TIME - DEPENDENT STOCHASTIC MODELS FOR FIRE RISK ASSESSMENT

THESIS

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by

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ABSTRACT

Put Out the Fire Before it Spreads

Tolstoy

Tolstoy's short story concerns a feud between neighbours which grows out of all proportion to the initiating event. His chosen title is indeed apposite, for even the most destructive of fires have generally begun as small, readily extinguishable flames. The research described in this thesis is, however, concerned not with methods of extinguishing a fire, but with methods of modelling the spread of a fire through a multi-compartment structure. Interest is centred exclusively on structures having one or more volumes which are critical in the sense that the consequences of a fire in those volumes are so calamitous as to render negligible any fire damage to other volumes in the structure. Several models are developed, and the relevance of models from related fields of application is discussed.

The First Chapter serves to provide both a general introduction to the research problem and a summary of pre-requisite information. Chapter Two offers a discussion of the relationship between this modelling problem and those that have arisen in other fields. Chapters Three and Four contain various models derived as pertinent to this application, whilst the Fifth and Sixth Chapters are both concerned with models relevant to the problem of fitting fire barriers to an otherwise complete structure. The aim is to elucidate how the available resources may best be used to fit barriers so as to minimize the likelihood of fire damage to any critical volume. Chapter Seven provides a summary of the whole work.

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I should also like to thank my friends and family for their encouragement, and particularly I am grateful to my brother Robert who has always been ready to read and discuss my work.

I am fortunate to have had the supervisory attention of three members of the academic staff of Liverpool University: - Dr Alan Veevers, Dr Brian Boffey and Dr Derek Yates, - as well as that of Mr Mike Finucane of the Fire Safety Section of the United Kingdom Atomic Energy Authority's Safety and Reliability Directorate (SRD). Mr Finucane has always been ready to advise on the practical aspects of the project and kindly secured a warm welcome to the Fire Safety Section for the periods of time I spent there. Whilst most of the week-by-week supervision was undertaken by Dr Veevers, Dr Yates has offered less-frequent but regular encouragement, and Dr Boffey supervised the work which appears in Chapters Five and Six. I have very much enjoyed working with Dr Veevers; I thank him heartily for his warmth, friendship and excellent supervision and I should like to take this opportunity to wish him well as he leaves to take up his new post in Australia.

Thanks are also due to the Science and Engineering Research Council, and to the SRD, who jointly funded this research project.

CONTENTS

CHAPTER 1

INTRODUCTION

Nam tua res agitur, paries cum proximus ardet.

1.1 General Introduction

'For it is your concern when your neighbour's wall is aflame'. Thus wrote Horace (65-08 BC) in Volume 1 of his Epistles. That maxim has lost none of its truth over the intervening years, for even if we are rather less concerned with our neighbour than were the first century BC Romans, we are certainly no less concerned for ourselves.

This thesis details the research carried out under an SERC (CASE) award for which the co-operating industrial sponsor has been the United Kingdom Atomic Energy Authority (UKAEA) Safety and Reliability Directorate (SRD). The SRD has reflected the general concern of the Atomic Energy Authority that nuclear installations be constructed and maintained to a very high safety standard so that the probability of critical events such as those at Three Mile Island and Chernobyl be made as small as possible.

1.2.1 The Genesis of the Research

The fire-safety section of the Safety and Reliability Directorate of the AEA is concerned with minimizing the damage, to both materials and lives, which can result from a serious fire at one of its establishments. It is recognized that simply taking precautions which seek to ensure that no fire ever occurs is not sufficient, for however unlikely the event may be considered, it would be erroneous to assign its occurrence a zero probability. Thus steps are taken to understand the nature of fire as a physical and chemical phenomenon, and psychological and mathematical techniques are used to model the behaviour which may be observed in humans and buildings in response to a fire.

1.2.2 Models of Fire Growth in a Single Volume

There is a substantial amount of published research into the nature of a fire's development within one room or 'volume', - examples include Quintiere (1981), Williamson (1981) and Thomas (1984). This is also clear from a glance through current textbooks about fire such as the now well-established texts by Lie (1972) and Read and Morris (1983) and a more recent, very thorough text by Shields and Silcock (1987), each of which, like others, have sections on the growth of 'enclosure' fires, but do not so much discuss the spread of a fire through a multi-compartment building. The knowledge about the behaviour of enclosure fires is grounded in an understanding of the complex chemistry and physics of the combustion process.

This process is now fairly well understood and may be summarized briefly as follows:

1) One item in the volume is ignited and flames spread over its surface.
2) Heat radiates to other items in the room which also ignite.
3) Provided that there is sufficient oxygen and combustible material, the heat build-up increases more and more rapidly.
4) Flashover occurs and the whole volume is involved in the fire.

The word 'flashover' is used to describe the transition from the 'growth stage' of a fire to the 'fully developed stage' and tends to be regarded as a turning point in the severity of a fire. It can be assumed that a pre-flashover fire in one volume does not present a significant threat of immediate spread to a neighbouring volume, whilst a fire which has reached flashover intensity does present such a threat as long as it continues to burn vigorously.

1.2.3 The Research Problem

The particular collaboration between the SERC and the AEA which has given rise to this thesis is rooted in a well-established association between the SRD and the University of Liverpool's Department of Statistics and Computational Mathematics and the Department of Computer Science. The emphasis of the joint research has been on the development of stochastic models for the spread of flames through a burning, multi-compartment, or 'segregated', structure (see, for example, Veevers *et al.* (1988)).

The SRD has expressed a special interest in the case in which a structure consisting of a number of volumes has a small number of particularly sensitive volumes for which protection from the effects of flames is especially critical. In this case any damage caused by a fire which does not affect these 'critical' volumes is unimportant in the context of a probabilistic risk assessment, PRA, involving these critical volumes. An example of such a situation is provided by a nuclear power-station having a control room which contains equipment vital to the normal functioning of the reactor. At the plant-design stage, a PRA of the reactor should include the risk arising from the threat of fire to the vital components.

It should be noted that since the work in this thesis is concerned with spread of flames rather than smoke or other combustion products, the word 'fire' will be used as a synonym for flames unless otherwise indicated. In addition, the word 'target' is used throughout to refer to a critical volume.

The research underpinning the thesis is allied to the collaborative work mentioned above, and as such it is concerned with the class of problems whose features are described as follows:

1) A piece of critical equipment located in one or more 'target' volumes of a given segregated structure is at risk from damage by flame (but not from smoke alone).

2) Initial ignition occurs in only one volume of the structure.

3) Following ignition the fire can spread to other volumes in the structure by breaching common barriers.

4) Each volume contains sufficient combustible material for a fire of flashover intensity to develop.

5) No significant fire-fighting action takes place.

The 'fire-fighting action' of specification (5) refers to safety mechanisms within the structure, such as water-sprinkler systems, and to the contribution from fire-fighting teams arriving at the scene of a fire from elsewhere. It is reasonable to analyze separately the contribution from these 'active' measures and that from the 'passive' fire-resisting measures integral to the structure since they are effectively functionally independent. Their combined effects may be considered subsequently, depending on the methodology of the PRA.

The specific advance which this work represents is the development of time-dependent stochastic modelling of fire spread.

1.2.4 Existing Models of Fire Spread Through a Segregated Structure

The collaboration between the University of Liverpool and the SRD has resulted in the production of a time-independent probabilistic fire spread model which uses graph-theoretical network techniques in its evaluation of the probabilities associated with fire reaching particular volumes in a given structure. The model, written in Fortran code and named ARSSUN, predicts fire spread simultaneously in all directions from the volume in which the fire starts and is able to highlight any routes through the structure along which the probability of spread is particularly large.

Details of one of the most comprehensive models of the spread of fire through a segregated structure are given by Elms and Buchanan (1981). The authors describe the development of a computer-based method for the fire safety assessment of a multi-volume structure. The computer program, FIRESPREAD, is, like ARSSUN, intended for use as a comparative fire-safety analytical tool. The introduction to Elms and Buchanan states explicitly that, because of the model's limitations, it is not intended for precise predictive analysis of a structure's behaviour in a severe fire, but rather its application is to quantify the relative effects of a variety of possible fire control strategies for a particular building. The use of all such models to compare different passive fire control measures provides a good reason for modelling separately the effects of the active and passive strategies.

ARSSUN relies on the ascription by expert assessors of a constant breach probability to every barrier in the structure under scrutiny, whereas for FIRESPREAD any door is considered to have a known probability of being open (thus offering no resistance to a fire's progress), this being specified in addition to the nominal fire resistance (*qv* §1.3.2) of the barrier. Furthermore, the fire resistance afforded in practice is regarded as a Gaussian distributed random variable with mean equal to the nominal fire resistance, and a constant coefficient of variation ($\sigma/\mu=0.15$). Similarly, ARSSUN requires that for each volume the probability of growth to flashover intensity be specified, whilst for FIRESPREAD a mean value of severity in each volume is given, and the true severity is again considered a Gaussian random variable with constant coefficient of variation $=0.15$.

Other published time-independent models of fire spread include most recently Colbourn *et al.* (1991), whilst there is work by Ling (1982), Ling and Williamson (1985) and Ramachandran (1985) on the time-dependent aspects of fire-spread modelling. Their respective contributions are discussed in later chapters of the thesis.

1.2.5 The Importance of Time-Dependent Modelling of Fire Spread

The features outlined in subsection 1.2.3 as being relevant to both ARSSUN and this work are very specific and are confined to fire-spread models formulated within, or in co-operation with, the nuclear industry. The particular feature which sets the modelling problem somewhat apart from those which encouraged the work of Elms and Buchanan and Ling and Williamson being the existence of a few critical volumes whose being damaged by fire could have such serious consequences as to render negligible any damage to the rest of the structure.

There is a sense in which this feature makes a time-dependent model even more important in this case than in the more general problem addressed by other authors. Whilst for them there must be a general awareness that as time since ignition elapses, the damage becomes worse, the modelling assumptions suggested by the SRD tend towards implying a situation in which there is at one moment negligible damage (ie, the fire has not reached a target volume) and the next moment the consequences are catastrophic. In either case, it is generally unrealistic to consider as a numerical constant the probability of a barrier being breached upon exposure to a severe fire, since the greater is the length of exposure to a flashover intensity fire, the less likely is a barrier to maintain its integrity. Furthermore, the inclusion of even crude measures of the time scale involved allows some consideration to be given to the likelihood of fire-fighting equipment being utilized effectively, and can provide a different perspective on ways to improve structure design. Some comparisons between time-dependent and time-independent models are given in Chapter 3.

1.2.6 Coupling Fire growth in One Volume with Fire Spread

There has been no attempt formally to incorporate the extensive knowledge about the development of fires within one volume into stochastic models of fire spread through several volumes. The two main reasons for this are likely to be

1) the models of fire spread are intended to be as flexible as possible, and are constructed as models of a stochastic process whose coupling with complex deterministic models of fire growth would severely limit their flexibility; and
2) the models of fire spread are intended primarily to provide a comparative tool, and an understanding of the relative merits of a variety of barrier-fitting strategies is possible without reference to the specifics of fire-growth modelling.

The methods used in FIRESPREAD and ARSSUN were described in sub-section 1.2.4., and the other models similarly use single values to represent the probabilities of a fire developing to flashover intensity in each volume. The probabilities are derived with the aid of information on the number of serious fires per annum for different types of structure. An example of such information is given in Baldwin (1974). The emphasis in this thesis is on modelling the breach characteristics of fire barriers and consequently reference is made throughout to barrier-breach times or probabilities. There is, nevertheless, nothing to prevent the time-dependent models presented being regarded and used as models incorporating the time taken for a fire in a volume to develop to flashover intensity.

1.3 British Standards on Fire Safety

In this section an introduction to some aspects of the British Standards on fire safety is given.

1.3.1 Overview of Fire-related Properties Tested

The British Standards on Fire safety are described to some extent in BS 6336 (1982) which offers some simple theory of fire behaviour and gives fairly general practical guidance on the development and use of fire tests. The individual Standards provide details of the fire-related properties which need to be examined for the different materials with which they are concerned. The main properties or functions which come under examination are given in Table 1.1.

Property	Interpretation
Ignitability	A measure of the ease with which a material (or product or component) may be ignited.
Flammability	The property that determines the rate at which fire develops in a material.
Surface flame spread	The spread of flame across the surface of a material - this may occur without flames engulfing the whole of the material.
Heat release	A measure of the contribution made by a burning material to the fire in progress.
Smoke (gas) release	A measure of smoke and/or gas release from a material subject to a source of heat or ignition.
Fire resistance	An assessment of the time-to-failure of a material under standardized fire-test conditions.
Flame penetration	Penetration by flame of a covering material or a protective structure.
Smoke (gas) penetration	Penetration by smoke and/or gas of a material not necessarily aflame.

Table 1.1 Measures of a material's response to fire, from BS 6336.

1.3.2 Details Pertinent to the Content of this Thesis

The most relevant British Standards specifications as far as this research is concerned are those contained in BS 476 parts 20-23 (1987). BS 476 part 20 gives the general test conditions for determining the fire resistance of 'elements of building construction', whilst parts 20-23 give respectively the detailed test requirements for loadbearing elements, non-loadbearing elements and components of construction. The 'elements of building construction' include walls and partitions, floors, flat roofs, columns, beams, door and shutter assemblies, glazing and ceiling membranes, whilst 'components' include, for example, suspended ceilings designed to protect beams and intumescent seals used to increase the fire resistance of doorsets.

The Standard indicates that any specimen which is tested should be of full size whenever possible, and should be representative of the particular element of construction as it would be in practice. Furthermore, the specimen should have approximately the strength and moisture content which the element is expected to have when in service in a building and the specimen should, prior to undergoing the fire test, be subjected to a load which would be equivalent to the maximum that may be borne by the corresponding element in a real structure.

The specimen is heated in a furnace whose temperature must be controlled to vary with time according to the standard time/temperature relationship

$$T = 345\log_{10}(8t+1) + 20$$

where:

T = mean furnace temperature in ^{o}C at time t, and
t = duration of the test in minutes, up to a maximum of 360 minutes.

Some small deviations from the standard time/temperature curve are allowed. The measure of deviation at time t, p(t), is taken to be the ratio of the difference between the area under the mean furnace temperature/time curve and the area under the standard curve, to the area under the standard curve, expressed as a percentage.

Thus

$$p(t) = \left| \frac{\int_0^t T\,dt - 345\int_0^t \{20 + \log_{10}(8t+1\ \}dt}{345\int_0^t \{20 + \log_{10}(8t+1)\}dt} \right| \times 100\ \%.$$

BS 476 part 20 states that p(t) must conform to the following:

$p(t) < 15$ for $t \leq 10$

$p(t) < 10$ for $10 < t \leq 30$

$p(t) < 5$ for $t > 30$.

and recommends that the integrals be evaluated using Simpson's rule!

The specimen is assessed according to a number of specific failure criteria. These and their interpretations are given in Table 1.2.

Criteria	Failure shall be deemed to occur when ...
Stability	collapse of the specimen, under any appropriate load, takes place.
Integrity	flaming exists on the unexposed surface of the specimen for a continuous period of ≥10 seconds, or cracks exist through which flames pass.
Insulation	the difference between the mean temperature on the unexposed surface and the initial temperature exceeds 140°C, or the difference between the maximum temperature on the unexposed surface and the initial temperature exceeds 180°C.

Table 1.2 Failure criteria for elements of building construction, from BS 476 part 20.

The test results are given as times in minutes from the start of the test until failure has occurred under one or all of the criteria in Table 1.1, or, if no failure has occurred, the time is given as the duration of the test. The fire resistance of an element of building construction is defined to be the time in minutes from the start of the test on the specimen until failure first occurs under any of the criteria given, or until the test is terminated, whichever is the smaller.

For each newly designed and constructed fire-resistant building element, one or two elements or models thereof are subjected to the fire tests, and the test results deemed to apply to that new element. Two tests are carried out when it is considered possible that the response of an element to fire will be different depending upon which surface is exposed to the flames and heat of the fire. It is worth noting that the American Standards are similar (ASTM 1982).

From a statistical sampling-theory point of view, it is inadequate to test only one specimen, and on the basis of the results of that test to assign the same fire resistance to all elements of which that one specimen is considered representative. That, however, is what is done and it has led to an exiguity of data and a paucity of knowledge on the times-to-failure of elements of building construction.

1.4 Introduction to Mathematical Concepts and Definitions

This section serves to bring together all those concepts and definitions which are used in the rest of the thesis and which require explicit explanation or clarification.

1.4.1 The Network Representation of a Structure

In the consideration of the spread of fire through a structure, the use of a network representation of that structure, rather than the traditional illustration, enables speedy identification of salient points without the distraction of too much information.

For example, the single storey structure shown in Figure 1.1 may be represented by the network (Figure 1.2) in which the nodes symbolize the volumes whilst the arcs, or links, represent the barriers between the volumes.

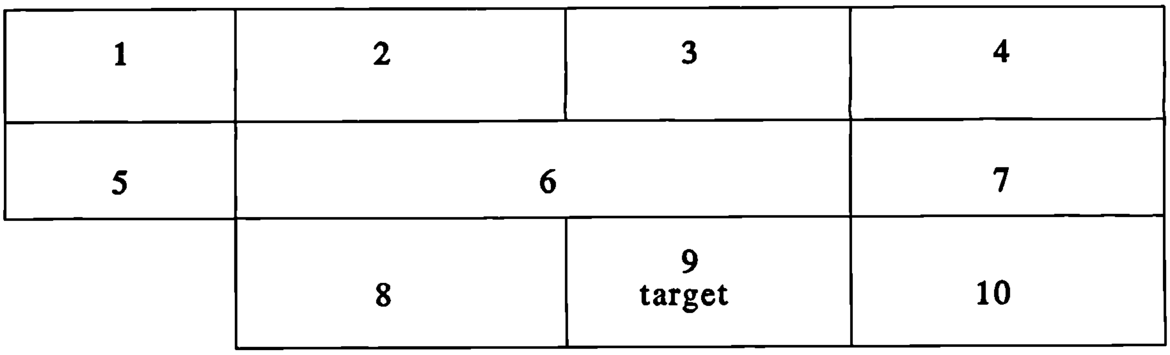


Figure 1.1 Plan of a single-storey, ten volume structure.

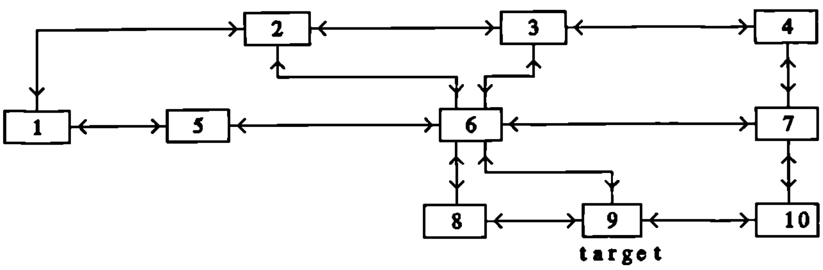


Figure 1.2 Network representation of Figure 2.1.

Double-headed arrows are used to emphasize that fire may breach a barrier in either direction, and that the breach-time characteristics of a given barrier may vary according to which face of the barrier is initially exposed to the fire.

Associated with the target and each of the other volumes is a number of routes, or paths, each of which defines a unique course which the fire may take from the ignition volume to the target volume, or from the source to the sink in network terminology. The number of links in a path from an ignition volume to the target is here referred to as the 'length' of the path, whilst the time taken for a fire to reach the target along a particular path is referred to as the 'travel time'. In general, the travel time of each path from a given source to the target may be calculated by summing the breach times associated with each arc in the path.

Thus for the structure depicted in Figure 1.3 which, for simplicity, is symmetrical in the sense that each barrier has a fixed failure time regardless of the face which is initially threatened by fire, the shortest-distance path from volume 1 to the target is of length 4.

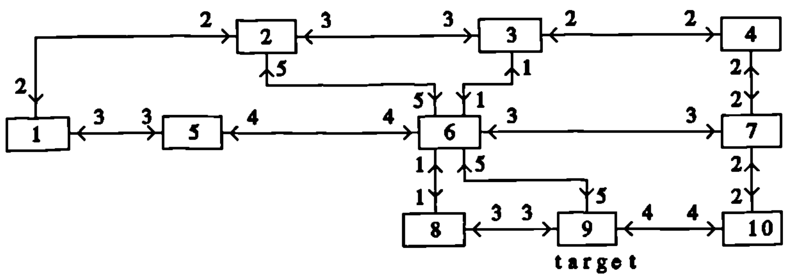


Figure 1.3 An example structure with constant barrier breach times displayed.

There are two such shortest-distance paths,

1) 1-2-6-9 and

2) 1-5-6-9.

The shortest-time path, however, is coincident with neither of these, but is 1-2-3-6-8-9 which is travelled in 10 units of time.

In the case of a stochastic network, in which the arc lengths are not constants but random variables, it becomes particularly important to emphasize the distinction between a shortest-length path (one with the smallest number of links), and a shortest-time path, since the identity and travel-time of the shortest-time path will vary with the values taken by the random variable.

1.4.2 Adjacency Matrix Representation

The network representation of a structure is not the only device which is used to assist in the safety assessment of a structure. One other often-used description is provided by an **adjacency matrix**, an example of which is given in Figure 1.4. The presence of a '1' in the matrix indicates that two volumes are adjacent, whilst a '0' indicates that they are not.

Volume	1	2	3	4	5	6	7	8	9	10
1	\	1	0	0	1	0	0	0	0	0
2		\	1	0	0	1	0	0	0	0
3			\	1	0	1	0	0	0	0
4				\	0	0	1	0	0	0
5					\	1	0	0	0	0
6						\	1	1	1	0
7							\	0	0	1
8								\	1	0
9									\	1
10										\

Figure 1.4 Adjacency Matrix for the structure of Figure 1.1

Clearly any adjacency matrix can be made more informative by replacing the simple 0/1 disjoint/connected classification with some more detail; - an example of this being the inclusion of barrier breach probabilities or fire-resistances.

1.5 Example Structure

The purpose of this research has been to tackle a practical problem arising from a specific need. It is therefore appropriate to take as a point of reference a realistic structure which encompasses all of the essential features of the structures on whose fire properties this work is intended to shed light. Figure 1.5 shows the plan of a structure which is similar to one for which the AEA made use of ARSSUN to carry out a fire safety assessment.

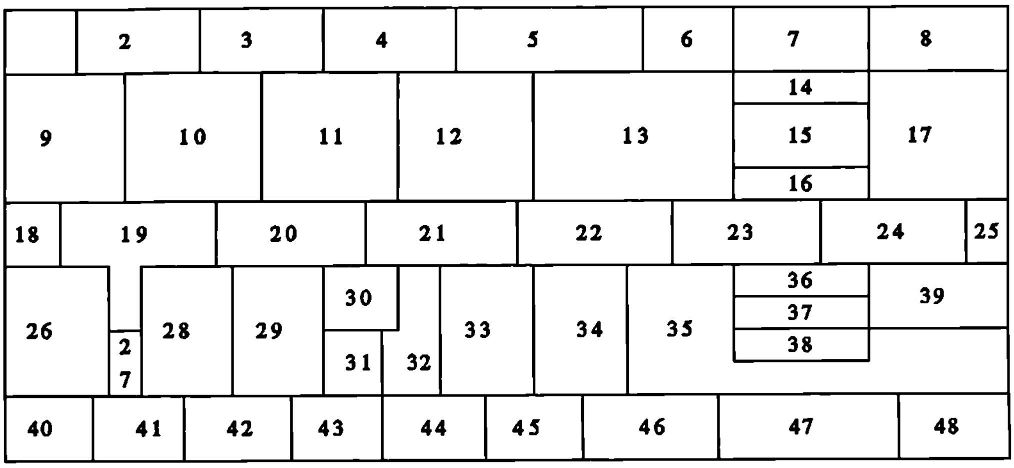


Figure 1.5 The floor-plan of a forty-eight volume structure.

CHAPTER 2

Some Related Stochastic Modelling Problems

Two roads diverged in a wood, and I -
I took the one less traveled by, ...

Robert Frost

2.1 Introduction

During the course of this research, especially in the earlier stages, the fire-spread modelling problem appeared to have much in common with other statistical modelling problems arising in a variety of research fields. These include, for example, problems in the analysis of spatial pattern or in the study of the spread of disease through a population or through an individual's cells. This Chapter provides a brief discussion of some models from these apparently related fields and explores their suitability as useful tools for the problem specified in Chapter 1.

2.2 Lattice Structures

Much use has been made of the concept of a lattice, - a regular set of points each equidistant from some number of its neighbours. A great deal of theoretical research has been published, upon which foundation rests a wide range of applications. Besag in particular, in papers published in the early 1970s (Besag (1972), Besag (1974)), and Bartlett (1971, 1974) contributed a great deal to the development of the theory underlying lattice modelling.

In the fire-spread application, if the volumes of a structure may be represented as the points on a square lattice, there is much to be gained by way of path identification and route-length determination. This is especially the case for shortest-length paths from an ignition volume to a target. Theorem 2.1 and its corollary below may not be original but I have not found the result stated elsewhere.

Theorem 2.1:

The length, n, of the shortest-length path between an ignition volume and a target is calculated as a function of their k-dimensional $(1 \leq k \leq 3)$ cartesian co-ordinates $\{(x_{1I}, x_{2I}, x_{3I})$ and $(x_{1T}, x_{2T}, x_{3T})\}$ on the lattice representation of the structure to be

$$n = \sum_{j=1}^{k} |x_{jI} - x_{jT}|.$$

Proof:

The proof is obvious from the two-dimensional case, - since 'diagonal' movement is not allowed, the total distance which the fire must travel to reach the target is the sum of the 'x' distance and the 'y' distance - ie $|x_I - x_T| + |y_I - y_T|$.

Corollary:

The *number* of shortest paths from an ignition volume to a target is similarly a function of their cartesian co-ordinates. In the two-dimensional case, the number of shortest routes may be determined from reference to a symmetrical rectangular version of Pascal's triangle. This is well illustrated by reference to Figure 2.1.

y: \ x:	0	1	2	3	4	5
0	0	1	1	1	1	1
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	10	20	35	56
4	1	5	15	35	70	126
5	1	6	21	56	126	252

Figure 2.1 A rectangular version of Pascal's triangle used to illustrate the number of shortest-length paths from one point to another on a 2-D square lattice.

Since the distance between any two volumes is dependent only on their relative positions and not on their absolute positions, the target volume may be considered to be at the point (0, 0). For any ignition volume in the same row (or column) as the target, there is only one shortest-length path - the one passing through all volumes which lie on that row (column) between the ignition volume and the target. The calculation of the number of shortest-length paths from other ignition volumes to the target is thereafter done recursively, since the number of shortest paths from any given point (x, y) to the point (0, 0) is simply the sum of the numbers of shortest paths from the two adjacent points (x-1, y) and (x, y-1).

In the three-dimensional case the distances may be calculated in the same way and a cuboid table of shortest path lengths may be constructed. Once again the target volume is considered to be at the origin. The method of illustration used here consists of the display of a series of tables, each successive table representing a unit increase in distance from the origin in the third (z) dimension. The tabulated values are the number of shortest-length paths to the origin from a point with the given co-ordinates. The value at the point (x, y, z) is calculated as the sum of the values at (x-1, y, z), (x, y-1, z) and (x, y, z-1). The first table in the series corresponds to z=0 and it contains values identical to those in the table in Figure 2.1.

1) z=0:

y \ x	0	1	2	3
0	0	1	1	1
1	1	2	3	4
2	1	3	6	10
3	1	4	10	20

2) z=1:

y \ x	0	1	2	3
0	1	2	3	4
1	2	6	12	20
2	3	12	30	60
3	4	20	60	140

3) z=2:

y \ x	0	1	2	3
0	1	3	6	10
1	3	12	30	60
2	6	30	90	210
3	10	60	210	560

4) z=3:

y \ x	0	1	2	3
0	1	4	10	20
1	4	20	60	140
2	10	60	210	560
3	20	140	560	1680

Figure 2.2 Tables of the numbers of shortest-length paths from the origin to various points on a 3-D square lattice.

Were the matter to be pursued, more concise tables could be produced by noting that the number of shortest-length routes from (x, y, z) is the same as the number from (x, z, y), (y, x, z), etc.

No such simple formulation arises, even in the two-dimensional case, when consideration is given to paths of length n+k ($k \geq 1$). The reason for this is that whilst no shortest length path will stray outside the rectangle defined by the set of four corner points $\{(x_I, y_I), (x_I, y_T), (x_T, y_I), \text{and } (x_T, y_T)\}$, many paths of length n+k will involve lattice points lying outside that rectangle.

The main objection to the use of a lattice arises when one reflects upon the feasibility of representing as a lattice the volumes of a segregated structure. It is self-evident that any such structure may, with the possible addition of some dummy volumes or partitions, be represented as a square lattice. The function of such a model is limited, since any of the usual 'rules' which may be applied to a complete lattice would not necessarily apply to a lattice which consisted of dummy nodes or points with missing neighbours. The gains made by transforming the structure into a neat, regular model are thus likely to be outweighed by the necessity of checking any of the assumptions which could normally be applied to that model.

2.3 Markovian Models

Markov processes, such as the simple random walk (see, for example, Feller (1968)), have found favour in a number of fields. The requisite feature is that at each 'step' of the process, knowledge only of the present state of the process (and not of any past states) is relevant in the prediction of the future behaviour of the process. The conjecture that the application of Markovian ideas may offer some assistance in the solution of the fire-spread modelling problem of this thesis is an attractive one. It is especially so if it is considered that a fire may, at each 'step' - in this case a discrete time interval - spread only from its current volume to an adjacent one. The initial complications arise first because the fire may in fact spread to more than one adjacent volume, and second because the fire may not spread to (ie revisit) a volume which it has once occupied. This second factor holds regardless of whether the fire is still burning in the volume, has burnt itself out or been otherwise extinguished, and that alone is sufficient to render inappropriate the 'memoryless' quality specified in a Markov process.

A related process is a branching process, application of which would allow a fire to spread from one volume to one or more adjacent volumes. Working with a discrete-time framework, branching process ideas could certainly be used as a basis for modelling the number of volumes aflame at any one time, but the fire-spread problem requires the additional information concerning the relative locations of each volume. Furthermore, the branching probabilities, describing the probabilities of fire breaching barriers to spread to adjacent volumes, would not in general be constant either over all volumes or over all discrete time periods.

2.4.1 Models of the Spread of Epidemics

Bailey (1975) states that the first published work on mathematical models of the spread of an infectious disease may be attributed to Daniel Bernoulli. It seems that in the time from the reading of that paper, 1760, until the mid-1950s, not a great deal had been published on the subject, but thereafter interest in the field began to grow and this application now receives quite some attention.

The apparent relationship with the modelling of fire spread is at once appealing and obvious. In each model attention is focused on a number of objects which, initially, are in a disease-free or fire-free condition, and which may subsequently become infected by disease or consumed by fire. Furthermore, it is often the case in a disease process that once an individual has been infected, and undergone a period of being infectious, that individual may not be infected again - either because immunity has been conferred by having had the disease, or because the individual has died. That situation finds echoes in the fire-spread modelling problem, for in this case once a fire is established in any volume it will continue to burn until it is extinguished or until there is nothing left to provide its fuel.

The fundamental difference between the two applications rests in the mobility, or otherwise, of the susceptible (never been on fire) and the infected/infectious (currently on fire) individuals. In the infectious disease application, the individuals generally form a unit, or group, (often closed) with infectious and susceptible individuals encountering one another according to some mixing rule (Bailey (1975)). The mixing rule is generally constant for a particular group of individuals over time, and the probability of a susceptible individual becoming infected is a function of the numbers of each class in the group, and the mixing rule to which they are subject, and it is the numbers in the two classes which change over time. For a segregated structure, the probability of fire reaching a volume in which it did not start is a function of the current location of the fire and the types of barrier in the structure. The volumes do not move with respect to one another, so there is no mixing rule and once ignition has first occurred, each volume may only catch fire if the fire breaches a barrier held in common with an adjacent volume which is already aflame.

2.4.2 Contact processes

Allied to the models of disease spread are contact processes. Mollison (1977) discusses a 'simple birth process' in which each individual is assigned, at 'birth', a location some distance from its 'parent'; the distance being a random variable following a particular probability distribution. Once again the emphasis in the model is not appropriate to the fire-spread case for similar reasons to those given in the previous subsection. The 'birth' event in fire spread represents the breaching of a barrier and the spread of flames to a previously un-visited volume. The distance of each new-born individual from its parent must, under the assumptions stated in the Chapter 1, be unity so that the variation arises not in the distance from parent component, but from the uncertainty first about whether the fire will breach a barrier at all, and second about which of the (several) barriers bounding a volume will be breached. Mollison (1978) discusses the related Markovian Contact Processes and cites their disease-spread applications, and similar models found in Bartholomew (1982) are applied to social and labour mobility.

