_1 + k\cdot z_2$, where $k = k_1 + i k_2$ $(k_1 = r\cdot\cos\theta,\ k_2 = r\cdot\sin\theta)$ is a fixed complex number. (This analysis closely follows that given in [Gibson&Newstead].) We may think of $\mathfrak{P}$ as having homogeneous co-ordinates p_1, p_2, p_3 with $p_1 = d_1x_1 - k_2y_2 + k_1x_2$, $p_2 = d_1y_1 + k_2x_2 + k_1y_2$ and $p_3 = w$; thus defining a natural linear projection $\pi_k : P\mathbb{C}^6 - \mathcal{V} \to P\mathbb{C}^2$ given by

$$(x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (d_1x_1 - k_2y_2 + k_1x_2,\ d_1y_1 + k_2x_2 + k_1y_2,\ w)$$

where $\mathcal{V}$ is the centre of projection i.e. the linear subspace defined by $p_1 = p_2 = p_3 = 0$. The images of the lines at infinity L and $\bar{L}$ are the circular points at infinity $I = (1,i,0)$ and $J = (1,-i,0)$. The image of the restriction map $\pi_k|_{\mathcal{R}'}$ of the projection π_k to the residual linkage variety $\mathcal{R}'$ is a curve $\mathcal{C}_k$ which we shall call the **complex coupler curve**.

Recall that the linkage variety is given by the three quadratic and two linear equations of (1.1). Let $\mathcal{W}$ denote the 4-space defined by the two linear equations. Then $\mathcal{W}$ and $\mathcal{V}$ span $P\mathbb{C}^6$ and intersect in a line $\mathcal{V}'$ meeting L and $\bar{L}$ in the points whose co-ordinates are

$$A = (kd_3,\ -ikd_3,\ -d_1d_3,\ id_1d_3,\ d_1(d_2-k),\ -id_1(d_2-k),0) \quad \text{and}$$
$$\bar{A} = (\bar{k}d_3,\ i\bar{k}d_3,\ -d_1d_3,\ -id_1d_3,\ d_1(d_2-\bar{k}),\ id_1(d_2-\bar{k}),0).$$

Thus provided $\mathcal{R}'$ does not meet $\mathcal{V}'$ we may factor the projection as the projection from $\mathcal{V}'$, followed by the projection $\pi_k : \mathcal{W} - \mathcal{V}' \to P\mathbb{C}^4$ given by the same forms p_1, p_2, p_3. We shall refer to π_k as the **coupler projection**. But $\mathcal{R}'$ meets $\mathcal{V}'$ only in the uninteresting cases when $k=0$ or $k=d_2$, i.e. the coupler point lies at one end of the coupler bar, when $\mathcal{C}_k$ is a circle. Thus it is sufficient to consider the variety in $P\mathbb{C}^4$ obtained from $\mathcal{R}$ by projecting from $\mathcal{V}'$. We shall denote the images of $\mathcal{R}$, $\mathcal{R}'$, L, $\bar{L}$ by the same symbol. Then $\mathcal{R}$ is projectively equivalent to the variety obtained by using the linear equations of (1.1) to eliminate two of the variables say, x_3 and y_3 giving $\mathcal{R}$ as the intersection of three quadrics

$$x_1^2 + y_1^2 = w^2, \qquad x_2^2 + y_2^2 = w^2$$
$$(d_4w - d_1x_1 - d_2x_2)^2 + (d_1y_1 + d_2y_2)^2 = d_3^2w^2$$

and $\mathcal{V}'$ is given by the equations $p_1 = p_2 = p_3 = 0$.

Write the three quadrics as Q_1, Q_2, Q_3 and consider the net $XQ_1 + YQ_2 + ZQ_3$. Choose a point P on $\mathcal{V}'$ not lying on L or $\bar{L}$. Then the condition for a quadric in the net to pass through this point is a linear condition on X, Y, Z thus defining a pencil in the net. Explicitly, it is easily checked to be given by

$$k\bar{k}X + d_1^2 Y + d_1^2(k-d_2)(\bar{k}-d_2)Z = 0.$$

We shall call this pencil the **associated pencil** of the coupler point. Any quadric in this pencil meets $\mathcal{V}'$ in three points $(A, \bar{A}$ and $P)$ and must therefore contain $\mathcal{V}'$. The pencil may be written as $XQ_1' + ZQ_2'$ where $Q_1' = [Q_1 - Q_2 k\bar{k}/d_1^2]$ and $Q_2' = [Q_3 - Q_2(k-d_2)(\bar{k}-d_2)]$. Then, using the identities $p_1 = d_1x_1 - k_2y_2 + k_1x_2$, $p_2 = d_1y_1 + k_2x_2 + k_1y_2$, $p_3 = w$, Q_1' and Q_2' may be written in the form

$$\left.\begin{aligned} A + Bx_2 + Cy_2 &= 0 \\ D + Ex_2 + Fy_2 &= 0 \end{aligned}\right\} \quad (1.6)$$

where

$A = p_1^2 + p_2^2 + (k_1^2 + k_2^2 - d_1^2)p_3^2$,
$B = -2(k_1p_1 + k_2p_2)$,
$C = -2(k_1p_2 - k_2p_1)$,
$D = p_1^2 + p_2^2 + (-d_3^2 + d_4^2 + k_2^2 + (d_2-k_1)^2)p_3^2 - 2d_4p_1p_3$,
$E = 2p_1(d_2-k_1) - 2d_4(d_2-k_1)p_3 - 2k_2p_2$,
$F = 2k_2p_1 - 2k_2d_4p_3 + 2p_2(d_2-k_1)$.

We shall now apply the results of §1.5.

We recall that the intersection of a generic pencil of quadrics is a Segre Quartic Surface $\mathcal{S}$ containing sixteen lines. Let us assume for the moment that the pencil is indeed generic; we shall consider this condition in more detail in Chapter 5 when we discuss the geometry of the real four-bar coupler curves. Writing the quadrics in the above manner, makes the projection more transparent. Fixing a point in the image of the projection, fixes the values of p_1, p_2, p_3. Observe that (1.6) may be considered as linear equations in x_2 and y_2. Thus by applying Cramer's Rule to equations (1.6), we may write x_2 and y_2 (and hence x_1 and y_1) uniquely in terms of p_1, p_2 and p_3, provided $\Phi = BF - CE \neq 0$. The condition $\Phi = 0$ defines a conic $\mathfrak{C}$ in the fixed plane consisting of the points on which the coupler projection fails to be 1-1. Explicitly, this is given by

$$-d_2k_2(p_1^2 + p_2^2) + d_2d_4k_2p_1p_3 + d_4(k_1^2 + k_2^2 - d_2k_1)p_2p_3 = 0.$$

A point on $\mathfrak{C}$ has *either* no pre-image *or* is one of five base points for which there is a line of pre-images. Since L and $\bar{L}$ meet $\mathcal{V}'$, two of the base points are the circular points I and J. Under our assumption that the pencil is generic $\mathfrak{C}$ cannot be reducible. Otherwise, three of the five base points would lie on a line, implying that three lines on $\mathcal{S}$ lie in a 3-space, contradicting the known configuration of lines. The conic is real, passes through I and J and hence is a circle. This is the **circle of singular foci** well known in the mechanisms literature (see for instance [Hunt 1978]).

We may now deduce some of the geometry of the coupler curves. Since the residual curve $\mathfrak{R}'$ does not meet the centre of projection and π_k has degree 1, we may apply the Projection Formula to deduce that the coupler curve has degree six. The projection is a generically 1-1 rational map and hence birational. In particular, this implies that $\mathfrak{C}_k$ has the same geometric genus as $\mathfrak{R}'$: thus for the generic mechanism $\mathfrak{C}_k$ has genus one and for the non-generic mechanisms the components all have genus zero i.e. they are rational.

In the non-generic cases any singular point of $\mathfrak{R}'$ is mapped by π_k to a point of $\mathcal{C}_k$ (off $\mathcal{C}$) with the same isomorphism type. Thus the singular points of $\mathcal{C}_k$ off $\mathcal{C}$ are necessarily ordinary double points. The remaining possible singular points of $\mathcal{C}_k$ are I, J and three other points lying on $\mathcal{C}$. We recall that $\mathfrak{R}'$ meets L and $\bar{L}$ in three distinct points and therefore $\mathcal{C}_k$ has three distinct branches at each of I, J. Thus I and J are singular points of $\mathcal{C}_k$. The remaining three points P_l have a line $\mathcal{L}_l$ as their pre-image on $\mathcal{S}$ meeting a third quadric in the net, not already in the pencil, *either* in two distinct points *or* in one point at which it is tangent. In the former case, $\mathfrak{R}'$ meets $\mathcal{L}_l$ in two distinct branches and maps to a point of $\mathcal{C}_k$ with two branches. In the latter case, $\mathcal{L}_l$ is tangent to the quadric at a critical point of the projection: hence its image is a singular point on $\mathcal{C}_k$. Let $\mathcal{C}_k = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ be the decomposition of the coupler curve into its irreducible components and apply the Genus Formula (Theorem A8),

$$p_a(\mathcal{C}_k) = \Sigma_{i=1}^{r} p_g(\mathcal{C}_i) + \Sigma^* \delta_p - (r-1)$$

where the * implies that the sum is taken over the singular points of $\mathcal{C}_k$. For planar curves of degree d the arithmetic genus is ½(d-1)(d-2); thus for sextics the genus is 10. For the generic mechanism $r = 1$, there are no singular points off $\mathcal{C}$ and the geometric genus is 1. Hence, $\Sigma^*\delta_p = 9$ implying that the circular points have $\delta_p = 3$ and the remaining three points have $\delta_p = 1$. In the non-generic cases, i.e. the circumscriptible, parallelogram/kite and rhombus cases, all components of $\mathfrak{C}_k$ are rational, i.e. the geometric genera are all zero. Moreover, there are r ordinary double points (with $\delta_p = 1$) off $\mathfrak{C}$. Hence, the sum of the δ_p's for the singular points of $\mathfrak{C}_k$ lying on $\mathfrak{C}$ is 9 and we may conclude that the circular points have $\delta_p = 3$ and that the remaining three points have $\delta_p = 1$. Thus $\mathfrak{C}_k$ has two ordinary triple points at I and J whilst the other three singular points on $\mathfrak{C}$ are *either* ordinary double points *or* ordinary cusps.

Finally, we wish to point out that there is an interesting open problem concerning the relation between the eight types of planar four-bars mentioned in §1.4 and the geometry of the associated coupler curves. Suppose we restrict ourselves to generic choices of coupler points in the sense that the associated pencil of quadrics discussed above is general, i.e. contains exactly five point-cones. We showed that the coupler curve has exactly three finite singular points, each an ordinary double point or a (real) cusp: these singular points can be real or complex and in the real case one can make the further distinction between crunodes and acnodes. In this way one obtains thirteen basic multi-singularity types. The problem is to determine for each of Hain's eight basic types which of these thirteen multi-singularity types can occur. This should yield a useful division of generic coupler curves into finitely many types, for each of which one could pursue in greater detail the real algebraic geometry and the differential geometry. We shall leave further discussion of this until Chapter 5 where we make the first steps to a solution of this problem.

CHAPTER 2. THE TOPOLOGY OF THE SPHERICAL FOUR-BAR MECHANISM

It was clear to the author that the techniques for determining the topology of the planar four-bar mechanism need not be restricted to planar mechanisms, but could be just as well applied to spatial mechanisms. We recall from §1.4 that for any mechanism M with linkage variety V, we have submechanisms M' with linkage variety V' and a natural projection $\pi: V \to V'$ of degree d. The philosophy is that for generic mechanisms M, M' of mobility one, their real linkage varieties are non-singular compact curves and hence diffeomorphic to a finite union of circles. Then the topology of V is related to d-fold coverings of circles, each a topological component of V', whose branching can be described in terms of the design parameters. For a suitable choice of M', for which one knows something of the topology of V', one hopes to obtain information about the topology of V. Moreover, there is no reason why one should restrict oneself even to mechanisms with mobility one. For instance, for mechanisms of mobility two, i.e. their linkage varieties are surfaces, there may be natural projections onto spheres which generalises the philosophy of calculating the topology of real linkage curves via projections onto circles initiated in Chapter 1.

Thus we shall determine the topology of the linkage variety for the first non-trivial spatial mechanism, namely the spherical four-bar. We will not be giving a full treatment of the algebraic geometry of the spherical four-bar linkage variety in this chapter - as this has been done in [Gibson&Selig] - we shall simply give a brief account of the results that we need. The result on the topology that we shall prove here does not already exist in the literature.

First we need to recall some of the basic facts established in [Gibson&Selig]. Just as the planar four-bar consists of four rigid bodies in 2-space jointed at points, the spherical four-bar consists of four rigid bodies in 3-space jointed along lines called the **joint axes** (as showed in fig 2.1).

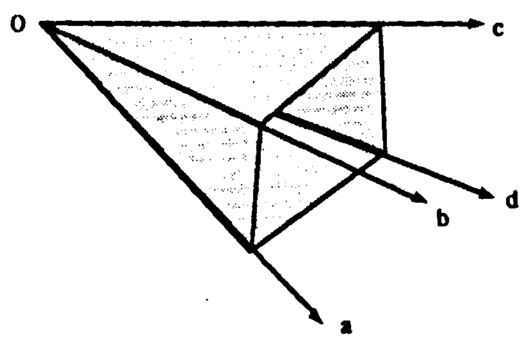


Fig. 2.1

The joint axes are to be represented by a cyclic sequence of four unit vectors a, b, c, d in $\mathbb{R}^3$ subject to the constraints that the angles A, B, C, D between adjacent pairs remain constant. Write $\langle\,,\rangle$ for the standard scalar product on $\mathbb{R}^3$ and set $\alpha = \cos A$, $\beta = \cos B$, $\gamma = \cos C$, $\delta = \cos D$ and $\alpha' = \sin A$, $\beta' = \sin B$, $\gamma' = \sin C$, $\delta' = \sin D$. Then the constraints are

$$\langle a,b\rangle = \alpha \qquad \langle b,c\rangle = \beta \qquad \langle c,d\rangle = \gamma \qquad \langle d,a\rangle = \delta$$
$$\langle a,a\rangle = 1 \qquad \langle b,b\rangle = 1 \qquad \langle c,c\rangle = 1 \qquad \langle d,d\rangle = 1$$

Since we are only interested in the relative motion of the joint axes, we can choose $a = (1,0,0)$, $b = (\alpha,\alpha',0)$. With these choices three of the above constraints are automatic and we are left with five equations defining a real affine variety which we can then complexify and projectivise to obtain the **linkage variety** $\mathfrak{R}$ in $P\mathbb{C}^6$. Explicitly, if we set $c = (x_1, x_2, x_3)$, $d = (y_1, y_2, y_3)$ and take w to be the homogenising parameter, then $\mathfrak{R}$ is defined by

$$\left.\begin{array}{c} y_1 = \delta w : \alpha x_1 + \alpha' x_2 = \beta w : x_1y_1 + x_2y_2 + x_3y_3 = \gamma w^2 \\ x_1^2 + x_2^2 + x_3^2 = w^2 \;:\; y_1^2 + y_2^2 + y_3^2 = w^2 \end{array}\right\} \quad (2.1)$$

The Jacobian matrix for this set of equations is

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -\delta \\ \alpha & \alpha' & 0 & 0 & 0 & 0 & -\beta \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & -2\gamma w \\ 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 & -2w \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 & -2w \end{bmatrix}$$

In discussing the spherical four-bar, it is necessary to distinguish the **antipodal** case when one (or more) of the angles A,B,C,D equals π. A very special case is the **antipodal rhombus** when $A = B = C = D = \pi$ and $\mathfrak{R}$ comprises two complex conjugate 2-planes intersecting in a single real point. In [Gibson&Selig] it is shown that, if we exclude this case, <u>$\mathfrak{R}$ is indeed a curve of degree eight</u>.

In the non-antipodal case $\mathfrak{R}$ meets the hyperplane $w = 0$ in two pairs of complex conjugate points

$$P = (-\alpha', \alpha, -\mathbf{i}, 0, 0, 0, 0) \;:\; \overline{P} = (-\alpha', \alpha, \mathbf{i}, 0, 0, 0, 0)$$
$$Q = (0, 0, 0, 0, 1, -\mathbf{i}, 0) \;:\; \overline{Q} = (0, 0, 0, 0, 1, \mathbf{i}, 0)$$

It is clear that the Jacobian matrix $\mathfrak{J}$ evaluated at these points has non-maximal rank. Hence all four points are singular on $\mathfrak{R}$. Since $\mathfrak{R}$ has degree eight, we may apply Bézout's Theorem to deduce that the hyperplane $w = 0$ meets $\mathfrak{R}$ with intersection multiplicity 2 at each point. Thus the multiplicity of P, $\bar{P}$, Q, $\bar{Q}$, equals the intersection multiplicity i.e. they are double points and, moreover, no tangent to $\mathfrak{R}$ lies in $w = 0$.

However, in the antipodal case the picture changes and $\mathfrak{R}$ intersects $w = 0$ in skew complex conjugate lines L (through P, Q) and $\bar{L}$ (through $\bar{P}, \bar{Q}$). We will assume that we are in the non-antipodal case. Indeed, in the antipodal case it follows from the discussion in [Gibson&Selig] that the real linkage variety $\mathfrak{R}$ is the union of two real conics and the topology is thereby determined.

Finite singular points occur when the Jacobian matrix with $w \neq 0$ has non-maximal rank. In the non-antipodal case the condition is that the spherical quadrilateral "collapses". In such a configuration the singular points satisfy $x_3 = 0$, $y_3 = 0$. A little further work yields a condition on the design parameters, namely

$$A \pm B \pm C \pm D \equiv 0 \pmod{2\pi}.$$

This condition is analogous to the one for the planar four-bar for a finite singularity to occur and so it seems appropriate to call these equations the **Grashof equalities**.

We shall assume henceforth, that we are in the constructible case, i.e. that the equations (2.1) have at least one real solution and that none of the Grashof equalities are satisfied. Thus <u>$\mathcal{R}$ has no singular points off the hyperplane</u> $w = 0$ and in view of the results of [Gibson&Selig] is <u>irreducible</u>.

For the spherical four-bar mechanism M we can realise the philosophy of §1.4 by taking M' to be one of the moving links. The configuration space for the first link is a complex projective space $P\mathbb{C}^3$ with co-ordinates x_1, x_2, x_3, w and the linkage variety is the non-singular conic X_1' given by $x_1^2 + x_2^2 + x_3^2 = w^2$ and $\alpha x_1 + \alpha' x_2 = \beta' w$. The natural projection $P\mathbb{C}^6 \to P\mathbb{C}^3$ given by $(x_1, x_2, x_3, y_1, y_2, y_3, w) \to (x_1, x_2, x_3, w)$ then restricts to a finite mapping $\pi_1 : \mathcal{R} \to X_1'$. Likewise, the configuration space for the third link is $P\mathbb{C}^3$ with co-ordinates y_1, y_2, y_3, w and the linkage variety is the non-singular conic X_3' given by $y_1^2 + y_2^2 + y_3^2 = w^2$ and $y_1 = \delta w$. Here again the natural projection $P\mathbb{C}^6 \to P\mathbb{C}^3$ defined by $\pi_3 : (x_1, x_2, x_3, y_1, y_2, y_3, w) \mapsto (y_1, y_2, y_3, w)$ restricts to a finite mapping $\pi_3 : \mathcal{R} \to X_3'$. Note that since $\mathcal{R}$ is irreducible, no component can map to a point. The centres of π_1, π_3, intersected with the 4-space defined by the linear equations in (2.1), are precisely the lines joining $Q, \bar{Q}$ and $P, \bar{P}$ respectively. The degrees d_1, d_3 of π_1, π_3 can be computed via the Projection Formula

$$\deg \mathcal{R} = d_1 \cdot \deg X_1' + v_1 + \bar{v}_1 \quad \text{and} \quad \deg \mathcal{R} = d_3 \cdot \deg X_3' + v_3 + \bar{v}_3$$

where $v_1, \bar{v}_1$ (respectively $v_3, \bar{v}_3$) are the intersection multiplicities at $P, \bar{P}$ (respectively $Q, \bar{Q}$) of $\mathcal{R}$ with a generic hyperplane containing the centre. Since $P, \bar{P}$, $Q, \bar{Q}$ are all double points on $\mathfrak{R}$ these multiplicities must be ≥ 2, the minimum values of 2 being attained for the hyperplane $w = 0$ (by Bézout's Theorem (A3)). As $\mathfrak{R}$, X_1' and X_3' have degrees 8, 2 and 2 respectively, we conclude that $\underline{d_1 = d_3 = 2}$.

We can now apply the description of §1.4 to the <u>real</u> mappings $\pi_1 : \mathfrak{R} \to X_1'$ and $\pi_3 : \mathfrak{R} \to X_3'$ the first step being to determine the number of real critical points. As in the case of the planar four-bar, critical points occur when a tangent line to $\mathfrak{R}$ meets the centre of projection. Note that the centre of projection for π_1 (resp. π_3) is given by $x_1 = x_2 = x_3 = 0$ (resp. $y_1 = y_2 = y_3 = 0$) and is the line joining $Q, \bar{Q}$ (resp. $P, \bar{P}$). The tangent line is given as the kernel of the Jacobian matrix $\mathfrak{J}$ of equations (2.1). Thus the condition for critical points is that the matrix, obtained by abutting $\mathfrak{J}$ with the Jacobian matrix of the centre of projection, has non-maximal rank. For projection π_1 (resp. π_3) this is equivalent to the matrix $\mathfrak{J}_1$ (resp. $\mathfrak{J}_3$), obtained from $\mathfrak{J}$ by deleting rows 1, 2, 3 and 7 (resp. 4, 5, 6 and 7), having non-maximal rank. These matrices are

$$\mathfrak{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 2y_1 & 2y_2 & 2y_3 \end{bmatrix} \qquad \mathfrak{J}_3 = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & \alpha' & 0 \\ y_1 & y_2 & y_3 \\ 2x_1 & 2x_2 & 2x_3 \\ 0 & 0 & 0 \end{bmatrix}$$

For π_1 the condition for critical points may be expressed algebraically as $x_2y_3 - x_3y_2 = 0$; and likewise for π_3 the condition is $\alpha(x_2y_3 - x_3y_2) + \alpha'(x_3y_1 - x_1y_3) = 0$. The condition may easily be described mechanically: for π_1 (resp. π_3) the condition is that the vectors a, c, d (resp. b, c, d) should be co-planar. It is now a matter of determining the number of <u>real</u> solutions of the equations (2.1) which satisfy one of these conditions. This provides another computation.

There are no real points at infinity so we are only interested in solutions with $w \neq 0$. We will do the calculation for projection π_1 and leave the calculation for π_3 to the reader. Firstly, we may use equation $y_1 = \delta$ to eliminate one of the variables in equations (2.1). Thus two of the conditions are

$$\left.\begin{aligned} x_2y_2 + x_3y_3 &= \gamma - \delta x_1 \\ x_2y_3 - x_3y_2 &= 0 \end{aligned}\right\}$$

Using Cramer's Rule we get

$$y_2 = \frac{(\gamma - \delta x_1)x_2}{x_2^2 + x_3^2} \quad : \quad y_3 = \frac{(\gamma - \delta x_1)x_3}{x_2^2 + x_3^2}$$

Note that $x_2^2 + x_3^2 = 0$ if and only if $a = c$. Substituting for y_2, y_3 in $y_2^2 + y_3^2 = 1 - \delta^2$ and using the identity $x_1^2 + x_2^2 + x_3^2 = 1$, we get

$$(\gamma - \delta x_1)^2 = (1 - \delta^2)(x_2^2 + x_3^2) = (1 - \delta^2)(1 - x_1^2).$$

It immediately follows that $x_1 = \gamma\delta + \varepsilon_1\gamma'\delta'$ (where $\varepsilon_1 = \pm 1$), a real number, and since $\alpha' \neq 0$ in the non-antipodal case, we get $x_2 = (\beta - \alpha x_1)/\alpha'$. To find the corresponding x_3 value we substitute for x_2 in $x_1^2 + x_2^2 + x_3^2 = 1$, which gives

$$x_3^2 = \frac{\alpha'^2\beta'^2 - (x_1 - \alpha\beta)^2}{\alpha'^2}.$$

We obtain real solutions for x_3 if and only if the numerator is positive i.e. if and only if $[x_1-\alpha\beta+\alpha'\beta']\cdot[x_1-\alpha\beta-\alpha'\beta']\leq 0$. But, $x_1=\gamma\delta+\varepsilon_1\gamma'\delta'=\cos(C-\varepsilon_1 D)$ and $\alpha\beta+\varepsilon_2\alpha'\beta'=\cos(A-\varepsilon_2 B)$, where $\varepsilon_2=\pm 1$. Hence, the condition is

$$[\cos(A+B) - \cos(C-\varepsilon_1 D)] \cdot [\cos(A-B) - \cos(C-\varepsilon_2 D)] \leq 0.$$

Alternatively, we may use the additive formulae for sines and write the condition as

$$S_1=\sin\frac{(A+B+C+\varepsilon_1 D)}{2}\cdot\sin\frac{(A+B-C-\varepsilon_1 D)}{2}\cdot\sin\frac{(A-B+C+\varepsilon_2 D)}{2}\cdot\sin\frac{(A-B-C-\varepsilon_2 D)}{2}$$

Thus the result is that for the projection π_1 we obtain two real critical points for each choice of sign for which $S_1 < 0$. For the projection π_3 the result is that we get two real critical points for each choice of sign for which $S_3 < 0$, where

$$S_3=\sin\frac{(A+\varepsilon_1 B+C+D)}{2}\cdot\sin\frac{(A-\varepsilon_1 B-C+D)}{2}\cdot\sin\frac{(A+\varepsilon_2 B+C-D)}{2}\cdot\sin\frac{(A-\varepsilon_2 B-C-D)}{2}$$

On this basis one obtains a finite set of inequalities involving the expressions $A\pm B\pm C\pm D$ which determine whether the projection in question has 0, 2 or 4 real critical points.

<u>Case 1: $0 < A, B, C, D < \frac{1}{2}\pi$</u>

The neatest formulation of the result seems to be when A, B, C, D all lie in the range $[0, \frac{1}{2}\pi]$ and cosine is a strictly decreasing function (i.e. $\theta > \tau \Rightarrow \cos\theta < \cos\tau$). Write the angles in <u>decreasing</u> order of magnitude as A', B', C', D' and set

$\mathcal{X} = A' - B' - C' + D'$. Then, by comparing the angles, it is easily checked that: <u>π_1 (resp. π_3) has four real critical points if and only if</u> $\mathcal{X} < 0$ <u>and the smallest angle is</u> C or D <u>(respectively</u> B or C); <u>has two real critical points if and only if</u> $\mathcal{X} > 0$; <u>and has no real critical points if and only if</u> $\mathcal{X} < 0$ <u>and the smallest angle is</u> A or B <u>(respectively</u> A or D).

We can now determine the real topology of $\mathcal{R}$ and answer the crank/rocker question, just as we did for the planar four-bar. When $\mathcal{X} > 0$ both projections π_1, π_3 have exactly two real critical points, so that the first and third links are rockers and $\mathcal{R}$ has just one connected component. Suppose $\mathcal{X} < 0$. If the smallest angle is one of B, C, D one of the projections π_1, π_3 has four critical points and the results of §1.4 tell us that $\mathcal{R}$ has two connected components. In fact, when the smallest angle is B or D the corresponding link cranks and the other rocks. But when C is the smallest angle the links corresponding to B and C both rock. It remains to discuss the case when A is the smallest angle, so that both projections π_1, π_3 have no real critical points and the links corresponding to B and D both crank. As in the case of the planar four-bar, the theory of §1.4 does not determine the topology of $\mathcal{R}$ in this situation and we have to argue further. The key observation (again) is that the curve $\mathcal{R}$ possesses a natural involution, given this time by reversing the signs of x_3, y_3. The effect of this involution on the determinants $\mathcal{D}_1 = \det(a,c,d)$ and $\mathcal{D}_3 = \det(b,c,d)$ is just to reverse their signs, so that $\mathcal{D}_1$, $\mathcal{D}_3$ assume both positive and negative values on $\mathcal{R}$. If $\mathcal{R}$ had just one connected component, both functions would vanish somewhere on $\mathcal{R}$ and hence both projections π_1, π_3 would have a critical point; thus giving a contradiction. We may conclude, therefore, that in the case when A is the smallest angle $\mathfrak{R}$ has two connected components. These results show that, when the angles A, B, C, D lie in the range $[0,\frac{1}{2}\pi]$, one has a perfect analogy with the planar four-bar, namely that the linkage curve $\mathfrak{R}$ has one/two connected components if and only if $\mathcal{X} > 0/\mathcal{X} < 0$.

It is worth remarking that this comparison with the four-bar has been noted by Gilmartin and Duffy [Gilmartin&Duffy] who calculated, not without some difficultly, the limiting positions for the spherical four-bar for this case using trigonometry. However they were unable to tie up this observation with the topology of the motion of the mechanism. Following Hain's classification for the four-bar, they labeled the four types according to the crank/rocker analysis. We append their table of types with a column, indicating the number of critical points for the projections and a column showing the corresponding topology.

Hain Type	criteria for determining type	* comp	* critical points of π_1	of π_3
CR_1,	$\mathcal{X} < 0$, B shortest	2	0	4
CR_2,	$\mathcal{X} < 0$, D shortest	2	4	0
DL,	$\mathcal{X} < 0$, A shortest	2	0	4
DR,	$\mathcal{X} < 0$, C shortest	2	4	0
R_{ii},	$\mathcal{X} > 0$, A longest	1	2	2
R_{io},	$\mathcal{X} > 0$, B longest	1	2	2
R_{oi},	$\mathcal{X} > 0$, C longest	1	2	2
R_{oo},	$\mathcal{X} > 0$, D longest	1	2	2

Case 1 $0 < A, B, C, D < \frac{1}{2}\pi$

Case 2: $\frac{1}{2}\pi < A, B, C, D < \pi$

The authors of [Gilmartin&Duffy] fail to see that analogous reasoning shows that, when A, B, C, D all lie in the range $[\frac{1}{2}\pi, \pi]$, on which cosine is strictly increasing, the linkage curve $\mathfrak{R}$ has one (resp. two) connected components if and only if $\mathfrak{X} < 0$ (resp. $\mathfrak{X} > 0$). The above crank/rocker analysis applies, provided we replace "shortest" by "longest" (and vice versa) and reverse the inequalities. We summarise the result in the form of a table

Hain Type	criteria for determining type	# comp	# critical points of π_1	of π_3
CR_1,	$\mathfrak{X} > 0$, B longest	2	0	4
CR_2,	$\mathfrak{X} > 0$, D longest	2	4	0
DL,	$\mathfrak{X} > 0$, A longest	2	0	4
DR,	$\mathfrak{X} > 0$, C longest	2	4	0
R_{ii},	$\mathfrak{X} < 0$, A shortest	1	2	2
R_{io},	$\mathfrak{X} < 0$, B shortest	1	2	2
R_{oi},	$\mathfrak{X} < 0$, C shortest	1	2	2
R_{oo},	$\mathfrak{X} < 0$, D shortest	1	2	2

Case 2 $\frac{1}{2}\pi < A, B, C, D < \pi$

Case 3: in general.

In the general case there is no neat formulation of the condition, thus it is necessary to calculate the signs of S_1 and S_3 directly from the formulae given above. For any numerical example, this is totally straightforward and determines the number of topological components. The eight crank/rocker types still provide a sensible classification of the linkage variety.

CHAPTER 3. GEARED FIVE-BAR MOTION WITH GEAR RATIO -1.

Introduction

A particularly attractive family of mechanisms is provided by the geared five-bar mechanisms with arbitrary gear ratio. Here the basic structure of the mechanism is that of a pentagon. In general, a pentagon has two degrees of freedom i.e. its linkage variety is given by (n-2) equations in n-space and is therefore a surface. To obtain a mechanism with just one degree of freedom we must provide one extra constraint. Theoretically, we could make a constraint from any polynomial condition expressing a relation between the bars and the result would most likely be interesting geometry; but in general such constraints would be mechanically impossible to achieve. There is, however, one method of providing a constraint which is often exploited in mechanisms: that of the 'gearing' two bars.

Consider then the kinematic chain consisting of five rigid bodies smoothly jointed to form a pentagon. Label the bars from 1 to 5 and let bar 5 be fixed. Consider the fixed plane as the complex numbers $\mathbb{C}$ and let bars 1 to 5 lie on vectors $z_1 = e^{i\theta_1}, \ldots, z_4 = e^{i\theta_4}$ and $z_5 = -1$ repectively (where $\mathbf{i} = \sqrt{-1}$). Then we say that bars i and j are 'geared' together when we have imposed the constraint that, whenever bar i moves through an arc of length α, then bar j moves through an arc of length $\pm k\alpha$ where k is a fixed real number called the **gear ratio** and the (fixed) choice of sign represents the direction in which bar i moves in relation to bar j. Thus $\theta_j = \pm k\theta_i + \gamma$ where γ is some (fixed) **phase angle** determined by the initial positions of the bars. Hence, we get $z_j = Az_i^k$ for the positive choice of sign and $z_j = A\bar{z}_i^k$ for the negative choice of sign, where $A = e^{i\gamma}$. Mechanically, rational gear ratios $k = \frac{p}{q}$ (where p and q are positive integers) are obtained, for instance, when bar i is attached to a gear with p teeth and bar j is attached to a gear with q teeth. This provides us with three types of mechanisms as showed in Fig. 3.1, depending upon, whether we

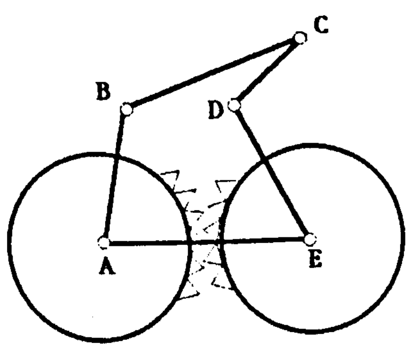


Fig. 3.1

fix bar AE, bar AB or DE, or bar BC or CD. Mechanically, the case when we choose the sign + presents an extra difficulty, since we need to reverse the direction of bar 4 with respect to bar 1. This is overcome by inserting an extra coupling device (an 'intermediate idler') in between the two gears.

Non-trivial coupler curves are obtained by all three mechanisms when the coupler point is rigidly attached to either of the **coupler bars**, i.e those which are non-adjacent to the fixed bar. The coupler point, which lies on both coupler bars (i.e. the hinge), is generally treated as a special case. It may come as a surprise to the reader that the coupler curve for the mechanism with bar AE fixed and gear ratio $\alpha = +1$, whose coupler point is the hinge, i.e bars 1 and 4 crank with the same speed at a fixed phase angle apart and in the same direction, is a four-bar coupler curve. Observe that, if one constructs the parallelograms CDED' and ABCB' as showed in fig 3.2, then we may form a new mechanism AB'CD'E such that bars B'C and CD' are 'geared together'. Indeed, just as bars 1 and 4 lie on vectors a fixed phase angle apart, bars B'C and CD' lie on vectors which are a fixed angle apart throughout the motion. Thus the mechanism is clearly seen to be a four-bar AB'D'E with coupler triangle B'CD. Conversely, we may obtain any four-bar in this manner. This result was observed in [Blokh].

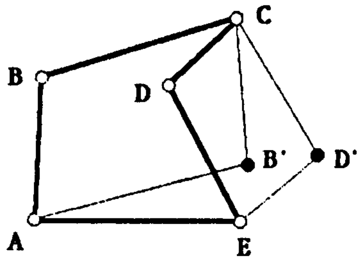


Fig. 3.2

It is a point of theoretical interest that, provided the gear ratio is a rational number, the corresponding motion is governed by polynomial equations defining a linkage variety which may be studied using algebraic-geometric techniques. In particular, we will be interested in the mechanism showed in figure 3.1 with bar AE fixed and when the gear ratio is equal to -1.

We saw in Chapter 1 that in the case of the planar four-bar the linkage curve lies in a 4-space and one is projecting from a line. The key observation is that both the linkage curve and the line lie on a Segre quartic surface, the line being one of precisely sixteen lines on that surface and meeting just five others. Under the projection these five lines map to five points in the ambient plane, determining a unique conic. In the generic case these five points are precisely the singular points of the coupler curve, the conic being the familiar circle of singular foci.

Given the above context, the discussion in [Freudenstein&Primrose] of geared five-bar motion assumes particular interest, in view of the authors' comment that here again the coupler curve, when the coupler point is a hinge, has just five singular points lying on a conic. The approach taken by the authors in [Freudenstein&Primrose] is to write down the equation for the coupler curve (for the hinge case) and to evaluate the lowest order terms of the polynomial: this determines a conic on which all singular points of the coupler must lie. The nett result of our approach is that one is able to say rather more about the geometry of coupler curves than appeared in [Freudenstein&Primrose]. We may summarise these as follows:

(1) In Chapter 1 it was showed that the Grashof equalities correspond precisely to the natural geometric condition that the linkage curve has a singularity off the hyperplane at infinity. The latter condition makes perfect sense for any planar mechanism and so provides a sensible general definition of the term Grashof equation. In particular, we can adopt this point of view for the geared five-bar and phrase the Grashof equations in terms of the design parameters. With this definition we then prove that for almost all design parameters (in a sense which we shall make precise) the Grashof equations do not hold.

(2) We are able to give a complete list of the possible reductions of the linkage curve, and hence the coupler curve, into

irreducible algebraic components. In particular, in the generic case the linkage curve is irreducible of degree 8 and genus 3, meeting the hyperplane at infinity in two ordinary double points and four other (non-singular) points.

(3) The geometry of the real linkage curve is of particular interest. In the generic case, just described, we can determine the number of topological components. The key idea here is to apply the philosophy indicated in §1.4 of studying the natural projections from the linkage curve to the circles representing the motions of the first and fourth links. That reduces the problem to one of counting the number of real intersections of a given circle with two explicitly given conics - an entirely practical procedure which could be carried out, for instance, by graphical means. The number in question is 1, 2, 3 or 4 and conforms with the bound given by Harnack's Theorem (A9). In this connection it is well worth pointing out that the Harnack bound is not always the best possible: indeed we shall see that for the Watt six-bar it fails to give a useful restriction.

(4) We can obtain the coupler curves from the linkage curve by linear projection from a line and thereby deduce their properties. As a consequence we can take the coupler point to be any point rigidly attached to a moving link, whereas the analysis in [Freudenstein&Primrose] is valid only when the coupler point is a hinge. An interesting facet of the present study is that we can explain an intriguing analogy between the planar four-bar and the geared five-bar with coupler point a hinge. Freudenstein and Primrose observe that in both examples the (complex) coupler

curves have, in general, just five singular points determining a unique conic; a circle for the planar four-bar and a hyperbola for the geared five-bar. This phenomenon is explained by the fact that in both cases the singular points correspond in a natural way to the five lines meeting a given line on a Segre quartic surface.

In §3.1 we set up the basic geometry of the linkage variety. We show that the linkage variety $\mathfrak{R}$ is isomorphic to an intersection of three quadric hypersurfaces in $P\mathbb{C}^4$ and is therefore of degree eight. The hyperplane at infinity meets $\mathfrak{R}$ in six points; four of these are always simple points of $\mathfrak{R}$, whilst the other two are ordinary double points, provided a certain condition does not hold. Further, we show that $\mathfrak{R}$ has no finite singular points in general. In §3.2 we give a complete list of the reductions of $\mathfrak{R}$, a completely new result, and in §3.3 we determine the number of connected components of the real linkage variety in terms of the design parameters. Finally, in §3.4 we discuss the coupler curves. As indicated above, we show that there is an analogy between the geared five-bar with coupler point a hinge and the planar four-bar. We describe the reductions of the coupler curves, in the general case, in detail.

§3.1 The Complex Linkage Curve

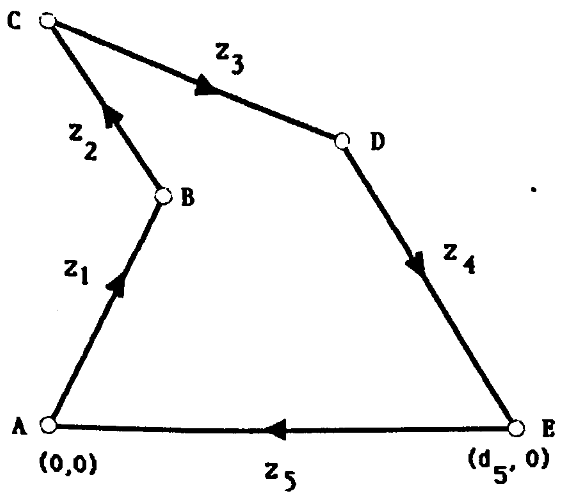


Fig. 3.3

We label the five bars of the mechanism as 1, 2, 3, 4, 5 with the last one fixed (as showed in Fig. 3.3). They have positive lengths d_1, d_2, d_3, d_4, d_5, and their directions are given by unit complex numbers z_1, z_2, z_3, z_4, z_5. It is no restriction to suppose that $z_5=-1$: indeed, we can suppose that the ends of the fixed bar are at the points $(0,0)$ and $(d_5,0)$. We remind the reader that we are going to consider the case when the fixed bar is AE as showed in Fig. 3.1, and when the gear ratio is -1. Therefore, the constraint imposed by the gearing of bars 1 and 4 is $z_4 = A\bar{z}_1$. The equations, which govern the motion, have the form

$$\left.\begin{aligned} & d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = d_5 \\ & z_4 = A\bar{z}_1 \\ & |z_1|^2 = |z_2|^2 = |z_3|^2 = |z_4|^2 = 1 \end{aligned}\right\} \quad (3.1)$$

where $A = A_1 + iA_2$ (with A_1, A_2 real) is a unit complex number. The first equation expresses the closure of the pentagon and the second equation expresses the fact that links 1 and 4 are geared together; with gear ratio -1 and phase angle γ, where $e^{i\gamma} = A$. There is a point of theoretical interest here, namely that the equations (3.1) are not in the shape required to associate naturally a Darboux variety in the general setting of [Gibson&Newstead] as explained in §1.2. This can be remedied by writing them in the shape

$$\left.\begin{aligned} &d_1z_1+d_2z_2+d_3z_3+d_4z_4=d_5 \\ &z_1z_4=A \\ &|z_1|^2=|z_2|^2=|z_3|^2=|z_4|^2=1 \end{aligned}\right\} \quad (3.1')$$

where now the first two equations are polynomials in z_1, z_2, z_3, z_4, z_5. In either case we can write $z_k=x_k+iy_k$ with x_k, y_k real and equate real and imaginary parts to obtain an algebraic variety in $\mathbb{R}^8$ defined by eight equations: as for the planar and spherical four-bars this can be complexified and projectivised (with w the homogenising parameter) to obtain complex projective varieties in $P\mathbb{C}^8$. We get

$$\left.\begin{aligned} &d_1x_1+d_2x_2+d_3x_3+d_4x_4=d_5w \\ &d_1y_1+d_2y_2+d_3y_3+d_4y_4=0 \\ &x_4=A_1x_1+A_2y_1 : \ y_4=A_2x_1-A_1y_1 \\ &x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2=x_4^2+y_4^2=w^2 \end{aligned}\right\} \quad (3.2)$$

corresponding to (3.1) and

$$\left.\begin{aligned} &d_1x_1+d_2x_2+d_3x_3+d_4x_4=d_5w \\ &d_1y_1+d_2y_2+d_3y_3+d_4y_4=0 \\ &x_1x_4-y_1y_4=A_1w^2 : \ x_1y_4+x_4y_1=A_2w^2 \\ &x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2=x_4^2+y_4^2=w^2 \end{aligned}\right\} \quad (3.2')$$

corresponding to (3.1').

The systems of equations (3.2), (3.2') define linkage varieties $\mathcal{R}$, $\mathcal{S}$ respectively. We need to be clear about the relation between these varieties. Certainly $\mathfrak{R} \subset \mathcal{S}$. For, if a point in $P\mathbb{C}^8$ satisfies (3.2), we have

$$x_4 + iy_4 = A(x_1 - iy_1) \ : \ x_4 - iy_4 = \bar{A}(x_1 + iy_1)$$

yielding

$$(x_1 + iy_1)(x_4 + iy_4) = A(x_1^2 + y_1^2) = Aw^2$$
$$(x_1 - iy_1)(x_4 - iy_4) = \bar{A}(x_1^2 + y_1^2) = \bar{A}w^2$$

and by adding and subtracting these relations we see that our point satisfies (3.2'). Conversely, note that provided $w \neq 0$ (and hence all of $x_1 \pm iy_1, x_4 \pm iy_4$ are $\neq 0$) we can reverse these steps. Thus $\mathfrak{R} - W = \mathcal{S} - W$, where W denotes the hyperplane at infinity defined by $w=0$. Thus $\mathfrak{R}$, $\mathcal{S}$ have the same finite points, but their intersections with W can, and do, differ. For our purposes it is sufficient to observe that the residual varieties, obtained from $\mathfrak{R}$, $\mathcal{S}$ by deleting irreducible components in W, are identical.

We need to study $\mathfrak{R}$ in more detail. Note first that the four linear equations in (3.2) are linearly independent, so $\mathfrak{R}$ is isomorphic to a variety in $P\mathbb{C}^4$. Moreover, the equation $x_4^2 + y_4^2 = w^2$ follows immediately from $x_1^2 + y_1^2 = w^2$ and two of the linear equations. Thus $\mathfrak{R}$ is defined by

$$\left.\begin{array}{l} d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = d_5w \\ d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 = 0 \\ x_4 = A_1x_1 + A_2y_1 \ : \ y_4 = A_2x_1 - A_1y_1 \\ x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = w^2 \end{array}\right\} \quad (3.3)$$

and is isomorphic to the intersection of a net of quadrics in $P\mathbb{C}^4$.

The intersections of $\mathfrak{R}$ with W are easily obtained. Setting $w=0$ in (3.3), we get $y_k = \pm i x_k$ for $k=1,2,3$ yielding eight possibilities. Two pairs of these give complex conjugate points

$$+++/-++ \qquad P_1 = (0,0,d_3,id_3,-d_2,-id_2,0,0,0)$$
$$+--/--- \qquad \overline{P_1} = (0,0,d_3,-id_3,-d_2,id_2,0,0,0)$$

The remaining four sign combinations give four distinct points

$$+-+ \qquad Q_1 = (1,\ i,\ -\tfrac{d_4}{d_2}A,\ \ i\tfrac{d_4}{d_2}A,\ \ -\tfrac{d_1}{d_3},\ \ -i\tfrac{d_1}{d_3},\ \ A,\ \ -iA,\ \ 0)$$
$$-+- \qquad \overline{Q}_1 = (1,\ -i,\ -\tfrac{d_4}{d_2}\overline{A},\ \ -i\tfrac{d_4}{d_2}\overline{A},\ \ -\tfrac{d_1}{d_3},\ \ i\tfrac{d_1}{d_3},\ \ \overline{A},\ \ i\overline{A},\ \ 0)$$
$$++- \qquad Q_2 = (1,\ i,\ -\tfrac{d_1}{d_2},\ \ -i\tfrac{d_1}{d_2},\ \ -\tfrac{d_4}{d_3}A,\ \ i\tfrac{d_4}{d_3}A,\ \ A,\ \ -iA,\ \ 0)$$
$$--+ \qquad \overline{Q}_2 = (1,\ -i,\ -\tfrac{d_1}{d_2},\ \ i\tfrac{d_1}{d_2},\ \ -\tfrac{d_4}{d_3}\overline{A},\ \ -i\tfrac{d_4}{d_3}\overline{A},\ \ \overline{A},\ \ i\overline{A},\ \ 0)$$

In particular, $\mathfrak{R}$ intersects W in a finite set, so it has no irreducible components of dimension ≥ 2 and must therefore be a curve. By Bézout's Theorem (A3) $\mathfrak{R}$ has degree 8, thus it meets W in eight points, counted with multiplicities. These multiplicities are soon determined. The singular points of $\mathfrak{R}$ are those points on $\mathfrak{R}$ where the Jacobian matrix of the equations (3.3) has rank <7. The Jacobian $\mathfrak{J}$ is

$$\mathfrak{J} = \begin{bmatrix} d_1 & 0 & d_2 & 0 & d_3 & 0 & d_4 & 0 & -d_5 \\ 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 & d_4 & 0 \\ A_1 & A_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ A_2 & -A_1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2x_1 & 2y_1 & 0 & 0 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 2x_2 & 2y_2 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 2x_3 & 2y_3 & 0 & 0 & -2w \end{bmatrix}$$

From this one sees immediately that P_1 and $\overline{P}_1$ are singular points of $\mathfrak{R}$ since in each case the fifth row is zero and therefore $\mathfrak{J}$ has non-maximal rank. Hence, P_1 and $\overline{P}_1$ have multiplicity ≥ 2 on $\mathfrak{R}$. It follows immediately from the above that P_1 and $\overline{P}_1$ have multiplicity equal to 2 on $\mathfrak{R}$, i.e. they are double points, whilst $Q_1, \overline{Q_1}, Q_2, \overline{Q_2}$ must have multiplicity equal to 1 on $\mathfrak{R}$, i.e. they are non-singular. Moreover, all the branches of $\mathfrak{R}$, centered at points on W, meet W transversally.

A natural question arising at this juncture is to ask for the analytic type of the singular points at $P_1, \overline{P_1}$. (We shall need to know this to compute the genus of the linkage curve.) To do this we render the equations (3.3) affine, translate the singularity to the origin in $\mathbb{C}^8$ and then smoothly eliminate all but two of the variables, via the Implicit Function Theorem, to obtain a plane curve in $\mathbb{C}^2$ with a singular point at the origin. We shall do this calculation explicitly for P_1 and deduce the result for $\overline{P}_1$ by complex conjugation. Make equations (3.3) affine by setting $x_2 = 1$. Then translate P_1 to the origin by making the affine change of co-ordinates $y_2 \mapsto y_2 + i$, $x_3 \mapsto x_3 - \frac{d_2}{d_3}$, $y_3 \mapsto y_3 - i\frac{d_2}{d_3}$ and leaving the other coordinates fixed. We obtain the equations

$$\left.\begin{aligned}
&d_1x_1 + d_3x_3 + d_4x_4 = d_5w \\
&d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 = 0 \\
&x_4 = A_1x_1 + A_2y_1 : \ y_4 = A_2x_1 - A_1y_1 \\
&x_1^2 + y_1^2 = y_2^2 + 2iy_2 = x_3^2 + y_3^2 - 2\frac{d_2}{d_3}(x_3 + iy_3) = w^2
\end{aligned}\right\}$$

Using the four linear equations, we may eliminate four of the variables, for instance, x_3, y_3, x_4, y_4, to give the equations

$$x_1^2+y_1^2 = y_2^2+2iy_2 = w^2$$

$$d_3^2w^2=\left(d_5w-(d_1+d_4A_1)x_1-d_4A_2y_1\right)^2+\left((d_1-d_4A_1)y_1+d_2y_2+d_4A_2x_1\right)^2$$
$$-2d_2(d_5w-(d_1+d_4A_1)x_1-d_4A_2y_1)+i2d_2((d_1-d_4A_1)y_1+d_2y_2+d_4A_2x_1)$$

Since the derivative of the second equation with respect to y_2 is non-zero at the origin, we may apply the Implicit Function Theorem which allows us to approximate y_2 as a Taylor series in w in a neighbourhood of the origin. Let $y_2 = aw + bw^2 + cw^3 + dw^4 + \cdots$. Then substituting into the second equation and evaluating the coefficients, we find that $a=0$, $b=-\frac{1}{2}i$, $c=0$, $d=\frac{1}{8}$. Thus we may eliminate y_2 in terms of w in the third equation. The new form of the third equation has non-zero derivatives at the origin with respect to w. Thus by the Implicit Function Theorem we may approximate w by a Taylor Series in x_1, y_1, in a neighbourhood of the origin. Let $w=ax_1+by_1+O(2)$. Then substituting into the third equation and evaluating the coefficients we find that $a=(d_1+d_4A)/d_5$, $b=i(d_1-d_4A)/d_5$ and using this series we may eliminate w. In this way we obtain a curve of the form $0=ax_1^2+2bx_1y_1+cy_1^2+O(3)$ where

$$a=d_5^2-(d_1+d_4A)^2 \;:\; b=-i(d_1^2-d_4^2A)^2 \;:\; c=d_5^2+(d_1-d_4A)^2$$

We <u>fail to obtain an ordinary double point</u> if and only if the discriminant of the quadratic part vanishes, i.e. <u>if and only if $A=1$ and $d_5^2=4d_1d_4$</u>. We shall refer to this as the **inverse condition**, since it is precisely the condition that there exists a position of the mechanism for which the points d_1z_1, d_4z_4 in the complex plane are inverse with respect to the circle of radius $\frac{1}{2}d_5$ centred at the origin. Thus, provided the inverse condition does not hold, the points P_1, $\overline{P_1}$ are ordinary double points on $\mathcal{R}$.

The next step is to ask when $\mathcal{R}$ can possess finite singularities, by which we mean singular points off W. Since $\mathcal{R}$ has no irreducible components of dimension ≥ 2, the conditions for this are that the Jacobian matrix $\mathcal{J}$ (with $w = 1$), given earlier, should have non-maximal rank. Clearly, these conditions are polynomial, both in the variables x_k, y_k and the design parameters d_1, d_2, d_3, d_4, d_5, A_1, A_2. Explicitly, we may make row and column operations on $\mathcal{J}$ to derive necessary and sufficient conditions for a finite point of $\mathcal{R}$ to be singular, namely (3.3) and the following

$$x_2y_3 = x_3y_2 \ : \ x_1[(d_1 - d_4A_1)y_3 + d_4A_2x_3] - y_1[(d_1 + d_4A_1)x_3 + d_4A_2y_3] = 0.$$

The first of these two equations together with the quadratic equations of (3.3) give $x_2 = \varepsilon x_3$, $y_2 = \varepsilon y_3$ so in the real case we may give the mechanical interpretation that bars 2 and 3 are parallel. This gives nine equations in $P\mathbb{C}^8$ and we now apply the following theorem from Elimination Theory.

Theorem [Hartshorne]
Let $f_1,\dots,f_r$ be homogeneous polynomials in $x_0,\dots,x_n$, having indeterminate coefficients a_{ij}. Then there is a set $g_1,\dots,g_t$ of polynomials in the a_{ij}, with integer coefficients, which are homogeneous in the coefficients of each f_i separately, with the following property: for any field k, and for any set of special

values of the $a_{ij} \epsilon k$ a necessary and sufficient condition for the f_i to have a common zero different from $(0,...,0)$ is that the a_{ij} are a common zero of the polynomials g_i.

The theorem applied to these nine equations yields that the condition for the existence of a finite singularity is a polynomial one on the design parameters. Moreover, this polynomial is not identically zero; for there certainly exist choices of design parameters for which $\mathfrak{R}$ has no finite singularities. I claim, for example, that the linkage variety of the mechanism with design parameters $d_1 = d_2 = d_3 = d_4 = d_5 = 1$ and $A = -1$ has no finite singular point.

Proof of claim: Any finite singular point satisfies

$$x_2 = \varepsilon_1 x_3 \;:\; y_2 = \varepsilon_2 y_3 \;:\; x_1[2y_3] = 0$$
$$x_1 + x_2 + x_3 + x_4 = w \;:\; y_1 + y_2 + y_3 + y_4 = 0$$
$$x_4 = -x_1 \;:\; y_4 = y_1 \;:\; x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = w^2$$

Thus by the third equation we have *either*

1) $y_3 = 0$. Then the second equation implies $y_2 = 0$. Substitute $y_2 = y_3 = 0$ into the fifth equation. Then the fifth and seventh equations yield $y_1 = y_4 = 0$. Thus the last three equations imply $\pm x_1 = \pm x_2 = \pm x_3 = \pm x_4 = w$ and a contradiction follows by substituting $x_k = \pm w$ into first equation *or*

2) $x_1 = 0$. Then $x_4 = 0$ by the sixth equation. Hence $y_1 = y_4 = \varepsilon w$ $(\varepsilon = \pm 1)$ by the seventh and eighth equation. Substituting for x_1, x_4, y_1, y_4 in equations four and five gives (i) $x_2 = w - x_3$ and (ii)

$y_2 = -2\varepsilon w - y_3$. Then, since $w \neq 0$, the first equation together with (i) yields $x_2 = x_3 = \frac{1}{2}w$ and the second equation together with (ii) yields $y_2 = y_3 = \varepsilon w$. Thus, substituting for x_3, y_3 in terms of w into the last equation, gives a contradiction.

