Finite polynomial maps and G-variant map germs

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Abstract

The first half of this thesis is devoted to the study of finite polynomial maps $\mathbf{C}^n \to \mathbf{C}^n$ and the use of Gröbner bases to determine if a given map is finite. We begin by examining those maps which have quasihomogeneous components, and give a simple condition for such maps to be finite. This condition is extended to those maps which are quasihomogeneous as above, but with extra lower order terms. Next, we give a general criterion for testing the finiteness of a given polynomial map and an implementation in the Maple computer algebra system. Our next step is to generalize our results to regular maps between affine varieties. Again, a finiteness criterion is given, plus its implementation in Maple. Lastly in this half, we consider the trace bilinear form associated with a finite map and show how it may be used to find real roots of a polynomial system.

The second half of the thesis is concerned with the study of G-variant map germs, which commute with the action of a finite group G on the source and target spaces. We give a relation between the G-variant degree associated with a map germ, bilinear forms on the local algebra and preimages of zero under a perturbation of the original map. We look at both the complex and real affine space situation. We then give the equivalent results when we do not have a 'good' deformation of the map, when we have two groups acting and when we use modular representations. Next, we give an invariant of G-variant maps which is stronger than G-degree, based upon a lattice of vector subspaces. Finally, we examine the structure of the class of G-variant maps and consider criteria for maps to have 'good' deformations and to be finite. We then give ways of determining generators for the class of maps by generalizing theorems of Noether and Molien.

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Dedicated to my parents

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Overview

The first five chapters of this thesis are devoted to the study of finite polynomial maps and the use of Gröbner bases to determine if a given map is finite.

The first chapter is an introduction to the theory of Gröbner bases, with results which will be used in later chapters. It includes the definition of monomial orders and Gröbner bases, some results from elimination theory and the application of Gröbner bases to modules and subrings.

The second chapter is devoted to results involving Gröbner bases and their implementation in Maple. These results include a solution to the problem of determining submodule membership, calculating Gröbner bases in a module and finding generators for a syzygy module.

In the third chapter, we begin by defining what we mean by a finite polynomial map $\mathbf{C}^n \rightarrow \mathbf{C}^n$ and give some simple algebraic and topological properties of these maps. The next step is to examine a particular class of maps, those which have quasihomogeneous components, as there is a very simple condition for such maps to be finte. We then consider maps which are finite maps with quasihomogeneous components, as above, plus extra lower order terms. Again, the condition required for these maps to be finite is simple. This allows us to consider the projective geometry of a finite map and to examine if a general polynomial map will be finite. Next, we look a little more closely at the ideal of leading terms, as this ideal is important for determining if a given map is finite or not. We give a general criterion for testing the finiteness of a given polynomial map and an implementation in the Maple computer algebra system. Finally, in this chapter, we look at the various degrees which can be associated to a finite map and how they are related to one another.

The fourth chapter generalises the results of chapter 3 to regular maps between affine varieties. The definition of finiteness in this case and a new version of the finiteness criterion are given, plus an implementation in Maple. We also consider the generalisation to varieties of the results on quasihomogeneity and on the question of finiteness (or not) of a general map.

The fifth chapter is devoted to the study of the trace bilinear form associated with a finite map. This is used to determine the real solutions of a system of equations. An algorithm for calculating the form is implemented in Maple and several examples are given of its use.

The second half of the thesis is concerned with the study of G-variant map germs, which in some sense commute with the action of a finite group G on the source and target spaces.

In chapter six, we define G-variant map germs and give the basic results from the literature. The G-variant degree associated with a map germ is defined. We consider the relation between this, bilinear forms on the local algebra and preimages of zero under a perturbation of the original map. We look at both the complex and real affine space situation and define G-signature in the latter case. Finally in this chapter, we apply the results obtained to various examples.

In the seventh chapter we look at further results related to G-variant maps. Firstly, we consider the relation between the G-variant degree and the preimages of zero under a deformation of the original map, where zero is not a regular value. We also consider the case when we have two groups acting naturally on the direct sum of two affine spaces. The following section is dedicated to the effect of using modular representations (i.e. representations in characteristic p) of G in the real case. Next, we consider a new invariant of G-variant maps which is stronger than G-degree. This is based upon a lattice of vector subspaces and the subgroups of

G which fix these subspaces. Finally, we briefly consider an alternative approach via the quotient of our affine space with respect to the group action.

Chapter eight is given over to the study of the class of G-variant maps, given the actions of G on two given spaces, designated as the source and target respectively. First, we examine the structure of this class of maps and consider criteria for a maps to have 'good' deformations and to be finite. We then go on to look at ways of determining generators for the class of maps (as a module) by generalizing the theorems of Noether and Molien on invariant polynomials.

Chapter 1

Gröbner bases

In this section, we describe Gröbner bases and give their basic properties. In general, the notation and definitions follow those of [CLO]. The polynomials in this section will be over **C**, although all results (except where stated) will hold over an arbitrary field.

Gröbner bases allow one to handle ideals of a polynomial ring in an algorithmic way. In particular they help to give solutions to the following problems:

- The Ideal Membership Problem: Given $\phi \in \mathbf{C}[x_1, \ldots, x_n]$ and an ideal $I = <f_1, \ldots, f_n>$, can we determine whether $\phi \in I$?

- Solving Polynomial Equations: Given a set of polynomials $\{f_1, \ldots, f_n\}$ in $\mathbf{C}[x_1, \ldots, x_n]$ can we describe all the solutions of

$$f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0$$

  in $\mathbf{C}^n$?

- The Implicitization Problem: Given a subset P of $\mathbf{C}^n$ parametrically as

$$P = \{(g_1(t_1, \ldots, t_s), \ldots, g_n(t_1, \ldots, t_s))\},$$

  where $g_1, \ldots, g_n$ are polynomials, can we describe it as an affine variety (or part of one)?

The first problem is the most important for our purposes, for as we shall see, it leads naturally to a description of the quotient by the given ideal. The second and third problems are, in effect, opposites of one another and their solution is known as Elimination Theory (for some results see Propositions 1.2.7 and 1.2.8). Before defining a Gröbner basis, we need to fix a monomial order on the polynomial ring, as explained below.

1.1 Monomial orders

To define an order on the monomials of $\mathbf{C}[x_1, \ldots, x_n]$, we firstly note that there is a 1-1 correspondence between the monomials of this ring and $\mathbf{N}^n$. The monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ naturally corresponds to $(\alpha_1, \ldots, \alpha_n)$. By identifying these two spaces, we can make the following definition:

Definition A *monomial ordering* on $\mathbf{C}[x_1, \ldots, x_n]$ is any relation $>$ on $\mathbf{N}^n$ satisfying:

(i) $>$ is a total (or linear) ordering on $\mathbf{N}^n$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbf{N}^n$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-ordering on $\mathbf{N}^n$.

We will now give examples of monomial orderings by describing some of the more common ones.

Lexicographic order (or simply 'lex') is defined as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$. We say $\alpha >_{lex} \beta$ if in the vector difference $\alpha - \beta \in \mathbf{Z}^n$, the leftmost non-zero entry is positive. (This will in fact usually be written as $x^\alpha >_{lex} x^\beta$).

Graded Lex order (or 'grlex') is as follows. Take $\alpha, \beta \in \mathbf{N}^n$ as before. We say $\alpha >_{grlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i$$

or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$.

Graded Reverse Lex order (or 'grevlex') is defined as follows. Let $\alpha, \beta \in \mathbf{N}^n$. Then $\alpha >_{grevlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i$$

or $|\alpha| = |\beta|$ and in $\alpha - \beta$, the right-most non-zero entry is negative.

In general, an ordering is called *graded* if $|\alpha| > |\beta|$ implies $\alpha > \beta$. Obviously, grlex and grevlex are examples of graded orders.