2.5 Summary

Whilst there are aspects of these models, and others, which suggest a useful application to the fire-spread modelling problem, the fact that they are ill-adapted results in all cases primarily from an emphasis inappropriate in the fire-spread case. This emphasis generally takes one of two forms. The first concerns the stochastic modelling - in these models the variation arises in the distance between individuals, or the numbers of progeny, and so on and not as factors readily comparable with barrier breach probabilities or times. The second is perhaps more subtle and concerns that which may be referred to as 'direction of inference'. In the fire-spread problem, the position is such that at any time, t_i, the status (ie aflame, etc) of each volume in the structure is known, and the quantity of interest is the status of any target volumes at time $t_j > t_i$. The probability of a status change from 'never-been-aflame' to 'burning' for any particular volume, i, varies over time. Before the fire has started, the probability for each volume is simply the probability of ignition in that volume. Since the chances of spontaneous ignition in more than volume are considered negligible, once the fire has started, the probability will increase from zero for each volume adjacent to one in which there is a fire. In many other applications, for example Ord (1975) and also Besag's lattice-based work (Besag (1972), (1974)) the intention is either to model the status of an individual at time t, knowing the status of neighbouring individuals at time t, or to model the spread process when the status of all volumes is known at times t_0, ..., t_j, ...

It is almost inevitable that simplifying assumptions are found to be necessary in the early stages of a mathematical modelling process. The construction of a new model allows greater freedom over the sorts of assumption that are made than does attempting to fit an existing model to a new modelling problem. The assumptions underlying the work developed in the next chapters are considered to be those which have the least impact on the integrity of the practical problem to which solutions are sought.

CHAPTER 3

Discrete-time Modelling of Fire Spread Through a Structure

A little fire is quickly trodden out,
Which being suffered, rivers cannot quench.

Shakespeare

3.1 Introduction

In this Chapter, attention is focused on modelling the spread of fire through structures whose barrier breach times are considered to be discrete random variables. A number of time-dependent models are developed and some comparison is made with the work of other authors. Since the structures' barrier breach times are random variables, these structures may be considered as stochastic networks. Proposed solutions to stochastic network problems have been given by, for example, Mirchandani (1976), Frank (1969) and Kulkarni (1986). Their work will be discussed later in the thesis.

3.2 Multinomial Models

The model considered here is based upon the allocation to each barrier of a discrete, multinomial breach-time probability distribution. In its simplest case, this model reduces to a deterministic one as demonstrated in sub-section 3.2.1 below. Wherever necessary, reference will be made to the simple structure whose network representation is shown in Figure 3.1.

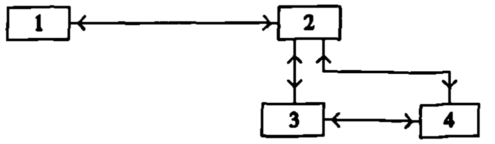


Figure 3.1 A simple example structure with four volumes.

3.2.1 Case 1

All barriers are identical and each barrier is bound to fail in exactly 1 unit of time. Thus

$$\Pr\{\text{fire breaches a given barrier in time } t=1\} = 1,$$

and the shortest-time path from any source to the target is simply that path (or those paths) having the fewest links, and is therefore also the shortest-length path. In the example of Figure 3.1, the shortest path is readily seen to be 1-2-4, with two arcs, so that the time to travel from ignition volume to target is $t=2$. In general, if there were n arcs in the shortest route from ignition volume to target, the traversal time would be $t=n$. Then, in the analysis of the routes to the target from each possible ignition volume, i, no paths other than the shortest ones of length n_i need be considered since the fire is bound to arrive at the target in exactly n_i units of time.

If the ignition volume is known to be volume i, the probability of arrival at the target within a time t may be seen to depend solely upon whether the value of t exceeds the number of barriers in the shortest path from i to the target. More specifically,

$$\Pr\{\text{fire arrives at target in time } T\leq t \mid \text{starts in volume } i\} =$$

$$\delta_i = \begin{cases} 1 \text{ if } n_i \leq t \\ 0 \text{ if } n_i > t. \end{cases}$$

For the structure as a whole, if N is the total number of volumes, and

P_i = Pr{ignition in vol i | ignition somewhere in the structure}

(i.e. P_i is a conditional ignition probability and $\sum P_i = 1$),

it follows that

Pr{fire reaches target in time $T \le t$ | ignition in the structure} =

$$\sum_{i=1}^{N} \delta_i P_i .$$

<u>Example</u>

For the illustrative structure shown in Figure 3.1, with conditional ignition probabilities P_1=0.6, P_2=0.3, P_3=0.08 and P_4=0.02, the distribution of arrival times both following ignition in a specific volume, and following ignition in an unspecified volume, are as given in Table 3.1.

Ignition Volume	Time	Probability	Comment
1	<2	0	
	$=2$	1	
2	<1	0	
	$=1$	1	
3	<1	0	
	$=1$	1	
4	$=0$	1	
Unspecified			
	0	0.02	Ign. in target volume.
	1	0.38	Ign. in volume 2 or 3.
	3	0.6	Ign. in volume 1.

Table 3.1 Distribution of times to arrival at the target following ignition anywhere in the structure - case 1.

3.2.2 Case 2

All barriers are identical and each barrier is bound to fail in at most 2 units of time. Thus

Pr{fire breaches a given barrier in time t=1} = p_1

Pr{fire breaches a given barrier in time t=2} = p_2 = $1-p_1$.

For the given target and any other ignition volume i, there will be one or more shortest paths of length n_i from the source to the target, so that the minimum possible travel time is n_i, and the maximum possible travel time is $2n_i$. The distribution of time to arrival at the target along a shortest path from any given ignition volume i may then be constructed as shown in Table 3.2.

No. time units	Probability	Comment
n_i	$\binom{n_i}{0} p_1^{n_i}$	travel all arcs in time t=1
n_i+1	$\binom{n_i}{1} p_1^{n_i-1} p_2^1$	travel any 1 arc in time t=2 & remainder in time t=1
n_i+2	$\binom{n_i}{2} p_1^{n_i-2} p_2^2$	travel any 2 arcs in time t=2 & remainder in time t=1
⋮		
n_i+j	$\binom{n_i}{j} p_1^{n_i-j} p_2^j$	travel any j arcs in time t=2 & remainder in time t=1
⋮		
$2n_i$	$\binom{n_i}{n_i} p_2^{n_i}$	all arcs travelled in time t=2.

[The notation $\binom{n}{j}$ is used to denote $\frac{n!}{(n-j)!j!}$.]

Table 3.2 Probabilities of times to arrival at the target along a shortest path from the ignition volume - case 2.

Thus the distribution of travel time along a shortest path may be written as $n_i + X$, where X has a binomial, $B(n_i, p_1)$, distribution.

It is, however, not necessarily the case that the fire will reach the target by breaching as few barriers as possible (ie along a shortest path); indeed it is possible for the fire first to reach the target by travelling along any of the paths whose length is less than twice that of the shortest.

If consideration be given to all the possible routes from a given ignition volume to the target, and if $P_{ij}(t)$ be defined as Pr{reach target in time t from vol i along a path of length j}, the expressions for the probabilities of fire first reaching the target in times n_i+m $(0 \leq m \leq n_i)$ may be seen to be as given in Table 3.3.

By way of example, it can be seen that in order for the fire to first reach the target in the shortest possible time, n_i, it must travel along the shortest route as quickly as possible and each of the n_i barriers must be breached in one unit of time. Similarly, if the fire is to first reach the target in time n_i+1, it must either travel the shortest path in time n_i+1, so that one of the n_i barriers is breached in time $t=2$, and the rest in time $t=1$; or it must travel a path of length n_i+1 breaching each barrier in time $t=1$ *having not first arrived by travelling a path of length* n_i. For the general arrival time, n_i+m $(m \leq n_i)$, all routes of length $1 \leq n_i+m$ must be considered, and in each case allowance must be made for the possibility of fire having previously arrived by a shorter route in time τ, $n_i \leq \tau \leq n_i+m$.

No. time units	Probability of 1st arrival at target in specified time
n_i	$P_{in_i}(n_i)$
n_i+1	$P_{in_i}(n_i+1)+P_{i(n_i+1)}(n_i+1)\{1-P_{in_i}(n_i)-P_{in_i}(n_i+1)\}$
n_i+2	$P_{in_i}(n_i+2)\{1-P_{i(n_i+1)}(n_i+1)\}$ + $[\{1-P_{in_i}(n_i)-P_{in_i}(n_i+1)-P_{in_i}(n_i+2)\}P_{i(n_i+1)}(n_i+2)]$ + $[\ \{1-P_{in_i}(n_i)-P_{in_i}(n_i+1)-P_{in_i}(n_i+2)\}\ x\ \{1-P_{i(n_i+1)}(n_i+1)-P_{i(n_i+1)}(n_i+2)\}P_{i(n_i+2)}(n_i+2)\ ]$
⋮	
n_i+m	See below
⋮	
$2n_i$	$P_{in_i}(2n_i)\prod_{r=1}^{n_i-1}\left\{1-\sum_{k=r}^{n_i-1}\left[P_{i(n_i+r)}(n_i+k)\right]\right\}$

Table 3.3 Probabilities of times to arrival at the target from a specified ignition volume - case 2.

The probability $P(n_i+m)$ corresponding to the general arrival time n_i+m may be derived as follows:

$$P(n_i+m) =$$

$$P_{in_i}(n_i+m)\prod_{r=1}^{m-1}\left\{1-\sum_{k=r}^{m-1}\left[P_{i(n_i+r)}(n_i+k)\right]\right\} +$$

$$\left\{1-\sum_{k=0}^{m}\left[P_{in_i}(n_i+k)\right]\right\}P_{i(n_i+1)}(n_i+m)\prod_{r=2}^{m-1}\left\{1-\sum_{k=r}^{m-1}\left[P_{i(n_i+r)}(n_i+k)\right]\right\} +$$

$$\left\{1-\sum_{k=0}^{m}\left[P_{in_i}(n_i+k)\right]\right\}\left\{1-\sum_{k=1}^{m}\left[P_{i(n_i+)1}(n_i+k)\right]\right\}P_{i(n_i+)2}(n_i+m)\ x$$

$$\prod_{r=3}^{m-1}\left\{1-\sum_{k=r}^{m-1}\left[P_{i(n_i+r)}(n_i+k)\right]\right\}+\ \ldots \qquad =$$

$$\sum_{s=0}^{m} P_{i(n_i+s)}(n_i+m)\prod_{\substack{r=0\\ r\neq s}}^{m-1}\left\{1-\sum_{k=0}^{mm}\left[P_{i(n_i+r)}(n_i+k)\right]\right\} \qquad (3.1)$$

where $mm = \begin{cases} m-1 \text{ if } r > s \\ m \text{ if } r < s \end{cases}$ and $P_{ij}(t)=0$ if $t<j$.

Care must be taken to ensure appropriate interpretation of sums and products at the extreme values of m.

If $F_{ij}(t)$ is defined to be the distribution function of time taken to reach the target from volume i along a path of length j, equation (3.1) may be expressed as

$$\sum_{s=0}^{m} P_{i(n_i+s)}(n_i+m)\prod_{\substack{r=0\\ r\neq s}}^{m-1}\left\{1-F_{i(n_i+r)}(n_i+mm)\right\}. \qquad (3.2)$$

Note that:-

1) The $P_{i(n_i+s)}(n_i+m)$ terms may be calculated with reference to the contents of Table 3.2 - even though Table 3.2 was constructed specifically to demonstrate the shortest-length path probabilities,

it also guides the calculation of the probabilities associated with longer routes.

2) The distribution function for a particular ignition volume is
Pr{fire arrives at target from vol. i in time $T \leq n_i + u$} =

$$\sum_{m=0}^{u} \sum_{s=0}^{m} P_{i(n_i+s)}(n_i+m) \prod_{\substack{r=0 \\ r \neq s}}^{m-1} \left\{ 1 - \sum_{k=0}^{mm} \left[P_{i(n_i+r)}(n_i+k) \right] \right\}.$$

3) For the structure as a whole, when the ignition volume is not specified in advance,
Pr{fire arrives at target from vol. i in time $T \leq t$} =

$$\sum_{i=1}^{\#vols} \left\{ \left[\sum_{m=0}^{u_i} \sum_{s=0}^{m} P_{i(n_i+s)}(n_i+m) \prod_{\substack{r=0 \\ r \neq s}}^{m-1} \left\{ 1 - \sum_{k=0}^{mm} \left[P_{i(n_i+r)}(n_i+k) \right] \right\} \right] \right\} P_i \qquad (3.3)$$

where all terms are as defined previously, and $u_i = t - n_i$.

Example

For the illustrative structure shown in Figure 3.1, with conditional ignition probabilities $P_1=0.6$, $P_2=0.3$, $P_3=0.08$ and $P_4=0.02$; and with all barriers having identical breach-time distributions so that $p_1=0.3$ and $p_2=0.7$, the distributions of arrival times following ignition both in a specified and an unspecified volume are given in Table 3.4.

Ignition Volume	Time	Probability	Cumulative Probability
1	<2	0	
	2	0.09	0.09
	3	0.43323	0.52323
	4	0.47677	1
2	0	0	
	1	0.3	0.3
	2	0.7	1
3	0	0	
	1	0.3	0.3
	2	0.7	1
4	0	1	1
Unspecified			
	0	0.02	0.02
	1	0.114	0.134
	2	0.32	0.454
	3	0.259938	0.713938
	4	0.286062	1

Table 3.4 Distributions of times to arrival at the target following ignition anywhere in the structure - case 2.

3.2.3 Case 3

All barriers are identical and each barrier is bound to fail in at most 3 units of time. Thus

$$\Pr\{\text{fire breaches a given barrier in time } t=1\} = p_1$$

$$\Pr\{\text{fire breaches a given barrier in time } t=2\} = p_2$$

$$\Pr\{\text{fire breaches a given barrier in time } t=3\} = p_3 = 1-p_1-p_2.$$

Once again for the given target and any other ignition volume, there will be one or more shortest paths of length n_i from the source to the target, with minimum travel time $t=n_i$; and as each barrier may take up to three time units until failure, the fire is bound to arrive at the target in time t no greater then $3n_i$. The distribution of time to arrival at the target along a shortest path from any given ignition volume i may then be specified as in Table 3.5.

No. time units	Probability	Comment
n_i	$\binom{n_i}{0} p_1^{n_i}$	travel all arcs in time t=1
n_i+1	$\binom{n_i}{1} p_1^{n_i-1} p_2^1$	travel any 1 arc in time t=2 remainder in t=1
n_i+2	$\binom{n_i}{2} p_1^{n_i-2} p_2^2 + \binom{n_i}{1} p_1^{n_i-1} p_3^1$	{ travel any 2 arcs in time t=2, remainder in t=1 OR travel any 1 arc in time t=3, remainder in t=1
⋮		
n_i+j	$\sum_{k=0}^{[\frac{j}{2}]} \frac{n_i!}{(n_i-j+k)!(j-2k)!k!} p_1^{(n_i-j+k)} \cdot p_2^{(j-2k)} \cdot p_3^{k}$	
		$[\frac{j}{2}] = \begin{cases} \frac{j}{2} & \text{if j even} \\ \frac{j-1}{2} & \text{if j odd} \end{cases}$

Table 3.5 Probabilities of times to arrival at the target along a shortest path from the ignition volume - case 3.

If consideration be paid to all the possible routes from a given ignition volume to the target, and if $P_{ij}(t)$ be defined as before, the probability distribution has precisely the form of that given in Table 3.3 of the previous sub-section. Furthermore, it is once again a straightforward matter to write down the distribution function for the structure as a whole as it is identical to the corresponding expression given as equation 3.3 above.

Example

In the example structure, suppose that the ignition probabilities are as given before, and that for each of the four identical barriers $p_1=0.2$, $p_2=0.5$ and $p_3=0.3$. Table 3.6 shows the distributions of arrival times following ignition both in a specified and an unspecified volume.

Ignition Volume	Time	Probability	Cumulative Probability
1	<2	0	
	2	0.04	0.04
	3	0.20608	0.24608
	4	0.39044	0.63652
	5	0.29634	0.93286
	6	0.06714	1
2	0	0	
	1	0.2	0.2
	2	0.512	0.712
	3	0.288	1
3	0	0	
	1	0.2	0.2
	2	0.512	0.712
	3	0.288	1
4	0	1	1
Unspecified	0	0.02	0.02
	1	0.076	0.096
	2	0.21856	0.31456
	3	0.233088	0.547648
	4	0.234264	0.78192
	5	0.177804	0.959716
	6	0.040284	1

Table 3.6 Distributions of times to arrival at target following ignition somewhere in the structure - case 3.

3.2.4 The General Case, Case r

These models may be developed by allowing a greater number of failure periods for each barrier, so that in general Case r may be discussed, where,

$$\Pr\{\text{fire spreads to neighbouring vol in time } t=j\} = p_j, \; j=1, 2, \ldots r,$$

and $\sum_{j=1}^{r} p_j = 1.$

It is, however, unlikely to be necessary to have r larger than perhaps five or six since the model would become cumbersome and the problem would be more naturally handled using continuous breach-time distributions.

The derivation of the arrival-time distribution from a specified ignition volume to the target is perhaps best done in two parts. The first stage consists of the identification of, and the calculation of probabilities pertaining to, distinct routes to the target. The complexity of the probability calculations increases with the value of r, and whilst a general formula is not included here, the general method of computation for a particular value of r is as given in Algorithm 3.1.

Algorithm 3.1

Having identified all the paths from ignition volume i to the target; for each path of length j, and for each value of t of interest,

1) Enumerate all possible combinations of barrier breach times which result in the path travel-time being t.
2) Calculate the probability of each combination in (1) using the probability function of the Multinomial distribution:-

$$f_T(t_1, \ldots, t_s) = n! \prod_{i=1}^{s} \frac{p_i^{t_i}}{t_i!} .$$

3) Sum the probabilities calculated in (2), to give expressions for the $P_{ij}(t)$.

The enumeration problem posed as (1) in the above algorithm may be expressed as follows:-

How may r non-negative integers, a_T, be selected (with replacement) so that

i) $\sum_{T=1}^{r} a_T = j$ and

ii) $\sum_{T=1}^{r} T.a_T = t$?

Here a_T denotes the number of barriers in the selected path which are breached in exactly time t=T. Thus the first constraint represents the requirement that there be j barriers in the path whilst the second ensures that the path travel time, evaluated as

$$\sum_{T=1}^{r} \left\{ \left[\text{\# barriers breached in time } t=T \right] . \left[T \right] \right\}$$

is equal to t as required.

A solution to the enumeration problem may therefore be found as the solution to the pair of simultaneous equations (i) and (ii) above, with the additional constraint that any a_T which satisfy the equations must be integers.

The second stage comprises the combination of the probabilities calculated in stage one, for which the method is the same whatever the value of r, being that as given in equation 3.1.

3.2.5 Extension to allow Non-breaching of Threatened Barriers

A further generalization of practical significance may be obtained by considering a modified Case r in which the probability p_{∞} is defined to represent the probability of a fire failing to breach the barrier at all. Such a modification results in this model more closely resembling a development of ARSSUN for which 'small' (typically of the order of 10^{-2} - 10^{-4}) time-independent barrier breach probabilities are generally supplied.

The most complex part of the model stems from its reliance on the identification of independent routes from ignition volume to target. In practice the way forward is as expressed in general terms in Algorithm 3.2.

Algorithm 3.2

For each potential ignition volume,

1) Identify all (feasible) routes to the target.

2) Partition the structure into a number of connected sub-structures so that each route through any given sub-structure has no barriers in common with any other route through that sub-structure.

3) Identify a new set of 'routes' from ignition volume to target - this time as routes through sub-structures rather than individual volumes.

4) Work backwards through the structure - from target to ignition volume - along each of the new routes, identifying as entry and exit volumes in each sub-structure those volumes which are common to that sub-structure and its neighbours on that new route.

5) Derive for each sub-structure the distribution of time to travel from entry to exit volumes.

6) Combine the distributions obtained in (5).

The algorithm is illustrated by the following example.

Example

For the example structure, let the conditional ignition probabilities be as before, and let r=2, $p_1 = 0.01$, $p_2 = 0.03$ and $p_\infty = 0.96$.

The potential ignition volumes and associated routes are given in Table 3.7.

Ign Vol	Route number	number of arcs	Volumes on route
1	1.1	2	1 2 4
	1.2	3	1 2 3 4
2	2.1	1	2 4
	2.2	2	2 3 4
3	3.1	1	3 4
	3.2	2	3 2 4
4	4.1	0	4

Table 3.7 Summary of basic information for the example.

If fire breaks out in volume 4, it reaches the target in time 0 with probability 1, and the contribution to the arrival time distribution from volume 4 is clearly

Pr{reach target in time 0} =

1 x Pr{ignition in volume 4} = 0.02,

whilst Pr{reach target in time$>0|$ignition in volume 4} = 0.

From volume 3 there are two independent routes to the target, the first of these is of length 1 and the distribution of time to arrival along that route, conditional on ignition in that volume, is

Pr{reach target in time t=1} = $P_{31}(1)$ = 0.01

Pr{reach target in time t=2} = $P_{31}(2)$ = 0.03

whilst for the second

$P_{32}(1) = 0$

$P_{32}(2) = 0.0001$

$P_{32}(3) = 0.0006$

$P_{32}(4) = 0.0009$

Similarly, for volume 2 as ignition volume,

$P_{21}(1) = 0.01$ $\quad$ $P_{21}(2) = 0.03$ $\quad$ $P_{22}(1) = 0$

$P_{22}(2) = 0.0001$ $\quad$ $P_{22}(3) = 0.0006$ $\quad$ $P_{22}(4) = 0.0009$

There are two routes from volume 1 to the target, and they share a common arc, so it is necessary to partition the structure unto two sub-structures. The first of these is a sub-structure containing only volume 1, whilst the other contains the remaining volumes. The only link between the sub-structures is provided by the barrier separating volume 1 from volume 2, so that the entry and exit volumes for the first sub-structure are each volume 1, and for the second are volume 2 and volume 4 respectively. The calculations concerning fire reaching the target from volume 2 have been made above, so all that is needed is some adjustment for the link between the two sub-structures.

The distribution appropriate to fire starting in volume 1 may be seen to be:

$P_{12}(1) = 0$

$P_{12}(2) = 0.01 \times P_{21}(1) = 0.0001$

$P_{12}(3) = 0.01 \times P_{21}(2) + 0.03 \times P_{21}(1) = 0.0006$

$P_{12}(4) = 0.03 \times P_{21}(2) = 0.0009$

$P_{13}(1) = 0$

$P_{13}(2) = 0$

$P_{13}(3) = 0.01 \times P_{22}(2) = 0.000001$

$P_{13}(4) = 0.01 \times P_{22}(3) + 0.03 \times P_{22}(2) = 0.000009$

$P_{13}(5) = 0.01 \times P_{22}(4) + 0.03 \times P_{22}(3) = 0.000027$

$P_{13}(6) = 0.03 \times P_{22}(4) = 0.000027$

Before the final probability distribution is calculated, the probabilities enumerated above must be aggregated as described earlier. The results of that process are shown in Table 3.8.

Ignition Volume	Time, t, after ignition, to first reach the target	Probability - Pr{T=t|ign in vol i}
1	2	0.0001
	3	0.000609993
	4	0.000908947
	5	0.0000269568
	6	0.0000269568
	∞	0.9983271087
2	1	0.01
	2	0.030096
	3	0.000576
	4	0.000864
	∞	0.958464
3	1	0.01
	2	0.030096
	3	0.000576
	4	0.000864
	∞	0.958464
4	0	1
	∞	0

Table 3.8 Aggregated travel time probabilities for the example.

The probabilities calculated above may then be multiplied by the corresponding conditional probabilities of ignition in each volume to yield the complete probability distribution as shown in Table 3.9.

Time, t, after ignition, to first reach the target	Probability - Pr{T=t}	Pr{T≤t}
0	0.02	0.02
1	0.0038	0.0238
2	0.011496860	0.035296860
3	0.0005848758	0.03588173580
4	0.0008737104	0.03675544620
5	0.00001617408	0.036771620280
6	0.00001617408	0.036787794360
fails to reach	0.96321220564	1

Table 3.9 Probability distributions for the example structure.

Appreciation of numerical results such as these is simplified by graphical illustration. Figure 3.2 overleaf shows the probability function as given in Table 3.9, with 'F' being used to indicate the event of fire not reaching the target. On the same axes is shown a similar graphical representation of the results which are obtained when the ARSSUN program is run on the same structure with the same ignition probabilities and the requisite Bernoulli B(1, 1-p_{∞}) breach distribution for each barrier. The ARSSUN results are classified in Figure 3.2 as 'R', for 'reaches the target' and 'N', for 'does not reach the target' and the probability axis is given on a logarithmic scale so as to facilitate the depiction of the small probabilities. The output from ARSSUN is reproduced in Appendix A.

There is a discrepancy in the results in that the probability of fire never reaching the target (and thus the probability of fire reaching the target) are not found to be the same by the two methods (0.03678... for the multinomial and 0.03681 for ARSSUN). This is accounted for by ARSSUN's neglecting of all second- and higher- order effects.

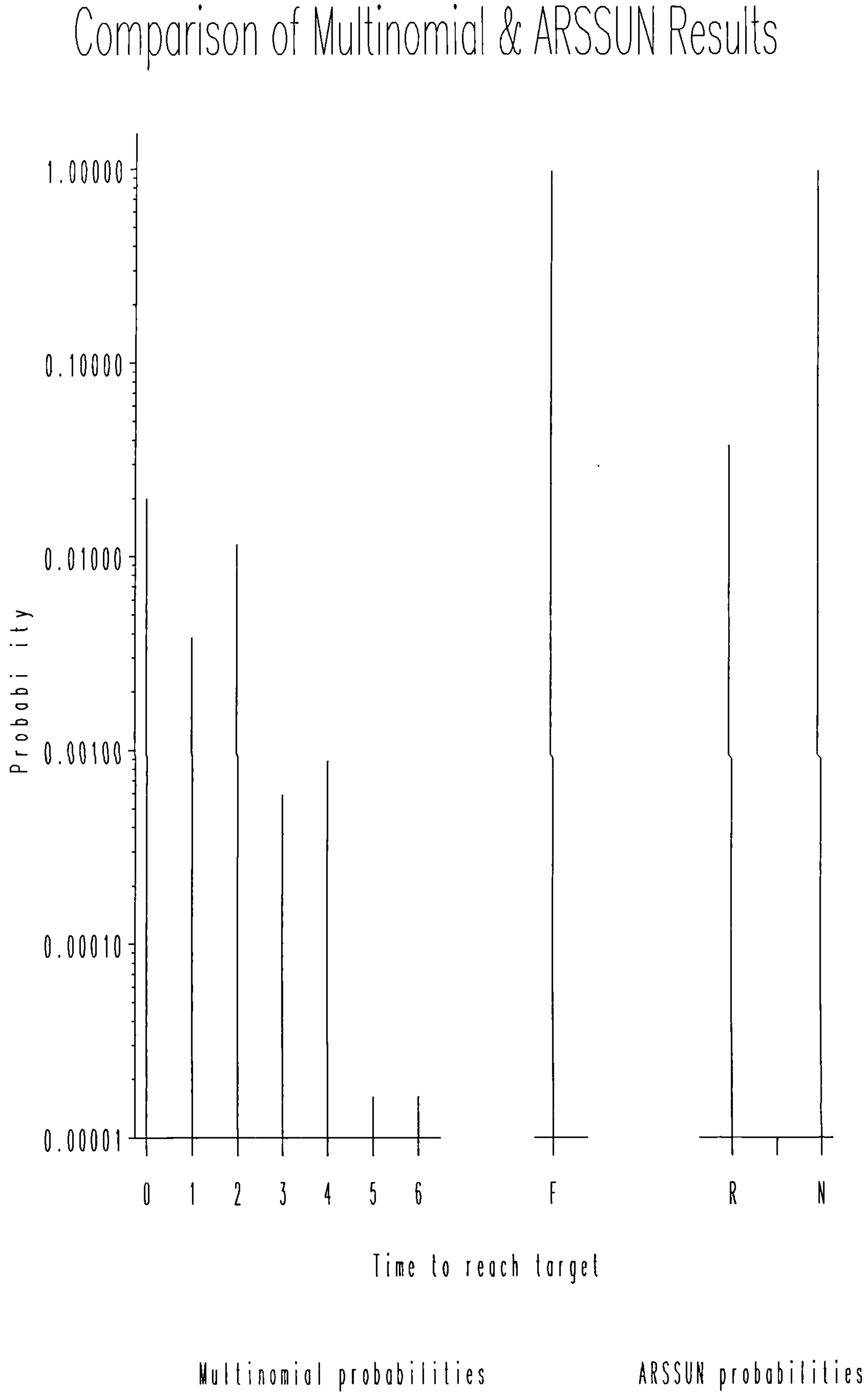


Figure 3.2 Graphical illustration of results for the example.

Once a reliability assessor has access to results such as these, he or she will generally wish to re-run the model with different parameter values representing different barriers in order to establish whether it is possible (usually within cost constraints) to further reduce the risk of fire damage to the contents of the critical target volume. It is here that the importance of a time-based model can be seen very clearly, for it is not just the reach target/fail to reach target probability masses as calculated by ARSSUN which are important, but the shape of the probability function, such as shown on the left hand side of Figure 3.2, which can offer much guidance to the engineers conducting the probabilistic risk assessment.

3.3 Structures with non-identical barriers

The single most restrictive element of these models is the requirement that all the barriers in a structure be considered identical. It is upon this premise that the notion of the application of the multinomial model is built, since this probability distribution arises from the situation in which a number of independent trials are carried out under the condition that at each trial the probability of a particular outcome (for example, burning through a barrier in 1 unit of time) remains the same throughout all trials.

If there are only two or three different types of barrier, which may be characterized as

type A : Pr{fire spreads to neighbouring volume in time $t=i$} = p_{Ai}

type B : Pr{fire spreads to neighbouring volume in time $t=i$} = p_{Bi}

type C : Pr{fire spreads to neighbouring volume in time $t=i$} = p_{Ci}, the model is still tractable. The way forward for each ignition volume, and for each route is to group together the barriers of each type and to consider repeated multinomial trials for the identical barriers in each group. For each route there will then be a number of multinomial expressions such as those shown earlier in the preceding section - for example in Table 3.3; and that number will not exceed the number of different barrier types. The resulting expressions may then be combined to provide an assessment of the structure as a whole, as illustrated by the following example. The chosen structure, shown in Figure 3.3, is slightly different from that used in earlier sections of this chapter as it facilitates a clearer illustration of the techniques.

Example

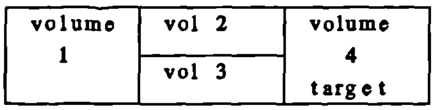


Figure 3.3 A simple four-volume structure.

Using the case r=2, suppose that barriers 1 (vol.1 ↔ vol.2) and 2 (vol.1 ↔ vol.3) are of type A whilst barriers 3 (vol.2 ↔ vol.3), 4 (vol.2 ↔ vol.4) and 5 (vol.3 ↔ vol.4) are of type B,

where,

for type A, Pr{breach in t=1} = p_{A1}, Pr{breach in t=2} = $1-p_{A1}$; and
for type B, Pr{breach in t=1} = p_{B1}, Pr{breach in t=2} = $1-p_{B1}$.

The routes to the target, with possible travel times and corresponding conditional probabilities are as shown in Table 3.10.

Parentheses, {}, are used to indicate theoretically possible travel times which would not arise in practice since any fire starting in the volume indicated would first have reached the target via a different route.

From Vol	Route no.	Via barriers	Possible travel times [and corr. probabilities | ignition in vol]
1	1.1	1,4	$2[p_{A1}p_{B1}]$, $3[p_{A2}p_{B1}+p_{A1}p_{B2}]$, $4[p_{A2}p_{B2}]$
	1.2	2,5	$2[p_{A1}p_{B1}]$, $3[p_{A2}p_{B1}+p_{A1}p_{B2}]$, $\{4[p_{A2}p_{B2}]\}$
	1.3	1,3,5	$3[p_{A1}p_{B1}^2]$, $\{4[2(p_{A1}p_{B1}p_{B2})+(p_{A2}p_{B1}^2)]\}$ $\{5[2(p_{A2}p_{B1}p_{B2})+(p_{A1}p_{B2}^2)]\}$, $\{6[p_{A2}p_{B2}^2]\}$
	1.4	2,3,4	$3[p_{A1}p_{B1}^2]$, $\{4[2(p_{A1}p_{B1}p_{B2})+(p_{A2}p_{B1}^2)]\}$ $\{5[2(p_{A2}p_{B1}p_{B2})+(p_{A1}p_{B2}^2)]\}$, $\{6[p_{A2}p_{B2}^2]\}$
2	2.1	4	$1[p_{B1}]$, $2[p_{B2}]$
	2.2	3,5	$\{2[p_{B1}^2]\}$, $\{3[2(p_{B1}p_{B2})]\}$, $\{4[p_{B2}^2]\}$
	2.3	1,2,5	$\{3[p_{A1}^2p_{B1}]\}$, $\{4[2(p_{A1}p_{A2}p_{B1})+(p_{A1}^2p_{B2})]\}$ $\{5[2(p_{A1}p_{A2}p_{B2})+(p_{A2}^2p_{B1})]\}$, $\{6[p_{A2}^2p_{B2}]\}$
3	3.1	5	$1[p_{B1}]$, $2[p_{B2}]$
	3.2	3,4	$\{2[p_{B1}^2]\}$, $\{3[2(p_{B1}p_{B2})]\}$, $\{4[p_{B2}^2]\}$
	3.3	2,1,4	$\{3[p_{A1}^2p_{B1}]\}$, $\{4[2(p_{A1}p_{A2}p_{B1})+(p_{A1}^2p_{B2}]\}$ $\{5[2(p_{A1}p_{A2}p_{B2})+(p_{A2}^2p_{B1})]\}$, $\{6[p_{A2}^2p_{B2}]\}$
4	4.1	-	0[1]

Table 3.10 Algebraic summary of information for the example.