It follows immediately from the claim that **generically**, by which we mean for almost all design parameters, in the sense of Lebesgue measure, the linkage curve has no finite singularity. Rather more intuitively, we can always avoid finite singularities on $\mathfrak{R}$ by arbitrarily small deformations of the design parameters.

The polynomial condition on the design parameters just derived should be called the **Grashof equality**, since it is the exact analogue of this concept for the planar four-bar in §1.1. One could certainly write down this condition quite explicitly, but it does not appear to adopt any particularly neat form. In any case, from a practical point of view it would hardly be worthwhile, since for any numerically given choice of design parameters one could perform the eliminations by hand to decide whether the choice is generic or not.

Let us finish this section by mentioning a point of theoretical interest concerning $\mathfrak{R}$. In §1.2 we outlined a general construction presented in [Gibson&Newstead], whereby to certain rather special complex projective varieties, a birationally isomorphic variety $\mathfrak{D}$, called the associated **Darboux variety**, could be assigned. We recall that the particular interest of this construction for the planar four-bar was, that the corresponding variety $\mathfrak{D}$ was precisely the Darboux curve studied originally in

[Darboux]. The birational isomorphism turned out to be an isomorphism, thus elucidating some remarkable connections between the residual linkage variety and $\mathfrak{D}$. It is interesting to point out here that this general construction applies equally well to the curve $\mathfrak{R}$ to produce a corresponding Darboux curve $\mathfrak{D}$ for the geared five-bar mechanism. To be perfectly explicit, $\mathfrak{D}$ is the curve in $P\mathbb{C}^4$ defined by the equations

$$\left.\begin{aligned} &d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = d_5w \\ &\frac{d_1}{z_1} + \frac{d_2}{z_2} + \frac{d_3}{z_3} + \frac{d_4}{z_4} = \frac{d_5}{w} \\ &z_1z_4 = Aw^2 \end{aligned}\right\} \quad (3.4)$$

and the residual curve $\mathfrak{D}'$ is obtained from $\mathfrak{D}$ by deleting any irreducible algebraic components which lie in the co-ordinate hyperplanes $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, $z_4 = 0$ or $w = 0$. We begin our analysis by finding the components of $\mathfrak{D}$ in the hyperplanes $z_k = 0$.

Suppose that $w = 0$. Then either $z_1 = 0$ or $z_4 = 0$. If $z_1 = 0$ then we get the line L_1 given by the equations $z_1 = w = d_2z_2 + d_3z_3 + d_4z_4 = 0$ which meets the hyperplane $z_2 = 0$ in the point $Q_1 = (0,0,-d_4,d_3,0)$ and the hyperplane $z_3 = 0$ in the point $Q_2 = (0,-d_4,0,d_2,0)$. If $z_4 = 0$ then we get the line L_4 given by the equations $z_4 = w = d_1z_1 + d_2z_2 + d_3z_3 = 0$ which meets the hyperplane $z_2 = 0$ in the point $Q_4 = (-d_3,0,d_1,0,0)$ and the hyperplane $z_3 = 0$ in the point $Q_5 = (-d_2,-d_1,0,0,0)$. The lines L_1 and L_4 meet in the point $Q_3 = (0,-d_3,d_2,0,0)$.

Now suppose that $w \neq 0$ so that $z_1 \neq 0$ and $z_4 \neq 0$. Then

$z_2 = 0$ if and only if $z_3 = 0$. Thus when $z_2 = z_3 = 0$ we get two further points P_1 and P_2 whose co-ordinates we will not write down.

Summarising, $\mathcal{D}$ meets $z_1 = 0$, $z_4 = 0$ in distinct intersecting lines L_1, L_4, meets each of $z_2 = 0$, $z_3 = 0$ in four points and meets $w = 0$ in the union of L_1, L_4. Note that $\mathcal{D}$ is always a curve; since any component of dimension ≥ 2 would meet each hyperplane $z_k = 0$ in infinitely many points, contradicting the analysis above.

Applying Bézout's Theorem (A3) to equations (3.4), we find that $\mathcal{D}$ is a curve of degree 8 in $P\mathbb{C}^4$. Removing the two components L_1 and L_4 from $\mathcal{D}$, we get the residual curve $\mathcal{D}'$ which has degree six. To obtain the defining equations of $\mathcal{D}'$ we multiply the second equation of (3.4) through by $z_1z_2z_3z_4w$ to remove the denominators giving the following set of equations

$$\left.\begin{array}{l} d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = d_5w \\ [d_1z_2z_3z_4 + d_2z_1z_3z_4 + d_3z_1z_2z_4 + d_4z_1z_2z_3]w = d_5z_1z_2z_3z_4 \\ z_1z_4 = Aw^2 \end{array}\right\} \quad (3.4')$$

Using the last equation of (3.4') to substitute for z_1z_4 in the right hand side of the second equation and eliminating w from both sides, yields the defining equations of $\mathcal{D}'$

$$\left.\begin{array}{l} d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = d_5w \\ [d_1z_2z_3z_4 + d_2z_1z_3z_4 + d_3z_1z_2z_4 + d_4z_1z_2z_3] = d_5Az_2z_3w \\ z_1z_4 = Aw^2 \end{array}\right\}$$

Thus we have showed that $\mathcal{D}'$ is isomorphic to the set theoretic complete intersection of a point-cone and a cubic surface in $P\mathbb{C}^3$.

We do not intend to study $\mathcal{D}$ any further in this chapter. The fact is that $\mathcal{R}$ is actually easier to study directly via the natural projections to be introduced in the next section (following the technique of §1.3). However, we feel it is worth pointing out that the Darboux construction can be applied to geared five-bar motion. The reader should also note that, unlike the planar four-bar, the varieties $\mathcal{R}$ and $\mathcal{D}'$ are birationally isomorphic but <u>not</u> isomorphic. This follows from the fact that $\mathcal{R}$ always has one more singular point than $\mathcal{D}'$: since $\mathcal{R}$ always has two singular points $P, \overline{P}$ in $w=0$, whilst $\mathcal{D}'$ has only the singular point Q_3 - we do not give the details here.

§3.2 Reductions of the Linkage Curve

An important mathematical question, which seems to have been invariably neglected in the mechanisms literature, is that of the irreducibility of the algebraic curves which arise naturally in the subject. The general situation is that any algebraic curve $\mathcal{R}$ (assumed to be in a complex projective space) is a union of finitely many irreducible algebraic curves $\mathcal{R}_1,...,\mathcal{R}_s$ called the **irreducible components** of $\mathcal{R}$: moreover, the degree of $\mathcal{R}$ is the sum of the degrees of $\mathcal{R}_1,...,\mathcal{R}_s$. In the case of a linkage curve $\mathcal{R}$ the key question is how this reduction into components depends on the design parameters.

For the planar four-bar the approach to this problem adopted in [Gibson&Newstead] is via the Genus Formula (A11) for an algebraic curve. However, in the present example this approach suffers two drawbacks, namely that the analytic types of the finite singularities of $\mathfrak{R}$ are not easily determined and that it is not clear, a priori, whether one may have the complication of repeated components. What we intend to do instead is to adopt the philosophy we initiated in §1.3 of studying the natural projections from the linkage curve $\mathfrak{R}$ down to the conics representing the motion of the four links. In this section we shall use this technique to understand the reductions of the complex curve $\mathfrak{R}$ and in §3.3 we shall use the main result of §1.4 to discuss the topology of the real curve $\mathfrak{R}$, at least in the generic case.

Formally we proceed as follows. For $j=1,2,3$ let $\mathfrak{C}_j$ be the conic in the complex projective plane with co-ordinates x_j, y_j, w defined by $x_j^2+y_j^2=w^2$. Since the four linear equations in (3.3) are independent, $\mathfrak{R}$ is isomorphic to a complete intersection of three quadrics in $P\mathbb{C}^4$. If we eliminate variables x_4,y_4, then we may take the equations to be

$$x_1^2+y_1^2=w^2 \ : \ x_2^2+y_2^2=w^2 \ : \ x_3^2+y_3^2=w^2$$

$$(d_1+d_4A_1)x_1+d_2x_2+d_4A_2y_1+d_3x_3=d_5w$$
$$(d_1-d_4A_1)y_1+d_2y_2+d_4A_2x_1+d_3y_3=0$$

We then have natural projections $\pi_j : P\mathbb{C}^4 \rightarrow P\mathbb{C}^2$ $(j=1,\ 2,\ 3)$ defined by $(x_1,y_1,x_2,y_2,x_3,y_3,w) \mapsto (x_j,y_j,w)$ which restrict to projections $\pi_j : \mathcal{R} \to \mathcal{C}_j$. For each $j = 1, 2$ or 3 the centre of π_j is the line L_j given by $x_j = y_j = w = 0$. Observe that L_1 is the only one of these lines which meets $\mathcal{R}$; indeed, it is precisely the line joining P_1, $\overline{P_1}$. Moreover, L_1, L_2 and L_1, L_3 are pairs of skew lines, whilst L_2, L_3 are skew lines if and only if $d_1 \neq d_4$, as a minor computation will verify.

Now let X be an irreducible component of the linkage curve $\mathcal{R}$. Then each π_j maps X *either* to a single point *or* to the whole of the conic $\mathcal{C}_j$. The first thing to be clear about is when the former possibility can arise. We fix x_j, y_j, w and ask when (3.3) admits infinitely many solutions. Since $\mathcal{R}$ has only finitely many points on $w = 0$, we can start by setting $w = 1$. For each j we obtain, after straightforward eliminations, two conics in $\mathbb{C}^2$, one representing the unit circle. Therefore we obtain infinitely many solutions if and only if the two conics coincide. When $j = 1$ the conics are

$$x_2^2 + y_2^2 = 1$$

$$\begin{aligned} d_2^2(x_2^2+y_2^2) &+ 2d_2y_2[(d_1-d_4A_1)y_1+d_4A_2x_1] + \\ &2d_2x_2[(d_1+d_4A_1)x_1+d_4A_2y_1-d_5] + [(d_1-d_4A_1)y_1+d_4A_2x_1]^2 + \\ &\qquad [(d_1+d_4A_1)x_1+d_4A_2y_1-d_5]^2-d_3^2 = 0. \end{aligned}$$

Both conics are circles, coinciding with the first, precisely when $\underline{d_2 = d_3}$ and $x_2 = \varepsilon x_3$, $y_2 = \varepsilon y_3$ where $\varepsilon = \pm 1$ i.e. there exists a (complex) configuration of the mechanism for which links 2, 3 are opposite and equal. Explicitly, the condition on the design parameters for a (complex) configuration is

$$A_1 = \frac{-(d_1^2 - d_4^2)^2 + (d_1^2 + d_4^2)d_5^2}{2d_1d_4d_5^2} \qquad (3.5)$$

and the configuration is real if and only if $|A_1| \le 1$. The physical situation is illustrated in Fig 3.4. <u>We have also established that $\pi_1 : \mathfrak{R} \to \mathcal{C}_1$ has degree 2.</u>

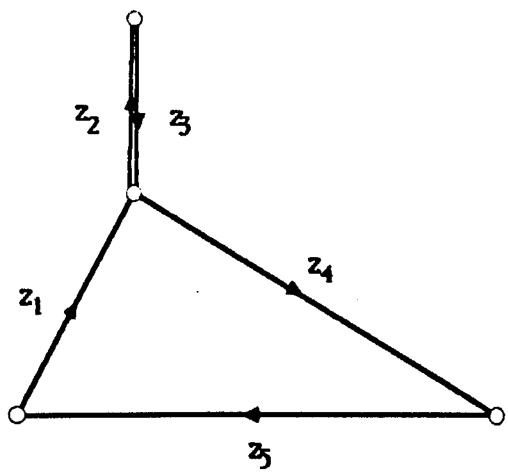


Fig. 3.4

When $j = 2, 3$ the two conics are

$$x_1^2 + y_1^2 = w^2$$

$$\begin{aligned}&[d_1^2 + d_4^2 + 2d_1d_4A_1]x_1^2 \quad + \quad 4d_1d_4A_2y_1^2 \quad + \quad [d_1^2 + d_4^2 - 2d_1d_4A_1]x_1y_1 \\ &-2[(d_1 + d_4A_1)x_1 + d_4A_2y_1][d_5 - d_jx_j] \quad + \quad 2d_jy_j[(d_1-d_4A_1)y_1 + d_4A_2x_1] \\ &= d_j^2w^2 - [d_5-d_jx_j]^2 - d_j^2y_j^2\end{aligned}$$

and the second conic is never a circle, thus no component X can map to a point and *therefore* $\pi_2 : \mathcal{R} \to \mathcal{C}_2$ *and* $\pi_3 : \mathcal{R} \to \mathcal{C}_3$ *have degree 4*.

The above considerations alone yield the useful fact that *any component* X *of* $\mathcal{R}$ *has even degree*. Indeed, since for $j = 2,3$ the centre L_j of π_j does not meet $\mathcal{R}$, the Projection Formula (Theorem A11) yields $\deg X = d_j \cdot \deg \mathcal{C}_j = 2d_j$ where d_j is the degree of π_j restricted to X. On this basis we see that the possible reductions of $\mathcal{R}$, given by the corresponding partitions of the degree, are among 8, $6+2$, $4+4$, $4+2+2$, $2+2+2+2$; allowing here the possibility of a repeated component.

The next step in our analysis is the proposition that *any conic component* X *of* $\mathcal{R}$ *must pass through both of the singular points* $P_1, \overline{P_1}$ *in* W. As a preliminary, note that if $\pi : X \to Y$ is a map of degree d between curves X, Y where Y is non-singular, then any point on Y has at most d pre-images (see §A7). Thus by the above analysis the restrictions of π_2, π_3 to X have degree 1 and hence any point in either image has exactly one pre-image in X. Now let $I = (1,i,0)$, $J = (1,-i,0)$ be the circular points at infinity in $P\mathbb{C}^2$ and note that under π_2 (resp. π_3) the points $P_1, \overline{Q_1}, Q_2$ (resp. $P_1, Q_1, \overline{Q_2}$) map to I, whilst $\overline{P_1}, Q_1, \overline{Q_2}$ (resp. $\overline{P_1}, \overline{Q_1}, Q_2$) map to J. It follows immediately that X passes through $P_1, \overline{P_1}$ or $Q_1, \overline{Q_1}$ or $Q_2, \overline{Q_2}$. We can exclude the last two cases by observing that the tangent lines to $\mathcal{R}$ at the non-singular points Q_1, Q_2, and similarly at $\overline{Q_1}$, $\overline{Q_2}$, are skew. Indeed, a straightforward computation shows that the tangent lines are defined respectively by linear relations of the form

$$x_1 + iy_1 = 0 : x_2 - iy_2 = 0 : \Sigma\alpha_j x_j + \Sigma\beta_j y_j + \gamma w = 0$$
$$x_1 - iy_1 = 0 : x_2 + iy_2 = 0 : \Sigma\bar{\alpha}_j x_j + \Sigma\bar{\beta}_j y_j + \bar{\gamma} w = 0$$

where the α_j, β_j, γ are complex scalars with $\gamma \neq 0$. The observation is immediate and establishes the proposition. Explicitly, the coefficients in the above equations are $\alpha_1 = d_1 + d_4 A$, $\alpha_2 = d_2$, $\beta_1 = i(d_1 - d_4 A)$, $\beta_2 = -id_2$, $\gamma = -d_5$ for Q_1, and $\alpha_1 = d_1 + d_4\bar{A}$, $\alpha_2 = d_2$, $\beta_1 = i(-d_1 + d_4\bar{A})$, $\beta_2 = -id_2$, $\gamma = -d_5$ for Q_2.

A first consequence of the proposition is that the possible reduction $2+2+2+2$ cannot occur, since then $\mathfrak{R}$ would fail to meet W at $Q_1, \overline{Q_1}, Q_2, \overline{Q_2}$. A second consequence is that $\mathfrak{R}$ has a conic component if and only if that component is projected by π_1 to a point (again by the Projection Formula), thus by the above $\mathfrak{R}$ has a conic component if and only if $d_2 = d_3$ and $x_2 = \varepsilon x_3$, $y_2 = \varepsilon y_3$ where $\varepsilon = \pm 1$ i.e. the mechanism has a (complex) configuration in which links 2, 3 are equal and opposite. In particular, identities $d_2 = d_3$ and (3.5) must be satisfied, thus such a reduction is exceptional. The distinction between the reductions $6+2$ and $4+2+2$ is easily described. For such a configuration the equations (3.3) reduce to

$$\left.\begin{array}{lcl} d_1x_1 + d_4x_4 = d_5 & : & x_4 = A_1x_1 + A_2y_1 \\ d_1y_1 + d_4y_4 = 0 & : & y_4 = A_2x_1 - A_1y_1 \\ x_2 = \varepsilon x_3 & : & y_2 = \varepsilon y_3 \\ x_1^2 + y_1^2 = 1 \ : & x_2^2 + y_2^2 = 1 & : \ x_3^2 + y_3^2 = 1 \end{array}\right\} \quad (3.6)$$

Provided $d_1 \neq d_4$ the first four linear equations in (3.6) give a unique solution for x_1, y_1, x_4, y_4 so we obtain a unique configuration giving a reduction $6+2$.

However, when $d_1 = d_4$ we obtain two (possibly coincident) solutions, leading to two possible configurations and a reduction $4+2+2$. The condition for a configuration given by equation (3.5) (necessarily real) becomes $A = 1$. Of course in this case the configurations are mirror images of each other in the fixed link; the physical situation is illustrated in Fig 3.5.

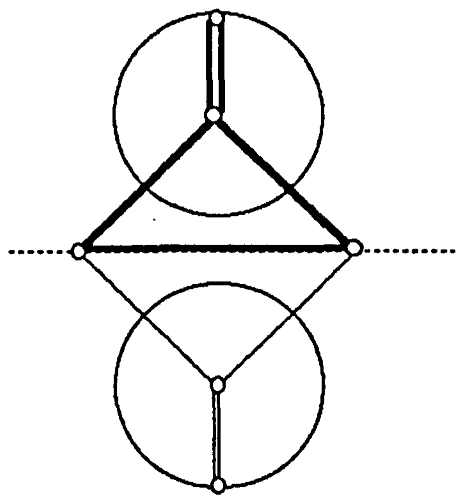


Fig. 3.5

Thus equations (3.3) with $d_1 = d_4$, $d_2 = d_3$, $A = 1$ may be seen to give two subvarieties

$$\left.\begin{aligned} x_1^2 + y_1^2 &= w^2 \\ x_2^2 + y_2^2 &= w^2 \\ d_5 x_1 - 2d_1 w &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} x_1^2 + y_1^2 &= w^2 \\ x_2^2 + y_2^2 &= w^2 \\ d_5 w - 2d_1 x_1 - 2d_2 x_2 &= 0 \end{aligned}\right\}$$

The first subvariety represents the two conics and the second subvariety represents the quartic component. It follows that, in general, the quartic component is isomorphic to a non-singular intersection of two quadrics in 3-space, hence an elliptic curve. Exceptionally, the quartic acquires a singular point and becomes rational. To find the condition for this to occur we need to determine when the Jacobian matrix has non-maximal rank. The matrix is

$$\begin{bmatrix} 2x_1 & 2y_1 & 0 & 0 & -2w \\ 0 & 0 & 2x_2 & 2y_2 & -2w \\ -2d_1 & 0 & -2d_2 & 0 & d_5 \end{bmatrix}$$

It is a straightforward exercise to show that the matrix has non-maximal rank if and only if $d_5 = 2\cdot|\varepsilon_1 d_1 + \varepsilon_2 d_2|$ (where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$) for which there is a singular point of the form $(\varepsilon_1, 0, \varepsilon_2, 0, 1)$. The quartic is irreducible, thus the singular point can only be an ordinary double point or a cusp. To determine which of these cases may occur, we shall make a local co-ordinate calculation. Make the above equations affine by setting $w = 1$ and translate the singular point to the origin by making the affine change of co-ordinates $x_1 \mapsto x_1 + \varepsilon_1$, $x_2 \mapsto x_2 + \varepsilon_2$ and leaving the remaining co-ordinates fixed. The resulting set of equations are

$$x_1^2 + y_1^2 + 2\varepsilon_1 x_1 = 0 \ : \ x_2^2 + y_2^2 + 2\varepsilon_2 x_2 = 0 \ : \ d_1 x_1 - d_2 x_2 = 0$$

We may use the third equation to eliminate x_2 from the second equation giving an equation in x_1 and y_2 for which the derivative with respect to x_1 is non-zero. Thus by the Implicit

Function Theorem we may approximate x_1 in a neighbourhood of the origin by a Taylor Series in y_2. Let $x_1 = ay_2 + by_2^2 + \cdots$. Then substituting for x_1 in the second equation and evaluating coefficients, yields that $a = 0$ and $b = \frac{\varepsilon_2}{2}\frac{d_2}{d_1}$. Substituting for x_1 in the first equation, shows that the quartic is isomorphic, near the origin, to a plane curve of the form $y_1^2 + \frac{\varepsilon_1 d_2}{\varepsilon_2 d_1} y_2^2 + O(3)$. In particular, we see that the curve has a double point at the origin with distinct tangents i.e. an ordinary double point.

Note the very special case when the two solutions of x_1, y_1, x_4, y_4 in (3.6) coincide and we obtain a reduction $4+2+2$ with a repeated conic: under the above hypotheses this happens precisely when $A = 1, d_1 = d_4, d_2 = d_3, d_5 = 2d_1$ (thus the inverse condition is satisfied). The physical situation is illustrated in Fig 3.6. In this case the reader may readily check that the quartic is elliptic in general, but exceptionally, when $d_2 = d_5$, the quartic acquires a singular point.

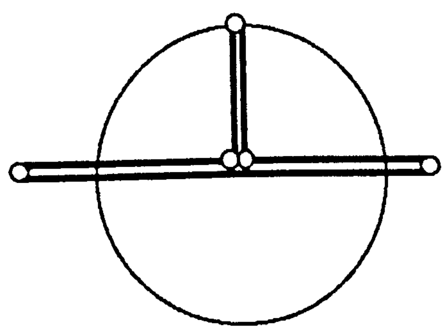


Fig. 3.6

The reduction $4+4$ will be discussed in more detail in §3.4. Recall first that π_1 maps Q_1, Q_2 to I and $\overline{Q_1}, \overline{Q_2}$ to J: further, π_2 (respectively π_3) maps $P_1, \overline{Q_1}, Q_2$ (respectively $P_1, Q_1, \overline{Q_2}$) to I and $\overline{P_1}, Q_1, \overline{Q_2}$ (respectively $\overline{P_1}, \overline{Q_1}, Q_2$) to J.

It follows immediately that the two quartic components are <u>real</u>, one passing through P_1, $\overline{P_1}$, Q_1, $\overline{Q_1}$ and the other through P_1, $\overline{P_1}$, Q_2, $\overline{Q_2}$. By the Projection Formula the restriction of π_1 to either component has degree 1; thus both components are mapped by π_1 birationally onto a circle, implying that they are rational curves. Further, since by the theory of projections (Theorem A12) multiplicity can only increase under a projection of degree 1, we see that both components have no finite singular points and must therefore be non-singular curves.

We can elicit further useful information about the above reductions via the Genus Formula (A8) for a (possibly reducible) algebraic curve. That formula states that

$$p_a(\mathcal{R}) = \Sigma_k p_a(\mathcal{R}_k) + \Sigma\delta_p - (r-1)$$

where p_a denotes the arithmetic genus, δ_p is the δ-invariant of a singular point P of $\mathcal{R}$ and $\mathcal{R}_1,\dots,\mathcal{R}_r$ are the normalisations of the irreducible components $\mathcal{R}_1,\dots,\mathcal{R}_r$ of $\mathcal{R}$. It follows from Theorem A5 that $p_a(\mathcal{R}) = 5$ for a complete intersection of three quadrics in 4-space. Further, the δ-invariants at $P_1,\overline{P_1}$ equal 1, since these points have been shown to be ordinary double points of $\mathcal{R}$. Thus we can re-write the Genus Formula as

$$r+2 = \Sigma p_a(\mathcal{R}_k) + \Sigma^*\delta_P \qquad (3.7)$$

where the * indicates that we sum the δ-invariant over the <u>finite</u> singular points of $\mathcal{R}$. I <u>claim</u> that in the <u>generic case</u>, i.e. when the Grashof equation is not satisfied, <u>$\mathcal{R}$ is an irreducible octic of</u>

geometric genus 3.

Proof of Claim: In the generic case $\Sigma^*\delta p = 0$. To show that $\mathfrak{R}$ is irreducible we shall consider each possible reduction in turn and derive a contradiction.

In the 4+4 case equations (3.7) yield $4 = p_a(\mathfrak{R}_1) + p_a(\mathfrak{R}_2)$ with $\mathfrak{R}_1$, $\mathfrak{R}_2$ rational non-singular quartics: that is impossible, since the arithmetic genus of such a curve is zero. In the 2 + 2 + 4 case, equations (3.7) yield $5 = p_a(\tilde{\mathfrak{R}}_1) + p_a(\tilde{\mathfrak{R}}_2) + p_a(\tilde{\mathfrak{R}}_3)$ where $\mathfrak{R}_1$, $\mathfrak{R}_2$ are non-singular conics and $\mathfrak{R}_3$ is a quartic. Since conics are rational, this reduces to $5 = p_a(\tilde{\mathfrak{R}}_3) \leq p_a(\mathfrak{R}_3)$ giving a contradiction, since the arithmetic genus of a quartic must be ≤ 3.

Finally, in the 2+6 case, (3.7) gives $4 = p_a(\tilde{\mathfrak{R}}_1) + p_a(\tilde{\mathfrak{R}}_2)$ where $\mathfrak{R}_1$ is a conic and $\mathfrak{R}_2$ is a sextic. Hence, $4 = p_a(\tilde{\mathfrak{R}}_2)$. To obtain a contradiction we argue in the following way. Recall first, that the conic $\mathfrak{R}_1$ must pass through P_1, $\overline{P}_1$ and hence $\mathfrak{R}_2$ must pass through all six points P_1, $\overline{P}_1$, Q_1, $\overline{Q}_1$, Q_2, $\overline{Q}_2$, else P_1, $\overline{P}_1$ fail to be singular on $\mathfrak{R}$. It follows easily that $\mathfrak{R}_2$ is non-degenerate, i.e. not contained in any 3-space: indeed any such 3-space would have to contain the six points just listed and therefore would coincide with the hyperplane $w=0$. This situation is an impossibility since $\mathfrak{R}$ only meets $w=0$ in finitely many points. The non-degeneracy of $\mathfrak{R}_2$ allows us to apply the Castelnuovo inequality

Castelnuovo Inequality [Griffiths]

The greatest possible genus of an irreducible non-degenerate curve C of degree d in $P\mathbb{C}^n$ is $\frac{1}{2}m(m-1) + m\varepsilon$, where m is the integer part of $(d-1)/(n-1)$ and $\varepsilon = (d-1)-m(n-1)$.

In the situation at hand $d = 6$, $n = 4$, $m = 1$, $\varepsilon = 2$ and we may deduce from the inequality that $p_a(\mathcal{R}_2) \leq 2$. Finally, we observe that by Bézout's Theorem (A3) $\mathcal{R}_2$ must be non-singular at all of the points P_1, $\bar{P}_1$, Q_1, $\bar{Q}_1$, Q_2, $\bar{Q}_2$ and, therefore, $\mathcal{R}_2$ is a non-singular curve with $\tilde{\mathcal{R}}_2 = \mathcal{R}_2$: that provides the requisite contradiction and the claim is proved.

The above analysis leaves open the question, whether the $4+4$ reduction actually occurs. An explicit example is obtained by choosing $d_1 = 2$, $d_2 = 4$, $d_3 = 1$, $d_4 = 2$, $d_5 = 3$, $A = -1$. Since $d_2 \neq d_3$ we cannot be in either of the cases $2+6$ or $2+2+4$. The key point in this example is that there are at least four finite singular points: $(\pm 1, 0, 1, 0, -1, 0, \mp 1, 0, 1)$ and $(0, \pm 5, 3, \mp 4, 3, \mp 4, 0, \pm 5, 5)$ thus we have necessarily $\Sigma^* \delta_p \geq 4$. However, when $\mathcal{R}$ is an irreducible octic, the genus formula (3.7) yields $\Sigma^* \delta_p \leq 3$: we must therefore be in the $4+4$ case.

§3.3 The Real Linkage Curve

Throughout this section we shall assume that we are in the generic constructible case, so that the complex projective curve $\mathcal{R}$ is an irreducible octic of geometric genus 3 whose only singular points are ordinary double points P_1, $\bar{P}_1$ and which possesses at least one real finite point. In this situation the real affine curve $\mathcal{R}$ is compact, non-singular and non-empty thus diffeomorphic to a finite disjoint union of circles. By Harnack's Theorem (A9) the number of topological components is ≤ 4. Our objective in this section is to show how, in principle, one can determine this number in terms of the design parameters via a technique introduced in §1.4.

In brief, the technique is as follows. We saw in §3.2 that the projection $\pi_1 : \mathcal{R} \to \mathcal{C}_1$ has degree 2 and that $\mathcal{R}$, $\mathcal{C}_1$ are irreducible non-singular curves. Let us now consider the corresponding real curves (without changing notation) and write $\mathcal{T}_1,\dots,\mathcal{T}_n$ for the topological components of $\mathcal{R}$. Then according to the main result of §1.4 there are just three possible qualitative pictures.

(I) There is just one component $\mathcal{T}_1$ mapped immersively onto $\mathcal{C}_1$ as a double cover.

(II) There are just two components $\mathcal{T}_1$, $\mathcal{T}_2$ each mapped diffeomorphically onto $\mathcal{C}_1$.

(III) There are n components $\mathcal{T}_1,\dots,\mathcal{T}_n$ mapping onto disjoint arcs $A_1,\dots,A_n$ of $\mathcal{C}_1$ with exactly 2n critical values, namely the end-points of these arcs.

Case (I) is the double crank, whilst case (II) corresponds to two single cranks. It is important to note that, although the absence of branch points for π_1 tells us that we must be in one of these cases, it does not tell us which one. By contrast the presence of branch points tells us that we must be in case (III) and their number completely determines the topology of $\mathcal{R}$. Case (III) corresponds to the engineering concept of a rocker.

Thus to apply the above result we must compute the number of real critical points of π_1. Critical points occur when the tangent line to the (complex) curve $\mathfrak{R}$ meets the centre of π_1, i.e. these projective subspaces fail to span a 7-space. Thus we have critical points, whenever the 7×6 matrix $\mathfrak{J}'$, obtained from the Jacobian matrix of the equations (3.3) by deleting the columns corresponding to the variables x_1, y_1, w, has rank <6.

$$\mathfrak{J}' = \begin{bmatrix} d_2 & 0 & d_3 & 0 & d_4 & 0 \\ 0 & d_2 & 0 & d_3 & 0 & d_4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2x_2 & 2y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_3 & 2y_3 & 0 & 0 \end{bmatrix}$$

The reader will readily check that $\mathfrak{J}'$ has non-maximal rank precisely when the vectors (x_2,y_2) and (x_3,y_3) are linearly dependent. Using the equations (3.3) we see that this is exactly the condition that $x_3 = \varepsilon x_2$, $y_3 = \varepsilon y_2$ with $\varepsilon = \pm 1$: thus in the real case the mechanical interpretation of a critical point is that links 2 and 3 are parallel. Substituting $x_3 = \varepsilon x_2$, $y_3 = \varepsilon y_2$ in (3.3), we obtain

$$\left.\begin{aligned} &(d_1 + d_4A_1)x_1 + d_4A_2y_1 + (d_2 + \varepsilon d_3)x_2 = d_5w \\ &d_4A_2x_1 + (d_1 - d_4A_1)y_1 + (d_2 + \varepsilon d_3)y_2 = 0 \\ &x_1^2 + y_1^2 = w^2 \quad : \quad x_2^2 + y_2^2 = w^2 \end{aligned}\right\} \qquad (3.8)$$

In the projective 4-space with homogeneous co-ordinates x_1, y_1, x_2, y_2, w the linear equations define a 2-plane in which the quadratic equations define two distinct conics intersecting in ≤ 4

(complex) points. Eliminating x_2, y_2, we have

$$x_1^2 + y_1^2 = w^2$$

$$2d_1d_4A_1(x_1^2 - y_1^2) + 4d_1d_4A_2x_1y_1 - 2d_5w[(d_1 + d_4A_1)x_1 + d_4A_2y_1] + [d_1^2 + d_4^2 + d_5^2 - (d_2 + d_3\varepsilon)^2]w^2 = 0 \qquad \text{where } \varepsilon = \pm 1.$$

Since there are two values of ε, we obtain in all ≤ 8 critical points and hence by the above theory ≤ 4 topological components agreeing with the estimate given by Harnack's Theorem (A9). More precisely, we need to calculate the number of real critical points, given by the number of real intersections of the conics. That is easily determined. If we rationally parameterise one conic by a parameter t and substitute in the equation of the other, we obtain a real quartic in t. Explicitly, we may parameterise

$x_1^2 + y_1^2 = w^2$ by $x = \dfrac{-2tw}{1+t^2}$, $\quad y = \dfrac{(1-t^2)w}{1+t^2}$ and substitute into the second equation to obtain a quartic $at^4 + bt^3 + ct^2 + dt + e = 0$ with

$$\begin{aligned}
a &= d_1^2 + d_4^2 - 2d_1d_4A_1 + 2d_4d_5A_2 + d_5^2 - (d_2 + d_3\varepsilon)^2 \\
b &= -4d_1d_4A_2 + 4d_5(d_1 + d_4A_1) \\
c &= 2(d_1^2 + d_4^2) + 12d_1d_4A_1 + 2(d_5^2 - (d_2 + d_3\varepsilon)^2) \\
d &= 4d_1d_4A_2 + 4d_5(d_1 + d_4A_1) \\
e &= d_1^2 + d_4^2 - 2d_1d_4A_1 - 2d_4d_5A_2 + d_5^2 - (d_2 + d_3\varepsilon)^2
\end{aligned}$$

Then the number of real roots of this quartic for each choice of sign is the required number of real critical points.

Suppose we obtain 2n real critical points. If $n \geq 1$ then we are in case (III) and have precisely n topological components. However, when there are no real critical points we must decide between cases (I) and (II). To this end consider the smooth function on $\mathcal{R}$ defined by $\varphi(P) = x_2y_3 - x_3y_2$, where $P = (x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4,w)$. By the above the zeros of φ are precisely the critical points of π_1. Assume that we are in case (I) so that over every point of $\mathcal{C}_1$ lie exactly two distinct points P, P' of $\mathcal{R}$. Tacitly, we use the fact here that the image $\pi_1(\mathcal{R})$ is non-singular so that the critical points of π_1 coincide with the branch points (see §A7). Geometrically, the point P' is the "reflection" of P, as indicated in Fig 3.7. The key observation is that we have $\varphi(P') = -\varphi(P)$, so φ assumes both positive and negative values. However in case (I) the real linkage curve $\mathcal{R}$ is connected, so φ would necessarily admit a zero and π_1 would have a critical point, contrary to the hypothesis. We may conclude that, when there are no critical points, we must be in case (II) when $\mathcal{R}$ has exactly two topological components.

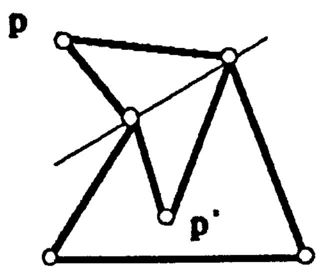


Fig. 3.7

§3.4 Projections to the Coupler Curve

In this section we shall apply the analyses of the preceding sections to understand the geometry of the family of coupler curves which are the loci of a point rigidly attached to link 2. With the notation of §3.1 we can write $P = d_1z_1 + k\cdot z_2$ where k is a fixed complex number. If we write $k = k_1 + ik_2$, with k_1, k_2

real, then we can think of P as the point in the projective plane with homogeneous co-ordinates p_1, p_2, p_3 where $p_1 = d_1x_1 - k_2y_2 + k_1x_2$, $p_2 = d_1y_1 + k_2x_2 + k_1y_2$, $p_3 = w$. These formulas define a projection $\tau_k : P\mathbb{C}^8 - L \to P\mathbb{C}^2$, where L is the centre of the projection i.e. the 5-dimensional projective subspace defined by the vanishing of p_1, p_2, p_3. We can restrict this projection to the (complex projective) linkage curve $\mathfrak{R}$ to obtain a rational mapping $\tau_k|_{\mathfrak{R}}$, the closure of whose image is an algebraic curve $\mathfrak{C}_k$ in $P\mathbb{C}^8$ which we refer to as the complex coupler curve.

Now let M be the 4-dimensional projective subspace of $P\mathbb{C}^8$ defined by the linear equations in (3.3). Then $\mathfrak{R} \subset M$ and L intersects M in a line L'. Thus it suffices to consider the projection $\tau_k : M - L' \to P\mathbb{C}^2$ given by the same forms p_1, p_2, p_3 and its restriction $\tau_k|_{\mathfrak{R}}$. It is an easy matter to check that L' fails to meet $\mathfrak{R}$ if and only if $k \neq d_2$, so that in that case $\tau_k|_{\mathfrak{R}}$ is a regular mapping.

We begin our analysis with the special case when $k = d_2$, i.e. the coupler point P is the hinge joining links 2 and 3. It is this case which was first studied in [Freudenstein]. The centre L' meets $\mathfrak{R}$ in precisely the points Q_2 and $\overline{Q}_2$. We are now in a situation very similar to that studied in the case of the planar four-bar so we shall proceed along the same lines. Write q_1, q_2, q_3 for the quadrics in M obtained by intersecting M with the quadrics in $P\mathbb{C}^8$ given by $x_1^2 + y_1^2 = w^2$, $x_2^2 + y_2^2 = w^2$, $x_3^2 + y_3^2 = w^2$ and consider the net $\lambda_1q_1 + \lambda_2q_2 + \lambda_3q_3$. The condition for a quadric in the net to pass through a given point is linear in $\lambda_1, \lambda_2, \lambda_3$, defining a pencil in the net. If, in particular, we choose a point on

L' distinct from Q_2, $\bar{Q}_2$ we see that any quadric in the pencil meets L' in three distinct points and hence contains L'. There is, therefore, a pencil of quadrics in the net containing L'. One can easily check that in the projective plane with homogeneous co-ordinates $\lambda_1, \lambda_2, \lambda_3$ the pencil is given by

$$d_2^2\lambda_1 + d_1^2\lambda_2 + d_2^2 d_4^2\lambda_3 = 0.$$

The intersection of the pencil is a Segre quartic surface $\mathcal{S}$, containing the line L'. We may write the pencil as

$$\alpha(d_4^2 q_1 - q_3) + \beta(d_2^2 d_4^2 q_2 - d_1^2 q_3).$$

The projection of $\mathcal{S}$ from L' onto a plane is well understood. Choosing the co-ordinate system x_1, y_1, w_o, w_1, w (where $w_o = d_1x_1 + d_2x_2$, $w_1 = d_1y_1 + d_2y_2$), so that the projection is onto the last three co-ordinates, any two quadrics in the pencil can be written in the form

$$x_1f_o + y_1f_1 + f_2 = 0 \quad : \quad x_1f'_o + y_1f'_1 + f'_2 = 0$$

where f_o, f_1, f'_o, f'_1 are homogeneous of degree 1 and f_2, f'_2 are homogeneous of degree 2 in w_o, w_1, w. Thus x_1, y_1 may be solved uniquely in terms of w_o, w_1, w off the conic F in the coupler plane defined by $f_of'_1 - f_1f'_o = 0$. Quite explicitly, we may write the generating quadrics in the above form with

$f_o = 2d_4(d_5A_1w - A_1w_0 - A_2w_1)$
$f_1 = 2d_4(d_5A_2w - A_2w_0 + A_1w_1)$
$f_2 = -w_2^2 - w_3^2 + 2d_5ww_0 + (d_3^2 - d_4^2 - d_5^2)w^2$
$f'_o = 2d_4(d_1^2d_5A_1w - d_1d_4w_0 - d_1^2A_1w_0 - d_1^2A_2w_1$
$f'_1 = 2d_4(d_1^2d_5A_2w - d_1d_4w_0 - d_1^2A_2w_0 + d_1^2A_1w_1$
$f'_2 = (d_4^2 - d_1^2)(w_0^2 + w_1^2) + (-d_2^2d_4^2 - d_1^2d_5^2 - d_4^2 + d_3^2)w^2 + 2d_1^2d_5w_0w.$

Thus F has the equation

$$-A_2w_o^2 + A_2w_1^2 + d_5A_2w_ow - d_5A_1w_1w + 2A_1w_1w_o = 0.$$

F is irreducible if and only if $A \neq 1$, reducing to a real line-pair ($w_1 = 0$ and $2w_0 = d_5w$) when $A = 1$. In the former case F meets the line at infinity $w = 0$ in the coupler plane in the two distinct real points $(\pm 1 + A_1, A_2, 0)$. Hence F is a hyperbola with centre $(\frac{1}{2}d_5, 0, 1)$ and asymptotes $\mp 2A_2w_0 + 2(1 \pm A_1)w_1 \pm d_5A_2w = 0$: indeed, precisely that discussed in [Freudenstein]. Points on F have *either* no pre-image *or* a line of pre-images in $\mathcal{S}$. Thus, provided there are only finitely many points on F of the latter type, we obtain an isomorphism between $\mathcal{S}$ (with a finite union of lines deleted) and the coupler plane (with F deleted).

At this stage it is interesting to determine precisely when there are only finitely many points on F common to $\mathcal{C}_k$, i.e when F and $\mathcal{C}_k$ have no common component. A sufficient condition for this is that F and $\mathcal{C}_k$ have no point of intersection on the line at infinity $w = 0$ in the coupler plane. Now F meets this line in real points and $\mathcal{C}_k$ meets it in the images under τ_k of P_1, $\bar{P}_1$, which are complex, and the images under τ_k of Q_1, $\bar{Q}_1$, which are real if and only if $A = 1$ and $d_1 = d_4$. When F is irreducible we have $A \neq 1$ and hence F, $\mathcal{C}_k$ have no common component. It remains to discuss the case when F reduces to a line-pair, in which case the lines are easily checked to be $w_1 = 0$ and $2w_0 = d_5 w$.

We claim first that the line $w_1 = 0$ cannot be a component of the coupler curve. Its pre-image under τ_k is the hyperplane H with equation $d_1 y_1 + d_2 y_2 = 0$: of the six points at infinity P_1, $\bar{P}_1$, Q_1, $\bar{Q}_1$, Q_2, $\bar{Q}_2$ on the linkage curve, H generally meets only Q_2, $\bar{Q}_2$, but exceptionally (when $A = -1$ and $d_1 = d_4$) it also meets Q_1, $\bar{Q}_1$. Now suppose $w_1 = 0$ is a component of the coupler curve, so H contains a component K of the linkage curve. Then generally, K meets the hyperplane at infinity in ≤ 2 points, so K would have to be a line or a conic: however we saw in §3.2 that $\mathcal{R}$ has no line components and that conic components have to pass through P_1, $\bar{P}_1$, so either way we have a contradiction. On the other hand, in the exceptional case, when it passes through all of Q_1, $\bar{Q}_1$, Q_2, $\bar{Q}_2$, one can check directly that H meets the linkage curve in only finitely many points and thus cannot contain a component of K. That establishes the claim that the line $w_1 = 0$ is never a component of the coupler curve.

We can discuss the line $2w_0 = d_5 w$ similarly, taking H to be the hyperplane defined by $2(d_1 x_1 + d_2 x_2) = d_5 w$. Again, the general situation is that the only points at infinity on the linkage curve, which lie on H, are $Q_2, \bar{Q}_2$ yielding a contradiction as above. Exceptionally, when $A = 1$ and $d_1 = d_4$, it also meets $Q_1, \bar{Q}_1$. In this exceptional case one easily checks that provided $d_2 \neq d_3$, H meets $\mathcal{R}$ in only finitely many points. When $d_2 = d_3$, H meets $\mathcal{R}$

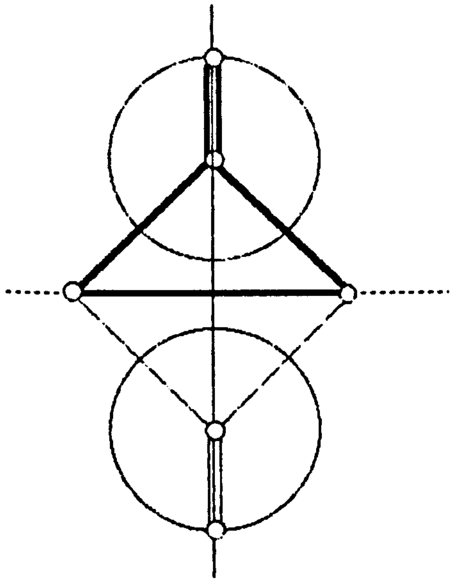


Fig. 3.8

in a quartic component. Going back to §3.2, we see that this is precisely the case when we have a reduction 2+2+4: what has happened in this case, is that although the two conic components project birationally to circles in the coupler plane, the quartic component projects with degree 2 to a line, namely the line $2w_0 = d_5 w$. One can see this very clearly in Fig 3.8.

Henceforth, we shall suppose that we are not in the 2+2+4 case, so that under the birational isomorphism between $\mathcal{S}$ and the coupler plane each component of $\mathfrak{R}$ is projected birationally onto a component of $\mathfrak{C}_k$, thus with degree 1. Since $\tau_k | \mathfrak{R}$ is not defined at Q_2, $\bar{Q}_2$ the image is not closed. The algebraic closure is obtained by adding the finite set of points which are images under τ_k of the tangents to $\mathfrak{R}$ at Q_2, $\bar{Q}_2$. As these points are non-singular on $\mathfrak{R}$ there are exactly two such points namely, $(d_5, \pm i d_5, 2)$, lying as one would expect on F. Note that P_1, $\bar{P}_1$ map respectively under τ_k to the circular points at infinity I, J; and since P_1, $\bar{P}_1$ are singular on $\mathfrak{R}$ it follows that I, J must be singular on the coupler $\mathfrak{C}_k$. In fact we can be more precise. As I, J do not lie on F the birational isomorphism between $\mathfrak{R}$ and the coupler plane will be an isomorphism close to P_1, $\bar{P}_1$ preserving the local analytic type of these singularities, so I, J will likewise be ordinary double points on $\mathfrak{C}_k$.

Principally, one is interested in the case when $\mathfrak{R}$ is irreducible, when $\mathcal{C}_k$ will be an irreducible circular sextic (by the Projection Formula). It is, however, not without interest to look at the reducible cases. In the $2+6$ case the conic component projects to a circle: the sextic component must pass through all six points at infinity on $\mathfrak{R}$ with multiplicity 1, thus by the Projection Formula projects to a circular quartic. The reduction $4+4$ is perhaps the most interesting case. Recall from §3.2 that the components are real, rational and non-singular, one passing through $P_1, \bar{P}_1, Q_1, \bar{Q}_1$; and the other through $P_1, \bar{P}_1, Q_2, \bar{Q}_2$; the former projects to a rational circular quartic, whilst the latter projects to a circle by the Projection Formula. Indeed, it is clear from the work in §3.2 that we can characterise the $4+4$ reduction by the conditions that $d_2 \neq d_3$ and that the coupler point can trace a circle. The example of a $4+4$ reduction given in §3.2 has design parameters $d_1 = d_4 = 2$, $d_2 = 4$, $d_3 = 1$, $d_5 = 3$, $A = -1$.

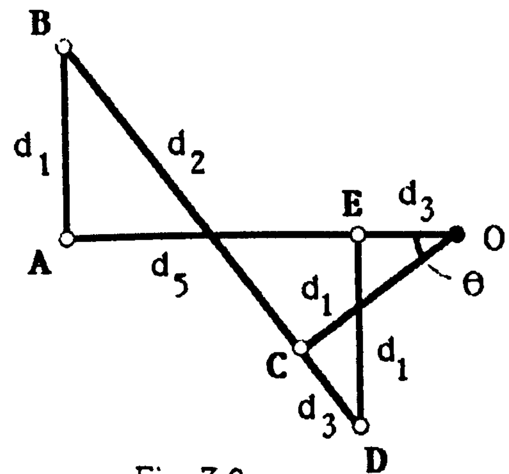


Fig. 3.9

In Fig 3.9 the reader can see how circular motion is obtained by adding links OE, OC of lengths 1, 2 respectively. Then the contraparallelograms ABCO and EOCD are similar throughout the motion (with a common angle θ) so that the coupler point C traces a circle of radius 2 with centre O.

Although the above discussion yields the broad underlying geometry of the coupler, it fails to reveal detailed information about the singularities. To make progress on this front we shall make an explicit and perfectly natural generality hypothesis, namely that the pencil of quadrics containing the line of projection L' is general i.e. contains five distinct point cones. The first point to make is that this condition holds for almost all design parameters, so that is useful. The pencil is general when its discriminant D, a binary quintic polynomial with coefficients polynomial in the design parameters, has five distinct roots. The condition for D to have coincident roots is a polynomial one in its coefficients. Thus the condition for the pencil to be non-general is a polynomial one in the design parameters. Moreover, the condition is a non-trivial one: for instance, one can check by explicit computation that the pencil is general in the case when $d_1 = \sqrt{2}$, $d_2 = \sqrt{2}$, $d_3 = 1$, $d_4 = 2$, $d_5 = 1$, $A_1 = \sqrt{0.5}$, $A_2 = \pm\sqrt{0.5}$. It follows immediately that the pencil is general for almost all design parameters. In this context it is worth remarking that when $A = 1$ one can easily check that the pencil fails to be general: certainly then, in the general case one must have $A \neq 1$ and hence F will be a hyperbola.

From now on <u>we shall assume that the pencil of quadrics containing L' is general</u>. Under that assumption we may apply the results of §1.5 that the surface $\mathcal{S}$ contains exactly sixteen lines, any one of which meets exactly five other mutually skew lines. Let $L'_1, L'_2, L'_3, L'_4, L'_5$ be the five lines on $\mathcal{S}$ meeting L' and let I_1, I_2, I_3, I_4, I_5 be the five distinct points on F which are their images under τ'_k. Each line L'_j meets the quadric q_1

either in two distinct points *or* in just one point at which it is tangent. L'_j cannot be contained in q_1, for then it would be a line component of $\mathfrak{R}$, a possibility excluded in §3.2. In the former case L'_j meets $\mathfrak{R}$ in two distinct simple points and $\mathcal{C}_k$ has two branches at I_j: and in the latter case L'_j is tangent to $\mathfrak{R}$ so $\mathcal{C}_k$ still has a singular point at L_j. Bézout's Theorem (A3) tells us that F meets $\mathcal{C}_k$ with total intersection multiplicity 12. However, we know from the above that F meets $\mathcal{C}_k$ in the five singular points I_1, I_2, I_3, I_4, I_5 and in the two closure points, so that it follows immediately that I_1, I_2, I_3, I_4, I_5 must be double points on $\mathcal{C}_k$, whose branches meet F transversally, and that the closure points are simple on $\mathcal{C}_k$. The only other singular points on $\mathcal{C}_k$ arise from the ordinary double points at $P_1, \overline{P}_1$, which map under τ_k to ordinary double points at I, J, and any finite singular points of $\mathfrak{R}$. Thus the singular points at I, J have δ-invariant 1, whilst the δ-invariants of any finite singular points of $\mathfrak{R}$ will be left invariant by the projection. Applying the Genus Formula to the curve $\mathcal{C}_k$, we find that the δ-invariants of the double points at I_1, I_2, I_3, I_4, I_5 all equal 1 and hence that each of these singularities is *either* an ordinary double point *or* a cusp.

One can say a little more about cusps. A cusp occurs if and only if L'_j is tangent to $\mathcal{S}$ (and therefore tangent to $\mathfrak{R}$), i.e. if and only if L'_j is the tangent to $\mathfrak{R}$ at a critical point of the restriction τ_k. Thus, the condition for a point P on $\mathfrak{R}$ to be a critical point is that the matrix obtained from the Jacobian matrix of equations (3.3) by abutting the Jacobian matrix of the projection has non-maximal rank ; since this is equivalent to determining the points of $\mathfrak{R}$ where the tangent lies in the kernel of the projection.

The matrix is

$$\begin{bmatrix} d_1 & 0 & d_2 & 0 & d_3 & 0 & d_4 & 0 & -d_5 \\ 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 & d_4 & 0 \\ A_1 & A_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ A_2 & -A_1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2x_1 & 2y_1 & 0 & 0 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 2x_2 & 2y_2 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 2x_3 & 2y_3 & 0 & 0 & -2w \\ d_1 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By elementary row and column operations one can show that the condition for the above matrix to have non-maximal rank is equivalent to the condition for the following matrix to have non-maximal rank

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ A_1x_3+A_2y_3 & A_2x_3-A_1y_3 \end{bmatrix}$$

Thus, the condition is $x_1y_2 = y_1x_2$ and $x_1[A_2x_3-A_1y_3] = y_1[A_1x_3+A_2y_3]$. Using the quadratic equations of (3.3), it can easily be showed that the conditions are equivalent to $x_1 = \varepsilon x_2$, $y_1 = \varepsilon y_2$, $x_3 = \varepsilon' x_4$, $y_3 = \varepsilon' y_4$ where $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$. Substituting for x_1, y_1, x_3, y_3 in equations (3.3), we observe first that such a configuration must be real, second that it must be finite and third that there are ≤ 4 such configurations. The mechanical interpretation of these conditions is that links 1,2 should be parallel and links 3,4 should be parallel. Eliminating all the configuration variables from the equations, one obtains four non-trivial polynomial conditions on the design parameters which must be satisfied for a cusp to appear. These conditions are

$$d_5^2[(d_1+\varepsilon_1 d_2)^2 + (\varepsilon_2 d_3+d_4)^2] - [(d_1+\varepsilon_1 d_2)^2 - (\varepsilon_2 d_3+d_4)^2]^2 = 2A_1 d_5^2[d_1+\varepsilon_1 d_2][\varepsilon_2 d_3+d_4]$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. In particular, therefore, cusps fail to appear for almost all design parameters. A careful study of these polynomial conditions (which may be found in [Freudenstein] who derives this condition from purely mechanical reasoning) reveals that in fact there are ≤ 3 cusp configurations: it would be interesting to have a geometric argument for this fact. Thus in the generic case the possible multi-singularity types of the coupler curve $\mathcal{C}_k$ (with $k = d_2$) are $7A_1$, $6A_1/A_2$, $5A_1/2A_2$, $4A_1/3A_2$ where we adopt the Arnold notation for simple singularities. In [Freudenstein] it is explicitly verified that all four types can occur. However, in the real case one can make the finer distinction in the A_1 case (ordinary double point) between an A_1^+ (acnode) and an A_1^- (crunode) giving rise to forty-one real multi-singularity types. It appears to be an open problem, whether all these types can occur.

We shall conclude our discussion of coupler curves by taking up the general case when $k \neq d_2$ so that $\tau_k|\mathcal{R}$ is a regular rational mapping. <u>We claim that this mapping is birational</u> (thus of degree 1) except when $\mathcal{R}$ has a repeated conic component.

Provided the inverse condition does not hold, the eight tangent lines to $\mathfrak{R}$ at the points P_1, $\overline{P}_1$, Q_1, $\overline{Q}_1$, Q_2, $\overline{Q}_2$ project to eight distinct tangents to $\mathfrak{C}_k$ at points on the line at infinity $w=0$ in the coupler plane. Thus a component K of $\mathfrak{R}$ with degree d has d distinct branches meeting the hyperplane $w=0$ and its image K_k under τ_k has d distinct branches meeting the line at infinity $w=0$ in the coupler plane. Thus the total multiplicity of points of K_k on $w=0$ is $\geq d$, hence K_k has degree $\geq d$. It follows immediately from the Projection Formula (Theorem A11) that K_k has degree d and that $\tau_k|K$ has degree 1.