To give an example of these orders, consider the monomials x^3, x^2yz^2 and xy^3z in $\mathbf{C}[x, y, z]$. Under the three orders above we get

$$\begin{array}{ccccc} x^3 & >_{lex} & x^2yz^2 & >_{lex} & xy^3z, \\ x^2yz^2 & >_{grlex} & xy^3z & >_{grlex} & x^3, \\ xy^3z & >_{grevlex} & x^2yz^2 & >_{grevlex} & x^3, \end{array}$$

giving three different permutations of the monomials.

Given two orders, each defined on a set of variables, we can combine the two as follows. Suppose that we are considering the ring $\mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and that there are orders $>_1$ and $>_2$ defined on the x and y variables respectively. The *product order* of these two, $>_p$, is defined by $x^\alpha y^\beta >_p x^\gamma y^\delta$ if $x^\alpha >_1 x^\gamma$ or $x^\alpha = x^\gamma$ and $y^\beta >_2 y^\delta$.

We now describe some notation used in relation to monomial orders. Let $\phi = \sum_\alpha a_\alpha x^\alpha$ be a non-zero polynomial in $\mathbf{C}[x_1, \ldots, x_n]$ and let $>$ be a monomial order. Then the *multidegree* of f is given by

$$\text{multideg}(f) = \max\{\alpha \in \mathbf{N}^n : a_\alpha \neq 0\}$$

where the maximum is with respect to $>$. We define the *leading coefficient* of f to be

$$\text{LC}(f) = a_{\text{multideg}(f)}$$

and the *leading monomial* to be

$$\text{LM}(f) = x^{\text{multideg}(f)},$$

which has coefficient 1. Finally, we define the *leading term* of f to be

$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

We will often write $\text{LT}(I)$ for an ideal I. This is simply shorthand for $\{\text{LT}(f) : f \in I\}$. We can now state the result which motivates the introduction of monomial orders:

Proposition 1.1.1 (The Division Algorithm in $\mathbf{C}[x_1, \ldots, x_n]$) *Suppose that we have fixed a monomial order $>$ and let $F = (f_1, \ldots, f_s)$ be an ordered s-tuple of polynomials in $\mathbf{C}[x_1, \ldots, x_n]$. Then every $\phi \in \mathbf{C}[x_1, \ldots, x_n]$ can be written as*

$$\phi = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in \mathbf{C}[x_1, \ldots, x_n]$ and either $r = 0$ or r is a linear combination of monomials, none of which is divisible by any of $LT(f_1), \ldots, LT(f_s)$. We will call r a remainder *of ϕ on division by F. Furthermore, if $a_i f_i \neq 0$, then we have*

$$\textit{multideg}(\phi) \geq \textit{multideg}(a_i f_i).$$

Proof This is straightforward. See [CLO, p63]. The algorithm is very similar to that of polynomials in a single variable. At each stage, we attempt to divide by each of the given divisors in turn. This is done by looking at the leading monomials of the divisor and the polynomial to be divided, to see if the former will divide the latter. In this way, we build up a quotient associated to each divisor (the a_i given above). When none of the divisors can divide what remains, this is the remainder (given by r above). □

We end this subsection by stating a small result

Lemma 1.1.2 *Let I be an ideal generated by some set of monomials $\{x^\alpha : \alpha \in A\}$. Then a monomial x^β lies in I if and only if x^β is divisible by x^α for some $\alpha \in A$.*

Proof See [CLO, p69]. □

1.2 Definition and properties of Gröbner bases

Definition Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a *Gröbner basis* if

$$<\mathrm{LT}(g_1), \ldots \mathrm{LT}(g_t)>=<\mathrm{LT}(I)> .$$

The following proposition guarantees the existence of a Gröbner basis:

Proposition 1.2.1 *Fix a monomial order. Let $I \subset \mathbf{C}[x_1, \ldots, x_n]$ be an ideal other than 0. Then I has a Gröbner basis $G = \{g_1, \ldots, g_t\}$. Furthermore, we have $I =<g_1, \ldots, g_t>$.*

Proof See [CLO, p76]. □

With the division algorithm as described in Proposition 1.1.1, the remainder is actually dependent on the order that the elements $f_1, \ldots, f_s$ are listed. This means that we can start with an element $\phi \in<f_1, \ldots, f_s>$ and obtain a remainder of zero (the intuitive answer) only for certain orderings of the f_i. Gröbner bases solve this problem as the next proposition demonstrates.

Proposition 1.2.2 *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ and let $\phi \in \mathbf{C}[x_1, \ldots, x_n]$. Then there is a unique element $r \in \mathbf{C}[x_1, \ldots, x_n]$ with the following properties:*

(i) No term of r is divisible by one of $LT(g_1), \ldots, LT(g_t)$.

(ii) There is some $g \in I$ such that $\phi = g + r$.

In particular, r is the remainder on division of ϕ by G, however the elements of G are listed.

Proof See [CLO, p81]. □

As a corollary, we obtain the solution to the Ideal Membership Problem:

Corollary 1.2.3 *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ and let $\phi \in \mathbf{C}[x_1, \ldots, x_n]$. Then $\phi \in I$ if and only if the remainder on division by G is zero.* □

Since the remainders are unique, we will write $\overline{\phi}^G$ for the remainder on dividing ϕ by the Gröbner basis G. Now Gröbner bases can be calculated using an algorithm called Buchberger's Algorithm. This depends for its operation on the following construction.

Let $f, g \in \mathbf{C}[x_1, \ldots, x_n]$ be non-zero polynomials. Let x^γ be the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$. Then the *S-polynomial* of f and g is the combination

$$S(f, g) = \frac{x^\gamma}{\mathrm{LT}(f)} \cdot f - \frac{x^\gamma}{\mathrm{LT}(g)} \cdot g.$$

The next proposition gives a method for testing if a given set of generators for I is in fact a Gröbner basis.

Proposition 1.2.4 *Let I be a polynomial ideal. Then a set of generators $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for I if and only if for all pairs $i \neq j$, $\overline{S(g_i, g_j)}^G = 0$.*

Proof See [CLO, p84]. □

The following proposition gives a description of the quotient and in fact, even yields a $\mathbf{C}$ vector space basis for this quotient. Note that this proposition requires the coefficient field to be algebraically closed. Since this result is so important, we also give its proof, following [CLO, p232].

Proposition 1.2.5 *Fix a monomial order on $\mathbf{C}[x_1, \ldots, x_n]$. Let I be an ideal in $\mathbf{C}[x_1, \ldots, x_n]$ with Gröbner basis G. Let V be the variety associated to I, i.e. the set of points in $\mathbf{C}^n$ at which every element of I vanishes. Then the following statements are equivalent.*

(i) V is a finite set.

(ii) For each $i \in \{1, \ldots, n\}$, there is some $m_i \in \mathbf{N}$ such that $x_i^{m_i} \in <LT(I)>$.

(iii) For each $i \in \{1, \ldots, n\}$, there is some $m_i \in \mathbf{N}$ such that $x_i^{m_i} = LM(g)$ for some $g \in G$.

(iv) The $\mathbf{C}$ vector space $Span(x^\alpha : x^\alpha \notin <LT(I)>)$ is finite dimensional.

(v) The $\mathbf{C}$ vector space $\frac{\mathbf{C}[x_1,\ldots,x_n]}{I}$ is finite dimensional.