Pr{fire reaches target in time $T \leq 3$ | starts in vol. 1} =

$$\frac{1}{2}\Big[2p_{A1}p_{B1} + 2(p_{A1}p_{B2} + p_{A2}p_{B1} + p_{A1}p_{B1}^2(1 - p_{A1}p_{B1} - p_{A1}p_{B2} - p_{A2}p_{B1})\Big] =$$

$$p_{A1} + p_{A2}p_{B1} + p_{A1}p_{B1}^2(1 - p_{A1} - p_{A2}p_{B1}).$$

Also, Pr{fire reaches target in time $T \le 4$ | starts in vol. 1} =

Pr{fire reaches target in time $T \le 3$ | starts in vol. 1} +

Pr{fire reaches target in time $T = 4$ | starts in vol. 1} =

$$p_{A1} + p_{A2}p_{B1} + p_{A1}p_{B1}^2(1 - p_{A1} - p_{A2}p_{B1}) +$$

$$p_{A2}p_{B2}(1 - p_{A1}p_{B1}^2) =$$

$$p_{A1} + p_{A2}p_{B1} + p_{A1}p_{B1}^2 - p_{A1}^2p_{B1}^2 - p_{A1}p_{A2}p_{B1}^3 + p_{A2}p_{B2} - p_{A1}p_{A2}p_{B1}^2p_{B2} =$$

$$1 + p_{B1}^2(p_{A1} - p_{A1}^2 - p_{A1}p_{A2}) =$$

$$1 + p_{B1}^2p_{A1}(1 - p_{A1} - p_{A2}) =$$

1.

For a numerical example, let the conditional ignition probabilities in volumes 1 to 4 be 0.8, 0.09, 0.07 and 0.04 respectively, and also let p_{A1}= 0.3 and p_{B1}=0.4, so that p_{A2}=0.7 and p_{B2}=0.6. Then the distribution of arrival times is as given in Table 3.11.

Ignition Volume	Time, t, after ignition, to first reach the target	Probability - Pr{T=t|ign in vol i}
1	2	0.12
	3	0.48016
	4	0.39984
2	1	0.4
	2	0.6
3	1	0.4
	2	0.6
4	0	1
Unspecified	0	0.04
	1	0.064
	2	0.192
	3	0.384128
	4	0.319872

Table 3.11 Aggregated travel time probabilities for the example.

3.4 Existing Approaches to the Solution of Stochastic Networks

Mirchandani (1976) proposed a method of calculating the expected shortest response time for a stochastic network. He describes his method as being composed of three steps as follows:-

1) transform the network to its 'emergency equivalent network' in which each link has an independent Bernoulli probability of operating and a deterministic operating time when the link is operative;
2) determine all possible routes from ignition volume to target;
3) make use of a recursive algorithm to compute both the expected response time and the reliability.

Mirchandani's language is based on that appropriate to communication networks, - thus in the present context the construction 'operative link' should be understood as indicating a barrier which is breached by fire whilst and the reliability of a path is the probability that all the barriers in the path are breached.

The principle is fairly straightforward as the simplified three-step algorithm suggests, but this belies some complexity in practice.

Step 1

A barrier (s, t) which has a discrete breach-time distribution which may be written as

$$\Pr\{T=t\} = \begin{cases} p_k \quad \text{for } t = t_k \text{ , } k = 1, 2, \ldots, r \text{ ; } t_1 < t_2 < \ldots < t_r \\ p_\infty = 1 - \sum_{i=1}^{r} p_i \quad \text{for } t = \infty \end{cases}$$

is represented in the 'emergency equivalent network' by r parallel links (s, k, t), k=1, ..., r; each link having an independent probability ρ_k of being 'travelled' and an associated travel (ie breach) time, so that

$$\rho_1 = p_1$$

$$\rho_k = p_k \left[1 - \sum_{i=1}^{k-1} p_i\right]^{-1}, \; k=2, 3, \ldots, r.$$

Illustration

If consideration be given to a structure fitted with the barriers described in sub-section 3.2.3, with a slight modification so that

$$\Pr\{T=1\} = p_1$$

$$\Pr\{T=2\} = p_2$$

$$\Pr\{T=3\} = p_3$$

$$\Pr\{T=\infty\} = p_4 = 1 - \sum_{i=1}^{3} p_i ,$$

the 'emergency equivalent network' representation of the structure would consist of a number of nodes, of which each pair representing adjacent volumes would be joined by three arcs with breach-times and Bernoulli breach probabilities,

for $t=1, \; \rho_1 = p_1$

$$t=2, \; \rho_2 = \frac{p_2}{1-p_1}$$

$$t=3, \; \rho_3 = \frac{p_3}{1-p_1-p_2} \ldots$$

Thus the number of links in the network representation of a structure whose barriers are all of the type described above will in general increase from m to 3m, whilst the number of routes from any particular ignition volume to target in that structure will increase from M to

$$\sum_{j=1}^{M_L} 3^j ,$$ where M_L is the number of barriers in route M.

Step 2

Identification of all feasible routes from all possible ignition volumes to the target is generally far from simple. It does of course become much more complex when the number of routes increases in the manner indicated above.

Step 3

The recursive algorithm is based upon the 'equivalent network', each of whose arcs has a deterministic travel time and a Bernoulli probability of being travelled. Thus when mention is made of routes, it should be remembered that a number of 'distinct' routes will be distinct as far as the 'equivalent network' is concerned, but will not appear to be distinct upon examination of the original structure. The algorithm is established as follows.

For a given ignition volume, i, and target, suppose that there are K distinct routes from i to the target and let these be denoted by R_j, where $j=1, 2, \ldots, K$. Define E_j to be the event that path R_j is connected - in other words that all the barriers in the path have been breached. Q_m is used to represent the *expression* for the probability that one or more of the paths $R_1, \ldots, R_m$ is connected. Furthermore, $P_{\cup/m}$ is used to denote the conditional probability that any of paths $R_1, \ldots, R_{m-1}$ is connected given that path R_m is connected. Thus

$$Q_m = \Pr\{E_1 \cup E_2 \cup \ldots \cup E_m\}$$
$$P_{\cup/m} = \Pr\{E_1 \cup E_2 \cup \ldots \cup E_{m-1} | E_m\}.$$

If $P_{\cup/o}$, $P_{\cup/1}$, & Q_0 are all defined to be $\equiv 0$, it follows that $P_{\cup/m}$ may be derived from Q_{m-1} by setting equal to unity all arc travel probabilities for those links which are common to both path R_j (for $j=1, \ldots, m-1$) and R_m. The relationship may be written as

$$P_{\cup/m} = \left[Q_{m-1} \right]_{P_m \rightarrow 1}. \tag{3.4}$$

If $\Pr\{R_m$ is connected$\}$ is written as P_m,

$$\begin{aligned} Q_m &= \Pr\{E_1 \cup E_2 \cup \ldots \cup E_m\} \\ &= \Pr\{[E_1 \cup E_2 \cup \ldots \cup E_{m-1}] \cup [E_m]\} \\ &= \Pr\{E_1 \cup E_2 \cup \ldots \cup E_{m-1}\} + \Pr\{E_m\} - \Pr\{[E_1 \cup E_2 \cup \ldots \cup E_{m-1}] \cap [E_m]\} \\ &= Q_{m-1} + P_m - (\Pr\{[E_1 \cup E_2 \cup \ldots \cup E_{m-1}] \mid [E_m]\}.\Pr\{E_m\}) \end{aligned}$$

$$Q_m = Q_{m-1} + P_m[1-P_{\cup/m}] \tag{3.5}$$

It is equations 3.4 and 3.5 which may be used recursively in the derivation of expressions for the $P_{\cup/i}$, $i=2, 3, \ldots$. The paths R_j should be arranged in order of increasing travel times, τ_j. The probability of fire reaching the target conditional upon its starting in a given volume is then

$$R = \sum_{j=1}^{K} P_j[1-P_{U/j}];$$

whilst the expected shortest path response time, conditional upon the fire reaching the target, is

$$\bar{\tau} = \frac{\sum_{j=1}^{K} \tau_j P_j [1-P_{U/j}]}{R}.$$

Ling and Williamson (1985), and prior to that in Castino and Harmathy (1982), made some use of this algorithm in their models of fire spread. Their particular application featured a fairly simple structure including self-closing doors, which, in the event of a fire, may close successfully with a probability p. Thus in the event of a fire, there is a probability p that a door has a fire resistance, R, and a probability 1-p that the door remains open and offers no fire resistance. The solution to that problem is well-addressed by Mirchandani's algorithm since each 'real' link (ie each barrier) is represented by only two in the emergency equivalent network, and the analysis does not become excessively complex.

One way of further investigating Mirchandani's algorithm algebraically is to use a Computer Algebra package. I chose REDUCE (Hearn (1985)) on the IBM mainframe at Liverpool University as a convenient package to use. A simple example structure is shown in Figure 3.4, with the corresponding network and 'equivalent network' in Figure 3.5 and Figure 3.6 respectively. The barriers are identical, each one failing in one, two or three units of time with probabilities p_1, p_2, and p_3 respectively. The probability that a barrier is not breached by fire at all is $1-p_1-p_2-p_3$.

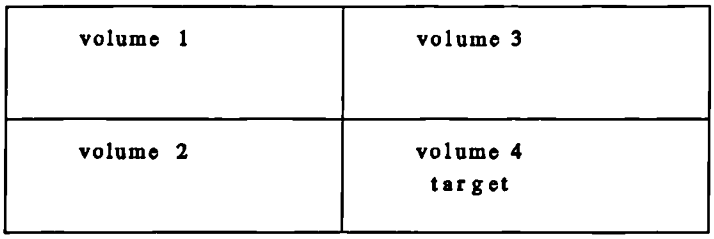


Figure 3.4 A single storey four-volume structure.

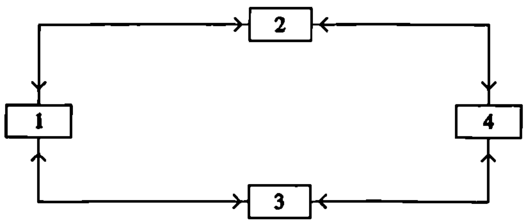


Figure 3.5 The network representation of Figure 3.3.

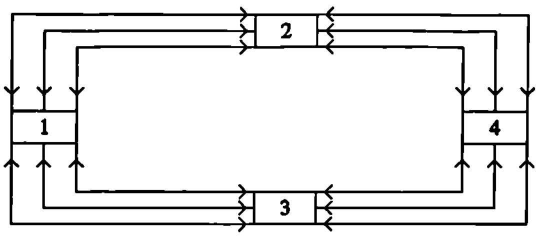


Figure 3.6 The 'emergency network' representation of Figure 3.3.

Corresponding to each link in Figure 3.6 is a Bernoulli distribution and an associated breach-time as given above. Writing these number pairs as $L(t_j, \rho_j)$, it can be seen that joining each pair of adjacent volumes is a trio of links;

$L(1, p_1)$,

$$L(2, \frac{p_2}{1-p_1}), \text{ and}$$

$$L(3, \frac{p_3}{1-p_1-p_2}).$$

The number of non-zero length routes from the possible ignition volumes (1, 2, 3 & 4) to the target is eighteen, thirty, thirty and nought respectively. Each volume was assigned a conditional ignition probability (their sum being unity so that the condition 'given that fire has started somewhere in the structure' is fulfilled).

REDUCE was used to verify that appropriate probabilities summed to unity, as well as to derive algebraic forms for the intermediate results. Some sample input and output is included in Appendix B. The selected output shows results evaluated for the event of fire starting in volume 1. REDUCE may be used both interactively and in batch mode, so in this case a file containing the input code was written and submitted. The printed results are provided as an illustration of the output which REDUCE can produce. The whole output file for this example is about 5000 lines long.

3.5 Conclusion

The new methodology introduced in this Chapter allows the application of a time-dependent model of fire spread in a fairly straightforward and flexible way. The principle results are those summarized in equation 3.3, the contents of Tables 3.2, 3.3, and 3.5, and the Algorithms in sub-sections 3.2.4 and 3.2.5. Sub-section 3.2.5 is especially important in that the particular model presented there is shown to provide an improvement on the ARSSUN method, as is illustrated in Figure 3.2. Whilst the restriction that the barrier breach-times have a discrete probability distribution is certainly a limitation to the models' realism, it is seen to be less so when it is remembered that a barrier's breach characteristics are generally given as a single nominal breach time. The particular values (in minutes) given to each time unit should be selected as those appropriate to the type of barrier under consideration, guidance being available from the relevant Standard Fire-test result. Other discrete probability distributions, such as the Poisson or the Negative Binomial, could also be used as models of barrier breach-time distributions, but this is not pursued further here.

In the next Chapter, consideration is given to a number of continuous distributions, each as possible models for barrier breach-time, which leads to an intuitively appealing solution to the time dependent fire-spread problem.

CHAPTER 4

Continuous-time Modelling of Fire Spread Through a Structure

There are three things that are never satisfied, yea, four things say not, It is enough:
The grave; and the barren womb; and the earth that is not filled with water; and the fire that saith not, It is enough.

Proverbs 30:15.

4.1 Introduction

Despite the appeal of some straightforward mathematical argument and the ensuing algorithms, the depiction of time-since-ignition as a discrete rather than a continuous variable is a somewhat unsatisfactory simplification. Although it is certainly more satisfactory than modelling fire spread without reference to time at all, it cannot represent accurately the real-life spread of a fire through a building. This Chapter is concerned with the development of continuous-time models.

4.2 Distributional Models of Barrier Failure Time

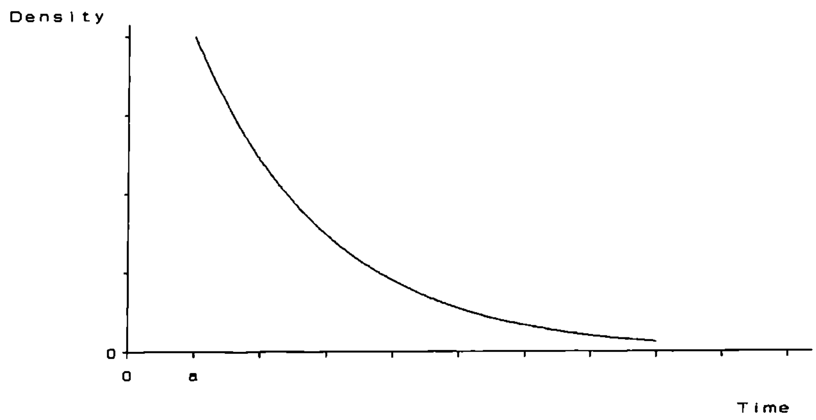
The fundamental difference between the models discussed in this chapter and those presented in Chapter 3 is that the breach-time distributions associated with each barrier are continuous rather than discrete. The fire resistance of a barrier bounding a volume in which there is a fire obviously has an extremely important influence on the time taken for a fire to spread from the burning volume to an adjacent one, and thus from the volume in which the fire starts to any target volume. It is apodictic that any stochastic models of barrier failure must characterize as accurately as possible the real-life response of a barrier to fire, especially as any small inaccuracies in modelling an individual barrier's performance may well be compounded when consideration is given to fire spread through a succession of barriers.

The role of the British Standards on fire resistance ratings of different types of barrier has been discussed in Chapter 1, in which it was mentioned that, for each particular type of element of interest, only one or two 'representative samples' undergo a fire-resistance test. Each fire test is sponsored by the company which manufactured the element and the test results become the property of that company, and are rarely available to the public.

Despite the inadequacy of the B.S.I. test results from a modelling perspective, it is often the case that those results are the most influential when a model of a barrier's breach characteristics is being prepared. Any decision concerning the acceptability of a particular breach-time distribution must be made by a qualified risk assessor who is able correctly to interpret the specifications of that particular model. Reasonable distributions for a barrier rated at α minutes include those in Figure 4.1, for which appropriate additional information may be found in Hastings and Peacock (1975) or Johnson and Kotz (1970).

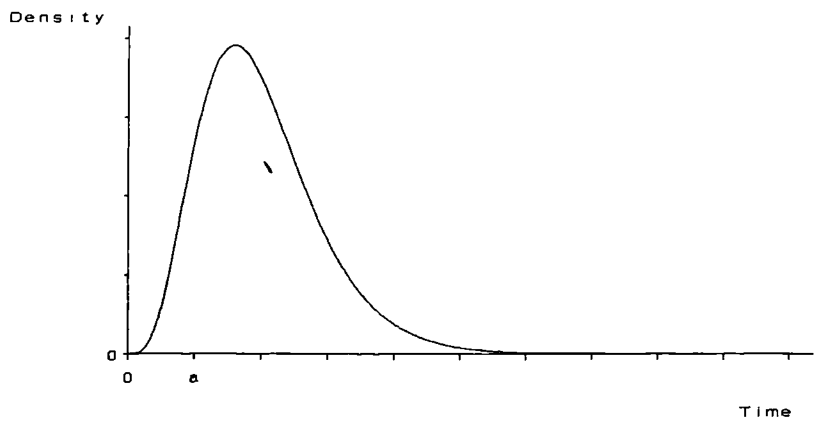
a) Two-parameter Exponential

$$f(t) = \lambda.\exp(-\lambda[t-\alpha])$$



b) Gamma

$$f(t) = \frac{\lambda^{\nu} t^{\nu-1}.\exp(-\lambda t)}{\Gamma(\nu)}$$



c) Weibull

$$f(t) = \frac{\beta}{t}[\frac{t}{\theta}]^{\beta}\exp(-[\frac{t}{\theta}]^{\beta})$$

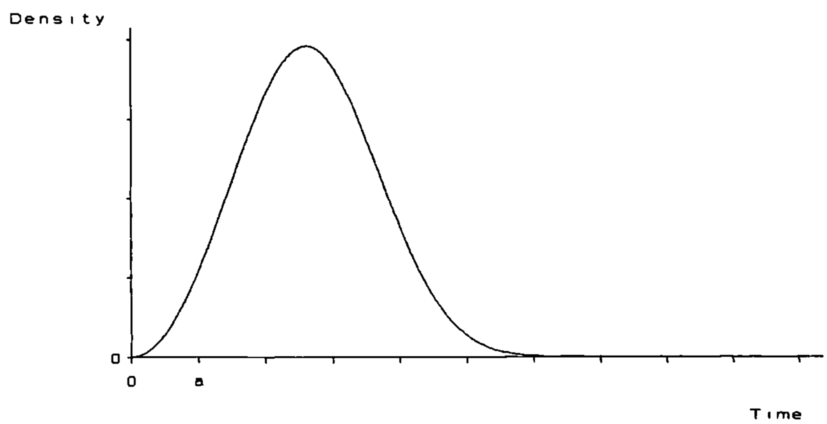
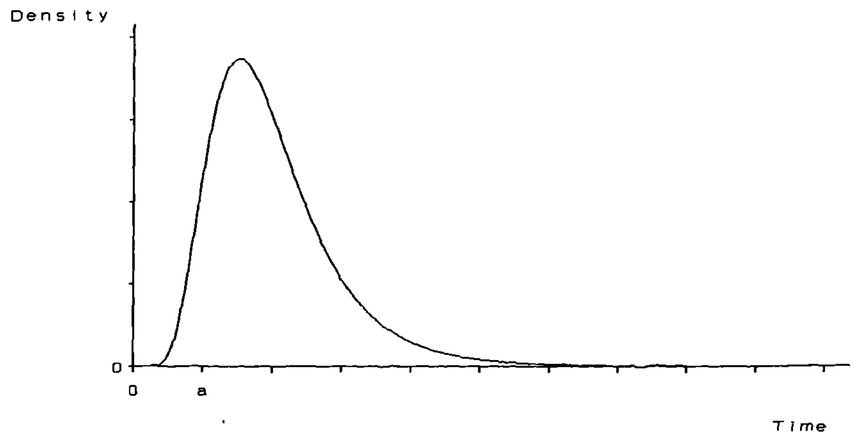


Figure 4.1a Some candidate Breach-time probability distributions for a barrier rated at a minutes.

d) Lognormal

$$f(t) = [\sigma\sqrt{2\pi}\,t]^{-1}\exp[-\frac{1}{2}(\frac{\ln t-\mu}{\sigma})^2]$$



e) Gaussian

$$f(t) = [\sigma\sqrt{2\pi}]^{-1}\exp[-\frac{1}{2}(\frac{t-\mu}{\sigma})^2]$$

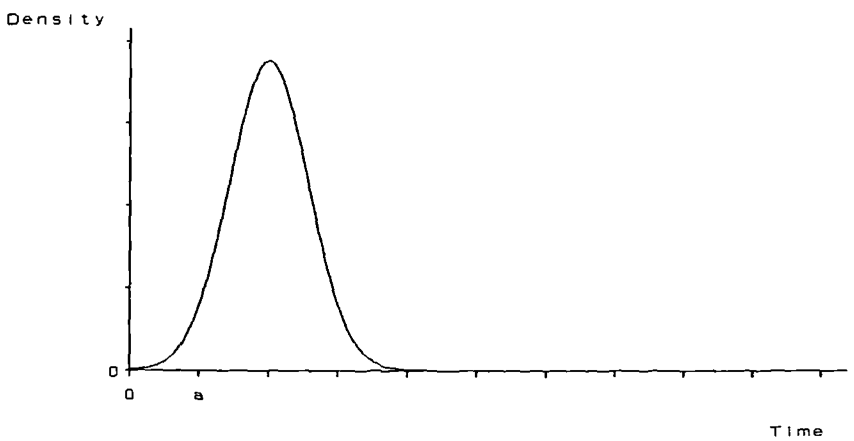


Figure 4.1b Some candidate Breach-time probability distributions fora barrier rated at a minutes.

It should be noted that the one parameter Exponential distribution is both a special case of the Gamma distribution ($\text{Exp}(\lambda)=\gamma(\lambda, 1)$) and a special case of the Weibull distribution ($\text{Exp}(\lambda)=W(\frac{1}{\lambda}, 1)$). The Erlang($\lambda$, c) distribution is the name given to the Gamma(λ, υ) when $\upsilon=c$ is an integer.

Model (a) represents the situation in which the barrier is secure up to the rating value and then burn-through may take place soon afterwards. Models (b), (c), (d) and (e) can all represent the cases in which there is a small, but non-zero, probability of a barrier being breached before the nominal rating value. Note that in (c) there can be a rapid drop in probability density to the right of the mode. There is some empirical support for the use of the Weibull distribution from data given by Ling and Williamson (1985) whilst the Lognormal distribution provides a good fit to data quoted in a confidential report from the Fire Research Station (Ramachandran (1972)). In (e) the variation in breach-time is symmetrically distributed about a nominal value which may, in some cases, be the rating value. Its applicability is supported in AEA funded work currently underway at Edinburgh University [Beard (1989)], and similarly finds support in the work of Buchanan and Elms (1981) described in Chapter 1. Further support for these models is provided by some unpublished data obtained from tests done in Australia [Veevers (1990)].

The data given in Ramachandran (1972) are scanty - the observations were measures of fire-resistance on sixteen doors, each door being assembled from a combination of one of four types of wood and one of four types of glue. It is fortunate that an analysis of variance performed on the logarithms of the fire-resistance values indicates nothing to suggest that the different woods or glues lead to significantly different fire resistances, for then both Ramachandran and his readers are able to treat the observations on the different doors as effectively being a random sample of sixteen 'identical' doors. That allows the construction of a distribution of fire resistances from which inferences may be made. Ramachandran states that the logarithms of the times follow a Gaussian distribution, and thus the actual fire resistance is Lognormally distributed, but the data support a Weibull distribution equally well.

The unpublished Australian data are reproduced in Table 4.1.

Type1	Type2	Type3	Type4	Type5	Type6	Type7	Type8	Type9
35.42	39.17	42.50	55.00	57.75	60.33	70.00	78.00	61.75
35.83	39.17	42.58	55.50	60.00	62.50	71.00	80.08	62.25
35.83	39.25	43.08	56.00	60.75	62.50	71.50	80.42	62.83
35.83	39.33	43.58	56.08	61.00	63.08	72.00	80.50	69.42
37.08	39.58	43.83	57.33	61.75	65.25	72.00	81.42	70.75
37.25	40.67	45.08	58.50	62.95	65.33	74.08	82.42	71.33
								71.83
								72.17
								72.17
								73.75
								78.42
								81.25

Type10	Type11	Type12	Type13	Type14	Type15	Type16
85.25	14.67	29.50	48.67	14.16	35.75	52.67
86.17	15.92	32.67	49.92	14.58	36.50	52.67
86.17	15.92	33.00	50.84	15.33	37.25	53.17
87.00	16.17	33.25	50.92	15.33	37.33	53.33
87.25	16.83	33.42	51.84	15.42	37.84	53.33
87.41	17.33	34.33	52.08	15.84	38.50	54.42

Table 4.1 Fire resistances (in minutes) for sixteen different types of door.

The door types all appear to be distinct in that their mean and median fire resistances differ, but the small number of observations on each type does not uniquely indicate any particular breach-time distribution. Probability plots of the data show support for the Gaussian and Lognormal distributions - see Figures 4.2a and 4.2b.

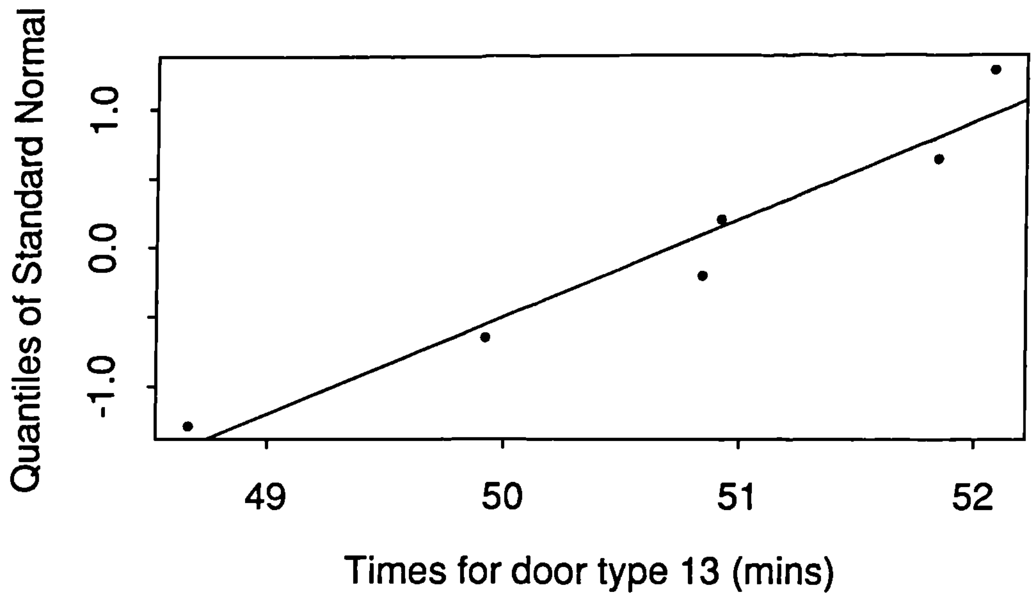


Figure 4.2a Fitting a Gaussian distribution to the data

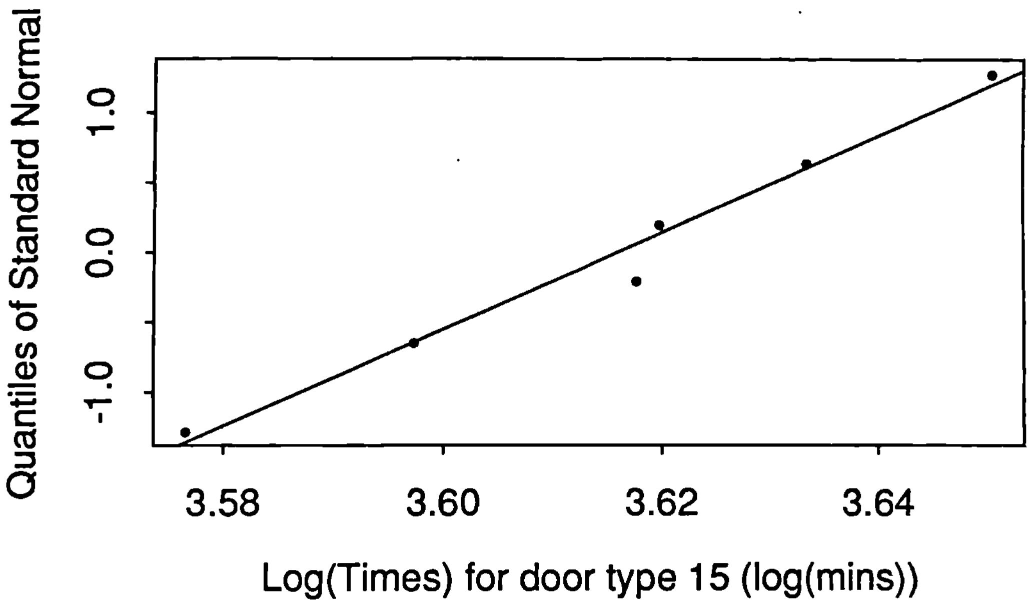


Figure 4.2b Fitting a Log-Normal distribution to the data

It is unfortunate that there is a lack of data from which to draw clear conclusions about the appropriateness of the various suggested breach-time distributions, and it is to be hoped that the need for representative modelling will one day overwhelm the limited testing and the trade secrecy which are currently features of the processes of manufacture and testing of fire-resistant materials.

4.3 Stochastic Networks

Once the breach-time distributions appropriate to each barrier in the structure have been decided upon, it is possible to construct a stochastic network representation of the structure and apply standard techniques to the solution of the 'shortest path' problem for that structure. The algorithm developed by Mirchandani (1976) for solving a stochastic network whose link travel times are discrete random variables may be applied to the continuous case if one is prepared to construct discrete approximations to the continuous link travel-time distributions.

4.3.1 Proposed Solutions to Stochastic Network Problems

Frank (1969) proposed an exact method of finding shortest-path probability distributions in networks whose link lengths are random variables. The basic principles of Frank's approach are presented below, and for the purposes of this exposition his ideas are couched in the language of the fire-spread modelling problem. Throughout the discussion, the assumption is made that each barrier is symmetrical in that its breach-time probability distribution is the same regardless of which face is first exposed to fire. This restriction on the link lengths is not imposed by the methodology, but its introduction does facilitate a simpler exposition of Frank's work.

Let G be the graph representation of a segregated structure comprising a total of n volumes and m barriers, $b_1, b_2, \dots b_m$.
Assigned to each barrier, j, is a breach time, - a continuous random variable, T_j.
Let the joint probability density function of the random vector of breach times $T=(T_1, T_2, \dots T_m)$ be $f_T(t_1, t_2, \dots t_m)$.

For any two volumes u, v of the structure, there is a set $\{R_1(u,v), R_2(u,v), \dots, R_q(u,v)\}$ of paths connecting those two volumes.

For each potential ignition volume, i, let Z be the random variable

$$Z_i = \min_k[|R_k(u,v)|] = \min_k\Big[\sum_{b_j \in R_k} T_j\Big],$$

so that Z corresponds to the travel-time of the shortest path from ignition volume to target. The idea is to calculate the probability distribution $F_Z(z)$.

If $R = (|R_1|, |R_2|, \dots |R_q|)$, the vector of travel-times along each of the paths, and the pdf of R is $f_R(x_1, x_2, \dots x_q)$; then
$Pr\{Z \le z\}$ =
$1-Pr\{Z > z\}$ =
1-Pr{fire doesn't reach target in time z by any of the q poss. paths}

$$\Rightarrow F_Z(z) = 1 - \int_z^\infty \dots \int_z^\infty f_R(x_1, x_2, \dots x_q)dx_1, \dots, dx_q. \tag{4.1}$$

Thus it is necessary to compute $f_R(x_1, x_2, \dots x_q)$ before $F_Z(z)$ can be found.