One can now pursue the kind of analysis we gave when $k=d_2$. The results are as follows. When $\mathfrak{R}$ is irreducible, $\mathfrak{C}_k$ is an irreducible octic, having the same geometric genus as $\mathfrak{R}$, with ordinary triple points at I and J. In the $4+4$ case $\mathfrak{C}_k$ is the union of two rational quartics, one circular and the other bicircular. In the $2+6$ case $\mathfrak{C}_k$ is the union of a circle and a bicircular sextic. And in the $2+2+4$ case $\mathfrak{C}_k$ is the union of two circles and generally an elliptic circular quartic, provided the conic components are distinct. Note that when the conic component is repeated the inverse condition holds and the above analysis no longer applies.

Provided we are careful we can still gain useful information even when the inverse condition does hold. For instance, when $\mathfrak{R}$ is irreducible it has six (instead of eight) distinct tangents at points in $w=0$ mapping to six distinct tangents to the coupler $\mathfrak{C}_k$ as above. We can argue via Bézout's Theorem (A3) and the Projection Formula (Theorem A11) that τ_k maps $\mathfrak{R}$ to

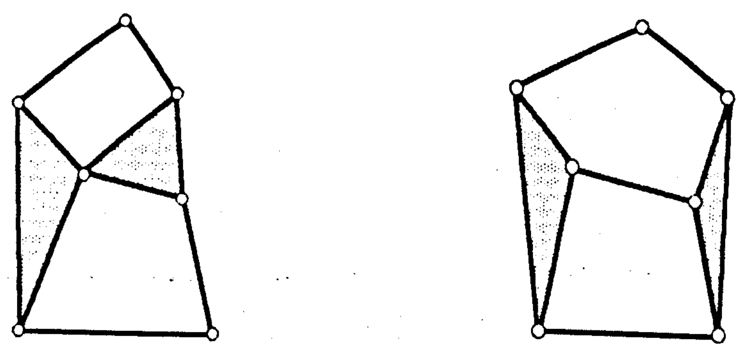
$\mathcal{C}_k$ with degree 1, and that $\mathcal{C}_k$ has degree 8 and the same geometric genus as $\mathcal{R}$. However, the triple points at I, J are necessarily non-ordinary. Likewise, in the 4+4 case, we find that $\mathcal{C}_k$ is still the union of two quartics (one circular and the other bicircular) touching at I, J. On the other hand the 2+6 case can no longer arise, nor can the general 2+2+4 case. One can still have the 2+2+4 case with a repeated conic component: the repeated conic projects to a circle, whilst generally the quartic component projects to an elliptic circular quartic.

One could pursue the geometry of the complex coupler curve $\mathcal{C}_k$ much further using little more than the techniques expounded in this chapter. However, a more profitable (and certainly more interesting) direction would be to elucidate the geometry of the real couplers $\mathcal{C}_k$, at least in the generic case. Note incidentally, that our work in §3.3 has automatically solved the problem of determining the number of real circuits for an arbitrary choice of coupler point. In principle, it should be possible to obtain a basic classification of real couplers in terms of the real multi-singularity type and the number of real critical points of the projection π_1 as discussed in §3.3.

CHAPTER 4. THE GEOMETRY OF THE WATT SIX-BAR MECHANISMS.

Introduction

The six-bar mechanisms are the first planar examples of linkages with more than one kinematic chain. Smoothly jointing together six rigid bodies, displays two possible kinematic chains and a total of five different mechanisms. The two six-bar chains are named after two great men of the steam-engine era, namely Watt and Stephenson. The Watt chain, as showed in Fig 4.1(a), consists of two ternary links with a common hinge and may be visualised as two four-bar chains rigidly connected. Indeed, this description of the Watt mechanism proves to be a useful one in describing the geometry of its motion. The Stephenson chain, see Fig 4.1(b), has two ternary links but it is distinct from the Watt chain since they have no common turning joint, and moreover, it only possesses one four-bar chain.



(a) (b)

Fig. 4.1

We may fix either a binary or a ternary link, thus we obtain two distinct Watt and three distinct Stephenson mechanisms. These are named the Watt I and II, and the Stephenson I, II and III mechanisms respectively, as showed in Fig 4.2.

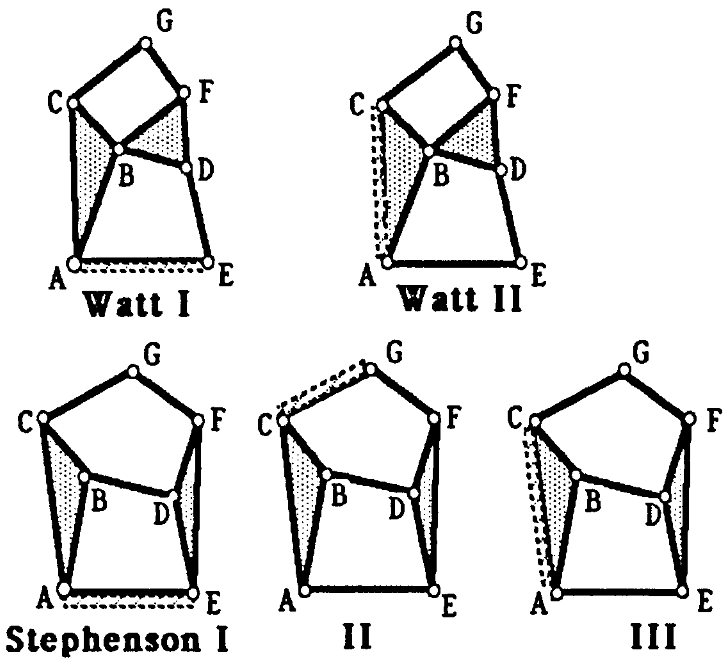


Fig. 4.2

For each mechanism there is no unique choice of coupler bar. In the Watt I, for instance, we may take either of the links CG and FG as our coupler bar and obtain six-bar curves i.e. curves which are not coupler curves of mechanisms with fewer links. Clearly, any other choice of coupler bar would give rise to arcs of four-bar coupler curves or arcs of circles. For the Watt II mechanism no choice of coupler bar gives rise to six-bar curves. The Stephenson I has two possible choices of coupler bar giving rise to six-bar curves namely, links CG and FG and likewise the Stephenson II (resp. III) has links BD and AE (resp. FG). Note that it is sufficient to consider bar CG for the Stephenson I and bar FG for the Stephenson II.

Applications of the coupler curves of six-bar mechanisms, despite their inherent complexity, are surprisingly diverse. Hunt [Hunt] gives an interesting example of a Watt I in the design of a wall-mounted desk-lamp which we reproduce here in Fig 4.3. Another example, where a six-bar mechanism is used as the take up lever on a sewing machine, may be found in [Ogawa].

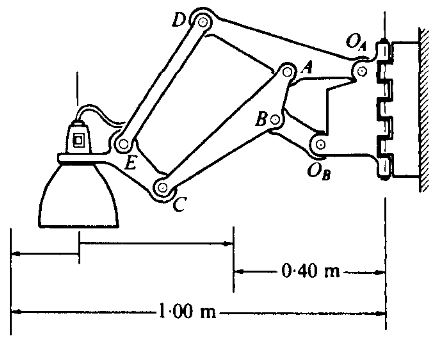


Fig. 4.3

In §4.1, 4.2 and 4.3 we set up the basic geometry for the Watt I and II linkage and Darboux varieties in the complex projective framework. In §4.4 and §4.5 we present a complete solution to the problem of determining the reductions of the linkage variety in terms of the design parameters and deduce all the possible reductions of associated coupler curves. Finally, in §4.6 we take up the geometry of the linkage variety in the real case by completely determining its topology. That enables us to deduce the number of real circuits of associated real coupler curves. In the very special case, when the coupler point is a hinge, we have therefore a formal mathematical proof of a result sketched in [Primrose].

In §4.1 we introduce formally the linkage varieties of the

Watt I and Watt II mechanisms and verify that they intersect the hyperplanes at infinity in the union of two skew planes and two skew lines: that implies that the varieties fail to be complete intersections. By this we mean that these varieties fail even to be set-theoretic complete intersections: the linkage variety for the planar four-bar is certainly a complete intersection in the set-theoretic sense, but fails to be so in the ideal-theoretic sense. Our interest centres around the residual linkage variety obtained by deleting the four irreducible components at infinity. The strategy for studying this latter variety is based on an observation that the residual linkage variety is birationally isomorphic to the residual Darboux variety and moreover the one can be obtained from the other by a linear projection. An illustration of this approach was given in §1.2 for the reduction of the four-bar linkage curve. Thus the next step, which we undertake in §4.2, is to study the Darboux varieties associated to the Watt six-bar mechanisms. That brings us to the fundamental observation that the Darboux varieties for the Watt I and II are the "same", in the sense that they can be described by the same set of equations. Further, we deduce that the residual linkage varieties for the Watt I and the Watt II are birationally isomorphic and we prove a more general result which implies that the real linkage varieties of mechanisms with the same kinematic chain are isomorphic.

It is well worth pointing out at this early stage that the residual linkage variety of the Watt I and the residual Darboux variety fail to be (complex) isomorphic - indeed they have different numbers of double points in the hyperplanes at infinity. But in contrast, we shall see in §4.6 that these linkage varieties <u>are</u>

isomorphic in the real case by our general result providing the key tool to studying their topology.

In §4.2 we show that the common Darboux variety of the Watt six-bars is projectively equivalent to an algebraic curve of degree 9 in projective 3-space which always reduces to a line and a curve of degree 8 called the residual Darboux curve. The underlying philosophy is now to utilise the birational isomorphism between the residual linkage and Darboux varieties, which is given by a perfectly explicit projection, to deduce the geometry of the former from the latter. This is explained in §4.3. In particular this enables us to show that the residual linkage variety is a curve of degree 16. It is important to note that this result would be difficult to obtain directly from the varieties, since they are not complete intersections and, therefore, we cannot apply Bézout's Theorem. Another consequence is that we can determine exactly when the linkage varieties have finite singularities, i.e. singular points off the hyperplanes at infinity. This condition ought to be called the Grashof equality, since it is the exact analogue for the Watt six-bars of the corresponding condition for the planar four-bar discussed in §1.1.

In §4.4 we give a full account of the reductions of the Darboux varieties from which we deduce the reductions of the Watt I linkage variety. A summary of this result may be found in tabular form at the end of §4.4. In §4.5 we discuss the complex geometry of the coupler curves.

Finally in §4.6, we determine the topology of the real linkage varieties. In particular, we show that there can be at most four connected components, thus giving a better upper bound than that which one can obtain from Harnack's Theorem.

§4.1 Introduction to the Complex Linkage Variety

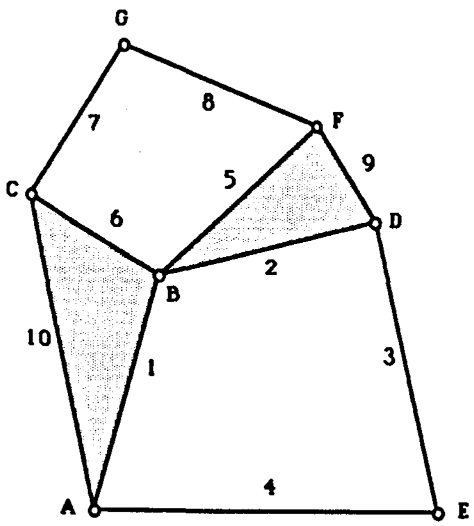


Fig. 4.4

We label the bars of the Watt I mechanism as 1,2,...,10 as showed in Fig.4.4. The quadrilateral formed by the bars 1, 2, 3, 4 is the **base quadrilateral**, whilst that formed by bars 5, 6, 7, 8 is the **upper quadrilateral**. When we speak of the **triangles** we mean the two triangles formed by bars 1, 6, 10 and bars 2, 9, 5. The bars have positive lengths $d_1, d_2, \ldots, d_{10}$ and their directions are given by unit complex numbers $z_1, z_2, \ldots, z_{10}$. Henceforth, we shall assume that bar 4 is the fixed bar of the base quadrilateral, so that z_4 will be constant: it will be no restriction to suppose that $z_4 = -1$. The constraints on the motion can be written as

$$\left.\begin{aligned} & d_1 z_1 + d_2 z_2 + d_3 z_3 - d_4 = 0 \\ & d_5 z_5 + d_6 z_6 + d_7 z_7 + d_8 z_8 = 0 \\ & z_5 = u z_2 \; : \quad z_6 = v z_1 \\ & |z_k|^2 = 1 \text{ for } 1 \le k \le 8, k \ne 4 \end{aligned}\right\} \quad (4.1)$$

where u,v are fixed unit complex numbers. We shall write $u = u_1 + iu_2$, $v = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are all real. It is natural to write $z_k = x_k + iy_k$, with x_k, y_k real and equate real and imaginary parts to obtain a real algebraic variety in $\mathbb{R}^{14}$ defined by 15 equations, two of which are redundant. This variety can be complexified and projectivised (with w the homogenising parameter) to obtain a complex projective variety $\mathfrak{R}$ in $P\mathbb{C}^{14}$ defined likewise by 13 equations. Explicitly

$$\left.\begin{array}{l} d_1x_1 + d_2x_2 + d_3x_3 - d_4w = 0 \\ d_1y_1 + d_2y_2 + d_3y_3 = 0 \\ d_5x_5 + d_6x_6 + d_7x_7 + d_8x_8 = 0 \\ d_5y_5 + d_6y_6 + d_7y_7 + d_8y_8 = 0 \\ x_5 = u_1x_2 - u_2y_2 \ : \ \ y_5 = u_2x_2 + u_1y_2 \\ x_6 = v_1x_1 - v_2y_1 \ : \ \ y_6 = v_2x_1 + v_1y_1 \\ x_k^2 + y_k^2 = w^2 \quad (k = 1,2,3,7,8) \end{array}\right\} \quad (4.2)$$

We shall refer to $\mathfrak{R}$ as the **linkage variety** of the Watt I mechanism. The linear equations in (4.2) are linearly independent, so define a 6-dimensional projective subspace of $P\mathbb{C}^{14}$. Thus $\mathfrak{R}$ is projectively equivalent to an intersection of 5 quadrics in $P\mathbb{C}^6$.

One might reasonably expect $\mathfrak{R}$ to be a curve intersecting the **hyperplane at infinity** (i.e. the hyperplane defined by $w = 0$) in a finite number of points. However, the situation is by no means so simple as we can verify by direct computation. Setting $w = 0$ in (4.2), we see that the intersection is given by the linear equations, augmented by the equations $y_k = \varepsilon_k i x_k$ $(k = 1,2,3,7,8)$ where $\varepsilon_k = \pm 1$. For each choice of signs of $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$ we obtain a projective subspace of $w = 0$, thus thirty-two in all, appearing in complex conjugate skew pairs. These are rather easily described. The sign choice +++++ yields a 2-plane W. The defining equations are (with x_5, y_5, x_6 and y_6 omitted for brevity)

W: $d_1x_1 + d_2x_2 + d_3x_3 = 0 \; : \; d_5\bar{u}x_2 + d_6\bar{v}x_1 + d_7x_7 + d_8x_8 = 0$

$y_j = ix_j$ (for j = 1,2,3,7 and 8) : $w = 0$

Taking the successive sign choices -++++, +-+++, etc., in which exactly one minus sign is chosen, we obtain the respective distinct lines L_1, L_2, L_3, L_4, L_5 in W, forming the configuration given in Fig.4.5 with L_1, L_2, L_3 concurrent.

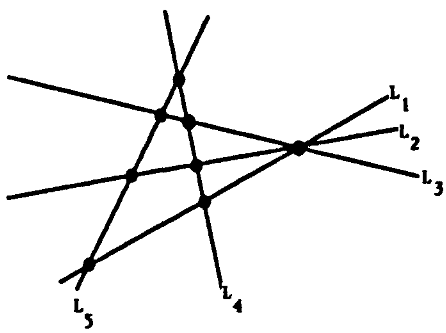


Fig. 4.5

These are defined by the equations

L_1: $d_2x_2 + d_3x_3 = 0 \; : \; d_5\bar{u}x_2 + d_7x_7 + d_8x_8 = 0$

$y_j = ix_j$ (for j = 2,3,7 and 8) : $x_1 = y_1 = w = 0$

L_2: $d_1x_1 + d_3x_3 = 0 \; : \; d_6\bar{v}x_1 + d_7x_7 + d_8x_8 = 0$

$y_j = ix_j$ (for j = 1,3,7 and 8) : $x_2 = y_2 = w = 0$

L_3: $d_1x_1 + d_2x_2 = 0 \; : \; d_5\bar{u}x_2 + d_6\bar{v}x_1 + d_7x_7 + d_8x_8 = 0$

$y_j = ix_j$ (for j = 1,2,7 and 8) : $x_3 = y_3 = w = 0$

L_4: $d_1x_1 + d_2x_2 + d_3x_3 = 0 \; : \; d_5\bar{u}x_2 + d_6\bar{v}x_1 + d_8x_8 = 0$

$y_j = ix_j$ (for j = 1,2,3 and 8) : $x_7 = y_7 = w = 0$

L_5: $d_1x_1 + d_2x_2 + d_3x_3 = 0 \ : \ d_5\bar{u}x_2 + d_6\bar{v}x_1 + d_7x_7 = 0$

$y_j = ix_j$ (for $j = 1,2,3,7$ and 8) : $x_8 = y_8 = w = 0$

Write J_{jk} for the point of intersection of L_j, L_k and J_{123} for the intersection of L_1, L_2 and L_3. Then it is easily verified that each J_{jk}, with the sole exception of J_{45}, is one of the subspaces obtained by choosing precisely two minus signs. Reversing the roles of the plus and minus signs in the above discussion we obtain analogously a complex conjugate 2-plane $\overline{W}$ containing complex conjugate lines $\overline{L}_1$, $\overline{L}_2$, $\overline{L}_3$, $\overline{L}_4$, $\overline{L}_5$ having the same configuration as in Fig.4.5 and intersecting in the points $\overline{J}_{jk}$. The equations of these subvarieties may be obtained from their complex conjugate varieties by simply conjugating the equations given above. Signs +--++ and +-+-- give identical points to -+-++ and -++--, respectively. The only sign choices which have not been covered so far are +++--, ---++ yielding complex conjugate skew lines M, $\overline{M}$, respectively. M meets $W,\overline{W}$ respectively in J_{45}, $\overline{J}_{123}$, whilst $\overline{M}$ meets $W,\overline{W}$ respectively in J_{123}, $\overline{J}_{45}$. The variety M is given by the following set of equations

M: $d_5\bar{u}x_2 = d_6\bar{v}x_1 \ : \ d_3d_5\bar{u}x_3 + (d_1d_5\bar{u} - d_2d_6\bar{v})x_1 = 0$

$d_7x_7 + d_8x_8 = 0$

$y_j = ix_j$ for $j = 1,2$ and 3 : $y_j = -ix_j$ for $j = 7,8$: $w = 0$

and we may obtain the defining equations of $\overline{M}$ by taking the complex conjugate set of equations. In this way we arrive at Fig.4.6 illustrating the intersection of the linkage variety $\mathfrak{R}$ with the hyperplane at infinity $w = 0$.

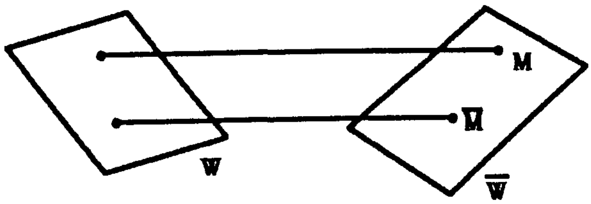


Fig. 4.6

To sum up: the desired intersection is the union of the 2-planes $W, \overline{W}$ and the lines $M, \overline{M}$, all of which must be irreducible components of $\mathfrak{R}$. In particular, $\mathfrak{R}$ fails to be a complete intersection. It is a tedious exercise to calculate the co-ordinates of the points J_{jk}: (omitting the x_5, y_5, x_6, y_6 co-ordinates for brevity) they are as follows:

$$J_{123} = (0,0,0,0,0,0,1,i,-\frac{d_7}{d_8}, -i\frac{d_7}{d_8},0)$$

$$J_{14} = (0,0,1,i,-\frac{d_2}{d_3},-i\frac{d_2}{d_3},0,0,-\bar{u}\frac{d_5}{d_8},-i\bar{u}\frac{d_5}{d_8},0)$$

$$J_{15} = (0,0,1,i,-\frac{d_2}{d_3},-i\frac{d_2}{d_3},-\bar{u}\frac{d_5}{d_7},-i\bar{u}\frac{d_5}{d_7},0,0,0)$$

$$J_{24} = (1,i,0,0,-\frac{d_1}{d_3},-i\frac{d_1}{d_3},0,0,-\bar{v}\frac{d_6}{d_8},-i\bar{v}\frac{d_6}{d_8},0)$$

$$J_{25} = (1,i,0,0,-\frac{d_1}{d_3},-i\frac{d_1}{d_3},-\bar{v}\frac{d_6}{d_7},-i\bar{v}\frac{d_6}{d_7},0,0,0)$$

$$J_{34} = (1,i,-\frac{d_1}{d_2},-i\frac{d_1}{d_2},0,0,0,0,-\frac{(d_2 d_5 \bar{u}-d_1 d_6 \bar{v})}{d_2 d_8},-i\frac{(d_2 d_5 \bar{u}-d_1 d_6 \bar{v})}{d_2 d_8},0)$$

$$J_{35} = (1,i,-\frac{d_1}{d_2},-i\frac{d_1}{d_2},0,0,-\frac{(d_2 d_5 \bar{u}-d_1 d_6 \bar{v})}{d_2 d_7},-i\frac{(d_2 d_5 \bar{u}-d_1 d_6 \bar{v})}{d_2 d_7},0,0,0)$$

$$J_{45} = (1,i,-\frac{d_6 \bar{v}}{d_5 \bar{u}},-i\frac{d_6 \bar{v}}{d_5 \bar{u}},-\frac{(d_1 d_5 \bar{u}-d_2 d_6 \bar{v})}{d_3 d_5 \bar{u}},-i\frac{(d_1 d_5 \bar{u}-d_2 d_6 \bar{v})}{d_3 d_5 \bar{u}},0,0,0,0,0)$$

The union of the irreducible components of $\mathfrak{R}$ distinct from $W, \overline{W}, M, \overline{M}$ will be called the **residual linkage variety** $\mathfrak{R}'$.

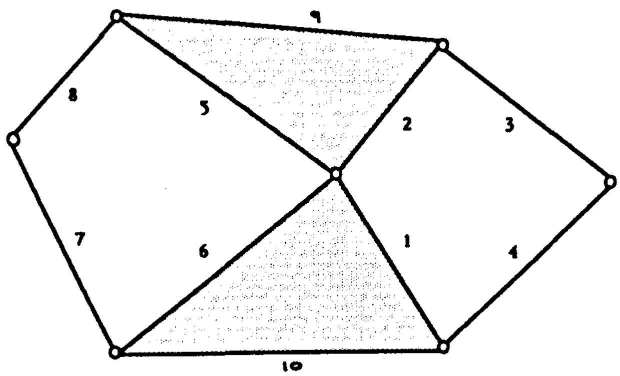


Fig. 4.7

There is a similar set up for the Watt II mechanism. Label the bars as 1,2,...,10 as showed in Fig.4.7. We shall refer to the quadrilateral formed by bars 1, 2, 3, 4 as the **base quadrilateral**, whilst that formed by bars 5,6,7,8 will be referred to as the **upper quadrilateral**. When we speak of the **triangles**, we mean the two triangles formed by bars 1,6,10 and bars 2,9,5. The bars have positive lengths $d_1,d_2,...,d_{10}$ and their directions are given by unit complex numbers $z'_1,z'_2,...,z'_{10}$. Without loss of generality we may assume that $z'_1=-1$. The constraints on the motion can be written as

$$\left.\begin{array}{l} -d_1 + d_2z'_2 + d_3z'_3 + d_4z'_4 = 0 \\ d_5z'_5 + d_6z'_6 + d_7z'_7 + d_8z'_8 = 0 \\ z'_5 = uz'_2 \;:\; z'_6 = -v \\ |z'_k|^2 = 1 \;\; (2 \le k \le 8) \end{array}\right\}$$

where u,v are fixed unit complex numbers. We shall write $u=u_1+iu_2$, $v=v_1+iv_2$, where u_1, u_2, v_1, v_2 are all real. It is natural to write $z'_k=x'_k+iy'_k$, with x'_k, y'_k real and equate real and imaginary parts to obtain a real algebraic variety in $\mathbb{R}^{14}$ defined by 15 equations, two of which are redundant. This variety can be complexified and projectivised (with w' the homogenising parameter) to obtain a complex projective variety $\mathcal{S}$ in $P\mathbb{C}^{14}$ defined by 13 equations. Explicitly

$$\left.\begin{array}{l} -d_1 w + d_2 x'_2 + d_3 x'_3 + d_4 x'_4 = 0 \\ d_2 y'_2 + d_3 y'_3 + d_4 y'_4 = 0 \\ d_5 x'_5 + d_6 x'_6 + d_7 x'_7 + d_8 x'_8 = 0 \\ d_5 y'_5 + d_6 y'_6 + d_7 y'_7 + d_8 y'_8 = 0 \\ x'_5 = u_1 x'_2 - u_2 y'_2 \ : \ y'_5 = u_2 x'_2 + u_1 y'_2 \\ x'_6 = v_1 w' \ : \ y'_6 = -v_2 w' \\ x'^2_k + y'^2_k = w^2 \quad (k = 2,3,4,7,8) \end{array}\right\} \quad (4.4)$$

We shall refer to $\mathcal{S}$ as the **linkage variety** of the Watt II mechanism. The linear equations in (4.4) are linearly independent, so define a 6-dimensional projective subspace of $P\mathbb{C}^{14}$. Thus, $\mathcal{S}$ is projectively equivalent to an intersection of 5 quadrics in $P\mathbb{C}^6$.

In a similar manner to the Watt I we may set $w' = 0$ in (4.4) to obtain the picture of the Watt II linkage variety at infinity. We see that the intersection is given by the linear equations of (4.4) augmented by the equations $y'_k = \varepsilon_k \mathbf{i} x'_k$ $(k = 2,3,4,7,8)$. As before each choice of signs yields a projective subspace of $w' = 0$, thirty-two in all. Their description is as follows. The sign choice +++++ yields a 2-plane W'. Taking the successive sign choices -++++, +-+++, ... in which exactly one minus sign is chosen we obtain five distinct lines L'_1, L'_2, L'_3, L'_4, L'_5 in W', forming a configuration identical to Fig. 4.5 with L'_1, L'_2, L'_3 concurrent. Write J'_{jk} for the point of intersection of L'_j, L'_k and write J'_{123} for the intersection of L'_1, L'_2, L'_3. Then it is easy to check that each J'_{jk}, with the exception of J'_{14} and J'_{15}, is one of the subspaces obtained by choosing precisely two minus signs. (Sign choices -+-++ and --+++ both yield J'_{123}, and -+++- and -++-+

both yield J'_{45}.) Reversing the roles of the plus and minus signs, yields the following complex conjugate subspaces: a 2-plane $\overline{W}'$ and lines $\overline{L}'_1$, $\overline{L}'_2$, $\overline{L}'_3$, $\overline{L}'_4$, $\overline{L}'_5$ in an identical configuration to Fig.4.5. The signs we still have to account for are +++--, -++-- which yield a line M' and ---++, +--++ which yield the complex conjugate line $\overline{M}'$. The overall picture is similar to Fig.4.6: M' passes through J'_{45} and $\overline{J}'_{123}$ and is skew to $\overline{M}'$ which passes through $\overline{J}'_{45}$ and $\overline{J}'_{123}$.

The complexity of the varieties $\mathcal{R}$ and $\mathcal{S}$ makes them difficult to study directly. Our approach to this problem is based on the fact, established in [Gibson&Newstead], that the residual linkage varieties are birationally isomorphic to the associated residual Darboux variety $\mathcal{D}'$, a statement which we shall amplify in the next two sections.

§4.2 The Associated Darboux Variety.

According to the general construction given in [Gibson&Newstead], as explained in §1.2, the **Darboux variety** $\mathcal{D}$ associated to the Watt I mechanism is obtained as follows. We start with the linear equations in (4.1). Then for each such equation we form a new equation obtained by replacing each z_k by $1/z_k$ and conjugating all the coefficients. We then homogenise the equations with z_4 the homogenising parameter. In this way we obtain the following system of equations

$$\left.\begin{aligned}
&d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = 0 \\
&\frac{d_1}{z_1} + \frac{d_2}{z_2} + \frac{d_3}{z_3} + \frac{d_4}{z_4} = 0 \\
&d_5z_5 + d_6z_6 + d_7z_7 + d_8z_8 = 0 \\
&\frac{d_5}{z_5} + \frac{d_6}{z_6} + \frac{d_7}{z_7} + \frac{d_8}{z_8} = 0 \\
&z_5 = uz_2 \ : \quad z_6 = vz_1
\end{aligned}\right\} \qquad (4.5)$$

These equations define an algebraic variety in $P\mathbb{C}^7$ with homogeneous co-ordinates $z_1,\ldots,z_8$ which is the required Darboux variety $\mathcal{D}$.

We may obtain the Darboux variety associated to the Watt II mechanism from equations (4.3) in a similar manner. We form two new equations by replacing each z'_k by $1/z'_k$ and conjugating all the coefficients. If we then homogenise the equations with z'_4 the homogenising parameter, then $z_j \mapsto z'_j$ defines a projective equivalence between $\mathcal{D}$ and the Watt II Darboux variety. Thus it makes sense to refer to *the* Darboux variety $\mathcal{D}$ of the Watt kinematic chain: for no matter which bar we fix, the corresponding Darboux variety is projectively equivalent to $\mathcal{D}$.

We begin the study of $\mathcal{D}$ by noting that the four linear equations in (4.5) are linearly independent, so define a 3-dimensional projective subspace; the remaining two equations define cubic surfaces in that 3-space whose intersection is $\mathcal{D}$. The **residual Darboux variety** $\mathcal{D}'$ is by definition the union of the irreducible components of $\mathcal{D}$ which do not lie in any of the **distinguished hyperplanes** $z_k = 0$, for $1 \le k \le 8$. Clearly, the first step in studying $\mathcal{D}'$ is to see how $\mathcal{D}$ intersects the hyperplanes $z_k = 0$. This requires no more than a few straightforward computations. Note first that, if we set $z_1 = 0$, $z_2 = 0$ in (4.5), we obtain a line L in $\mathcal{D}$, namely, the line joining the points $Q_1 = (0,0,0,0,0,0,d_8,-d_7)$ and $Q_2 = (0,0,d_4,-d_3,0,0,0,0)$ given by

$$z_1 = z_2 = z_5 = z_6 = 0 \quad : \quad d_3z_3 + d_4z_4 = 0 \quad : \quad d_7z_7 + d_8z_8 = 0$$

In the hyperplane $z_1 = 0$ we obtain the line L and four points

$$P_1 = (0,1,0,-\tfrac{d_2}{d_4},v,0,-v\tfrac{d_5}{d_7},0) \quad : \quad P_2 = (0,1,0,-\tfrac{d_2}{d_4},v,0,0,-v\tfrac{d_5}{d_8})$$

$$P_3 = (0,1,-\tfrac{d_2}{d_3},0,v,0,-v\tfrac{d_5}{d_7},0) \quad : \quad P_4 = (0,1,-\tfrac{d_2}{d_3},0,v,0,0,-v\tfrac{d_5}{d_8})$$

whilst in the hyperplane $z_2 = 0$ we obtain the line L and four more points

$$P_5 = (1,0,0,-\tfrac{d_1}{d_4},0,v,-v\tfrac{d_6}{d_7},0) \quad : \quad P_6 = (1,0,0,-\tfrac{d_1}{d_4},0,v,0,-v\tfrac{d_6}{d_8})$$

$$P_7 = (1,0,-\tfrac{d_1}{d_3},0,0,v,-v\tfrac{d_6}{d_7},0) \quad : \quad P_8 = (1,0,-\tfrac{d_1}{d_3},0,0,v,0,-v\tfrac{d_6}{d_8})$$

Note that the points $P_1, P_2, \ldots, P_8$ are distinct and that none lie on the line L. To obtain the remaining intersections of $\mathcal{D}$ with the distinguished hyperplanes we can assume henceforth, that $z_1 \neq 0$, $z_2 \neq 0$. Under that condition $z_3 = 0$ if and only if $z_4 = 0$. Assuming these conditions hold, (4.5) yields a binary quadratic $az_1^2 + bz_1z_7 + cz_7^2 = 0$ where $a = d_7p$, $b = pq + d_7^2 - d_8^2$ and $c = d_7q$ with $p = (d_2d_6v - d_1d_5u)/d_2$ and $q = (d_1d_6u - d_2d_5v)/d_1uv$. It is easily checked that this quadratic is identically zero if and only if $d_1 = d_2$, $d_5 = d_6$, $d_7 = d_8$ and $u = v$; that is, therefore, the condition for

$\mathcal{D}$ to intersect $z_3=0$, $z_4=0$ in a line component. In mechanical terms this means that the upper quadrilateral should be a kite and that the two triangles should be congruent. Generally this condition fails and the binary quadratic has two roots yielding points E_1, E_2. The two points coincide when the discriminant of the quadratic is zero. The condition for this to occur is:

$$\text{(C1)} \qquad (d_2d_6v-d_1d_5u)(d_1d_6u-d_2d_5v) = (d_7 \pm d_8)^2\, d_1d_2uv$$

(which is also satisfied in the case of the line component). The condition may be considered as a binary quadratic in u and v thus the condition has the form $u=Cv$ where C is a unit complex number whose value is one of two roots of a quadratic (which is easily determined). We shall assume from now on that <u>condition (C1) is not satisfied</u>.

Let us now impose the further condition that $z_3 \neq 0$, $z_4 \neq 0$. Under that hypothesis $z_7=0$ if and only if $z_8=0$. Assuming these conditions hold, (4.5) yields a binary quadratic $az_1^2 + bz_1z_3 + cz_3^2$, where $a = d_3p$, $b = pq + d_3^2 - d_4^2$ and $c = d_3q$ with $p=(d_1d_5u-d_2d_6v)/d_5u$ and $q=(d_1d_6v-d_2d_5u)/d_6v$. The quadratic is identically zero if and only if $d_1=d_2$, $d_3=d_4$, $d_5=d_6$ and $u=v$: that is, therefore, the condition for $\mathcal{D}$ to intersect $z_7=0$, $z_8=0$ in a line component. In mechanical terms this means that the lower quadrilateral should be a kite and that the two triangles should be congruent. Generally, this condition fails and the binary quadratic has two roots yielding points F_1, F_2. The two points coincide when the discriminant of the quadratic is zero. The condition for this to occur is:

(C2) $(d_1d_5u-d_2d_6v)(d_1d_6v-d_2d_5u) = (d_3 \pm d_4)^2 d_5d_6uv$

(which is also satisfied in the case of the line component). As for condition (C1) the condition may be considered as a binary quadratic in u and v thus, the condition has the form $u = Cv$, where C is a unit complex number whose value is one of two roots of a quadratic. We shall assume for the remainder of the paper that condition (C2) is not satisfied.

On the above basis we can already gain useful information about the geometry of $\mathcal{D}$. Firstly, $\mathcal{D}$ has no irreducible component of dimension ≥ 3; indeed, such a component would intersect every hyperplane in a variety of dimension ≥ 2 which we now know not to be the case. Secondly, a component of dimension 2 has to intersect every hyperplane in a curve: in particular, it has to intersect the hyperplanes $z_3 = 0$ ($z_4 = 0$) and $z_7 = 0$ ($z_8 = 0$) in the line components described above, so must coincide with the unique plane containing these lines. Moreover, the condition for $\mathcal{D}$ to have such a component implies that both quadrilaterals are kites and that the triangles are congruent. In general this condition fails and the intersections of $\mathcal{D}$ with the distinguished hyperplanes are as follows.

$z_1 = 0$: $P_1, P_2, P_3, P_4, Q_1, Q_2$ and L

$z_2 = 0$: $P_5, P_6, P_7, P_8, Q_1, Q_2$ and L

$z_3 = 0$: $P_1, P_2, P_5, P_6, Q_1, E_1, E_2$

$z_4 = 0$: $P_3, P_4, P_7, P_8, Q_1, E_1, E_2$

$z_7 = 0$: $P_2, P_4, P_6, P_8, Q_2, F_1, F_2$

$z_8 = 0$: $P_1, P_3, P_5, P_7, Q_2, F_1, F_2$.

It is of course conceivable that L is a <u>repeated</u> component: that happens if and only if every point on L is singular on $\mathcal{D}$. A point $(z_1,\dots,z_8)$ on L is singular precisely when the Jacobian matrix of the equations (4.5) has non-maximal rank. The Jacobian $\mathfrak{J}$ is

$$\mathfrak{J} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 & 0 \\ -\frac{d_1}{z_1^2} & -\frac{d_2}{z_2^2} & -\frac{d_3}{z_3^2} & -\frac{d_4}{z_4^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5 & d_6 & d_7 & d_8 \\ 0 & 0 & 0 & 0 & -\frac{d_5}{z_5^2} & -\frac{d_6}{z_6^2} & -\frac{d_7}{z_7^2} & -\frac{d_8}{z_8^2} \\ 0 & -u & 0 & 0 & 1 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

An easy computation shows that for points on L the matrix $\mathfrak{J}$ has non-maximal rank exactly when

$$d_1 d_5 v z_3^2 z_7^2 = d_2 d_6 u z_3^2 z_7^2$$

Now the points Q_1, Q_2 are precisely the points on L for which either $z_3 = 0$ or $z_7 = 0$: thus both these points are singular on $\mathcal{D}$. And <u>the condition for every point on</u> L <u>to be singular</u> is

$$\text{(C3)} \quad u = v,\ d_1 d_5 = d_2 d_6.$$

In more mechanical terms, that is the condition that the triangles are similar. We shall assume from now on that <u>condition (C3) is not satisfied</u>.

Although we know that Q_1, Q_2 are singular on $\mathcal{D}$ it is not clear, whether they are singular on $\mathcal{D}'$. A lengthy, but straightforward local co-ordinate calculation, provides that answer. We shall give the details for the point Q_1 and leave the calculation for Q_2 to the reader. First make equations (4.5) affine by setting $z_7 = 1$ and translate the point to the origin by making the affine change of co-ordinates $z_i \mapsto z_i$ for $i = 1,..,6$ and $z_8 \mapsto z_8 - \frac{d_7}{d_8}$. Then the equations (4.5) become

$$\left.\begin{aligned}
&d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = 0 \\
&\frac{d_1}{z_1} + \frac{d_2}{z_2} + \frac{d_3}{z_3} + \frac{d_4}{z_4} = 0 \\
&d_5z_5 + d_6z_6 + d_8z_8 = 0 \\
&\frac{d_5}{z_5} + \frac{d_6}{z_6} + d_7 + \frac{d_8^2}{d_8z_8 - d_7} = 0 \\
&z_5 = uz_2 \ : \quad z_6 = vz_1
\end{aligned}\right\}$$

Eliminating variables z_4, z_5, z_6, z_8 and using the linear equations, we obtain the two equations

$$-(d_1z_2z_3 + d_2z_1z_3 + d_3z_1z_2)(d_1z_1 + d_2z_2 + d_3z_3) + d_4^2z_1z_2z_3 = 0$$

$$-d_5d_7vz_1 - d_6d_7uz_2 - (d_5vz_1 + d_6vz_2 + d_7uvz_1z_2)(d_5uz_2 + d_6vz_1) - (d_7^2 - d_8^2)uvz_1z_2 = 0$$

Since the derivative of the second equation with respect to z_2 is non-zero, we may use the Implicit Function Theorem to write z_2 as a power series in z_1 near the origin. Let $z_2 = az_1 + bz_1^2 + \cdots$ be its Taylor Series, then substituting for z_2 in this equation and equating coefficients we find that $a = -\frac{d_5u}{d_6v}$. Finally, we substitute the series for z_2 in the first equation to obtain the affine local co-ordinates of the curve near the origin

$$z_1\left[d_3pz_3^2 + \left(pq + \frac{(d_3^2 - d_4^2)d_5v}{d_4d_6u}\right)z_1z_3 + \frac{d_3d_5vqz_1^2}{d_4d_6u} \right] + O(3) = 0$$

where $p = \dfrac{(d_1d_5v - d_2d_6u)}{d_4d_6u}$ and $q = \dfrac{(d_1d_6u - d_2d_5v)}{d_6u}$.

We may now deduce that Q_1 is an ordinary triple point provided conditions (C1), (C2) and (C3) do not hold. In particular, $\mathcal{D}'$ has a double point at Q_1 with distinct tangents i.e. Q_1 and likewise Q_2 is an ordinary double point on $\mathcal{D}'$ provided conditions (C1), (C2) and (C3) do not hold.

For the remainder of this paper we shall assume that the mechanism is **general** by which we mean that conditions (C1), (C2), (C3) are not satisfied.

Under that hypothesis $\mathcal{D}$ is a curve projectively equivalent to a complete intersection of two cubic surfaces in 3-space. Thus, by Bézout's Theorem, $\mathcal{D}$ has degree 9. Applying the well known formula for computing the arithmetic genus of an intersection of hypersurfaces (Theorem A5), yields that $\mathcal{D}$ has arithmetic genus 10. $\mathcal{D}$ has only one irreducible component in a distinguished hyperplane, namely the line L. Further, $\mathcal{D}'$ is the union of the irreducible components of $\mathcal{D}$ distinct from L. Since under our assumption L will not be a repeated component the residual Darboux curve $\mathcal{D}'$ will have degree 8. From these facts we can immediately deduce further useful information about the points where $\mathcal{D}'$ meets the distinguished hyperplanes. Indeed, applying Bézout's Theorem to each intersection in turn, we see that, since E_1,E_2 and F_1,F_2 are distinct pairs of points, all the points $P_1,\ldots,P_8$, E_1,E_2, F_1,F_2 are simple points on $\mathcal{D}'$: moreover, at any of these points and at Q_1, Q_2, the intersections are transverse.

The next step is to determine when finite singular points can occur, by which we mean singularities off the hyperplanes $z_k = 0$ $(k = 1, \ldots, 8)$. Since under our assumptions $\mathcal{D}'$ has no components of dimension ≥ 2, the conditions for this to occur are that the Jacobian matrix $\mathfrak{J}$ with $z_k \neq 0$ for all k should have non-maximal rank. By elementary row and column operations and deleting linearly independent rows and columns (e.g. $u \times$col5+col2; $v \times$col6+col1; delete independent rows 5 & 6 and columns 1 & 2; col1$\div d_3$; col2$\div d_4$; col3$\div d_2$; col4$\div d_1$; col5$\div d_7$; col8$\div d_8$; col1-col2; col3-col2; col4-col2; (note that $z_8 \neq 0$ so...) $z_8^2 \times$row4+row3; delete independent rows 1 & 3 and columns 2 & 6) we may reduce the problem to determining when the following matrix has non-maxinal rank.

$$\begin{bmatrix} \dfrac{z_4^2 - z_3^2}{z_3^2} & \dfrac{z_4^2 - z_2^2}{z_2^2} & \dfrac{z_4^2 - z_1^2}{z_1^2} & 0 \\ 0 & \dfrac{d_5 u(z_8^2 - z_5^2)}{d_2 z_5^2} & \dfrac{d_6 v(z_8^2 - z_6^2)}{d_1 z_6^2} & \dfrac{z_8^2 - z_7^2}{z_7^2} \end{bmatrix}$$

It follows that the matrix has non-maximal rank if and only if *either* one of the two rows is zero, giving cases (i) and (ii) below *or* neither row is zero implying that columns one and four are zero and the determinant of the 2×2 minor, consisting of columns two and three is zero thus giving case (iii) below. Hence a point $P = (z_1, \ldots, z_8)$ is a finite singularity if and only if at least one of the following three conditions is satisfied:

(i) $z_1 = \varepsilon_2 z_2 = \varepsilon_3 z_3 = \varepsilon_4 z_4$ $(\neq 0)$, where $\varepsilon_2 = \pm 1, \varepsilon_3 = \pm 1, \varepsilon_4 = \pm 1$.

Substituting for z_2,z_3,z_4 into the first equation of (4.5), we find that this implies a condition of the form

$$d_1+\varepsilon_2 d_2+\varepsilon_3 d_3+\varepsilon_4 d_4=0 \qquad (4.6)$$

The condition is precisely that the four-bar mechanism, obtained by removing bars 5,6,7 and 8, satisfies a Grashof equality as described in §1.1. Eliminating all but z_1, z_7 in equations (4.5), we get a binary quadratic in z_1, z_7

$$d_7 P z_1^2 + (PQ + d_7^2 - d_8^2) z_1 z_7 + d_7 Q z_7^2 = 0$$

where $P=(\varepsilon_2 d_5 v+d_6 u)/uv$ and $Q=(\varepsilon_2 d_5 u+d_6 v)$, giving two values of z_7 (in terms of z_1) and therefore two singular points in general. Mechanically, the lower quadrilateral has flattened and the two points correspond to the two positions of the mechanism as indicated in Fig.4.8 which differ only by the position of bars 7 and 8. Thus generally, for each choice of sign $(\varepsilon_2,\varepsilon_3,\varepsilon_4)$ for which a condition of the form (4.6) is satisfied we obtain two distinct finite singular points of the form $(1,\varepsilon_2,\varepsilon_3,\varepsilon_4,*,*,*,*)$ on $\mathcal{D}$.

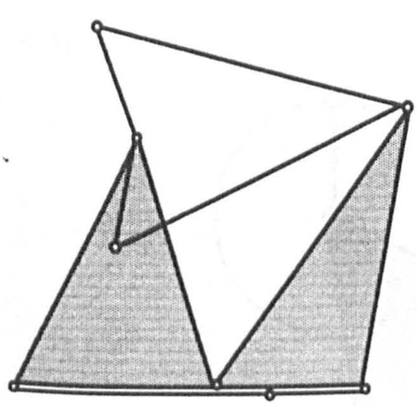


Fig 4.8

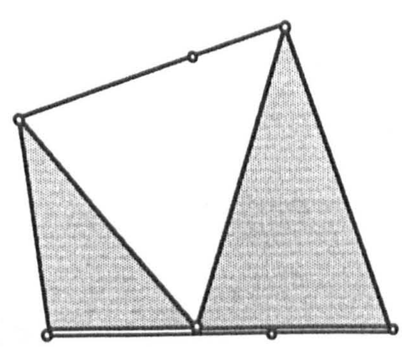


Fig 4.9

Exceptionally, the discriminant of the quadratic may vanish. But this occurs if and only if $PQ=(d_7 \pm d_8)^2$ in which case the two singularities coincide. Rewriting the condition as a quadratic in u and v, we have

$$\varepsilon_2 d_5 d_6 u^2 + [d_5^2 + d_6^2 - (d_7 \pm d_8)^2]uv + \varepsilon_2 d_5 d_6 v^2 = 0$$

Thus we may write the condition as $u=\varepsilon_2 B_\pm v$; where $B_\pm$ are the two complex conjugate roots (real if and only if $B_\pm = \pm 1$ since u and v are unit complex numbers) of the quadratic in u obtained from the above quadratic by setting $\varepsilon_2 = 1$ and $v=1$. In this case $z_7 = \pm z_8$, so bars 7 and 8 are parallel and correspond mechanically to the case showed in Fig.4.9.

More degenerately, the quadratic may be identically zero giving a singular line. This can only occur if $P=Q=0$, $d_7=d_8$, that is, $d_5=d_6$, $d_7=d_8$, $u=-\varepsilon_2 v$ implying that condition (C2) holds: indeed, the line meets $z_7=0$ and $z_8=0$ in a singular point ($\neq Q_2$), so that F_1 and F_2 must coincide. The mechanical interpretation here is that bars 5 and 6 overlap, hinges C and F coincide and bars 7 and 8 move with one degree of freedom as showed in Fig. 4.10.

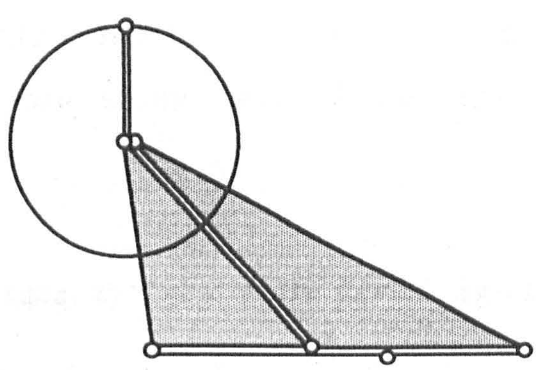


Fig 4.10

(ii) $z_5 = \varepsilon_6 z_6 = \varepsilon_7 z_7 = \varepsilon_8 z_8$, where $\varepsilon_6 = \pm 1, \varepsilon_7 = \pm 1, \varepsilon_8 = \pm 1$. Substituting into the third equation of (4.5), we see that this implies a condition of the form

$$d_5 + \varepsilon_6 d_6 + \varepsilon_7 d_7 + \varepsilon_8 d_8 = 0 \qquad (4.7)$$

This condition is precisely the Grashof equality for the four-bar obtained by removing bars 1,2,3 and 4 as described in §1.2. The argument follows closely that of case (i). Generally, we have two distinct singular points on $\mathcal{D}$ for each condition of the form (4.7) that is satisfied. When $P'Q' = (d_3 \pm d_4)^2$ (where $P' = \varepsilon_6 d_1 v + d_2 u$ and $Q' = (\varepsilon_6 d_1 u + d_2 v)/uv$) the two singularities coincide. Rewriting this condition as a binary quadratic in u and v, we have

$$\varepsilon_6 d_1 d_2 u^2 + [d_1^2 + d_2^2 - (d_3 \pm d_4)^2]uv + \varepsilon_6 d_1 d_2 v^2 = 0$$

Thus we may write the condition for the coincidence of the singular points as $u = \varepsilon_6 A_\pm v$; where $A_\pm$ are the two complex conjugate roots (real if and only if $A_\pm = 1$, since u and v are unit length) of the quadratic in u obtained by setting $\varepsilon_6 = 1$ and $v = 1$ in the given binary quadratic. More degenerately, when $d_1 = d_2$, $d_3 = d_4, u = -\varepsilon_6 v$, we get a singular line. In the latter case condition (C1) necessarily holds. The corresponding mechanical interpretations are easily deduced and are similar to those of case (i).

(iii) $z_3 = \varepsilon_4 z_4$, $z_7 = \varepsilon_8 z_8$ with $\varepsilon_4 = \pm 1$, $\varepsilon_8 = \pm 1$ and

$$\frac{d_6 v(z_8^2 - z_6^2)(z_4^2 - z_2^2)}{d_1 z_6^2 z_2^2} = \frac{d_5 u(z_8^2 - z_5^2)(z_4^2 - z_1^2)}{d_2 z_5^2 z_5^2} \qquad (4.8)$$

Using the linear equations above together with the linear equations of (4.5), we may eliminate several of the variables. One can then show the essential redundancy of the third equation and derive the necessary condition on the parameters for this type of finite singular point to arise. The reasoning is as follows. Substitute $z_3 = \varepsilon_4 z_4$, $z_7 = \varepsilon_8 z_8$ in equations (4.5) and then eliminate z_4 and z_7 using the linear equations thus obtained. This leaves two quadratic equations in the remaining variables

$$d_1 d_2 z_1^2 + (d_1^2 + d_2^2 - (\varepsilon_4 d_3 + d_4)^2) z_1 z_2 + d_1 d_2 z_2^2 = 0 \quad (4.9)$$
$$d_5 d_6 z_5^2 + (d_5^2 + d_6^2 - (\varepsilon_8 d_7 + d_8)^2) z_5 z_6 + d_5 d_6 z_6^2 = 0 \quad (4.10)$$

Then we may write z_1 and z_5 in terms of z_2 and z_6 respectively in two ways, provided the design parameters do not satisfy a condition of the form (4.6) or (4.7). Let $z_1 = A_\pm z_2$ and $z_5 = B_\pm z_6$, thus it follows from $z_5 = u z_2$ and $z_6 = v z_1$ that $u = A_\pm B_\pm v$ is a necessary condition for this type of finite singular point. (The reader should note that the constants $A_\pm$ and $B_\pm$ are identical to those labelled in cases (i) and (ii).) We will now show the redundancy of the third equation of (4.8). Suppose P is a point satisfying $z_3 = \varepsilon_4 z_4$, $z_7 = \varepsilon_8 z_8$. Then from the above analysis we have $z_1 = A_\pm z_2$ and $z_5 = B_\pm z_6$ and $u = A_\pm B_\pm v$. But $A_\pm \neq 0$ so we may use (4.9) to give the identity

$$(\varepsilon_3 d_3 + d_4)^2 = \frac{(d_1 \pm d_2 A_\pm)(d_1 A_\pm \pm d_2)}{A_\pm} \quad (4.11)$$

Hence (4.5) gives $z_4 = -\dfrac{(d_1 \pm d_2 A_\pm) z_2}{(\varepsilon_3 d_3 + d_4)}$. Thus, $z_4 = \dfrac{(d_1 \pm d_2 A_\pm)^2 z_2^2}{(\varepsilon_3 d_3 + d_4)^2}$ and it follows from (4.9) that $z_4^2 = \dfrac{(d_1 A_\pm \pm d_2) A_\pm}{(d_1 + d_2 A_\pm)} z_2^2$.

Hence,

$$z_2^2 - z_4^2 = \frac{(d_1 - d_1 A_\pm^2) z^2{}_2}{(d_1 + d_2 A_\pm)} \quad \text{and} \quad z_1^2 - z_4^2 = -\frac{(d_2 - d_2 A_\pm^2) A_\pm z^2{}_2}{(d_1 + d_2 A_\pm)}.$$

Thus,

$$(z_2^2 - z_4^2) = \frac{-d_1}{d_2 A_\pm}(z_1^2 - z_4^2).$$

Similarly we have the identity

$$(z_6^2 - z_8^2) = \frac{-d_5}{d_6 B_\pm}(z_5^2 - z_8^2).$$

It is now an easy matter to see that the third equation is satisfied: for using $z_5 = uz_2$ and $z_6 = vz_1$ this reads

$$d_2 d_6 u (z_2^2 - z_4^2)(z_6^2 - z_8^2) z_1 z_2 - d_1 d_5 v (z_1^2 - z_4^2)(z_5^2 - z_8^2) z_1 z_3 = 0$$

Thus the condition is satisfied if and only if

$$\left[d_2 d_6 u \cdot \frac{-d_1}{d_2\ A_\pm} \cdot \frac{-d_5}{d_6\ B_\pm} - d_1 d_5 v\right][z_1^2 - z_4^2][z_5^2 - z_8^2] z_1 z_2 = 0$$

Since $z_1 \neq 0$, $z_2 \neq 0$, $z_1 \neq \pm z_4$, $z_5 \neq \pm z_8$ this is the case if and only if $u = A_\pm B_\pm$. But we know this to be the case and our result is proved. Thus $u = A_\pm B_\pm$ is a necessary and sufficient condition for this type of finite singular point. The mechanical interpretation here is simply that bars 3 and 4, and bars 7 and 8, are parallel as indicated in Fig. 4.11.

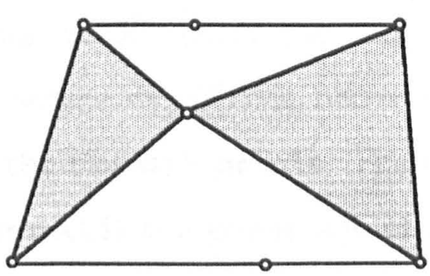


Fig. 4.11

Note that the first two equations of (4.8) are satisfied in the exceptional cases of (i) and (ii) above, when a pair of singular points coincide, but that we can also have mechanisms giving rise to points satisfying $u = A_\pm B_\pm$ that do not satisfy the conditions of cases (i) or (ii).

§4.3. Geometry of the Linkage Variety.