Proof (i)⇒(ii) If $V = \emptyset$, then $1 \in I$ by the Weak Nullstellensatz ([CLO, p169]). In this case, we can take $m_i = 0$ for all i. If V is non-empty, then for a fixed i, let a_j, $j = 1, \ldots, k$, be the distinct complex numbers appearing as i^{th} coordinates of points of V. Form the one variable polynomial

$$\psi(x_i) = \prod_{j=1}^{k} (x_i - a_i).$$

By construction, ψ vanishes at every point in V, so ψ lies in the ideal of V. Thus by the Nullstellensatz ([CLO, p172]), there is some $m \geq 1$ such that $\psi^m \in I$. But this means that the leading monomial of ψ^m is in $<\mathrm{LT}(I)>$. Examining the expression defining ψ, we see that $x_i^{km} \in<\mathrm{LT}(I)>$.

(ii)$\Leftrightarrow$(iii) Let $x_i^{m_i} \in<\mathrm{LT}(I)>$. Since G is a Gröbner basis of I, $<\mathrm{LT}(I)> = <\mathrm{LT}(g) : g \in G>$. By Lemma 1.1.2, there is some $g \in G$ such that $\mathrm{LT}(g)$ divides $x_i^{m_i}$. But this implies that $\mathrm{LT}(g)$ is a power of x_i as claimed. The opposite implication follows from the definition of $<\mathrm{LT}(I)>$.

(ii)$\Rightarrow$(iv) If some power of x_i, $x_i^{m_i}$ lies in $<\mathrm{LT}(I)>$ for each i, then the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for which some $\alpha_i \geq m_i$ are all in $<\mathrm{LT}(I)>$. The monomials in the complement must have $\alpha_i \leq m_i - 1$ for each i. As a result, the number of monomials in the complement of $<\mathrm{LT}(I)>$ can be at most $m_1 m_2 \cdots m_n$.

(iv)$\Leftrightarrow$(v) Consider the map

$$\Psi : \mathbf{C}[x_1, \ldots, x_n] \rightarrow \mathrm{Span}(x^\alpha : x^\alpha \notin<\mathrm{LT}(I)>)$$

given by $\Psi(\phi) = \overline{\phi}^G$. This is well-defined by Proposition 1.2.2 and is quite easily seen to be a linear map with $\ker\Psi = I$.

(v)$\Rightarrow$(i) To show that V is finite, it suffices to show that for each i there can be only finitely many distinct i^{th} coordinates for the points of V. Fix i and consider the classes $[x_i^j]$ in $\frac{\mathbf{C}[x_1,\ldots,x_n]}{I}$, where $j = 0, 1, 2, \ldots$. Since $\frac{\mathbf{C}[x_1,\ldots,x_n]}{I}$ is finite dimensional, the $[x_i^j]$ must be linearly dependent in $\frac{\mathbf{C}[x_1,\ldots,x_n]}{I}$. That is, there exist constants c_j (not all zero) and some m such that

$$\sum_{j=0}^{m} c_j [x_i^j] = \left[\sum_{j=0}^{m} c_j x_i^j\right] = 0.$$

However, this implies that $\sum_{j=0}^{m} c_j x_i^j \in I$. Since a non-zero polynomial can have only finitely many roots in $\mathbf{C}$, this shows that the points of V have only finitely many different i^{th} coordinates. $\square$

The vector spaces in items (iv) and (v) above are naturally isomorphic. For if we take the cosets corresponding to the elements of the set $L = \{x^\alpha : x^\alpha \notin <\mathrm{LT}(I)>\}$, then these span the space given by the quotient in item (v). The set L above will be known as the *LT monomial basis.* Given that a variety is finite, we can say more about the dimension of the associated vector space with the following proposition.

Proposition 1.2.6 *Let $V \subset \mathbf{C}^n$ be a variety consisting of points $\{p_1, \ldots, p_m\}$, with ideal I. Then*

$$dim_c \frac{\mathbf{C}[x_1, \ldots, x_n]}{I} = m$$

Proof Firstly, we define a map

$$\Phi : \frac{\mathbf{C}[x_1, \ldots, x_n]}{I} \to \mathbf{C}^m$$

by

$$\Phi([f]) = (f(p_1), \ldots, f(p_m))$$

where $[f]$ denotes the class of f in the given quotient. Now if $[f] = [g]$, then $f = g + h$ for some $h \in I$. So

$$\begin{aligned} \Phi([f]) &= (g(p_1) + h(p_1), \ldots, g(p_m) + h(p_m)) \\ &= (g(p_1) + 0, \ldots, g(p_m) + 0) \\ &= \Phi([g]) \end{aligned}$$

so Φ is well-defined. Also,

$$\begin{aligned} \Phi([\mu f + \lambda g]) &= \Phi(\mu[f] + \lambda[g]) \\ &= (\mu f(p_1) + \lambda g(p_1), \ldots, \mu f(p_m) + \lambda g(p_m)) \\ &= \mu(f(p_1), \ldots, f(p_m)) + \lambda(g(p_1), \ldots, g(p_m)) \\ &= \mu\Phi([g]) + \lambda\Phi([f]) \end{aligned}$$

so Φ is linear. Now suppose $\Phi([f]) = \Phi([g])$, then

$$(f(p_1) - g(p_1), \ldots, f(p_m) - g(p_m)) = (0, \ldots, 0).$$

Thus $f - g \in I$, or in other words $[f] = [g]$. So Φ is injective. Now for each i, define $W_i = \{p_j : j \neq i\}$. We know that $\mathbf{I}(\{p_i\})$ (i.e. the ideal corresponding to this point) is maximal (see [CLO, p200]) and we next wish to show that

$$\mathbf{I}(\{p_i\}) + \mathbf{I}(W_i) = <1> . \tag{1.1}$$

If $W_i = \emptyset$ then $\mathbf{I}(W_i) = <1>$ and we are done. Otherwise, if W_i is non-empty, then W_i must contain some $p_j \neq p_i$, p_j and p_i differing in some coordinate x_k. If p_i has k^{th} coordinate a, then $x_k - a$ lies in $\mathbf{I}(\{p_i\})$ but not in $\mathbf{I}(W_i)$. Thus, by the maximality of $\mathbf{I}(W_i)$, (1.1) must hold.

From (1.1), we know that there exist polynomials $f_i \in \mathbf{I}(W_i)$ and $g_i \in \mathbf{I}(\{p_i\})$ such that $f_i + g_i = 1$. So

$$\begin{aligned} \Phi(f_i + g_i) &= (f_i(p_1), \ldots, f_i(p_i), \ldots, f_i(p_m)) + (g_i(p_1), \ldots, g_i(p_i), \ldots, g_i(p_m)) \\ &= (1, \ldots, 1, \ldots, 1). \end{aligned}$$

Now we know that $f_i(p_j) = 0$ for $j \neq i$ and $g_i(p_i) = 0$, so

$$\begin{aligned} \Phi(f_i) &= (0, \ldots, 0, 1, 0, \ldots, 0) \\ \Phi(g_i) &= (1, \ldots, 1, 0, 1, \ldots, 1) \end{aligned}$$

and hence we find that Φ is onto. Thus Φ is an isomorphism and so

$$\dim_c \frac{\mathbf{C}[x_1, \ldots, x_n]}{I} = m$$

as required. □

To end this section, we give a couple of results from Elimination Theory.

Proposition 1.2.7 (The Elimination Theorem) *Let $I \in \mathbf{C}[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis for I with respect to lex order with $x_1 > \cdots > x_n$. Then for every $0 \leq k \leq 0$, the set*

$$G_k = G \cap \mathbf{C}[x_{k+1}, \ldots, x_n]$$

is a Gröbner basis for the ideal $I_k = I \cap \mathbf{C}[x_{k+1}, \ldots, x_n]$ in $\mathbf{C}[x_{k+1}, \ldots, x_n]$.