Frank shows that manipulation of the characteristic function of the vector of breach times, T, yields an expression for the characteristic function, $\phi(s)$, of **R**, the vector of travel times along the different paths. The pdf of **R** may then be derived from $\phi(s)$ using the result (see for example Cramer (1974)) which states that in general for a distribution $f_X(x)$ with characteristic function $\phi(s)=\phi(s_1, \dots s_n)$,

$$f_X(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi(s)\exp(-is^T x)ds_1 \dots ds_n.$$

A further multiple integration of the pdf of **R** yields the required distribution function $F_Z(z)$.

The methodology is, by Frank's own admission, very complex and, I would suggest, virtually intractable as a means of solution to a realistic fire-spread modelling problem. A similarly complex multiple-integral solution is presented by Sigal et al (1980).

Kulkarni (1986) develops methods for the evaluation of the distribution of the length of the shortest path between two nodes in a stochastic network whose arc lengths are independently exponentially distributed.

4.4 Distributions of Time to reach the Target Volume

The method presented in this section relies first on the allocation to each barrier of a continuous breach-time distribution and second on the identification of all routes from each possible volume of ignition to the target. Reference to sub-section 1.2.5 (of Chapter 1) indicates that the time taken to reach the target along any path may be considered to be the sum of the times taken to breach the barriers in that path.

In the general case, the barrier breach times are independently, but not necessarily identically, distributed random variables $T_{m,n}$ with distribution functions $F_{m,n}(t)$, where m and n are adjacent volumes.

Thus the time taken for fire to travel a particular route, k, from an ignition volume, i, is itself a random variable, $T_{i,k}$, which is the sum of the values of $T_{m,n}$ corresponding to the barriers defining that path. For each ignition volume, i, there will be K_i random variables, $T_{i,k}$, each with distribution function $F_{i,k}(t)$.

The primary value of interest is, of course, the time to first arrival at the target, a time which may be achieved by fire travelling any of the routes from the specified ignition volume. It may, for example, be that the shortest travel time is consistently associated with one particular path, a situation which would indicate either that steps ought to be taken to increase the quality of barriers in that path, or that extra barriers be added to impede to a greater degree the progress of the fire.

4.5 Extreme Value Theory

The fire spread model introduced in the previous section may be re-expressed as follows:-

What is the distribution of the smallest order statistic drawn from a number of not necessarily independent distributions, each of which is in turn the sum of independent (breach-time) random variables?

4.5.1 Introduction of Standard Theory

Were all the paths from a particular ignition volume to the target independent, so that no two paths had any barriers in common, standard theoretical results, - see for example David (1982), would be applicable. Those results are presented here and the theory is applied to some of the cases which may arise from the breach-time distributions suggested in Table 4.1. It should be noted that the discussions here concern themselves with the distribution of the smallest order statistic arising from a small number of distributions, whose derivation and range of application differ from those pertinent to the asymptotic theory of extremes.

Let T_i be the smallest order statistic associated with K_i distributions, $F_{i,k}(t)$. The distribution of T_i is given by

$$\Pr\{T_i \leq t\} = 1 - \prod_{k=1}^{K_i} [1 - F_{i,k}(t)]. \qquad (4.5)$$

For a structure with V volumes, there will be V such statistics, each associated with one ignition volume.

The time, T, from ignition to fire first reaching the target, conditional on fire breaking out somewhere in the structure, is distributed as:-

$$\Pr\{T \leq t\} = \sum_{i=1}^{V} \left\{ \left[1 - \prod_{k=1}^{K_i}\left[1 - F_{i,k}(t)\right]\right] \times P_i \right\} \qquad (4.6)$$

where P_i is the conditional probability of ignition in each volume, so that $\sum_{i=1}^{V} P_i = 1$.

4.5.2 Application to Suggested Barrier Breach-time Distributions

Since the complexity of the mathematics increases with the diversity of the individual independent barrier breach-time distributions, it seems reasonable to separate the possible relationships between the individual independent barrier breach distributions according to the following:-

Category ...	the breach-time distributions are ...
1	identically distributed (iid)
2	same distribution, different parameters
3	different distributions.

The first category comprises all cases in which each of the barriers in the structure, and thus all the barriers in any particular path, have the same breach-time distribution, $F_T(t)$. In this case, the travel time, $T_{i,k}$, for a path of length n is distributed as the sum of n iid random variables. When $F_T(t)$ is the Gamma (Exponential) distribution, or the Normal distribution, equation (4.5) has a simple, closed form as the distribution of the sum of the breach times is of the same family as the common barrier breach-time distribution. This also holds for the three-parameter Gamma distribution, of which the two-parameter Exponential is a special case. If the barrier breach-time distributions are all iid Weibull(β, θ), or iid Lognormal(μ, σ), an expression for their sums, corresponding to the path travel times, may be found by making use of the characteristic function of the distribution. For the Weibull distribution this is:

$$\psi(s) = \int_{-\infty}^{\infty} f(t)e^{ist}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{\beta}{t} \left[\frac{t}{\theta} \right]^{\beta} \exp\left\{ -\left[\frac{t}{\theta} \right]^{\beta} \right\} e^{ist}\, dt.$$

The characteristic function of the sum of n iid such random variables is

$$\phi(s) = \beta^{n}\theta^{-n\beta} \left[\int_{-\infty}^{\infty} t^{\beta-1} \exp\left\{ ist - \left[\frac{t}{\theta} \right]^{\beta} \right\} dt \right]^{n},$$

and the corresponding density function is

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}\, \beta^{n}\theta^{-n\beta} \left[\int_{-\infty}^{\infty} t^{\beta-1} \exp\left\{ ist - \left[\frac{t}{\theta} \right]^{\beta} \right\} dt \right]^{n} ds.$$

The characteristic function of a Lognormal(μ, σ) is

$$\int_{-\infty}^{\infty} e^{ist}\,[\sigma\sqrt{2\pi}\;t]^{-1}.\exp\left[-\frac{1}{2}.\left\{\frac{\ln(t)-\mu}{\sigma}\right\}^2\right]dt$$

and the density function of the sum of n iid Lognormal(μ, σ) random variables may be written as

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n e^{-ist}\left\{\int_{-\infty}^{\infty} t^{-1}\exp\left[e^{ist}-\frac{1}{2}.\left\{\frac{\ln(t)-\mu}{\sigma}\right\}^2\right]dt\right\}^n ds.$$

In the case of a Lognormal distribution, it is usually easier to transform the random variable to a Gaussian variate.

Table 4.2 shows the distribution of the smallest order statistic of path travel times along any path, k, from ignition volume i to the target, when the common barrier breach-time distribution is as specified. In each case, $n=c_{i,k}$ is the number of barriers in path k from volume i.

Breach - times	Distribution function of smallest order statistic
$\exp(\lambda,\ \alpha)$	$1 - \prod_{k=1}^{K_i}\left[\exp[-\lambda(t-\alpha c_{i,k})]\sum_{j=0}^{c_{i,k}-1}\frac{\lambda^j}{j!}(t-\alpha c_{i,k})^j\right].$
$\gamma(\lambda,\ \upsilon)$	$1 - \prod_{k=1}^{K_i}\left[1-\int_0^t \frac{(\lambda u)^{J-1}.e^{-\lambda u}}{\Gamma(J)}du\right]$ (where $J=\upsilon c_{i,k}$) if υ not an integer.
or	$1 - \prod_{k=1}^{K_i}\left[\exp(-\lambda t)\sum_{j=0}^{J-1}\frac{(\lambda t)^j}{j!}\right]$ if υ is an integer.
$W(\beta,\ \theta)$	$1-\prod_{k=1}^{K_i}\left[1-\int_0^{\tau}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ist}\beta^n\theta^{-n\beta}\left[\int_{-\infty}^{\infty}t^{\beta-1}\exp\left\{ist-\left(\frac{t}{\theta}\right)^{\beta}\right\}dt\right]^n ds\ dt\right].$
$L(\mu,\ \sigma)$	$1-\prod_{k=1}^{K_i}\left[1-\int_0^{\tau}\frac{1}{2\pi}\int_{\infty}^{\infty}\left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n e^{-ist}\left\{\int_{-\infty}^{\infty}t^{-1}\exp\left[e^{ist}-\frac{1}{2}\left\{\frac{\ln(t)-\mu}{\sigma}\right\}^2\right]dt\right\}^n ds\ dt\right].$
$N(\mu,\ \sigma^2)$	$1-\prod_{k=1}^{K_i}\left[1-\Phi\left[\frac{t-\mu c_{i,k}}{\sigma.\sqrt{c_{i,k}}}\right]\right]$

Table 4.2 Distributions of the smallest order statistic of travel time along any of the K_i paths from ignition volume i to the Target for cases falling into category 1.

The second category comprises the slightly less restrictive cases in which the barrier breach times in each path are all independently, but not necessarily identically, distributed random variables from the same family. Once again, closed forms of the expressions for the smallest order statistic, T_i, do not exist when the barrier breach times are Weibull or Lognormally distributed.

If a path contains $c_{i,k}$ barriers, each of whose breach-time distribution is two-parameter Exponential(λ_j, α_j), the sum will only be distributed as a Gamma random variable if the λ_j are identical. In other cases, an expression for the distribution of the sum must be derived by other means.

The characteristic function $\psi_j(s)$ of an Exp(λ_j) random variable is

$$\psi_j(s) = \frac{\lambda_j}{\lambda_j - is}$$

and the characteristic function $\phi(s)$ of the sum of n exponentially distributed random variables is

$$\phi(s) = \prod_{j=1}^{n} \psi_j(s) = \prod_{j=1}^{n} \left\{ \frac{\lambda_j}{\lambda_j - is} \right\},$$

which it is sometimes useful to write as

$$\prod_{j=1}^{n} \left\{ \sum_{k=0}^{\infty} \frac{i^k s^k}{\lambda^k} \right\}.$$

The density function of the sum of the random variables is given by

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(s) e^{-ist} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \left\{ \sum_{k=0}^{\infty} \left[\frac{is}{\lambda_j} \right]^k \right\} e^{-ist} ds.$$

Thus the distribution function of the sum may be seen to be

$$F_T(t) = \int_{-\infty}^{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \left[\sum_{k=0}^{\infty} \left[\frac{is}{\lambda_j} \right]^k \right] e^{-ist} \, ds dt.$$

If the breach-time random variables corresponding to the barriers in a path are all Gamma (λ_j, υ_j) distributed, the same methodology as used above for the Exponential case can be used to show that the density of the sum is

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{c_{i,k}} \left[\frac{\lambda_j}{\lambda_j - is} \right]^{\upsilon_j} e^{-ist} \, ds$$

since the characteristic function of a $\gamma(\lambda, \upsilon)$ is $\left[\frac{\lambda}{\lambda - is} \right]^{\upsilon}$.

The density function of the sum of n independent, non-identically distributed Weibull random variables, $W(\beta_j, \theta_j)$, is

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist} \prod_{j=1}^{n} \left[\beta_j \theta_j^{-\beta_j} \left[\int_{-\infty}^{\infty} t^{\beta_j - 1} \exp\left\{ ist - \left(\frac{t}{\theta_j} \right)^{\beta_j} \right\} dt \right] \right] ds.$$

The density function of the sum of n independent, non-identically distributed Lognormal random variables, $L(\mu_j, \sigma_j)$, is

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist} \prod_{j=1}^{n} \left[\left[\frac{1}{\sigma_j \sqrt{2\pi}} \right] \left\{ \int_{-\infty}^{\infty} t^{-1} \exp\left[e^{ist} - \frac{1}{2} \left\{ \frac{\ln(t) - \mu_j}{\sigma_j} \right\}^2 \right] dt \right\} \right] ds.$$

The distribution of the smallest order statistic in each case is readily derived from the densities as given.

As the sum of a number of independent Normally distributed random variables is itself Normally distributed, the distribution of the sum of n $N(\mu, \sigma^2)$ random variables is simply

$$N\left(\sum_{j=1}^{n} \mu_j, \ \sum_{j=1}^{n} \sigma_j^2 \right).$$

In respect of category 2, Table 4.3 has been constructed. Unless otherwise indicated, J is used to denote c_{ik}, the number of barriers in path k from volume i to the target.

Exponential(λ_j):

$$1-\prod_{k=1}^{K_i}\left[1-\int_{-\infty}^{T}\frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{j=1}^{J}\left[\sum_{k=0}^{\infty}\left[\frac{i\,s}{\lambda_j}\right]^{k}\right]e^{-ist}\,ds\,dt\right].$$

Gamma(λ_j, υ_j):

$$1-\prod_{k=1}^{K_i}\left[1-\int_{-\infty}^{T}\frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{j=1}^{J}\left[\sum_{m=0}^{\infty}\frac{i^{m}\,(\upsilon_j+m-1)!}{\lambda_j^{m}\,m!(\upsilon_j-1)!}\right]e^{-its}.s^{m}\,ds\,dt\right].$$

Weibull(β_j, θ_j):

$$1-\prod_{k=1}^{K_i}\left[1-\int_{-\infty}^{T}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ist}\prod_{j=1}^{J}\left[\beta_j\theta_j^{-\beta_j}\left[\int_{-\infty}^{\infty}t^{\beta_j-1}\exp\left\{ist-\left(\frac{t}{\theta_j}\right)^{\beta_j}\right\}dt\right]\right]ds\,d\tau\right].$$

Lognormal(μ_j, σ_j):

$$1-\prod_{k=1}^{K_i}\left[1-\int_{-\infty}^{T}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ist}\prod_{j=1}^{J}\left[\left[\frac{1}{\sigma_j\sqrt{2\pi}}\right]\left\{\int_{-\infty}^{\infty}t^{-1}\exp\left[e^{ist}-\frac{1}{2}\left\{\frac{\ln(t)-\mu_j}{\sigma_j}\right\}^{2}\right]dt\right\}\right]ds\,d\tau\right].$$

N(μ_{ij}, σ^2_{ij}):

$$1-\prod_{k=1}^{K_i}\left[1-\phi\left(\frac{t-\mu_L}{\sigma_L}\right)\right].$$

where $\mu_L = \sum\mu_{mn}$
and $\sigma_L^2 = \sum\sigma^2_{mn}$
over all barriers mn in path k

Table 4.3 Distributions of the smallest order statistic of travel time along any of the K_i paths from ignition volume i to the Target for cases falling into category 2.

When a structure contains barriers whose breach-time distributions belong to a number of different families, the mathematics becomes rather more complex. Consideration is given to pairs of different distributions, from which results may be seen the germane methodology for the combination of more than two families of distributions.

1) Normal $N(\mu, \sigma^2)$ and Gamma $\gamma(\lambda, \upsilon)$

The characteristic functions being

$$\psi_N(s) = \exp(\mu is - \frac{s^2\sigma^2}{2}) \qquad \text{and}$$

$$\psi_\gamma(s) = \left[\frac{\lambda}{\lambda - is}\right]^\upsilon$$

respectively, the characteristic function of the sum is

$$\phi(s) = \exp(\mu is - \frac{s^2\sigma^2}{2})\left[\frac{\lambda}{\lambda - is}\right]^\upsilon .$$

The density of this sum of the two random variables is

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(s)e^{-ist}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist}.\left\{\exp\left[\mu is - \frac{s^2\sigma^2}{2}\right]\right\}\left[\frac{\lambda}{\lambda - is}\right]^\upsilon ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ e^{-ist} \cdot \left\{ \exp\left[\mu is - \frac{s^2\sigma^2}{2} \right] \right\} \cdot \left[\sum_{k=0}^{\infty} \frac{i^k(\upsilon+k-1)!\,s^k}{k!\,(\upsilon-1)!\lambda^k} \right] \right\} ds$$

$$= \frac{1}{2\pi} \left\{ \sum_{k=0}^{\infty} \frac{i^k(\upsilon+k-1)!}{k!(\upsilon-1)!\lambda^k} \int_{-\infty}^{\infty} s^k\, e^{-ist} \cdot \left\{ \exp\left[\mu is - \frac{s^2\sigma^2}{2} \right] \right\} \right\} ds$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(\upsilon+k-1)!}{k!(\upsilon-1)!\lambda^k} \int_{-\infty}^{\infty} (is)^k \exp\left[-\left(\frac{\sigma}{\sqrt{2}} \right)^2 s^2 - is(t-\mu) \right] ds. \quad (4.7)$$

Clearly $\mathrm{Re}\left[\frac{\sigma}{\sqrt{2}} \right] > 0$, and $\mathrm{Re}(k) > -1$. Furthermore, since

$\arg(is) = \theta$

where θ is defined such that $r\cos\theta = 0$ and $r\sin\theta = s$, so that

if $r=s$, $\sin\theta=1$ and $\theta=\frac{\pi}{2}$, whilst

if $r=-s$, $\sin\theta=-1$ and $\theta=-\frac{\pi}{2}$;

it follows that $\arg(is) = \frac{\pi}{2}\,\mathrm{sign}(s)$.

Thus, from Gradshteyn et al (1972) page 338, equation (4.7) may be written in terms of a parabolic cylinder function as

$$\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(\upsilon+k-1)!}{k!(\upsilon-1)!\lambda^k} 2^{\frac{-k}{2}} \sqrt{\pi} \left(\frac{\sigma}{\sqrt{2}} \right)^{-k-1} \exp\left[\frac{-(t-\mu)^2}{8\frac{\sigma^2}{2}} \right] D_k\left[\frac{(t-\mu)^2}{\frac{\sigma}{2}\sqrt{2}} \right]$$

$$= \frac{\exp\left[\frac{-(t-\mu)^2}{4\sigma^2} \right]}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(\upsilon+k-1)!}{k!(\upsilon-1)!\lambda^k} 2^{\frac{-k}{2}} \left(\frac{\sigma}{\sqrt{2}} \right)^{-k-1} D_k\left[\frac{(t-\mu)^2}{\frac{\sigma}{2}\sqrt{2}} \right].$$

Other representations in terms of degenerate hypergeometric functions or Hermite polynomials are also possible.

2) Normal $N(\mu, \sigma^2)$ and Exponential(λ)

The derivation of an expression for a combination of a Gaussian and an Exponential distribution is similar to that detailed above.

The characteristic function of the sum is

$$\phi(s) = \exp(\mu is - \frac{s^2\sigma^2}{2})\left[\frac{\lambda}{\lambda - is}\right].$$

The corresponding density is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist} . e^{\mu is - \frac{s^2\sigma^2}{2}} . \left[\frac{\lambda}{\lambda - is}\right] ds.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ e^{-ist} . e^{\mu is - \frac{s^2\sigma^2}{2}} . \left[\sum_{k=0}^{\infty} \frac{i^k s^k}{\lambda^k}\right]\right\} ds.$$

$$= \frac{1}{2\pi}\left\{\left[\sum_{k=0}^{\infty}\frac{i^k}{\lambda^k}\right] \int_{-\infty}^{\infty} s^k\, e^{-ist} . e^{\mu is - \frac{s^2\sigma^2}{2}} . \right\} ds.$$

$$= \frac{1}{2\pi}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}\right)^k \int_{-\infty}^{\infty} (is)^k \exp\left[-\left(\frac{\sigma}{\sqrt{2}}\right)^2 s^2 - is(t-\mu)\right] ds. \qquad (4.8)$$

Clearly $\mathrm{Re}\left(\frac{\sigma}{\sqrt{2}}\right) > 0$, $\mathrm{Re}(k) > -1.$ and $\arg(is) = \frac{\pi}{2}\,\mathrm{sign}(s).$

As before, this may be written in terms of a parabolic cylinder function as

$$\frac{\exp\left[\frac{-(t-\mu)^2}{4\sigma^2}\right]}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} 2^{\frac{-k}{2}} \left(\frac{\sigma}{\sqrt{2}}\right)^{-k-1} D_k\left[\frac{(t-\mu)^2}{\frac{\sigma}{2}\sqrt{2}}\right]$$

and again once it is possible to express this density in terms of degenerate hypergeometric functions or Hermite polynomials.

3) Normal N(μ, σ^2) and Lognormal(μ, σ)

The characteristic functions being

$$\psi_N(s) = \exp(\mu is - \frac{s^2\sigma^2}{2})$$

and

$$\psi_L(s) = \int_{-\infty}^{\infty} e^{ist} [\sigma\sqrt{2\pi}\ t]^{-1}.\exp\left[- \frac{1}{2}.\left\{\frac{\ln(t) - \mu}{\sigma}\right\}^2\right] dt$$

respectively, the characteristic function of the sum is

$$\phi(s) = \exp(\mu is - \frac{s^2\sigma^2}{2}). \int_{-\infty}^{\infty} e^{ist} [\sigma\sqrt{2\pi}\ t]^{-1}.\exp\left[- \frac{1}{2}.\left\{\frac{\ln(t) - \mu}{\sigma}\right\}^2\right] dt.$$

The density of this sum is

$f_T(t) =$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}\left\{\exp(\mu is - \frac{s^2\sigma^2}{2}) \int_{-\infty}^{\infty} e^{ist}[\sigma\sqrt{2\pi}\ t]^{-1}.\exp\left[-\frac{1}{2}\left\{\frac{\ln(t) - \mu}{\sigma}\right\}^2\right]dt\right\} ds.$$

4) Gamma $\gamma(\lambda, \upsilon)$ and Lognormal(μ, σ)

The characteristic function of the sum is

$$\phi(s) = \left[\frac{\lambda}{\lambda - is} \right]^{\upsilon} \int_{-\infty}^{\infty} e^{ist} \, [\sigma\sqrt{2\pi} \; t]^{-1} . \exp\left[- \frac{1}{2} \left\{ \frac{\ln(t) - \mu}{\sigma} \right\}^{2} \right] dt$$

and the corresponding density function is

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \left\{ \left[\frac{\lambda}{\lambda - is} \right]^{\upsilon} \int_{-\infty}^{\infty} e^{ist} [\sigma\sqrt{2\pi} \; t]^{-1} . \exp\left[-\frac{1}{2} \left\{ \frac{\ln(t) - \mu}{\sigma} \right\}^{2} \right] dt \right\} ds$$

These results are collated in Table 4.4.

$N(\mu,\ \sigma^2)$ and $\gamma(\lambda,\ \upsilon)$ (or Exponential (λ)):

$$\int_{-\infty}^{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}.\left\{\exp\left[\mu is - \frac{s^2\sigma^2}{2}\right]\right\}\left[\frac{\lambda}{\lambda - is}\right]^{\upsilon} ds\ d\tau$$

$N(\mu,\ \sigma^2)$ and Lognormal$(\mu,\ \sigma)$:

$$\int_{-\infty}^{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}\left\{\exp(\mu is - \frac{s^2\sigma^2}{2}) \int_{-\infty}^{\infty} e^{ist}[\sigma\sqrt{2\pi}\ t]^{-1}.\exp\left[-\frac{1}{2}\left\{\frac{\ln(t) - \mu}{\sigma}\right\}^2\right]dt\right\}dsd\tau$$

$\gamma(\lambda,\ \upsilon)$ and Lognormal$(\mu,\ \sigma)$

$$\int_{-\infty}^{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}\left\{\left[\frac{\lambda}{\lambda - is}\right]^{\upsilon} \int_{-\infty}^{\infty} e^{ist}[\sigma\sqrt{2\pi}\ t]^{-1}.\exp\left[-\frac{1}{2}\left\{\frac{\ln(t)-\mu}{\sigma}\right\}^2\right]\ dt\right\}dsd\tau$$

Table 4.4 Distribution functions of the sums of distributions pertinent to category 3.

It is likely that any particular route through a structure will contain a number of barriers with identical or similar breach-time distributions. Furthermore, since, in terms of deriving a distribution for the time taken to reach the target along a given path, the order in which the barriers are considered is irrelevant, it ought to be possible to group a path's barriers into sets so that each set contains barriers whose breach distributions are identical or at least of the same family. The breach-time distribution for each set of barriers could then be derived fairly readily, and the number of 'different-family' combinations would be restricted to the number of 'same family' sets, and would perhaps be relatively small.

Example

Consideration is given to the structure depicted in Figure 4.3. Volume 5 is the target volume and the barrier breach-time distributions are coded by letters.

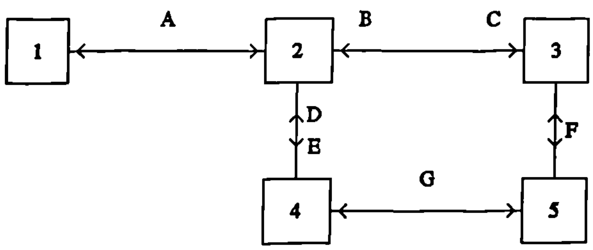


Figure 4.3 Network representation of a five-volume structure.

The barriers with only one breach-time distribution are coded thus since only one direction of burn-through is of interest in those cases. The time units are minutes, and the breach-time distributions and their abbreviations are:

Barrier:	Distribution:	Abbreviation:
A	Exponential (0.1, 10)	E1
B	Lognormal (3.22, 0.15)	L1
C	Gamma (0.1, 5)	G1
D	Lognormal (3.0, 0.12)	L2
E	Normal (20, 36)	N1
F	Gamma (0.1, 8)	G2
G	Normal (30, 49)	N2.

A summary of the information about the routes is given in Table 4.5.

Ign. vol.	(prob)	Route	Barriers	Time distributed as
1	(0.35)	1.1	A E G	E1 + N1 + N2
		1.2	A C F	E1 + G1 + G2
2	(0.25)	2.1	C F	G1 + G2
		2.2	E G	N1 + N2
3	(0.15)	3.1	F	G2
		3.2	B E G	L1 + N2 + N2
4	(0.2)	4.1	G	N2
		4.2	D C F	L2 + G2 + G2
5	(0.05)	5	none	

Table 4.5 Summary of path and travel-time information.

More specifically, using the results derived above, the path travel times are seen to be as given in Table 4.6.

Route	Distribution of travel time
1.1	$\frac{1}{2\pi}\int_{-\infty}^{\tau}\int_{-\infty}^{\infty}\left\{e^{-ist}.\exp\left[50is - \frac{85s^2}{2}\right]\left[\sum_{k=0}^{\infty}(10is)^k\right]\right\}ds\ dt+10$
1.2	$\gamma(0.1,\ 14)+10$
2.1	$\gamma(0.1,\ 13)$
2.2	N(50, 85)
3.1	$\gamma(0.1,\ 8)$
3.2	See below
4.1	N(30, 49)
4.2	See below

Table 4.6 Probability distributions of path travel times.

3.2 is ...

$$\frac{1}{2\pi}\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} e^{-ist}\left\{\exp\left[50is-\frac{85s^2}{2}\right].\int_{-\infty}^{\infty} e^{ist}[t\sqrt{4.8\pi}]^{-1}.\exp\left[-\frac{1}{2}\left\{\frac{\ln(t)-3.2}{1.55}\right\}^2\right]dt\right\}dsdt$$

4.2 is ...

$$\frac{1}{2\pi}\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} e^{-ist}\left\{\left[\frac{0.1}{0.1-is}\right]^{13}\int_{-\infty}^{\infty} e^{ist}[t\sqrt{4.8\pi}]^{-1}.\exp\left[-\frac{1}{2}\left\{\frac{\ln(t)-3.2}{1.55}\right\}^2\right]dt\right\}ds.dt$$

Some simulations using the Minitab statistical computer package suggest that the distributions given in Table 4.7 serve as good approximations to the corresponding distributions in Table 4.6.

Route	Approximate distribution of travel time
1.1	N(70, 175)
1.2	γ(0.1, 14)+10
2.1	γ(0.1, 13)
2.2	N(50, 85)
3.1	γ(0.1, 8)
3.2	N(76, 98)
4.1	N(30, 49)
4.2	N(151, 1265)

Table 4.7 Approximate distributions of path travel times.

The final results for the distributions of the smallest order statistics given in Table 4.8 overleaf are straightforwardly derived from the application of equation 4.5.

Ign vol.(prob) and	Required distribution
Ignition in volume 1 (p=0.35):	$1 - \left[\exp(0.1(t+10)) \left(\sum_{i=0}^{13} \frac{(0.1(t+10))^i}{i!} \right) \right] \left[1 - \phi\left(\frac{t-70}{\sqrt{175}} \right) \right]$
Ignition in volume 2 (p=0.25):	$1 - \left[\exp(0.1t) \left(\sum_{i=0}^{12} \frac{(0.1t)^i}{i!} \right) \right] \left[1 - \phi\left(\frac{t-50}{\sqrt{85}} \right) \right]$
Ignition in volume 3 (p=0.15):	$1 - \left[\exp(0.1t) \left(\sum_{i=0}^{7} \frac{(0.1t)^i}{i!} \right) \right] \left[1 - \phi\left(\frac{t-76}{\sqrt{98}} \right) \right]$
Ignition in volume 4 (p=0.20):	$1 - \left[1 - \phi\left(\frac{t-30}{7} \right) \right] \left[1 - \phi\left(\frac{t-151}{35.57} \right) \right]$
Ignition in volume 5 (p=0.05):	0.

Table 4.8 Distributions of smallest order statistics of arrival times for each volume..

Suppose a fire breaks out in volume 4. The distribution of the smallest order statistic of time to reach the target is readily calculated from the formula in Table 4.8. The distribution function is plotted as Figure 4.4 overleaf. From the graph it can be seen, for example, that the chances of fire reaching the target within nine minutes of the ignition in volume 4 are very small, whilst the target will almost inevitably be ablaze forty-five minutes after ignition.

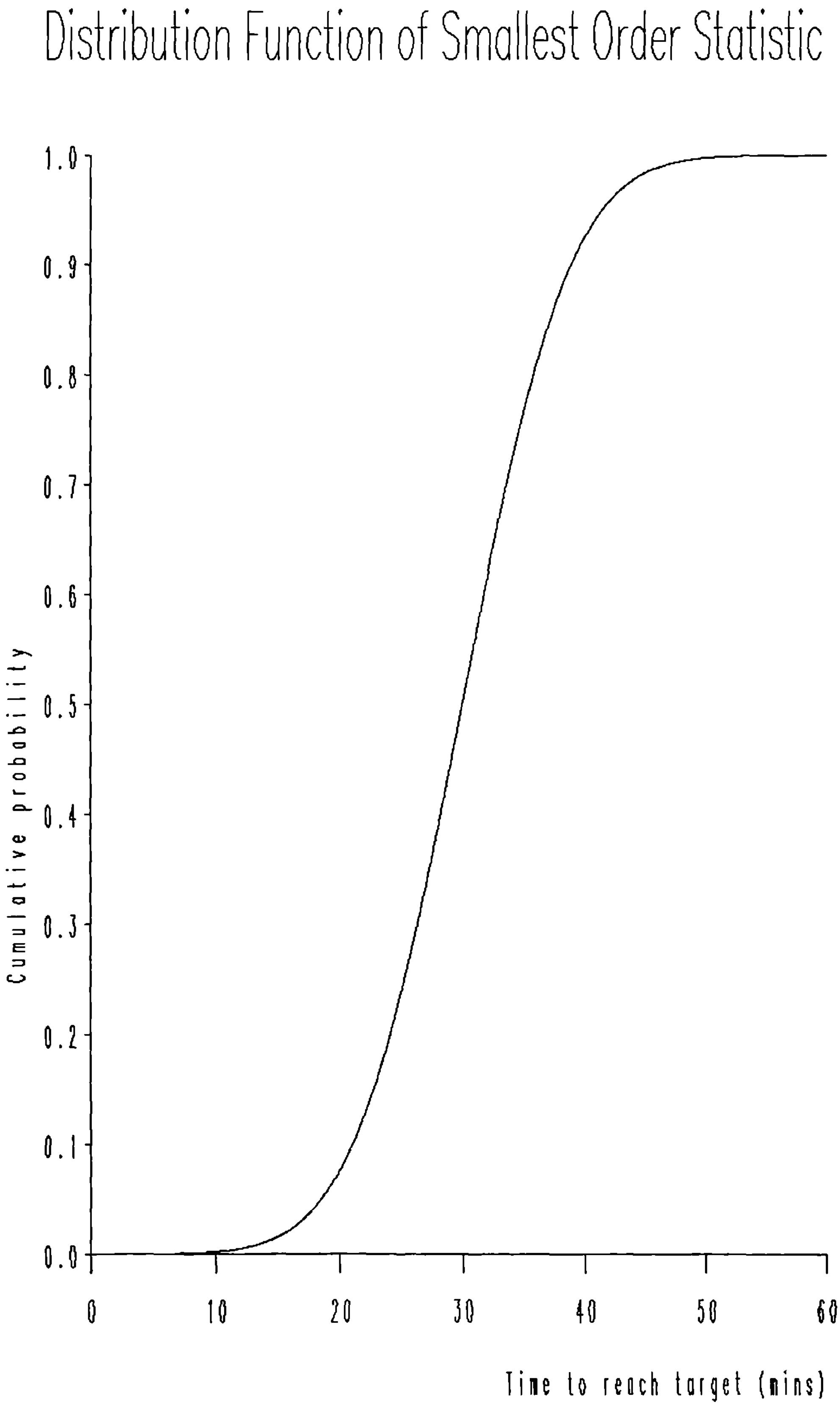


Figure 4.4 Distribution function of smallest order statistic of time to arrival at target from volume 4.