We shall now describe the geometry of the Watt I residual linkage variety $\mathcal{R}'$ in more detail. Its relation with the residual Darboux variety $\mathcal{D}'$ is as follows. Consider the projection $\pi:P\mathbb{C}^{14} \rightarrow P\mathbb{C}^{7}$ which maps $(x_1,y_1,\ldots,x_8,y_8,w)$ to $(z_1,\ldots,z_8)$ where $z_k = x_k + iy_k$ for $k=1,2,3,5,6,7,8$ and $z_4 = -w$. The centre of π is the 6-dimensional subspace of $P\mathbb{C}^{14}$ defined by $x_k = -iy_k$ for $k \neq 4$, and $w=0$, thus contains the 2-plane W. As explained in §1.2, the projection π defines a 1-1 correspondence between the points of $\mathcal{R}$ with $w \neq 0$ and the <u>finite</u> points of $\mathcal{D}'$ i.e. those with $z_k \neq 0$ for all k. Thus the restriction $\pi|_{\mathcal{R}'}$ is a generically 1-1 rational map and is therefore a <u>birational map between $\mathcal{R}'$ and $\mathcal{D}'$</u> failing to be regular only where $\mathcal{R}$ meets W. Thus the open set $\pi|_{\mathcal{R}'}(\mathcal{R}')$ contains all finite points of $\mathcal{D}'$ but not all of the points with $z_k=0$ for some k. It is clear then, that $\mathcal{D}'$ is the Zariski closure of the set $\pi|_{\mathcal{R}'}(\mathcal{R}')$. More precisely, the image of $\mathcal{R}'$ only fails to be closed because points of $\mathcal{R}'$ meet the centre of projection i.e. at the points of $\mathcal{R}'$ where $\pi|_{\mathcal{R}'}(\mathcal{R}')$ is non-regular. Let us call the points $\mathcal{D}'-\pi|_{\mathcal{R}'}(\mathcal{R}')$ the **closure points**. From the general theory of projections (Theorem A13) the pre-image of a point on Y under a linear projection $\pi:X \rightarrow Y$ between two varieties X and Y is either a point on X or has as its pre-image the osculating n-plane to X of some point where X meets the centre of π. Thus the set of points, which makes up the closure points of $\pi|_{\mathcal{R}'}(\mathcal{R}')$, may be obtained by taking the images of osculating n-planes. For well behaved projections of curves (by which we mean, those projections, whose centre does not contain any tangent of $\mathcal{R}'$) we need only consider the images of tangent lines i.e. osculating 1-planes. Thus, provided π is well behaved any point of $\mathcal{D}'$, which does not have a pre-image on $\mathcal{R}'$ must be a closure point and is the image of the tangent line to $\mathcal{R}'$ at a point where $\mathcal{R}'$ meets the centre of projection. We wish to show that no point of $\mathcal{D}'$ with $z_4=0$ is a closure point and hence has a pre-image on $\mathcal{R}'$.

We begin by noting that the equations defining $\mathcal{R}$ have real coefficients and therefore any point P is either real i.e. $P=\overline{P}$, or complex i.e. $P \neq \overline{P}$ in which case the complex conjugate point $\overline{P}$ also lies on $\mathcal{R}$ and has the same singularity type as P.

First we shall show that π is well behaved. Suppose P is a point of $\mathcal{R}'$ whose tangent line T lies in the centre of projection. Then in particular, T lies in $w=0$. But then this implies that the point $\overline{P}$ on $\mathcal{R}'$ has a tangent $\overline{T}$ also lying in $w=0$. We now obtain our required contradiction by noting that $\overline{T}$ does not meet the centre of projection and hence $\overline{P}$ maps to a point of $\mathcal{D}'$ whose tangent lies in $z_4=0$: which we know is never the case.

A similar argument will yield that no closure point of $\mathcal{D}'$ lies in the hyperplane $z_4=0$. Let P' be a point on $\mathcal{D}'$ with $z_4=0$ and suppose that it is a closure point. Then there is a tangent T to $\mathcal{R}'$ at a point P lying on W (and so meeting the centre of projection) which maps to P'. Since W does not meet its complex conjugate plane $\overline{W}$, the point P is complex and therefore has a complex conjugate point $\overline{P}$ (lying on $\overline{W}$) with tangent $\overline{T}$ to $\mathcal{R}'$. Thus, T maps to a point in the hyperplane $z_4=0$, implying that T lies in the pre-image of $z_4=0$, i.e. the hyperplane $w=0$. But the tangent $\overline{T}$ to $\mathcal{R}'$ at $\overline{P}$ is complex conjugate to T and therefore lies in $w=0$ too. The projection is regular at all points of $\mathcal{R}'$ in $\overline{W}$ so $\overline{T}$ maps *either* to a point implying that $\overline{P}$ is critical and that $\pi(\overline{P})$ is cuspidal on $\mathcal{D}'$ *or* to a line contained in $z_4=0$ implying that $\mathcal{D}'$ touches $z_4=0$ at $\pi(\overline{P})$. In both cases we contradict the way in which $\mathcal{D}'$ meets $z_4=0$ described in §4.2. We deduce then that <u>every point of</u> $\mathcal{D}'$ <u>in</u> $z_4=0$ <u>has a pre-image on</u> $\mathcal{R}'$.

It is easily checked that the pre-images of P_3,P_4,P_7,P_8 are the points $\overline{J}_{15}, \overline{J}_{14}, \overline{J}_{25}, \overline{J}_{24}$ and that the points E_1,E_2 have unique pre-images $\overline{E}'_1,\overline{E}'_2$ lying on $\overline{L}_3$ (which, generically, are not the points $\overline{J}_{34}$, $\overline{J}_{35}$). For example, the pre-image of the point $P_3=(0,1,-\frac{d_2}{d_3},0,u,0,-\frac{d_5}{d_7}u,0)$ satisfies $w=0$ thus $x_j+iy_j=0$ for $j=1,6$ and 8, and $x_j-iy_j=0$ (since $x_j+iy_j\neq 0$) for $j=2,3,5$ and 7. Thus $x_2+iy_2=2x_2=1$, $x_3+iy_3=2x_3=-\frac{d_2}{d_3}$, $x_5+iy_5=2x_5=u$, $x_7+iy_7=2x_7=-\frac{d_5}{d_7}u$ and hence $x_2=\frac{1}{2}$, $y_2=-\frac{i}{2}$, $x_3=\frac{-d_2}{2d_3}$, $y_3=\frac{i\,d_2}{2d_3}$, $x_5=\frac{u}{2}$, $y_5=-\frac{iu}{2}$, $x_7=-\frac{d_5}{2}\frac{u}{d_7}$, $y_7=i\frac{d_5}{2}\frac{u}{d_7}$. Applying equations (4.2), yields $x_1=y_1=x_8=y_8=0$ and hence we have defined all the co-ordinates uniquely giving the pre-image

$$\overline{J}_{15}=(0,\ 0,\ 1,\ -i,\ -\frac{d_2}{d_3},\ i\frac{d_2}{d_3},\ u,\ -iu,\ 0,\ 0,\ -\frac{d_5}{d_7}u,\ i\frac{d_5}{d_7}u,\ 0,\ 0,\ 0)$$

The pre-image of Q_1 under the mapping π is the line M (whose equation we gave earlier) which we know must meet the curve $\mathfrak{R}'$ in at least one point. We shall show that $\mathfrak{R}'$ meets M in just one point $\bar{J}_{13}$.

Suppose then that $\mathfrak{R}'$ meets M at a point $P \neq J_{13}$. Since M does not meet its complex conjugate line $\overline{M}$, P must be complex with a conjugate point $\overline{P}$ on $\mathfrak{R}'$ lying on $\overline{M}$. But $\overline{M}$ is contained in the hyperplanes $x_7 + iy_7 = 0$, $x_8 + iy_8 = 0$, $w = 0$ and meets the centre of projection in a point as the reader may readily check. Thus $\overline{M}$ maps to a point in the hyperplanes $z_4 = 0$, $z_7 = 0$ and $z_8 = 0$. In particular, $\overline{P}$ maps to this point implying that $\mathfrak{D}'$ contains a point lying on all three hyperplanes; a clear contradiction, since we know that no such point exists by the results of §4.2.

We have showed, therefore, that $\mathfrak{R}'$ meets the plane $\overline{W}$ in the set $\overline{A} = \{\bar{J}_{123}, \bar{J}_{15}, \bar{J}_{14}, \bar{J}_{25}, \bar{J}_{24}, \bar{E}'_1, \bar{E}'_2\}$ and hence $\mathfrak{R}'$ meets the complex conjugate plane W in the set $A = \{J_{123}, J_{15}, J_{14}, J_{25}, J_{24}, E'_1, E'_2\}$. In fact, we have established a stronger result. <u>The projection</u> $\pi|_{\mathfrak{R}'}$ <u>only fails to be defined at points in</u> A <u>and is a</u> 1-1 <u>correspondence between the open sets</u> <u>$\mathfrak{R}\backslash A$ and $\mathfrak{R}'\backslash\pi|_{\mathfrak{R}'}(A)$: implying that they are isomorphic sets</u>. Firstly, this implies that P_3, P_4, P_7, P_8, E_1, E_2, and Q_1 are points with the same singularity type as $\bar{J}_{15}$, $\bar{J}_{14}$, $\bar{J}_{25}$, $\bar{J}_{24}$, $\bar{E}'_1$, $\bar{E}'_2$ and $\bar{J}'_{123}$ respectively. Thus <u>with our assumptions</u> Q_1 <u>is an ordinary double point on</u> $\mathfrak{R}$. Secondly, this implies that any finite point P on $\mathfrak{R}'$ has the same singularity type as the finite point $\pi|_{\mathfrak{R}'}(P)$ on $\mathfrak{D}'$ so that the condition on the design parameters for $\mathfrak{R}'$ to possess a finite singular point is identical to the condition derived in §4.2 for $\mathcal{D}'$ to possess a finite singular point. This condition should be called the **Grashof equality**, since it is the exact analogue of this concept for the planar four-bar (in §1.2). We recall that this condition, phrased in terms of the design parameters, is a polynomial one (not identically zero), so it follows that generically, by which we mean for almost all design parameters in the sense of Lebesgue measure, the Darboux and linkage varieties have no finite singularity; and in particular, we can always avoid finite singularities by small deformations of the design parameters. Henceforth, we shall refer to a Watt I mechanism as **generic** when the Grashof equality does not hold.

We assert that the degree of the curve $\mathcal{R}'$ is sixteen. For a given hyperplane H, the degree of a curve is equal to the sum of all intersection multiplicities $i(P,H\cap\mathcal{R}')$ for points P lying in the intersection of H and $\mathcal{R}'$. If we take H to be the hyperplane given by $w=0$ then we have

$$\deg \mathcal{R}' = \Sigma_{P\in A} i(P,H\cap\mathcal{R}') + \Sigma_{\overline{P}\in\overline{A}} i(\overline{P},H\cap\mathcal{R}')$$

But under the involution of complex conjugation, intersection multiplicity is preserved, thus $i(P,H\cap\mathcal{R}') = i(\overline{P},H\cap\mathcal{R}')$. Hence

$$\deg \mathcal{R}' = 2\cdot\Sigma_{\overline{P}\in\overline{A}} i(P,H\cap\mathcal{R}')$$

and now we need only calculate the intersection multiplicities $i(\overline{P},H\cap\mathcal{R}')$. To do this we use the fact from the theory of projections (Theorem A12) that under a degree one linear projection π, intersection multiplicity does not decrease

$$i(\bar{P},H\cap\mathcal{R}') \leq i(\pi(\bar{P}),\pi(H)\cap\pi(\mathcal{R}'))$$

for any curve $\mathcal{R}'$, any point $\bar{P}$ on $\mathcal{R}'$ and a given hyperplane H containing the centre of projection. Now let $\mathcal{R}'$ and H be as above and let $\bar{P}$ be any point in $\bar{A}$, then $\pi(\bar{A})=\{Q_1,P_3,P_4,P_7,P_8,E_1,E1\}$, $\pi(H)$ is the hyperplane $z_4=0$ and $\pi(\mathcal{R}')=\mathcal{D}'$. Denote by p_k,e_k,q_k, the intersection multiplicities of $\mathcal{D}'$ with the hyperplane $z_4=0$ at the points P_k,E_k,Q_k and denote by j_{kl}, j_{123} the intersection multiplicity of $\mathcal{R}'$ with the hyperplane $w=0$ at the points $\bar{J}_{kl}$, $\bar{J}_{123}$. We recall that $\mathcal{D}'$ has degree 8 so that the total intersection multiplicity of $\mathcal{D}'$ with $z_4=0$ is 8 by Bézout's theorem. Further, we recall that Q_1 is a double point and that the remaining intersections P_3, P_4, P_7, P_8, E_1, E_2 of $\mathcal{D}'$ with $z_4=0$ are all simple points (with our assumptions). So $p_k=1$ for $k=3,4,7,8$, $e_1=e_2=1$ and $q_1=q_2=2$. It follows that $j_{kl}=1$ and that $j_{123}=2$ since $j_{123}\geq \text{mult}(\bar{J}_{123})=2$. The required result that <u>the degree of $\mathcal{R}$ is 16</u> now follows.

For the Watt II mechanism we have a projection $\pi':P\mathbb{C}^{14}\rightarrow P\mathbb{C}^7$ which maps $(x'_2,y'_2,\ldots,x'_8,y'_8,w')$ to $(z_1,\ldots,z_8)$ where $z_k=x'_k+iy'_k$ for $k\neq1$, and $z_1=-w'$, so the restriction $\tau=\pi'|_{\mathcal{S}'}$ defines a birational map between $\mathcal{S}'$ and $\mathcal{D}'$ failing to be regular only where $\mathcal{S}'$ meets W'. In a similar manner to the Watt I we can show that points of $\mathcal{D}'$ with $z_1=0$ are not closure points and hence have a pre-image on $\mathcal{S}'$. It is easily verified that P_1, P_2, P_3, P_4 have pre-images $\bar{J}'_{25}$, $\bar{J}'_{24}$, $\bar{J}'_{35}$, $\bar{J}'_{34}$. The points P_5, P_6, P_7, P_8, E_1, E_2, F_1, F_2 have no pre-image so they <u>must</u> be closure points implying that there are eight distinct branches of $\mathcal{S}'$ passing through the centre of projection and therefore meeting W. These branches have complex conjugate branches meeting $\overline{W}$ which will project to branches through points in $z_1 = 0$. Then four of these branches must pass through P_1, P_2, P_5, P_6, two branches pass through Q_1 and the remaining two pass through Q_2. But the pre-image on $\mathcal{R}$ of Q_1 (resp. Q_2) is M' (resp. $\overline{M}'$) which meets $\overline{W}$ in just one point J'_{123} (resp. J'_{45}). Therefore, the pre-images of the two branches through Q_1 (resp. Q_2) must pass through J'_{123} (resp. J'_{45}). Then it follows by similar arguments to the Watt I case that P_1, P_2, P_3, P_4, Q_1, Q_2, are points with the same singularity type as $\overline{J}'_{25}$, $\overline{J}'_{24}$, $\overline{J}'_{35}$, $\overline{J}'_{34}$, $\overline{J}'_{123}$, $\overline{J}'_{45}$, respectively. In particular, $\overline{J}'_{123}$ and $\overline{J}'_{45}$ are ordinary double points under the assumption that conditions (C1),(C2),(C3) do not hold. Moreover, we find that the degree of the residual linkage variety $\mathcal{S}'$ is 16. It is worthwhile noting that, although $\mathcal{R}'$ and $\mathcal{S}'$ have the same degree and are birationally isomorphic, they are not projectively equivalent, since $\mathcal{R}'$ and $\mathcal{S}'$ have the same number of finite singularities, but a different number of double points in the hyperplane at infinity (i.e. 2 and 4 respectively).

§4.4. The Reductions of the Watt Darboux Variety.

In the first three sections of this chapter we set up the basic geometry of the linkage varieties for the Watt I and Watt II mechanisms. These varieties are birationally isomorphic, each comprising two skew lines, two skew planes and a curve of degree 16, called the residual linkage curve. In general, one expects this curve to be irreducible, but there are certainly degenerate situations when it can reduce. And correspondingly any associated coupler curve, which is a projection of the residual linkage curve, will also reduce. Ideally, one would like a complete list of the possible reductions in terms of the design parameters. That problem was effectively solved for the planar four-bar over a century ago, but has never been discussed for more complex mechanisms, despite the fact that one can gain considerable insight into coupler curves by effecting small perturbations of reducible cases (see, for example, [Fichter]).

This section is devoted to presenting a complete solution to this problem. There are two key observations. The first is that both the residual linkage curves for the Watt I and the Watt II are birationally equivalent to the same Darboux residual curve. The details of this relation were set out in §1.2. Thus, in principle, it suffices to determine the reductions of the residual Darboux curve - an easier problem since that curve is only of degree 8 and lives naturally in a 3-space. We remind the reader that this was done for the four-bar in §1.2 and provides a simple example of the approach that we shall adopt. The second key observation follows from the results of §1.4 that there is a natural projection from the linkage variety of a mechanism onto that of any sub-mechanism: one can paraphrase this in more mechanical terms by saying that any configuration of the mechanism determines uniquely a configuration of a given sub-mechanism. For either Watt six-bar there are two natural sub-mechanisms, namely, the two underlying planar four-bars, so that we have two natural projections to consider. Moreover, both of these projections have degree 2 in the sense of general algebraic geometry: in mechanical terms that means that for a given general configuration of one four-bar there are two distinct corresponding configurations of the Watt six-bar. Phrased in terms of the corresponding Darboux curves this means that there are two natural projections from the Darboux curve for the six-bar onto those for the two planar four-bars. The latter curves are plane cubics whose geometry is very well understood; in particular, one knows exactly how the Darboux cubic of a planar four-bar reduces in terms of the design parameters (§1.1). From this point on it is a technical exercise in algebraic geometry, using the Genus Formula for a curve and the Projection Formula, to determine the reductions of the residual Darboux curve corresponding to the types of the underlying four-bars. The passage from the residual Darboux curve to the residual linkage curve proceeds via the following proposition: each irreducible component of the residual Darboux curve corresponds to a birationally equivalent component of the residual linkage curve of twice the degree. Finally, one can deduce the possible reductions of coupler curves.

We begin by recalling that the four linear equations in (4.5) are linearly independent defining a three dimensional subspace of $P\mathbb{C}^7$ so $\mathcal{D}'$ is isomorphic to the intersection of two cubic surfaces K_B and K_U in $P\mathbb{C}^3$. Explicitly, we may eliminate z_4, z_5, z_6, z_8, so that K_B and K_U are given by equations (4.12) and (4.13) respectively

$$\left.\begin{array}{l} d_1d_3z_1^2z_2 + d_2d_3z_1z_2^2 + d_1d_2z_1^2z_3 + d_1d_2z_2^2z_3 + \\ d_1d_3z_2z_3^2 + d_2d_3z_1z_3^2 + (d_1^2 + d_2^2 + d_3^2 - d_4^2)z_1z_2z_3 = 0 \end{array}\right\} \quad (4.12)$$

$$\left.\begin{array}{l} d_6d_7uv^2z_1^2z_2 + d_5d_7u^2vz_1z_2^2 + d_5d_6v^2z_1^2z_7 + d_5d_6u^2z_2^2z_7 + \\ d_6d_7uz_2z_7^2 + d_5d_7vz_1z_7^2 + (d_5^2 + d_6^2 + d_7^2 - d_8^2)z_1z_2z_7 = 0 \end{array}\right\} \quad (4.13)$$

For the remainder of this section we shall refer to the isomorphic curve as the Darboux curve without change of notation. The reader may easily check that K_B is a point cone with vertex $Q_1 = (0,0,0,1)$ over a plane cubic curve B and K_U is a point cone with vertex $Q_2 = (0,0,1,0)$ over a plane cubic curve U, where B and U are given by an identical set of equations (4.12) and (4.13). If we consider the two quadrilaterals in the Watt mechanism as "submechanisms", then we find that the Darboux cubics corresponding to the base and upper quadrilaterals are projectively isomorphic to the cubics B and U respectively. Again we shall refer to the isomorphic curves as the Darboux varieties without change of notation. The Darboux varieties of quadrilaterals are well understood. In §1.1 we showed that the cubics could be classified into four types namely: generic, circumscriptible, parallelogram/kite, rhombus, depending on the design parameters. However, we shall make two cases from the previous kite/parallelogram case, so that we distinguish five types

(1) $d_1+d_2 \neq d_3+d_4$, $d_1+d_3 \neq d_2+d_4$ and $d_1+d_4 \neq d_2+d_3$: B is a non-singular cubic.

(2) $d_1+d_2 = d_3+d_4$ or $d_1+d_3 = d_2+d_4$ or $d_1+d_4 = d_2+d_3$ (but no two hold simultaneously) : B is a nodal cubic.

(3) $d_1 = d_2 \neq d_3 = d_4$: B reduces to a conic and chord (with two real double points).

(4) $d_1 = d_4 \neq d_2 = d_3$ or $d_1 = d_3 \neq d_2 = d_4$: B reduces to a conic and chord (with two real double points).

(5) $d_1 = d_2 = d_3 = d_4$: B is the union of three distinct lines (with three real double points).

In cases (2)-(5) the singular points are of the form $(\pm 1, \pm 1, 1)$.

<u>Remark</u>: For each Grashof condition of the Darboux cubics which is satisfied, there is a Grashof condition of the Watt linkage variety. Indeed, for each singular point $P = (\pm 1, \pm 1, 1)$ of B (resp. U), π_B (resp. π_U) maps two distinct singular points of $\mathcal{R}'$ onto P. Indeed, the singular points of B have the form $(\pm 1, \pm 1, 1, *)$ and the singular points of U have the form $(\pm 1, \pm 1, *, 1)$. The reader may wish to refer to §4.2.

Let $1 \leq i \leq 5$ denote the above type of base quadrilateral in the Watt mechanism and let $1 \leq j \leq 5$ denote the the analogous type for the upper quadrilateral (replacing 1,2,3,4,B by 5,6,7,8,U respectively in the above list) then we will write i/j to describe the Watt mechanism, whenever the appropriate conditions on the d_k's occur. The reader should note that it is necessary to split up the parallelogram/kite case into two types (3) and (4) for the following discussion: for it will soon become clear that the two types give rise to distinct geometries of $\mathcal{D}'$.

For a given reduction of B (resp. U) into linear and quadratic components we get a corresponding reduction of K_B (resp. K_U) into linear and quadratic components by taking the appropriate point cone over each component. Then for a given mechanism of type i/j we know the components of K_B and K_U from which we can immediately deduce a reduction by taking each component of K_B and intersecting it with each component of K_U. The reader should note that the resulting subvarieties will not, in general, be the irreducible components of $\mathcal{D}'$, since each intersection may reduce further. Thus, for instance, if K_B is a plane and quadric and K_U is three planes the intersection will yield three lines and three conics, where the conics may be reducible. It would suffice then, to determine which of the conics are irreducible in order to establish the reduction of $\mathcal{D}'$ in this case.

It is natural to consider the projection $\pi_B:(z_1,z_2,z_3,z_7)\rightarrow(z_1,z_2,z_3)$ on the Darboux variety $\mathcal{D}$ (remembering that we have eliminated variables z_4,z_5,z_6,z_8). Let us for the moment assume simply that $\mathcal{D}$ is a curve, i.e. that the condition $d_1=d_2,d_3=d_4,d_5=d_6,d_7=d_8,u=v$ does not hold. Later in this section we shall make further restrictions. Clearly the image of the projected curve is precisely the Darboux curve for the lower quadrilateral B. The centre of the projection is the subspace defined by $z_1=z_2=z_3=0$ and it intersects $\mathcal{D}$ in the point Q_1. Thus, we are projecting $\mathcal{D}$ from one of its singular points Q_1. The picture one has in mind (see Fig. 4.12) is that the cone K_B represents this projection. $\mathcal{D}$ lies on K_B and passes through the vertex Q_1. A point P on B has as its pre-image a line L_P on the cone through Q_1 and P. L_P generally meets the curve $\mathcal{D}$ in a finite number of points, but exceptionally it may be a component.

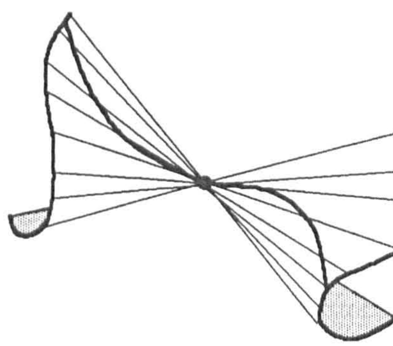


Fig. 4.12

The first step is to find the inverse image of a point (z_1,z_2,z_3) on the Darboux curve of the base quadrilateral B. There are six distinguished points, namely those which lie on one of the hyperplanes $z_k=0$ $(k=1,2,3)$. We will consider these separately later, but for the moment we may assume that $z_k\neq 0$ $(k=1,2,3)$. Assume then, that the point satisfies equation (4.12), then we require the z_7 co-ordinate satisfying the equation (4.13). If we let $P=d_5uz_2+d_6vz_1$ and $Q=(d_5vz_1+d_6uz_2)/uv$, then we obtain a quadratic in z_7

$$d_8Qz_7^2+(PQ-d_7^2+d_8^2)z_7+d_8P=0$$

In general, we expect to get two points in the pre-image, indeed the mechanics of the Watt strongly suggest that. For fixing z_1, z_2, z_3, fixes a position of the lower quadrilateral. Then there are two choices of z_7, z_8 as indicated in Fig. 4.13.

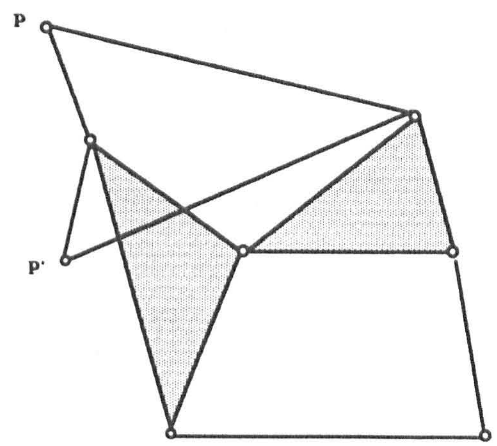


Fig. 4.13

But exceptionally, we could get a line. This happens if and only if $P=0$, $Q=0$ and $d_7=d_8$. It follows from $P=0$ and $Q=0$ that $d_5=d_6$, and $uz_2+vz_1=0$ i.e. the upper quadrilateral collapses (see Fig.4.14). Conversely, the conditions $d_5=d_6$ and $d_7=d_8$ imply that there is a point on B whose pre-image is a line.

Thus, provided the condition $d_5=d_6$, $d_7=d_8$ does not hold, we have just a finite number of points in the pre-image of a point on the Darboux curve of the base quadrilateral. Indeed, we have showed that in this case we have two points on a Zariski open subset of B and therefore <u>the projection</u> $\pi_B|_{D'}$ <u>has degree two.</u>

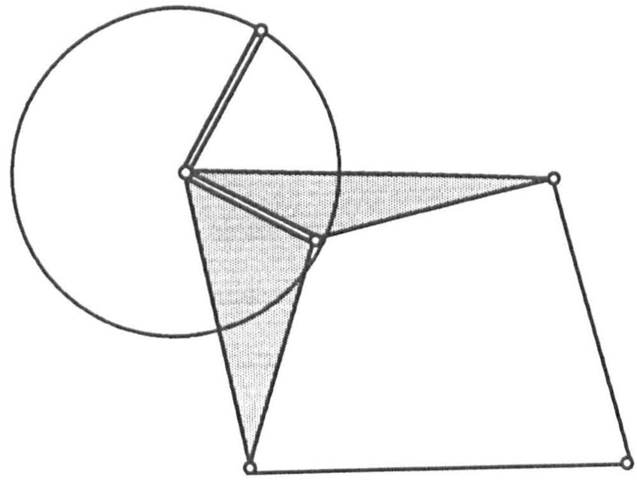


Fig. 4.14

If the condition $d_5 = d_6$, $d_7 = d_8$ does hold, we hope that there are only finitely many points on B for which the pre-image is a line, so that we get a finite map. To this end, note that the condition $uz_2 + vz_1 = 0$ defines a line $z_1 = \lambda z_2$ (where $\lambda = -u/v$) in the plane of the Darboux cubic. We require the points of intersection of this line with B: in general, there will be only three, but there is the possibility that the line $z_1 = \lambda z_2$ is a component of B. Certainly that could only be the case when the base quadrilateral is of the parallelogram/kite type by the analysis of the four-bar. Substituting $z_1 = \lambda z_2$ in equation (4.12), we find that

$$z_2^3(d_1 d_3 \lambda^2 + d_2 d_3 \lambda) + z_2^2 z_3(d_1 d_2 \lambda^2 + [d_1^2 + d_2^2 + d_3^2 - d_4^2]\lambda + d_1 d_2) + z_2 z_3^2(d_1 d_3 + d_2 d_3 \lambda) = 0.$$

Thus we get a line component if and only if all the coefficients are zero. Hence, we have $d_1d_3\lambda^2 + d_2d_3\lambda = 0$ $(\lambda \neq 0)$ giving $\lambda = -\frac{d_2}{d_1}$, and $d_1d_3 + d_2d_3\lambda = 0$ giving $\lambda = -\frac{d_1}{d_2}$. Thus $\underline{d_1 = d_2}$ and $\lambda = -1$. Hence, $d_1d_2\lambda^2 + [d_1^2 + d_2^2 + d_3^2 - d_4^2]\lambda + d_1d_2 = 0$, which implies that $\underline{d_3 = d_4}$. But combining this with the conditions above yields $d_1 = d_2$, $d_3 = d_4$, $d_5 = d_6$, $d_7 = d_8$, $u = v$. This condition clearly implies conditions (C1),(C2),(C3). Since we are assuming that these conditions do not hold, we see that there are at most three points on B for which the pre-image could be a line. The reader may readily check that all three points have a line of pre-images on $\mathcal{D}$. <u>One point is (0,0,1) and has as its pre-image the line L</u>. <u>The other two points</u>, whose co-ordinates we shall not write down, <u>have pre-images the lines</u> Q_1E_1, Q_1E_2, <u>which are distinct, provided condition (C2) is not satisfied</u>.

It now remains to find the pre-images of the points with $z_k = 0$ for some k. They are (0,0,1), (0,1,0), (1,0,0), $(0,-d_3,d_2)$, $(-d_3,0,d_1)$, $(-d_2,d_1,0)$ with pre-images L, P_1 and P_2, P_5 and P_6, P_3 and P_4, P_7 and P_8, E_1 and E_2 respectively. Thus, even in the case when $d_5 = d_6$, $d_7 = d_8$ and we have line components, the projection π_B defines a degree two mapping from the subvariety of $\mathcal{D}$ obtained by removing the lines Q_1E_1, Q_1E_2, L to the Darboux variety B.

In an identical manner we may consider the effect of the projection $\pi_U:(z_1,z_2,z_3,z_7) \to (z_1,z_2,z_7)$ on $\mathcal{D}$. We find that we are projecting $\mathcal{D}$ from the singular point Q_2 to the Darboux variety U for the upper quadrilateral.

Proceeding as for the previous projection, we find that a point on U with no z_k co-ordinate zero has, in general two distinct pre-images implying <u>that π_U defines a degree two mapping between $\mathcal{D}'$ and U.</u> Exceptionally, we could get a line. In this case we must have $d_1 = d_2$, $d_3 = d_4$: moreover, the identities imply that $uz_2 + vz_1 = 0$ and so the lower quadrilateral collapses. Conversely, if the base quadrilateral does collapse in this manner, then U does have points whose pre-images are lines.

If $d_1 = d_2$, $d_3 = d_4$, then we still hope that there are finitely many points on U for which the pre-image is a line. The result is that if there are infinitely many points whose pre-image is a line then $d_5 = d_6$, $d_7 = d_8$, $u = v$ but combining these with the above condition we get $d_1 = d_2$, $d_3 = d_4$, $d_5 = d_6$, $d_7 = d_8$, $u = v$. As we pointed out earlier this contradicts our assumptions. In the case when only finitely many points have line pre-images we find that $(0,0,1)$ has the pre-image L and there are two other points with the lines Q_2F_1, Q_2F_2 as their pre-images. Thus even in this case π_U defines a degree two mapping from the subvariety of $\mathcal{D}$ obtained by removing the lines Q_2F_1, Q_2F_2, L to the Darboux variety U.

It is also worth noting that any component of $\mathcal{D}'$, lying in a linear or quadratic component of K_B (resp. K_U), will map into the linear or quadratic component of B (resp. U) under the projection π_B (resp. π_U). So for example, a conic component of $\mathcal{D}'$, which is the intersection of a linear component of K_B and a quadratic component of K_U, will map into the linear component of B under π_B and the quadratic component of U under π_U. The significance of this fact will soon become clear.

For a curve C we will write the symbol $f_1+f_2+...+f_r$ to represent its reduction into components $C_1,C_2,...,C_r$ where f_k is the degree of the k^{th} component C_k and the sum of the f_k is equal to the degree of C. In particular, we shall use this notation for the reductions of the Watt I residual Darboux variety.

Before we continue we note that by the symmetry of our situation the residual Darboux varieties of the mechanisms i/j and j/i have the same symbol $f_1+f_2+...+f_r$ so we need only consider those i/j with $j \le i$.

At a number of points in the following calculation we will apply the Genus Formula, but first we shall need to know the arithmetic genus of $\mathcal{D}$. The variety $\mathcal{D}$ is the intersection of two cubic surfaces in $P\mathbb{C}^3$ thus we may apply Theorem (A5) which gives a formula for determining the arithmetic genus of an intersection of hypersurfaces.

Theorem Let $V_1,...,V_{n-1}$ be hypersurfaces of degrees $d_1,...,d_{n-1}$ in $P\mathbb{C}^n$ intersecting in a curve, then the arithmetic genus p_a of that curve is given by the formula

$$2-2p_a = d.\{n + 1 - \Sigma_{i=1}^{n} d_i\} \quad \text{where } d = \Pi_{i=1}^{n} d_i.$$

Then $n = 3$, $d_1 = d_2 = 3$ and it follows from the formula that <u>$\mathcal{D}$ has arithmetic genus 10</u>.

To determine the genera of the irreducible components of a connected curve we have the **Genus Formula** (Theorem A8), which states that, if a connected curve C has r components $C_1,\ldots,C_r$, and $\tilde{C}_1,\ldots,\tilde{C}_r$ are their normalisations, then their genera are related by the following formula

$$p_a(C) = \Sigma_{i=1}^{n} p_a(\tilde{C}_i) + \Sigma^* \delta_p - (r-1)$$

where the * denotes that the sum is taken over the singular points of C and δ_p is the delta invariant of the singular point P.

Under our assumptions Q_1 and Q_2 are ordinary triple points on $\mathcal{D}$, so the δ-invariant of these points is 3. Then $p_a(\mathcal{D}) = 10$ and if $\mathcal{D}$ has r components $C_1,\ldots,C_r$ other that L the genus formula yields

$$r = g(C_1) + \ldots + g(C_r) + \Delta - 4$$

where $g(C_i) = p_a(\tilde{C}_i)$ is the geometric genus of C_i and Δ is the sum of the δ-invariants of all finite singular points. We shall now proceed with the reductions. The results are summarised in the table which may be found at the end of the section.

We begin with the most degenerate situation 5/5, where B and U are triangles. Since K_B and K_U are point cones over these curves, they are unions of three planes. Thus their intersection is nine lines, immediately giving the required reduction. B and U each have three double points so $\mathcal{R}'$ has twelve singular points (recall that $\mathcal{D}'$ has two singular points for each one on B and U). Applying the Genus Formula, we find that each of the twelve singular points on $\mathcal{R}'$ has $\delta = 1$.

For the cases 5/4 and 5/3 K_B is the union of three planes and K_U is the union of a quadric cone and a plane. Thus they intersect in three lines and three (possibly reducible) conics. Under the projection π_U the conic component of U must be double covered so at least two of the conics must be irreducible. Further, we have three singular points on B and two on U, so that $\mathcal{R}'$ has ten finite singular points implying that $\Delta \geq 10$. Applying the Genus Formula, we find that $r \geq 6$: so the remaining conic reduces, giving the reduction 1+1+1+1+2+2 and implying that each finite singular point has $\delta = 1$. But $j=5$ so $d_1=d_2$, $d_3=d_4$ and therefore two of the lines are Q_2E_1, Q_2E_2. Also when $i=3$, we have $d_5=d_6, d_7=d_8$: therefore, we have the lines Q_1F_1, Q_1F_2. So in the case 5/3 the four lines are $Q_2E_1, Q_2E_2, Q_1F_1, Q_1F_2$ and the conics do not meet Q_1 or Q_2, whereas in the case 5/4 both conics pass through Q_1, two lines pass through Q_2 and the remaining lines do not meet Q_1 or Q_2.

In the cases 5/2, 5/1 B is three lines and U is irreducible so K_B is the union of three planes and K_U is an irreducible cubic cone. Thus the intersection of K_B, K_U is the union of three (possibly reducible) plane cubics. But U must be double covered by π_U so at least two of the cubics are irreducible. Further, we know that $\mathcal{D}'$ contains the line L and the two lines Q_2E_1, Q_2E_2 (since $d_1 = d_2$, $d_3 = d_4$) : so these must be contained in the third cubic. Hence the reduction is 1+1+3+3. The projection π_B maps the cubics to lines, since they lie in planes meeting the centre of projection. Applying the Projection Formula, we find that the cubics must meet the centre of projection Q_1 and map onto the lines with degree 2. The projection π_U can only map the cubics birationally onto U. Thus the cubics have genus zero in case 5/2 and genus one in case 5/1. The reader may now verify, using the genus formula, that in case 5/2 $\mathcal{D}'$ has eight finite singular points with $\delta=1$ and that in case 5/1 $\mathcal{D}'$ has six finite singular points with $\delta=1$.

Next, we consider the case 4/4. K_B and K_U are both the union of a plane and a quadric cone intersecting in a line L_1, two conics and a quartic (possibly reducible). In particular, the conics lie, firstly, on a plane which is mapped into a line under one projection and secondly, on a quadric which is mapped into a conic under the other projection. But the conics cannot map to points, so they are irreducible and the projection π_B (resp. π_U) maps one onto the conic and the other onto the linear component of B (resp. U). So the conics provide a single covering of the line and conic components of B and U. But the line components of B and U need one further covering, so there must be a component of $\mathcal{D}'$ lying in the plane components of K_B, K_U mapping to a line : the only candidate is $L_1 = L$. But $\mathcal{D}'$ has eight singular points, so $\Delta \geq 8$ and the Genus Formula implies that $r \geq 4$. Since L must be a component of the quartic, we may now deduce that the quartic reduces. The quartic cannot reduce to 1+1+2 or 1+1+1+1, since the lines would map to lines (no components map to points in case 4/4) implying that the line components of B and U have at least two further coverings thus giving a contradiction. The only other reduction, which includes a line, is 1+3. The Projection Formula implies that the cubic maps birationally onto the conic components of B and U, thus has genus zero. The Genus Formula implies that the eight finite singular points of $\mathcal{D}'$ have $\delta = 1$.

In the case 4/3, K_B and K_U are both the union of a plane and a quadric cone intersecting in a line L_1, two conics and a quartic (possibly reducible). Since $d_5=d_6, d_7=d_8$ we have two line components Q_1F_1, Q_1F_2 which are mapped to points by π_B and mapped to U_1 (the line component of U) by π_U, giving a double covering of U_1. Thus $\mathcal{D}'$ has no other lines, since they would give a third covering of U_1. Suppose then that the quartic reduces, then it is the union of two conics. Hence, $\mathcal{D}'$ is the union of a line and four conics. But no conic passes through Q_2 for otherwise π_B would map it to a line thus giving a third covering of U_1. Thus no component of $\mathcal{D}'$ passes through Q_2 giving a contradiction. Therefore the quartic is <u>irreducible</u> and passes through Q_2. Applying the Projection Formula, we find that the quartic must have a double point at Q_2 and maps birationally onto U_2, the conic component of U. Thus the quartic has genus zero. Hence, the required reduction is 1+2+2+4. We may now apply the Genus Formula to show that the finite singular points have $\delta=1$.

The next cases to consider are 4/2 and 4/1 where K_B is the union of a plane and quadric cone and K_U is an irreducible cubic surface. Their intersection is the union of a plane cubic and a sextic, both of which are mapped <u>onto</u> the irreducible cubic U by π_U. Thus, all components of $\mathcal{D}'$ have degree ≥ 3, since no components map to points. It follows that L must be contained in the sextic and therefore the only possible reduction of $\mathcal{D}'$ is 3+5. The cubic maps birationally onto U and the quintic must give a further covering. The Projection Formula shows that this can only be the case if the quintic has a double point at Q_2 and maps birationally onto U. Mapping this time with the projection π_B, we find that the only way a cubic and quintic can double cover the line and conic components of B is when both map with degree two. It follows that both components pass through the centre of projection Q_1. We observe that the genus of the cubic, quintic and U are identical (since they are birationally equivalent) and that $g(U)=0$ for $j=2$ and $g(U)=1$ for $j=1$. Applying the Genus Formula, we find that in the case 4/2 the six finite singular points of $\mathcal{D}'$ (two for each singular point of U and B) have $\delta=1$ and that in the case 4/1 the four finite singular points of $\mathcal{D}'$ (two for each singular point of B, U non-singular) have $\delta=1$.

In the case 3/3, K_B and K_U are both the union of a plane and a quadric cone intersecting in a line L_1, two conics and a quartic (possibly reducible). Since $d_1=d_2, d_3=d_4, d_5=d_6, d_7=d_8$, we have the four line components, $Q_1F_1, Q_1F_2, Q_2E_1, Q_2E_2$. Under the projection π_U (resp. π_B), Q_1F_1, Q_1F_2 (resp. Q_2E_1, Q_2E_2) map to the line component of U (resp. B) implying that they lie in the plane component of K_U (resp. K_B). Thus at least one of the two lines lies in the plane component of K_U (resp. K_B) and the quadric component of K_B (resp. K_U), while one of the four lines might be L_1. Thus both conics reduce giving four lines and a (possibly reducible) quartic. Thus the line components of B and U are double covered implying that there are no more lines in $\mathcal{D}'$, since they would necessarily map to lines giving a third covering. We deduce that if the quartic reduces, it does so to two conics. We shall obtain a contradiction to this by observing that the eight singular points on $\mathcal{D}'$ lie on the lines $Q_1F_1, Q_1F_2, Q_2E_1, Q_2E_2$ (two on each) as an easy computation shows. Then, since no two of the lines meet in a finite point, the singular points must occur where the conics meet the line. Thus, if the conics meet at all, they must meet at one of these points giving a <u>triple</u> point on $\mathcal{D}'$. Applying the Genus Formula, we find $\Delta=10$. So there could be a triple point with $\delta=3$ (i.e. with distinct tangents) implying that the conics would meet with distinct tangents. We now recall that the quartic is given as the intersection of two quadrics in $P\mathbb{C}^3$ which is known to have arithmetic genus 1. Applying the Genus Formula here, we find that the quartic can only reduce to two conics, if there is a singular point with $\delta=2$ i.e. the conics meet with <u>non</u>-distinct tangents giving the required contradiction. Thus, the quartic is irreducible and the reduction of $\mathcal{D}'$ is 1+1+1+1+4.

The next cases to consider are 3/2, 3/1. Then K_B is the union of a plane and quadric intersecting the irreducible cubic surface K_U in a plane cubic and sextic (both possibly reducible). Since $d_1=d_2$, $d_3=d_4$, the lines Q_2E_1, Q_2E_2 are components of $\mathcal{D}$. The lines are mapped by π_B to the line component of B, so they lie in the plane component of K_B. Thus the cubic contains Q_2E_1, Q_2E_2 and one other line : namely L, since any other line would give a third covering of the line component of B. Therefore the sextic double covers the irreducible cubic U and since any component must have degree ≥ 3, the sextic is *either* irreducible *or* it is the union of two cubics mapping birationally onto U. Let us suppose then, that the sextic reduces to two cubics. First note that Q_1 is simple on both cubics; since, if it was a double point on one of them, then π_B would map that cubic to a line, which together with Q_2E_1, Q_2E_2, would give three coverings of the line component of B, a clear contradiction. Thus, the cubics are mapped birationally onto the conic component of B. But multiplicity does not decrease under birational projection (Theorem A12), thus we may deduce that the cubics have no finite singular point and, moreover, that the cubics are non-singular with genus zero (equal to that of a conic). Hence, for the case 3/1 we have an immediate contradiction, since the cubics are also birationally equivalent to U, a cubic with genus one. For the case 3/2, we note that π_U defines an isomorphism between an open set of either cubic and an open set of U. Indeed, the node on U has two pre-images on $\mathcal{D}$ which are singular points of $\mathcal{D}$ by the remark made at the beginning of the section. Thus, even if the pre-images lie on Q_2E_1, Q_2E_2 they also lie on one of the two cubics. Hence the node is contained in the isomorphic sets and it follows that one of the cubics has an ordinary double point. This contradicts the fact showed above that the cubics are non-singular.

To complete the list of reductions we must consider the cases 2/2, 2/1, 1/1. In any of these cases, no components map to points hence any component of $\mathcal{D}'$ maps onto the irreducible cubics B and U. This gives us three possible reductions: 8 or 3+5 or 4+4. If the reduction is 3+5, then the cubic maps birationally onto B and U and the quintic has double points at Q_1, Q_2 also mapping birationally onto B and U. But then $\pi_U(Q_1) = (0,0,1)$ must be singular on U contradicting the fact that all singular points of U are of the form $(\pm1,\pm1,1)$.

Consider now the reduction 4+4. Each quartic must meet both Q_1 and Q_2 and map birationally onto B and U. For the case 2/1 we have an immediate contradiction, since this would imply that B and U are birationally isomorphic, contrary to the fact that they have different genera. In the case 2/2 we note that for at least one of the quartics, the birational map π_U defines an isomorphism between an open set of that quartic and an open set of U containing its unique singular point. By the Remark made at the beginning of the section, the pre-image of a node has the form $(\pm1,\pm1,p,1)$ and is one of the finite singular points of $\mathcal{D}'$. Thus, the pre-image is an ordinary double point of the quartic and is mapped by π_B onto a singular point of B. But the singular point of B has the form $(1,\pm1,\pm1)$ and the point $(\pm1,\pm1,p,1)$ can map to this point if and only if $p = \pm1$. However, under our hypothesis such a point is never a singular point of $\mathcal{D}'$ giving the required contradiction. Finally, in the 1/1 case, $\mathcal{D}$ has no finite singular points. Applying the genus formula with $r = 4$ and genera equal to 1, yields that $\Delta \geq 4$ contradicting the fact that there is no finite singular point.

We now draw the reader's attention to the fact that the generic mechanism is of type 1/1 and hence <u>the residual Darboux variety $\mathcal{D}'$ is irreducible with no finite singular points</u>.

<u>We complete this section by proving that each component on the residual Darboux variety $\mathcal{D}'$ has as its pre-image on the Watt I residual linkage variety $\mathfrak{R}'$ a birationally isomorphic component of twice the degree</u>.

Let us continue to denote the projection from the residual linkage variety to the residual Darboux variety by π. Let C_d be a component of degree d of $\mathcal{D}'$. Then C_d has d distinct branches meeting the hyperplane $z_4 = 0$ (under our assumptions E_1 and E_2 are distinct and Q_1 is an ordinary double point). Thus its pre-image (also an irreducible variety since the projection π is 1-1 and C_d is irreducible) meets the plane $\overline{W}$ in d distinct branches and let us suppose that t branches meet W. But W lies in the centre of projection, so each branch meeting W is mapped by π to a branch through a closure point of C_d. But points with $z_k \neq 0$ for all k are not closure points as we showed earlier, so these branches pass through points with $z_k = 0$ for some $k \neq 4$ i.e through points in $\Phi = \{P_1, P_2, P_5, P_6, Q_2, F_1, F_2\}$. We now <u>claim</u> that any component C_d has at least d branches through points in Φ so that $t \geq d$. Therefore, since it is so for all components of $\mathcal{D}'$, we have

$$8 \geq \Sigma t_i \geq \Sigma d_i = 8$$

the last equality being true by the definition of degree. Thus $t_i = d_i$ and C_{d_i} has a pre-image of degree $2d_i$.

<u>It remains to prove the claim</u>. Suppose there are s branches of C_d passing through the points Q_1, E_1, E_2: so that $0 \leq s \leq 4$, since two branches may pass through Q_1. Then there are d-s branches passing through the points P_1, P_2, P_5 or P_6 and d-s branches passing through P_3, P_4, P_7, P_8 so that C_d meets the hyperplanes $z_3 = 0$ and $z_4 = 0$ in d points (counting multiplicities): thus satisfying Bézout's Theorem. Note that $0 \leq d-s \leq 4$.

If $d-s \leq 2$, then C_d passes through at most two of P_3,P_4,P_7,P_8. If C_d passes through just P_3 or P_7, then C_d meets $z_8 = 0$ in d branches through points in $\{P_2,P_6,Q_2,F_1,F_2\} \subset \Phi$. Similarly, if C_d passes through just P_4 or P_8, then C_d meets $z_7 = 0$ in d branches through points in $\{P_1,P_5,Q_2,F_1,F_2\} \subset \Phi$. The only other possibility is when $d-s = 2$ and C_d passes through one of P_3 or P_4 and through one of P_4 or P_8. C_d will meet two of P_1,P_2,P_5,P_6 and has $d-1$ branches through points in $\{P_1,P_5,Q_2,F_1,F_2\} \subset \Phi$ and $d-1$ branches through points in $\{P_2,P_6,Q_2,F_1,F_2\}$, so that C_d has d branches meeting $z_7 = 0$ and $z_8 = 0$. Hence, C_d has branches through <u>one</u> of P_1,P_5, one of P_2,P_6 and $d-2$ of Q_2,F_1,F_2 i.e. d branches through points in Φ as required.

If $d-s = 3$ then we must have one of the following possibilities

(i) C_d passes through both P_4 and P_8 and one of P_3,P_7. Hence, C_d has $d-2$ branches through P_2,P_6,Q_2,F_1,F_2 and has $d-2$ branches through P_1,P_5,Q_2,F_1,F_2 (so that C_d meets $z_7 = 0$ and $z_8 = 0$ in the correct number of branches). But C_d may only meet three of P_1,P_2,P_5,P_6, so C_d must pass through P_1,P_5 one of P_2,P_6 and $d-3$ branches passing through Q_2,F_1,F_2. Thus C_d has d branches passing through Φ as required.

(ii) C_d passes through both P_3 and P_7 and one of P_4,P_8. A similar argument to (i) gives the required result.

Finally, if $d-s=4$, C_d then passes through all P_j $1 \le j \le 8$ and $d-4$ branches through Q_2, F_1, F_2 (so that C_d meets $z_7=0$ correctly) thus giving d branches through points in Φ as required.

lower quadrilateral

upper quad	(1)	(2)	(3)	(4)	(5)
(1)	8	8	1+1+6	3+5	1+1+3+3
(2)	8	8	1+1+6	3+5	1+1+3+3
(3)	1+1+6	1+1+6	1+1+1+1+4	1+1+2+4	1+1+1+1+2+2
(4)	3+5	3+5	1+1+2+4	1+2+2+3	1+1+1+1+2+2
(5)	1+1+3+3	1+1+3+3	1+1+1+1+2+2	1+1+1+1+2+2	1+1+1+1+1+1+1+1

Table of reductions according to type i/j

§4.5. The Watt I Coupler Curves.

It is easily seen that any coupler curve for the Watt II mechanism is either the arc of a circle or an arc of a four-bar coupler curve and therefore of no interest to us here. Thus we devote this section to the study of the coupler curves of the Watt I mechanism.

Let us assume that $\mathfrak{R}'$ is an irreducible curve - in particular this is the case when the mechanism is either generic or one of the types 1/2, 2/1 or 2/2. For the Watt I mechanism there are two families of coupler curves (which are not coupler curves of lower order mechanisms, for example four-bars or a single link): one family comprises the loci of points S rigidly attached to link 7 and the other the loci of points T rigidly attached to link 8. Let us first consider S. With the notation of §4.1 we can write $S = d_1 z_1 + d_6 z_6 + s.z_7$ where s is a fixed complex number. As we vary s we move through the first family of coupler curves. If we write $s = s_1 + is_2$ with s_1, s_2 real, then we can think of S as a point in the projective plane with homogeneous co-ordinates P_1, P_2 and P_3 given by $P_1 = d_1x_1 + d_6x_6 + s_1x_7 - s_2y_7$: $P_2 = d_1y_1 + d_6y_6 + s_2x_7 + s_1y_7$: $P_3 = w$, thus defining a projection $\tau_s : P\mathbb{C}^{14} \backslash V_s \to P\mathbb{C}^2$ with V_s the centre of projection, that is, the projective subspace defined by the vanishing of P_1, P_2 and P_3. The restriction $\varphi_s = \tau_s|_{\mathcal{R}'}$ to the residual linkage curve is a rational mapping. The Zariski closure of $\varphi_s(\mathcal{R}')$ is an algebraic curve C_s in $P\mathbb{C}^2$ which we shall refer to as the **complex coupler curve**. The centre V_s meets $\mathcal{R}'$ in two points J_{14}, $\bar{J}_{14}$ in general. Exceptionally, we have the following cases giving additional points:

(i) J_{123}, $\bar{J}_{123}$, J_{15}, $\bar{J}_{15}$, if and only if $s = 0$. Then S is positioned at the hinge C (see Fig.4.4) and the locus will be an arc of a circle.

(ii) J_{24}, $\bar{J}_{24}$ if and only if $d_1 = d_6$, $v = -1$. Then hinge C coincides with hinge A and S traces an arc.

(iii) J_{25}, $\bar{J}_{25}$ if and only if $s = d_7(1 + \bar{v}d_1/d_6)$. Then triangles SGC and ABC are similar.

(iv) E_1, $\bar{E}_1$, (resp. E_2, $\bar{E}_2$) if and only if $s = -(d_1 + vd_6)/E_+$

(resp. $s = -(d_1 + vd_6)/E_-$) where $E_\pm$ are the roots of the quadratic in X whose coefficients of X^2, X^1, X^0 are d_7Q, $PQ + d_7^2 - d_8^2$, d_7P respectively, where $P = (d_2d_6v - d_1d_5u)/d_2$, $Q = (d_1d_6u - d_2d_5v)d_1uv$.

The reader may find it interesting to note that cases (iii) and (iv) give rise to coupler curves whose degrees are smaller than that of the general member of the family; but, unlike cases (i) and (ii), are not coupler curves of lower order mechanisms.

Let us assume that we do not have the exceptional cases (i)-(iv). We note that this does not exclude the case when S is the hinge G. We may determine the degree of C_S by considering the images in the coupler plane of tangents to $\mathcal{R}'$ at points meeting $w = 0$. We will not list the tangents to $\mathcal{R}'$ here as they are quite lengthy. However, we will list their images. The reader may readily check that points of $\mathcal{R}'$ lying in the two-planes W, $\overline{W}$ map to the circular points at infinity $I = (1,i,0)$ and $J = (1,-i,0)$ respectively. Therefore I and J are the only points of C_S with $P_3 = 0$. The tangents to C_S at I are of the form $P_1 + iP_2 = \sigma_k P_3$ with

(i) $\sigma_1 = 0$,
(ii) $\sigma_2 = d_4(1 + vd_6/d_1)$,
(iii) $\sigma_3 = -usd_4d_5/d_2d_7$,
(iv) $\sigma_4, \sigma_5 = -ud_5(d_1 + vd_6)/(d_2d_6v - d_1d_5u + d_3d_6vF_\pm)$ where $F_\pm$ are the roots of the quadratic polynomial in X whose coefficients of X^2, X^1, X^0 are d_3Q, $PQ + d_3^2 - d_4^2$, d_3P respectively (where $P = (d_1d_5u - d_2d_6v)/d_5u$, $Q = (d_1d_6v - d_2d_5u)/d_6v$).

Note that four of the five tangents remain fixed as s varies. The tangents to C_s at J are the complex conjugates of the above lines.