Proof See [CLO, p114]. □

Proposition 1.2.8 (The Closure Theorem) *Let V be an affine variety in $\mathbf{C}^n$ given by the ideal $I = <f_1, \dots, f_s>$. If $I_k = I \cap \mathbf{C}[x_{k+1}, \dots, x_n]$ and $\pi_k : \mathbf{C}^n \to \mathbf{C}^k$ is the projection onto the last $n-k$ variables, then the following is true.*

(i) The variety $\mathbf{V}(I_k)$ corresponding to I_k is the smallest containing $\pi_k(V) \subset \mathbf{C}^{n-k}$. In other words, it is the Zariski closure of $\pi_k(V)$.

(ii) When $V \neq \emptyset$, there is a proper affine subvariety $W \subset \mathbf{V}(I_k)$ such that $\mathbf{V}(I_k) - W \subset \pi_k(V)$.

Proof See [CLO, p123]. □

1.3 Ring and module applications

We now turn our attention to the use of Gröbner bases for deciding questions about rings and modules.

Proposition 1.3.1 *Suppose that $f_1, \dots, f_m \in \mathbf{C}[x_1, \dots, x_n]$ are given. Fix a monomial order in $\mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ where any monomial involving one of the x variables is greater than all monomials in $\mathbf{C}[y_1, \dots, y_m]$. Let G be a Gröbner basis of the ideal $< f_1 - y_1, \dots, f_m - y_m > \subset \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_m]$. Given $\phi \in \mathbf{C}[x_1, \dots, x_n]$, let $\psi = \overline{\phi}^G$ be the remainder of ϕ on division by G. Then*

(i) $\phi \in f^*\mathbf{C}[y_1, \dots, y_m]$ if and only if $\psi \in \mathbf{C}[y_1, \dots, y_m]$.

(ii) If $\phi \in f^*\mathbf{C}[y_1, \dots, y_m]$, then $\phi = f^*(\psi)$.

Proof See [CLO, p329] □

Next we examine the extension of Gröbner bases to modules. Suppose M is a finitely generated free module over $\mathbf{C}[x_1, \ldots, x_n]$, with basis $\{e_i\}$. A *monomial* in M is an element of the form $m = x^\alpha e_i$ and a *term* is simply a monomial multiplied by a scalar. Let $\lambda x^\alpha e_i, \mu x^\beta e_j$ be terms, where $\lambda, \mu \in \mathbf{C}$. We say that $\lambda x^\alpha e_i$ is *divisible* by $\mu x^\beta e_j$ if $i = j$ and λx^α is divisible by μx^β. The quotient is then $\frac{\lambda x^\alpha}{\mu x^\beta}$.

Definition A *monomial order*, $>$, on M above is one which satisfies

(i) $>$ is a total order.

(ii) If m_1, m_2 are monomials in M and $m_3 \neq 1$ is a monomial of $\mathbf{C}[x_1, \ldots, x_n]$, then

$$m_1 > m_2 \Rightarrow m_1 m_3 > m_2 m_3 > m_2.$$

It is simple to extend a monomial order on $\mathbf{C}[x_1, \ldots, x_n]$ to one on M. In order to do this, we simply consider the component number first. So, if we have an order $>_R$ on $\mathbf{C}[x_1, \ldots, x_n]$, we can extend this to M by defining $>_M$ as $x^\alpha e_i >_M x^\beta e_j$ if

$$i > j \quad \text{or} \quad i = j \text{ and } x^\alpha >_R x^\beta.$$

With a monomial order defined, the definition of leading terms, the division algorithm etc. follow as in the ring case. The definition of a Gröbner basis is also very similar:

Definition Fix a monomial order on M. A finite subset $G = \{g_1, \ldots, g_t\}$ of a submodule $N \subset M$ is said to be a *Gröbner basis* if

$$<\mathrm{LT}(g_1), \ldots \mathrm{LT}(g_t)>=<\mathrm{LT}(N)>$$

where '<>' denotes here the generation of a submodule of M.

We can also define the S-polynomial in an analogous way to that seen previously. Let f, g be elements of M. If $\mathrm{LT}(f)$ and $\mathrm{LT}(g)$ are multiples of different basis elements of M, then define $S(f,g) = 0$. Otherwise, let $x^\gamma e_k$ be the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ and define

$$S(f,g) = \frac{x^\gamma e_k}{\mathrm{LT}(f)} \cdot f - \frac{x^\gamma e_k}{\mathrm{LT}(g)} \cdot g$$

as before. Note that in the above expression, we have divided by an element of M. This would appear to be undefined, but since they are both mutiples of the same basis element, e_k, we can "cancel" e_k to obtain a quotient in $\mathbf{C}[x_1, \ldots, x_n]$. The following generalisation of Proposition 1.2.4 also carries over.

Proposition 1.3.2 *Let N be a submodule of M. Then a set of generators $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for N if and only if for all pairs $i \neq j$, $\overline{S(g_i, g_j)}^G =$ 0.*

Proof See [E, Prop 6.8]. □

One of the reasons for looking at Gröbner bases on modules is that it allows us to calculate *syzygy modules.* A syzygy module on a set of polynomials $\phi_1, \ldots, \phi_n$ is the module generated by the n-tuples of the form $(\alpha_1, \ldots, \alpha_n)$ such that $\sum_i \alpha_i \cdot \phi_i = 0$. In other words, it is the module of relations between the elements. The next proposition demonstrates the calculation of these modules.

Proposition 1.3.3 *Let N be a submodule of M, the free $\mathbf{C}[x_1, \ldots, x_n]$-module generated by $\{e_1, \ldots e_r\}$, with $G = \{g_1, \ldots, g_t\}$ a Gröbner basis for N. Let M' be a module over $\mathbf{C}[x_1, \ldots, x_n]$, freely generated by $e'_1, \ldots, e'_t$, one basis element for each of the g_i. For each $i < j$ with $LT(g_i)$ and $LT(g_j)$ multiples of the same basis element of M, define*

$$S'(f,g) = \frac{x^\gamma e_k}{LT(g_i)} \cdot e'_i - \frac{x^\gamma e_k}{LT(g_j)} \cdot e'_j$$

where $x^\gamma e_k$ *is the least common multiple of* $LM(g_i)$ *and* $LM(g_j)$. *On dividing* $S'(g_i, g_j)$ *by* G, *we obtain an expression*

$$S'(g_i, g_j) = \sum_{l=1}^{t} \epsilon_l^{(i,j)} g_l,$$

where $\epsilon_l^{(i,j)} \in \mathbf{C}[x_1, \ldots, x_n]$, *since the remainder must be zero. Now define*

$$T_{i,j} = S'(f, g) - \sum_{l=1}^{t} \epsilon_l^{(i,j)} e_l'$$

and induce a monomial order, $>_s$ *on* M' *from that on* M *as follows. Define* $x^\alpha e_i' >_s x^\beta e_j'$ *if*

$$LM(x^\alpha g_i) > LM(x^\beta g_j) \quad or \quad LM(x^\alpha g_i) = LM(x^\beta g_j) \text{ but } i < j.$$

Under this order, the set $\{T_{i,j}\}$ *is a Gröbner basis for (and hence generates) the syzygies on the* g_i.

Proof See [E, Theorem 6.10]. □

Chapter 2

Additional Gröbner basis methods

This section contains results involving Gröbner bases which are of practical use when considering finite maps.

2.1 Determining submodule membership

This is a generalization of Propostion 1.3.1, to (not necessarily free) modules. We will define the v-degree of a polynomial in variables $x_1, \ldots, x_n, v_1, \ldots, v_r, z_1, \ldots, z_m$ to be its degree considered as a polynomial in the v variables. An application of this result is to be found in §8.2, where this method can be used to write a polynomial G-variant map as an element of the module of such maps.