4.5.3 Extreme Value Theory Applied to Non-independent Distributions

The work presented in the previous section was based on the assumption that, for a specified ignition volume and target, the path travel-time distributions are independent. This is equivalent to the assumption that no two routes from the ignition volume have a barrier in common. That restriction is clearly unrealistic as most buildings are not constructed in such a limited way.

Standard extreme value theory is derived on the basis that the contributing distributions are independent. Since Gumbel (1958), the interest in the distribution of extremes has greatly increased, other texts being written by Galambos (1978) and Leadbetter *et al.* (1983) amongst others. Leadbetter seeks to bring together existing material as well as to introduce new theory, particularly that pertinent to the consideration of extreme values arising from combinations of non-independent distributions. Much of the work is concerned with identifying the situations in which a distribution of extremes arising from non-independent distributions may be well-approximated by that which would be obtained were the contributing distributions actually independent.

4.5.4 Dependence Assumptions

Whilst the concept of independence is simple to grasp and may be quite straightforwardly defined, the quantification of dependence is more difficult. There are various levels of dependence, at one extreme are Markov processes (in which only the present state and no previous one is relevant in the prediction of a future state) and at the other are processes for which the dependence structure can be extremely involved.

The term m-dependent (Watson (1954)) is used to describe a sequence of random variables, $\{t_i\}$, which has the property that if $|j-k| > m$, t_j and t_k are independent. A slightly less rigid dependence structure is that described by *strong mixing*, which allows that any event A, "based on the past up to time t_0" is "nearly independent" of any event B "based on the future from time t_0+k+1 onwards" when k is large (Leadbetter *et al.* (1983)). More formally, for the two events A and B, and a function g(k), strong mixing requires that

$$|Pr(A \cap B) - Pr(A)Pr(B)| < g(k).$$

An even more general model of the dependence structure of a stationary sequence of random variables is provided by considering the behaviour of

$$r_n \log(n) \text{ as } n \to \infty$$

where r_n is the covariance of a pair of random variables ξ_j and ξ_{j+n}.

Much of the classical theory of extremes is concerned with the asymptotic behaviour of a process, and this emphasis is reflected and magnified in the literature about the behaviour of extremes under less rigid distributional and independence assumptions. The particular interest in asymptotic results is understandable since it serves to shed light on some practical modelling problems, - such as those arising from our still largely unpredictable weather, whilst also restricting the complexity of the mathematics.

4.6 Conclusion

The application to the modelling of fire-spread is not best characterized by recourse to asymptotic results derived for sequences with clearly definable dependence and distributional features. The dependence in the fire-modelling problem arises because, in general, a number of paths from an ignition volume to the target will have some barriers in common, and whilst there may, in some cases, be a relationship between, for example, path length and the number of common barriers, there is little to suggest precisely what form that relationship takes. During the course of the research, several computer programs were written to investigate, using simulation techniques, whether m-dependence or other restricted-dependence constructs were appropriate to the fire-spread modelling problem discussed here, but these provided no clear indication that the relationship demonstrates any fixed dependence structure. The best that can be done, short of developing a non-asymptotic theory of the extremes of non-independent and not identically distributed random variables, is to apply the standard theory and to be aware of its limitations. In probabilistic risk assessments it is common engineering practice to regard numerical calculations as providing a good estimation rather than a precise figure. This is often interpreted as meaning 'within an order of magnitude'. With the current fire-resistance classification system, there can be little doubt that any small inaccuracies arising from the statistical modelling are negligible compared with the inaccuracies implicit in fire-resistance specifications. Thus, from the fire-safety engineering point of view, the method proposed here is entirely satisfactory.

CHAPTER 5

Optimization of Segregated Structure Design
- Time-independent Models

But to us, *probability is the very guide of life.*
Bishop Joseph Butler

5.1 Introduction

In this chapter consideration is given to a somewhat different problem, - that of fitting fire barriers to an otherwise complete structure. For the purposes of this discussion, each fire barrier is considered to have been assigned a single-value breach probability rather than a breach-time distribution.

Suppose that the layout of a proposed building has been decided upon, and that the likely contents of each volume are known. Then, each volume may be assigned a realistic (conditional) ignition probability, any 'target' volumes may be identified, and questions only remain as to the types of barrier to be used. Specifically, this may be thought of as a question concerning the type of door to be fitted to each doorway since, in the absence of a glazing forming a partition between two volumes, a door is likely to constitute the most readily penetrable part of a barrier. The question may then be posed: how may a door be selected for each doorway so that the probability of any fire in the building reaching the target is minimized, and so that the budget available for expenditure on doors is not exceeded?

5.2 Simple Linear Structures

Initially, the special case of a straightforward linear structure is considered as this provides an understanding of the building blocks of some larger, non-linear structures. A simple example as given in Figure 5.1 serves as an illustration.

Target Volume P_T	Volume 4 P_4	Volume 3 P_3	Volume 2 P_2	Volume 1 P_1

Figure 5.1 A five-volume linear chain structure with a target at one end.

The conditional ignition probability, P_i, for each volume, i, is known, as is the budget which limits the total spending on doors.

The optimization problem is expressed as:

For each pair of adjacent volumes fit a door so that:

Pr{a fire which has started in the building reaches the target}

is minimized subject to the cost constraint:

Total cost of doors $\leq$ Budget.

In practice a limited range of doors will be available, the doors being classified initially according to cost and breach probability, but in the event of a cost/probability relationship being available only one parameter need be specified.

5.2.1 Models of a Barrier's cost/breach probability Relationship

It seems that a reasonable relationship between the cost, c_j and breach probability, p_j of any fire door, j, ought to satisfy the criteria outlined below:-

1) p_j may be expressed as a function of c_j: that is $p_j=f(c_j)$;
2) f is monotonically decreasing;
3) $p_j \to 0$ as $c_j \to \infty$;
4) $c_j=0 \Rightarrow p_j=1$.

From the infinite number of parametric functions which satisfy these criteria, two one-parameter functions are selected as being appropriate to the present application.

Model 1: Exponential Function: $p_j = \exp(-kc_j)$, $k>0$

Model 2: Inverse Function: $p_j = \dfrac{a}{c_j + a}$, $a>0$

These Models can be justified on the following grounds.

Supposing that for a particular material, a solid door of thickness d has $p_j=p_d$, and $c_j=c_d$, what values of cost and breach probability (c_{2d} and p_{2d}) might be expected for a door of thickness 2d? Neglecting accessories and fitting costs, it seems reasonable to expect that $c_{2d}=2c_d$ (or perhaps c_{2d} is slightly less than $2c_d$ to allow for 'bulk order' reductions!) since twice as much material will be required. If the new, thick door be regarded as two independent, immediately adjacent thin doors, $p_{2d}= p_d x p_d$. Thus in general, for a door of thickness nd, $p_{nd}=p_d^n$, and, letting $p_j= e^{-k}$; $k>0$, Model 1 arises.

It is, however, rather simplistic to treat one door of thickness 2d as equivalent to two independent doors each of thickness d. It is clear that once a fire has burned a depth d through the door it will already have impinged on the remaining half of the door, thus violating any independence assumption. This suggests that perhaps $p_{2d} > p_d^2$, so that p_d^2 provides a lower bound for p_{2d}, and as the following argument indicates, an upper bound for p_{2d} is perhaps provided by $\frac{p_d}{2}$.

Writing p_{2d} as $p_a p_b$, where p_a is the probability that fire breaches the 'first half' of the door - in other words that it penetrates to a depth d, and considering the suggested upper bound for p_{2d}, it follows that

$p_a p_b \leq \frac{p_d}{2}$, but $p_a = p_d$, so this expression simplifies to

$p_b \leq \frac{1}{2}$, which does support the hypothesis that $\frac{p_d}{2}$ does indeed provide a (conservative) upper bound for p_{2d}.

Since it is usual in practice for values of p_d to lie in the range $10^{-2} \geq p_d \geq 10^{-4}$, it is not unreasonable to impose the constraint

$p_d \leq 0.5$,

and then p_{2d} may be said to lie in the range

$$p_d^2 \leq p_{2d} \leq \frac{p_d}{2}.$$

Taking $P_{2d} = \frac{p_d}{2}$ suggests that for any particular door, j,

$$p_j = \frac{1}{c_j}.$$

This may be modified slightly to:

$$p_j = \frac{a}{c_j + a}, \; a > 0$$

so that condition (4) is satisfied and Model 2 arises.

Both the exponential function and the inverse function will be considered, and it is expected that in practice a function which assigns p_j somewhere between the values provided by these methods might be applicable. Maskell and Baldwin (1972) considers the relationship between the fire resistance of a steel or concrete column and the cost of its construction. A relationship of the form:

Cost = A + BR

where A, B are positive constants and R is fire resistance,

is found for columns whose fire resistance exceeds 30 minutes.

5.2.2 Application of Model 1 to Simple Structures

Throughout Section 5.2, the simplifying assumption is made that an infinite variety of doors is available. Figure 5.2 shows a linear structure with a target volume at one end and conditional ignition probabilities P_1, P_2 and P_T. The problem is to fit two doors, one between volumes 1 and 2, and the other between volume 2 and the target, so that the probability of any fire reaching the target is minimized. The breach probabilities and costs of the two doors are denoted by p_1, p_2 and c_1, c_2 respectively. The total budget available is C.

Target Volume P_T	Volume 2 P_2	Volume 1 P_1

Figure 5.2 A linear structure with a target at one end.

Model 1

Exponential Function: $p_j = \exp(-kc_j)$.

For the simple structure of Figure 5.2,

let Φ =

Pr{fire reaches the target | ignition somewhere in the structure}, so

$$\Phi = P_T + P_2 p_2 + P_1 p_1 p_2$$
$$= P_T + P_2 e^{-kc_2} + P_1 e^{-k(c_1+c_2)},$$

and, since an infinite variety of doors is available and there is nothing to be gained by not spending all the available budget,

$$\Phi = P_T + P_2 e^{-kc_2} + P_1 e^{-kC}$$

and

$$\frac{\partial \Phi}{\partial c_2} = -kP_2 e^{-kc_2} .$$

Thus $\frac{\partial \Phi}{\partial c_2} < 0$ so that Φ is minimized by setting c_2 to be as large as possible. Thus $c_2 = C$ = total budget available, and $c_1 = 0$.

In the light of the knowledge that the derivation of the Model 1 follows from the premise that it is total door thickness which is important, the result obtained above is not surprising.

The structure shown in Figure 5.3 is similar to that of Figure 5.2, but has n+1 volumes.

barrier n	barrier n-1		barrier 2	barrier 1	
Target Volume P_T	Volume n P_n	....	Volume 3 P_3	Volume 2 P_2	Volume 1 P_1

Figure 5.3 A general linear structure with a target at one end.

For the larger structure in Figure 5.3, it is clear that if fire starts in volume 1, it does not matter whether material be devoted to barrier 1 or 2 or 3 or ... , since it will be equally effective as a barrier to fire starting in volume 1 regardless of its location. Similarly, for a fire starting in volume i it does not matter whether cost be devoted to door i or i+1 or ... . Following this reasoning, it is clear that it is optimal to put all available resources towards a very solid door adjacent to the target.

Thus Model 1 represents one 'extreme' model in that the optimal solution is dependent on no information other than the total budget available and the costs of individual doors.

5.2.3 Application of Model 2 to Simple Structures

Inverse Function: $$p_j = \frac{a}{c_j + a}, \quad a>0$$

With this relationship established it is possible to solve the minimization problem defined in section 5.2 above.

Let g_i = Pr{fire reaches volume i} so that

g_T = Pr{fire reaches target}

= Pr{ignition in target or ign. in vol 4 and barrier 4 breached}

and

g_4 = Pr{fire reaches volume 4} =

= Pr{ignition in vol 4 or ign. in vol 3 and barrier 3 breached}

etc.

The probability of fire reaching the target is then

$\Phi = g_T$

$= P_T + g_4 p_4$

$$= P_T + p_4(P_4 + g_3 p_3)$$
$$= P_T + p_4(P_4 + p_3(P_3 + g_2 p_2))$$
$$= P_T + p_4(P_4 + p_3(P_3 + p_2(P_2 + g_1 p_1))), \qquad (5.1)$$

and the problem is

$$\text{MIN } \Phi = P_T + p_4(P_4 + p_3(P_3 + p_2(P_2 + P_1 p_1))) \qquad (5.2)$$
$$\text{subject to} \quad c_1 + c_2 + c_3 + c_4 \leq \text{Budget}$$

The above formulation implicitly assigns a probability of zero to the event of fire spontaneously breaking out in more than one volume during the course of an event involving the spread of fire. If the assumption were not made, the minimization problem would be as follows:-

$$\text{MIN } \Phi = g_T$$
$$= P_T + g_4 p_4 - P_T g_4 p_4 \; - \; P_T + g_4 p_4 (1 - P_T)$$
$$= P_T + (1-P_T) p_4 (P_4 + g_3 p_3 (1-P_4))$$
$$= P_T + (1-P_T) p_4 (P_4 + (1-P_4) p_3 (P_3 + (g_2 p_2 (1-P_3))))$$
$$= P_T + (1-P_T) p_4 (P_4 + (1-P_4) p_3 (P_3 + (1-P_3) p_2 (P_2 + g_1 p_1 (1-P_2))))$$
$$= P_T + (1-P_T) p_4 (P_4 + (1-P_4) p_3 (P_3 + (1-P_3) p_2 (P_2 + P_1 p_1 (1-P_2)))) \qquad (5.3)$$

subject to the constraint $c_1 + c_2 + c_3 + c_4 \leq$ Budget.

The objective function is of the same form as in (5.2) but with modified coefficients of p_i. Consequently, the following development for the solution of (5.2) is also applicable in this case. The reason that (5.2) is preferred to (5.3), apart from its relative simplicity, is that in practice the ignition frequencies are so small (of the order of 10^{-4}) that the probability of ignition in more than one volume may be considered negligible.

Returning to consideration of equation (5.2), application of Lagrange's method of undetermined multipliers (see for example Walsh (1975)) leads to the Lagrangian function, where μ is the multiplier and $K_1=c_1+c_2+c_3+c_4$.

$$L = P_T+p_4(P_4+p_3(P_3+p_2(P_2+P_1p_1))) - \mu(K_1-c_1-c_2-c_3-c_4)$$

$$= P_T+p_4(P_4+p_3(P_3+p_2(P_2+P_1p_1))) - \mu\left[K_1-(\frac{a}{p_1} - a)-(\frac{a}{p_2} - a)-(\frac{a}{p_3} - a)-(\frac{a}{p_4} - a)\right]$$

$$= P_T+p_4(P_4+p_3(P_3+p_2(P_2+P_1p_1))) - \mu\left[K_2- \frac{a}{p_1} - \frac{a}{p_2}- \frac{a}{p_3}- \frac{a}{p_4}\right]$$

where $K_2= K_1+ 4a$

and finally,

$$L = P_T+p_4(P_4+p_3(P_3+p_2(P_2+P_1p_1))) - \lambda\left[K - \frac{1}{p_1} - \frac{1}{p_2}- \frac{1}{p_3}- \frac{1}{p_4}\right]$$

where $K = \frac{K_2}{a}$ and $\lambda = \mu a$.

Taking partial derivatives and setting them equal to zero yields

$$\frac{\partial L}{\partial p_1} = P_1p_2p_3p_4 - \frac{\lambda}{p_1^2} = 0$$

$$\frac{\partial L}{\partial p_2} = P_1p_1p_3p_4 + P_2p_3p_4 - \frac{\lambda}{p_2^2} = 0$$

$$\frac{\partial L}{\partial p_3} = P_1p_1p_2p_4 + P_2p_2p_4 + P_3p_4 - \frac{\lambda}{p_3^2} = 0$$

$$\frac{\partial L}{\partial p_4} = P_1p_1p_2p_3 + P_2p_2p_3 + P_3p_3 + P_4 - \frac{\lambda}{p_4^2} = 0,$$

and solving each of the above for λ,

$$\lambda = P_1p_1^2p_2p_3p_4$$

$$\lambda = P_1p_1p_2^2p_3p_4 + P_2p_2^2p_3p_4$$

$$\lambda = P_1p_1p_2p_3^2p_4 + P_2p_2p_3^2p_4 + P_3p_3^2p_4$$

$$\lambda = P_1p_1p_2p_3p_4^2 + P_2p_2p_3p_4^2 + P_3p_3p_4^2 + P_4p_4^2.$$

From these it follows that for optimality

$$\lambda = P_1p_1^2p_2p_3p_4 \;=\; P_1p_1p_2^2p_3p_4 + P_2p_2^2p_3p_4$$

$$\Rightarrow \quad P_1p_1^2 = P_1p_1p_2 + P_2p_2$$

$$\Rightarrow \quad P_1p_1^2 = p_2(P_1p_1 + P_2)$$

$$\Rightarrow \quad p_2 = \frac{P_1p_1^2}{P_1p_1 + P_2} \tag{5.4}$$

Similarly,

$$p_3 = \frac{P_1^2p_1^4}{P_1^2p_1^3 + P_1P_2p_1^2 + P_1P_3p_1 + P_2P_3} \tag{5.5}$$

$$p_4 = \frac{P_1^4p_1^8}{D}$$

where $D = P_1^4p_1^7 + P_1^3P_2p_1^6 + P_1^3P_3p_1^5 + (P_1^3P_2 + P_1^2P_2P_3)p_1^4 + 2P_1^2P_2P_4p_1^3$
$+ (P_1^2P_3P_4 + P_1P_2^2P_4)p_1^2 + 2P_1P_2P_3P_4p_1 + P_2^2P_3P_4$

The corresponding results for p_2 and p_3 in terms of c_1 are

$$p_2 = \frac{a^2P_1}{a^2(P_1+P_2) + ac_1(P_1+2P_2)+c_1^2P_2}$$

$$p_3 = \frac{a^4P_1^2}{d}$$

where $d = a^4(P_1P_3+P_1P_2+P_3P_2+P_1^2) + a^3c_1(3P_1P_3+2P_1P_2+4P_2P_3+P_1^2) +$
$a^2c_1^2(3P_1P_3+P_1P_2+6P_2P_3+P_1^2) + ac_1^3(P_1P_3+4P_2P_3) + c_1^4P_2P_3.$

The expression for p_4 is similar to those given for p_2 and p_3, but involves considerably more terms (53 in the denominator) and in consequence is informative as an indicator of the rapid increase in analytic complexity which follows an increase in the size of a linear structure.

For structures with a large number of volumes, the evaluation of expressions like those above will clearly be time consuming and computationally inefficient. More efficient, recursive forms of these expressions are derived below, and it is these which are used in the computer program to be introduced in sub-section 5.2.5.

From equation (5.4) it can be seen that, for any two adjacent volumes i, i+1 (with volume i+1 nearer to the target),

$$p_{i+1} = \frac{g_i p_i^2}{g_i p_i + P_{i+1}} \qquad (5.6)$$

where, as before, g_k= Pr{there is a fire in vol.k}.

Thus, when i=1, i+1=2, and $g_1 = P_1$ equation (5.6) simplifies to equation (5.4).

When i=2,

$$p_3 = \frac{g_2 p_2^2}{g_2 p_2 + P_3}$$

where $g_2 = P_1 p_1 + P_2$ and $p_2 = \dfrac{P_1 p_1^2}{P_1 p_1 + P_2}$,

so that,

$$p_3 = \frac{(P_1 p_1 + P_2)\left[\dfrac{P_1 p_1^2}{P_1 p_1 + P_2}\right]^2}{(P_1 p_1 + P_2)\left[\dfrac{P_1 p_1^2}{P_1 p_1 + P_2}\right] + P_3}$$

$$= \frac{\dfrac{P_1^2 p_1^4}{P_1 p_1 + P_2}}{P_1 p_1^2 + P_3}$$

$$= \frac{P_1^2 p_1^4}{(P_1 p_1^2 + P_3)(P_1 p_1 + P_2)}$$

and finally, -

$$p_3 = \frac{P_1^2 p_1^4}{P_1^2 p_1^3 + P_1 P_2 p_1^2 + P_1 P_3 p_1 + P_2 P_3} \qquad \text{as in equation (5.5).}$$

p_4 may be treated similarly.

Thus the optimal quality of door required for each doorway may be identified once the quality of door for the doorway next furthest from the target has been identified.

5.2.4 Two Important Theorems

Theorem 5.1 introduces a swapping rule applicable to structures of the type presented in this section.

Theorem 5.1:

For a given linear structure with doors already in place, if $i<k$ (so that door i is further from the target than door k) and $p_i<p_k$; the configuration of doors is sub-optimal and an improvement may be made by swapping the two doors, one for the other.

Proof:

Equation (5.6) states that for two adjacent doors i, i+1, with door i+1 nearer the target,

$$p_{i+1} = p_i\left[\frac{g_i p_i}{g_i p_i + P_{i+1}}\right],$$

where p_j is the breach probability associated with door j
P_j is the ignition probability associated with volume j
and g_j = Pr{there is a fire in volume j}.

As $g_i p_i \leq g_i p_i + P_{i+1}$, it is clear that $p_k \leq p_i$.

This demonstrates the monotonicity property of these particular linear structures as stated in the theorem, in that at optimality each door breach probability is less than (or equal to) that of any door further from the target.

It is essential to establish that the local stationary solution found using the above method is also a global optimum. A sufficient condition, provided that the solution is at an interior point of the feasible region, is that the function Φ is a convex function of $c_1, c_2, \ldots, c_n$ over the convex region $c_1 \geq 0$, $c_2 \geq 0$, ..., $c_n \geq 0$; $c_1+c_2+ \ldots +c_n \leq C$. The proof of Theorem 5.2 establishes this condition.

Theorem 5.2:

Consider again the general linear structure of Figure 5.3, reproduced as Figure 5.4 below, and let

$$\Phi = \Pr\{\text{fire reaches the target volume}\}$$
$$= P_n p_n + P_{n-1} p_n p_{n-1} + P_{n-2} p_n p_{n-1} p_{n-2} + \ldots + P_1 p_1 p_2 \ldots p_n$$
$$= P_n \phi_n + P_{n-1} \phi_{n-1} \quad + P_{n-2} \phi_2 \qquad + \ldots + P_1 \phi_1$$

where $\phi_i = p_i p_{i+1} \ldots p_n \quad 1 \leq i \leq n$,

then Φ is a convex function of the costs, c_i.

	barrier n	barrier n-1		barrier 2	barrier 1
Target Volume P_T	Volume n P_n	....	Volume 3 P_3	Volume 2 P_2	Volume 1 P_1

Figure 5.4 A general linear structure with n+1 volumes.

Proof:

As the sum of a set of convex functions is itself a convex function, and as a positive multiple of a convex function is a convex function, the convexity of Φ may be established by proving that the general term ϕ_i is convex.

Substituting $p_i = \dfrac{a}{c_i + a}$ into ϕ_i gives

$$\phi_i = \left[\frac{a}{c_i + a}\right]\left[\frac{a}{c_{i+1} + a}\right]\left[\frac{a}{c_{i+2} + a}\right] \cdots \left[\frac{a}{c_n + a}\right] \quad 1 \leq i \leq n$$

$$= \frac{a^{(n-i+1)}}{\prod_{j=i}^{n}(c_j + a)} .$$

The convexity of each ϕ_i may be proven by demonstrating that all the principal determinants of the Hessian H_i of ϕ_i, are non-negative. (See for example Walsh (1975).)

The Hessian of ϕ_i is

$$H_i = \begin{bmatrix} \dfrac{\partial^2 \phi_i}{\partial c_i^2} & \dfrac{\partial^2 \phi_i}{\partial c_i \partial c_{i+1}} & \cdots & \dfrac{\partial^2 \phi_i}{\partial c_i \partial c_n} \\ \dfrac{\partial^2 \phi_i}{\partial c_{i+1} \partial c_i} & \dfrac{\partial^2 \phi_i}{\partial c_{i+1}^2} & \cdots & \dfrac{\partial^2 \phi_i}{\partial c_{i+1} \partial c_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^2 \phi_i}{\partial c_n \partial c_i} & \dfrac{\partial^2 \phi_i}{\partial c_n \partial c_{i+1}} & \cdots & \dfrac{\partial^2 \phi_i}{\partial c_n^2} \end{bmatrix}$$

Where $$\frac{\partial^2 \phi_i}{\partial c_j^2} = \frac{2a^{(n-i+1)}}{\left[\prod_{k=i}^{j-1}(c_k + a)\right](c_j + a)^3\left[\prod_{k=j+1}^{n}(c_k + a)\right]}$$

and $$\frac{\partial^2 \phi_i}{\partial c_j \partial c_m} = \frac{a^{(n-i+1)}}{\left[\prod_{k=i}^{j-1}(c_k+a)\right](c_j+a)^2\left[\prod_{k=j+1}^{m-1}(c_k+a)\right](c_m+a)^2\left[\prod_{k=m+1}^{n}(c_k+a)\right]}$$

(with the condition that all products whose ranges are empty evaluate to unity).

Alternatively, $\frac{\partial^2 \phi_i}{\partial c_j^2} = \frac{2\phi_i}{(c_j+a)^2}$ and $\frac{\partial^2 \phi_i}{\partial c_j \partial c_m} = \frac{\phi_i}{(c_j+a)(c_m+a)}$

where $j=i, i+1, \ldots, n$.

Thus the r^{th} principal determinant of the Hessian of ϕ_i may be written as

$$PD_r(i) = \begin{vmatrix} \frac{2\phi_i}{(c_i+a)^2} & \frac{\phi_i}{(c_i+a)(c_{i+1}+a)} & \cdots & \frac{\phi_i}{(c_i+a)(c_{i+r-1}+a)} \\ \frac{\phi_i}{(c_{i+1}+a)(c_i+a)} & \frac{2\phi_i}{(c_{i+1}+a)^2} & \cdots & \frac{\phi_i}{(c_{i+1}+a)(c_{i+r-1}+a)} \\ \vdots & \vdots & & \vdots \\ \frac{\phi_i}{(c_{i+r-1}+a)(c_i+a)} & \frac{\phi_i}{(c_{i+r-1}+a)(c_{i+1}+a)} & \cdots & \frac{2\phi_i}{(c_{i+r-1}+a)^2} \end{vmatrix}$$

$r=1, 2, \ldots, n-i+1$.

Extracting the factor $\phi_i/(c_{i+p-1}+a)$ from each row p, $1 \le p \le r$ and the factor $(c_{i+p-1}+a)$ from each column q, $1 \le q \le r$ gives

$$PD_r(i) = \frac{\phi_i^r}{\prod_{j=i}^{i+r-1}(c_j+a)^2} \begin{vmatrix} 2 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 2 \end{vmatrix} = \frac{\phi_i^r}{\prod_{j=i}^{i+r-1}(c_j+a)^2} D_r$$

Subtracting row 1 from each of rows 2, 3, ..., r of D_r:

$$D_r = \begin{vmatrix} 2 & 1 & 1 & 1 & \cdots & & 1 \\ -1 & 1 & 0 & 0 & \cdots & & 0 \\ -1 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & & 1 \end{vmatrix}$$

and expanding by the bottom (r^{th}) row:

$$D_r = (-1)^{r-1}\left[-1 \begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 \\ 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 \end{vmatrix} + (-1)^{r-1}\, 1 \begin{vmatrix} 2 & 1 & 1 & 1 & \cdots & & 1 \\ -1 & 1 & 0 & 0 & \cdots & & 0 \\ -1 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \right]$$

For each minor subtract each of rows 2, 3, ..., r-1 in turn from row 1 to get

$$D_r = (-1)^{r-1}\left[-1 \begin{vmatrix} 0 & 0 & 0 & 0 & \cdots & & & 1 \\ 1 & 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 \end{vmatrix} + (-1)^{r-1}\, 1 \begin{vmatrix} r & 0 & 0 & 0 & \cdots & & 0 \\ -1 & 1 & 0 & 0 & \cdots & & 0 \\ -1 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \right]$$

and writing the mxm identity matrix as I_m:

$$D_r = (-1)^{r-1}\left[(-1)^{r-1} \left| I_{r-1} \right| + (-1)^{r-1}\, r \left| I_{r-1} \right| \right]$$

$$= (-1)^{r-1}\left[(-1)^{r-1}\left[1+r \right] \right]$$

$$= 1+r.$$

$$\text{So, } PD_r(i) = \frac{(r+1)\,\phi_i^r}{\prod_{j=i}^{i+r-1}(c_j+a)^2}$$

$$= \frac{\left[\dfrac{a^{(n-i+1)}}{\prod_{j=i}^{n}(c_j+a)}\right]^r (r+1)}{\prod_{j=i}^{i+r-1}(c_j+a)^2}$$

$$= \frac{\left[a^{(n-i+1)}\right]^r (r+1)}{\left[\prod_{j=i}^{n}(c_j+a)\right]^r \left[\prod_{j=1}^{i+r-1}(c_j+a)^2\right]}.$$

Finally,

$$PD_r(i) = \frac{\left[a^{(n-i+1)}\right]^r (r+1)}{\left[\prod_{j=i}^{i+r-1}(c_j+a)^{r+2}\right]\left[\prod_{j=i+r}^{n}(c_j+a)^r\right]}$$

This is always non-negative as $a>0$ and $c_i \geq 0$ for all i.

Thus every principal determinant of the Hessian of each ϕ_i is non-negative, implying that each function ϕ_i is convex and the desired result :

$$\Phi = \sum_{i=1}^{n} \phi_i P_i \text{ is a convex function}$$

is proven.

Figure 5.5 overleaf illustrates the relationship between the theoretical optimum breach probabilities of the five adjacent doors in a six-volume linear structure for the Inverse Model (Model 2) with a=1 and identical ignition probabilities. The values on the horizontal axis denote possible values for p_1, the failure probability of the door fitted furthest from the target volume, whilst the plotted curves depict the corresponding failure probabilities for the four doors fitted nearer to the target. These probabilities were calculated using equation 5.6 in sub-section 5.2.3. The curve lying closest to the horizontal axis shows the optimal values for p_5, the breach probability of the door bounding the target volume. The graph demonstrates clearly the dramatic increase in quality from one door to the next demanded by the theoretical solution.

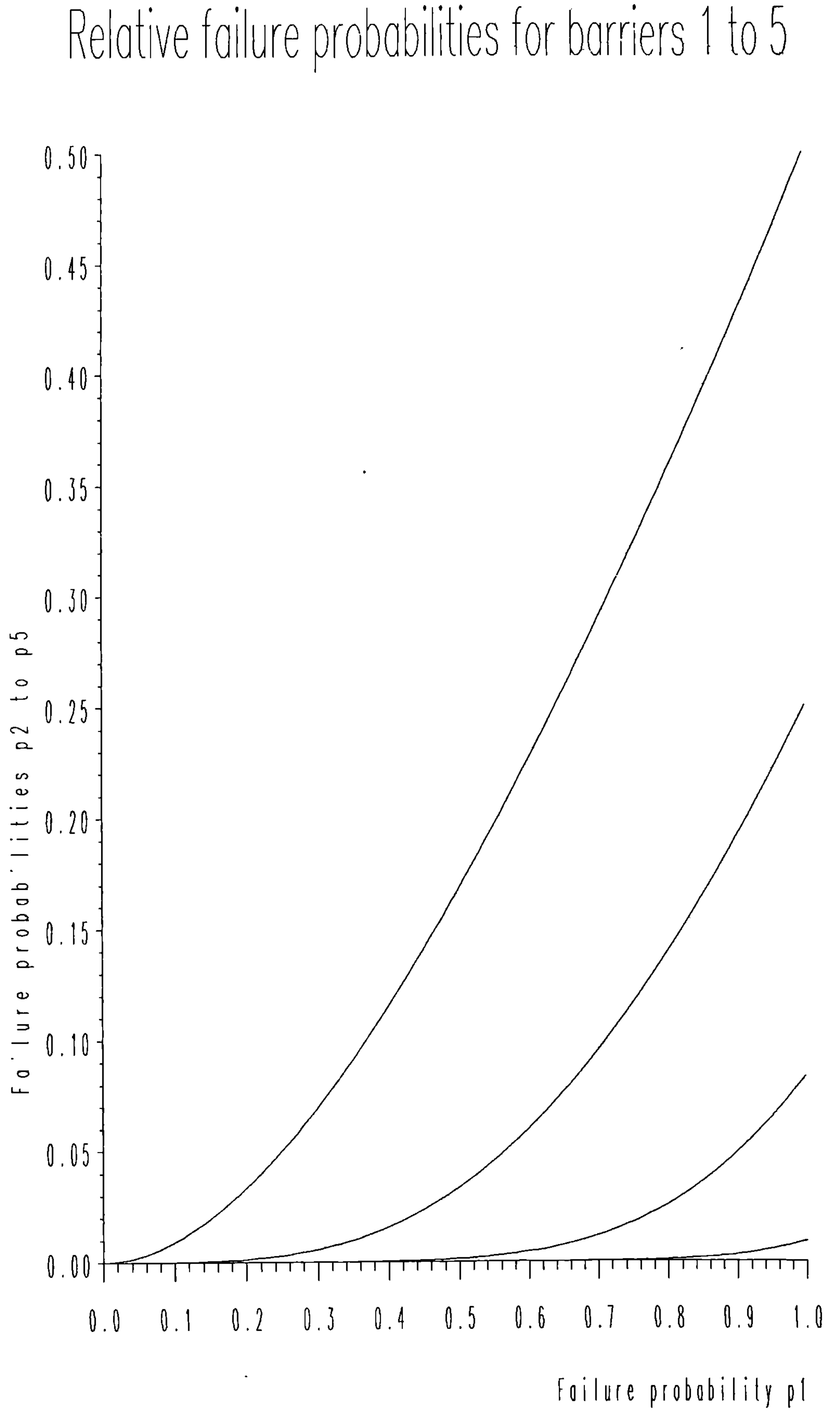


Figure 5.5 Optimal relative failure probablities of barriers 1 to 5.