Let d, e and f be the degrees of τ_s, $\mathfrak{R}'$ and C_s respectively. Then they are related by the Projection Formula (Theorem A11), which yields $e - \Phi = d.f$, where Φ is the sum of intersection multiplicities of $\mathfrak{R}'$ with a generic hyperplane containing V_s, at points in V_s. In particular, the hyperplane $w = 0$ contains V_s and intersects $\mathfrak{R}'$ transversally at the points J_{14}, $\bar{J}_{14}$ implying $\Phi = 2$. But the multiplicity of C_s at I and J is at least five, since there are five distinct tangents. Thus the total intersection multiplicity of C_s with the line $P_3 = 0$ is at least ten and therefore, by Bézout's theorem, C_s has degree >10; that is $f \geq 10$. Hence, $d = 1$ and the degree of C_s is 14. Moreover, τ_s is a generically 1-1 rational map and therefore birational. Since the geometric genus is a birational invariant, this implies that $\mathfrak{R}'$ and C_s have identical genera: thus for a generic mechanism C_s has genus five.

The real singular foci (see §A4 for definition) of C_s are easily derived from the list of tangents at I and J. Three of them have an easy description. One focus is the hinge $A = (0,0)$. Now let U be the point such that triangles EAT and DBF are similar. Then the second focus is the point H such that triangles UAH and GCS are similar. The third focus K is the point such that triangles AEK and ABC are similar.

The analysis for the locus of T is similar to that of S. We may write $T = d_1z_1 + d_6z_6 + d_7z_7 + t.z_8$, where t is a fixed complex number. If we write $t = t_1 + it_2$ with t_1, t_2 real, then we can think of T as a point in the projective plane with homogeneous co-ordinates P_1, P_2, P_3 where $P_1 = d_1x_1 + d_6x_6 + d_7x_7 + t_1x_8 - t_2y_8$, $P_2 = d_1y_1 + d_6y_6 + d_7y_7 + t_2x_8 + t_1y_8$ and $P_3 = w$. Thus we have defined a projection τ_t whose centre V_t generally does not meet $\mathcal{R}'$. Exceptionally, however, V_t may meet $\mathcal{R}'$ in the following ways

(i) $J_{123}, \bar{J}_{123}$ if and only if $t = d_8$. Thus T is the hinge F and the locus is an arc of a four-bar coupler curve.

(ii) $J_{14}, \bar{J}_{14}$ if and only if $t = 0$. Then T is the hinge G the locus of which we have considered in the family of loci of S.

(iii) $J_{24}, \bar{J}_{24}$ if and only if $t = (1 + \bar{v}d_1/d_6)d_8$. Then T is positioned so that the triangles ABC and GFT are similar.

(iv) $E_1, \bar{E}_1$ (resp. $E_2, \bar{E}_2$) if and only if $t = d_2d_8(d_1 + d_6v + d_7E_+)/(-d_1d_5u + d_2d_6v + d_2d_7E_+)$ (resp. $t = d_2d_8(d_1 + d_6v + d_7E_-)/(-d_1d_5u + d_2d_6v + d_2d_7E_-)$) where $E_\pm$ are the roots of the quadratic in X whose coefficients of X^2, X^1, X^0 are d_7Q, $PQ + d_7^2 - d_8^2$, d_7P, where $P = (d_2d_6v - d_1d_5u)/d_2$ and $Q = (d_1d_6\bar{v} - d_2d_5\bar{u})/d_1$.

Let us assume that we do not have cases (i)-(iv). We shall follow the line of argument for S. Tangents to C_t at I have the form $P_1 + iP_2 = \sigma_k P_3$ with

(i) $\sigma_1 = 0$,

(ii) $\sigma_2 = d_4$,

(iii) $\sigma_3 = -ud_4d_5/d_2$,

(iv) $\sigma_4 = -tud_4d_5/d_2d_8$,

(v) $\sigma_5 = d_4(d_1 + d_6v[1 + t/d_8])/d_1$, and

(vi) $\sigma_6, \sigma_7 = -(d_1 + d_6v)d_5u/(d_2d_6v - d_1d_5u + d_3d_6vF_\pm)$ where $F_\pm$ are as indicated above.

The tangents to C_t at J are the complex conjugates of these lines. Thus the multiplicity of C_t at I and J is ≥ 7, implying that the degree of C_t is ≥ 14. Applying the Projection Formula, we find that τ_t has degree 1 and that C_t has degree 16. Then τ_t is a generically 1-1 rational map and therefore birational. Since the geometric genus is a birational invariant, this implies that $\mathcal{R}'$ and C_t have identical genera: thus for a generic mechanism C_t has genus five.

The real singular foci are easily obtained from the above. Five of them are easily described geometrically. Two real foci are the hinges $A = (0,0)$ and $E = (d_4,0)$. A third focus H is such that triangles EAH and DBF are similar and a fourth K is such that triangles HAK and FGT are similar. Let U be a point such that triangles AEU and ABC are similar. Then a fifth focus is the point V such that triangles EUV and FGT are similar.

§4.6 The Topology of the Watt I Real Linkage Varieties

In the first four sections of this chapter we described the basic geometry of the linkage varieties for the Watt six-bar mechanisms and determined exactly how these varieties reduce in terms of the design parameters. The next natural step in this programme is to study in detail the real geometry in the general case. In this final section of the chapter we take up the study of the real linkage variety, a compact real affine curve of genus 5 and degree 16. In general, when the Grashof equality is not satisfied, this curve has no real singularities: thus its topology is completely determined by the number of connected components, each diffeomorphic to a circle. Thus, one is faced with the problem of determining this number in terms of the design parameters. Part of the interest here is that Harnack's Theorem (A9), which gives an upper bound of 6 for the number of connected components, is not the best possible. In fact the number is 1,2,3 or 4. In particular, this number determines the number of real circuits of associated coupler curves, since these appear as projections (of degree 1) of the real linkage curve.

It was in this latter context that the problem was first investigated [Primrose] in 1967 by Primrose, Freudenstein and Roth. These authors were concerned with the very special case when the coupler point is a hinge and produced intuitive arguments to show that the required upper bound is 4. The arguments appear to contain gaps and in some measure this work arose from trying to bridge these gaps. More to the point, the aim was to develop a formal argument which laid bare the general principles and which might be extended to other mechanisms where the answer to this problem is unknown. It appears that the planar four-bar and the Watt six-bars are the only linkages for which the topology has been studied: even for the Stephenson six-bars the problem remains open. However, [Primrose] does contain the germ of an interesting idea which dovetails the technique expounded in §1.4: it is significant that one needs the concept of the linkage variety to lend mathematical expression to this idea. The mechanical expression is that kinematic inversion gives rise to a one-to-one correspondence between the configurations of the Watt I and the Watt II mechanisms. One therefore expects a natural bijection between the associated real linkage curves. It is by no means clear, how one should set out about writing down such a mapping. The key to this problem lies in the fact that both (complex projective residual) linkage curves are birationally isomorphic to the same residual Darboux curve, hence birationally isomorphic to each other. One has to note here that, although all three curves are real, the birational isomorphisms with the Darboux curve are complex. However, the composite birational isomorphism between the real linkage curves is actually real: better still it is an isomorphism - a consequence of the fact that the complex linkage varieties fail to meet the hyperplanes at infinity in real points. That produces an explicit polynomial diffeomorphism between the real linkage curves. The importance of this step lies in the fact that it reduces the problem for the Watt I mechanism to the more tractable Watt II.

The remainder of the argument follows the philosophy explained and exploited in §1.4. We project the real linkage curve for the Watt II onto the real linkage curves associated to the underlying planar four-bars. These projections are of degree 2 and fit into the general framework explained in §1.4. Effectively, the results of that section reduce the problem of determining the topology of the domain to that of counting the number of critical points of the projection. That in turn reduces to a simple geometric problem, which one can solve completely, ending the sequence of ideas.

We noted in §4.3 that the residual curves $\mathcal{R}'$ and $\mathcal{S}'$ for given d_k's, u, v are birationally isomorphic. However, the connection between the curves is much stronger: the real linkage curves are real isomorphic curves. We shall show a stronger result.

Consider the following set of constraints

$$\left.\begin{array}{l} \ell_j(z_1,\dots,z_s) = \Sigma_{i=1}^{s} a_{ij} z_i + a_{s+1\,j} = 0, \quad 1 \le j \le r \\ |z_i|^2 = 1, \quad 1 \le i \le s, \quad a_{ij} \in \mathbb{C} \end{array}\right\} \qquad (4.14)$$

By the general construction (see §1.2) we may associate a Darboux variety from these constraints in the following manner. Denote by $\overline{\ell}_j$ the polynomial obtained from ℓ_j by conjugation. Since the vectors z_i are unit length, we have $\overline{z}_j = \frac{1}{z_j}$, so we substitute for $\overline{z}_j$ in $\overline{\ell}_j$ to give an equation in z_j. We make the polynomials $\ell_j, \overline{\ell}_j$ homogeneous by introducing the variable z_{s+1} giving equations (4.15).

$$\left.\begin{aligned} \ell_j(z_1,\dots,z_{s+1}) &= \Sigma_{i=1}^{s+1} a_{ij}z_i = 0, \quad 1\le j\le r \\ \overline{\ell}_j(z_1,\dots,z_{s+1}) &= \Sigma_{i=1}^{s+1} \frac{\bar{a}_{ij}}{\bar{z}_i} = 0, \quad 1\le j\le r \end{aligned}\right\} \quad (4.15)$$

Clearing denominators in the second equation gives two homogeneous equations which define the Darboux variety. Write $a_{ij} = a^1_{ij} + i a^2_{ij}$, where $a^1_{ij}, a^2_{ij} \in \mathbb{R}$. Then we may construct a variety S in $P\mathbb{R}^{2s}$ by setting $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$ in the set of equations (4.14) and equating real and imaginary parts. Then S is given by

$$\Sigma_{i=1}^{s}[a^1_{ij}x_i - a^2_{ij}y_i] + a^1_{s+1\,j} = 0,$$

$$\Sigma_{i=1}^{s}[a^2_{ij}x_i + a^1_{ij}y_i] + a^2_{s+1\,j} = 0, \quad 1\le j\le r.$$

$$x_i^2 + y_i^2 = 1 \qquad \text{for } 1\le i\le s.$$

Complexifying the equations by allowing the variables to take complex values and homogenising by introducing the variable w, gives a variety in $P\mathbb{C}^{2s}$ which we continue to denote by S given by

$$\Sigma_{i=1}^{s}[a^1_{ij}x_i - a^2_{ij}y_i] + a^1_{s+1\,j}w = 0,$$

$$\Sigma_{i=1}^{s}[a^2_{ij}x_i + a^1_{ij}y_i] + a^2_{s+1\,j}w = 0, \quad 1\le j\le r.$$

$$x_i^2 + y_i^2 = w^2 \qquad \text{for } 1\le i\le s.$$

Now let us fix a t, $1\le t\le s$, and let us consider the set of constraints

$$\left.\begin{aligned} &\ell_j(z_1,\dots,z_{t-1},z_{t+1},\dots,z_s) = \Sigma_{\substack{1\le i\le s+1\\ i\ne t}} a_{ij}z_i + a_{tj} = 0 \quad 1\le j\le r \\ &|z_i|^2 = 1, \quad 1\le i\le s+1, i\ne t \qquad a_{ij}\in\mathbb{C} \end{aligned}\right\}$$

where s, a_{ij} are the same as in constraints (4.14). In a similar manner to (4.14), we may construct a Darboux variety which is easily seen to be projectively equivalent to $\mathbb{D}$, since it may be given by an identical set of equations. To construct the linkage variety T of these equations we may set $z_j = x'_j + iy'_j$ $(x'_j, y'_j \in \mathbb{R})$ in the equations to give a variety T in $P\mathbb{R}^{2s}$ defined by

$$\Sigma_{i \neq t}[a^1_{ij}x'_i - a^2_{ij}y'_i] + a^1_{tj} = 0,$$

$$\Sigma_{i \neq t}[a^2_{ij}x'_i + a^1_{ij}y'_i] + a^2_{tj} = 0, \qquad \text{for} \quad 1 \leq j \leq r.$$

$$x'^2_i + y'^2_i = 1 \qquad \text{for} \quad 1 \leq i \leq s+1, \ i \neq t.$$

Complexifying the equations by allowing the variables to take complex values and homogenising by introducing the variable w', gives a variety in $P\mathbb{C}^{2s}$ which we continue to denote by T whose equations are

$$\Sigma_{i \neq t}[a^1_{ij}x'_i - a^2_{ij}y'_i] + a^1_{tj}w' = 0,$$

$$\Sigma_{i \neq t}[a^2_{ij}x'_i + a^1_{ij}y'_i] + a^2_{tj}w' = 0, \qquad \text{for} \quad 1 \leq j \leq r.$$

$$x'^2_i + y'^2_i = w'^2 \qquad \text{for} \quad 1 \leq i \leq s+1, \ i \neq t.$$

Then we have the following theorem.

Theorem The two complex residual varieties S' and T', obtained from S and T by removing any subvarieties in the hyperplanes at infinity, are birationally equivalent. The two real residual varieties S' and T' are real isomorphic.

Proof : By the result of [Gibson&Newstead] as explained in §1.2 both varieties S' and T' are birationally equivalent to the residual linkage variety $\mathcal{D}'$ (obtained from $\mathcal{D}$ by removing all components lying in any hyperplane of the form $z_j = 0$). Thus S' and T' are clearly birationally equivalent. Let $\varphi : S' \to \mathcal{D}'$ and $\gamma : T' \to \mathcal{D}'$ be the equivalences. Then the second part of the theorem follows from two further facts. Firstly, the compositions $\pi = \gamma^{-1} \circ \varphi$ and $\pi' = \varphi^{-1} \circ \gamma$ only fail to be regular on S' and T' at points on the hyperplane at infinity; in particular, only at complex points of S' and T'. Secondly, despite the fact that the maps φ and γ are given by complex polynomials, the maps π and π' are easily showed to be given by real polynomials. Thus π and π' define real isomorphisms between S' and T'. Quite explicitly the maps are

$$\pi : (x_1, y_1, \ldots, x_s, y_s, w) \to (x'_1, y'_1, \ldots, x'_{t-1}, y'_{t-1}, x'_{t+1}, y'_{t+1}, \ldots, x'_{s+1}, y'_{s+1}, w'),$$

where $x'_i = x_i x_t + y_i y_t$, $y'_i = y_i x_t - x_i y_t$, $x'_{s+1} = x_t w$, $y'_{s+1} = -y_t w$, $w' = w^2$, and

$$\pi' : (x'_1, y'_1, \ldots, x'_{t-1}, y'_{t-1}, x'_{t+1}, y'_{t+1}, x'_{s+1}, y'_{s+1}, w') \to (x_1, y_1, \ldots, x_s, y_s, w),$$

where $x_i = x'_i x'_{s+1} + y'_i y'_{s+1}$, $y_i = y'_i x_{s+1} - x'_i y'_{s+1}$, $x_t = -x'_{s+1} w'$, $y_t = y'_{s+1} w'$ and $w = -w'^2$.

■

The consequence of the theorem for planar mechanisms is as follows. Let K be a kinematic chain - by which we mean a finite number of rigid bodies smoothly jointed together. Let M_t be the n-bar mechanism, obtained from K by fixing bar t, $1 \le t \le n$; then the mechanisms M_t, for $1 \le t \le n$, are called the **kinematic inversions** of the kinematic chain. Then writing down the set of constraints of M_t, we may form the Darboux variety $\mathcal{D}_t$ in the manner explained in §1.2. The first point to note is that the Darboux varieties $\mathcal{D}_i$ are projectively equivalent. This follows immediately from the fact that, if the set of constraints of M_t has the form

$$\left.\begin{array}{l} \ell_j(z_1,\dots,z_s) = \Sigma_{i\neq t} a_{ij} z_i + a_{tj} = 0, \quad 1 \le j \le r \\ |z_i|^2 = 1, \quad 1 \le i \le s,\ i \neq t, \quad a_{ij} \in \mathbb{C} \end{array}\right\}$$

then the associated Darboux varieties $\mathcal{D}_t$ have the form

$$\left.\begin{array}{l} \ell_j(z_1,\dots,z_s,w) = \Sigma_{i\neq t} a_{ij} z_i + a_{tj} w = 0, \quad 1 \le j \le r \\ \bar{\ell}_j(z_1,\dots,z_s,w) = \Sigma_{i\neq t} \dfrac{\bar{a}_{ij}}{z_i} + \dfrac{\bar{a}_{tj}}{w} = 0, \quad 1 \le j \le r \end{array}\right\}$$

Thus the Darboux varieties are projectively equivalent; indeed, for the varieties $\mathcal{D}_{t_1}$ and $\mathcal{D}_{t_2}$ we have the equivalence $w \mapsto z_{t_1}$, $z_{t_2} \mapsto w$ and $z_i \mapsto z_i$ for $i \neq t_1, t_2$. We may now apply the above theorem to their associated linkage varieties V_t. This yields that the residual linkage varieties V'_t are birationally isomorphic and the real linkage varieties are real isomorphic. We may deduce the following facts:

1) The number of irreducible components of the residual varieties is invariant under kinematic inversion. However, the degrees of the components are not necessarily the same, since degree is not a birational invariant.

2) The number of connected components of the real residual linkage variety is invariant under kinematic inversion. Further, that number depends only upon the design parameters. Therefore, it is sufficient to determine this number for just one inversion.

3) For coupler projections of degree one, all coupler curves have the same number of circuits equal to the number of connected components of the residual linkage varieties. Further, that number depends only upon the design parameters.

4) The Zariski open sets consisting of the residual varieties, with all points lying in the hyperplanes at infinity removed, are isomorphic. Hence, the finite singular points of the residual linkage varieties have the same isomorphism type. It follows that the Grashof Equality (i.e. the condition for a finite singular point) is the same for all inversions.

5) Since the Darboux varieties are birationally isomorphic, the geometric genus of the Darboux variety is an invariant of the kinematic chain.

Quite explicitly for the Watt kinematic chain we have two distinct kinematic inversions namely the Watt I and II mechanisms. The isomorphism between the linkage varieties φ is given by $x'_j = x_j x_1 + y_j y_1$, $y_j' = y_j x_1 - x_j y_1$ for $j = 2,3,5,6,7,8$ and $x_4' = -x_1 w$, $y'_4 = y_1 w$, $w' = -w^2$ and φ' is given by $x_j = x'_j x'_4 + y'_j y'_4$, $y_j = y'_j x'_4 - x'_j y'_5$ for $j = 2,3,5,6,7,8$ and $x_1 = -x'_4 w'$, $y_1 = y'_4 w'$, $w = -w'^2$. The composites fail to be defined at points in the hyperplanes at infinity $w = 0$, $w' = 0$. So φ, φ' define mutually inverse rational maps, thus a birational map which only fails to be a biregular correspondence (i.e. an isomorphism) at the (finite number of) points in the hyperplanes at infinity. But, since there are no real points at infinity, the map does define a real biregular correspondence between the real parts of $\mathcal{R}'$ and $\mathcal{S}'$. This fact yields two useful corollaries. Firstly, the singularities of the two residual curves are in 1-1 correspondence (real singularities on $\mathcal{R}'$ correspond to real singularities on $\mathcal{S}'$) and have identical singularity types; a fact which can be deduced from the property that the finite parts of $\mathcal{R}'$ and $\mathcal{S}'$ are isomorphic to the finite parts of the Darboux variety. Secondly, φ is a real polynomial isomorphism and thus a diffeomorphism. The two real curves therefore have the same number of connected components and so it suffices to indicate the number for just one of the two Watt curves. We should point out, incidentally, that these remarks imply the statement in [Primrose], namely, that the number of real circuits of the associated couplers are invariant under kinematic inversion. Indeed, the number of real circuits of a coupler is, by definition, the number of connected components of any real normalisation and we only need to observe from our work in §4.1 that the real linkage curve is a normalisation of the coupler.

The aim of the remainder of this section is to establish this number in terms of the design parameters.

Let us assume from this point on that we have a generic and general constructible mechanism, i.e. that the Grashof equality and conditions (C1)-(C3) do not hold. Thus both complex projective curves $\mathcal{R}'$, $\mathcal{S}'$ are irreducible, have at least one real point and the only singular points are non-real ordinary double points in the hyperplane at infinity. The real affine curves are non-empty, smooth and compact, and thus diffeomorphic to a finite disjoint union of circles. By Harnack's Theorem (see §A9) the number of topological components is ≤6, but we shall show that the maximum number obtained by a Watt mechanism is 4. By the above remarks it suffices to determine this number for the Watt II linkage curve.

Consider the projections π (resp. π') $:P\mathbb{C}^{14} \rightarrow P\mathbb{C}^{6}$ defined by mapping onto the x'_2, y'_2, x'_3, y'_3, x'_4, y'_4, w' (resp. x'_5, y'_5, x'_6, y'_6, x'_7, y'_7, x'_8, y'_8, w') co-ordinates. The restriction $\pi|_{\mathcal{S}}$ maps $\mathcal{S}$ into a curve T and the restriction $\pi'|_{\mathcal{S}}$ maps $\mathcal{S}$ into a curve Z. The curve T is defined by those equations of (4.4) involving only x'_2, y'_2, x'_3, y'_3, x'_4, y'_4, w' and is the set of equations defining the linkage curve for the four-bar obtained from the Watt mechanism by removing bars 5,6,7 and 8. The defining equations of Z are obtained by taking those equations of (4.4) involving only x'_5, y'_5, x'_6, y'_6, x'_7, y'_7, x'_8, y'_8, w and using the equations expressing x'_6, y'_6 in terms of w' to eliminate x'_6, y'_6. We are left with five equations in seven unknowns: these equations define a variety projectively equivalent to the linkage curve of the four-bar obtained from the Watt mechanism by removing bars 1, 2, 3 and 4. Our aim is to deduce properties of $\mathcal{S}'$ from properties of the four-bars.

For a generic Watt mechanism the planar four-bars are generic in the sense of §1.1 for, as we remarked in §4.2, if one of the four-bars flattens, then the Grashof equality for the Watt mechanism is immediately satisfied.

Before proceding we recall some facts about four-bars from Chapter 1. The linkage varieties T, Z have degree 8 meeting the hyperplane at infinity $w' = 0$ in two skew complex conjugate line components. The residual curves T', Z' are obtained by removing these lines. Thus T', Z' have degree 6 and meet the hyperplane $w' = 0$ in six points (complex conjugate pairs), three on each of the lines. For a generic four-bar T', Z' are irreducible and non-singular and the real curves may have one or two connected components. Since π will map components of $\mathcal{S}$ with $w' = 0$ into components of T with $w' = 0$, π maps $\mathcal{S}'$ into T'. Moreover, $\pi|_{\mathcal{S}'}$ is a <u>finite</u> map, since $\mathcal{S}'$ and T' are irreducible, and $\mathcal{S}'$ doesn't map into a point. We recall that for finite maps there exists an integer $d \geq 1$, called the <u>degree</u> of the mapping, such that all but a finite number of points on T' have exactly d pre-images on $\mathcal{S}'$. Since T' is non-singular, we know by a general result of finite mappings (see §A7) that all points on T' have $\leq d$ pre-images. The points on T' with $< d$ pre-images are called **branch points**. We claim that the <u>degree of</u> $\pi|_{\mathcal{S}'}$ <u>and</u> $\pi'|_{\mathcal{S}'}$ <u>is two</u>.

The claim follows from the Projection Formula (Theorem

A11), which states that the degrees s, t, d of $\mathcal{S}'$, T' and the mapping π are related by the formula $s - \delta = d.t$, where δ is the total intersection multiplicity of $\mathcal{S}'$ with a generic hyperplane H containing the centre V of π. We showed in §4.3 that $s = 16$ and in §1.1 that $t = 6$. We obtain δ by observing that V meets $\mathcal{S}'$ in two points J_{123}, $\bar{J}_{123}$ and that the hyperplane $w' = 0$ contains V and meets $\mathcal{S}'$ at both of these points with intersection multiplicity two, implying that $\delta = 4$. It now follows that $d = 2$. The proof of the result for $\pi'|_{\mathcal{S}'}$ follows analogously.

We now apply the technique developed in §1.4. Briefly, the technique is as follows. Let us consider the real curve $\mathcal{S}'$ (without changing notation) writing $C_1,\ldots,C_n$ for its topological components. Let V be a topological component of the real curve T' with at least one point which has a real pre-image on $\mathcal{S}'$ and let π be a degree two map from (the complex curves) $\mathcal{S}'$ to T'. Then, according to Proposition 1.1 there are just three possible qualitative pictures for each V.

(I) There is just one component C_1 in the pre-image of V mapped by π immersively onto V as a double cover. π has no real critical points.

(II) There are two components C_1, C_2 in the pre-image of V, each mapped by π diffeomorphically onto V. π has no real critical points.

(III) There are n components $C_1,\ldots,C_n$ in the pre-image of V, mapping onto disjoint arcs $A_1,\ldots,A_n$ of V with exactly 2n critical values, namely, the endpoints of $A_1,\ldots,A_n$. π has 2n real critical points.

To apply the above result, we must compute the number of real critical points of π. Critical points occur when the tangent line to the (complex) curve meets the centre of π i.e. these projective subspaces fail to span a 9-space. Thus we have critical points whenever the 15×8 matrix, obtained from the Jacobian matrix of equations (4.4) by deleting the columns corresponding to the variables x'_2, y'_2, x'_3, y'_3, x'_4, y'_4, w', has rank <8. The resulting matrix (with five zero rows removed) is

$$\begin{bmatrix} d_5 & 0 & d_6 & 0 & d_7 & 0 & d_8 & 0 \\ 0 & d_5 & 0 & d_6 & 0 & d_7 & 0 & d_8 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2x_5 & 2y_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_6 & 2y_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_7 & 2y_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2x_8 & 2y_8 \end{bmatrix}$$

It is a straightforward exercise to show that this matrix has non-maximal rank if and only if the vectors (x'_7, y'_7) and (x'_8, y'_8) are linearly dependent. Using equations (4.4), we see that this is precisely the condition that $x'_8 = \varepsilon x'_7$, $y'_8 = \varepsilon y'_7$ where $\varepsilon = \pm 1$: thus in the real case the mechanical interpretation of a critical point is that links 7 and 8 are parallel (see Fig.4.15).

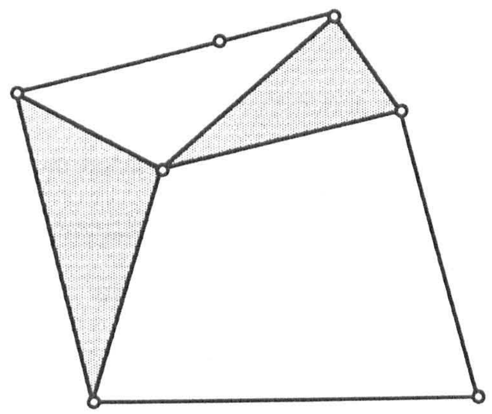


Fig. 4.15.

Since there are no real points with $w' = 0$, we may assume $w' \neq 0$. Then take the projective subspace defined by the two equations $x'_8 = \varepsilon x'_7$, $y'_8 = \varepsilon y'_7$ together with equations (4.15). Then the reader may check that the variety reduces to two subvarieties, one lying in $w' = 0$ and another which is projectively equivalent to the intersection of three quadrics in $P\mathbb{C}^3$. Therefore, by Bézout's Theorem (A3) we have at most eight critical points with $w' \neq 0$ and, in particular, we cannot have more than eight <u>real</u> critical points. This corresponds to what we would expect mechanically.

Let V_j be a real connected component of T' and suppose that there are $2n_j$ real critical points of π which project into V_j. Then if $n_j \geq 1$ we are in case (III). However, when there are no critical points and V_j does contain a point with a real pre-image, we must decide between cases (I) and (II). Consider then the smooth function F on $\mathcal{S}'$ defined by $x'_7y'_8 - x'_8y'_7$. By the above, the zeros of F are precisely the critical points of π. Assume we are in case (I) so that over every point of V_j lie exactly two distinct points P, P' at C_1. (We use here the fact that the critical points of π coincide with the branch points.) Geometrically, P' is the "reflection" of P, as indicated in Fig. 4.13. The key observation is that we must have $F(P) = -F(P')$, so F assumes both positive and negative values. However, in case (I) C_1 is connected, so F would necessarily admit a zero i.e. π would have a critical point on C_1 contrary to our hypothesis. We conclude, therefore, that when there are no real critical points lying over a V_j (but there are real pre-images), then we must be in case (II) when V_j has exactly two components lying over it, each mapping diffeomorphically.

We can now continue our analysis of the case in hand working with the real varieties. First, recall from §1.4 that T' has one or two (topological) components. Suppose first that all the components of $\mathcal{S}'$ map into just one component of T'. In particular, this applies when T' has just one component. Then *either* there are no critical points, so we are in case (II) and $\mathcal{S}'$ has just two components *or* there are 2n critical points (n=1,2,3 or 4) and $\mathcal{S}'$ has n components. The situation is more complex when T' has two components and at least one component of $\mathcal{S}'$ maps into each. The simplest case is when there are 2m real critical points over one component and 2n real critical points over the other, so $\mathcal{S}'$ has m+n components. In all these cases it is clear that $\mathcal{S}'$ has at most four components. That is, however, rather less clear in the remaining case when there are no real critical points over one component and 2n over the other (n=1,2,3 or 4) yielding n+2 components for $\mathcal{S}'$. We claim that this last case cannot arise for n=3 or n=4, so that indeed $\mathcal{S}'$ has never more than four components.

To this end we need to recall more detail from Chapter 1 concerning the planar four-bar. Write e_1,e_2,e_3,e_4 (resp. e_5,e_6,e_7,e_8) for d_1,d_2,d_3,d_4 (resp. d_5,d_6,d_7,d_8) in increasing order of magnitude and set $E=e_1+e_4-e_2-e_3$, $E'=e_5+e_8-e_6-e_7$. Then T' (resp. Z') has one component if and only if $E>0$ (resp. $E'>0$) and two components if and only if $E<0$ (resp. $E'<0$). In fact we showed in §1.4 that the natural projections π_j of T' (resp. Z') into the circles $x_j'^2+y_j'^2=1$ with $j=1,2,3$ (resp. $j=5,7,8$) have degree two and that we have the following possibilities.

(a) $E<0$ and neither d_j nor d_1 is the shortest of d_1,d_2,d_3,d_4 (resp. $E'<0$ and neither d_j nor d_6 is the shortest of d_5,d_6,d_7,d_8). The two components of T' (resp.Z') map onto disjoint arcs A_1, A_2 of the circle and there are exactly two critical points on each component, mapped under π_j to the end-points of the arcs.

(b) $E<0$ and either d_j or d_1 is the shortest of d_1,d_2,d_3,d_4 (resp. $E'<0$ and either d_j or d_6 is the shortest of d_5,d_6,d_7,d_8). The two components of T' (resp.Z') map diffeomorphically onto the circle and there are no critical points.

(c) $E>0$ (resp. $E'>0$) and the one component of T' (resp. Z') maps onto an arc A_1, with exactly two critical points mapping to the end-points of the arc.

In all the above cases the condition for a critical point of π_j

is that $(x'_i,y'_i)=\pm(x'_k,y'_k)$ for $i,k\neq j$ or 1 (resp. $i,k\neq j$ or 6).

We are now in a position to complete the proof that $\mathcal{S}'$ cannot have more than four topological components. In view of our previous remarks we can assume that T' has two topological components T_1',T_2' and that Z' has two topological components Z_1',Z_2'. We can assume that there are at least three components of $\mathcal{S}'$ mapping under π (resp. π') into T_1' (resp. Z_1') with exactly two critical points on each component. Moreover, we can assume that over T_2' (resp. Z_2') there are exactly two components of $\mathcal{S}'$, each projecting diffeomorphically onto that component.

The first observation is that the condition for a point of $\mathcal{S}'$ to be a real critical point of π' is precisely that its image under π is a real critical point of π_2 ; indeed, in both cases the condition is that bars 3 and 4 should have equal or opposite directions. An immediate consequence is that π_2 has at least one critical point. In fact we must be in case (a) above; the two components of T' are mapped under π_2 to two disjoint arcs of the circle with four critical points (two on each component) mapping to the end-points of the arcs. Likewise, the condition for a point of $\mathcal{S}'$ to be a real critical point of π is precisely that its image under π' is a real critical point of π_5; indeed, in both cases the condition is that bars 7 and 8 should have equal or opposite directions. Thus we deduce that π_5 also has at least one real critical point, so that the two components of Z' map under π_5 to two disjoint arcs of the circle with four critical points (two on each component) mapping to the end-points of the arcs. That brings us to the crux of the proof. The critical points of π (of which there are at least six) must map to the critical points of π_5 (of which there are exactly two on each component of Z'). However, since π' is a degree 2 mapping, a point in Z' has at the most two pre-images in $\mathcal{S}'$, so that at the most, four critical points of π' can map to Z_1'. That means that at the most two components of $\mathcal{S}'$ can map under π' into Z_1', providing a contradiction and establishing the desired result.

CHAPTER 5. THE REAL GEOMETRY OF THE FOUR-BAR MECHANISM AND THE CLASSIFICATION OF COUPLER CURVES.

Introduction

In this final chapter we shall restrict our attention to the real geometry of the four-bar mechanism. Following a comment made in §1.6, that there is a natural classification of the generic coupler curves by the Hain/singularity types, we present an initial investigation of this difficult problem. In [Hain 1964] Hain distinguishes eight types of four-bar mechanisms. Firstly, we can partition the mechanisms into two groups distinguished by the number of circuits of the coupler curves which is one/two depending on whether $E>0/E<0$ (where E=sum of the longest and shortest lengths minus the sum of the remaining lengths). Secondly, we can further subdivide each group into four distinct types by the way in which the bars crank or rock during the motion of the mechanism, for which there is a simple criterion, namely, that in the one circuit case we have four cases depending on whether d_1, d_2, d_3, or d_4 is the longest link and in the two component case depending on whether d_1, d_2, d_3, or d_4 is the shortest link. This analysis has been given in §1.4. The further partitioning by singularity types is based on the result (§1.6) that generic coupler curves have three finite double points, at least one of which is real, lying on the circle of singular foci. Provided the pencil in the associated net of quadrics determined by any given coupler point (see §1.6) is generic, the singularities are either ordinary double points (A_1) or cusps (A_2). Thus, for each of the Hain types, there are four distinct types of complex coupler curves which could conceivably arise, depending upon the combination of double points, namely, $3A_1$, $2A_1/A_2$, $A_1/2A_2$, $3A_2$. In the real case, we can make the further distinction of a real A_1 being an acnode (A_1^+) when its tangents are complex or a crunode (A_1^-) when its tangents are real. Of course, if the double point is complex (which we shall denote by A_1^*) there is no further distinction to be made. Note that cusps are always real whenever they occur. This yields thirteen distinct possible types of real coupler curves for each Hain type, namely, $3A_1^-$, $2A_1^-/A_2$, $A_1^-/2A_2$, $3A_2$, $2A_1^-/A_1^+$, $A_1^+/A_1^-/A_2$, $A_1^+/2A_2$, $A_1^-/2A_1^+$, $2A_1^+/A_2$, $A_1^-/2A_1^*$, $2A_1^*/A_2$, $3A_1^*$, $A_1^+/2A_1^*$.

It might seem an impossible task to determine theoretically how many of the $8 \times 13 = 104$ Hain/singularity types actually occur. For convenience we shall call these the **A-types**. However, we shall show (see §2 of this chapter) that several of the Hain types have identical geometries leaving just four Hain types with distinct geometries: thus reducing the number of A-types to fifty-two.

This still leaves a considerable program of work (at least too large for us to attempt here) the first step of which is to decide which of the possible singularity types can occur for each Hain type in the complex case. We shall give a complete answer to this problem in §5.2. The second step is to determine the underlying geometry which distinguishes between a coupler curve having one or three real singular points and which distinguishes between the real double points being A_1^+ or A_1^-. We give one such account in

§5.4 and §5.6. The third step is to determine, perhaps by computer graphics, which of the A-types actually do occur. The final step is to prove, mathematically, the existence of the cases which we have showed to occur graphically. It is probably too much to expect one coherent argument showing the existence for all of the Hain types because their geometries differ considerably. But it is likely that there are arguments covering clusters of cases and with some luck we might just cover all of the cases in this manner. This final step seems to be the most difficult and is a considerable program of work.

It is unfortunate that the author was unable to complete this program due to lack of time. However, it is clear from the results that have been already obtained, that this is a profitable direction of research and that the techniques necessary to complete this work are available and comprise little more than is indicated in this chapter. The author hopes to complete this work at a later stage.

In §5.1 we will give a proof of Robert's Theorem and indicate how this can be used to reduce the number of A-types that need to be considered. Section 5.2 is dedicated to the complex classification of the coupler curves. We shall determine which of the possible A-types can occur in the complex case (i.e. when we do not distinguish A_1^-, A_1^+, A_1^*).

§5.1. Roberts' Triple Generation Theorem and the Classification of Four-bar Mechanisms

Perhaps the most celebrated theorem on the planar four-bar is Roberts' Triple Generation Theorem [Roberts]. Whilst working on problems related to the motion of the planar four-bar, Roberts discovered that the circle of singular foci was intimately related to the singular points of coupler curves and, moreover, the three singular foci held a special significance in the motion of any given four-bar. Indeed, Roberts observed one very important property of the three foci, namely, that they are the fixed hinges of three four-bar mechanisms which possess coupler points drawing identical coupler curves. The literature refers to the three mechanisms as **cognates**. We shall now present a new proof (although the underlining principle is the same as that given by Roberts) in terms of the linkage variety $\mathfrak{R}$.

Roberts' Triple Generation Theorem : The coupler curve of any planar four-bar mechanism may be obtained as the coupler curve of two other four-bars.

Proof : Suppose that we have a mechanism $\mathfrak{M}$ whose linkage variety is given by the set of equations (1.1) and that we fix a coupler point P, uniquely defining a coupler projection $\pi:(x_1,y_1,x_2,y_2,x_3,y_3,w) \mapsto (d_1x_1-k_2y_2+k_1x_2,\ d_1y_1+k_2x_2+k_1y_2,\ w)$. Let $\mathcal{S}_3$ be the symmetric group on three elements and let it act on the ambient space of the linkage variety by permuting the indices 1, 2, 3 of the variables $x_1, y_1, x_2, y_2, x_3, y_3, w$ leaving w unchanged. Then for any permutation σ in $\mathcal{S}_3$ we may define a new variety $\mathcal{R}^{\sigma}$ which is the linkage variety for the mechanism $\mathfrak{M}^{\sigma}$ obtained from $\mathfrak{M}$ by making the obvious swapping of bars as prescribed by the permutation σ. So, for example, if $\sigma = (1\ 2)$ then we would swap bars 1 and 2 to give a new mechanism with the same set of lengths as the original but with a new ordering. The new linkage variety $\mathcal{R}^{\sigma}$ is defined by the equations

$$d_1 x_{\sigma(1)} + d_2 x_{\sigma(2)} + d_3 x_{\sigma(3)} - d_4 w = 0$$
$$d_1 y_{\sigma(1)} + d_2 y_{\sigma(2)} + d_3 y_{\sigma(3)} = 0$$
$$x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = w^2$$

If we also allow the permutation to "act" on the projection (i.e. we permute the indices in the formula for the projection) to define a new projection $\mathcal{P}^{\sigma}: (x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (d_1 x_{\sigma(1)} - k_2 y_{\sigma(2)} + k_1 x_{\sigma(2)}, d_1 y_{\sigma(1)} + k_2 x_{\sigma(2)} + k_1 y_{\sigma(2)}, w)$, then it is clear that we have $\mathcal{P}(\mathcal{R}) = \mathcal{P}^{\sigma}(\mathcal{R}^{\sigma})$. Thus, Roberts' Theorem is proved, if we can show that three of the projections $\mathcal{P}^{\sigma}$ defined in this way are indeed coupler projections for the mechanism $\mathfrak{M}^{\sigma}$. This is a straightforward exercise and the result is that those permutations which are in the Alternating Group (i.e. the identity, $(1\,2\,3)$ and $(1\,3\,2)$) describe coupler projections, and the remaining permutations describe projections of linkage varieties whose images are reflections of coupler curves. We shall now describe how we must position the three mechanisms, so that they produce identical coupler curves.

Let $\mathfrak{M}$ be the mechanism with the usual notation (i.e. bar lengths d_1, d_2, d_3, d_4 and coupler triangle given by the complex number $k = k_1 + ik_2$) and let $\sigma = (1\,2\,3)$. Then the corresponding coupler projection is $\mathfrak{P}^\sigma : (x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (d_1x_2 - k_2y_3 + k_1x_3, d_1y_2 + k_2x_3 + k_1y_3, w)$. Using the linear equations of (1.1), we may rewrite the projection as

$$\mathfrak{P}^\sigma : (x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto$$

$$\left(\frac{d_4k_1w}{d_2} - \frac{d_3k_1x_1}{d_2} + \frac{d_3k_2y_1}{d_2} + K_1x_2 - K_2y_2,\; \frac{d_4k_2w}{d_2} - \frac{d_3k_2x_1}{d_2} - \frac{d_3k_1y_1}{d_2} + K_2x_2 + K_1y_2,\; w\right)$$

where $K_1 = \frac{d_1}{d_2}(d_2 - k_1)$ and $K_2 = -\frac{d_1}{d_2}.k_2$. Write $r = (k_1^2 + k_2^2)^{1/2}$ and $s = (k_1^2 + k_2^2 - 2d_2k_2 - d_2^2)^{1/2}$. The reader should now observe that this is the coupler projection for the mechanism with bars 1, 2, 3, 4 of length $r.\frac{d_3}{d_2}$, $r.\frac{d_1}{d_2}$, r, and $r.\frac{d_4}{d_2}$ respectively; with the coupler point given by the complex number $K = K_1 + iK_2$; and for which the fixed bar has the origin and the point $(k_1.r.\frac{d_4}{d_2}, k_2.r.\frac{d_4}{d_2})$ as its endpoints. This is showed in Fig. 5.1. We note that the ratios between the bar lengths of this mechanism are $\underline{d_3{:}d_1{:}d_2{:}d_4}$ and that the new coupler triangle is similar to the original. In particular, <u>the invariant</u> E <u>(= longest + shortest - sum of remaining lengths) for the cognate</u> $\mathfrak{M}^\sigma$ <u>is identical to that of the original mechanism</u> $\mathfrak{M}$: since the set of bar lengths have remained unchanged. Alternatively, one may observe that the number of circuits of any coupler curve of a given mechanism is independent of the choice of coupler point and, therefore, if two mechanisms draw identical curves, then both mechanisms draw coupler curves with an identical number of circuits for any coupler point.

A similar calculation may be made for the permutation

$\tau = (1\,3\,2)$. The result is that we obtain a coupler projection of the mechanism with bars 1, 2, 3, 4 of lengths s, $s.\frac{d_3}{d_2}$, $s.\frac{d_1}{d_2}$ and $s.\frac{d_4}{d_2}$, repectively with $(d_4,0)$ and $(k_1.r.\frac{d_4}{d_2}, k_2.r.\frac{d_4}{d_2})$ as the endpoints of its fixed bar. This is showed in Fig.5.1. Note that the ratios between the bar lengths of this mechanism are $\underline{d_2:d_3:d_1:d_4}$ and that the new coupler triangle is similar to the original. But since the set of bar lengths is unchanged, <u>the invariant</u> E <u>for the cognate</u> $\mathfrak{M}^\sigma$ <u>is identical to that of the original mechanism</u> $\mathfrak{M}$.

■

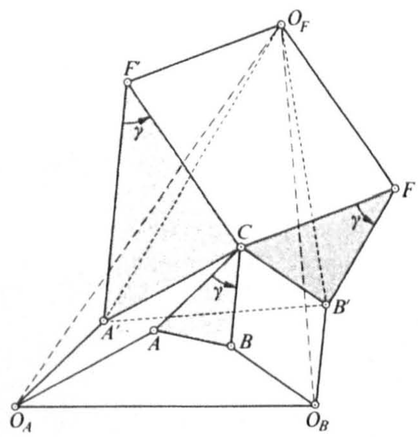


Fig. 5.1.

Before continuing, let us recall that Hain classifies the generic four-bar mechanism into the following eight types:

(i) R_{ii} : $E>0$, d_4 longest; bar 1 and 3 rock inwardly.

(ii) R_{oo} : $E>0$, d_2 longest; bars 1 and 3 rock outwardly

(iii) R_{oi} : $E>0$, d_3 longest; bar 1 rocks outwardly and 3 rocks inwardly.

(iv) R_{io}: $E>0$, d_1 longest; bar 1 rocks inwardly and 3 rocks outwardly.

(v) CR_1 : $E<0$, d_1 shortest; bar 1 cranks, bars 2 and 3 rock.

(vi) CR_2: $E<0$, d_3 shortest; bar 3 cranks, bars 1 and 2 rock.

(vii) DR: $E<0$, d_2 shortest; bar 2 cranks, bars 1 and 3 rock.

(viii) DL: $E<0$, d_4 shortest; bars 1, 2 and 3 crank.

Then the connection between the Hain types of a mechanism and its cognates is as follows

Hain Type of Mechanism	Hain Type of (123)-Cognate	Hain Type of (132)-Cognate
CR_1	DR	CR_2
CR_2	CR_1	DR
DR	CR_2	CR_1
DL	DL	DL
R_{ii}	R_{ii}	R_{ii}
R_{oo}	R_{oi}	R_{io}
R_{oi}	R_{io}	R_{oo}
R_{io}	R_{oo}	R_{oi}

Thus, for any mechanism of Hain type DL or R_{ii} the corresponding cognates have identical Hain types. Whilst mechanisms, which are of one of the types CR_1, CR_2, DR, have cognate mechanisms of the remaining two types. Hence, the geometry of each of these types is identical: because any coupler curve of one of these types can be drawn by a cognate mechanism with either of the other types. Similarly, any mechanism of one of the types R_{oo}, R_{oi}, R_{io} has cognate mechanisms of the remaining two types; implying, as before, that any coupler curve of one of these types can be drawn by a cognate mechanism with either of the other types.

We may conclude, therefore, that in order to study the geometry of the coupler curves it is sufficient to consider just <u>four</u> types of mechanisms instead of the original eight. We shall relabel these types as

I : R_{00}, R_{01}, R_{10};   I' : R_{11};   II : CR_1, CR_2, DR;   II': DL;

where I and II indicate that the corresponding coupler curves have one and two circuits, respectively; each of these types is divided into two subtypes.

Finally, we shall say something about the cognates of the degenerate mechanisms.

(1) The cognates of the rhombus are identical rhombuses.

(2) The (123)- and (132)- cognates of a parallelogram give the two types of kite i.e. one with $d_1 = d_2$ and one with $d_1 = d_4$.

(3) The (123)- and (132)- cognates of a kite give the parallelogram and a kite of the opposite type.

(4) The cognates of a circumscriptible are circumscriptibles.

Thus we may define the above analysis as the classification of the degenerate cases. Note that in cases (2) and (3) the ratio between the longest and shortest lengths is an invariant of the class.

§5.2.The Complex Classification of Four-bar Coupler Curves.

In §1.6 it was showed that a generic coupler curve has three finite double points. The condition for a cusp is easily obtained. We shall repeat the method used in [Marsh]. The result follows from the observation that a point P on a curve $\mathcal{C}$, which is the image under linear projection π of a non-singular curve $\mathcal{R}$, is a cusp if and only if the pre-image of P is a critical point of the projection. Thus, the problem of determining when the coupler curve has cusps is equivalent to determining when the coupler projection π_k has a finite critical point. The condition for this is that the matrix $\mathfrak{J}$, obtained from the Jacobian matrix of equations (1.1) by abutting the Jacobian matrix of the projection, has non-maximal rank. The matrix $\mathfrak{J}$ is

$$\mathfrak{J} = \begin{bmatrix} d_1 & 0 & d_2 & 0 & d_3 & 0 & -d_4 \\ 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 \\ 2x_1 & 2y_1 & 0 & 0 & 0 & 0 & -2w \\ 0 & 0 & 2x_2 & 2y_2 & 0 & 0 & -2w \\ 0 & 0 & 0 & 0 & 2x_3 & 2y_3 & -2w \\ d_1 & 0 & k_1 & -k_2 & 0 & 0 & 0 \\ 0 & d_1 & k_2 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By elementary row and column operations it is sufficient to determine the condition for the following matrix to have non-maximal rank

$$\begin{bmatrix} -\frac{k_1}{d_1}x_1 - \frac{k_2}{d_1}y_1 & \frac{k_2}{d_1}x_1 - \frac{k_1}{d_1}y_1 \\ x_2 & y_2 \\ -\frac{(d_2-k_1)}{d_3}x_3 + \frac{k_2}{d_3}y_3 & -\frac{k_2}{d_3}x_3 - \frac{(d_2-k_1)}{d_3}y_3 \end{bmatrix}$$

The matrix has non-maximal rank if and only if

$$(x_2,y_2) = \ell\left(-\frac{k_1}{d_1}x_1-\frac{k_2}{d_1}y_1, \frac{k_2}{d_1}x_1-\frac{k_1}{d_1}y_1\right)$$

$$= m\left(-\frac{(d_2-k_1)}{d_3}x_3+\frac{k_2}{d_3}y_3 - \frac{k_2}{d_3}x_3-\frac{(d_2-k_1)}{d_3}y_3\right.$$

These conditions yield

$$\left.\begin{array}{ll} x_1 = -\varepsilon_1(k_1x_2 - k_2y_2)/\alpha, & y_1 = -\varepsilon_1(k_2x_2 + k_1y_2)/\alpha \\ x_3 = -\varepsilon_2((d_2-k_1)x_2 + k_2y_2)/\beta, & y_3 = (k_2x_2 - (d_2-k_1)y_2)/\beta \\ \text{where } \alpha = (k_1^2+k_2^2)^{1/2} \text{ and } \beta = (k_2^2+(d_2-k_1)^2)^{1/2}. & \end{array}\right\} \quad (5.1)$$

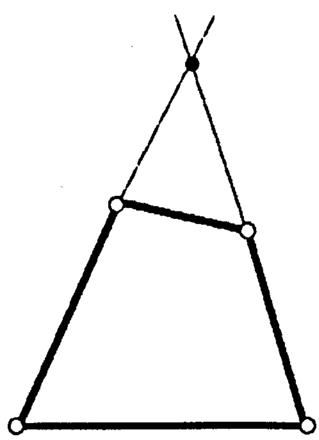


Fig. 5.2

Mechanically, this means that the coupler point gives rise to a coupler curve with a cusp, if and only if it is the intersection of two lines: one lying on bar 1 and the other on bar 3 as showed in Fig. 5.2. Thus, for any position of the mechanism such that bars 1 and 3 are non-parallel, there is a unique coupler point giving rise to a coupler curve with a cusp. Furthermore, the locus of such points in the **coupler plane**, that is, the (k_1,k_2)-plane, is a curve. We will discuss this curve in more detail later in the chapter.

We may use equations (5.1) to express x_1, y_1, x_3, y_3, w in terms of x_2, y_2 and eliminate them in equations (1.1) giving two linear equations in x_2, y_2, w

$$Ax_2 + By_2 = C \quad \text{and} \quad -Bx_2 + Ay_2 = 0$$

where

$$A = -\varepsilon_1 d_1 k_1/\alpha - \varepsilon_2 d_3(d_2-k_1)/\beta + d_2,$$

$$B = \varepsilon_1 d_1 k_2/\alpha - \varepsilon_2 d_3 k_2/\beta \quad \text{and} \quad C = d_4 w \quad (\varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1).$$

Note that A and B are always real. Applying Cramer's Rule to the two linear equations, we find that $x_2 = AC/(A^2+B^2)$ and $y_2 = BC/(A^2+B^2)$. This implies that $A^2+B^2 = d_4^2$ and substituting for A and B it now follows that the condition for a coupler point $k = k_1 + ik_2$ to give rise to a coupler curve with a cusp has the form

$$\left.\begin{aligned} &\frac{d_1^2k_1^2}{k_1^2+k_2^2} + \frac{2d_1d_3k_1(d_2-k_1)}{\varepsilon_1(k_1^2+k_2^2)^{1/2}.\varepsilon_2(k_2^2+(d_2-k_1)^2)^{1/2}} + \frac{d_3^2(d_2-k_1)^2}{k_2^2+(d_2-k_1)^2} \\ &-2d_2\left[\frac{d_1k_1}{\varepsilon_1(k_1^2+k_2^2)} + \frac{d_3(d_2-k_1)}{\varepsilon_2(k_2^2+(d_2-k_1)^2)}\right] + d_2^2 + \frac{d_1^2k_1^2}{k_1^2+k_2^2} \\ &\frac{-2d_1d_3k_2^2}{\varepsilon_1(k_1^2+k_2^2)^{1/2}.\varepsilon_2(k_2^2+(d_2-k_1)^2)^{1/2}} + \frac{d_3^2k_2^2}{k_2^2+(d_2-k_1)^2} = d_4^2 \end{aligned}\right\} \quad (5.2)$$

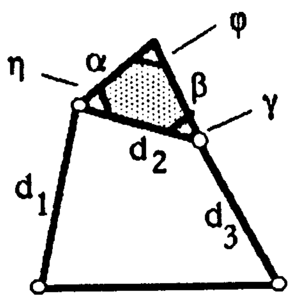


Fig. 5.3.

But the given **coupler triangle**, by which we mean the triangle with vertices the coupler point and the two hinges on the coupler bar, has sides of lengths d_2, α, β as showed in Fig. 5.3.

Therefore, $\cos(\varphi) = -\varepsilon_1\varepsilon_2(d_2k_1 - k_1^2 - k_2^2)/\alpha\beta$, $\cos(\gamma) = \varepsilon_2(d_2-k_1)/\beta$ and $\cos(\eta) = \varepsilon_1 k_1/\alpha$ yielding

$$d_1^2 + d_2^2 + d_3^2 - d_4^2 - 2d_1d_3\varepsilon_1\varepsilon_2\cos(\varphi) - 2d_1d_2\varepsilon_1\cos(\eta) - 2d_2d_3\varepsilon_2\cos(\varphi) = 0 \qquad (5.3)$$

i.e. there are four equations, one for each choice of signs $(\varepsilon_1,\varepsilon_2)$. This condition was obtained as the necessary condition for a cusp by mechanical means [Cayley]. The author of [Cayley] uses this condition to show the existence of a three-cuspidal coupler curve. Further, for any choice of coupler point k satisfying any one of the equations (5.3), we can deduce that the coupler curve does possess a cusp. For, whenever (5.3) is satisfied for some choice of sign $(\varepsilon_1,\varepsilon_2)$, x_2, y_2 are uniquely defined as the solution of two linear equations. The remaining variables are then determined uniquely by equations (5.1). The cusp on the coupler curve is easily checked to be (p_1,p_2,w) where

$$d_4p_1 = \frac{-\varepsilon_1 d_1(k_1A-k_2B)w}{\alpha - k_2B + k_1A}, \qquad d_4p_2 = \frac{-\varepsilon_1 d_1(k_2A-k_1B)w}{\alpha - k_1B + k_2A}$$

It is perhaps surprising to find that, whenever a cusp does occur, it is always <u>real</u>. However, since A and B are real, it follows that the cusp is determined as the intersection of two real lines and therefore giving a real point. The locus of all points in the fixed plane, which are cusps for some coupler point, is a curve and is called the **fixed centrode.** The locus of all coupler points in the coupler plane, which give rise to a curve with a cusp, is called the **moving centrode**.

It is conceivable that a coupler point satisfies more than one equation of the form (5.3). The reader may check that at most three of the four conditions can hold simultaneously for any coupler point. Thus the above analysis shows that there may be coupler curves with one, two or three cusps depending on whether one, two or three conditions of the form (5.3) hold simultaneously. (Of course we knew this fact for the generic case from the fact, proved in §1.6, that there are at most three finite singular points lying on the circle of singular foci and the general case follows from the fact that any singular point not on the circle of singular foci is an ordinary double point.) It can then be showed, that only two of the four Hain types have mechanisms such that there exist coupler points which give rise to a coupler curve with three cusps. Yet, in all but one of the Hain types, there exist mechanisms with coupler points which draw coupler curves with two cusps. Coupler curves with one cusp may be obtained by any mechanism: for we have showed that for any position of the mechanism, for which bars 1 and 3 are non-parallel, there is a unique coupler point which gives rise to a curve with a cusp.