Proposition 2.1.1 *Let $f_1, \ldots, f_m, b_1, \ldots, b_r, \phi$ be polynomials in $\mathbf{C}[x_1, \ldots, x_n]$. Fix a product monomial order on $\mathbf{C}[x_1, \ldots, x_n, v_1, \ldots, v_r, z_1, \ldots, z_m]$ with $x_i > v_j > z_k$ for all i, j, k. Let $I = <b_1 - v_1, \ldots, b_r - v_r, f_1 - z_1, \ldots, f_m - z_m>$ and let G be its Gröbner basis with respect to the given monomial order. Let $\varphi = \bar{\phi}^G$ (the remainder of ϕ on division by G) and let M be the $\mathbf{C}[f_1, \ldots f_n]$ module generated by $\{1, b_1, \ldots, b_r\}$. Then*

- *(i) $\phi \in M$ if and only if $\varphi \in \mathbf{C}[v_1, \ldots, v_r, z_1, \ldots, z_m]$ and v-deg$(\varphi) \leq 1$.*

- *(ii) If $\phi \in M$ then $\phi = \psi(b_1, \dots, b_r, f_1, \dots, f_m)$ expresses ϕ as an element of M.*

Proof Suppose $\psi \in \mathbf{C}[v_1, \dots, v_r, z_1, \dots, z_m]$ and v-deg$(\psi) \leq 1$. Then we may write

$$\phi = \sum_{i=1}^{r} \alpha_i(b_i - v_i) + \sum_{j=1}^{m} \beta_j(f_j - z_j) + \psi$$

where $\alpha_i, \beta_j \in \mathbf{C}[x_1, \dots, x_n, v_1, \dots, v_r, z_1, \dots, z_m]$ for all i, j. Substituting $v_i = b_i$ and $z_j = f_j$ throughout gives

$$\phi = \psi(b_1, \dots, b_r, f_1, \dots, f_m)$$

as required. The condition on the v-degree ensures that this expresses ϕ as an element of M.

Conversely, suppose $\phi \in M$. Then $\phi = \phi'(b_1, \dots, b_r, f_1, \dots, f_m)$ for some polynomial $\phi' \in \mathbf{C}[v_1, \dots, v_r, z_1, \dots, z_m]$. Now we see, by Proposition 1.3.1, that $\phi = \psi(b_1, \dots, b_r, f_1, \dots, f_m)$, with $\psi = \bar{\phi}^G$ as before. But what is the v-degree of ψ? Suppose v-deg$(\psi) > 1$ and hence $LT(\psi) > LT(\phi')$. Now $\phi' \equiv \phi$ modulo I and so by Lemma 2.1.2, $LT(\phi') \geq LT(\bar{\phi}^G) = LT(\psi)$. This is a contradiction, so v-deg$(\psi) \leq 1$ as required.□

Lemma 2.1.2 *Let I be an ideal with Gröbner basis G. Suppose $\phi' \equiv \bar{\phi}^G$ modulo I. Then $LT(\phi') \geq LT(\bar{\phi}^G)$.*

Proof Now $LT(\phi') \geq LT(\bar{\phi'}^G)$ by the definition of division. Also, $\bar{\phi'}^G = \bar{\phi}^G$ since this determines a unique representative of the coset. Thus

$$LT(\phi') \geq LT(\bar{\phi'}^G) = LT(\bar{\phi}^G)$$

as required. □

The following routine implements the submodule membership test in Maple, using the built in Gröbner basis package. Here g is the element to be tested (written as a polynomial), F is the list of generators of the ring, B is the list of generators of the module and vars is the list of variables. If a fifth argument is given, this is assigned an expression giving g as an element of the module.

```
inmodule:=proc(g,F,B,vars)
local gg,i,vv,FF,G,_V,_Z,L,_VLIST;
  vv:=vars;
  FF:=[];
  _VLIST:=[];
    for i from 1 to nops(B) do
    vv:=[op(vv),_V[i]];
    FF:=[op(FF),B[i]-_V[i]];
    _VLIST:=[op(_VLIST),_V[i]];
  od;
  for i from 1 to nops(F) do
    vv:=[op(vv),_Z[i]];
    FF:=[op(FF),F[i]-_Z[i]];
  od;
  G:=grobner[gbasis](FF,vv,plex);
  gg:=grobner[normalf](g,G,vv,plex);
  L:=grobner[leadmon](gg,vv,plex)[2];
  for i from 1 to nops(vars) do
    if divide(L,vars[i]) then RETURN(false);
    fi;
  od;

  if degree(L,_VLIST)>1 then RETURN(false);
```

```
  fi;
  if nargs=5 then assign(args[5],gg);
  fi;
  RETURN(true);
end:
```

Here is an example of the routine being used:

```
>vars:=[x,y];
                              vars := [x, y]
---------------------------------------------------------------
> F:=[x^2,y^2];
                                   2    2
                             F := [x  , y ]
---------------------------------------------------------------
> B:=[1,x,y];
                             B := [1, x, y]
---------------------------------------------------------------
> inmodule(x*y,F,B,vars);
                                 false
---------------------------------------------------------------
>inmodule(8*x^2*y+3*x,F,B,vars,'q');
                                  true
---------------------------------------------------------------
> q;
                     3 _V[2] + 8 _Z[1] _V[3]
```

Now _V refers to elements of B and _Z refers to elements of F (very much as in Proposition 2.1.1 above). Thus the final line of output states that

$$8x^2y + 3x = 3 \cdot x + (8x^2) \cdot y$$

which expresses it as an element of the module.

2.2 Calculating module Gröbner bases

The following proposition gives a method for calculating Gröbner bases in a free module $\mathbf{C}[x_1, \ldots, x_n]^s$ by calculating a related basis in the ring $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_s]$. This is of practical use, as the ring case is supported on more computer algebra packages. In fact, the implementation of this result, `mgbasis`, is used in §2.3 for determining generators for syzygy modules.

Proposition 2.2.1 *Suppose M is a submodule of $\mathbf{C}[x_1, \ldots, x_n]^s$ generated by $b_1, \ldots, b_r$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 appearing in the i^{th} position. Suppose we have an order $>_x$ on $\mathbf{C}[x_1, \ldots, x_n]$. We can extend this to an order on $\mathbf{C}[x_1, \ldots, x_n]^s$ by considering the number of the component first and taking $e_1 > \ldots > e_s$. Let us call this (module) monomial order $>_1$. Now define $Z \subset \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_s]$ to be the set of polynomials all of whose monomials are of the form $\mathbf{x}^\alpha z_j$ for some α, j. Let $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_s]$ have an order $>_2$ fixed on it, which is a product of lex order on the z variables and $>_x$ on the x variables.*

Let $\Omega : \mathbf{C}[x_1, \ldots, x_n]^s \to Z$ be the map obtained by rewriting e_i as z_i and let G be a (ring) Gröbner basis for $<\Omega(b_1), \ldots, \Omega(b_r)>$. Then $G' = \Omega^{-1}(G \cap Z)$ is a (module) Gröbner basis for M under $>_1$.