It should be noted that the monotonicity property holds even when the ignition frequencies are chosen so that the relative frequencies for volumes furthest from the target are considerably larger than those nearer the target. This result is apparent from the algebra, but the numerical results are interesting in that they illustrate how the available funds are drawn away from the critical door and distributed slightly more evenly.

This decrease in probability, or increase in cost, is demonstrated numerically in the results of the Basic program which are summarized in Table 5.1 below. The results are derived from analysis of a six-volume structure with a target volume at one end, such as that depicted in Figure 5.4. Three cases are cited, Case A in which the ignition probabilities are equal, Case B in which the ignition probabilities increase with increasing distance from the target in the ratio 1:2:3:4:5:6 and Case C in which they increase similarly in the ratio 1:5:25:125:625:3125. In each case the parameter a=1 and the total budget available is 1000 units. They immediately show that however large the budget, most of it should be spent on the provision of a relatively excellent critical door between the target and its adjacent volume, and that a weighting of the ignition probabilities does not lead to a completely different pattern in the results.

	Barrier no.	Cost	Fail prob	Pr{reaches volume}	
Case A	1	0.2206	0.8193	0.1667	
	2	1.7104	0.3690	0.3032	
	3	5.7482	0.1482	0.2785	
	4	32.9965	0.0294	0.2079	
	5	959.3457	0.0010	0.1727	
	Total cost:	1000.0214		Pr{reach target}:	0.1668
Case B	1	0.5063	0.6639	0.2857	
	2	2.3971	0.2944	0.4270	
	3	7.5358	0.1172	0.3164	
	4	40.4332	0.0241	0.1799	
	5	949.1264	0.0011	0.0996	
	Total cost:	999.9988		Pr{reach target}:	0.0477
Case C	1	4.2468	0.1906	0.8001	
	2	9.7525	0.0930	0.3125	
	3	21.5924	0.0443	0.0611	
	4	75.0913	0.0131	0.0091	
	5	889.2496	0.0011	0.0014	
	Total cost:	999.9326		Pr{reach target}:	0.0003

Table 5.1 Optimal door breach probabilities and costs.

5.3 Linear Structures with a Limited Range of Doors

In reality, of course, only a few different types of door will be available, and there is certainly a limit on the best door which is available at any time. The theoretical solution discussed above is useful in that it provides a lower bound on the probability of fire reaching the target, but it does not in general provide a very tight bound because the solution allows an infinite variety of doors, including unrealistically fire-resistant ones. A much improved bound is provided by evaluating the theoretical optimum with the additional constraint that the 'best' door which may be fitted is restricted to a realistic value. This is done by the addition to the program of a simple loop feature which causes any door whose optimal cost is greater than the maximum allowed to be replaced by a 'best' door, and the solution to the whole optimization problem is re-evaluated in the light of that allocation. The theoretical optimal solution then still allows an infinite variety of doors, but within a reasonable price limitation. A discrepancy between the theoretical optimum and that obtainable in practice still exists, and arises from the tremendous flexibility provided by the infinite selection of doors from which the theoretical solution may make its choice.

5.3.1 Methods of Solution with a Limited Range of Doors

The optimal solution in a real-life situation is dependent upon the kinds of door which are available. The optimization problem is as detailed in section 5.1, with the additional constraint that only certain specific door types may be fitted to each doorway. Thus the optimization is now over a finite set of discrete values. It is well known that an optimal solution to the discrete problem may not simply be derived from a solution to a corresponding continuous optimization problem (see for example Taha (1987)) by perhaps choosing for each doorway that door whose characteristics most closely resembled those of the door required by the optimal solution in the continuous case.

In general, the discrete optimization problem may be solved using either the Branch and Bound technique (see, for example, Boffey(1982)), or complete enumeration, and it is the latter which is used in the computer programs developed for this work. Thus the discrete optimization in the examples that follow was carried out by evaluating all possible combinations of different doors in doorways, subject to the budget constraint, and selecting that permutation which yielded the smallest probability of fire reaching the target.

It seems sensible always to allow for the possibility that no door be fitted, - this is equivalent to fitting a door with unit breach probability and zero cost. Other realistic failure probabilities are available from various sources including the Safety and Reliability Directorate of the U.K. Atomic Energy Authority.

5.3.2 Application of the two models

Model 1

Under the assumptions of the Exponential model, the addition of the 'best available door' constraint, and indeed the limited range of doors constraint, simply lead to the modification of the original solution so that the probability of fire reaching the target is minimized by carrying out the following given in Algorithm 5.1.

Algorithm 5.1

1) Start at barrier n (adjacent to target)
2) Fit the best door which available resources allow
3) Move on to consideration of the next barrier
4) Repeat steps (2) & (3) until either
   a) total expenditure = budget constraint
   or b) all barriers have been considered
5) Stop.

Model 2

Solutions to the optimization problem were explored using Basic computer code on a BBC computer. The results from running the Basic program show the theoretical solution both with and without the 'best door' constraint, and also display optimal policy when the types of available door are known and specified. The first few runs of the program were used to determine which of the variables (number of doors to fit, number and types of available door, budget, and relative ignition frequencies) have a significant bearing on the result of the minimization problem, and at which values of the remaining variables these effects are important. It is clear from the analytic work earlier in the chapter that a substantial portion of the budget is best devoted to the door immediately adjacent to the target volume, and that very little is available to those doors which are some four or five volumes distant from the target. This result is well illustrated by numerical example. When the 'best door' constraint is active, there is a value of the budget for which no improvement may be made regardless of any extra resources made available. This situation arises since, if there are n doors to be fitted, and the best door available costs x units, a budget in excess of nx units will confer no advantage over a budget of exactly nx units as this amount is sufficient to furnish the structure with the best available doors. Thus the best measure of available budget is not the absolute amount available, but the proportion of the amount which would be required to fit top-price doors in all possible locations in the structure.

Figure 5.6 below shows graphically the results of fitting five doors to a six-volume linear structure such as that of Figure 5.4. The value of the parameter a is again taken to be unity, and the ignition probability is identical for each volume. The graph shows the effect, on the probability of fire reaching the target, of varying the budget available from 10% to 100% of that necessary to fit all best doors to the structure (the 'limiting budget'). The bottom curve shows the case where there is no restriction at all on the types of door available whilst the middle curve represents the case in which the best-available door constraint is active and the top curve represents the case in which the types of available door are specified. For this example, it was considered that six different types of door were available, with costs 0, 1, 3, 6, 10 and 15 units. These costs correspond respectively to the following breach probabilities: 1, 0.5, 0.25 ,0.1429, 0.0909 and 0.0625.

The results illustrated are typical of those realized when other parameters are chosen. The two particular features to note are perhaps first, spending more than about 40% of the limiting budget is not justifiable in terms of reduced risk; and second, imposition of the best-door constraint has a negligible effect when the financial allocation is a small percentage of the limiting budget.

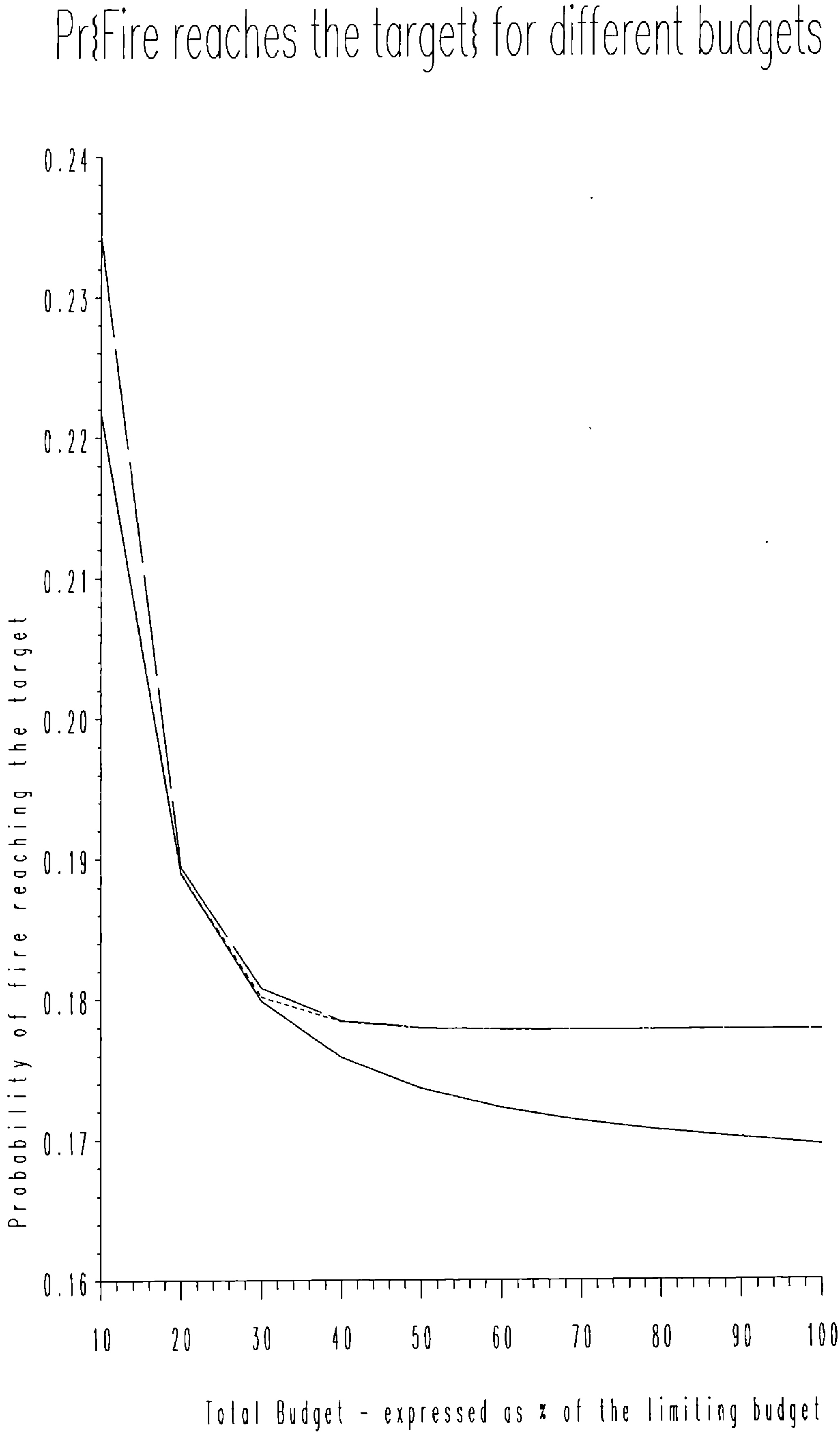


Figure 5.6 Plot of arrival probabilities vs budget.

The effects of varying the ignition probabilities (so that there is an increased relative risk of ignition in volumes further from the target) in the two constrained door-types cases are similar to those shown in Table 5.1 above for the unconstrained case. The pattern of resources being drawn away from the doors nearer to the target volume to those further away is repeated, but is less marked. If there are but few different types of door available, it is likely that, except in cases of extreme difference in ignition probability, no differences will be observed. It should be noted that if the ignition probabilities tend to be larger in volumes nearer the target, the optimal solution requires that even greater resources be devoted to the doors nearer the target volume.

The choice of door costs/breach probabilities for these examples has been subjective, but the selection chosen does facilitate an understanding of the relationships involved and the patterns which emerge from this model.

5.4 More Complex Linear Structures

In the previous sections, a very simple linear structure with a target volume at one end was considered. A natural extension of this basic work is the study of a similar linear chain, but with the target volume in such a position that fire may approach from both left and right. Such a structure is shown in Figure 5.7. The assumption that a door has a constant breach probability, regardless of the side from which it is threatened, is made throughout this section.

barrier r   barrier s

Volume 1 P_1	. . .	volume r P_r	target volume	Volume s P_s	. . .	Volume 2 P_2

Figure 5.7 A linear chain with a single interior target volume.

5.4.1 Application of Model 1

$$p_i = \exp(-kc_i) \Rightarrow c_i = \frac{-1}{k}\ln(p_i)$$

It follows from the symmetry of the structure that each side, left and right, may be treated individually as a structure having a target volume at one end. The implication of the results from above is that the optimal solution is to allocate all the resources available to the left hand side to the door between volume r and the target, and all the resources available to the right hand side to the door between volume s and the target.

This may be verified using the Lagrangian function in the usual way.

$$\phi = P_1 \prod_{\substack{i=1 \\ i \text{ odd}}}^{r} p_i + P_3 \prod_{\substack{i=3 \\ i \text{ odd}}}^{r} p_i + \ldots + P_r p_r + P_T + P_s p_s + \ldots + P_4 \prod_{\substack{i=4 \\ i \text{ even}}}^{s} p_i + P_2 \prod_{\substack{i=2 \\ i \text{ even}}}^{s} p_i$$

$$L = \sum_{\substack{j=1 \\ j \text{ odd}}}^{r} P_j \prod_{\substack{i=j \\ i \text{ odd.}}}^{r} p_i + P_T + \sum_{\substack{k=2 \\ k \text{ even}}}^{s} P_k \prod_{\substack{i=k \\ i \text{ even}}}^{s} p_i - \lambda\left[K + \frac{\ln p_1}{k} + \frac{\ln p_2}{k} + \ldots + \frac{\ln p_n}{k}\right]$$

where n=total number of non-target volumes.

Differentiating gives the following set of equations;

$$\frac{\partial L}{\partial p_1} = P_1 p_3 \cdots p_r - \frac{\lambda}{kp_1}$$

$$\frac{\partial L}{\partial p_2} = P_2 p_4 \cdots p_s - \frac{\lambda}{kp_2}$$

$$\frac{\partial L}{\partial p_3} = P_1 p_1 p_5 \cdots p_r + P_3 p_5 \cdots p_r - \frac{\lambda}{kp_1}$$

$$\frac{\partial L}{\partial p_4} = P_2 p_2 p_6 \cdots p_s + P_4 p_6 \cdots p_s - \frac{\lambda}{kp_2}$$

Setting these derivatives equal to zero and solving for $\frac{\lambda}{k}$ gives, for the general odd number, u, $(1 \leq u \leq r)$,

$$\frac{\lambda}{k} = \left[\sum_{\substack{j=1 \\ j \text{ odd}}}^{u} P_j \prod_{\substack{i=j \\ i \text{ odd} \\ i \neq u}}^{r} p_i \right] p_u$$

$$= \sum_{\substack{j=1 \\ j \text{ odd}}}^{u} P_j \prod_{\substack{i=j \\ i \text{ odd}}}^{r} p_i \qquad (5.7)$$

and for the general even number, v, $(2 \leq v \leq s)$,

$$\frac{\lambda}{k} = \sum_{\substack{j=2 \\ j \text{ even}}}^{v} P_j \prod_{\substack{i=j \\ i \text{ even}}}^{s} p_i \qquad (5.8)$$

The results for each side are equivalent to those derived in the single-sided case, with the additional equation which is formed by letting u and v in equations (5.7) and (5.8) take the values r and s respectively.

Thus

$$\sum_{\substack{j=1 \\ j\ odd}}^{r} P_j \prod_{\substack{i=j \\ i\ odd}}^{r} p_i = \sum_{\substack{j=2 \\ j\ even}}^{s} P_j \prod_{\substack{i=j \\ i\ even}}^{s} p_i. \tag{5.9}$$

The optimal solution requires that $p_i=1$ for all doors i; $i \neq r$, $i \neq s$, so that $p_1 = p_3 = \ldots p_{r-2} = p_2 = p_4 = \ldots p_{s-2} = 1$ may be substituted into equation (5.9) which then simplifies to

$$\left[\sum_{\substack{j=1 \\ j\ odd}}^{r} P_j \right] p_r = \left[\sum_{\substack{j=2 \\ j\ even}}^{s} P_j \right] p_s$$

and $$p_r = p_s \left[\frac{P_2 + P_4 + \ \ldots \ + P_s}{P_1 + P_3 + \ \ldots \ + P_r} \right] \tag{5.10}$$

and the ratio of p_r to p_s depends upon the total ignition probability on each side.

This relationship may be expressed in terms of the costs, c_r and c_s, and the budget. As $p_i = \exp(-kc_i)$, it follows from equation (5.10) that

$$\left[\sum_{\substack{j=1 \\ j\ odd}}^{r} P_j \right] \exp(-kc_r) = \left[\sum_{\substack{j=2 \\ j\ even}}^{s} P_j \right] \exp(-kc_s) \Rightarrow$$

$$kc_r = kc_s - \left\{ \ln\left[\sum_{\substack{j=2 \\ j\ \text{even}}}^{s} P_j \right] - \ln\left[\sum_{\substack{j=1 \\ j\ \text{odd}}}^{r} P_j \right] \right\} \tag{5.11}$$

and from equation (5.11) and the budget constraint c_r+c_s=Budget;

$$2c_s - \frac{1}{k}\left\{ \ln\left[\sum_{\substack{j=2 \\ j\ \text{even}}}^{s} P_j \right] - \ln\left[\sum_{\substack{j=1 \\ j\ \text{odd}}}^{r} P_j \right] \right\} = \text{Budget} \Rightarrow$$

$$c_s = \frac{\text{Budget}}{2} + \frac{1}{2k}\left\{ \ln\left[\sum_{\substack{j=2 \\ j\ \text{even}}}^{s} P_j \right] - \ln\left[\sum_{\substack{j=1 \\ j\ \text{odd}}}^{r} P_j \right] \right\}$$

$$c_r = \frac{\text{Budget}}{2} - \frac{1}{2k}\left\{ \ln\left[\sum_{\substack{j=2 \\ j\ \text{even}}}^{s} P_j \right] - \ln\left[\sum_{\substack{j=1 \\ j\ \text{odd}}}^{r} P_j \right] \right\}. \tag{5.12}$$

The above solution is based on the assumption that an infinite variety of doors is available. In practice, when a limited set of doors is available, this problem may be solved using either complete enumeration or branch and bound techniques. Use could also be made of Algorithm 5.2 given at the end of the following sub-section. The difficulty arises because the precise relative allocation of the budget to each side of the structure may not be feasible with a particular set of doors.

5.4.2 Application of Model 2

Initially consider a smaller structure as illustrated in Figure 5.8.

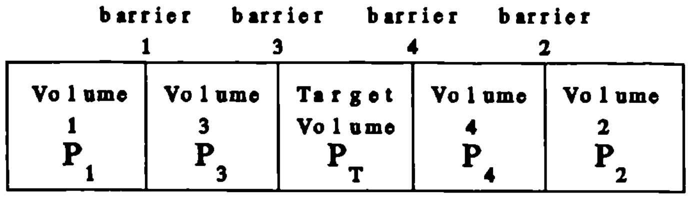


Figure 5.8 A five-volume structure with a centrally located target.

In this case the Lagrangian function is

$$L = P_T + P_1 p_1 p_3 + P_3 p_3 + P_2 p_2 p_4 + P_4 p_4 - \lambda\left[K - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p_4}\right].$$

Notice that the Lagrangian is separable, and so the results on each side of the target are equivalent to those of the single-sided case.

Taking partial derivatives and setting them equal to zero yields

$$\frac{\partial L}{\partial p_1} = P_1 p_3 - \frac{\lambda}{p_1^2} = 0$$
$$\frac{\partial L}{\partial p_2} = P_2 p_4 - \frac{\lambda}{p_2^2} = 0$$
$$\frac{\partial L}{\partial p_3} = P_1 p_1 + P_3 - \frac{\lambda}{p_3^2} = 0$$
$$\frac{\partial L}{\partial p_4} = P_2 p_2 + P_4 - \frac{\lambda}{p_4^2} = 0,$$

and solving each of the above for λ

$$\lambda = P_1 p_1^2 p_3$$
$$\lambda = P_2 p_2^2 p_4$$
$$\lambda = P_1 p_1 p_3^2 + P_3 p_3^2$$
$$\lambda = P_2 p_2 p_4^2 + P_4 p_4^2$$

and, as for the single-sided problem

$$p_3 = \frac{P_1 p_1^2}{P_1 p_1 + P_3} \tag{5.13}$$

and

$$p_4 = \frac{P_2 p_2^2}{P_2 p_2 + P_4} \tag{5.14}$$

There is of course a third identity, namely

$P_1 p_1^2 p_3 = P_2 p_2^2 p_4$
(since $\lambda = P_1 p_1^2 p_3 = P_2 p_2^2 p_4$)

and so

$$P_1 p_1^2 \left[\frac{P_1 p_1^2}{P_1 p_1 + P_3} \right] = P_2 p_2^2 \left[\frac{P_2 p_2^2}{P_2 p_2 + P_4} \right]. \tag{5.15}$$

In the event of the ignition frequencies being identical, equation (5.15) simplifies to

$$\frac{p_1^4}{p_1 + 1} = \frac{p_2^4}{p_2 + 1} \tag{5.16}$$

Solving for p_1 yields two complex roots and two identical real roots which are $p_1 = p_2$.

A solution to the more general problem may be found using Algorithm 5.2.

Algorithm 5.2

1) Allocate a proportion, C_L of the total budget to the left hand side of the structure, and the remaining $C-C_L$ to the right hand side,

2) Solve each single-sided problem,

3) Repeat for a range of values of C_L,

4) Plot the final values of ϕ and interpolate as necessary to find the minimum.

5.5 A Constrained Linear Structure with Two Target Volumes

A logical development of the previous work is the consideration of a linear structure which has two target volumes, an example of which is shown in Figure 5.9.

barrier 1 barrier 3 barrier 5 barrier 4 barrier 2

target volume T_1	Volume 1 P_1	volume 3 P_3	. . .	Volume 4 P_4	Volume 2 P_2	target volume T_2

Figure 5.9 A general two-target, n-volume structure.

It is now important to restate explicitly an assumption implied by the concept of a relationship between a door's cost and its breach probability. The existence of such a relationship means that each door can have only one breach probability, as mentioned at the beginning of this Chapter. The single breach probability does not allow a door to display different characteristics when threatened by a fire on one side from those displayed when threatened by a fire on the other. This feature has been of no consequence in the single-target structures considered heretofore, but now that there are two target volumes, this restriction does have an effect.

5.5.1 Application of Model 1

$p_i = \exp(-kc_i) \Rightarrow c_i = \frac{-1}{K} \ln(p_i)$

Using the same reasoning as employed previously, it follows that for a fire starting in volume 1, any resources devoted to door 3 may be equally well devoted to door 2; for a fire starting in volume 3, resources devoted to door 3 may be equally well devoted to door 1, and resources devoted to door 5 may equally be devoted to door 2; and so on for the whole structure. Thus the probability of fire reaching either of the two targets, given that it starts somewhere in the structure, is minimized by setting $p_i = 1$ for all $i \neq 1$, $i \neq 2$, and allocating all the available resources to the provision of doors 1 and 2.

In this case, the probability of fire threatening door 1, given that it breaks out in any of the volumes 1, 2, 3, ... , n is equal to the probability of fire threatening door 2, given that it breaks out in any of the volumes 1, 2, 3, ... , n.

This probability is

$$F_m = \sum_{i=1}^{n} P_i.$$

Thus the question of the relative allocation of resources to doors 1 and 2 is settled as

$$F_m p_1 = F_m p_2$$

so that the result

$$p_1 = p_2 \qquad (5.17)$$

is obtained and resources are divided equally between the two doors which bound the target volumes.

This result may be confirmed by the setting-up and solving of the Lagrangian in the usual way. As an example, classical methodology is used to find a solution in the case of the structure of Figure 5.10.

	barrier 1	barrier 3	barrier 4	barrier 2

target volume T_1	Volume 1 P_1	volume 3 P_3	Volume 2 P_2	target volume T_2

Figure 5.10 A two-target, five-volume structure.

Minimize $\phi = P_{T1} + P_{T2} + P_1p_1 + P_1p_3p_4p_2 + P_2p_2 + P_2p_4p_3p_1 + P_3p_3p_1 + P_3p_4p_2$

subject to $c_1 + c_2 + c_3 + c_4 \leq$ Budget=K, and $p_i = \exp(-kc_i)$.

$$
\begin{aligned}
L = P_{T1} &+ P_{T2} + P_1e^{-kC1} + P_1e^{-kC2}.e^{-kC3}.e^{-kC4} \\
&+ P_2e^{-kC2} + P_2e^{-kC1}.e^{-kC3}.e^{-kC4} \\
&+ P_3e^{-kC1}.e^{-kC3} + P_3e^{-kC2}.e^{-kC4} \\
&- \lambda(K - c_1 - c_2 - c_3 - c_4).
\end{aligned}
$$

Partial differentiation with respect to each of the c_i, setting equal to zero and solving for λ results in the following equalities:

$$
\begin{aligned}
& kP_1e^{-kC1} + kP_2e^{-kC1}.e^{-kC3}.e^{-kC4} + kP_3e^{-kC1}.e^{-kC3} \\
&= kP_1e^{-kC2}.e^{-kC3}.e^{-kC4} + kP_2e^{-kC2} + kP_3e^{-kC2}.e^{-kC4} \\
&= kP_1e^{-kC2}.e^{-kC3}.e^{-kC4} + kP_2e^{-kC1}.e^{-kC3}.e^{-kC4} + kP_3e^{-kC1}.e^{-kC3} \\
&= kP_1e^{-kC2}.e^{-kC3}.e^{-kC4} + kP_2e^{-kC1}.e^{-kC3}.e^{-kC4} + kP_3e^{-kC2}.e^{-kC4}.
\end{aligned}
$$

The solution of these is straightforward and it is readily seen that

$$e^{-kC3} = e^{-kC4} = 1 \qquad \text{so that } p_3 = p_4 = 1 \text{ and } c_3 = c_4 = 0$$

$$\text{and } e^{-kC1} = e^{-kC2} \qquad \text{so that} \qquad c_1 = c_2 = \frac{K}{2}.$$

Thus the application of standard methodology confirms the results obtained using the ad-hoc argument above.

5.5.2 Application of Model 2

$$p_j = \frac{a}{c_j + a} \Rightarrow c_j = \frac{a}{p_j} - a$$

Consideration is again given to the structure depicted in Figure 5.10.

The algebraic solution for a structure as simple as that of Figure 5.10 proves to be rather complex. The Lagrangian may be solved in the usual way, and quadratic or higher order equations may be derived for the various p_j, but beyond that stage numerical methods of solution must be employed.

$$L = P_{T_1} + P_{T_2} + P_1\left[p_1 + p_3p_4p_2\right] + P_2\left[p_2 + p_4p_3p_1\right] + P_3\left[p_3p_1 + p_4p_2\right] - \lambda\left[K - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}\right]$$

After partial differentiation, setting equal to zero and solving for λ, the following set of equations is found

$$\lambda = p_1^2\left[P_1 + P_3p_3 + P_2p_3p_4\right] \qquad \lambda = p_2^2\left[P_2 + P_3p_4 + P_1p_3p_4\right]$$

$$\lambda = p_3^2\left[P_3p_1 + P_2p_1p_4 + P_1p_2p_4\right] \quad \lambda = p_4^2\left[P_3p_2 + P_2p_1p_3 + P_1p_2p_3\right].$$

Such a set of equations may be solved in practice using multi-dimensional search techniques (see for example Walsh (1975)).

It is worth noting that a 'symmetrical' solution, - in other words one in which $p_1=p_2$, $p_3=p_4$, etc. only occurs when the ignition probabilities are similarly symmetrical.

5.6 A Non-linear Structure with Exponential Cost Function

In the introduction to this Chapter, Model 1 was derived from the following assumption:

Given that a door costing c units has breach probability p,

a door costing mc units will have breach probability p^m.

It was shown that the number of spaces, in excess of one (or two), into which to fit doors is irrelevant since all the available resources should be devoted to the one (or two) doors bounding the target volume(s).

It is this property which allows the theory to be extended readily to the consideration of non-linear structures as long as those non-linear structures may be decomposed into separate linear components.

The following structure, shown in Figure 5.11, is fairly common in practice, the volumes being arranged in a rectangle enclosing an open space such as a lawn or gardens.

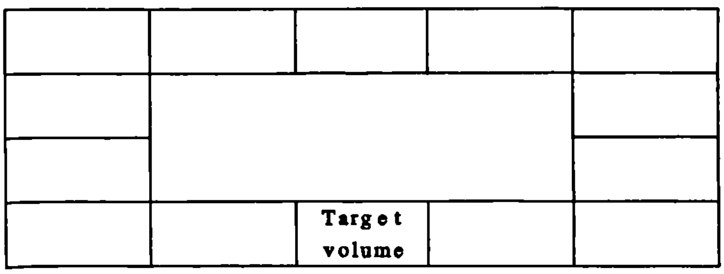


Figure 5.11 A number of volumes arranged in a quadrangle.

This structure is exactly equivalent to a fifteen volume linear chain which has a target at each end. Furthermore, from the results obtained above, that chain may be represented as the one in Figure 5.12 below whose solution is known to be $c_1 = c_2 = \frac{K}{2}$, where K is the limiting budget.

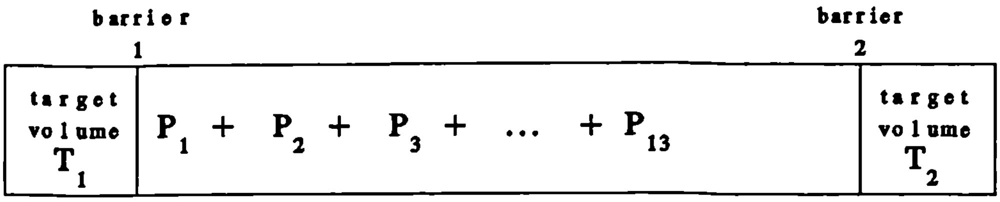


Figure 5.12 An equivalent representation of Figure 5.11.

Similarly, a structure such as that in Figure 5.13 is essentially constructed from two linear chains, the quadrangle (Chain A) and the spur (Chain B); and is identically equivalent to the structure shown in Figure 5.14.

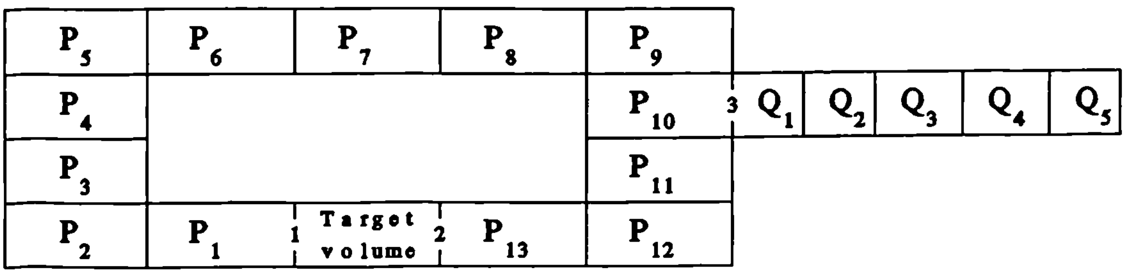


Figure 5.13 A quadrangle+spur arrangement of volumes, which may connect via the spur with other similar structures.

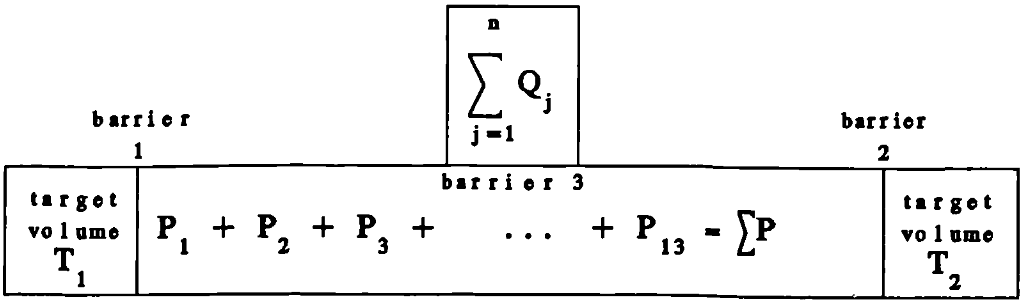


Figure 5.14 An equivalent representation of Figure 5.13.