Let us write the symbol ±±, whenever we refer to the equation (5.3) with the signs of ε_1, ε_2 given by the symbol. Then, any pair of these conditions, which hold simultaneously, yield the following set of conditions

+- and -+ : $d_3\cos\gamma = d_1\cos\eta, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 + 2d_1d_3\cos\varphi$

+- and -- : $d_3\cos\varphi = d_2\cos\eta, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 + 2d_2d_3\cos\gamma$

-+ and -- : $d_2\cos\gamma = d_1\cos\varphi, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 + 2d_1d_2\cos\eta$

++ and +- : $d_1\cos\varphi = -d_2\cos\gamma, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 - 2d_1d_2\cos\eta$

++ and -+ : $d_3\cos\varphi = -d_2\cos\eta, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 - 2d_2d_3\cos\gamma$

++ and -- : $d_3\cos\gamma = -d_1\cos\eta, \quad d_4^2 = d_1^2 + d_2^2 + d_3^2 - 2d_1d_3\cos\varphi$

For simplicity we shall consider separately coupler triangles for which the interior angles are acute (i.e. an acute triangle) and those for which there is an angle greater than a right angle.

Case 1: Acute Coupler Triangles.

Let us assume that C is a coupler point whose coupler triangle is acute. Then C gives rise to a coupler curve with two cusps if and only if C satisfies two conditions of the form (5.3). For such a coupler triangle the angles φ, γ, η, (using the same terminology as in Fig. 5.3) are all less than a right angle, implying that $\cos(\varphi)$, $\cos(\gamma)$, $\cos(\eta)$, have values in the interval [0,1]. In particular, the cosines of the angles are all positive, thus those pairs of equations containing a minus sign cannot occur, leaving the three possibilites +- & -+, +- & --, -+ & --. From the second of each pair of equations we may easily deduce that d_4 is the longest of the bars. Thus, for the Hain types with one circuit, only group I', with d_4 the longest, has an acute coupler triangle for which the coupler curve has two cusps. Likewise, Hain types with two circuits cannot have d_4 as the shortest length, implying that only type II, with d_1, d_2, or d_3 the shortest, can produce two cuspidal coupler curves. Conversely, we need to show that there do exist two-cuspidal coupler curves for these types. These are easily constructed: for type I', choose $d_1=d_3=3$, $d_2=1$, $d_4=\sqrt{28}$ with an equilateral coupler triangle i.e. $k_1=0.5$, $k_2=\sqrt{0.75}$ and for type II choose $d_1=d_3=2$, $d_2=\sqrt{0.5}$, $d_4=\sqrt{8.5}$.

Case 2. Obtuse Coupler Triangles.

For obtuse coupler triangles we need to consider all pairs of conditions. Only one angle in a triangle can be obtuse. Let us assume that angle φ is obtuse. Then, the only pairs of conditions that can hold simultaneously are +- & -+, ++ & +- and ++ & -+. Similarly, whenever angle γ (resp. η) is obtuse the possible pairs are +- & --, ++ & +-, and ++ & -- (resp. -+ & --, ++ & -+, and ++ & --). By Roberts' Theorem it is sufficient to consider those coupler triangles for which angle φ is obtuse, so that $-1 \le \cos\varphi \le 0$. Then, for the case +- & -+, the second condition yields

$$d_4^2 \ge (d_1 - d_3)^2 + d_2^2$$

In particular, $\underline{d_4 \ge d_2}$ implying that for the two circuited coupler curves d_4 is never the smallest length, i.e. type II' mechanisms do not satisfy these conditions. Similarly, in cases ++ & +- and ++ & -- the second of the pair of conditions yields $\underline{d_4 \ge d_3}$ and $\underline{d_4 \ge d_1}$. Thus type II' mechanisms never satisfy a pair of conditions of the form (5.3), and therefore can never give rise to two-cuspidal coupler curves. We note that type I has two-cuspidal coupler curves only when the coupler triangle is obtuse.

For a coupler curve to have three cusps, three pairs of equations of the form (5.3) must be satisfied simultaneously. Suppose then that the coupler triangle is <u>acute</u>. Then all three conditions +-, -+ and -- hold simultaneously. This yields

$$\frac{d_1}{\cos\gamma} = \frac{d_2}{\cos\varphi} = \frac{d_3}{\cos\eta} = \lambda$$

Substituting for d_1, d_2, d_3 in the second equation of +- and -+, we find

$$d_4^2 = (\cos^2\varphi + \cos^2\gamma + \cos^2\eta + 2.\cos\varphi.\cos\gamma.\cos\eta).\lambda^2 = \lambda^2$$

Thus $\lambda = d_4$, yielding $d_1 = d_4.\cos\gamma$, $d_2 = d_4.\cos\varphi$ and $d_3 = d_4.\cos\eta$. Further, the design parameters must satisfy the condition

$$(d_1^2 + d_2^2 + d_3^2 - d_4^2).d_4 + 2d_1d_2d_3 = 0.$$

But cosine takes 1 as its maximum value, so it is clear from these equations that d_4 is the longest length, thus excluding types I and II' as possible candidates for three-cuspidal coupler curves (with coupler triangle acute). The remaining two types do occur; examples are: $d_1 = d_2 = d_3 = 1$, $d_4 = 2$, with an equilateral coupler triangle for type I', and $d_1 = d_3 = \cos 50°$, $d_2 = \cos 80°$, $d_4 = 1$, and a coupler triangle with angles $\gamma = \eta = 50°$, $\varphi = 80°$.

Now suppose that the coupler triangle is obtuse. Then the combination of three pairs of conditions of the form (5.3), which may hold simultaneously, depends entirely on which of the angles is obtuse. If φ (resp. γ, η) then only the three conditions ++, +- and -+ (resp. ++, +-, -- and ++,-+,--) may hold simultaneously. Then, a condition of the following form is satisfied

$$\left.\frac{\varepsilon_1 d_1}{\cos\gamma} = \frac{\varepsilon_2 d_2}{\cos\varphi} = \frac{\varepsilon_3 d_3}{\cos\eta} = \lambda \right\} (5.4),$$

where the triple of signs $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, whenever the angle φ (resp. γ and η) is obtuse, is (+,-,+) (resp. (-,+,+) and (+,+,-)). The reader may check that, as for the acute angle case, we have $\lambda = d_4$. It follows, therefore, that d_4 is the longest. Furthermore, the design parameters must satisfy the conditon

$$(d_1^2 + d_2^2 + d_3^2 - d_4^2).d_4 - 2d_1d_2d_3 = 0.$$

But, whenever angle φ is obtuse, we have $-\cos\varphi < \cos\gamma$ and $-\cos\varphi < \cos\eta$. For, if $0 > \cos\varphi + \cos\tau$ (where $\tau = \gamma$ or η) then by the addition formula for cosine we have $0 > \cos\frac{1}{2}(\varphi+\tau).\cos\frac{1}{2}(\tau-\varphi)$: contradicting the fact that $\frac{1}{2}\pi > \frac{1}{2}(\varphi+\tau), \frac{1}{2}(\tau-\varphi) > 0$. From equations (5.4) we deduce that d_2 is the shortest length. Similarly, when angle γ (resp. η) is obtuse we may deduce that d_1 (resp. d_3) is the shortest length.

By Roberts' Theorem we may assume without loss of generality that angle φ is obtuse, so that d_4 is the longest and d_2 is the shortest. Then $E = d_4 + d_2 - d_1 - d_3 = d_4.(1 - \cos\varphi - \cos\gamma - \cos\eta)$. But, $\varphi = \pi - \gamma - \eta$. Thus, $\cos\eta = -\cos(\gamma+\eta)$ and

$$E = d_4.([\cos 0 + \cos(\gamma+\eta)] - [\cos\gamma + \cos\eta])$$

Using the addition formula for cosines, we find

$$E = 2.d_4.\cos\tfrac{1}{2}(\gamma+\eta).[\cos\tfrac{1}{2}(\gamma+\eta) - \cos\tfrac{1}{2}(\eta-\gamma)]$$

and applying the addition formula once more, yields

$$E = -2.d_4.\cos\tfrac{1}{2}(\gamma+\eta).\sin\eta.\sin\gamma.$$

We easily deduce, therefore, that $E < 0$. Hence, only type II four-bars can give rise, by obtuse coupler triangles, to coupler curves with three cusps. We may take, as an example of this, the mechanism with $d_1 = d_3 = \frac{1}{2}\sqrt{3}$, $d_2 = \frac{1}{2}$, $d_4 = 1$ and coupler triangle with angles $\gamma = \eta = 30°$ and $\varphi = 120°$.

Thus the <u>complex</u> classification of four-bar coupler curves is complete. For, we have showed that for each type the following possibilities can occur:

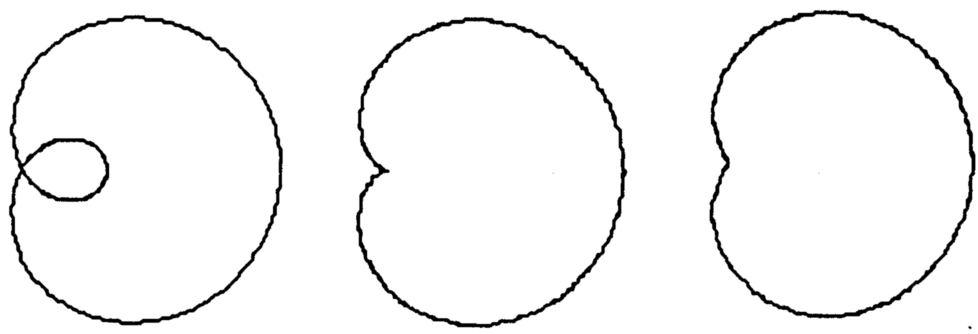
Type I: $3A_1$, $2A_1/A_2$, $A_1/2A_2$
Type I': $3A_1$, $2A_1/A_2$, $A_1/2A_2$, $3A_2$
Type II: $3A_1$, $2A_1/A_2$, $A_1/2A_2$, $3A_2$
Type II': $3A_1$, $2A_1/A_2$

§5.3. The Problems of Surveying Four-bar Coupler Curves by Graphical Means.

The aim of the remainder of this chapter, and indeed of the thesis, is to survey the possible types of real coupler curves which can occur for each A-type. This is not just a simple task in computer graphics; the ultimate step must be to understand mathematically how and why certain types of coupler curves occur for one A-type and not for another. However, computer drawings do give an incredible insight into the motions and coupler curves of the four-bar. It should be noted that even before we begin a computer analysis, there are obstructions to overcome. Fortunately, these can be resolved using the geometry that we will develop. There are two natural methods of drawing coupler curves by a computer.

Method (1): We may parameterise each circuit (since they are diffeomorphic to a circle) of the coupler curve and then program a computer to draw the locus via the explicit parameterisation. This has the disadvantage that not all of the Zariski closure of the coupler curve is drawn. We have worded this very carefully, because the curve traced by the physical model (which we will refer to henceforth as the **physical coupler curve**) may not be an algebraic curve, but only a semi-algebraic curve failing to be algebraic only by the omission of finitely many points. This phenomenon is perhaps more easily illustrated to the reader by a more familiar example.

Consider the locus of a point P lying on a circle C_1 rolling on the outside of another fixed circle C_2. The locus is well known to be a limacon. Let C_2 be a circle, centre the origin, with radius a and let C_1 have radius b. Then the locus may be parameterised as $x = b\cos(t) + a\cos(2t)$, $y = b\cos(t) + a\sin(2t)$ and the reader may easily check that the locus lies on a bicircular quartic curve. For $b < 2a$ the locus has one real ordinary double point as showed in Fig. 5.4(a). When $b = 2a$ the curve acquires a cusp (Fig. 5.4(b)) and the locus in a cardioid and when $b > 2a$ the locus has no real singular point (Fig. 5.4(c)). The singular points of the quartic, however, are two ordinary double points at the circular points at infinity and a (real) double point at the origin. It follows then that in the case $b > 2a$ the quartic has an isolated double point at the origin (i.e. an acnode) which is not attained by the point P during the motion of the circle. We conclude then that in the same manner the coupler curve may have acnodes which are not attained by the locus of the physical mechanism.



(a) (b) (c)

Fig. 5.4

Every real point P on $\mathfrak{R}'$ corresponds to a true position of the mechanism, hence the image of P is a point on the physical coupler curve. Conversely, each point of the physical coupler curve is attained by a real position of the mechanism and therefore must be the image of a real point on $\mathfrak{R}'$. Thus, the physical coupler curve is the image (under the coupler projection) of the set of real points on $\mathfrak{R}'$. But, whilst the image of a complex projective algebraic variety is a complex projective algebraic variety by Elimination Theorem, the image of a real projective algebraic variety is not necessarily a real projective algebraic variety, but only semi-algebraic (i.e. subsets of $P\mathbb{R}^n$, which are the union, intersection or complement of sets of the form $\{(x_1,...,x_{n+1}) \in P\mathbb{R}^n \mid f(x_1,...,x_{n+1}) \geq 0\}$ where the f is a homogeneous polynomial, so any algebraic set is clearly semi-algebraic but not conversely). Thus, the image of the real residual linkage curve is a semi-algebraic subset of the real coupler curve C (i.e. the real part of the complex coupler curve). In the generic case we can deduce that any isolated point of C cannot come from a real point on $\mathfrak{R}'$. Recall that the real residual curve is non-singular and hence diffeomorphic to a disjoint union of circles. Since all coupler projections are finite maps, no "circle" can map to a point and hence the image of the real linkage variety does not possess an isolated point. In particular this provides a proof that no isolated point of the real coupler curve is attained physically.

Method (2): We may write down the defining equation of the coupler curve and program a computer to calculate the solutions of the equation and plot them. Drawing the coupler curve in this manner often guarantees the inclusion of the isolated points. The drawback here is that such programs tend to be very slow, inefficient, sometimes unreliable and often produce poor quality hardcopy (i.e. pictures on paper) depending on how well the program is written (good drawing quality generally involves more computer time and hence we lose out on speed of execution). This method of analysing coupler curves, when one may wish to draw a large number of curves, is impractical on a microcomputer (unless you have plenty of time and patience) and generally little gain for speed is attained on a mainframe or mini-computer in a multi-user environment. For the survey we will need to produce a large number of curves.

As the reader has most likely already guessed, the author has used approach number one. To make any progress, however, we have to overcome the problem that, if we have in front of us a drawing of a (physical) coupler curve with no (resp. one) real double point present, we will be unable to decide whether the Zariski closed coupler curve has three real acnodes or one real acnode and two complex conjugate double points (resp. one real ordinary double point and two real acnodes, or one real ordinary double point and two complex conjugate double points). The answer is non-trivial and leads to an interesting answer in terms of the real geometry of the Segre quartic surface which gives considerable illumination to the motion of the four-bar.

Recall that for a given planar four-bar its linkage variety is isomorphic to an intersection of a net of quadrics in $\mathbb{PC}^4$. The key idea is based on the geometry described in [Gibson&Newstead] and explained in §1.6 that choosing a coupler point determines a unique pencil in that net. In general, coupler points give rise to a general pencil so that the intersection is a Segre quartic surface. The coupler projection corresponds to projecting from a real line L on that surface and thus defines a birational correspondence with the projective plane branched over a conic; the well known circle of singular foci. The (complex) coupler curve is the image of the residual linkage curve under this projection. The five lines L_i $(i = 1,...,5)$ meeting L map to the five base points. Two of the five lines are complex conjugates, meet the residual linkage curve in three points and map to the circular points at infinity which lie on the coupler curve. Each of the remaining three lines meets a third quadric in the net, not already in the given pencil, in two points. Thus, each such line meets the linkage variety in two points and maps to a double point of the coupler curve, namely, one of the base points. The key point is that the problem of determining the number of real double points on a coupler curve is equivalent to determining the number of real lines on the associated Segre quartic surface.

We shall describe the real geometry of the Segre quartic surface (§5.5) and then apply these results to the geometry of the associated pencil of a coupler point (§5.6). This yields a method for determining the number of real double points on the coupler curve. Moreover, we will show that there is a curve $\mathcal{T} = 0$ which partitions the coupler plane into two regions. The nett result is that points P with $\mathcal{T}(P) < 0 / \mathcal{T}(P) > 0$ give rise to coupler curves with 1/3 real double points.

But first we pursue the geometry of coupler curves with cusps. The gain here is that there is a curve $\mathfrak{C}$ which partitions the coupler plane. If we consider the coupler plane with this curve removed, then we obtain finitely many connected regions. We will say that two regions are neighbours, if there exists a continuous path in the coupler plane between a point in one region and a point in the other which meets $\mathfrak{C}$ in only one point. Let us consider a continuous path $\mathfrak{P}$ between two points in neighbouring regions close to $\mathfrak{C}$. Let this path meet $\mathfrak{C}$ in the point Q. Then we may observe the phenomenon that one of the double points of the coupler curve of a coupler point P on $\mathfrak{P}$ as it approaches Q, *either* transforms from a crunode into a cusp when $P = Q$ and then into an acnode when it has passed through Q, *or* vice versa. Finally, we may consider the partition of the coupler plane obtained by removing both $\mathfrak{C}$ and $\mathfrak{T}$ into connected components. Then clearly, any two points in a component give coupler curves whose double points have identical singularity types. This will form the basis for our classification of four-bar curves.

In §5.4 we will study the curve $\mathfrak{C}$ (described above) followed in §5.5 by an analysis of the real Segre Quartic Surface in preparation for §5.6, when we describe the geometry of $\mathfrak{T}$ (described above). In §5.7 we conduct the survey of four-bar coupler curves.

§5.4. The Geometry of the Cusp Curve (Moving Centrode).

In §5.2 we showed that the necessary and sufficient condition for cusps to occur on four-bar coupler curves is:

$$\frac{d_1^2k_1^2}{k_1^2+k_2^2} + \frac{2d_1d_3k_1(d_2-k_1)}{\varepsilon_1(k_1^2+k_2^2)^{1/2}.\varepsilon_2(k_2^2+(d_2-k_1)^2)^{1/2}} + \frac{d_3^2(d_2-k_1)^2}{k_2^2+(d_2-k_1)^2}$$

$$-2d_2\left[\frac{d_1k_1}{\varepsilon_1(k_1^2+k_2^2)} + \frac{d_3(d_2-k_1)}{\varepsilon_2(k_2^2+(d_2-k_1)^2)}\right] + d_2^2 + \frac{d_1^2k_1^2}{k_1^2+k_2^2}$$

$$\frac{-2d_1d_3k_2^2}{\varepsilon_1(k_1^2+k_2^2)^{1/2}.\varepsilon_2(k_2^2+(d_2-k_1)^2)^{1/2}} + \frac{d_3^2k_2^2}{k_2^2+(d_2-k_1)^2} = d_4^2$$

It can be easily showed that the points satisfying this condition lie on the curve of degree eight (which henceforth we shall refer to as the **cusp curve** $\mathfrak{C}$) whose equation is:

$$\Big([d_1^2+d_2^2+d_3^2-d_4^2]d_2^2[k_1^2+k_2^2][k_1^2+k_2^2-2d_2k_1+d_2^2]+d_1^2d_2^2d_3^2[2(k_1^2+k_2^2)-2d_2k_1]^2$$
$$- 4d_2^4d_3^2[k_1^2+k_2^2][d_2-k_1]^2-4d_1^2d_2^4k_1^2[k_1^2+k_2^2-2d_2k_1+d_2^2]\Big)^2$$
$$-4d_1^2d_2^4d_3^2[k_1^2+k_2^2][k_1^2+k_2^2-2d_2k_1+d_2^2]\Big([d_1^2+d_2^2+d_3^2-d_4^2][2(k_1^2+k_2^2)-2d_2k_1]$$
$$+4d_2^2k_1[d_2-k_1]\Big)^2 = 0$$

This curve is obtained by Müller in [Müller] who shows, by mechanical means, that this is the condition for a cusp to occur. He gives an analysis of the curve for the degenerate cases, a straightforward exercise.

Circumscriptible Case:

1) We will not write down the equation of the curve but simply note that $k_2 = 0$ is a component of the curve twice repeated and that the degree of the remaining curve is six and has cusps at the

circular points at infinity and double points at $(0,0)$ and $(d_4,0)$.

Kite Case:

1) $d_1 = d_4$, $d_2 = d_3$

$k_2 = 0$ repeated four times and the quartic curve

$$[d_1^2-d_2^2][k_1^2+k_2^2]^2 - 4d_1^2d_2[d_1^2-d_2^2][k_1^2+k_2^2]k_1 + 4d_1^2d_2^2[d_1^2-d_2^2]k_1^2 - 4d_1^2d_2^4k_2^2 = 0$$

The quartic is rational. Indeed, it is a limacon with the parameterisation $x = \cos(t)\cdot(2a\cos(t) + b)$ $y = \sin(t)\cdot(2a\cos(t) + b)$ where $a = d_1^2d_2/[d_1^2-d_2^2]$ and $b = -2d_1d_2^2/[d_1^2-d_2^2]$ so that the origin is an acnode (resp. crunode) if and only if $d_2 > d_1$ (resp. $d_2 < d_1$).

2) $d_1 = d_2$, $d_3 = d_4$. Similarly, we may show that the curve is the line $k_2 = 0$ repeated four times and a quartic curve which like the previous case is a limacon. The parameterisation is given by $x = d_2 - \cos(t)\cdot(2a\cos(t) + b)$ and $y = \sin(t)\cdot(2a\cos(t) + b)$ where $a = d_3^2d_2/[d_3^2-d_2^2]$ and $b = -2d_3d_2^2/[d_3^2-d_2^2]$. The point $(d_4,0)$ is an acnode (resp. crunode) if and only if $d_2 > d_3$ (resp. $d_2 < d_3$).

Parallelogram Case: $d_1 = d_3$, $d_2 = d_4$

Then the curve is the line $k_2 = 0$ repeated four times and the conic whose equation is

$$4(d_1^2-d_2^2)k_1^2 + 4d_1^2k_2^2 - 4d_2(d_1^2-d_2^2)k_1 + 4d_1^2(d_2^2-d_1^2) = 0,$$

which is an ellipse if and only if $d_1 > d_2$ and an parabola if and only if $d_2 > d_1$.

Rhombus Case: The coupler curve can have no cusps.

Recall that for the four-bar mechanism the linkage variety $\mathfrak{R}$ is isomorphic to a net of quadrics $\mathfrak{U}$ in $P\mathbb{C}^4$ and that to any coupler point we may associate a pencil in that net. Whenever the pencil is generic the intersection of that net is a Segre quartic surface $\mathcal{S}$. The coupler projection given by such a coupler point is from a line on that surface and thus defines a birational map between $\mathcal{S}$ and the complex plane, branched over a circle. The image of the restriction of the coupler projection to the residual linkage curve $\mathfrak{R}'$ is the (complex) coupler curve $\mathfrak{C}_k$. The pre-image on $\mathcal{S}$ of a double point P on $\mathfrak{C}_k$ is a line L_1 meeting the centre of projection $\mathfrak{L}_k$. In general, the line meets a third quadric q in the associated net, not already contained in the associated pencil, in two points. Exceptionally, L_1 may be tangent to the quadric and therefore meets $\mathfrak{R}'$ in just one point whose image is a cusp on the coupler curve.

The cusp curve, which we will denote by $\mathfrak{C}$, plays a special role here. Observe that the intersection points of L_1 and q, whose coefficients are in terms of the design parameters, may be obtained by taking the resultant, a quadratic equation whose coefficients are likewise in terms of the design parameters. If we further suppose that L_1 is real, then the resultant is a real quadratic with coefficients in the design parameters. Let $\mathfrak{D}$ be the discriminant of the quadratic, again a polynomial in the design parameters, then $\mathfrak{D}>0/\mathfrak{D}=0/\mathfrak{D}<0$ if and only if the intersections are real/coincident/complex. In the real (resp. complex) case L_1 meets $\mathfrak{R}'$ in two real (resp. complex) branches and they map to real (resp. complex) branches of the coupler curve through P; thus P is a crunode (resp. acnode). Finally, in the coincident case we know that L_1 is tangent to $\mathcal{R}'$ and P is a cusp. Thus, the coupler point must lie on the cusp curve. Thus, the discriminant vanishes if and only if the coupler point lies on $\mathfrak{C}$. It now follows that, as a coupler point approaches the cusp curve and passes through it, one of the real double points on the coupler curve must make the transition acnode-cusp-crunode or vice versa.

We can study the geometry of the cusp curve in an alternative manner to that of looking at its defining equation. Recall that the condition for the coupler projection to have a critical point is

$$\left.\begin{aligned}
&y_2[-k_1x_1-k_2y_1]-x_2[k_2x_1-k_1y_1] = 0\\
&x_2[-k_2x_3-(d_2-k_1)y_3]-y_2[-(d_2-k_1)x_3+k_2y_3] = 0\\
&[k_1x_1+k_2y_1]\cdot[k_2x_3+(d_2-k_1)y_3]-[k_2x_1-k_1y_1]\cdot[-(d_2-k_1)x_3+k_2y_3]=0
\end{aligned}\right\} \quad (5.5)$$

Combining these equations together with the real affine linkage variety equations

$$\begin{aligned}
&d_1x_1 + d_2x_2 + d_3x_3 - d_4 = 0\\
&d_1y_1 + d_2y_2 + d_3y_3 = 0\\
&x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = 1,
\end{aligned}$$

gives a variety in the affine 8-space with co-ordinates $x_1, y_1, x_2, y_2, x_3, y_3, k_1, k_2$. Let us make this variety complex and projective by introducing the complex homogenising variable w, defining the **(complex) centrode variety** $\mathcal{V}$ in $P\mathbb{C}^8$. Let the union of components of $\mathcal{V}$, not lying entirely in the hyperplane $w = 0$, be called the residual centrode variety and let it be denoted by $\mathcal{V}'$. Then there are two linear projections from $P\mathbb{C}^8$, namely, $\pi_{\mathcal{R}}: (x_1, y_1, x_2, y_2, x_3, y_3, k_1, k_2, w) \mapsto (x_1, y_1, x_2, y_2, x_3, y_3, w)$ and $\pi_{\mathcal{C}}: (x_1, y_1, x_2, y_2, x_3, y_3, k_1, k_2, w) \mapsto (k_1, k_2, w)$ whose restrictions to $\mathcal{V}'$ are, respectively, the linkage variety $\mathcal{R}$ (<u>thus $\mathcal{V}'$ is a curve</u>) and a curve $\mathcal{C}$ in the (complex projective) coupler plane which is clearly the cusp curve.

It is worth noting at this point that we are using a standard technique in algebraic geometry. The cusp curve may be obtained from the residual linkage variety as the image of a rational map, defined by expressing the first two conditions of (5.5) as linear identities in k_1, k_2 and applying Cramer's rule to express k_1, k_2 as rational functions in $x_1, y_1, x_2, y_2, x_3, y_3, w$. Thus we define a rational map $\nu: \mathcal{V}' \to \mathcal{C}$ given by $\nu:(x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (k_1, k_2, w)$ where

$$k_1 = \frac{d_2[x_2y_3-y_2x_3][x_1x_2+y_1y_2]}{[x_2^2 + y_2^2][x_1y_3-y_1x_3]} \quad \text{and} \quad k_2 = \frac{-d_2[x_2y_3-y_2x_3][x_1y_2-x_2y_1]}{[x_2^2 + y_2^2][x_1y_3-y_1x_3]}.$$

Note that any component satisfying $[x_1y_3-y_1x_3] = 0$ has no image. The reason for this is clear, since the condition is satisfied if and only if the component is a conic. Then the image on the coupler curve is a circle and can never acquire a cusp. Thus we are only interested in components of $\mathcal{R}'$ which are not conics.

We could now argue carefully and obtain a result for all the degenerate cases (except the rhombus case when $\mathcal{V}'$ is empty). However, we will only consider the generic case. Then, in the above description we are factoring the rational map $\pi_{\mathcal{C}}$ as indicated in the following diagram

$$\begin{array}{ccc} & \mathcal{V}' \subset P\mathbb{C}^8 & \\ \pi_{\mathcal{R}} \swarrow & & \searrow \pi_{\mathcal{C}} \\ \mathcal{R}' \subset P\mathbb{C}^6 & \xrightarrow{\ \nu\ } & \mathcal{C} \subset P\mathbb{C}^2 \end{array}$$

Thus provided we can establish the geometry of $\mathcal{V}'$, then $\mathcal{C}$ is obtained by a linear projection of $\mathcal{V}'$ and then we can determine the geometry of $\mathcal{C}$ more easily than with the rational map ν.

The restriction $\pi_{\mathcal{R}}|_{\mathcal{R}'}$ to the residual linkage variety is easily checked to be generically 1-1 and so defines a birational map. Hence $\mathcal{V}'$ and $\mathcal{R}'$ have the same geometric genus. Indeed, we can see this mechanically. Any real point on the linkage variety determines a unique configuration of the mechanism. We then recall that the condition for a cusp is that the coupler point is the unique point of intersection of the two lines passing through bars 1 and 3 (see Fig .5.2). The inverse rational map $\mu: \mathcal{R}' \to \mathcal{V}'$ is defined by $\mu:(x_1, y_1, x_2, y_2, x_3, y_3, w) \mapsto (x_1, y_1, x_2, y_2, x_3, y_3, k_1, k_2, w)$ with k_1 and k_2 identical to the values given above.

The intersection of $\mathcal{V}$ with the hyperplane at infinity is given by $d_1x_1+d_2x_2+d_3x_3=w=0$ and $y_j=\varepsilon_j i x_j$ for $j = 1,2,3$, (where $\varepsilon_j = \pm 1$) giving two complex conjugate 3-planes W, $\overline{W}$, when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \pm 1$ (necessarily components of $\mathcal{V}'$) and 2-planes lying on these 3-planes for the other choices of sign. The two 3-planes meet in the real L line given by $x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = w = 0$. It is clear that L is the centre of projection $\pi_{\mathcal{C}}$.

We note that any finite point with $[x_1y_3 - y_1x_3] \neq 0$ is mapped by μ to a finite point of $\mathcal{V}'$. Now suppose that $(x_1, y_1, x_2, y_2, x_3, y_3, w)$ is a point on $\mathcal{R}'$ with $[x_1y_3 - y_1x_3] = 0$. Then we may recall, that this is the condition for the projection of $\mathcal{R}'$ onto one of the circles $x_2^2 + y_2^2 = w^2$ to have a critical point. We found in §1.4 that, in the generic case, there are just four finite points on $\mathcal{R}'$ satisfying this condition and that in the real case they correspond to the limiting positions so familiar to mechanism theorists. These map to points on $\mathcal{V}'$ of the form $(0, 0, 0, 0, 0, 0, k_1, k_2, 0)$ and hence lie on the line L. Explicitly, we may rewrite the rational map by multiplying through by $[x_1y_3 - y_1x_3]$. Then, since all but the last two co-ordinates of the image of one of the four points vanish, the rational map is defined at these points and it is a simple exercise to show that $k_1 = \varepsilon_1(4d_4^2[d_1+\varepsilon_2 d_3]^2 - (d_2^2 - d_4^2 - [d_1+\varepsilon_2 d_3]^2))^{1/2}$ and $k_2 = d_2^2 - d_4^2 - [d_1+\varepsilon_2 d_3]^2$ where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

If $(x_1, y_1, x_2, y_2, x_3, y_3, w)$ is a point on $\mathcal{R}'$ with $[x_2^2 + y_2^2] = 0$ then $w = 0$ and this is only satisfied by the six points of $\mathcal{R}'$ in the hyperplane at infinity listed in §1.1. However, in this case we cannot find a simple re-expression of the rational map, which is defined at these points, although we know that such a map does exist. Thus, it is necessary to calculate their images directly.

To calculate the images directly, we take a local parameterisation of the residual linkage variety at each point in turn. Substituting for each variable in the rational map in terms of the parameterisation, we obtain a parameterisation for the centrode variety at the image point. In particular, the constant term determines the co-ordinates of the image point. The calculations are laborious, but straightforward.

1) $P = (0,0,d_3,-id_3,-d_2,id_2,0)$. Let us make the residual linkage variety affine by setting $x_2 = 1$ and then let us apply an affine change of co-ordinates taking P to the origin. We then calculate the local parameterisation at the origin and, finally, apply the inverse affine transformation returning the origin to P to obtain the following local parameterisation at P

$$x_1 = \frac{(d_1^2 + d_4^2)w}{2d_1d_4} + \{\text{HOT in } w\} \quad : \quad y_1 = -\frac{i(-d_1^2 + d_4^2)}{2d_1d_4} + \{\text{HOT in } w\}$$

$$x_3 = -\frac{d_2}{d_3} + \frac{(-d_1^2 + d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\} \; : \; y_3 = \frac{id_2}{d_3} - \frac{i(-d_1^2+d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\}$$

$$y_2 = -i + iw^2 + \{\text{HOT in } w\}$$

Substituting for x_1, y_1, y_2, x_3, y_3 into the rational map, cancelling out any common factors of w in the numerator and denominator (this is allowable since $w \neq 0$ in a neighbourhood) and then setting $w = 0$ to get the constant term of the parameterisation of $\mathcal{V}'$ at the image, we find that $k_1 = k_2 = 0$. In particular, the linear term has non-zero coefficient, thus the image is the simple point $P' = (0,0,d_3,-id_3,-d_2,id_2,0,0,0)$ on $\mathcal{V}'$.

2) $Q = (-d_3,id_3,0,0,-d_1,id_1,0)$. Let us make the residual linkage variety affine by setting $x_1 = 1$ and then let us apply an affine change of co-ordinates taking Q to the origin. Then we calculate the local parameterisation at the origin and, finally, we apply the inverse affine transformation returning the origin to Q to obtain

the following local parameterisation of $\mathcal{R}'$ at Q

$$x_2 = \frac{(d_2^2 + d_4^2)w}{2d_2d_4} + \{\text{HOT in } w\} \quad : \quad y_2 = \frac{-i(-d_2^2 + d_4^2)w}{2d_2d_4} + \{\text{HOT in } w\}$$

$$x_3 = \frac{-d_1}{d_3} + \frac{(-d_2^2 + d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\} : y_3 = \frac{id_1}{d_3} - \frac{i(-d_2^2 + d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\}$$

$$y_1 = -i + iw^2 + \{\text{HOT in } w\}$$

Rewrite the rational map by multiplying through by $[x_2^2 + y_2^2][x_1y_3 - y_1x_3]$ to obtain an equivalent rational map to the original. Then substitute for the variables y_1, x_2, y_2, x_3, y_3, using the parameterisation, into the rational map. Then set $w = 0$ to get the constant term of the parameterisation of $\mathcal{V}'$ at the image. This yields that the co-ordinate of the image is $Q' = (0,0,0,0,0,0,1,-i,0)$ and that the linear term has non-zero coefficient. Thus Q' is simple on $\mathcal{V}'$. Furthermore, Q' lies on the centre of the $\pi_{\mathcal{R}}$ thus by general theory (Theorem A13) Q is the image of an osculating n-plane $\mathcal{K}$. Since Q lies on the hyperplane $w = 0$, $\mathcal{K}$ is contained in $w = 0$ (in the ambient space of the centrode variety); in particular, its tangent lies in $w = 0$ and thus $\mathcal{V}'$ touches $w = 0$ at Q'.

3) $R = (d_2, -id_2, -d_1, id_1, 0, 0, 0)$. Let us make the residual linkage variety affine by setting $x_1 = 1$ and then let us apply an affine change of co-ordinates taking R to the origin. Then calculate the local parameterisation at the origin and, finally, we apply the inverse affine transformation returning the origin to R to obtain the following local parameterisation of $\mathcal{R}'$ at R

$$x_2 = \frac{-d_1}{d_2} + \frac{(d_3^2 + d_4^2)w}{2d_2d_4} + \{\text{HOT in } w\} : y_2 = \frac{id_1}{d_2} - \frac{i(-d_3^2 + d_4^2)w}{2d_2d_4} + \{\text{HOT in } w\}$$

$$x_3 = \frac{(d_3^2 + d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\} : y_3 = \frac{i(-d_3^2 + d_4^2)w}{2d_3d_4} + \{\text{HOT in } w\}$$

$$y_1 = -i + iw^2 + \{\text{HOT in } w\}$$

Then we substitute for the variables y_1, x_2, y_2, x_3, y_3 using the above parameterisation into the rational map and we cancel out any common factors of w in the numerator and denominator (allowable since $w \neq 0$ in a neighbourhood). We set $w = 0$ to get the constant term. The linear term has non-zero coefficient, hence the image is the simple point $R' = (d_2, -id_2, -d_1, id_1, 0,0,0,0,0)$.

4) The images P', $\bar{Q}'$ and $\bar{R}'$ of the points $\bar{P}$, $\bar{Q}$ and $\bar{R}$ are obtained by taking the complex conjugates of the points above.

Summarising, $\mathcal{V}'$ meets $w = 0$ in six simple points on L and a further four points, two on W and two on $\bar{W}$, not lying on L.

Note that the projection $\pi_{\mathcal{R}}$ defines an isomorphism between the sets $\mathcal{R}'$-{points such that $w \cdot [x_1y_3 - y_1x_3] = 0$} and $\mathcal{V}'$-{points with $w = 0$}. Thus, in the generic case, $\mathcal{V}'$ is a non-singular curve; since there are no finite singular points and, as we found above, there are no singular points of $\mathcal{V}'$ in $w = 0$. But any birational map between two non-singular curves is an isomorphism (Theorem A7); moreover, $\pi_{\mathcal{R}}$ and its inverse μ are given by real polynomials, thus the real residual linkage curve and the real centrode curve are real isomorphic. It follows that the residual centrode variety $\mathcal{V}'$ has one or two connected

<u>components precisely when the linkage variety has one or two components</u>.

To determine the degrees of $\mathcal{V}'$ and $\mathcal{C}$ we consider the projection $\pi_{\mathcal{R}'}$. Its centre is the line L. Then considering the higher order terms of the local parameterisation of $\mathcal{V}'$ at points lying in $w = 0$ (the details of which we spare the reader), we may deduce that no tangent to $\mathcal{V}'$ coincides with L. Let H be a generic hyperplane through L and suppose that H meets $\mathcal{V}'$ in the points P_j, $j = 1,\ldots,m$. Then, the sum $\Sigma_j i(P_j, H \cap \mathcal{V}')$ of the intersection multiplicities of H and $\mathcal{V}'$ equals the sum of the multiplicities of P_j on $\mathcal{V}'$; the sum is easily checked to be six. Applying the Projection Formula, yields $\text{degree}(\mathcal{V}') - \Sigma_j i(P_j, H \cap \mathcal{V}') = \text{degree}(\mathcal{R}')$. Then the fact from §1.1 that $\mathcal{R}'$ has degree six, yields that <u>$\mathcal{V}'$ has degree twelve</u>.

We may now apply the Projection Formula to $\pi_{\mathcal{C}}$. The centre of projection M given by $k_1 = k_2 = w = 0$ meets $\mathcal{V}'$ in the four points P', $\bar{P}'$, R', $\bar{R}'$ whose tangents to $\mathcal{V}'$ do not lie in $w = 0$ (else the tangents to $\mathcal{R}'$ at P, $\bar{P}$, R, $\bar{R}$ lie in $w = 0$). Thus, in particular, they do not lie on M. Let H be a generic hyperplane through M. Suppose H meets $\mathcal{V}'$ in the points P_j, $j = 1,\ldots,m$. Then the sum $\Sigma_j i(P_j, H \cap \mathcal{V}')$ of the intersection multiplicities of H and $\mathcal{V}'$ equals the sum of the multiplicities of P_j on $\mathcal{V}'$. It is easily checked that the images of the six points on L map to six distinct points on the line at infinity, thus the degree of $\mathcal{C}$ is at least six by Bézout's Theorem. But the sum of the multiplicities is four, thus the Projection Formula yields that the <u>degree of the projection $\pi_{\mathcal{C}}$ is one and that $\mathcal{C}$ has degree eight</u>.

We should perhaps emphasise that, of the twelve 'limiting position' points, the four critical points of the projection π_1 (resp. π_3) map to (0,0) (resp. $(d_4,0)$), whilst the four critical points of π_2 map to distinct points on the hyperplane at infinity. Hence, the points (0,0) and $(d_4,0)$ have multiplicity ≥ 4 on $\mathfrak{C}$. But the line $k_2 = 0$ passes through both of these points and hence, by Bézout's Theorem, both points must have exactly multiplicity four and $\mathfrak{C}$ does not touch the line. We may deduce that the Hain type will determine how many real branches pass through (0,0), $(d_4,0)$ and how many real branches meet $w = 0$ (in other than I and J); for we have established a correspondence between the number of real branches and the number of real critical points of π_j.

We recall that a circuit of a real planar curve is defined to be the image of a connected component of any real desingularisation. Then we may take the curve $\mathcal{V}'$ (or $\mathcal{R}'$) as a desingularisation of the cusp curve $\mathfrak{C}$. Thus we find that <u>the cusp curve $\mathfrak{C}$ has one or two circuits</u> - the same number as its family of coupler curves determined entirely by the design parameters.

Studying the centrode variety, rather than the cusp curve via its equation, has yielded several new results. Firstly, we have showed that the curve is birationally isomorphic to the linkage variety from which it follows that the geometric genus is one. Secondly, we have been able to determine the number of circuits of the cusp curve in the generic case. Thirdly, we have been able to determine the multiplicity of (0,0) and $(d_4,0)$; moreover, we showed that the number of real branches through these points is determined by the Hain type.

Perhaps the most important result about the cusp curve, yet to be established, is that (in the generic case) double (resp. triple) points on the cusp curve correspond to coupler curves with two (resp. three) cusps. This very intuitive result is another of the 'folklore' results for which no proof exists in the literature. From our point of view this is very easy to establish. If P is a double (triple) point on the cusp curve, then, since $\mathcal{V}'$ has no singular points, P has two (resp. three) pre-images P_j on the centrode variety. These points are mapped by $\pi_{\mathcal{R}'}$ to points on the residual linkage variety and, since $\mathcal{R}'$ is non-singular, these points are necessarily distinct. Further, these points are critical points of the coupler projection and hence their images on the coupler curve traced by P are cusps.

I have not showed that the circular points at infinity are cusps (a result established by Müller). However, assuming that they are cusps, we may procede to show that there can be at most six finite singular points. The sum of the delta invariants of the singular points on a plane curve of degree eight and genus one is equal to $\frac{1}{2}(8-1)(7-1)-1=20$. But, whenever P = (0,0) or $(d_4,0)$, we have $\delta_P \geq 6$ and whenever P=I or J, we have $\delta_P=1$. Thus the sum of the δ_P 's of the remaining singular points is 6. Hence there can be at most six other double points, each with $\delta_P = 1$. Observe that the cusp curve is symmetrical about the line $k_2 = 0$.

If the cusp curve of a mechanism has a triple point, then it has two triple points symmetrically placed about the line $k_2 = 0$ and no other finite singular points. In particular, by disturbing the coupler point, one cannot obtain a coupler curve with two cusps.

However, by disturbing the mechanism slightly, by which I mean a small deformation of the design parameters, one hopes to get a mechanism whose cusp curve has three double points. Indeed, we showed in §5.2 that a mechanism possesses a coupler curve with three cusps if and only if the point (d_1,d_2,d_3,d_4) lies on a hypersurface H in the parameter space. We recall that H is the hypersurface defined by

$$[(d_1^2 + d_2^2 + d_3^2 - d_4^2)d_4 + 2d_1d_2d_3][(d_1^2 + d_2^2 + d_3^2 - d_4^2)d_4 - 2d_1d_2d_3] = 0$$

Therefore, the necessary and sufficient condition for the cusp curve to have a triple point is that the point (d_1,d_2,d_3,d_4) lies on H. Thus for almost all small deformations (d_1',d_2',d_3',d_4') of (d_1,d_2,d_3,d_4) (or more precisely all points in an ε-neighbourhood of (d_1,d_2,d_3,d_4) not lying on H), it is reasonable to expect the triple point on the cusp curve to "unfold" into three ordinary double points on the cusp curve of the mechanism with design parameters (d_1',d_2',d_3',d_4').

§5.5. The Geometry of the Real Segre Quartic Surface.

In chapter one we described how the intersection of a general pencil of quadrics in $P\mathbb{C}^4$ is a Segre Quartic Surface $\mathcal{S}$ containing sixteen lines with the properties that any given line meets exactly five other lines and that any pair of lines has two transversals (i.e. lines meeting both of them). Projecting from one of the lines L defines a birational map between $\mathcal{S}$ and the projective plane whose branch locus (i.e. the points of the projective plane which do not have a unique pre-image on the surface) is a conic passing through the five base points. The base points are the images of the five lines on $\mathcal{S}$ meeting the centre of projection L. Further, we showed that the image of any line, other than L and the five lines meeting it, is a line passing through two of the five base points; we recall that, if L_1 is such a line, then two of the five lines meeting L are transversals of L and L_1 and therefore the image of L_1 meets the images of the transversals namely, two of the base points.

We shall now make some further remarks in the form of the following lemmas about the birational correspondence in the case when L and the surface are real (i.e. $\mathcal{S} = \overline{\mathcal{S}}$), in preparation for the case when the Segre quartic surface is the intersection of the pencil of quadrics associated with a coupler projection of the planar four-bar. Indeed, the geometry of the real Segre quartic surface gives considerable illumination to the real geometry of four-bar coupler curves.

Lemma 5.1 Suppose that the surface $\mathcal{S}$ is real and let L be any real line on it. Then a line on the surface meeting L is real if and only if the corresponding base point is real.

Proof : Trivial.

∎

Lemma 5.2 Suppose that the surface $\mathcal{S}$ is real and let L be any real line on it. Then any other line L_1 on $\mathcal{S}$ is real if and only if the transversals of L and L_1 are either real or complex conjugates.

Proof : Clearly, if the two transversals T_1, T_2 are either real lines or a pair of complex conjugate lines, then the hyperplane H spanning L, T_1 and T_2 is real. Thus H intersects $\mathcal{S}$ in a real quartic curve consisting of L, T_1, T_2 and one other real line, namely, L_1. Conversely, if L_1 is real then the hyperplane spanning L and L_1 is real intersecting $\mathcal{S}$ in a real quartic curve. Thus it follows from a reason given in §1.5 that the quartic reduces to four lines L, L_1 and their two transversals. Since the quartic curve is real, it follows that the transversals are either real or complex conjugates. The lemma is now proved.

■

Lemma 5.3 A real Segre quartic surface $\mathcal{S}$ can possess no, four, eight or sixteen real lines .

Proof : Certainly, real surfaces with no real lines exist, for we may take as an example any empty intersection of two real quadrics which generate a general pencil. Suppose that $\mathcal{S}$ has at least one real line L, then it is sufficient to show that $\mathcal{S}$ has four, eight or sixteen lines.

Claim: If $\mathcal{S}$ is a real Segre quartic surface containing one real line L, then there are at least four real lines on $\mathcal{S}$.

Proof: Note that, if L' is a complex line lying on $\mathcal{S}$, then its conjugate $\bar{L}'$ also lies on $\mathcal{S}$. Thus of the five lines meeting L, one must be real and the remaining four are either pairs of real lines or pairs of complex conjugate lines. Thus, *either* the lines meeting L are two pairs of complex conjugate lines and one real line L_1 *or* there are three or five real lines meeting L. In the latter case the claim immediately follows. In the former case we can prove the claim by observing that each pair of (skew) conjugate lines span a real hyperplane containing L and meeting $\mathcal{S}$ in their two transversals, namely, L and one other (necessarily) real line L_2 distinct from L and L_1. Similarly, the other pair of conjugate lines have as their transversals L and a real line L_3 distinct from L and L_1. Thus, the result is proved if we can show that L_2 and L_3 are distinct. But, if this were the case, then the hyperplane spanning L and L_2 contains two pairs of conjugate lines, clearly contradicting the geometry of the surface $\mathcal{S}$ as described in §1.5.

We may now complete the proof geometrically. Recall that projecting the surface from L gives a birational correspondence with a plane $\mathcal{K}$ branched over a (real) conic. The base points are the images of the five lines meeting L. Further, we recall (from §1.5) that the images of the remaining ten lines on $\mathcal{S}$ are precisely the set of ten lines passing through any pair of base points. Thus, there are three possibilities.

(1) Only one real line L_1 meets L. Then there is one real base point P and two pairs $Q,\bar{Q}$ and $R,\bar{R}$ of complex conjugate base points lying on the conic. The lines through $Q,\bar{Q}$ and $R,\bar{R}$ are real and have real lines as their pre-images on $\mathcal{S}$ (see Fig. 5.5(a)). The remaining lines through any other pairs of base points are clearly complex. Thus, there are exactly <u>four</u> real lines on $\mathcal{S}$.

(2) Three real lines L_1, L_2, L_3 meet L. Then there are three real base points P,Q,R and one pair of complex conjugate points $S,\bar{S}$. The lines through the pairs $\{P,Q\}$, $\{P,R\}$, $\{Q,R\}$ and $\{S,\bar{S}\}$ are real with real pre-images on $\mathcal{S}$ (see Fig. 5.5(b)). The remaining six lines through pairs of points are complex. Thus, together with the four real lines L_1, L_2, L_3 and L and we have <u>eight</u> real lines in all.

(3) Five real lines L_1, L_2, L_3, L_4, L_5 meet L. Then all five base points are real. Thus, the ten lines passing through any pair of them are real (see Fig. 5.5(c)) implying that all <u>sixteen</u> lines on the surface are real.

The Lemma is now proved.

■

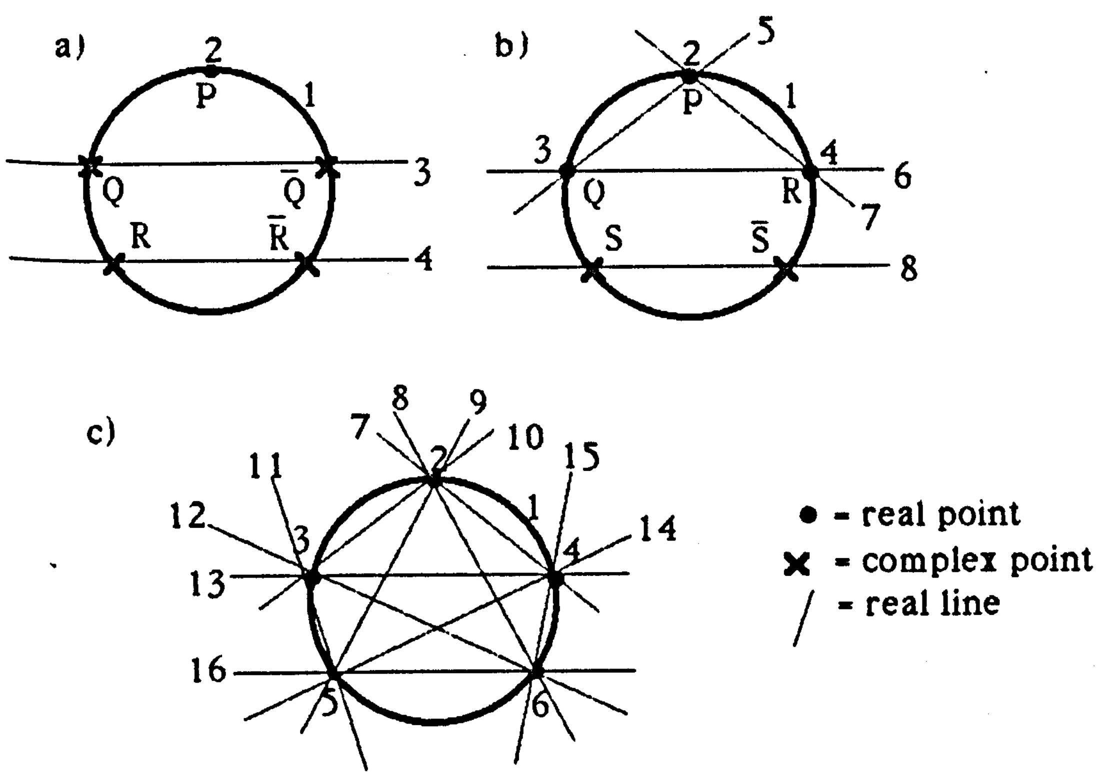


Fig. 5.5.

Finally, we need a method by which we can calculate the number of real base points, so that we can count the number of real double points of the four-bar coupler curves. This is provided by the following Lemma.

Lemma 5.4 Suppose $\mathcal{S}$ is a real Segre quartic surface, i.e. given as the complete intersection of two real quadrics q_1 and q_2 in $P\mathbb{C}^4$. Then the (real binary quintic polynomial) discriminant of the pencil $\alpha q_1 + \beta q_2$ has five distinct roots. Further, suppose that the surface has a real line L. Then the following are equivalent

(1) The discriminant has n real roots and (5-n) complex roots, occuring in conjugate pairs,
(2) There are n real, and (5-n) complex lines occuring in conjugate pairs, on $\mathcal{S}$ meeting L,
(3) The projection from L onto a plane $\mathcal{K}$ has n real, and (5-n) complex base points occuring in conjugate pairs.

Proof : The equivalence of (2) and (3) follows from Lemma 5.3. Thus, it is sufficient to show the equivalence of (1) and (2). Let the quadrics q_1 and q_2 be written in their matrix forms x^tAx and x^tBx where A and B are real symmetric matrices. The discriminant is the real binary quintic polynomial given as the vanishing of the determinant of the matrix $\alpha A + \beta B$. For each root (α_i,β_i) $i = 1,...,5$ of the discriminant, the corresponding quadric $\alpha_i A + \beta_i B$ in the pencil is a point cone whose vertex V_i is real if and only if the root is real. To complete the proof we will show that each point cone with a real vertex gives rise to exactly one real line meeting L.

We recall some facts from §1.5. Let C_i be the cone with vertex V_i and let $\mathcal{K}_i$ be the plane which is the join of L and V_i. Then $\mathcal{K}_i$ is the join of a line on C_i with the vertex and hence $\mathcal{K}_i$ lies on C_i; indeed the reader may recall that $\mathcal{K}_i$ lies in one of the two families of 2-planes on C_i. But $\mathcal{S}$ may be given as the intersection of C_i and any other quadric in the pencil q. Thus, the intersection of $\mathcal{K}_i$ with $\mathcal{S}$ is equal to the intersection of $\mathcal{K}_i$ with q i.e. a <u>conic</u> consisting of L and one other line L_i. Let us now suppose that V_i is real, then the plane $\mathcal{K}_i$ is real and meets $\mathcal{S}$ in a <u>real</u> conic. But as we have just showed, the conic is made up of the real line L and one other line L_i; thus L_i is real. Conversely, if L_i is a real line meeting L, then the join of L and L_i is a real plane $\mathcal{K}_i$. By the result of §1.5, $\mathcal{K}_i$ contains the vertex V_i of a point cone in the pencil and, moreover, only <u>one</u> such vertex lies on $\mathcal{K}_i$. But, if V_i is complex, then the complex conjugate point $\overline{V}_i$ lies on $\mathcal{K}_i$ and is the vertex of a point cone in the pencil giving the required contradicton. Thus V_i is real and the Lemma is proved.

■

We will now apply the above results to the geometry of the planar four-bar.

§5.6. The Geometry of the Tacnode Curve (Transition Curve).

The aim of this section is to determine the number of real double points on a given four-bar coupler curve. By the results of the previous section this is equivalent to counting the number of real roots of the discriminant of the associated pencil for a fixed coupler point. The first step we shall make in this direction follows the approach used in [Gibson&Newstead] for determining when the associated pencil is non-general. The authors of [Gibson&Newstead] show that the set of coupler points (k_1,k_2), which give rise to a non-general pencil, lie on an algebraic curve and describe this curve completely in the more degenerate cases (when the Grashof equality holds) and partly in the generic and circumscriptible cases. We shall show that this curve is the union of two curves $\mathcal{T}'$ and $\mathcal{T}''$, such that almost all points (k_1,k_2) on $\mathcal{T}'$ (resp. $\mathcal{T}''$) are coupler points which trace curves with a tacnode (resp. triple point). We shall refer to $\mathcal{T}'$ and $\mathcal{T}''$ as the **tacnode curve** and the **triple point curve**, respectively. Finally, we shall show that the tacnode curve $\mathcal{T}'(P) = 0$ partitions the coupler plane so that points P for which $\mathcal{T}'(P) < 0/\mathcal{T}'(P) > 0$, correspond to coupler curves with $1/3$ real double points. The curve $\mathcal{T}$ is the so called transition curve (Übergangskurve) of Müller [Müller] who derives (essentially) the same conclusion by purely mechanical means.