Proof Firstly note that Ω is a 1-1 correspondence, so Ω^{-1} is well-defined. Also, Ω respects the orders $>_1$ and $>_2$, so $m >_1 n$ if and only if $\Omega(m) >_2 \Omega(n)$. Let h be an arbitrary element of M. Thus we may write

$$h = \sum_{i=1}^{r} \alpha_i b_i$$

where $\alpha_i \in \mathbf{C}[x_1, \ldots, x_n]$. So

$$\Omega(h) = \sum_{i=1}^{r} \alpha_i \Omega(b_i) \in <\Omega(b_1), \ldots, \Omega(b_r)>$$

Therefore, $\mathrm{LT}_2(\Omega(h)) = \mathbf{x}^\alpha \mathbf{z}^{\alpha'} \cdot \mathrm{LT}_2(\gamma)$, for some α, α', with $\gamma \in G$, since $<\mathrm{LT}_2(G)>$ is a monomial ideal (see Lemma 1.1.2). Now every monomial of γ must contain a z variable, since $<G>$ is generated by polynomials from Z. Now $LT_2(\Omega(h))$ is linear in a single z variable, this means $\mathrm{LT}_2(\gamma) \in Z$ and thus $\gamma \in G \cap Z$. Hence

$$\mathrm{LT}_2(\Omega(h)) = \mathbf{x}^\alpha \cdot \mathrm{LT}_2(\gamma)$$

and so

$$\Omega^{-1}(\mathrm{LT}_2(\Omega(h))) = \mathbf{x}^\alpha \cdot \Omega^{-1}(\mathrm{LT}_2(\gamma)).$$

Now Ω respects the orders, so this becomes

$$\mathrm{LT}_1(h) = \mathbf{x}^\alpha \mathrm{LT}_1(\Omega^{-1}(\gamma))$$

in other words,

$$\mathrm{LT}_1(h) \in <\mathrm{LT}_1(G')> . \tag{2.1}$$

So if G' generates M, then it is a Gröbner basis. Now every element of $G \cap Z$ is expressible in terms of the $\Omega(b_i)$, thus $G' \subset M$. We proceed as in [E, Lemma 6.5]. Let N be the submodule generated by G' and let k be the element of $M - N$ with smallest leading term. Now using (2.1), there must be $l \in N$ such that $\mathrm{LT}_1(l) = \mathrm{LT}_1(k)$. Then $k - l \in M - N$ and has a smaller leading term. This contradicts the definition of k. Thus $M = N$ and G' is a Gröbner basis as required.□

This routine calculates a module Gröbner basis for the submodule generated by the list F using variables v, where F is written as a list of vectors with polynomial components. The order used is lexicographic, with component number considered first.

```
mgbasis:=proc(F,v)
local _ZLIST,i,g,_Z,s,Fring,G,Gdash;
  s:=nops(F[1]);
```

```
    for i from 2 to nops(F) do
    if nops(F[i])<>s
    then ERROR(`All vectors must be of equal length`)
    fi;
  od;
  _ZLIST:=[];
  for i from 1 to s do
    _ZLIST:=[op(_ZLIST), _Z[i]];
  od;
  for i from 1 to nops(F) do
    Fring[i]:=into_ring(F[i],v,_ZLIST,s)
  od;
  Fring:=convert(Fring,list);
  G:=grobner[gbasis](Fring,[op(v),op(_ZLIST)],plex);
  Gdash:=[];
  for i from 1 to nops(G) do
    g:=from_ring(G[i],_ZLIST);
    if g<>0 then Gdash:=[op(Gdash), g];
    fi;
  od;
RETURN(Gdash);
end:
```

It requires the subroutines `into_ring` and `from_ring` which are given in Appendix B.

2.3 Finding syzygy module generators

We have seen a method for determining generators for the syzygies on a Gröbner basis in Proposition 1.3.3. We are interested in looking at syzygies as we are going to be interested in finding a basis for a free module (i.e. a set of generators with trivial syzygy module). The following gives a method for determining generators for syzygies on any set of polynomials:

Proposition 2.3.1 *Let $\{\alpha_1, \ldots, \alpha_s\}$ be a set of polynomials in $\mathbf{C}[x_1, \ldots, x_n]$, generating an ideal I_α. Let $\alpha_{s+1}, \ldots, \alpha_m$ be additional polynomials in I_α such that $\{\alpha_1, \ldots, \alpha_s, \alpha_{s+1} \ldots, \alpha_m\}$ is a Gröbner basis for I_α under the given order on $\mathbf{C}[x_1, \ldots, x_n]$. Now each α_j for $j = s+1, \ldots, m$ may be written as*

$$\alpha_j = \sum_{i=1}^{s} \beta_{ij} \alpha_i$$

for some $\beta_{ij} \in \mathbf{C}[x_1, \ldots, x_n]$. Define

$$\rho : \mathbf{C}[x_1, \ldots, x_n]^m \longrightarrow \mathbf{C}[x_1, \ldots, x_n]^s$$

by

$$(\sigma_1, \ldots, \sigma_s, \ldots, \sigma_m) \mapsto (\bar{\sigma_1}, \ldots, \bar{\sigma_s})$$

where

$$\bar{\sigma_i} = \sigma_i + \sum_{j=s+1}^{m} \sigma_j \beta_{ij}.$$

Then if Σ is a generating set for the syzygies on $\{\alpha_1, \ldots, \alpha_s, \ldots, \alpha_m\}$, $\rho(\Sigma)$ generates the syzygies on $\{\alpha_1, \ldots, \alpha_s\}$.

Proof Firstly we need to show that the elements of $\rho(\Sigma)$ are syzygies on $\{\alpha_1, \ldots, \alpha_s\}$. Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s, \ldots, \sigma_m) \in \Sigma$. Now

$$\begin{aligned} \sum_{i=1}^{s} \bar{\sigma_i} \alpha_i &= \sum_{i=1}^{s} \left(\sigma_i \alpha_i + \sum_{j=s+1}^{m} \sigma_j \beta_{ij} \alpha_i \right) \\ &= \sum_{i=1}^{s} \sigma_i \alpha_i + \sum_{j=s+1}^{m} \sigma_j \sum_{i=1}^{s} \beta_{ij} \alpha_i \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{s} \sigma_i \alpha_i + \sum_{j=s+1}^{m} \sigma_j \alpha_j \\ &= \sum_{i=1}^{m} \sigma_i \alpha_i \\ &= 0. \end{aligned}$$

Hence $\rho(\boldsymbol{\sigma})$ is a syzygy on $\{\alpha_1, \ldots, \alpha_s\}$.

But does $\rho(\Sigma)$ generate all such syzygies? Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_s)$ be an arbitrary syzygy on $\{\alpha_1, \ldots, \alpha_s\}$. We can associate this with the syzygy $\boldsymbol{\tau}' = (\tau_1, \ldots, \tau_s, 0, \ldots, 0)$ on $\{\alpha_1, \ldots, \alpha_s, \ldots, \alpha_m\}$. Since Σ generates all such syzygies, we may write

$$\boldsymbol{\tau}' = \sum_{\sigma \in \Sigma} \delta_\sigma \boldsymbol{\sigma}$$

for some $\delta_\sigma \in \mathbf{C}[x_1, \ldots, x_n]$. Applying ρ gives

$$\boldsymbol{\tau} = \sum_{\sigma \in \Sigma} \delta_\sigma \rho(\boldsymbol{\sigma})$$

since it is a module homomorphism. Thus $\rho(\Sigma)$ is a generating set as required.□

The above proposition is implemented by the function **syz** (see Appendix B).

The following proposition can be used to restrict syzygies to a subring of the form $f^*\mathbf{C}[y_1, \ldots, y_m]$, so that we find generators for $S \cap f^*\mathbf{C}[y_1, \ldots, y_m]^s$.

Proposition 2.3.2 *Let S be a submodule of $\mathbf{C}[x_1, \ldots, x_n]^s$ generated by the (finite) set Σ. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i^{th} place. Let $\Gamma = \{(f_j - z_j)e_i : i \in \{1, \ldots, s\}, j \in \{1, \ldots, r\}\}$ and define $M = <\Sigma, \Gamma> \cap \mathbf{C}[z_1, \ldots, z_m]^s$ and $N = <\Sigma> \cap f^*\mathbf{C}[y_1, \ldots, y_m]^s$. If we define*

$$\Phi : \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_r]^s \to \mathbf{C}[x_1, \ldots, x_n]^s$$

to be substitution of f_i for each z_i, then $\Phi(M) = N$.