With the non-degenerate doors being those numbered 1, 2 and 3, the solution to the optimization problem is obtained from consideration of the Lagrangian.

The relative magnitudes of $\sum P$ and $\sum Q$ are important and three cases must be considered, leading to:-

$$\text{Case 1) } \sum P < \sum Q \begin{cases} c_1 = c_2 = \frac{1}{2}\left[K - c_3 \right] \\ \exp(-kc_3) = \dfrac{\sum P}{\sum Q} \quad \text{so that } c_3 = -\frac{1}{K} \ln\left[\dfrac{\sum P}{\sum Q} \right] \end{cases}$$

$$\text{Case 2) } \sum P = \sum Q \begin{cases} c_1 = c_2 = \frac{1}{2}\left[K - c_3 \right] = \frac{K}{2} \\ \exp(-kc_3) = 1 \quad \text{so that } c_3 = 0 \end{cases}$$

$$\text{Case 3) } \sum P > \sum Q \begin{cases} c_1 = c_2 = \frac{1}{2}\left[K - c_3 \right] \\ \exp(-kc_3) = \dfrac{\sum P}{\sum Q} \quad \text{so that } c_3 = -\frac{1}{K} \ln\left[\dfrac{\sum P}{\sum Q} \right] \end{cases}$$

Case 3 is an infeasible solution since if $\sum P > \sum Q$, the logarithm will be strictly positive which in turn implies that $c_3 < 0$. Thus the door 3 above will only be fitted when the sum of the ignition probabilities in Chain B (the spur) exceeds that of the ignition probabilities in Chain A (the quadrangle).

5.7 Conclusion

Whilst none of the structures considered in this Chapter is particularly complex, each has served to illustrate the nature of a solution to a real problem with unfortunately ill-defined parameters. It was stated earlier in the Chapter that it is perhaps likely that any 'true' relationship between a door's cost and its nominal breach probability may lie somewhere between those described by the two models discussed here. Recommendations based on the work described here must emphasize the importance of bounding any target volumes with extremely good doors, generally at the expense of doors further from the target, since this was shown to be an appropriate course of action under both models. Such extreme results suggest that for the consideration of this aspect of the fire-spread modelling problem, it may be appropriate to revise the assumptions which indicate that any damage to non-target volumes may be considered negligible. Some further applications of this work are discussed in Chapter 7.

CHAPTER 6

Optimization of Segregated Structure Design - Time-dependent Models

Sed fugit interea, fugit inreparabile tempus.

Virgil

6.1 Introduction

A natural extension of the discussion in the last Chapter is the formulation of an optimal design strategy which is based not just upon minimizing the probability of fire reaching a target, but upon minimizing the probability that the time to arrival at a target is less than some value, t. In this case breach-time probability distributions are once again assigned to each of the available barriers.

The optimization problem is formulated as follows,

For a given value of time-since-ignition, t, fit a barrier to each pair of adjacent volumes so as to minimize

$\phi(t) = \Pr\{\text{Time to arrival at target} < t \mid \text{fire has started}\}$

subject to

Total cost of doors $\leq$ Budget.

6.2 Simple Linear Structures

The now familiar linear structure with a target at one end and ignition probabilities P_i, is considered, a five-volume example being shown in Figure 6.1.

	barrier 4	barrier 3	barrier 2	barrier 1
Target Volume P_T	Volume 4 P_4	Volume 3 P_3	Volume 2 P_2	Volume 1 P_1

Figure 6.1 A five-volume linear structure.

It is clearly not sufficient to attempt to specify a relationship between cost and breach probability, as was done in the last chapter, because each barrier's breach characteristics are expressed not in terms of a single probability but in terms of a breach-time probability distribution. What can be done instead is to establish a relationship between a barrier's cost, its mean failure time and the variance of that failure time.

6.2.1 A First Approach

The work presented in earlier chapters of this thesis indicates that of all candidate continuous breach-time distributions, the Gaussian distribution is the most convenient. Thus, for the following discussion, the assumption is made that each door's breach time is Normally distributed as $N(\mu_j, \sigma_j^2)$, and that the μ_j may take any positive value. Furthermore, let the relationship between cost and mean breach time be $\mu_j = kC_j$, where k is a positive constant. This relationship finds support in Maskell and Baldwin (1972).

If the further simplifying assumption of homoscedacity is made, so that $\sigma_j^2 = S^2$, for all j, the optimization for the five-volume structure illustrated above may be expressed as:

Min $\phi(t)$ =

Pr{Time for a fire to reach target $<t$ | it does not start in target}=

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{(S^2+S^2+S^2+S^2)^{0.5}}\right]+P_2\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{(S^2+S^2+S^2)^{0.5}}\right]$$
$$+P_3\Phi\left[\frac{t-\mu_3-\mu_4}{(S^2+S^2)^{0.5}}\right]+P_4\Phi\left[\frac{t-\mu_4}{S}\right]$$

subject to the constraints $\mu_j=kC_j$, $C_1+C_2+C_3+C_4\leq$ Budget. (6.1)

$$\left[\Phi[z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right]dt \text{ is the cumulative distribution function of the N(0, 1) distribution.}\right]$$

The general optimization problem is not easily solved using standard methodology, so a more ad-hoc approach is first adopted.

Assuming an infinite variety of doors to be available, it follows that it is possible to select four doors for the structure shown in Figure 6.1 so that the total cost of the doors is identically equal to the budget. Any choice of doors which did not make full use of the available budget would be sub-optimal because the particular relationship between cost and mean breach-time necessarily implies that spending more money yields a greater resistance to the spread of fire.

Thus the budget constraint:

Total cost of doors $\leq$ Budget

may be replaced by:

$C_1+C_2+C_3+C_4=$ Budget.

Now, let $\mu_3+\mu_4= \theta$, so that (6.1) may be written as

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-(\theta)}{2S}\right] +P_2\Phi\left[\frac{t-\mu_2-(\theta)}{\sqrt{3}S}\right] +P_3\Phi\left[\frac{t-(\theta)}{\sqrt{2}S}\right] +P_4\Phi\left[\frac{t-\mu_4}{S}\right] \qquad (6.2)$$

From (6.2) is is clear that, with respect to all the terms save the last, the partitioning of θ between μ_3 and μ_4 is arbitrary as far as the minimization is concerned. It is an examination of the fourth term which reveals that as Φ is a cdf, and therefore continuous and strictly monotonically increasing, μ_4 must take as large a value as possible for optimality to be attained. Thus $\mu_4=\theta$, $\mu_3= \theta-\mu_4=0$, and equation (6.1) simplifies to

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_4}{2S}\right] +P_2\Phi\left[\frac{t-\mu_2-\mu_4}{\sqrt{3}S}\right] +P_3\Phi\left[\frac{t-\mu_4}{\sqrt{2}S}\right] +P_4\Phi\left[\frac{t-\mu_4}{S}\right]. \qquad (6.3)$$

Applying a similar argument to that used in the last paragraph, $(\mu_2+\mu_4)$ is set equal to θ, so that the optimal partitioning of θ between μ_2 and μ_4 is dictated by consideration of the last two terms of (6.3). Reasoning as above leads to the conclusion that, to minimize $\phi(t)$, $\mu_4=\theta$ and $\mu_2=0$. Finally equation (6.1) reduces to

$$P_1\Phi\left[\frac{t-\mu_1-\mu_4}{2S}\right] + P_2\Phi\left[\frac{t-\mu_4}{\sqrt{3}S}\right] + P_3\Phi\left[\frac{t-\mu_4}{\sqrt{2}S}\right] + P_4\Phi\left[\frac{t-\mu_4}{S}\right], \qquad (6.4)$$

from which, proceeding as before yields the result that, at optimality, $\mu_i=0$ for $i=1, 2, 3$ and all available resources are devoted to the barrier immediately adjacent to the target volume. As in Model 1 (the Exponential case) discussed in the last Chapter, this result holds regardless of the values of the ignition probabilities. Furthermore, it is clear that, as long as the outlined assumptions are not violated, the above argument generalizes to a linear chain of any length.

6.2.2 A Model with More Realistic Assumptions

It should be noted that whilst the homoscedacity premise is attractive in that it admits a fairly simple introduction to a complex problem, it is unfortunately of not much use in practice! The theory which maintains that all doors of a particular design and material, but of different thickness, have a common variance is perhaps tenable when the mean burn through times are (i) fairly similar, and (ii) not close to zero; but under other circumstances there can be little justification for such a hypothesis.

A less-restrictive and more realistic constraint is provided by the specification that the coefficient of variation be constant over all doors. Such a constraint requires that the larger a door's expected breach time, the larger the variance of that time. It finds support in the work of Elms and Buchanan (1981). The optimization problem for the example structure then becomes

Minimize

Pr{Time for a fire to reach target $<t$ | it does not start in target}=

$\phi(t)$ =

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2)^{0.5}}\right]+P_2\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{(\sigma_2^2+\sigma_3^2+\sigma_4^2)^{0.5}}\right]+$$

$$P_3\Phi\left[\frac{t-\mu_3-\mu_4}{(\sigma_3^2+\sigma_4^2)^{0.5}}\right]+P_4\Phi\left[\frac{t-\mu_4}{\sigma_4}\right]$$

subject to the constraints $\frac{\sigma_i}{\mu_j} = K,\ \mu_j = kC_j,\ \sum C_j \leq$ Budget. (6.5)

Substituting $\sigma_i = K\mu_i$ into the objective function gives

$$\phi(t) = P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{K(\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2)^{0.5}}\right]+P_2\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{0.5}}\right]$$

$$+P_3\Phi\left[\frac{t-\mu_3-\mu_4}{K(\mu_3^2+\mu_4^2)^{0.5}}\right]+P_4\Phi\left[\frac{t-\mu_4}{K\mu_4}\right]. \qquad (6.6)$$

Let $\mu_3+\mu_4=\theta$, so that all numerators save the last are unaffected by the partitioning of θ between μ_3 and μ_4. Then $\mu_3^2+\mu_4^2=\theta^2-2\mu_3\mu_4$.

This in turn implies that the maximum value of $\mu_3^2+\mu_4^2$ occurs at

both of $\begin{cases} \mu_3=\theta,\ \mu_4=0; \\ \mu_3=0,\ \mu_4=\theta. \end{cases}$

Thus whilst the denominators do not remain constant regardless of the partitioning of θ, there is no conflict as the partitioning which minimizes the critical numerator $(t-\mu_4)$ also maximizes each of the denominators. Clearly it is μ_4, rather than μ_3, which must be set equal to θ, and the argument then proceeds as before. The conclusion, as in the homoscedastic case, is that

$\phi(t) = \Pr\{\text{time for fire to arrive at target} \leq t\}$

is minimized by allocating all available resources to fitting the best possible barrier immediately adjacent to the target, so that no barriers are fitted elsewhere.

6.2.3 Application of Standard Methodology

The standard method of solution of an optimization problem of this type is provided by consideration of an extended Lagrangian function and the application of the Kuhn-Tucker conditions (see for example Walsh (1975)). The methodology is here demonstrated for the homoscedastic case of the example structure illustrated at the start of the chapter.

Min $\phi(t)$ =

Pr{Time for a fire to reach target $<t$ | it does not start in target}=

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{(S^2+S^2+S^2+S^2)^{0.5}}\right]+P_2\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{(S^2+S^2+S^2)^{0.5}}\right]+$$

$$P_3\Phi\left[\frac{t-\mu_3-\mu_4}{(S^2+S^2)^{0.5}}\right]+P_4\Phi\left[\frac{t-\mu_4}{S}\right]$$

subject to the constraints $\mu_i = kC_i$, $\sum C_i \leq$ Budget $=$ K. (6.7)

The extended Lagrangian is

L =

$$P_1\Phi\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{2S}\right]+P_2\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{\sqrt{3}S}\right]+P_3\Phi\left[\frac{t-\mu_3-\mu_4}{\sqrt{2}S}\right]+P_4\Phi\left[\frac{t-\mu_4}{S}\right]$$

$$-\lambda(K-\frac{\mu_1}{k}-\frac{\mu_2}{k}-\frac{\mu_3}{k}-\frac{\mu_4}{k}) - (\theta_1\mu_1+\theta_2\mu_2+\theta_3\mu_3+\theta_4\mu_4) \quad (6.8)$$

Noting that

$$\frac{\partial \Phi(z)}{\partial z} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right),$$

the Lagrangian may be differentiated with respect to each of the parameters to yield this set of equations (6.9):

$$\frac{\partial L}{\partial \mu_1} = \frac{-P_1}{2S} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t-\mu_1-\mu_2-\mu_3-\mu_4}{2S}\right]^2\right\} + \frac{\lambda}{k} - \theta_1 = 0$$

$$= -P_1A + \frac{\lambda}{k} - \theta_1 = 0$$

$$\frac{\partial L}{\partial \mu_2} = -P_1A + \frac{-P_2}{\sqrt{3}S} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t-\mu_2-\mu_3-\mu_4}{\sqrt{3}S}\right]^2\right\} + \frac{\lambda}{k} - \theta_2 = 0$$

$$= -P_1A - P_2B + \frac{\lambda}{k} - \theta_2 = 0$$

$$\frac{\partial L}{\partial \mu_3} = -P_1A - P_2B + \frac{-P_3}{\sqrt{2}S} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t-\mu_3-\mu_4}{\sqrt{2}S}\right]^2\right\} + \frac{\lambda}{k} - \theta_3 = 0$$

$$= -P_1A - P_2B - P_3C + \frac{\lambda}{k} - \theta_3 = 0$$

$$\frac{\partial L}{\partial \mu_4} = -P_1A - P_2B - P_3C + \frac{-P_4}{S} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t-\mu_4}{S}\right]^2\right\} + \frac{\lambda}{k} - \theta_4 = 0$$

$$= -P_1A - P_2B - P_3C - P_4D + \frac{\lambda}{k} - \theta_4 = 0.$$

Also

$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad -\left[K - \frac{\mu_1}{k} - \frac{\mu_2}{k} - \frac{\mu_3}{k} - \frac{\mu_4}{k}\right] = 0 \qquad (6.10)$$

$$\frac{\partial L}{\partial \theta_i}\theta_i = 0 \quad \Rightarrow \quad \theta_1\mu_1 = \theta_2\mu_2 = \theta_3\mu_3 = \theta_4\mu_4 = 0. \qquad (6.11)$$

Solving equations (6.9) for $\frac{\lambda}{K}$ gives

$$\frac{\lambda}{K} = P_1A+\theta_1 = P_1A+P_2B+\theta_2 = P_1A+P_2B+P_3C+\theta_3 = P_1A+P_2B+P_3C+P_4D+\theta_4$$

$$\Rightarrow \quad \theta_1 = P_2B+\theta_2$$
$$\text{and } \theta_2 = P_3C+\theta_3 = P_3C+P_4D+\theta_4$$
$$\text{and } \theta_3 = P_4D+\theta_4 \qquad (6.12)$$

Since P_1A, P_2B, P_3C and P_4D must be non-zero (unless the corresponding ignition probabilities are zero in which case the problem is trivial), and as $\theta_j \geq 0$ for all j; θ_1, θ_2 and θ_3 must also be non-zero.

Then, from (6.11)

$$\mu_1 = \mu_2 = \mu_3 = 0. \qquad (6.13)$$

Finally, from (6.10) and (6.12)

$$K - \frac{\mu_4}{k} = 0 \Rightarrow \frac{\mu_4}{k} = K$$

and, as $\mu_i = kC_i$, this implies that

$C_4 = K$, which is the total budget available.

The result is identical to that achieved using the ad-hoc approach.

6.3 More Complex Linear Structures

The development of this chapter is similar to that of the last. Again, progressively more complex linear structures are introduced with the intention of demonstrating their flexibility as 'building blocks' of larger, less straightforward structures. Throughout this section results are derived under the assumption of a constant coefficient of variation, the results for the homoscedastic supposition being less useful in practice and in any case sufficiently similar to render unnecessary any separate treatment.

6.3.1 Structures with an Interior Target Volume

Consideration is first given to the case in which the target is in an interior volume, as illustrated in Figure 6.2.

	barrier 1		barrier 3		barrier 4		barrier 2	
Volume 1 P_1		Volume 3 P_3		Target Volume P_T		Volume 4 P_4		Volume 2 P_2

Figure 6.2 A five-volume structure with a centrally-located target volume.

Min $\phi(t)$ =

Pr{Time for a fire to reach target <t | it does not start in target}=

$$P_1\Phi\left[\frac{t-\mu_1-\mu_3}{(\sigma_1^2+\sigma_3^2)^{0.5}}\right] + P_2\Phi\left[\frac{t-\mu_2-\mu_4}{(\sigma_2^2+\sigma_4^2)^{0.5}}\right] + P_3\Phi\left[\frac{t-\mu_3}{\sigma_3}\right] + P_4\Phi\left[\frac{t-\mu_4}{\sigma_4}\right]$$

subject to the constraints $\frac{\sigma_i}{\mu_i} = K$, $\mu_i = kC_i$, $\sum C_i \leq$ Budget. (6.14)

The objective function clearly consists of two independent sub-expressions - one for fire starting to the left of the target

$$P_1\Phi\left[\frac{t-\mu_1-\mu_3}{(\sigma_1^2+\sigma_3^2)^{0.5}}\right] + P_3\Phi\left[\frac{t-\mu_3}{\sigma_3}\right]$$

and one for fire starting to the right of the target

$$P_2\Phi\left[\frac{t-\mu_2-\mu_4}{(\sigma_2^2+\sigma_4^2)^{0.5}}\right] + P_4\Phi\left[\frac{t-\mu_4}{\sigma_4}\right].$$

Each of these is similar to equation (6.5) in section 6.2. The argument applied there may be repeated here so that

$$\mu_1 = \mu_2 = 0,\ C_3 + C_4 = \text{Budget}$$

and, as was the case in Chapter 5, the resources are allocated for the provision of two doors bounding the target; the relative allocation being dependent upon the ratio of the sums of the ignition frequencies on each side.

6.3.2 Structures with Two Target Volumes

Suppose now that there are two target volumes, one at each end of the structure, as shown in Figure 6.3.

	barrier 1	barrier 3	barrier 4	barrier 2
target volume T_1	Volume 1 P_1	volume 3 P_3	Volume 2 P_2	target volume T_2

Figure 6.3 A five-volume linear structure with two target volumes.

Min $\phi(t)$ =

Pr{Time for a fire to reach target <t | it does not start in target}=

$$P_1\Phi\left[\frac{t-\mu_1}{\sigma_1}\right] + P_1\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{(\sigma_2^2+\sigma_3^2+\sigma_4^2)^{0.5}}\right] + P_3\Phi\left[\frac{t-\mu_1-\mu_3}{(\sigma_1^2+\sigma_3^2)^{0.5}}\right] +$$
$$P_3\Phi\left[\frac{t-\mu_2-\mu_4}{(\sigma_2^2+\sigma_4^2)^{0.5}}\right] + P_2\Phi\left[\frac{t-\mu_2}{\sigma_2}\right] + P_2\Phi\left[\frac{t-\mu_1-\mu_3-\mu_4}{(\sigma_1^2+\sigma_3^2+\sigma_4^2)^{0.5}}\right]$$

subject to the constraints $\frac{\sigma_i}{\mu_i} = K,\ \mu_i = kC_i,\ \sum C_i \leq$ Budget. (6.15)

Substituting $\sigma_i = K\mu_i$ gives

Min $\phi(t)$ =

$$P_1\Phi\left[\frac{t-\mu_1}{K\mu_1}\right] + P_1\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{0.5}}\right] + P_3\Phi\left[\frac{t-\mu_1-\mu_3}{K(\mu_1^2+\mu_3^2)^{0.5}}\right] +$$
$$P_3\Phi\left[\frac{t-\mu_2-\mu_4}{K(\mu_2^2+\mu_4^2)^{0.5}}\right] + P_2\Phi\left[\frac{t-\mu_2}{K\mu_2}\right] + P_2\Phi\left[\frac{t-\mu_1-\mu_3-\mu_4}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{0.5}}\right] \quad (6.16)$$

The extended Lagrangian is

$$L = P_1\Phi\left[\frac{t-\mu_1}{K\mu_1}\right] + P_1\Phi\left[\frac{t-\mu_2-\mu_3-\mu_4}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{0.5}}\right] + P_3\Phi\left[\frac{t-\mu_1-\mu_3}{K(\mu_1^2+\mu_3^2)^{0.5}}\right] +$$
$$P_3\Phi\left[\frac{t-\mu_2-\mu_4}{K(\mu_2^2+\mu_4^2)^{0.5}}\right] + P_2\Phi\left[\frac{t-\mu_2}{K\mu_2}\right] + P_2\Phi\left[\frac{t-\mu_1-\mu_3-\mu_4}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{0.5}}\right] -$$
$$\lambda\left(K-\frac{\mu_1}{K}-\frac{\mu_2}{K}-\frac{\mu_3}{K}-\frac{\mu_4}{K}\right) - (\theta_1\mu_1+\theta_2\mu_2+\theta_3\mu_3+\theta_4\mu_4).$$

Define:-

$$A = \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left[\frac{t-\mu_1}{K\mu_1}\right]^2\right\}, \qquad B = \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left[\frac{t-\mu_2}{K\mu_2}\right]^2\right\},$$

$$C = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t - \mu_1 - \mu_3}{K(\mu_1^2+\mu_3^2)^{0.5}}\right]^2\right\}, \quad D = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t - \mu_2 - \mu_4}{K(\mu_2^2+\mu_4^2)^{0.5}}\right]^2\right\},$$

$$E = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t - \mu_1 - \mu_3 - \mu_4}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{0.5}}\right]^2\right\}, \quad F = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-1}{2}\left[\frac{t - \mu_2 - \mu_3 - \mu_4}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{0.5}}\right]^2\right\}.$$

Then, differentiation of the Lagrangian gives

$$\frac{\partial L}{\partial \mu_1} = -\frac{P_1 t}{K\mu_1^2}.A - \frac{P_2(\mu_1(t-\mu_1-\mu_3-\mu_4)+\mu_1^2+\mu_3^2+\mu_4^2)}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{3/2}}.E$$

$$-\frac{P_3(\mu_1(t-\mu_1-\mu_3)+\mu_1^2+\mu_3^2)}{K(\mu_1^2+\mu_3^2)^{3/2}}.C + \frac{\lambda}{K} - \theta_1 = 0.$$

$$\frac{\partial L}{\partial \mu_2} = -\frac{P_1(\mu_2(t-\mu_2-\mu_3-\mu_4)+\mu_2^2+\mu_3^2+\mu_4^2)}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{3/2}}.F - \frac{P_2 t}{K\mu_2^2}.B$$

$$-\frac{P_3(\mu_2(t-\mu_2-\mu_4)+\mu_2^2+\mu_4^2)}{K(\mu_2^2+\mu_4^2)^{3/2}}.D + \frac{\lambda}{K} - \theta_2 = 0.$$

$$\frac{\partial L}{\partial \mu_3} = -\frac{P_1(\mu_3(t-\mu_2-\mu_3-\mu_4)+\mu_2^2+\mu_3^2+\mu_4^2)}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{3/2}}.F$$

$$-\frac{P_2(\mu_3(t-\mu_1-\mu_3-\mu_4)+\mu_1^2+\mu_3^2+\mu_4^2)}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{3/2}}.E$$

$$-\frac{P_3(\mu_3(t-\mu_1-\mu_3)+\mu_1^2+\mu_3^2)}{K(\mu_1^2+\mu_3^2)^{3/2}}.C + \frac{\lambda}{K} - \theta_3 = 0.$$

$$\frac{\partial L}{\partial \mu_4} = - \frac{P_1(\mu_4(t-\mu_2-\mu_3-\mu_4)+\mu_2^2+\mu_3^2+\mu_4^2)}{K(\mu_2^2+\mu_3^2+\mu_4^2)^{3/2}}.F$$

$$- \frac{P_2(\mu_4(t-\mu_1-\mu_3-\mu_4)+\mu_1^2+\mu_3^2+\mu_4^2)}{K(\mu_1^2+\mu_3^2+\mu_4^2)^{3/2}}.E$$

$$- \frac{P_3(\mu_4(t-\mu_2-\mu_4)+\mu_2^2+\mu_4^2)}{K(\mu_2^2+\mu_4^2)^{3/2}}.D + \frac{\lambda}{K} - \theta_4 = 0.$$

(6.17)

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow - [C - \frac{\mu_1}{K} - \frac{\mu_2}{K} - \frac{\mu_3}{K} - \frac{\mu_4}{K}] = 0. \quad (6.18)$$

$$\frac{\partial L}{\partial \theta_i}.\theta_i = 0 \Rightarrow \theta_1\mu_1 = \theta_2\mu_2 = \theta_3\mu_3 = \theta_4\mu_4 = 0. \quad (6.19)$$

In view of the complexity of these equations the following discussion is heuristic. From the work in Chapter 5 it seemed sensible to try setting $\mu_3 = \mu_4 = 0$, so that all the budget be devoted to the two barriers adjacent to the target volumes, and from there to derive the corresponding values of μ_1 and μ_2.

Setting $\mu_3 = \mu_4 = 0$, yields

C = A, D = B, E = A, and F = B; so that

$$\frac{\partial L}{\partial \mu_1} = - \frac{P_1 t}{K\mu_1^2}.A - \frac{P_2 t}{K\mu_1^2}.A - \frac{P_3 t}{K\mu_1^2}.A + \frac{\lambda}{K} - \theta_1 = 0.$$

$$\frac{\partial L}{\partial \mu_2} = - \frac{P_1 t}{K\mu_2^2}.B - \frac{P_2 t}{K\mu_2^2}.B - \frac{P_3 t}{K\mu_2^2}.B + \frac{\lambda}{K} - \theta_2 = 0.$$

$$\frac{\partial L}{\partial \mu_3} = - \frac{P_1}{K\mu_2}.B - \frac{P_2}{K\mu_1}.A - \frac{P_3}{K\mu_1}.A + \frac{\lambda}{K} - \theta_3 = 0.$$

$$\frac{\partial L}{\partial \mu_4} = -\frac{P_1}{K\mu_2}.B - \frac{P_2}{K\mu_2}.A - \frac{P_3}{K\mu_2}.B + \frac{\lambda}{k} - \theta_4 = 0.$$

Solving each of the above for $\frac{\lambda}{K}$ gives

$$\frac{\lambda}{K} = \frac{tA}{K\mu_1^2}.\left[P_1 + P_2 + P_3\right] + \theta_1 = \frac{tB}{K\mu_2^2}.\left[P_1 + P_2 + P_3\right] + \theta_2 =$$

$$\frac{P_1 B}{K\mu_2} + \frac{A}{K\mu_1}.\left[P_2 + P_3\right] + \theta_3 = \frac{B}{K\mu_2}.\left[P_1 + P_3\right] + \frac{P_2 A}{K\mu_1} + \theta_4.$$

For μ_1 and μ_2 to be non-zero, $\theta_1 = \theta_2 = 0$, and the results of Chapter 5 suggest the investigation of the behaviour of $\phi(t)$ when $\mu_1 = \mu_2$.

Lemma 6.1

$$\mu_1 = \mu_2 = \frac{k.\text{Budget}}{2}, \quad \mu_3 = \mu_4 = 0$$

is one solution of the minimization problem.

It is straightforward to demonstrate that this solution is not, in practical cases, a maximum (so that it is a saddle point or a minimum) - it need only be shown that the value of $\phi(t)$ is increases when the μ_i take values other than those derived above.

For example, with $\mu_1 = \mu_2 = \dfrac{k\ \text{Budget}}{2}$, $\mu_3 = \mu_4 = 0$;

$$\phi(t) = \left[P_1 + P_2 + P_3\right].\left\{ \Phi\left[\frac{t - \mu_1}{K\mu_1}\right] + \Phi\left[\frac{t - \mu_2}{K\mu_2}\right] \right\}$$

$$= \left\{ \Phi\left[\frac{t-\mu}{K\mu}\right] + \Phi\left[\frac{t-\mu}{K\mu}\right] \right\} \cdot \sum_{i=1}^{3} P_i \qquad (6.20)$$

whereas, were the resources not equally divided between barrier 1 and barrier 2 (but still devoted to those two, to the exclusion of barriers 3 and 4), so that $\mu_1 = \mu+\delta, \mu_2 = \mu-\delta, \mu_3 = \mu_4 = \sigma_3 = \sigma_4 = 0$, then

$$\phi_1(t) = \left[P_1 + P_2 + P_3\right] \cdot \left\{ \Phi\left[\frac{1}{K}\left(\frac{t}{\mu+\delta} - 1\right)\right] + \Phi\left[\frac{1}{K}\left(\frac{t}{\mu-\delta} - 1\right)\right]\right\}$$

and

$$\left[\frac{1}{(P_1+P_2+P_3)}\right] \cdot \frac{\partial\phi_1}{\partial\delta} =$$

$$\frac{-t}{\sqrt{2\pi}\,K}\left\{\frac{1}{(\mu+\delta)^2}\exp\left[\frac{-1}{2K^2}\left(\frac{t}{\mu+\delta} - 1\right)^2\right] - \frac{1}{(\mu-\delta)^2}\exp\left[\frac{-1}{2K^2}\left(\frac{t}{\mu-\delta} - 1\right)^2\right]\right\}$$

so that one solution of $\frac{\partial\phi_1}{\partial\delta} = 0$ is $\delta = 0$.

Taking the second derivative,

$$\left[\frac{1}{(P_1+P_2+P_3)}\right] \cdot \frac{\partial^2\phi_1}{\partial\delta^2} =$$

$$\frac{-t}{\sqrt{2\pi}\,K}\left\{\exp\left[\frac{-1}{2K^2}\left(\frac{t}{\mu+\delta} - 1\right)^2\right] \cdot \left[\frac{-2}{(\mu+\delta)^3} + \frac{1}{(\mu+\delta)^2}\,\frac{2t}{2K^2(\mu+\delta)^2}\left(\frac{t}{\mu+\delta} - 1\right)\right]\right\}$$

$$+\frac{t}{\sqrt{2\pi}\,K}\left\{\exp\left[\frac{-1}{2K^2}\left(\frac{t}{\mu-\delta} - 1\right)^2\right] \cdot \left[\frac{2}{(\mu-\delta)^3} + \frac{1}{(\mu-\delta)^2}\,\frac{-2t}{2K^2(\mu+\delta)^2}\left(\frac{t}{\mu+\delta} - 1\right)\right]\right\}.$$

Set $\delta=0$, then

$$\left.\frac{\partial^2 \phi_1}{\partial \delta^2}\right|_{\delta=0} = \frac{-2t}{\sqrt{2\pi}\,K} \exp\left[\frac{-1}{2K^2}\left[\frac{t}{\mu}-1\right]^2\right] \cdot \left\{\frac{t}{K^2\mu^4}\left[\frac{t}{\mu}-1\right] - \frac{2}{\mu^3}\right\} \qquad (6.21)$$

$\phi_1(t)$ is minimized when (6.21) > 0, and thus when

$$\left\{\frac{t}{K^2\mu^4}\left[\frac{t}{\mu}-1\right] - \frac{2}{\mu^3}\right\} > 0.$$

This inequality leads to the constraint

$$t \leq \frac{\mu}{2}\left[1 + (1+8K^2)^{0.5}\right]. \qquad (6.22)$$

Since K^2 is always non-negative (as is K since $K = \frac{\sigma}{\mu}$), even in the most restrictive case ($K \simeq 0$) $\phi(t)$ is likely to be minimized by setting $\mu_1 = \mu_2 = \mu$ and $\mu_3 = \mu_4 = 0$ as long as $t \leq \frac{k.\text{Budget}}{2}$.

As K increases, the critical value of t increases and the constraint (6.22) becomes less restrictive.

Thus, in general, for an n-volume linear structure with two targets, one at each end,

when $t \leq \frac{\mu}{2}\left[1 + (1+8K^2)^{0.5}\right]$,

$\phi(t)$ =

Probability that {Time to arrival at target <t, given that fire has

broken out in one of the non-target volumes of the structure}
is likely to be minimized as

$$\phi(t) = 2\Phi\left[\frac{t - \mu}{K\mu}\right] . \sum_{i=1}^{n} P_i \quad \text{where } \mu = \frac{k.\text{Budget}}{2}, \quad K = \frac{\sigma}{\mu}, \tag{6.23}$$

$$= 2\Phi\left[\frac{2t - k.\text{Budget}}{2\sigma}\right] . \sum_{i=1}^{n} P_i . \tag{6.24}$$

It should be noted that the constraint on the parameter t is not as restrictive as may first be thought, in practical terms it simply ensures that its value is specified in a manner which is consistent with the nature of the particular problem. In other words, there is not much point in trying to minimize $\phi(50)$ when the budget is sufficient only to provide up to two doors each having a mean failure time of 10 minutes. Furthermore, it is in general the smaller values of t, relative to the budget, which will be of interest to building designers and engineers.