In this section we will use the notation of chapter one. We recall from [Gibson&Newstead], as explained in §1.6, that the linkage variety of the four-bar is isomorphic to the intersection of three quadrics in $P\mathbb{C}^4$. Explicitly, the quadrics are

$$Q_1: \quad x_1^2 + y_1^2 = w^2$$
$$Q_2: \quad x_2^2 + y_2^2 = w^2$$
$$Q_3: \quad (d_4w - d_1x_1 - d_2x_2)^2 - (d_1y_1 + d_2y_2)^2 = d_3^2w^2$$

The linkage variety $\mathfrak{R}$ is a curve consisting of two lines L and $\bar{L}$ in the hyperplane at infinity $w = 0$ and a sextic curve $\mathfrak{R}'$ which meets $w = 0$ in three pairs of complex conjugate points lying on L and $\bar{L}$. The curve $\mathfrak{R}'$ is called the residual linkage variety. Let us fix a coupler point $P = d_1z_1 + k \cdot z_2$, where $k = k_1 + ik_2$ is a complex number. Then the coupler curve is the image of the residual linkage variety under the linear projection given by $\pi_k : (x_1,y_1,x_2,y_2,w) \mapsto (P_1,P_2,w)$, where $P_1 = d_1x_1 - k_2y_2 + k_1x_2$, $P_2 = d_1y_1 + k_2x_2 + k_1y_2$ and $P_3 = w$. The centre of projection is the line $\mathfrak{L}$ given by $P_1 = P_2 = P_3 = 0$. It is easily checked that $\mathfrak{L}$ is a transversal of L and $\bar{L}$.

Let us denote by $\mathfrak{N}$ the net of quadrics $XQ_1 + YQ_2 + ZQ_3$ and let S be any point not lying on the intersection of the net i.e. the linkage variety. Then the subset of quadrics in $\mathfrak{N}$ passing through S is given by a linear condition in the variables X, Y, Z, thus defining a pencil of quadrics in $\mathfrak{N}$. In particular, we may choose a point S on $\mathfrak{L}$. Then any quadric in the pencil meets $\mathfrak{L}$ in S and two other points, one lying on L and one lying on $\bar{L}$. Hence, every quadric in the pencil contains $\mathfrak{L}$. Thus the corresponding pencil $\mathfrak{P}$ in $\mathfrak{N}$ comprises those quadrics in the net <u>containing $\mathfrak{L}$</u>. Quite explicitly, the pencil is given by

$$k\bar{k}X + d_1^2Y + d_1^2(k-d_2)(\bar{k}-d_2)Z = 0 \qquad (5.6)$$

and the pencil may be written

$$X\left(Q_1 - \frac{k\bar{k}}{d_1^2}Q_2\right) + Z(Q_3 - (k-d_2)(\bar{k}-d_2)Q_1) = 0$$

We may represent the generators of the net in matrix form x^tAx, x^tBx and x^tCx. The discriminant is the polynomial in X, Y and Z given by the vanishing of the determinant of the matrix $\mathcal{M} = XA+YB+ZC$. The matrix $\mathcal{M}$ is easily showed to be

$$\mathcal{M} = \begin{bmatrix} X+d_1^2Z & 0 & d_1d_2Z & 0 & -d_1d_4Z \\ 0 & X+d_1^2Z & 0 & d_1d_2Z & 0 \\ d_1d_2Z & 0 & Y+d_2^2Z & 0 & -d_2d_4Z \\ 0 & d_1d_2Z & 0 & Y+d_2^2Z & 0 \\ -d_1d_4Z & 0 & -d_2d_4Z & 0 & -X-Y+(d_4^2-d_3^2)Z \end{bmatrix}$$

The discriminant of the net determines a plane quintic curve. It is easily seen from the matrix that the curve reduces into a conic and cubic given by

$$XY + d_1^2YZ + d_2^2XZ = 0 \ : \ (X + Y + d_3^2Z)(XY + d_1^2YZ + d_2^2XZ) = 0 \quad (5.7)$$

It is showed in [Gibson&Newstead] that the conic and cubic touch at the three points (1,0,0), (0,1,0) and (0,0,1). We may deduce the condition for the pencil $\mathcal{P}$ to be general i.e. to have five distinct point cones or equivalently for its discriminant to have five distinct roots. The pencil $\mathcal{P}$ defines a line in the (X,Y,Z)-plane. Thus the condition is that the pencil is general if and only if that line meets the discriminant curve of the net in five distinct points. The conditions for the failure of the pencil to be general are given in

[Gibson&Newstead] and we shall give the details here.

Conditions for the associated pencil to be non-general.

Firstly, the pencil can fail to be general, whenever the line passes through one of the points (1,0,0), (0,1,0) and (0,0,1). But this is easily showed to be the case if and only if $k = 0$ or $k = d_2$ i.e. the coupler point is one of the hinges on the coupler bar.

Secondly, the pencil fails to be general, whenever the line is tangent to the conic. Substituting for Y, using (5.6), into the defining equation of the conic (5.7), gives a binary quadratic in X and Z whose discriminant is $-4d_1^4d_2^2k_2^2$. The line is tangent to the conic if and only if the discriminant vanishes i.e. $k_2 = 0$. Moreover, in general the line meets the conic in two complex conjugate points.

Finally, the pencil fails to be general, whenever the line is tangent to the cubic or (if the Grashof condition holds) the line passes through a double point of the cubic. We shall now summarise the results of [Gibson&Newstead] for this last case.

Generic Case:

The cubic is non-singular so that the condition is that the line is tangent to the cubic. Thus the condition is equivalent to saying that the point $(k_1^2 + k_2^2, d_1^2, d_1^2[k_1^2 + k_2^2 - 2d_2k_1 + d_2^2])$ lies on the dual curve of the cubic. The dual of a cubic is a sextic curve, thus, substituting for the co-ordinates of the point, we find that the point (k_1,k_2) lies on an algebraic curve of degree less than or equal to twelve. We shall need to be more precise than the authors of [Gibson&Newstead], but we shall leave the details until later in this section.

Circumscriptible Case:

The condition for the line to pass through node on the cubic is that the point (k_1,k_2) lies on one of the following circles

$$d_1+d_4 = d_2+d_3 : (d_1-d_3)(k_1^2 + k_2^2) - 2d_1d_2k_1 + d_1d_2(d_2 + d_3)$$
$$d_1+d_3 = d_2+d_4 : (d_1+d_3)(k_1^2 + k_2^2) - 2d_1d_2k_1 + d_1d_2(d_2 - d_3)$$
$$d_1+d_2 = d_3+d_4 : (d_1-d_3)(k_1^2 + k_2^2) - 2d_1d_2k_1 + d_1d_2(d_2 - d_3)$$

The line is tangent to the cubic if and only if the point (k_1,k_2) lies on a curve of degree ≤ 8 (not given in [Gibson&Newstead]).

Parallelogram/Kite Case:

1) $d_1 = d_2 \neq d_3 = d_4$: The condition for the line to pass through one of the two nodes on the cubic (which in this case is a conic and chord) is that (k_1,k_2) satisfies

$$(d_1-d_3)(k_1^2 + k_2^2) - 2d_1^2k_1 + d_1^2(d_1 + d_3) = 0 \text{ or}$$
$$(d_1+d_3)(k_1^2 + k_2^2) - 2d_1^2k_1 + d_1^2(d_1 - d_3) = 0$$

The line is never a component of the cubic, thus the condition for the line to be tangent to the cubic is equivalent to the line being tangent to the conic component. The result is that (k_1,k_2) must lie on the curve given by

$$d_3^2k_1^2 + (d_3^2 - d_1^2)k_2^2 = 0$$

Thus, when $d_3 > d_1$, we obtain only the point $k = 0$, whilst, when

$d_3 < d_1$, we get two lines through the origin; indeed, they are tangent to the above circles.

2) $d_1 = d_4 \neq d_2 = d_3$: The condition for the line to pass through one of the two nodes on the cubic (which in this case is a conic and chord) is that (k_1,k_2) satisfies

$$(d_1-d_2)(k_1^2 + k_2^2) + 2d_1d_2k_1 = 0 \quad \text{or}$$
$$(d_1+d_2)(k_1^2 + k_2^2) - 2d_1d_2k_1 = 0$$

The line is never a component of the cubic thus the condition for the line to be tangent to the cubic is equivalent to the line being tangent to the conic component. The result is that (k_1,k_2) must lie on the curve given by

$$d_1^2(k_1-d_2)^2 + (d_1^2 - d_2^2)k_2^2 = 0$$

Thus, when $d_1 > d_2$, we obtain only the point $k = d_2$, whilst, when $d_1 < d_2$, we get two lines through the origin; the lines are tangent to the above circles.

3) $d_1 = d_3 \neq d_2 = d_4$: The condition for the line to pass through one of the two nodes on the cubic (which in this case is a conic and chord) is that

$$k_1 = \tfrac{1}{2}(d_1 + d_2) \quad \text{or} \quad k_1 = \tfrac{1}{2}(d_2 - d_1)$$

The line is never a component of the cubic thus the condition for the line to be tangent to the cubic is equivalent to the line being tangent to the conic component. The result is that (k_1,k_2) must satisfy

$$(k_1^2 + k_2^2 - d_2k_1)^2 + (d_2^2 - d_1^2)k_2^2 = 0$$

Whenever $d_2 > d_1$, we get only the points $k = 0$ and $k = d_2$. When $d_2 < d_1$, we have two circles given by

$$k_1^2 + k_2^2 - d_2k_1 + \sqrt{d_2^2 - d_1^2}\, k_2 = 0$$
$$k_1^2 + k_2^2 - d_2k_1 - \sqrt{d_2^2 - d_1^2}\, k_2 = 0$$

Rhombus Case:

The condition for the line to pass through one of the three nodes on the cubic (which in this case is a triangle) is

$$k_1 = d_1, \quad k_1 = 0, \quad \text{or} \quad (k_1 - \tfrac{d}{2})^2 + k_2^2 = \tfrac{d^2}{4}$$

where $d = d_1 = d_2 = d_3 = d_4$. Note that, since the line is never a component of the cubic, the line is never tangent to the cubic.

Note that in the above analysis we have been careful to separate the two distinct conditions when the pencil passes through a node of the cubic and when the line touches the cubic. The reason for this will soon become clear.

In each of the above cases let us write $\mathcal{T}$ for the union of the varieties defining the set of points (k_1,k_2), in the coupler plane, for which the associated pencil either touches the cubic or passes through a double point of the cubic.

So far we have only considered the geometry of the quartic surface given as the intersection of a pencil of quadrics in $P\mathbb{C}^4$ when the pencil is general. However, even when the pencil is not general, we can still describe the geometry of the surface (no longer called a Segre surface). Of course, for a bad choice of quadrics the corresponding surface of intersection can be very degenerate.

The first level of degenericity occurs when the discriminant of the pencil no longer has distinct roots, but the singular quadrics corresponding to each root continue to be point cones and not worse. We could then repeat the analysis of the configuration of the lines on the surface and deduce that some of them must coincide. A complete description is given in [Jessop 1916]. We will not give the details here, but simply quote the result that in this case there can be twelve, nine, eight, six or four lines on the surface (when the pencil has the Segre symbol 1112, 122, 113, 23 or 14, respectively). It should be clear to the reader that, if we take a line L on the surface, there are no longer five distinct lines on the surface meeting it. Recall that the five lines meeting a given line L are constructed in the following manner. Let $\mathcal{K}$ be the plane spanning L and one of the vertices of the cones. Then $\mathcal{K}$ meets the surface in a conic, consisting of L and one other line. Thus, in the general case, we can do this for each of the five vertices giving the desired five lines. However, in the non-general case there are less than five distinct point vertices and hence there are less than five lines meeting L.

More degenerately, we could have singular quadrics with vertices that are lines. The result here is that there are *either* at most eight lines on the surface (whenever the Segre symbols of the pencil is one of (11)111, (11)21, (21)11, (21)2, (31)1, or (41)) *or* there are infinitely many lines and the surface is ruled (whenever the Segre symbols of the pencil is either (22)1 or (23)).

However, we can eliminate the possibility of infinitely many lines on the surface in the situation at hand, because such surfaces may only occur when the discriminant has one or two distinct roots (this is a consequence of the results in [Jessop 1916], in fact it follows from the Segre symbol). But this is never the case. The pencil would necessarily be tangent to the conic component of the discriminant of the net $\mathfrak{N}$, implying $k_2 = 0$, and the pencil would have to have 3-point contact with the cubic i.e. be an inflexional tangent. However, we will soon show that points (k_1,k_2) with $k_2 = 0$ have an associated pencil with two or three point contact with the cubic when $k_1 = 0$ or $k_1 = d_2$: the coupler points we have excluded from our discussion.

Thus, whenever the associated pencil is non-general, some of the five lines meeting the centre of the coupler projection $\mathfrak{L}$ coincide and this occurs if and only if some of the base points coincide. Therefore, the coupler curve has a coincidence of double points.

In the case of a generic mechanism the coupler curve C is a sextic with geometric genus one. We showed in §1.6 that, no matter which coupler point is chosen, C has ordinary triple points at I and J (we are excluding from the discussion the cases when the coupler point is one of the endpoints of the coupler bar). Thus, if there is any coincidence of double points, then it occurs among the finite singular points. We also showed in §1.6 that the sum of the delta invariants of the singular points on C is three for any choice of coupler point. Thus, there can be *either* just one singular point P with $\delta_P = 3$ and hence a triple point or higher-order cusp *or* one ordinary double point and a double point P with $\delta_P = 2$ i.e. a tacnode or ramphoid cusp. However, we can easily show that finite triple points cannot occur on the coupler curve in the generic case.

Suppose that P is a triple point of a coupler curve of any mechanism. Then, since there are no finite singular points on the residual linkage curve, P must lie on the circle of singular foci. Its pre-image, on the Segre quartic surface corresponding to the coupler point, is a line M meeting the centre of projection. M meets any other quadric in the net $\mathfrak{N}$ (and hence the residual linkage variety which we continue to denote by $\mathcal{R}'$), distinct from the ones in the associated pencil, in one of the following three ways:

1) M touches the quadric in one point P' lying on the residual linkage variety. Clearly, if this is the case, then P' is a critical point of the coupler projection. If P' is simple on $\mathcal{R}'$, then it follows that P is a cusp. This is necessarily the case for a generic mechanism. Whilst for a degenerate mechanism, P' could be an ordinary double point implying that P is a singular point worse than an ordinary double point or cusp. Indeed, P must have two branches, one simple and one non-simple. Hence, P is a triple point with non-distinct tangents. Thus $\delta_P \geq 3$, implying that there are no other finite singular points on the coupler curve.

Any coupler point giving rise to such a curve must lie on the cusp curve and hence does not occur in general.

2) M meets the quadric in two distinct points P' and P''. Then M is non-critical at P' and P'' and therefore each branch through P' and P'' maps as a local immersion onto the coupler curve. Thus, if P' and P'' are simple points on $\mathfrak{R}'$, then P is an ordinary double point on the coupler curve; in particular, this is the case for the generic mechanism. If the mechanism is degenerate, then one of the points can be an ordinary double point implying that P is an ordinary triple point. The situation when P' and P'' are both singular (in which case the mechanism is either a parallelogram/kite or rhombus) cannot arise, since the double points lie on a conic component of $\mathfrak{R}'$; indeed they are the intersection of components namely, a conic and quartic in the parallelogram/kite case and three conics in the rhombus case. Thus this would imply that two points on the conic map to the same point on the coupler curve giving an obvious contradiction.

3) M meets the surface in more than two points. Thus M necessarily lies on the surface implying that M is a component of the linkage curve. The only line components are those in the hyperplane at infinity, but we know that these map to I and J. Thus this situation can never arise.

Thus we have showed that in the generic case no coupler curve has a finite triple point. This leaves the possibilities of a higher-order cusp with $\delta p = 3$, a ramphoid cusp with $\delta p = 2$ or a tacnode with $\delta p = 2$. However, we showed in §5.3 that cusps do not occur in general. Indeed, for a ramphoid cusp or higher-order cusp to occur, it is necessary that the coupler point lies both on the cusp curve and on the curve $\mathcal{T}$. But in the generic case the cusp curve is an irreducible curve of degree twelve in the coupler plane. Comparing this with the curve $\mathcal{T}$, a curve of degree ≤ 12, we find that no component of $\mathcal{T}$ can be a component of the cusp curve. Thus, provided $\mathcal{T}$ is not identical to the cusp curve, we may apply Bézout's Theorem to show that they meet in finitely many points. We can easily exclude the possibility that $\mathcal{T}$ is the cusp curve; we will show soon that, in fact, $\mathcal{T}$ has only degree ten. Thus almost all points on $\mathcal{T}$ correspond to coupler curves with tacnodes. Hence, we may call $\mathcal{T}$ the **tacnode curve**.

In the degenerate cases the curve $\mathcal{T}$ corresponds not only to coupler points whose loci possess a tacnode, but also to coupler curves whose loci possess a triple point. We recall that the cusp curves for the circumscriptible, kite and parallelogram cases consist of the line $k_2 = 0$ and an irreducible curve of degree 8, 4 and 2 respectively. Note that the cusp curve for the rhombus case is empty. On the other hand for the circumscriptible, kite and parallelogram cases, the curve $\mathcal{T}$ is a circle and an octic; two circles and two (possibly complex) lines; and two lines and (if $d_2 > d_1$) two circles, respectively. Comparing $\mathcal{T}$ and the cusp curves, it is clear that no component of $\mathcal{T}$ can lie on the cusp curve, except possibly in the parallelogram case, when one of the two circles could coincide with the conic cusp curve; but in this case we recall that the cusp curve is an ellipse or parabola, so they cannot coincide. Thus the occurence of a ramphoid or higher-order cusp on a coupler curve can only occur for coupler points lying on both the cusp curve and on $\mathcal{T}$ and, moreover, it follows from Bézout's Theorem that there are only finitely many such points.

We will conclude our discussion of the degenerate four-bars by noting that the curve $\mathcal{T}$ is the union of two subvarieties $\mathcal{T}'$ and $\mathcal{T}''$, corresponding to the set of coupler points whose loci possess a tacnode and triple point, respectively. The two subvarieties are distinguished by the fact that the **tacnode curve** $\mathcal{T}'$ is the set of coupler points whose associated pencil touches the cubic, whilst the **triple point curve** is the line $k_2 = 0$ union the set of coupler points whose associated pencil passes through a double point of the cubic. We shall prove this quite explicitly. Let P be an ordinary triple point (we may exclude non-ordinary triple points, because we have showed that there are only finitely many of them) on a (necessarily) degenerate four-bar coupler curve and let M be its pre-image on the Segre quartic surface associated to the fixed coupler point. Then, by the above analysis, M passes through a double point on $\mathcal{R}'$ i.e. one of the points $(\varepsilon_1,0,\varepsilon_2,0,\varepsilon_3,0,1)$, where $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (1,1,-1)$, $(1,-1,1)$ or $(-1,1,1)$ and, moreover, the image of such a point must lie on the circle of singular foci. Thus, the singular point has the form $(P_1,P_2,P_3) = (d_1\varepsilon_1 + k_1\varepsilon_2,\ k_2\varepsilon_2,\ 1)$ and must lie on the circle of singular foci given in §1.6. We recall that the equation of the circle is

$$-d_2k_2(P_1^2 + P_2^2) + d_2d_4k_2P_1P_3 + d_4(k_1^2 + k_2^2 - d_2k_1)P_2P_3 = 0$$

Substituting the co-ordinates of the point into the equation, we find that

$k_2 = 0$ <u>or</u> $[\varepsilon_2 d_4 - d_2][k_1^2 + k_2^2] - 2d_1 d_2 \varepsilon_2 k_1 + d_1 d_2[\varepsilon_1 d_4 - d_1] = 0$

Thus, the line $k_2 = 0$ always gives rise to triple points in the degenerate cases. We can now take each degenerate case one by one and apply the equality $\varepsilon_1 d_1 + \varepsilon_2 d_2 + \varepsilon_3 d_3 - d_4 = 0$:

<u>Circumscriptible Case</u>: $d_1 + d_4 = d_2 + d_3$ then $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,1,1)$. Thus the two equalities $(\varepsilon_2 d_4 - d_2) = (d_3 - d_1)$, $(\varepsilon_1 d_4 - d_1) = (d_2 + d_3)$ and the second equality above gives the condition, derived earlier, for the pencil to pass through the node. The other two circumscriptible cases follow similarly.

<u>Kite Case</u>: $d_1 = d_2 \neq d_3 = d_4$ then $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,-1,1)$ <u>or</u> $(-1,1,1)$. Thus the three equalities: $(\varepsilon_2 d_4 - d_2) = -(d_1 + d_3)$; $(\varepsilon_1 d_4 - d_1) = (d_3 - d_2) = (d_3 - d_1)$ <u>or</u> $(\varepsilon_2 d_4 - d_2) = (d_3 - d_1)$; $(\varepsilon_1 d_4 - d_1) = -(d_2 + d_3) = -(d_1 + d_3)$; and the second equality above gives the two conditions, derived earlier, for the pencil to pass through one of the two nodes on the cubic. The other kite case follows similarly.

<u>Rhombus Case</u>: $d_1 = d_2 = d_3 = d_4$ then $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,1,-1)$, $(1,-1,1)$ <u>or</u> $(-1,1,1)$. Thus: $(\varepsilon_2 d_4 - d_2) = (\varepsilon_1 d_4 - d_1) = 0$ <u>or</u> $(\varepsilon_2 d_4 - d_2) = -2d_1$; and $(\varepsilon_1 d_4 - d_1) = 0$ <u>or</u> $(\varepsilon_2 d_4 - d_2) = -2d_1$; and $(\varepsilon_1 d_4 - d_1) = -2d_1$. It is easily checked that this gives the three conditions given earlier for the associated pencil to pass through one of the three nodes on the cubic.

<u>We shall now discuss the geometry of the tacnode curve $\mathcal{T}$ in the generic case and determine a procedure for deciding, whether a four-bar coupler curve has one or three double points.</u>

We have described the geometry of the tacnode curve in the parallelogram/kite and rhombus cases quite explicitly above. We shall now procede to describe this curve in the generic case, as this was not done in [Gibson&Newstead]. We recall that the tacnode curve is the set of points (k_1,k_2), in the coupler plane, for which the associated pencil touches the cubic component of the discriminant of the net $\mathfrak{N}$ of quadrics whose intersection is the linkage variety. The pencil is given by

$$k\bar{k}X + d_1^2Y + d_1^2(k-d_2)(\bar{k}-d_2)Z = 0$$

and the cubic by

$$(X + Y\,d_3^2Z)(XY + d_1^2YZ + d_2^2XZ) - d_4^2XYZ = 0$$

We wish to know when the pencil touches the cubic, or equivalently, when the pencil meets the cubic in two points instead of the general three points. We may eliminate the variable Y from the cubic using the equation of the pencil. The resulting equation is a binary cubic in X and Z. Then, the necessary and sufficient condition for the tangency of the pencil to the cubic is that the discriminant of the cubic is zero; for this is the criterion for the cubic to possess less than three zeroes (X,Z) and hence the pencil intersects the cubic in less than three points. Explicitly the binary cubic is coeff. of X^3: $-k\bar{k}(d_1^2 - k\bar{k})$

coeff. of X^2Z: $d_1^2(d_1^2 - k\bar{k})(d_2^2 - k\bar{k} - (k - d_2)(\bar{k} - d_2))$
$- d_1^2 k\bar{k}(d_3^2 - (k - d_2)(\bar{k} - d_2)) + d_1^2 d_4^2 k\bar{k}$

coeff. of XZ^2: $-d_1^4(d_1^2 - k\bar{k})(k - d_2)(\bar{k} - d_2) + d_1^4 d_4^2(k - d_2)(\bar{k} - d_2)$
$+ d_1^4(d_3^2 - (k - d_2)(\bar{k} - d_2))(d_2^2 - k\bar{k} - (k - d_2)(\bar{k} - d_2))$

coeff. of Z^3: $-d_1^6(k - d_2)(\bar{k} - d_2)(d_3^2 - (k - d_2)(\bar{k} - d_2))$

It is well known that the discriminant of a real binary cubic of the form $ax^3 + bx^2z + cxz^2 + dz^3 = 0$ $(a,b,c,d \in \mathbb{R})$ is

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

and that the cubic has

three distinct real roots	$\Leftrightarrow$	$\Delta < 0$
one real, two complex conjugate roots	$\Leftrightarrow$	$\Delta > 0$
one real, two coincident roots	$\Leftrightarrow$	$\Delta = 0$

For a fixed coupler point let Δ be the discriminant of the cubic expressed above. Then it is a straightforward matter to compute the sign of Δ and hence determine precisely how many of the roots (X_i,Z_i) $i = 1,2,3$ are real. Each real root of the cubic gives a real value of (X_i,Y_i,Z_i) and corresponds, therefore, to a real root of the discriminant of the associated pencil for the given coupler point. Finally, we have the required condition for a four-bar coupler curve to have one or three real double points. The necessary numerical computations can easily be done on a computer. Thus, together with a computer drawing of a coupler curve and a knowledge of how many of the three double points are real, the A-type can easily be established. We are now able to make progress with the survey of four-bar coupler curves which will form the final section of this chapter.

The equation of the tacnode curve is obtained by substituting the coefficients of the cubic into the formula for the discriminant. The equation is extremely complicated and the author (somewhat regretfully) includes it here for completeness sake and to confirm that it is indeed the transition curve (übergangskurve) of [Müller] (substitute $a^2=[k_1^2+k_2^2-2d_2k_1+d_1^2]$, $b^2=[k_1^2+k_2^2]$, $r=d_1$, $c=d_2$, $s=d_3$, $m^2=[d_1^2+d_2^2+d_3^2-d_4^2]$ in his equation).

$$
\begin{aligned}
&27[k_1^2+k_2^2-2d_2k_1+d_1^2]^2[k_1^2+k_2^2]^2[k_1^2+k_2^2-2d_2k_1+d_1^2-d_3^2]^2[k_1^2+k_2^2-d_1^2]^2\\
&+4[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2-2d_2k_1+d_1^2-d_3^2]\Big(2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]+\\
&\qquad [k_1^2+k_2^2]^2-d_1^2[k_1^2+k_2^2-2d_2k_1+d_1^2]-[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2]+d_1^2d_2^2\Big)^3\\
&+4[k_1^2+k_2^2][k_1^2+k_2^2-d_1^2]\Big([k_1^2+k_2^2-2d_2k_1+d_1^2]^2+2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]\\
&\qquad -d_3^2[k_1^2+k_2^2]-[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2-2d_2k_1+d_1^2]+d_3^2d_2^2\Big)^3\\
&-18[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2][k_1^2+k_2^2-2d_2k_1+d_1^2-d_3^2][k_1^2+k_2^2-d_1^2]\times\\
&\qquad\Big([k_1^2+k_2^2-2d_2k_1+d_1^2]^2+2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]-d_3^2[k_1^2+k_2^2]\\
&\qquad -[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2-2d_2k_1+d_1^2]+d_3^2d_2^2\Big)\times\\
&\qquad\Big(2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]+[k_1^2+k_2^2]^2-d_1^2[k_1^2+k_2^2-2d_2k_1+d_1^2]\\
&\qquad -[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2]+d_1^2d_2^2\Big)\\
&-\Big([k_1^2+k_2^2-2d_2k_1+d_1^2]+2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]-d_3^2[k_1^2+k_2^2]\\
&\qquad -[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2-2d_2k_1+d_1^2]+d_3^2d_2^2\Big)^2\times\\
&\qquad\Big(2[k_1^2+k_2^2-2d_2k_1+d_1^2][k_1^2+k_2^2]+[k_1^2+k_2^2]^2-d_1^2[k_1^2+k_2^2-2d_2k_1+d_1^2]\\
&\qquad -[d_1^2+d_2^2+d_3^2-d_4^2][k_1^2+k_2^2]+d_1^2d_2^2\Big)^2\\
&=0
\end{aligned}
$$

We may make the equation homogeneous using the coordinate w and consider the complexified projectivised curve given by the obvious equation so formed. The highest order terms in this equation have degree twelve, but it can easily be showed (see Müller's paper for some detail) that the coefficients of the eleventh and twelfth degree terms are zero. Thus the tacnode curve has degree ten. Müller also shows that the curve has quadruple points at the circular points at infinity. Further, the endpoints of the coupler bar $(0,0)$ and $(d_4,0)$ are ordinary double points. The origin is an A_1^+ (acnode) if and only if $d_1 < d_4$, and an A_1^- (crunode) if and only if $d_1 > d_4$. Similarly, the other endpoint $(d_4,0)$ is an A_1^+ (acnode) if and only if $d_3 < d_4$, and an A_1^- (crunode) if and only if $d_3 > d_4$. The line at infinity meets the curve in I and J, and two other points (by Bézout's Theorem, since the curve has degree ten and the circular points each have multiplicity four). It can also be showed that these points are distinct if and only if $d_2 \neq d_4$, and are real if and only if $d_2 > d_4$ (whilst complex conjugates when $d_4 > d_2$).

It now follows, that for the Hain group I', where d_4 is the longest, $\mathcal{T}$ has acnodes at the endpoints and meets the line at infinity in imaginary points. In particular, the tacnode curve is closed and finite. In the Hain group II', where d_4 is the shortest, $\mathcal{T}$ meets the endpoints in crunodes and meets the line at infinity in two real points. Hence, the curve has two asymptotes. The remaining two groups can have various combinations of acnodes and crunodes at the endpoints.

Combining the tacnode/triple point curve $\mathcal{T}$ and the cusp curve $\mathcal{C}$, we obtain a curve $\mathcal{T} \cup \mathcal{C}$ in the coupler plane. Indeed, the union stratifies the coupler plane where the zero-dimensional strata are the points where $\mathcal{T}$ meets $\mathcal{C}$, the one-dimensional strata are the connected components of $\mathcal{T} \cup \mathcal{C}$ with the zero-dimensional strata removed and the two-dimensional strata are the connected components of the coupler plane with $\mathcal{T} \cup \mathcal{C}$ removed. On each stratum it is clear that the number of crunodes, acnodes, complex nodes, tacnodes, cusps and triple points of the coupler curves corresponding to coupler points in the stratum, remains constant. It is a fundamental problem of the subject to determine precisely which stratifications may occcur. Of course, we could choose other stratifications by introducing further properties of the coupler curves; for instance, real Plucker numbers or vertices. For a generic mechanism the A-types correspond to the strata which do not lie on the tacnode curve.

§5.7. Survey Four-bar Coupler Curves.

This section needs little explanation. The aim is to show for each Hain type which of the singularity types can occur by computer graphics. This should not be viewed as a haphazard atlas of pictures (as it may appear to the reader!) but as the result of an extensive study of coupler curves, the gains of which in terms of intuition to the author, are more than can be described here in words or pictures. We shall only survey the coupler curves of the generic mechanism and coupler points for which the associated pencil is general; thus by the results of the previous section our curves will not possess tacnodes.

I will, however, try to give some indication of how the survey took place. The wrong approach to the problem would be to draw numerous curves and hope that all possible types turn up. The nett gain may be to solve the problem, but the nett loss is an enormous amount of intuition. More sensibly, one should search for one of the following two transitions:

1) acnode-cusp-crunode, when the coupler point passes through the cusp curve and

2) two real double points-tacnode-two complex conjugate double points, when the cusp curve passes through the tacnode curve.

One doesn't need to know where the coupler point for these curves lies in the coupler plane. One may simply start with any given curve and deform the coupler point until it reaches the cusp or tacnode curve; with some experience one knows how to deform the coupler point in order to move towards these curves. The circle of singular foci is invaluable here. If one starts, for instance, with a crunode, then one should try to shrink the loop into a cusp lying on the circle and then into an acnode. Conversely, if one knows that there is an acnode, then one can deform the coupler curve until it meets the circle of singular foci (in a cusp) and then passes through it to give a crunode. In this way we may begin with a curve with one singularity type and hope to find two further types.

We tabulate the results of the survey (unfortunately at present incomplete by a few cases). For each A-type we give the dimensions of the mechanism and the coupler point. This is followed by drawings of the curve obtained, using a Pascal program on an Apple Macintosh Plus and printed on an Imagewriter dot matrix printer. The bold circles represent the circle of singular foci; on which all real double points of the coupler curve lie.

The first four dimensions are d_1, d_2, d_3, d_4 and the next two dimensions are the modulus and argument of k.

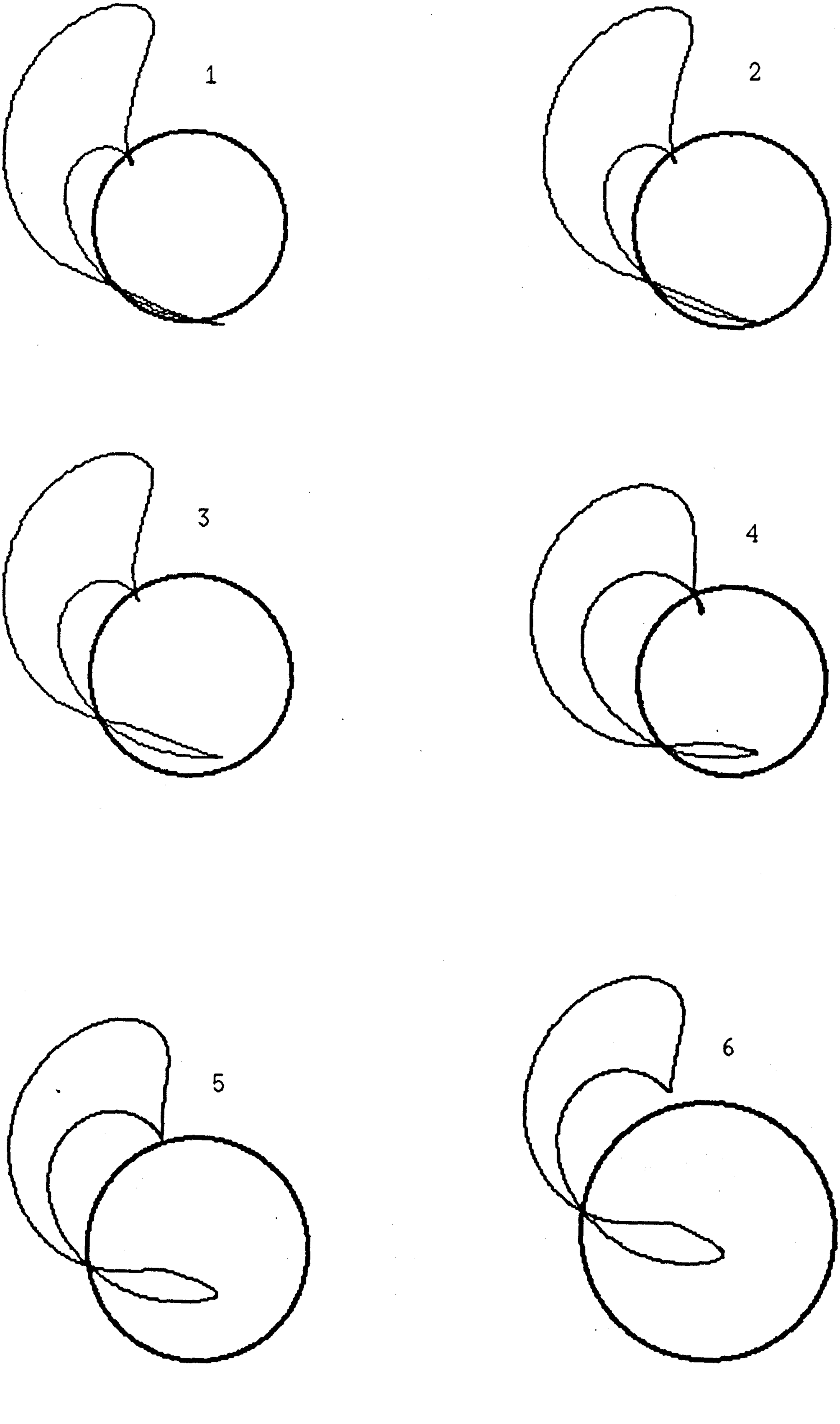
Hain Type I	dimensions	drawing no.
$3A_1^-$	0.45 0.5 0.89 0.71 0.3 0.57	1
	0.57 0.5 0.5 0.5 0.52 1.05	7
$2A_1^-/A_2$	0.45 0.5 0.89 0.71 0.29 0.57	2
$2A_1^-/A_1^+$	0.45 0.5 0.89 0.71 0.27 0.57	3
	0.45 0.5 0.89 0.71 0.19 0.81	4
$A_1^-/A_1^+/A_2$	0.45 0.5 0.89 0.71 0.19 0.6	5
$A_1^-/2A_1^+$	0.45 0.5 0.89 0.71 0.19 0.49	6
	0.45 0.5 1.27 0.71 0.19 0.49	13
$2A_1^+/A_2$	0.45 0.5 1.38 0.71 0.19 0.49	14
$3A_1^+$	0.45 0.5 1.41 0.71 0.19 0.49	15
$A_1^-/2A_1^*$	0.57 0.5 0.5 0.5 0.3 1.05	9
	0.21 0.5 0.27 0.19 0.25 0.93	10
	0.83 0.3 0.63 0.4 0.39 0.98	16
$2A_1^*/A_2$	0.21 0.5 0.27 0.19 0.37 0.93	11
	0.83 0.3 0.63 0.4 0.35 0.98	17
$A_1^+/2A_1^*$	0.21 0.5 0.27 0.19 0.43 0.93	12
	0.83 0.3 0.63 0.4 0.33 0.98	18
$A_1^-/2A_2$	0.75 0.43 0.25 0.5 0.25 0.53	19
$3A_2$	impossible	
$A_1^+/2A_2$	?	

Hain Type I'	dimensions	drawing no.
$3A_1^-$	0.3 0.33 0.44 0.59 0.31 0.7	20
$2A_1^-/A_2$	0.3 0.33 0.44 0.59 0.25 0.7	21
$2A_1^-/A_1^+$	0.3 0.33 0.44 0.59 0.18 0.7	22
	0.2 0.4 0.24 0.5 0.5 0.52	23
$A_1^-/A_1^+/A_2$	0.2 0.4 0.24 0.5 0.6 0.52	24
$A_1^-/2A_1^+$	0.2 0.4 0.24 0.5 0.64 0.52	25
	0.31 0.28 0.28 0.63 0.3 1.4	26
$2A_1^+/A_2$	0.31 0.28 0.28 0.63 0.3 1.22	27
$3A_1^+$	0.31 0.28 0.28 0.63 0.3 1.5	28
$A_1^-/2A_1^*$	?	
$2A_1^*/A_2$	?	
$A_1^+/2A_1^*$	?	
$A_1^-/2A_2$	0.31 0.28 0.28 0.63 0.42 1.25	29
$3A_2$	0.5 0.5 0.5 1.0 0.5 1.05	30
$A_1^+/2A_2$	0.3 0.1 0.3 0.53 0.1 1.05	31

Hain Type II	dimensions	drawing no.
$3A_1^-$	0.23 0.37 0.5 0.57 0.46 0.51	32
$2A_1^-/A_2$	0.23 0.37 0.5 0.57 0.5 0.51	33
$2A_1^-/A_1^+$	0.23 0.37 0.5 0.57 0.52 0.51	34
$A_1^-/A_1^+/A_2$	0.23 0.37 0.5 0.57 0.6 0.51	35
$A_1^-/2A_1^+$	0.23 0.37 0.5 0.57 0.64 0.51	36
	0.75 0.27 0.75 1.1 0.12 0.58	37
$2A_1^+/A_2$	0.75 0.27 0.75 1.1 0.12 0.52	38
$3A_1^+$	0.75 0.27 0.75 1.1 0.12 0.45	39
$A_1^-/2A_1^*$	0.6 0.23 0.37 0.5 0.24 0.54	40
$2A_1^*/A_2$	0.6 0.23 0.37 0.5 0.19 0.54	41
$A_1^+/2A_1^*$	0.6 0.23 0.37 0.5 0.16 0.54	42
$A_1^-/2A_2$	0.22 0.41 0.45 0.55 0.6 0.53	43
$3A_2$	0.64 0.17 0.64 1.0 0.14 0.87	44
$A_1^+/2A_2$	0.16 0.32 0.4 0.4 0.42 0.31	45

Hain Type II'	dimensions	drawing no.
$3A_1^-$	0.5 0.5 0.5 0.49 0.45 0.1	46
$2A_1^-/A_2$	0.5 0.5 0.5 0.49 0.35 0.1	47
$2A_1^-/A_1^+$	0.5 0.5 0.5 0.49 0.25 0.1	48
	0.27 0.23 0.4 0.06 0.92 0.17	52
$A_1^-/A_1^+/A_2$	?	
$A_1^-/2A_1^+$	?	
$2A_1^+/A_2$	?	
$3A_1^+$	?	
$A_1^-/2A_1^*$	0.33 0.38 0.47 0.17 0.37 1.05	49
$2A_1^*/A_2$	0.33 0.38 0.47 0.17 0.48 1.05	50
$A_1^+/2A_1^*$	0.27 0.23 0.4 0.06 0.7 0.17	54
	0.33 0.38 0.47 0.17 0.48 0.92	51
$A_1^-/2A_2$	impossible	
$3A_2$	impossible	
$A_1^+/2A_2$	impossible	

Drawings 8 and 53 possess tacnodes and are included to show the transition of two real points coalescing and becoming complex conjugates.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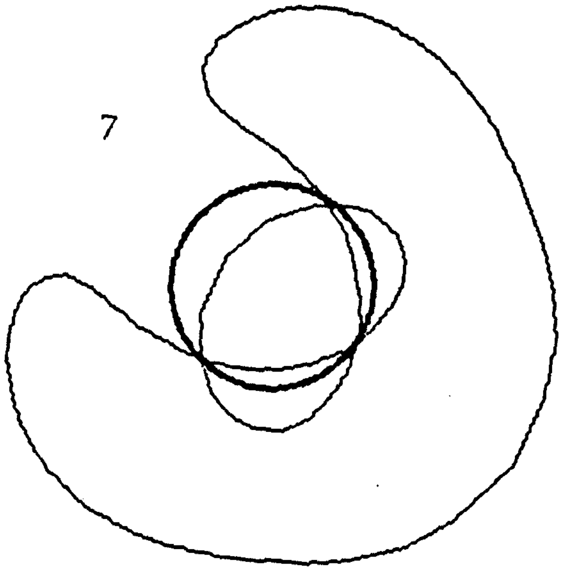

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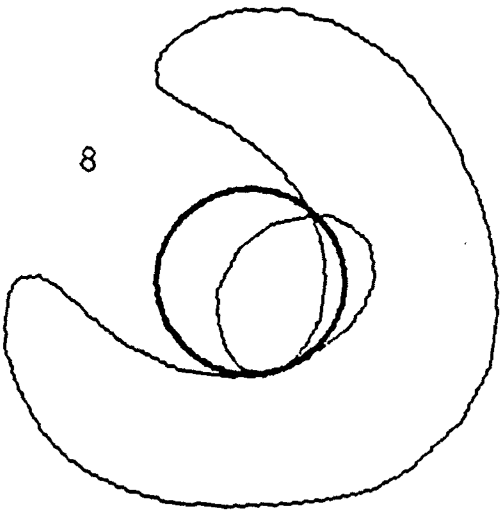

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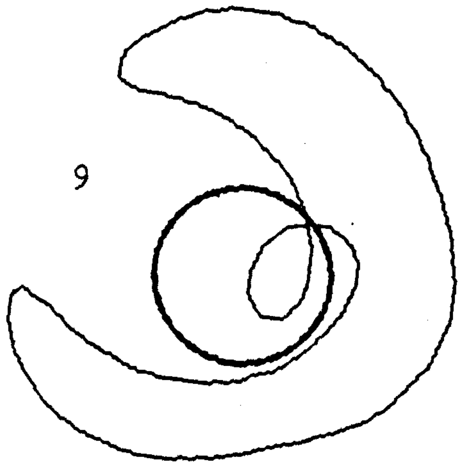

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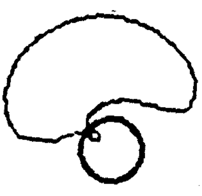

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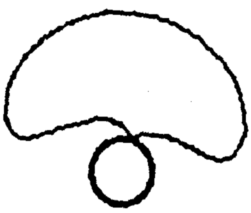

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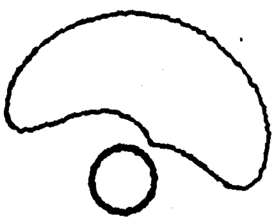

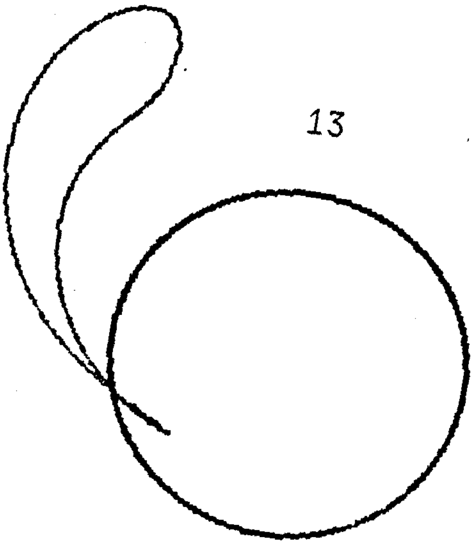

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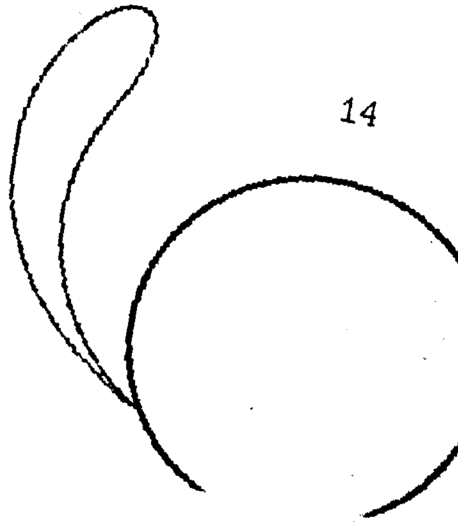

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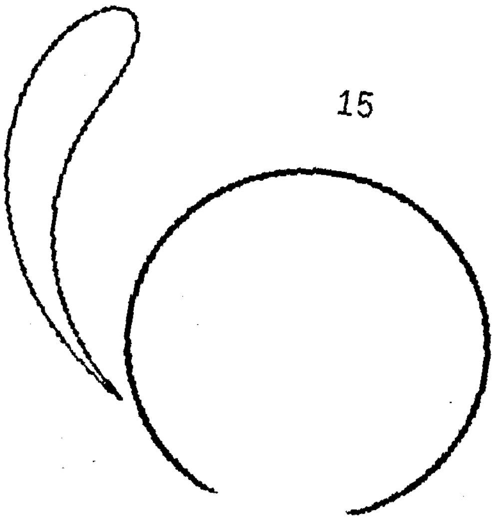

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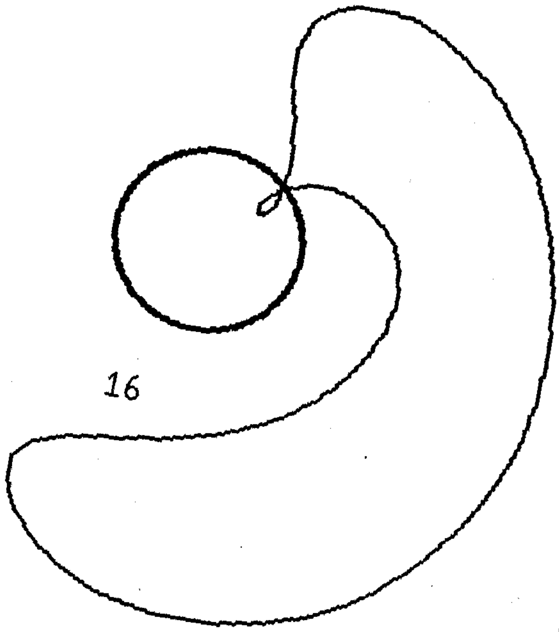

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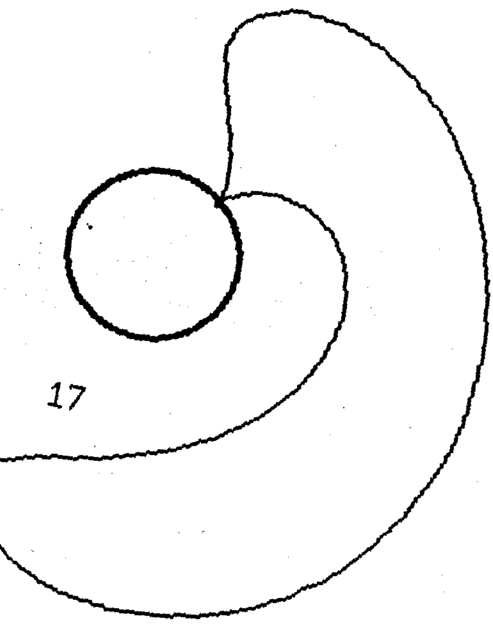

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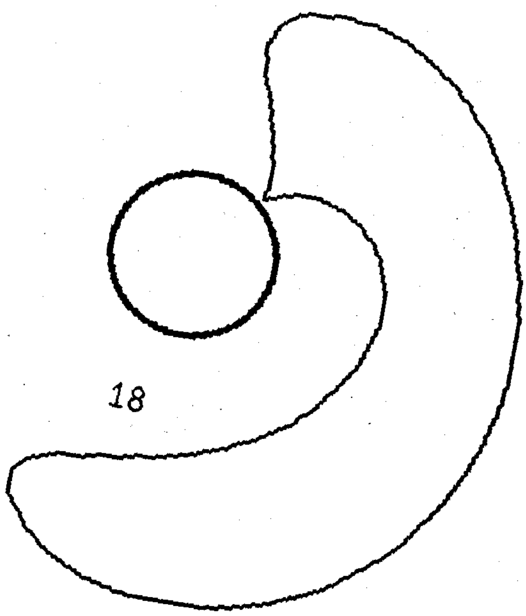

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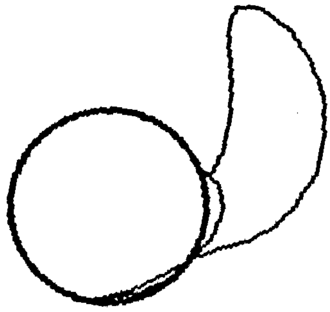

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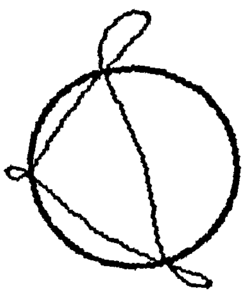

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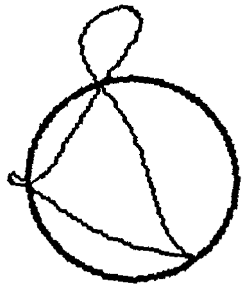

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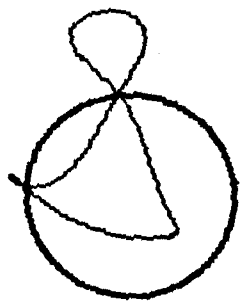

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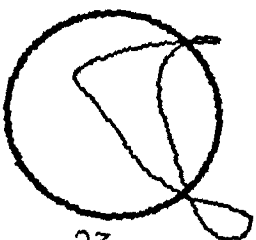

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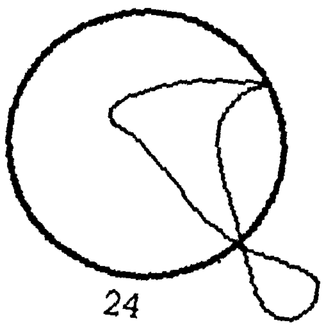

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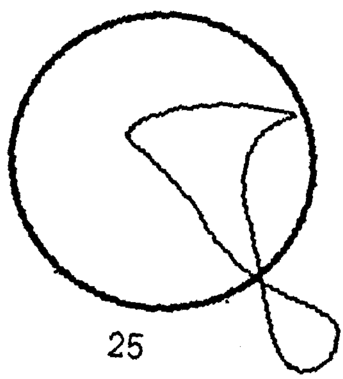

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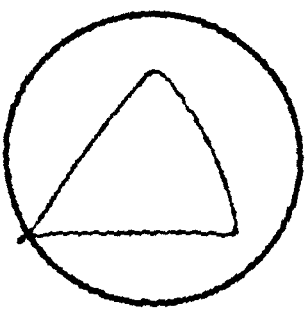

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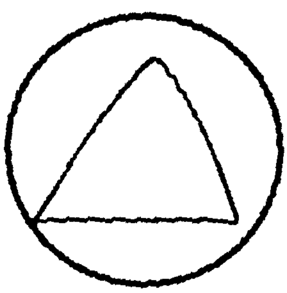

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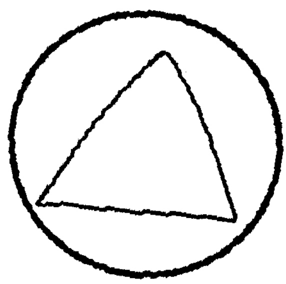

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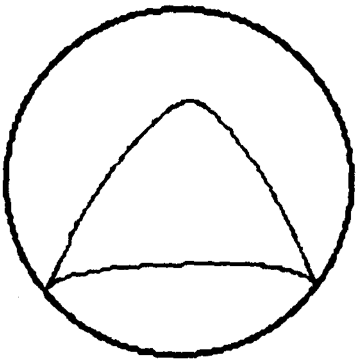

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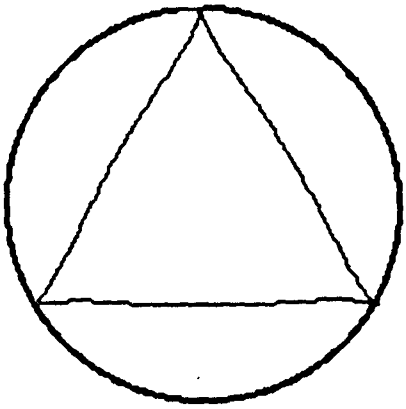

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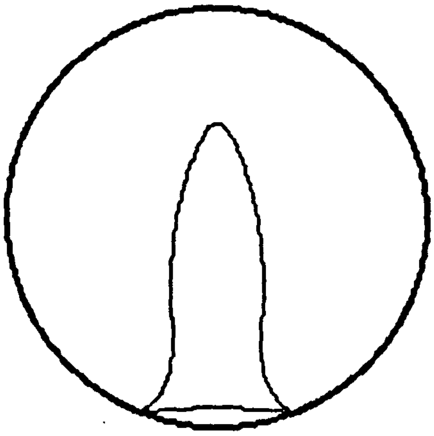

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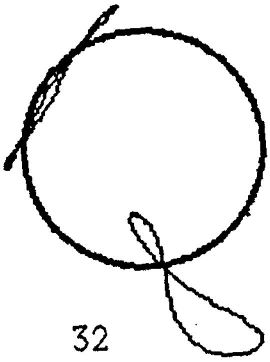

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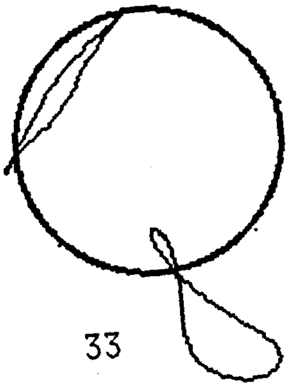

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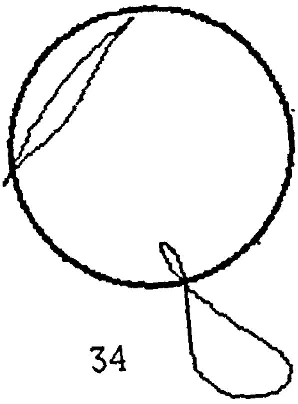