Proof Any element of $<\Sigma, \Gamma>$ will lie in $<\Sigma>$ under the map Φ and anything in $\mathbf{C}[z_1, \ldots, z_m]^s$ will be mapped into $f^*\mathbf{C}[y_1, \ldots, y_m]^s$. Thus $\Phi(M) \subset N$.

Now let n be an arbitrary element of N. Write $n_1, \ldots, n_s$ for its components. so $n = \sum e_k n_k$. Let G be a Gröbner basis for $<f_1 - z_1, \ldots, f_m - z_m>$ under lex order with $x_1 > \cdots > x_n > z_1 > \cdots > z_m$. By Proposition 1.3.1, $\overline{n_k}^G = \beta_k(\mathbf{z}) \in \mathbf{C}[z_1, \ldots, z_m]$, such that $\beta_k(f_1, \ldots, f_m) = n_k$. In other words, we may write

$$n_k = \sum_{i=1}^{m} \alpha_{k,i}(\mathbf{x}, \mathbf{z})(f_i - z_i) + \beta_k(\mathbf{z})$$

where $\alpha_{k,i} \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_r]$ and so

$$n = \sum_{k=1}^{s} e_k n_k = \underbrace{\sum_{k=1}^{s} \sum_{i=1}^{m} e_k \alpha_{k,i}(\mathbf{x}, \mathbf{z})(f_i - z_i)}_{d} + \underbrace{\sum_{k=1}^{s} e_k \beta_k(\mathbf{z})}_{c}.$$

Now $\Phi(c) = \sum_{k=1}^{s} e_k n_k = n$, but is $c \in M$? It is certainly in $\mathbf{C}[z_1, \ldots, z_m]^s$. Now $n, d \in <\Sigma, \Gamma>$, thus $c \in <\Sigma, \Gamma>$ too. Hence $c \in M$ and so $\Phi(M) \supset N$.□

Corollary 2.3.3 *If $\{m_1, \ldots, m_t\}$ generate M as a $\mathbf{C}[z_1, \ldots, z_m]$ module, then N is generated as an $f^*\mathbf{C}[y_1, \ldots, y_m]$ module by $\{\Phi(m_1), \ldots, \Phi(m_t)\}$.*

Proof Given an element $n \in N$, there is $m \in M$ such that $\Phi(m) = n$. Now we may write

$$m = \sum_{i=1}^{t} \delta(\mathbf{z}) m_i$$

where $\delta(\mathbf{z}) \in \mathbf{C}[z_1, \ldots, z_m]$ and so

$$n = \Phi(m) = \sum_{i=1}^{t} \delta(\mathbf{f}) \Phi(m_i).$$

Thus we have expressed n as an element of the $f^*\mathbf{C}[y_1, \ldots, y_m]$ module generated by $\{\Phi(m_1), \ldots, \Phi(m_t)\}$. □

The next result helps to find the generators of N in practice. (This is a slight generalization of Proposition 1.2.7.)

Lemma 2.3.4 *Define $I = <\Sigma, \Gamma>$ (with Σ, Γ as before) and $I_z = I \cap \mathbf{C}[z_1, \ldots, z_m]^s$. Let G be a Gröbner basis for I under a product order $>$ with $x_i > z_j$ for all i, j and let $G_z = G \cap \mathbf{C}[z_1, \ldots, z_m]^s$. Then G_z generates I_z and is in fact a Gröbner basis under the order inherited from $>$.*

Proof Write $G = \{\gamma_1, \ldots, \gamma_r\}$ and assume that $G_z = \{\gamma_1, \ldots, \gamma_q\}$. Now each element of G_z lies in I_z so $<G_z> \subset I_z$. Let ϕ be any element of I_z and divide it by G. This gives

$$\phi = \epsilon_1\gamma_1 + \cdots + \epsilon_q\gamma_q + 0 \cdot \gamma_{q+1} + \cdots + 0 \cdot \gamma_r$$

since $\phi \in I$ and because the leading terms of $\gamma_{q+1}, \ldots, \gamma_r$ must contain an x variable by virtue of the given order. (Note that this shows that dividing an element of I_z by G is the same as dividing it by G_z.) Now we have shown that $\phi \in <G_z>$, i.e. G_z generates I_z. But is it a Gröbner basis?

If S denotes the S-polynomial then we need to show that $\overline{S(\gamma_i, \gamma_j)}^{G_k} = 0$ for all $\gamma_i, \gamma_j \in G_k$ by Proposition 1.3.2. But $S(\gamma_i, \gamma_j) \in I_k$ since $\gamma_i, \gamma_j \in I_k$ and so, by the fact noted above,

$$\overline{S(\gamma_i, \gamma_j)}^{G_k} = \overline{S(\gamma_i, \gamma_j)}^{G} = 0$$

Thus G_k is a Gröbner basis for I_k. □

So in order to find the generators of N, we need to calculate a Gröbner basis for $<\Sigma, \Gamma>$ under a suitable product order and pick out those elements which lie in $\mathbf{C}[z_1, \ldots, z_m]$. Substitution of f_i for each z_i will give the required generators.

Corollary 2.3.5 *Let $\mathbf{C}[x_1, \ldots, x_n]$ be an $f^*\mathbf{C}[y_1, \ldots, y_m]$ module generated by $\alpha_1, \ldots \alpha_s$. Let Σ generate the syzygy module of $\alpha_1, \ldots \alpha_s$ and let Γ be as before. Let G be a Gröbner basis for $<\Sigma, \Gamma>$ under a product order with $x_i > z_j$ for all i, j. Then $\mathbf{C}[x_1, \ldots, x_n]$ is a free $f^*\mathbf{C}[y_1, \ldots, y_m]$ module if and only if $G \cap \mathbf{C}[z_1, \ldots, z_m] = \emptyset$.*

Proof This is equivalent to the fact that $\mathbf{C}[x_1, \ldots, x_n]$ is a free module if and only if

$$\sum_{i=1}^{s} \epsilon_i(\mathbf{f})\alpha_i = 0$$

for $\epsilon_i \in f^*\mathbf{C}[y_1, \ldots, y_m]$ implies $\epsilon_i = 0$ for all i. □

The following Maple routine, **subring_syz**, uses the above results to find generators for the syzygies in a subring. `G` is the list of elements on which to find the syzygies, `F` is the list of generators of the subring and `v` is the list of variables in the base ring. The routines `mgbasis`, `into_ring`, `from_ring`, `syz` from Appendix B and Albert Lin's routines from Appendix C are all required.

```
subring_syz:=proc(F,G,v)
local i,j,Sig,Gam,s,ModBasis,Inter;
  Sig:=convert(syz(G,v),listlist);
  s:=nops(G);
  Y:=[];
  for i from 1 to s do
    Y:=[op(Y), _Y[i]];
  od;
  Gam:=[];
  for i from 1 to nops(F) do
    for j from 1 to s do
      Gam:=[op(Gam), [0 $ k=1..j-1 , F[i]-Y[i] , 0 $ k=j+1..s]];
    od;
  od;
  ModBasis:=mgbasis([op(Sig),op(Gam)],[op(v),op(Y)]);
  Inter:=[];
  for i from 1 to nops(ModBasis) do
    if type(ModBasis[i],list(polynom(constant,Y)))
    then Inter:=[op(Inter),ModBasis[i]] fi;
  od;
```

```
  for i from 1 to nops(F) do
    Inter:=subs(_Y[i]=F[i],Inter);
  od;
RETURN(Inter);
end:
```

An example of `subring_syz` in use:

```
> F:=[x^2,y^2];
                                          2    2
                                   F := [x , y ]

-----------------------------------------------------------------
> G:=[y,y^2,x^4,x^4*y^6];
                                          2    4    4  6
                              G := [y, y , x , x  y ]

-----------------------------------------------------------------
> subring_syz(F,G,[x,y]);
                              [        4      2       ]
                              [ 0   x      - y     0  ]
                              [                       ]
                              [               6       ]
                              [ 0   0      y      -1  ]
```

Thus $(0, x^4, -y^2, 0)$ and $(0, 0, y^6, -1)$ generate the syzygies on $\{y, y^2, x^4, x^4y^6\}$ restricted to the ring of polynomials in $\{x^2, y^2\}$ (over C).