6.4 Linear Structures with a Limited Range of Doors

The results obtained in the previous sections have been based upon the premise that an infinite range of doors is available, subject to the mean breach times being non-negative, and there being some constraint on the variances of those breach times. How is progress made in the light of the knowledge that an infinite variety of doors is not available? In the case of a structure with a single target volume at one end, the principle of allocating all available resources to the barrier adjacent to the target is not changed; and Algorithm 5.1, presented in sub-section 5.3.2 of Chapter 5, may again be employed. The same is true for a structure such as that shown in Figure 6.4.

	barrier 1	barrier 3	barrier 5		barrier 4	barrier 2
target volume T_1	Volume 1 P_1	volume 3 P_3	. . .	Volume 4 P_4	Volume 2 P_2	target volume T_2

Figure 6.4 A general two-target volume linear structure.

For a structure such as this, subject to inequality (6.22), it is desirable that the available budget be divided equally between barrier 1 and barrier 2, and failing that, it be divided as evenly as possible between the 'odds' and the 'evens', with as much weight as possible to the lower-numbered barriers on each side.

When a linear structure is such that access to a single target is from both sides, Algorithm 5.2, presented in sub-section 5.4.2 of Chapter 5, is again found to be appropriate.

The solutions of optimization problems for which the types of available door are known and specified may be found using branch and bound techniques or complete enumeration, each of which in turn requires the solution of a number of sub-problems using methodology such as the above.

6.5 Conclusion

An optimization problem, concerning how doors may be best fitted to a structure so as to minimize the probability of fire reaching a target volume in time $T<t$, has been introduced. Several methods of solution have been discussed for the case when each of the doors has a Gaussian breach-time distribution. The general result that it is the barriers adjacent to the target volumes to which the bulk of the

financial resources must be allocated is undoubtedly intuitive and is complementary to the results discovered in Chapter 5. Were the problem to be structured in such a way that fire damage to non-target volumes were not disregarded, different conclusions would be drawn.

The principles of the solution germane to linear structures are also applicable to non-linear structures which may be decomposed into linear components. A general discussion of the applicability of these models is given in Chapter 7.

CHAPTER 7

Summary and General Discussion

To put one brick upon another,
Add a third, and then a fourth,
Leaves no time to wonder whether
What you do has any worth.

But to sit with bricks around you
While the winds of heaven bawl
Weighing what you should or can do
Leaves no doubt of it at all.

Philip Larkin

7.1 Conclusions

This thesis represents the first attempt to model the spread of flames through a segregated structure whose barriers are not constrained to have either constant breach probabilities or constant coefficient-of-variation Gaussian breach-time distributions.

It has sought to explore the relationship between the modelling of fire-spread and other related modelling problems, and to identify those aspects which render this fire-spread modelling problem distinct. The thesis suggests some new approaches which offer a contribution to the development of an understanding of the problems involved, and which facilitate the development of some solutions.

There is more work to be done. On the theoretical side, the value of further exploration of non-asymptotic extreme value theory as applied to random variables that are not independent and identically distributed has been identified. The development of such theory could enhance the work contained here, and would undoubtedly be found pertinent to other fields of applied research. The relative simplicity of the discrete-time models discussed in Chapter 3 lends them a certain attraction despite the shortcomings imposed by the assumptions, and it may be that in practice a discrete-time model is preferred, and perhaps consideration should be given to models other than the Multinomial discussed here, with preference accorded to the Poisson distribution or the Negative-binomial distribution.

The application of the models presented in Chapter 3 would be eased by the methodology being incorporated into some suitable software. Many of the necessary features, such as route identification, are already to be found in the ARSSUN code, and the recursive calculations described in Algorithm 3.1 and Algorithm 3.2 should present no difficulty to the experienced programmer. The main advantage which the Multinomial model has over ARSSUN is that instead of providing a single-value 'fire reaches target' probability, it yields a 'time-to-reach-target' probability distribution, as is illustrated in sub-section 3.2.5. This, I believe, is sufficient to justify the SRD devoting resources to the writing of some software so that the model's advantages be readily accessible.

The fundamental aspects of the continuous-time models developed in Chapter 4 were presented, by this author, at a conference on reliability technology (Veevers and Manasse (1990)) and the subject matter was well-received. The main reason for the enthusiasm probably arose from the barriers' continuous breach-time probability distributions giving the models an intuitive appeal not shared by time-independent models. It is worth re-iterating that whilst the candidate distributions were framed in terms of barrier breach times, there is no evidence to suggest that suitable parameter selection would not allow the distributions to model accurately the time for a fire's development to flashover intensity plus the time required to breach a barrier. The application of Extreme Value Theory, as detailed in section 4.5, is an important aspect of this fire-spread model whose consideration is only made possible as a result of the time-dependency which has been introduced. Once again, there can be no doubt that the utility of the models would be enhanced by their being coded up, a task which is unfortunately not straightforward, but one whose successful undertaking would be of great benefit to a fire-safety engineer wishing to utilize the methodology.

The work contained in Chapters Five and Six represents the results of an initial exploration of a previously un-explored problem. Once again, the absence of any substantial information about accurate quantification of fire-resistance, whether as a single probability (or time-value), or as a breach-time probability distribution, is to be lamented since there is much to be said for not constructing a model until there is sufficient data to support it. On the other hand, the two models of Chapter 5 provide some reasonable measure of the real-life situation, and both Chapter 5 and Chapter 6 demonstrate that even in the absence of reliable data, some headway may be made. It has been demonstrated that the focusing on linear structures is justifiable in the light of their usefulness as the building blocks of larger structures. It should also be noted that they have an applicability in their own right. Both the Manchester Airport fire and the King's Cross London Underground fire of November 1987 have demonstrated the devastating effects of fire growth in a single volume, and each raises the question: what would have happened had there been some sort of fire barriers partitioning the volumes into compartments? In the case of an aircraft, it is self-evident that fitting high-resilience fire-resisting foam seating is not sufficient if a fire starting in the body of the aircraft has ready access to the fuel in pipes or tanks. On a driverless train, especially one transporting non-flammable freight, the controlling computer is certainly critical, and the work establishes that resources should be devoted to fitting a very good fire-resisting barrier between the control room and the rest of the train, rather than to the fitting of a number of lower quality fire barriers at intervals along the train's length. Another linear structure which is the focus of current attention is the Channel Tunnel, in which the signal and communications systems are of prime importance.

Whilst the REDUCE computer algebra package was discussed only in the context of Chapter 3, it has also been put to good use in the exploration of some of the topics in other Chapters. It provides a useful tool for the simplification of algebraic constructs, and for the evaluation of numerical results derived from substitutions into those algebraic equations.

7.2 Final Example

Perhaps all that remains is to return to the structure presented in Chapter 1, and to examine whether the techniques of Chapter 4 may fructiferously be brought to bear. The structure is reproduced as Figure 7.1 below.

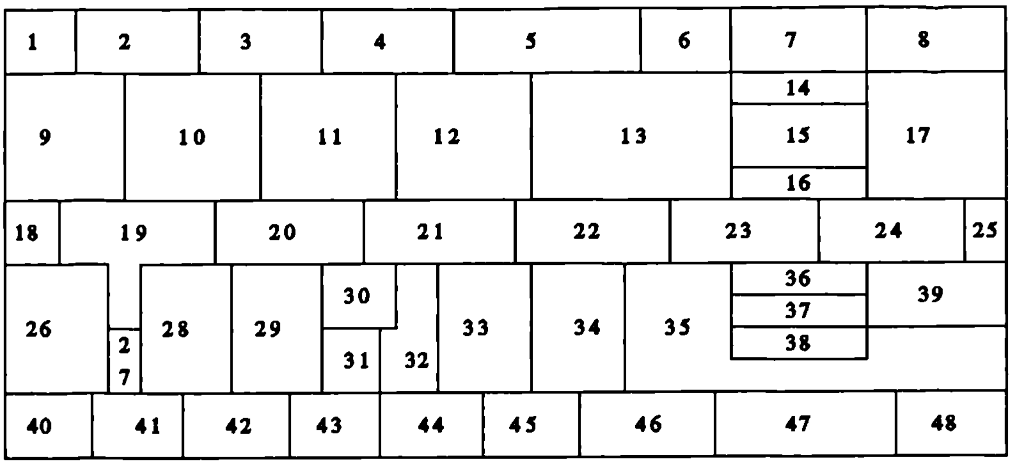


Figure 1.5 The floor-plan of a forty-eight volume structure.

The target volume is the room numbered 37, as was the case in the original structure. For the sake of simplicity in the absence of computer code for the methods of Chapter 4, attention is restricted to the possibility of a fire starting in the volume numbered 13, and all the barriers are considered to be symmetrical with identical Gaussian $N(20, 2^2)$ breach-time distributions. The time unit is the minute.

The shortest path is described as 13-23-36 and is of length 2. In order to eliminate the need to identify routes whose associated travel-to-target times are likely to be large relative to that expected if the fire first reaches the target along the shortest path, the following approach is adopted.

The probability of any one barrier being breached in time $T \leq t$ is given by

$$\Phi\left[\frac{t-20}{2}\right] = \alpha.$$

Let $\alpha=0.999968$, then $\Phi^{-1}(\alpha)=4.0$ and the probability of a barrier being breached in time $T \leq 28$ is $\alpha=0.999968$, and the probability of a barrier being breached in time $T \leq 12$ is $1-\alpha=0.000032$.

Then, since the shortest path is of length 2, it is extremely likely that a fire starting in volume 13 will have reached the target in time $T \leq 56$. (Probability is 0.999936).
Since each barrier is very unlikely to be breached in time $T \leq 12$, an idea of the length of the longest route which need be considered is provided by

$$\frac{56}{12} = 4.667 \simeq 5.$$

Thus no paths from volume 13 to volume 36 whose traversal requires the breaching of more than 5 barriers are considered.

The details of the paths are not reproduced here; that information is summarized as:

There are ... paths	of length ...
1	2
6	3
20	4
57	5.

Reference to Table 4.2 indicates that the distribution of the smallest order statistic of time to reach the target is given by

$$1 - \prod_{k=1}^{K} \left[1 - \Phi\left(\frac{t - \mu c_k}{\sigma . \sqrt{c_k}} \right) \right] \tag{7.1}$$

where c_k is the number of barriers in each path k from volume 13, and K is the total number of paths being considered.

From the information above, equation 7.1 is evaluated as

$$1 - \left[\left[1 - \Phi\left(\frac{t-40}{2\sqrt{2}}\right)\right]^{1} \times \left[1 - \Phi\left(\frac{t-60}{2\sqrt{3}}\right)\right]^{6} \times \left[1 - \Phi\left(\frac{t-80}{2\sqrt{4}}\right)\right]^{20} \times \left[1 - \Phi\left(\frac{t-100}{2\sqrt{5}}\right)\right]^{57}\right].$$

The cumulative distribution function is illustrated in Figure 7.2.

Figure 7.2 illustrates the important features extremely clearly. Once a fire is established in volume 13, the probability of the target being reached within 30 minutes is negligible, whilst it has a 50% chance of being reached in 40 minutes and is almost bound to have been reached within 45 minutes.

A risk assessment of the whole structure which facilitated the derivation of results such as these would enhance the fire risk assessment methodology, and for that reason their incorporation into appropriate software is again recommended.

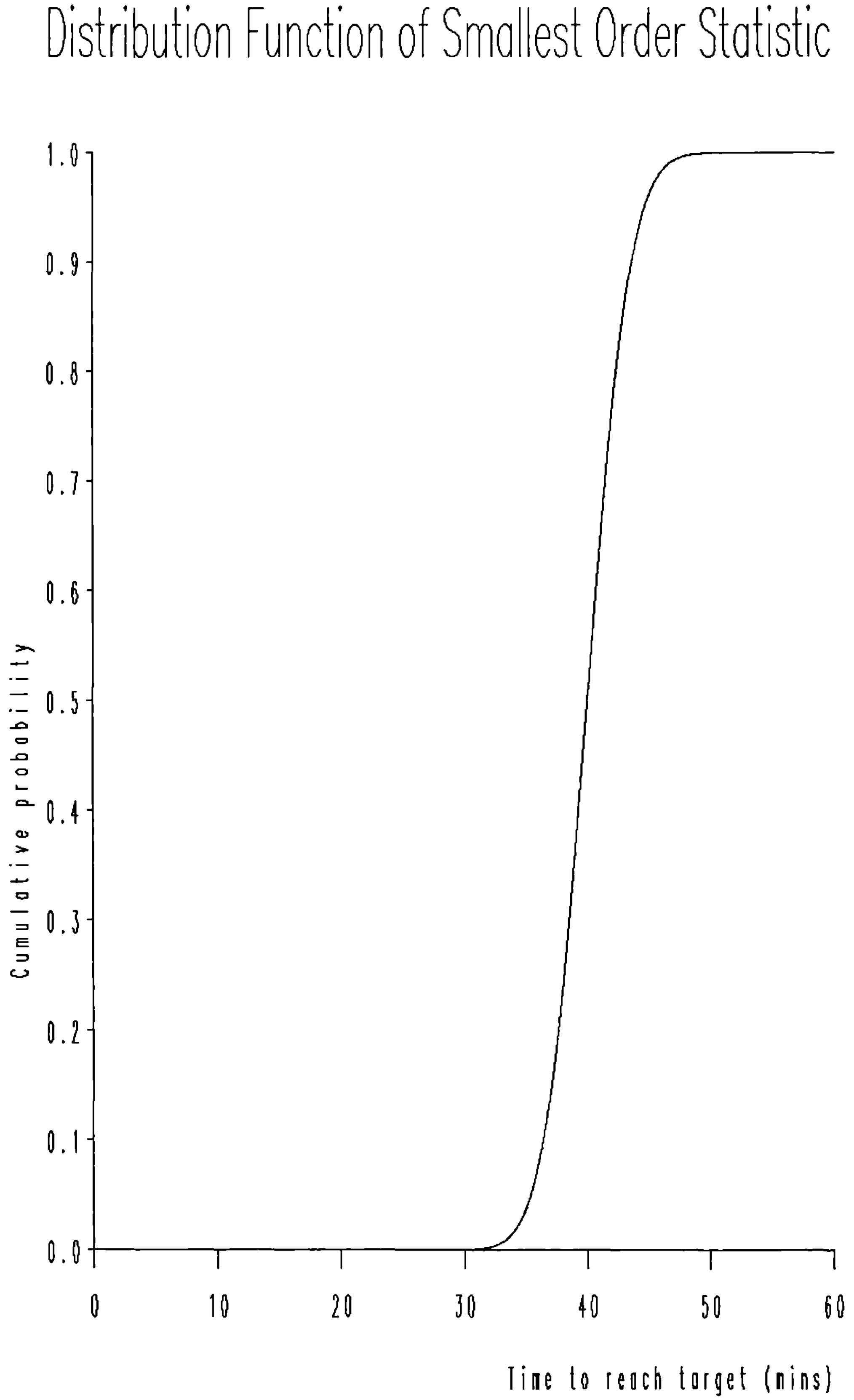


Figure 7.2 Distribution function of smallest order statistic of time to arrival at target from volume 13.

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APPENDIX A

Output From running the ARSSUN Code

on the

Example Structure in Chapter 3

```
*******************
* FIRE  PROPAGATION *
*******************

THE SYSTEM HAS  4 VOLUMES WITH PROBABILITIES
OF FIRE INITIATION AND OF FIRE SPREAD AS FOLLOWS:-

  VOLUME                  PROBABILITY OF
 NO. DECRIPTION            FIRE   SPREAD

  1. VOLUME 1             .6000  1.0000
  2. VOLUME 2             .3000  1.0000
  3. VOLUME 3             .0800  1.0000
  4. VOLUME 4             .0200  1.0000

THE VOLUMES ARE CONNECTED BY THE FOLLOWING FIRE PATHS
WITH THE GIVEN PROBABILITIES OF FIRE
SPREAD IN THE DIRECTION STATED

        FIRE PATH         PROBABILITY
         (a->b)            (a->b)

         1 -> 2             .0400
         2 -> 1             .0400
         2 -> 3             .0400
         2 -> 4             .0400
         3 -> 2             .0400
         3 -> 4             .0400
         4 -> 2             .0400
         4 -> 3             .0400

THERE ARE  1 SAFETY SYSTEMS WHICH
COVER THE VOLUMES AS SHOWN

        SAFETY
        SYSTEM            VOLUMES

    A            .          4                 {ie vol4 is target}

WEAK------------------->.200000E-01  4

WEAK------------------->.320000E-02  3  4
```

WEAK--------------------> .480000E-03 2 3 4

WEAK--------------------> .384000E-04 1 2 3 4

WEAK--------------------> .120000E-01 2 4

WEAK--------------------> .128000E-03 3 2 4

WEAK--------------------> .960000E-03 1 2 4

CUTOFF PROB IS .1000E-12

WARNING PROB IS .0000E+00

TOTAL PROB IS .3681E-01

ESTIMATE PROB IS .1554E-03

APPENDIX B

Computer Code and selected Output

for the

REDUCE Computer Algebra Application of Chapter 3

```
OUT RESULTS;
LET U1=0;
U1;
LET Q1=P1*P13;
Q1;
LET U2=SUB(P4=1,P16=1,Q1);
U2;
LET Q2=Q1+P4*P16*(1-U2);
Q2;
LET U3=SUB(P1=1,P14=1,Q2);
U3;
LET Q3=Q2+P1*P14*(1-U3);
Q3;
LET U4=SUB(P1=1,P7=1,P16=1,Q3);
U4;
LET Q4=Q3+P1*P7*P16*(1-U4);
Q4;
LET U5=SUB(P2=1,P13=1,Q4);
U5;
LET Q5=Q4+P2*P13*(1-U5);
Q5;
LET U6=SUB(P4=1,P17=1,Q5);
U6;
LET Q6=Q5+P4*P17*(1-U6);
Q6;
LET U7=SUB(P4=1,P10=1,P13=1,Q6);
U7;
LET Q7=Q6+P4*P10*P13*(1-U7);
Q7;
LET U8=SUB(P5=1,P16=1,Q7);
U8;
LET Q8=Q7+P5*P16*(1-U8);
Q8;
LET U9=SUB(P1=1,Q8);
U9;
LET Q9=Q8+P1*(1-U9);
Q9;
LET U10=SUB(P1=1,P7=1,P17=1,Q9);
U10;
LET Q10=Q9+P1*P7*P17*(1-U10);
Q10;
LET U11=SUB(P1=1,P8=1,P16=1,Q10);
U11;
LET Q11=Q10+P1*P8*P16*(1-U11);
Q11;
LET U12=SUB(P2=1,P14=1,Q11);
U12;
```

```
LET Q12=Q11+P2*P14*(1-U12);
Q12;
LET U13=SUB(P2=1,P7=1,P16=1,Q12);
U13;
LET Q13=Q12+P2*P7*P16*(1-U13);
Q13;
LET U14=SUB(P13=1,Q13);
U14;
LET Q14=Q13+P13*(1-U14);
Q14;
LET U15=SUB(P4=1,Q14);
U15;
LET Q15=Q14+P4*(1-U15);
Q15;
LET U16=SUB(P4=1,P10=1,P14=1,Q15);
U16;
LET Q16=Q15+P4*P10*P14*(1-U16);
Q16;
LET U17=SUB(P4=1,P11=1,P13=1,Q16);
U17;
LET Q17=Q16+P4*P11*P13*(1-U17);
Q17;
LET U18=SUB(P5=1,P17=1,Q17);
U18;
LET Q18=Q17+P5*P17*(1-U18);
Q18;
LET U19=SUB(P5=1,P10=1,P13=1,Q18);
U19;
LET Q19=Q18+P5*P10*P13*(1-U19);
Q19;
LET U20=SUB(P16=1,Q19);
U20;
LET Q20=Q19+P16*(1-U20);
Q20;
LET U21=SUB(P1=1,P7=1,Q20);
U21;
LET Q21=Q20+P1*P7*(1-U21);
Q21;
LET U22=SUB(P1=1,P8=1,P17=1,Q21);
U22;
LET Q22=Q21+P1*P8*P17*(1-U22);
Q22;
LET U23=SUB(P1=1,P16=1,Q22);
U23;
LET Q23=Q22+P1*P16*(1-U23);
Q23;
LET U24=SUB(P2=1,Q23);
U24;
LET Q24=Q23+P2*(1-U24);
Q24;
LET U25=SUB(P2=1,P7=1,P17=1,Q24);
U25;
LET Q25=Q24+P2*P7*P17*(1-U25);
Q25;
LET U26=SUB(P2=1,P8=1,P16=1,Q25);
U26;
LET Q26=Q25+P2*P8*P16*(1-U26);
```

```
Q26;
LET U27=SUB(P14=1,Q26);
U27;
LET Q27=Q26+P14*(1-U27);
Q27;
LET U28=SUB(P7=1,P16=1,Q27);
U28;
LET Q28=Q27+P7*P16*(1-U28);
Q28;
LET U29=SUB(P4=1,P10=1,Q28);
U29;
LET Q29=Q28+P4*P10*(1-U29);
Q29;
LET U30=SUB(P4=1,P11=1,P14=1,Q29);
U30;
LET Q30=Q29+P4*P11*P14*(1-U30);
Q30;
LET U31=SUB(P4=1,P13=1,Q30);
U31;
LET Q31=Q30+P4*P13*(1-U31);
Q31;
LET U32=SUB(P5=1,Q31);
U32;
LET Q32=Q31+P5*(1-U32);
Q32;
LET U33=SUB(P5=1,P10=1,P14=1,Q32);
U33;
LET Q33=Q32+P5*P10*P14*(1-U33);
Q33;
LET U34=SUB(P5=1,P11=1,P13=1,Q33);
U34;
LET Q34=Q33+P5*P11*P13*(1-U34);
Q34;
LET U35=SUB(P17=1,Q34);
U35;
LET Q35=Q34+P17*(1-U35);
Q35;
LET U36=SUB(P10=1,P13=1,Q35);
U36;
LET Q36=Q35+P10*P13*(1-U36);
Q36;
LET U37=SUB(P1=1,P8=1,Q36);
U37;
LET Q37=Q36+P1*P8*(1-U37);
Q37;
LET U38=SUB(P1=1,P17=1,Q37);
U38;
LET Q38=Q37+P1*P17*(1-U38);
Q38;
LET U39=SUB(P2=1,P7=1,Q38);
U39;
LET Q39=Q38+P2*P17*(1-U39);
Q39;
LET U40=SUB(P2=1,P8=1,P17=1,Q39);
U40;
LET Q40=Q39+P2*P8*P17*(1-U40);
Q40;
```

```
LET U41=SUB(P2=1,P16=1,Q40);
U41;
LET Q41=Q40+P2*P16*(1-U41);
Q41;
LET U42=Q41;
U42;
LET Q42=Q41+(1-U42);
Q42;
LET U43=SUB(P7=1,P17=1,Q42);
U43;
LET Q43=Q42+P7*P17*(1-U43);
Q43;
LET U44=SUB(P8=1,P16=1,Q43);
U44;
LET Q44=Q43+P8*P16*(1-U44);
Q44;
LET U45=SUB(P4=1,P11=1,Q44);
U45;
LET Q45=Q44+P4*P11*(1-U45);
Q45;
LET U46=SUB(P4=1,P14=1,Q45);
U46;
LET Q46=Q45+P4*P14*(1-U46);
Q46;
LET U47=SUB(P5=1,P10=1,Q46);
U47;
LET Q47=Q46+P5*P10*(1-U47);
Q47;
LET U48=SUB(P5=1,P11=1,P14=1,Q47);
U48;
LET Q48=Q47+P5*P11*P14*(1-U48);
Q48;
LET U49=SUB(P5=1,P13=1,Q48);
U49;
LET Q49=Q48+P5*P13*(1-U49);
Q49;
LET U50=Q49;
U50;
LET Q50=Q49+(1-U50);
Q50;
LET U51=SUB(P10=1,P14=1,Q50);
U51;
LET Q51=Q50+P10*P14*(1-U51);
Q51;
LET U52=SUB(P11=1,P13=1,Q51);
U52;

ARRAY X(53);
X(1):=U1;
X(2):=U2;
X(3):=U3;
X(4):=U4;
X(5):=U5;
X(6):=U6;
X(7):=U7;
X(8):=U8;
X(9):=U9;
```

```
X(10):=U10;
X(11):=U11;
X(12):=U12;
X(13):=U13;
X(14):=U14;
X(15):=U15;
X(16):=U16;
X(17):=U17;
X(18):=U18;
X(19):=U19;
X(20):=U20;
X(21):=U21;
X(22):=U22;
X(23):=U23;
X(24):=U24;
X(25):=U25;
X(26):=U26;
X(27):=U27;
X(28):=U28;
X(29):=U29;
X(30):=U30;
X(31):=U31;
X(32):=U32;
X(33):=U33;
X(34):=U34;
X(35):=U35;
X(36):=U36;
X(37):=U37;
X(38):=U38;
X(39):=U39;
X(40):=U40;
X(41):=U41;
X(42):=U42;
X(43):=U43;
X(44):=U44;
X(45):=U45;
X(46):=U46;
X(47):=U47;
X(48):=U48;
X(49):=U49;
X(50):=U50;
X(51):=U51;
X(52):=U52;
ARRAY W(53);

FOR I:=1:52 DO<<W(I):=SUB(P1=R1,P2=R2,P4=R1,P5=R2,P7=R1,P8=R2,P10
 =R2,P13=R1,P14=R2,P16=R1,P17=R2,X(I))>>;

ARRAY V(53);

FOR I:=1:52 DO<<V(I):=1-W(I)>>;

ARRAY PR(53);
PR(1):=R1*R1*V(1);
PR(2):=R1*R1*V(2);
PR(3):=R1*R2*V(3);
PR(4):=R1*R1*R1*V(4);
```

```
PR(5):=R1*R2*V(5);
PR(6):=R1*R2*V(6);
PR(7):=R1*R1*R1*V(7);
PR(8):=R1*R2*V(8);
PR(9):=R1*V(9);
PR(10):=R1*R1*R2*V(10);
PR(11):=R1*R1*R2*V(11);
PR(12):=R2*R2*V(12);
PR(13):=R1*R1*R2*V(13);
PR(14):=R1*V(14);
PR(15):=R1*V(15);
PR(16):=R1*R1*R2*V(16);
PR(17):=R1*R1*R2*V(17);
PR(18):=R2*R2*V(18);
PR(19):=R1*R1*R2*V(19);
PR(20):=R1*V(20);
PR(21):=R1*R1*V(21);
PR(22):=R1*R2*R2*V(22);
PR(23):=R1*R1*V(23);
PR(24):=R2*V(24);
PR(25):=R1*R2*R2*V(25);
PR(26):=R1*R2*R2*V(26);
PR(27):=R2*V(27);
PR(28):=R1*R1*V(28);
PR(29):=R1*R1*V(29);
PR(30):=R1*R2*R2*V(30);
PR(31):=R1*R1*V(31);
PR(32):=R2*V(32);
PR(33):=R1*R2*R2*V(33);
PR(34):=R1*R2*R2*V(34);
PR(35):=R2*V(35);
PR(36):=R1*R1*V(36);
PR(37):=R1*R2*V(37);
PR(38):=R1*R2*V(38);
PR(39):=R1*R2*V(39);
PR(40):=R2*R2*R2*V(40);
PR(41):=R1*R1*V(41);
PR(42):=1*V(42);
PR(43):=R1*R2*V(43);
PR(44):=R1*R2*V(44);
PR(45):=R1*R2*V(45);
PR(46):=R1*R2*V(46);
PR(47):=R1*R2*V(47);
PR(48):=R2*R2*R2*V(48);
PR(49):=R1*R2*V(49);
PR(50):=1*V(50);
PR(51):=R1*R2*V(51);
PR(52):=R1*R2*V(52);

ARRAY TOTPR(7);
TOTPR(1):=FOR J:=1:2 SUM PR(J);
TOTPR(2):=FOR J:=3:8 SUM PR(J);
TOTPR(3):=FOR J:=9:20 SUM PR(J);
TOTPR(4):=FOR J:=21:36 SUM PR(J);
TOTPR(5):=FOR J:=37:52 SUM PR(J);
TOTPR(6):=TOTPR(5)+TOTPR(4)+TOTPR(3)+TOTPR(2)+TOTPR(1);
FACTORIZE TOTPR(6);
```

```
ARRAY TOTPP(7);
FOR J:=1:6 DO<<TOTPP(J):=SUB(R1=OP1,R2=OP2/(1-OP1),TOTPR(J))>>;
TOTPP(1);
TOTPP(2);
TOTPP(3);
TOTPP(4);
TOTPP(5);
TOTPP(6);
FACTORIZE TOTPP(6);

ARRAY ET(53);
FOR I:=1:2 DO <<ET(I):=2*PR(I)>>;
FOR I:=3:8 DO <<ET(I):=3*PR(I)>>;
FOR I:=9:20 DO <<ET(I):=4*PR(I)>>;
FOR I:=21:36 DO <<ET(I):=5*PR(I)>>;
FOR I:=37:52 DO <<ET(I):=6*PR(I)>>;

ARRAY TOTET(7);
TOTET(1):=FOR J:=1:2 SUM ET(J);
TOTET(2):=FOR J:=3:8 SUM ET(J);
TOTET(3):=FOR J:=9:20 SUM ET(J);
TOTET(4):=FOR J:=21:36 SUM ET(J);
TOTET(5):=FOR J:=37:52 SUM ET(J);
TOTET(6):=TOTET(5)+TOTET(4)+TOTET(3)+TOTET(2)+TOTET(1);

ARRAY TOTT(7);
FOR I:=1:6 DO <<TOTT(I):=SUB(R1=OP1,R2=OP2/(1-OP1),TOTET(I))>>;
TOTT(1);
TOTT(2);
TOTT(3);
TOTT(4);
TOTT(5);
TOTT(6);
SHUT;
BYE;
```

```
TOTPR(1):=FOR J:=1:2 SUM PR(J);

                 2    2
TOTPR(1) :=  - R1 *(R1  - 2)

TOTPR(2):=FOR J:=3:8 SUM PR(J);

                     2   4        2   3        2   2        2
TOTPR(2) := 2*R1*(R2 *R1  - 4*R2 *R1  + 5*R2 *R1  - 2*R2 *R1 - 2*R2*

           4         3          2                        4
         R1  + 6*R2*R1  - 4*R2*R1  - 2*R2*R1 + 2*R2 + R1  - 2*

           3    2
         R1  + R1 )

TOTPR(3):=FOR J:=9:20 SUM PR(J);

                  4   4        4   3        4   2        4        4
TOTPR(3) :=  - (R2 *R1  - 4*R2 *R1  + 6*R2 *R1  - 4*R2 *R1 + R2 + 2*

            2   5         2   4         2   3         2   2       2
          R2 *R1  - 10*R2 *R1  + 18*R2 *R1  - 16*R2 *R1  + 8*R2

                  2            5            4            3
          *R1 - 2*R2  - 4*R2*R1  + 12*R2*R1  - 8*R2*R1  - 4*R2*

            2                  5       4       3       2
          R1  + 4*R2*R1 + 2*R1  - 4*R1  - 2*R1  + 8*R1  - 4*R1)

TOTPR(4):=FOR J:=21:36 SUM PR(J);

                     2   4        2   3        2   2        2       2
TOTPR(4) :=4*R2*(R2 *R1  - 4*R2 *R1  + 6*R2 *R1  - 4*R2 *R1 + R2 - 2

               4          3           2                        4
         *R2*R1  + 8*R2*R1  - 12*R2*R1  + 8*R2*R1 - 2*R2 + R1  -

             3       2
          4*R1  + 6*R1  - 4*R1 + 1)

TOTPR(5):=FOR J:=37:52 SUM PR(J);

              4   4        4   3        4   2        4        4     3
TOTPR(5) := R2 *R1  - 4*R2 *R1  + 6*R2 *R1  - 4*R2 *R1 + R2 - 4*R2 *

        4         3   3        3   2         3         3        2
      R1  + 16*R2 *R1  - 24*R2 *R1  + 16*R2 *R1 - 4*R2  + 6*R2 *

        4         2   3        2   2         2          2
      R1  - 24*R2 *R1  + 36*R2 *R1  - 24*R2 *R1 + 6*R2  - 4*R2*

        4           3           2                        4
      R1  + 16*R2*R1  - 24*R2*R1  + 16*R2*R1 - 4*R2 + R1  - 4*

        3       2
      R1  + 6*R1  - 4*R1 + 1

TOTPR(6):=TOTPR(5)+TOTPR(4)+TOTPR(3)+TOTPR(2)+TOTPR(1);
TOTPR(6) := 1
```