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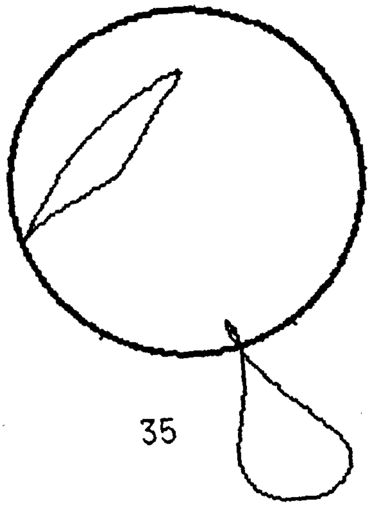

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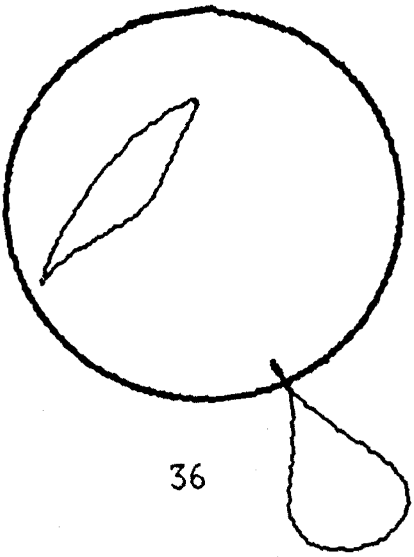

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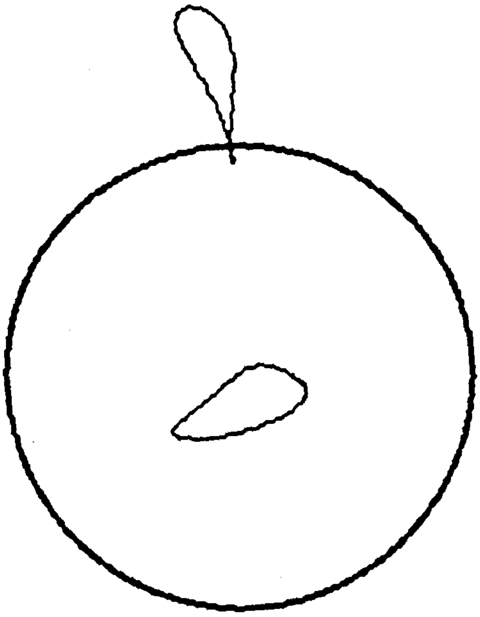

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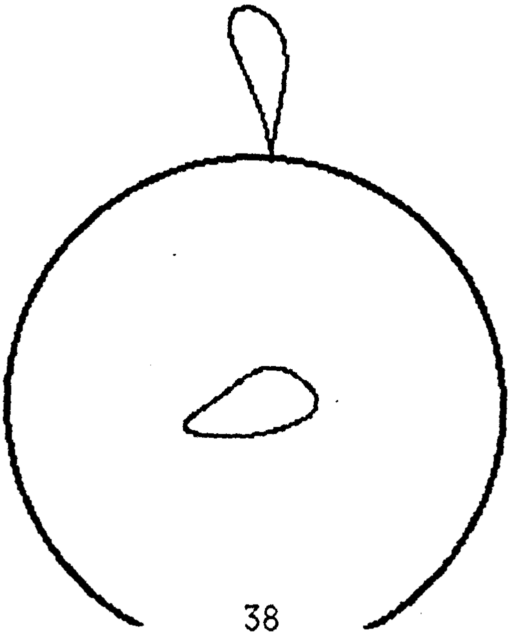

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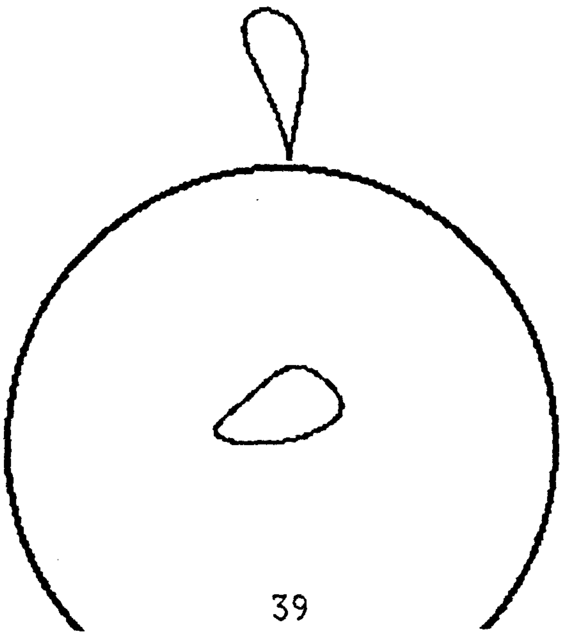

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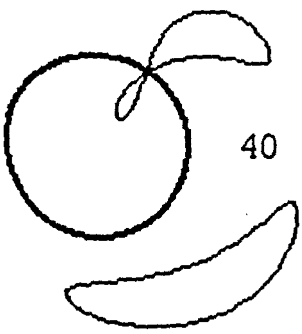

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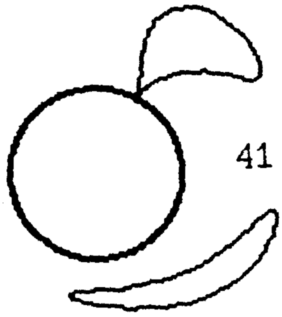

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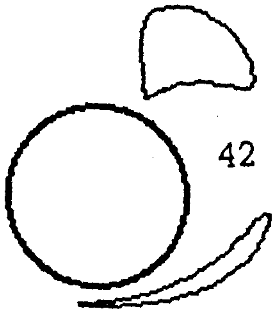

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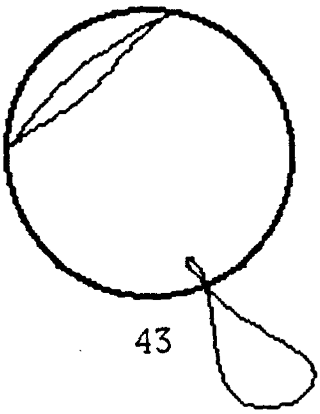

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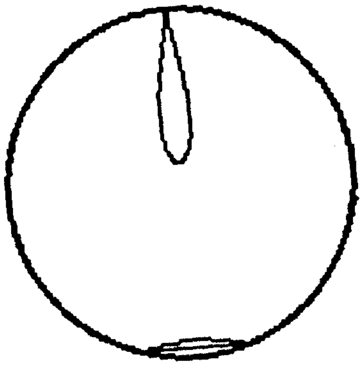

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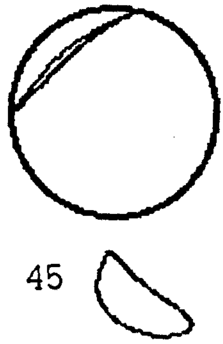

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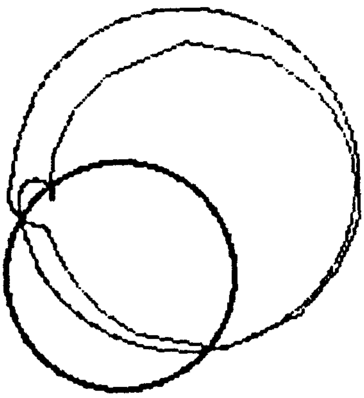

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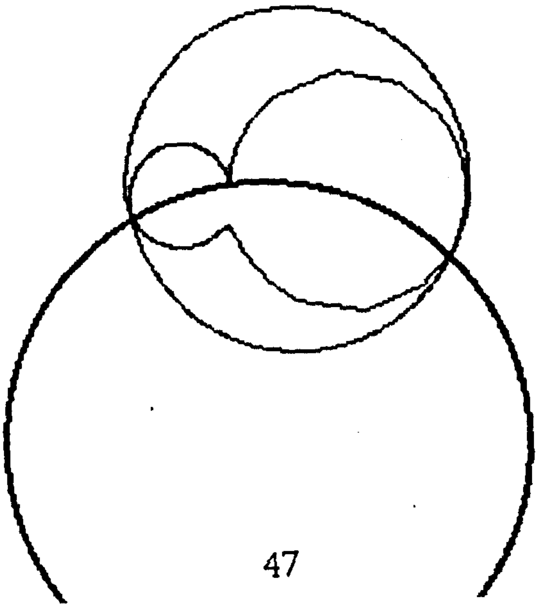

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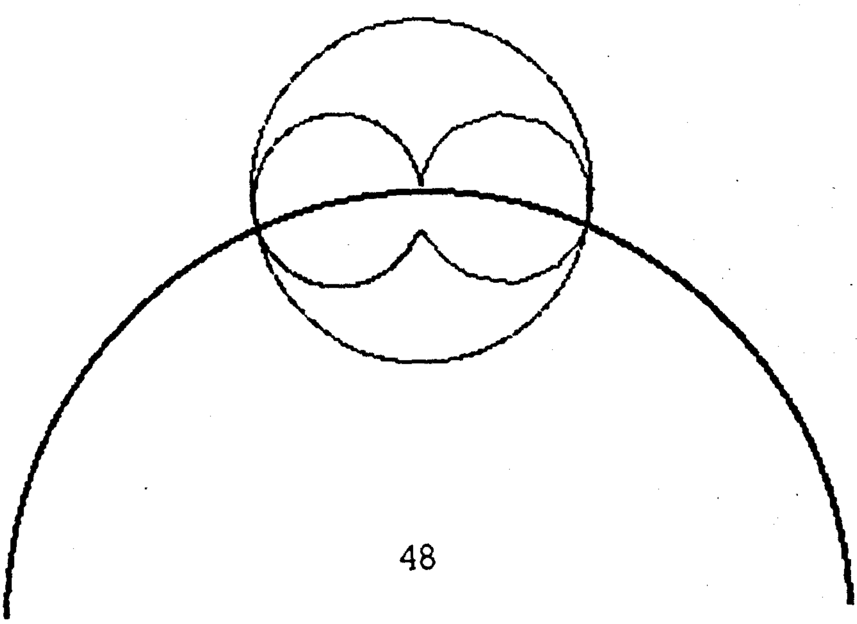

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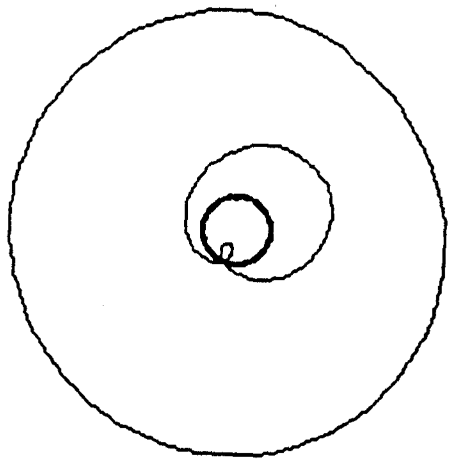

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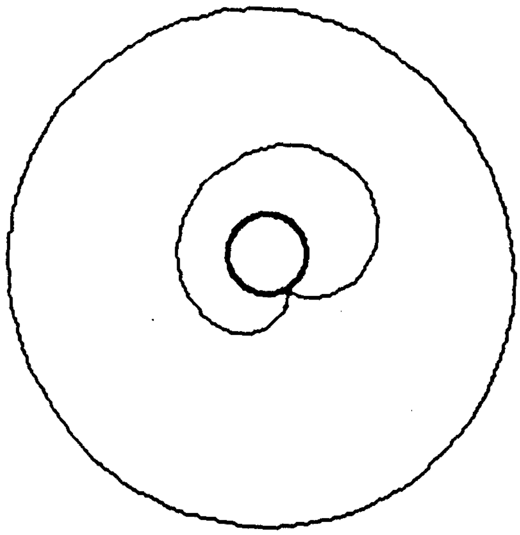

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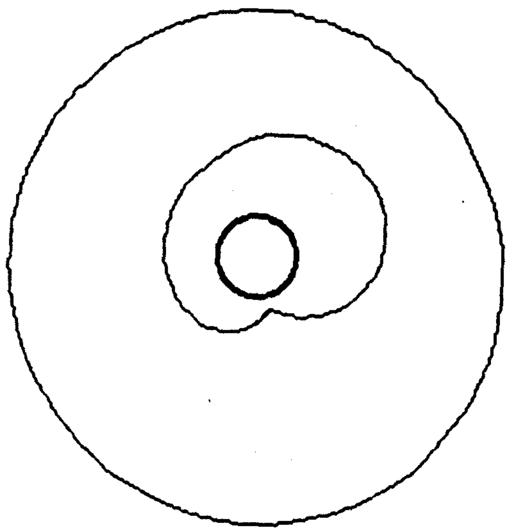

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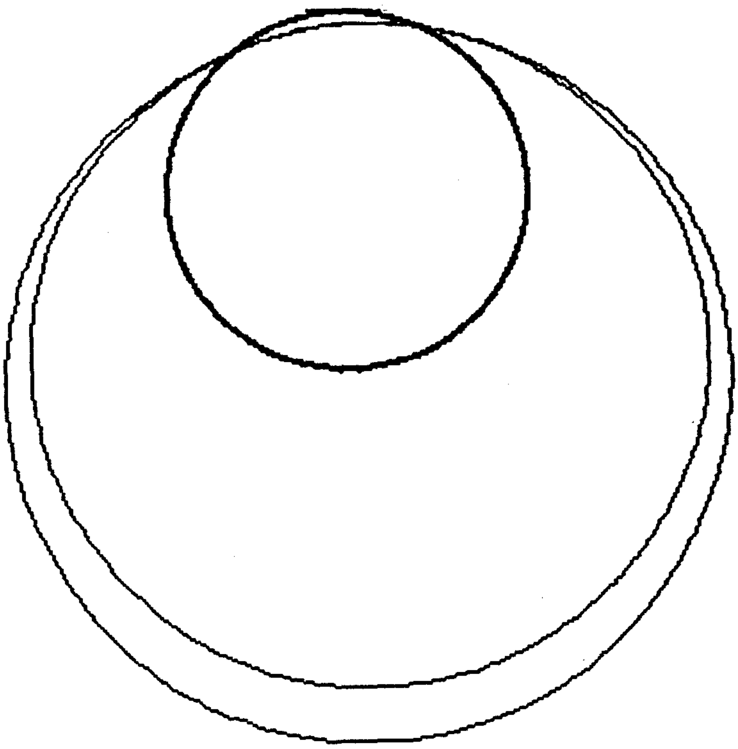

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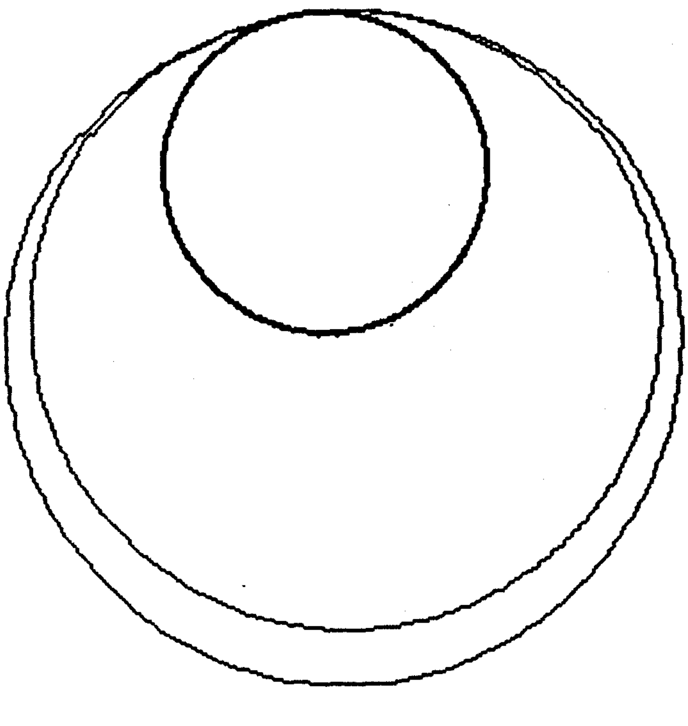

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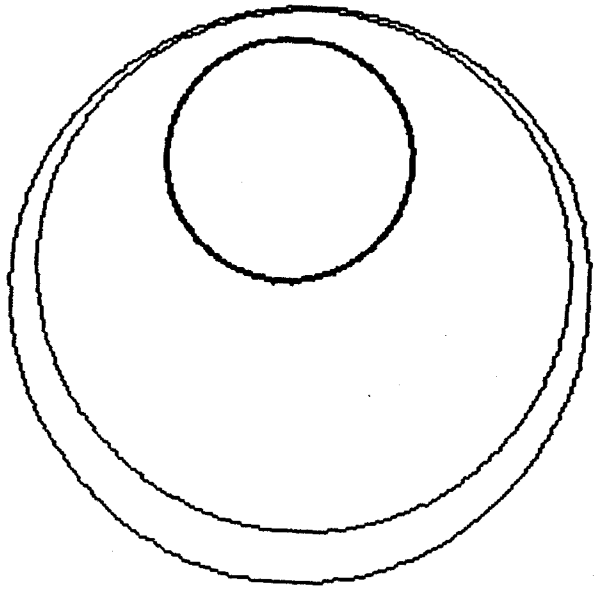

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APPENDIX: Introduction to the Algebraic Geometry

In this appendix we shall give a summary of the results which we need during the thesis. Primarily, we will be familiarising the reader with the notation and language of the geometric concepts. On the one hand, many of the results we shall be using may be found in most standard (final year undergraduate/graduate) textbooks, whilst on the other hand we will be using theorems which are simple to state yet hard to prove and involve concepts far beyond the scope of this appendix (for instance, the birational invariance of the arithmetic genus). Although it is not important that we understand the proofs, it is essential that we understand how such theorems can be applied in practice. Thus, we confine ourselves to collecting the results and theorems with references which we will often use, so that the reader will get a taste of the type of geometry that we will be using throughout the thesis. Some results, which are used only once, are not described here, but are stated in full in the text.

§A1 Affine Geometry

Let K be any field. Denote by K^n the set of n-tuples of elements in K, i.e. $K^n = \{(x_1,...,x_n) | x_i \in K\}$. Then K^1 is called the affine line, K^2 is called the affine plane and K^n is called **affine n-space**. Elements of K^n are called **points**. Let F be any set of polynomials in the n variables $x_1,...,x_n$, then we denote by $V(F) = \{(x_1,...,x_n) \in K^n | f(x_1,...,x_n) = 0$ for all f in $F\}$ the set of common zeroes of the polynomials in F. Sets of the form $V(F)$, where F is a finite set of polynomials, are called **varieties** and are the main objects which we shall study in affine space. Let X be a variety, then any subset X' of X that is a variety is called a **subvariety** of X.

Let $\mathfrak{U}$ be the ideal in the polynomial ring $K[x_1,...,x_m]$ generated by the polynomials $F_1,...,F_m$. Then it is clear that $V(\mathfrak{U}) = V(F)$. Indeed, by Hilbert's Basis Theorem [Fulton] we know that, if R is a Noetherian ring, then the ring of polynomials $R[x_1,...,x_n]$ is Noetherian; thus, any ideal $\mathfrak{U}$ in $R[x_1,...,x_n]$ is finitely generated. Thus, if $\mathfrak{U} = (F_1,...,F_m)$, then $V(\mathfrak{U}) = V(F_1) \cap .. \cap V(F_m)$. We have the following elementary lemma

Lemma A1 [Fulton]

1) If $\mathfrak{U} \subset \mathfrak{V}$ are ideals then $V(\mathfrak{V}) \subset V(\mathfrak{U})$.

2) $V(\mathfrak{U}_1 \cdot ... \cdot \mathfrak{U}_m) = V(\mathfrak{U}_1) \cup ... \cup V(\mathfrak{U}_m)$

3) $V(\Sigma_{\alpha \epsilon I} \mathfrak{U}_\alpha) = \cap_{\alpha \epsilon I} V(\mathfrak{U}_\alpha)$

4) $V(0) = K^n$, $V(1) = \emptyset$

Definition With the properties of Lemma A1 the sets of the form V(F) satisfy the axioms of the closed sets of a topology. We shall call this particular topology the **Zariski topology**. In general, we shall use this topology in preference to any other.

If F is a non-constant polynomial, then we shall call V(F) a **hypersurface**. If, in particular, F is linear, then we shall call V(F) a **hyperplane**. Likewise, if F is a polynomial of degree two, three etc., then we shall call V(F) a quadric, cubic, etc., hypersurface. If $n = 2$, the hypersurfaces are called (affine) plane curves.

Definition We say that a variety X is **irreducible**, whenever it cannot be written as the union of two subvarieties $X = X_1 \cup X_2$, where $X_1 \neq X$ and $X_2 \neq X$.

Lemma A2 [Fulton]

Any variety X may be written as the union of irreducible subvarieties $X = X_1 \cup \ldots \cup X_m$, such that $X_i \not\subset X_j$ for $i \neq j$. The X_i are uniquely determined and are called the **irreducible components** of X.

Definitions

1) Let $X = V(F_1,\ldots,F_m)$ be an irreducible variety in K^n and let $a = (a_1,\ldots,a_n)$ be a point on X. Then we define the **tangent space** $T_a(X)$ to X at a to be the linear subspace given by

$$\Sigma_i \frac{\partial F_j}{\partial x_i}(a)(x_i - a_i) = 0, \qquad 1 \leq j \leq m.$$

The dimension of the tangent space is equal to the corank of the **Jacobian matrix** (m_{ij}), where $m_{ij} = \frac{\partial F_i}{\partial F_j}(a)$ $1 \leq i \leq m$, $1 \leq j \leq n$.

2) We define the **dimension** $\dim(X)$ of the variety X to be the smallest dimension of the tangent spaces occuring at points of X.

3) We say that a point x on X is **simple**, if $\dim(X) = \dim T_x(X)$ and **singular**, if $\dim(X) > \dim T_x(X)$. If all points of X are simple, then we say that X is a non-singular variety.

4) For reducible varieties the above definitions make sense on each of its components. We define its dimension to be the maximum

dimension of any of its components. Then the Jacobian matrix at a point $x \in X'$ on a component X' of X has non-maximal rank if and only if *either* a component of X of dimension $> \dim(X')$ passes through x *or* x is singular on X'.

We shall consider a number of maps between affine varieties X and Y.

(i) Polynomial maps: maps $\varphi: X \subset K^n \to Y \subset K^m$, such that at each point $x = (x_1,...,x_n)$ on X, $\varphi = (\varphi_1(x_1,...,x_n),...,\varphi_m(x_1,...,x_n))$ for some polynomial functions $\varphi_i: K^n \to K$.

(ii) Isomorphisms: A polynomial map φ is an isomorphism, if there exists a polynomial map η, such that $\varphi \circ \eta$ and $\eta \circ \varphi$ are the identity maps on X and Y.

(iii) Affine change of co-ordinates : The polynomial map $\varphi: K^n \to K^n$ is called an affine change of co-ordinates, if it is bijective and given by linear polynomials at each point.

§A2 Projective Geometry

Let K be any field. Then we will define **projective n-space**, denoted by PK^n, to be the set of equivalence classes of points in K^{n+1} under the equivalence relation $(x_1,...,x_{n+1}) \sim (\lambda x_1,...,\lambda x_{n+1})$ for all $\lambda \neq 0$. Geometrically, we may think of PK^{n+1} to be the set of lines in K^{n+1} through the origin. The equivalence classes are called **points** and a representative of the class $(x_1,...,x_{n+1})$ is called its homogeneous co-ordinates. The spaces PK^1, PK^2 are usually called the projective line and the projective plane, respectively.

We may now proceed in an analogous way to affine spaces. Let F be any set of homogeneous polynomials in the $n+1$ variables $x_1,...,x_{n+1}$. Denote by $V(F) = \{(x_1,...,x_{n+1}) \in PK^n | f(x_1,...,x_{n+1}) = 0$ for all f in $F\}$ the set of common zeroes of the polynomials in F. Sets of the form $V(F)$, where F is a finite set of polynomials, are the main objects that we shall study in projective space and we will call them **varieties**. Let X be a variety, then any subset X' of X that is a variety is called a **subvariety** of X. Note that we need homogeneous polynomials (i.e. polynomials, whose monomials all have the same degree or, equivalently, $F(\lambda x_1,...,\lambda x_{n+1}) = \lambda^d F(x_1,...,x_{n+1})$ where d is the degree of F), for we need $F(x_1,...,x_{n+1}) = 0$ if and only if $F(\lambda x_1,...,\lambda x_{n+1}) = 0$. This makes sense for homogeneous polynomials, since $F(\lambda x_1,...,\lambda x_{n+1}) = \lambda^d F(x_1,...,x_{n+1})$.

Let $\mathfrak{U}$ be the homogeneous ideal (i.e. an ideal $\mathfrak{U}$ is homogeneous, if every polynomial $F = F_0 + ... + F_d$ in $\mathfrak{U}$ written as a sum of forms of degree $1 \le i \le d$, has the property that F_i is a member of $\mathfrak{U}$), in the polynomial ring $K[x_1,...,x_{m+1}]$, generated by the homogeneous polynomials $F_1,...,F_m$. Then it is clear that $V(\mathfrak{U}) = V(F)$.

Note that homogeneous ideals are the correct set of polynomials, for, if $x = (x_1,...,x_{n+1})$ is a zero of a polynomial F, then $F(x) = F_0(x_1,...,x_{n+1}) + ... + F_d(x_1,...,x_{n+1}) = 0$ and, moreover, $F(\lambda x) = \lambda F_0(x_1,...,x_{n+1}) + ... + \lambda^d F_d(x_1,...,x_{n+1}) = 0$. Hence, $F_i(x_1,..., x_{n+1}) = 0$ for all $0 \le i \le d$.

There is an analogous Basis Theorem for the projective case, implying that any homogeneous ideal $\mathfrak{U}$ is finitely generated. Further, any homogeneous $\mathfrak{U}$ can be generated by a set of forms. Thus, if $\mathfrak{U} = (F_1,\dots,F_m)$, then $V(\mathfrak{U}) = V(F_1)\cap\dots\cap V(F_m)$.

Lemma A1'

1) If $\mathfrak{U} \subset \mathfrak{V}$ are homogeneous ideals then $V(\mathfrak{V}) \subset V(\mathfrak{U})$.

2) $V(\mathfrak{U}_1\cdot\ldots\cdot\mathfrak{U}_m) = V(\mathfrak{U}_1)\cup\dots\cup V(\mathfrak{U}_m)$

3) $V(\Sigma_{\alpha\in I}\,\mathfrak{U}_\alpha) = \cap_{\alpha\in I}\, V(\mathfrak{U}_\alpha)$

4) $V(0) = PK^n$, $V(1) = \emptyset$

Definition With the properties of Lemma A1' the sets of the form V(F) satisfy the axioms of the closed sets of a topology. We shall call this particular topology the **Zariski topology**.

As in the affine case, whenever F is a non-constant polynomial, we shall call V(F) a **hypersurface**. If, in particular, F is linear we shall call V(F) a **hyperplane** and if F is a polynomial of degree two, three etc., we shall call V(F) a quadric, cubic, etc., hypersurface. For $n = 2$ the hypersurfaces are called (projective) plane curves.

Definition We say that a variety X is **irreducible**, whenever it cannot be written as the union of two subvarieties $X = X_1 \cup X_2$, where $X_1 \neq X$ and $X_2 \neq X$.

Lemma A2'

Any variety X may be written as the union of irreducible subvarieties $X = X_1 \cup ... \cup X_m$, such that $X_i \not\subset X_j$ fo $i \neq j$. The X_i are uniquely determined and are called the **irreducible components** of X.

Definitions

1) Let $X = V(F_1,...,F_m)$ be an irreducible variety in PK^n and let $a = (a_1,...,a_{n+1})$ be a point on X. Then we define the **tangent space** $T_a(X)$ to X at a to be the linear subspace given by

$$\Sigma_j \frac{\partial F_i}{\partial x_j}(a) x_j = 0, \qquad 1 \leq i \leq m.$$

The dimension of the tangent space is equal to the corank of the **Jacobian matrix** (m_{ij}), where $m_{ij} = \frac{\partial F_i}{\partial F_j}(a)$ $1 \leq i \leq m$, $1 \leq j \leq n+1$.

2) We define the **dimension** $\dim(X)$ of the variety X to be the smallest dimension of any tangent space at points of X. We say that a point x on X is **simple**, when $\dim(X) = \dim T_x(X)$ and **singular**, when $\dim(X) > \dim T_x(X)$. If all points of X are simple, then we say that X is a non-singular variety. Varieties of dimension one are called curves and varieties of dimension two are called surfaces.

3) For reducible varieties the above definitions make sense on each of its components. We define its dimension to be the maximum dimension of its components. Then the Jacobian matrix at a point $x \in X'$ on a component X' of X has non-maximal rank if and only if *either* a component of X of dimension $> \dim(X')$ passes through

x *or* x is singular on X'.

4) Let $X = V(F)$ be a hypersurface and suppose that a is a point of multiplicity k on X; by which we mean that the first k-1 partial derivatives, evaluated at a, are zero. Then we shall define the **tangent cone** $C_a(X)$ to be the variety given by

$$\sum_{k_1+..+k_{n+1}=k} \frac{\partial^k F(a)}{\partial^{k_1} X_{k_1} \dots \partial^{k_{n+1}} X_{k_{n+1}}} \cdot X_1^{k_1} \cdot \dots \cdot X_{n+1}^{k_{n+1}} = 0$$

For a general variety $X = V(\mathfrak{U})$ and any point a on X, the tangent cone $C_a(X)$ is obtained by taking the intersection of all tangent cones at a to hypersurfaces V(F) (containing a) for all $F \in \mathfrak{U}$.

Affine and projective varieties are inter-related. For we may cover projective n-space by n+1 affine "pieces" via the following correspondence. Let $K_i = \{(x_1,\dots,x_{n+1}) \in PK^n | \ x_i = 0\}$, then each point $(x_1,\dots,x_{n+1})$ has a homogeneous co-ordinate of the form $(x'_1,\dots,x'_{i-1},1,x'_{i+1},\dots,x'_{n+1})$ thus the maps $F_i: K_i \to K^n$, defined by

$$F_i:(x'_1,\dots,x'_{i-1},1,x'_{i+1},\dots,x'_{n+1}) \mapsto (x'_1,\dots,x'_{i-1},x'_{i+1},\dots,x'_{n+1})$$

are bijections and the whole of PK^n is covered by the n+1 sets K_i. Conversely, given an affine n-space K^n, we may "projectivise" embedding K^n in PK^n by the map $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,1)$. We call the hyperplane $x_{n+1} = 0$ the **hyperplane at infinity**.

We shall be interested in the following maps on projective varieties X and Y.

(i) Polynomial maps: maps $\varphi: X \subset PK^n \to Y \subset PK^m$, such that at each point x on X $\varphi = (\varphi_1(x_1,...,x_{n+1}),...,\varphi_{m+1}(x_1,...,x_{n+1}))$, where the φ_i are polynomials.

(ii) Projective change of co-ordinates : The polynomial map $\varphi: PK^n \to PK^n$ is called a projective change of co-ordinates if it is bijective and given by linear polynomials at each point.

(iii) Rational maps: Let $X \subset PK^n$ and $Y \subset PK^m$ be two varieties. Let $\varphi: U \to Y$ be a map from a Zariski open subset U of X into Y, such that $\varphi = (\varphi_1(x_1,...,x_{n+1}),...,\varphi_{m+1}(x_1,...,x_{n+1}))$ is given by the regular functions φ_i on U, i.e. at every point $a \in U$, $\varphi_i = f/g$, where f,g are polynomial maps with $g(a) \neq 0$. We say that two forms $\varphi: U \to Y$, $\eta: V \to Y$ satisfying the above conditions are equivalent, whenever $\varphi(a) = \eta(a)$ for all $a \in U \cap V$. An equivalence class Φ of such maps is called a **rational map**. Thus, a representative of a rational map is not defined on the whole of X, but only on a Zariski open subset; whereas the domain of a rational map Φ is the union of all Zariski open subsets U of X, where Φ has a representative defined on U. If its domain covers the whole of X, then we say that Φ is a **regular map**.

(iv) Birational maps: A rational map $\Phi: X \to Y$ is **birational**, if there exists a representative $\varphi: U \to Y$ of Φ which is an isomorphism between U and an open set V of Y. Or, equivalently, there is a map $\lambda: V \to X$, which is the representative of a rational map $\Psi: Y \to X$, such that $\varphi \circ \lambda$ and $\lambda \circ \varphi$ are the identity maps where defined.

Lemma A3 [Harris]

A rational map $\Phi: X \to Y$ which is generically 1-1 (i.e. there exists a representative $\varphi: U \to Y$ of Φ, which is bijective between the sets U and $\varphi(U)$) is a birational map.

Theorem A1 [Shafarevich]

1) A rational map $\varphi: X \to Y$ from a non-singular variety X to any variety Y is regular.

2) A birational map $\varphi: X \to Y$ between two non-singular varieties is biregular (i.e. an **isomorphism**).

From this point on we shall only be considering the case when the base field K is either the set of real numbers, denoted by $\mathbb{R}$, or the set of complex numbers, denoted by $\mathbb{C}$.

Theorem A2 [p70,Mumford]

For all varieties X of dimension r in $P\mathbb{C}^n$ there exists an integer $d \geq 1$ such that: if L is an (n-r)-linear subspace satisfying

a) $L \cap X = \{x_1, \ldots, x_k\}$

b) for all i, x_i is a simple point on X and the two tangent spaces $T_{x_i}(X)$ and $T_{x_i}(L)$ (consider as a subspace of $T_{x_i}(P\mathbb{C}^n)$) meet only in the origin,

then $k = d$.

Definition For a variety X in $P\mathbb{C}^n$ we define the **degree** to be the number of points in which almost all linear subspaces of complementary dimension meet it. This number is well defined by Theorem A2.

We shall now define the multiplicity of a point of a curve and the intersection multiplicity of a curve with a hypersurface.

We shall need the following theorem.

Theorem [Milnor]
Let x_0 be an isolated point of a real (or complex) curve C. Then a suitably chosen neighbourhood of x_0 in C is the union of finitely many 'branches' which intersect only at x_0. Each branch is homeomorphic to an open interval of real numbers (or to an open disc of complex numbers) under a homeomorphism $x = P(t)$ which is given by a power series

$$P(t) = x_0 + a_1 t + a_2 t^2 + \ldots$$

convergent for $|t|<\varepsilon$.

Further, let k be the smallest index, so that C is not contained in a co-ordinate hyperplane $x_k = \text{constant}$. Then the parameterisation P can always be chosen, so that $x_k = \text{constant} \pm t^m$ $(m \geq 1)$. P can also be chosen, so that the collection $\{i \mid a_i \neq 0\}$ of exponents has greatest common divisor equal to 1. Then the power series P is uniquely determined up to the sign of the parameter t (or upto multiplication of T by roots of unity in the complex case).

Thus, each branch of C may be parameterised

$$x = P(t) = x_0 + (0, \ldots, 0, t^m, \Sigma_{i \geq 0} a_{k+1,i} t^i, \ldots, \Sigma_{i \geq 0} a_{n,i} t^i)$$

■

Suppose a curve $C \subset P^n$ intersects a hypersurface $H \subset P^n$ in a point P. By the theorem a neighbourhood of P in C is the union of finitely many branches β_j parameterised as

$$(x_0^j(t),\dots,x_n^j(t)), \text{ where } x_\ell^j = \Sigma_{i=n_\ell}^{\infty} a_{\ell,i}^j t^i.$$

If H is given by a polynomial $H(x_0,\dots,x_n)=0$, then we define $i(x_0,H\cap\beta_j) = \text{ord}_t H(x_0^j(t),\dots,x_n^j(t))$ and we define the **intersection multiplicity** of C with H at P to be $i(P,H\cap C) = \Sigma_j i(P,H\cap\beta_j)$.

Intersection multiplicity satisfies the following properties:

1) $i(P,H\cap C) \geq 0$.

2) $i(P,H\cap C) = 0$ if and only if P does not lie on the intersection of C and H.

3) when $n=2$, $i(P,H\cap C) = i(P,C\cap H)$.

4) $i(P,H\cap C)$ is a projective invariant,
i.e. $i(P,H\cap C) = i(\varphi(P),\varphi(H)\cap\varphi(C))$ for any projective change of co-ordinates φ.

We define **multiplicity** of a point P on a curve C to be $\min\{i(P,H\cap C) \mid \text{for any hyperplane } H \text{ passing through } P\}$.

Let $V\subset P^n$ be a variety. Then we define the codimension of V to be $\text{cod}(V) = n - \dim(V)$. We say that a variety V has pure dimension when every irreducible component of V has the same dimension. We shall use the following theorem to calculate the degrees of varieties which are defined as the intersections of other varieties.

Bézout's Theorem A3

Let $V_1, \ldots, V_m$ be algebraic sets of pure dimension in $P\mathbb{C}^n$ intersecting properly, that is, $\mathrm{cod}(V_1 \cap \cdots \cap V_m) = \mathrm{cod}(V_1) + \ldots + \mathrm{cod}(V_m)$. Then

$$\mathrm{degree}(V_1 \cap \cdots \cap V_m) = \mathrm{degree}(V_1) \cdot \ldots \cdot \mathrm{degree}(V_m).$$

We shall also use the following variation of the theorem:

Let C be a curve and H any hypersurface in $P\mathbb{C}^n$. Let $P_1, \ldots, P_m$ be the set of points in the intersection of C and H. Then

$$\mathrm{degree}(C) = \Sigma_j i(P_j, H \cap C).$$

§A3 Linear Systems

We will often wish to consider special families of hypersurfaces and thus it will be an advantage to have a language in which to describe certain objects common to these types of families.

To any hypersurface $X = V(F) \subset P\mathbb{C}^n$ of degree d, where the form F is given by

$$F = \sum_{r_1 + \ldots + r_{n+1} = d} a_{r_1 \ldots r_{n+1}} X_1^{r_1} \cdot \ldots \cdot X_{n+1}^{r_{n+1}},$$

we may associate a point $(\ldots, a_{r_1 \ldots r_{n+1}}, \ldots)$ in the projective space $P\mathbb{C}^{N-1}$, where $N = \binom{n+d}{d}$. This makes sense, since $(\ldots, a_{r_1 \ldots r_{n+1}}, \ldots)$

corresponds to F for all $\lambda \neq 0$. Thus the family of all hypersurfaces of a given degree d in $P\mathbb{C}^n$ forms an N-dimensional projective space which we shall denote by $H(n,d)$. The m-dimensional subspaces of an $H(n,d)$ are called **linear systems** of hypersurfaces of degree d in $P\mathbb{C}^n$. Linear systems of dimension one, two, and three are called **pencils, nets and webs,** respectively.

We may describe a linear system $\mathcal{L}$, either by giving the defining equations of the m-dimensional subspace in $H(n,d)$ or by giving a set of generators for the system, i.e. a set of (m+1) linearly independent points in $H(n,d)$ whose span is the subspace $\mathcal{L}$. In the latter case, if the points correspond to the forms $F_1,\dots,F_{m+1}$, then the linear system is the set of forms $\lambda_1F_1+\dots+\lambda_{m+1}F_{m+1}$.

Example $H(2,2)$=set of conics in the plane. Then $H(2,2)$ is a five dimensional projective space. The general form of a conic in the projective space with co-ordinates x,y,z may be written $C=a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$. Thus, each co-ordinate of $H(2,2)$ corresponds to a monomial in the general form. For instance, the pencil $\lambda_1(xy + z^2) + \lambda_2(x^2 + yz)$ corresponds to the line in $P\mathbb{C}^5$ through the points (0,1,0,0,0,1) and (1,0,0,0,1,0).

Definition The (possibly empty) intersection of all hypersurfaces in a linear system is called the **base variety**.

Linear systems of quadrics

Let $\lambda_1Q_1+\dots+\lambda_{m+1}Q_{m+1}$ be a linear system of quadrics in $P\mathbb{C}^n$. Write the generating quadrics in matrix form $Q_i=xA_ix^t$, where

$x = (x_1,\dots,x_{n+1})$ and A_i is the $(n+1)\times(n+1)$ symmetric matrix (a_{ij}), whose elements are $a_{ij} = \frac{1}{2}\frac{\partial Q}{\partial x_i \partial x_j}$ for $i \neq j$ and $a_{ii} = \frac{\partial^2 Q}{\partial x^2}$. Then the determinant Δ of the matrix $\mathfrak{M} = \lambda_1 A_1 + \dots + \lambda_{m+1} A_{m+1}$ is an invariant of the pencil called the **discriminant**. Then $\Delta = 0$ is a homogeneous polynomial in $\lambda_1,\dots,\lambda_{m+1}$ and thus defines a discriminant variety $\mathcal{D}$ in $P\mathbb{C}^m$. A point $(\lambda_1,\dots,\lambda_{m+1}) \in \mathcal{D}$ if and only if $Q = \lambda_1 Q_1 + \dots + \lambda_m Q_m$ is a singular quadric.

For a pencil the discriminant is a binary polynomial of degree $(N-1)!$, where $N = \frac{1}{2}(n+2)(n+1)$, and for a net the discriminant is a plane curve of degree $(N-1)!$.

§A4 Foci

The significance of the foci of a plane curve is classical. When we deal with plane curves we may mention the foci and perhaps it is worthwhile discussing them. In the proof of Roberts' Triple Generation Theorem in Chapter 5 we shall see that the foci of the coupler curve of the four-bar play a somewhat mysterious role in the geometry.

Let C be a real plane curve of **class** m, i.e. the number of lines tangent to C through a general point P. Denote the circular points at infinity by I and J. Then there are m (complex) lines through I tangent to C and m conjugate lines through J tangent to C. If T is one of them, then it meets its conjugate in a real point P which we will call a **focus** of C. If the curve is circular, i.e. it passes through I and J, then there are tangents to C at I. If T is a tangent to C at I, then it meets its conjugate line $\bar{T}$ in a real point which we distinguish from the above foci by calling it a **singular foci**. We need not say anything more about foci.

§A5 Desingularisations of a Curve, Definition of the Genus and the Genus Formula

Following the excellent exposition given in [Mumford] leading to the definition of the arithmetic genus, we summarise the results as follows.

Theorem A4 [Mumford]
Let M be a finitely generated graded module over the ring $\mathbb{C}[x_1,...x_{n+1}]$ i.e. $M=\oplus_{k=1}^{\infty}M_k$, such that for all homogeneous polynomials F of degree d, $F\cdot M_k \subset M_{k+d}$. Then there is a polynomial $P_M(t)$ of degree at most n with rational coefficients, such that $\dim_{\mathbb{C}} M_k = P_M(k)$ for all sufficiently large k.

Thus, if $X \subset P\mathbb{C}^n$ is a variety and $\mathcal{I}(X)$ is the homogeneous ideal of polynomials vanishing on X, then we may apply the theorem to the module $\mathbb{C}[x_1,...x_{n+1}]/\mathcal{I}(X)$. Thus, the theorem tells us that there is a polynomial $P_X(t)$, such that the dimension of the "degree k^{th} piece" equals $P_X(k)$ for k sufficiently large. We shall call $P_X(k)$ the Hilbert polynomial.

The constant term of the Hilbert polynomial is very important. Classically, we do not use the constant $P_X(0)$ but the integer $p_a(X) = (-1)^r(P_X(0)-1)$ (recall that r is the dimension). We call $p_a(X)$ the **arithmetic genus.** For a plane curve X of degree d the arithmetic genus is given by $p_a(X) = \frac{1}{2}(d-1)(d-2)$. In general, we calculate the arithmetic genus via the formula of Theorem A5.

Theorem A5 [Gibson&Newstead]
Let C be a curve in $P\mathbb{C}^n$ given as the intersection of (n-1) hypersurfaces of degrees $d_1,\dots,d_{n-1}$ then the arithmetic genus of C is given by

$$2p_a(C) - 2 = \left(\sum_{i=1}^{n} d_i - n - 1\right)\prod_{i=1}^{n} d_i.$$

Examples:
1) For the intersection C of three quadric hypersurfaces in $P\mathbb{C}^4$ we have $n = 4$ and $d_1 = d_2 = d_3 = 2$ giving $p_a(C) = 5$.
2) For the intersection C of two cubics hypersurfaces in $P\mathbb{C}^3$ we have $n = 3$ and $d_1 = d_2 = 3$ giving $p_a(C) = 10$.

When classifying curves up to birational equivalence, the following theorem reduces the problem to one of studying non-singular curves upto birational equivalence and hence by Theorem A1 upto isomorphism.

Theorem A6
Every singular curve X is birationally isomorphic to a non-singular curve $\tilde{X}$.

Definition

1) The curve $\tilde{X}$ of the theorem is called the **desingularisation** (or **normalisation**) of X.

2) We define another genus, the **geometric genus**, for a curve X denoted by $p_g(X)$ to be arithmetic genus of the desingularisation of X. Thus for non-singular curves X we have $p_a(X) = p_g(X)$.

Then we have the following very important results.

Theorem A7

1) The arithmetic genus (for any variety) is invariant under isomorphism i.e. if X and Y are isomorphic, then $p_a(X) = p_a(Y)$.

2) The geometric genus for curves is a birational invariant i.e. if X and Y are birationally isomorphic, then $p_g(X) = p_g(Y)$.

The second result follows from the first, for, if X and Y are birationally isomorphic, then so are $\tilde{X}$ and $\tilde{Y}$. But, since $\tilde{X}$ and $\tilde{Y}$ are non-singular curves, they are isomorphic by Theorem A1 and thus have the same arithmetic genus by result 1). The proof of 1) is a very deep result. The theorem tells us that the geometric genus is an invariant of the equivalence class under birational isomorphisms.

To compute the genus of subvarieties of curves we shall find the following theorem indispensible.

Theorem A8 (The Genus Formula) [Gibson&Newstead]

Let X be a connected (but possibly reducible) complex projective curve and $\varphi: X' \to X$ a birational map. Then

$$p_a(X) = \sum_{i=1}^{t} p_a(X'_i) + \sum_{P \in C} \delta p - (t-1)$$

where $X'_1,\dots,X'_t$ are the irreducible components of X'. In particular, suppose that $X = X_1 \cup .. \cup X_t$ is connected. Denote by $\tilde{X}_i$ the desingularisation of X_i then, since $p_g(X_i) = p_a(\tilde{X}_i)$, we can take $\varphi: \tilde{X} \to X$ so that

$$p_a(X) = \sum_{i=1}^{t} p_g(X_i) + \sum_{P \in C} \delta p - (t-1)$$

We have yet to define δp. The non-negative integer δp is zero at a simple point P, whilst at a singular point P, provides us with a useful invariant under isomorphism. Its formal definition may be found in [Hartshorne], but we shall only need to know that for an m-tuple point P we have $\delta p \geq \frac{1}{2}m(m-1)$, with equality, whenever the multiple point is ordinary, i.e. it has distinct tangents. Further, we may find the following values useful

1) If P is an ordinary double point then $\delta p = 1$
2) If P is a cusp then $\delta p = 1$
3) If P is a tacnode then $\delta p = 2$
4) If P is a ramphoid cusp then $\delta p = 2$

(Indeed appealing to the classification of double points on curves up to isomorphism, which states that any double point may be put in the form $y^2 = x^r$ $(r \geq 2)$, then δp may be showed to be the integer part of $\frac{1}{2}r$. See [Hartshorne]).

§A6 Real Geometry, Circuits and Harnack's Theorem

Let X be a variety of dimension r in $P\mathbb{R}^n$. Then, whenever X is non-singular, the underlying set of X may be given the structure of a real r-manifold. In the case when X is a curve, then X is a compact 1-manifold and hence isomorphic to a disjoint union of circles [Milnor]. The only topological invariant here is the number of connected components. The number is bound by the genus of the complex curve X' (i.e. $X = V(\mathfrak{U})$, where the elements of $\mathfrak{U}$ are polynomials with real coefficients. Thus, we can also consider the set X' of real and complex zeroes in $P\mathbb{C}^n$). Indeed, the bound remains true even when X is singular.

Harnack's Theorem A9 [Shafarevich]
Let $X = V(\mathfrak{U})$ be a curve in $P\mathbb{C}^n$ given as the intersection of polynomials with real coefficients, then the number of connected components of the real curve is less than or equal to $p_g(X) + 1$.

We showed earlier that any complex curve X has a desingularisation $\tilde{X}$. If C is a real curve, then there is a real birational isomorphism $\varphi: \tilde{C} \to C$ between C and a non-singular curve $\tilde{C}$ ([Shafarevich]). If C is a plane curve, then we will call the image of each connected component of $\tilde{C}$ under the map φ a **circuit**. If C is a non-singular plane curve, the circuits are generally called **ovals**. (However, the plane curves which we shall meet are singular, so we shall not meet any ovals.) Thus it follows from Harnack's Theorem that the number of circuits of a (possibly singular) plane curve C is less than or equal to $p_g(C) + 1$.

§A7 Finite Mappings

A quasi-projective variety is an open subset of a projective variety. Let $\varphi:X\to Y$ be a regular map between two irreducible quasi-projective varieties X, Y of the same dimension, such that $Y=\varphi(X)$. Then φ satisfies the condition of finiteness, i.e. every point of Y has at most finitely many pre-images on X. Moreover, there is an integer d such that for all points P in a Zariski open subset of Y the number of pre-images of P on X is equal to d. The number d is called the **degree** of the map φ.

Now suppose that $\varphi:X\to Y$ is a rational map between two irreducible curves X and Y. Then the set of points X', where φ is regular and the set $Y'=\varphi(X')$ are quasi-projective varieties. Thus $\varphi: X'\to Y'$ is a finite map. Hence, it makes sense to say that φ is a map of degree d, because it is clear that we shall mean the degree of φ restricted to Zariski open subset on which it is regular.

If $\varphi:X\to Y$ is regular and Y is non-singular, then it is a fact ([Shafarevich]) that the number of pre-images of <u>any</u> point P on Y is $\leq d$. Points on Y with fewer than d pre-images are called **branch points**, whilst the pre-images of branch points are called **ramification points**. One of the main theorems on rational maps between non-singular curves is the Hurwitz Theorem.

Hurwitz's Theorem A10 [Hartshorne]

Let $\varphi:X\to Y$ be a rational (and hence regular) map of degree d between two non-singular complex projective curves X and Y. Then their genera are related by the following formula

$$2p_g(X) - 2 = d.(2p_g(Y) - 2) + \Sigma_{P\epsilon X}(e_P-1).$$

The positive integer $e_P=1$ if and only if P is not a ramification point, whereas; when P is a ramification point, e_P is, intuitively, a measure of the ramification. Furthermore, the e_P's satisfy $\Sigma e_P = d$. We shall only use the theorem for mappings of degree two. Thus, if P is a ramification point then $e_P = 2$ and a branch point has just one pre-image. Hence the formula reads

$$2p_g(X) - 2 = 4.(p_g(Y) - 1) + \{\text{number of branch points}\}.$$

Let $\varphi:X\rightarrow Y$ be a map of degree d between two varieties and suppose that Y is non-singular. We say that a point P on X is a **critical point** of φ, whenever $\dim T_P(X)\neq\dim T_{\varphi(P)}(Y)$ and we say that $\varphi(P)$ is a **critical value**. Then a point P on X is a ramification point of φ if and only if it is a critical point. Thus, whenever Y is non-singular we may reduce the problem of determining the branch points of a map to that of determining the dimension of a linear subspace, a task in linear algebra.

§A8 The Projection Formula

Consider the polynomial map $\pi:P\mathbb{C}^n-\mathfrak{L}\rightarrow P\mathbb{C}^m$ given by

$$\pi:(x_1,..,x_{n+1})\mapsto(\Sigma_i a_{1i}x_i,...,\Sigma_i a_{(m+1)i}x_i)$$

These maps are called **projections**. The linear subspace $\mathfrak{L}$ of $P\mathbb{C}^n$ given by $\Sigma_i a_{1i}x_i = ... = \Sigma_i a_{(m+1)i}x_i = 0$ is called the **centre** of projection; the map π is undefined at all points on $\mathfrak{L}$.

Theorem A11 The Projection Formula

Let $\pi_{\mathcal{L}}:P\mathbb{C}^n \to P\mathbb{C}^m$ be the projection with centre $\mathcal{L}$ given by $(x_1,\dots,x_{n+1}) \mapsto (\Sigma_i a_{1i}x_i,\dots,\Sigma_i a_{(m+1)i}x_i)$ and let X be a curve in $P\mathbb{C}^n$. Denote by $\varphi = \pi_{\mathcal{L}}|_X$ the restriction of the projection to X and let $Y = \varphi(X)$. Then φ is a rational map and by the results of the previous section is a finite map. Let the degree of φ be d. Suppose that the centre $\mathcal{L}$ meets X in the points $x_1,\dots,x_r$. Then we may relate the degree of X and the degree of Y, via the degree of the restriction map, by the formula

$$\deg X - \Sigma_j\, i(x_j, X \cap \mathcal{L}) = \deg\varphi \cdot \deg Y$$

where $\Sigma_j\, i(x_j, X \cap \mathcal{L})$ is the sum of all intersection multiplicities of a generic hyperplane through $\mathcal{L}$, with X at the points x_j, $1 \le j \le m$.

Proof (adaptation of [p76,Mumford] for our purposes)

For any hyperplane H we have by Bézout's Theorem, $\deg X = \Sigma_{x \in X \cap H}\, i(x, X \cap H)$, where $i(x, X \cap H)$ is the usual intersection multiplicity of the hyperplane H with X at the point x. Then, if $M \subset P\mathbb{C}^m$ is an (m-1)-dimensional linear subspace satisfying

(i) M meets Y transversally,

(ii) M does not meet Y in a branch point of φ , and

(iii) M does not meet Y in any point in the set $Y - \varphi(X)$ (i.e. in the closure of the image of X under the projection),

then its pre-image $M' = \varphi^{-1}(M) \cup \mathcal{L}$ meets X transversally in $\deg Y . \deg\varphi$ points of X and non-transversally in $x_1,\dots,x_m$. Thus $\deg X - \Sigma_{x \in X \cap M'}\, i(x, X \cap M') = \deg Y . \deg\varphi$ and the result follows from the fact that almost all linear subspaces of dimension (m-1) satisfy properties (i)-(iii). Moreover, it can be showed that the minimum possible value of the sum $\Sigma_{x \in X \cap M'} i(x, X \cap M')$ is attained by generic hyperplanes i.e. the sum is larger for non-generic hyperplanes.

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Theorem A12

Let $i(x, X \cap H)$ be the intersection multiplicity of a curve $X \subset P\mathbb{C}^n$ with a hyperplane H at a point x on X. Let $\pi : P\mathbb{C}^n \to P\mathbb{C}^m$ be a projection, let $Y = \pi|_X(X)$ and suppose that the degree of $\varphi = \pi|_X$ is one. Then, "intersection multiplicity does not decrease under the projection" i.e. $i(x, X \cap H) \le i(\varphi(x), Y \cap \varphi(H))$. In particular, multiplicity does not decrease under projection i.e. $mp(x) \le mp(\varphi(x))$.

Proof

Without loss of generality, we may assume that the projection is onto the first $m+1$ co-ordinates. Parameterise X at x. Let the parameterisation be $(x_0^j(t), \ldots, x_n^j(t))$, where x_i^j is a power series in t. Then, since π is one to one, π is a birational map implying that $(x_0^j(t), \ldots, x_m^j(t))$ is a parameterisation of Y at $\pi(x)$. Suppose that H is any hyperplane in $P\mathbb{C}^n$ containing $\mathcal{L}$, given by the homogeneous polynomial G, and suppose that $H' = \pi(H)$ is given by the polynomial G'. Then we can easily deduce that $G = G'$; since, if G vanishes at a point P, then certainly G' vanishes at $\pi(P)$ implying that $G'|G$ and we obtain $G = G'$, since G is linear. Thus

$i(\pi(x), H' \cap Y) = \Sigma_j \mathrm{ord}_t G(x_0^j(t), \ldots, x_m^j(t))$ + {sum of multiplicities at other branches through $\pi(x)$}

Therefore, $i(\pi(x), H' \cap Y) \ge i(x, H \cap C)$.

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Theorem A13 [Walker]

Let X be a curve in $P\mathbb{C}^n$ and let $Y = \pi(X)$, where π is a linear projection $\pi : X \to Y$. Then any point P on X *either* is the projection of a point of X *or* has as its pre-image a linear subspace which is the join of the centre of projection $\mathcal{L}$ and an osculating r-plane to some point of X on $\mathcal{L}$.

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