Chapter 3

Finite polynomial maps $\mathbf{C}^n \to \mathbf{C}^n$

This chapter is devoted to the definition of finite polynomial maps and a description of some of their properties. As we shall see, each such map is associated with a free module over a polynomial ring, namely $\mathbf{C}[x_1, \ldots, x_n]$ over its subring $f^*\mathbf{C}[y_1, \ldots, y_n]$. Our first aim will be to find a method for determining algorithmically if a given polynomial map is finite. If we have shown that a map is finite, we would then like to give generators for the associated module. Thirdly, we would like, if possible, to describe a free basis for the module.

We look at a simple class of maps, with components which are quasihomogeneous with respect to a fixed set of weights. We find a simple test for finiteness and a method for determining module generators, based upon testing if the quotient

$$\frac{\mathbf{C}[x_1, \ldots, x_n]}{<f_1, \ldots, f_n>}$$

is finite dimensional as a $\mathbf{C}$ vector space. These results are then extended to those maps which are quasihomogeneous maps with extra lower order terms added. Finally, we describe a general criterion for a map to be finite and a method for finding generators for the module. This time we need to test if a $\mathbf{C}$ space basis (defined by a monomial order) for the quotient

$$\frac{\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]}{<f_1 - z_1, \ldots, f_n - z_n>}$$

contains finitely many monomials in the x variables. We also have a condition, which if satisfied, shows that this set of generators is a basis.

Next we define three different degrees which can be associated with a finite map. These are given by the dimension of a vector space, degree of a field extension and the size of a basis for the free module mentioned above. These three numbers are shown to coincide.

In Chapter 4, The results on polynomial maps between affine spaces are applied to regular maps between affine varieties. Again, we find a general criterion for such a map to be finite and a method for determining generators for the associated module.

Finally in this half of the thesis, we look at the trace bilinear form of a finite map. This gives the number of roots of the system $f = c$ for a point c in the target, as well as the number of real roots in an algebraically defined region.

3.1 Definitions

Let $B \supset A$ be rings. An element $b \in B$ is said to be *integral* over A if it satisfies an equation of the form $b^m + a_1 b^{m-1} + \cdots + a_m = 0, a_i \in A$. The ring B is called integral over A if every element of B is integral over A. If B is finitely generated (as a ring) over A then B is integral over A if and only if it is a finitely generated A-module (See [ZS, ch V §1]).

Let $f = (f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map. The pullback map $f^* : \mathbf{C}[y_1, \ldots, y_n] \to \mathbf{C}[x_1, \ldots, x_n]$, associated with f, is defined by

$$f^*(\phi(y_1, \ldots, y_n)) = \phi(f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$$

Now f is said to be *quasifinite* if every point has a finite number of preimages, i.e. the set $f^{-1}(y)$ is finite for every $y \in \mathbf{C}^n$. On the other hand, f is said to be *finite* if the ring $\mathbf{C}[x_1, \ldots, x_n]$ is integral (and hence a finitely generated module) over the ring $f^*\mathbf{C}[y_1, \ldots, y_n]$. The definitions of finiteness and quasifiniteness are very similar and the difference between the two is quite subtle. In essence, they both imply that every point has a finite number of preimages, but for a finite map none of the roots in the set $f^{-1}(y)$ can go off to infinity as y moves around $\mathbf{C}^n$. Hence finiteness implies quasifiniteness but not vice versa. Suppose f is a finite polynomial map and we wish to show quasifiniteness. It suffices to prove that each coordinate x_i can only take a finite number of values on the set $f^{-1}(y)$, where y lies in the target space. By the integral property, x_i, considered as an element of $\mathbf{C}[x_1, \ldots, x_n]$, must satisfy an equation $x_i^m + \alpha_1 x_i^{m-1} + \cdots + \alpha_m = 0, \alpha_i \in f^*\mathbf{C}[y_1, \ldots, y_n]$. Thus, to be the i^{th} coordinate of $z = (z_1, \ldots, z_n) \in f^{-1}(y)$, x_i must satisfy

$$x_i(z)^m + \alpha_1(f(z))x_i(z)^{m-1} + \cdots + \alpha_m(f(z)) = 0$$

$$z_i^m + \alpha_1(y)z_i^{m-1} + \cdots + \alpha_m(y) = 0$$

which is a non-zero polynomial and hence z_i can only take finitely many values. An example of a map which is quasifinite but not finite is $f = (x, xy^2 + y)$. This result will be proved later.

We now give some useful properties of finite maps.

Proposition 3.1.1 *If $f : \mathbf{C}^n \to \mathbf{C}^n$ is finite and $f(\mathbf{C}^n)$ is dense in the target space (under the Zariski topology), then :*

1. *$f^*\mathbf{C}[y_1, \ldots, y_n]$ is isomorphic to $\mathbf{C}[y_1, \ldots, y_n]$ via f^*.*

2. *The map f is surjective (i.e. onto the target space).*

Proof See [Sh, p21] and [Sh, p48] respectively. □

In fact, the condition that the image is dense in the target will always hold as a result of the following:

Proposition 3.1.2 *Let $f : X \to Y$ be a regular mapping of affine varieties with Y irreducible. Furthermore, let both Y and the irreducible components of X have dimension n. Suppose that for some $y_0 \in Y$ we have $f^{-1}(y_0)$ finite, non-empty, then $f(X)$ is dense in Y*

Proof We wish to show that $\overline{f(X)} = Y$ (where $\overline{f(X)}$ denotes the Zariski closure of $f(X)$). By Theorem 1, [Sh, p54], it is enough to show that $\dim \overline{f(X)} \geq n$. We now replace X by an irreducible component containing some point of $f^{-1}(y_0)$. But $f : X \to \overline{f(X)}$ is a regular mapping. Thus $\dim f^{-1}(y) \geq n - \dim(\overline{f(X)})$ for all $y \in f(X)$ by [Sh, p60]. But if we take $y = y_0$ then $\dim f^{-1}(y_0) = 0$, so $\dim(\overline{f(X)}) \geq n$. □

Another property of finite mappings is the following:

Proposition 3.1.3 *If $f : \mathbf{C}^n \to \mathbf{C}^n$ is finite, then it is also proper. (In other words, the inverse image of a compact set is compact under the standard metric topology on $\mathbf{C}^n$.)*

Proof We wish to show that for any compact set C, $f^{-1}(C)$ is also compact, i.e. bounded. Now since f is finite, for all i there exist $\beta_j \in f^*\mathbf{C}[y_1, \dots, y_n]$ and m such that

$$x_i^m + \beta_{m-1}(f)x_i^{m-1} + \cdots + \beta_0(f) = 0.$$

If we choose $\boldsymbol{x} = (x_1, \dots, x_n)$ such that $f(\boldsymbol{x}) \in C$, where C is a given compact set, then

$$x_i^m + \kappa_{m-1}x_i^{m-1} + \cdots + \kappa_0 = 0 \tag{3.1}$$

where the $\kappa_j \in \beta_j(C)$ and hence are bounded. If we rewrite (3.1) as

$$x_i^m(1 + \frac{\kappa_{m-1}}{x_i} + \cdots + \frac{\kappa_0}{x_i^m}) = 0$$

then we see that for x_i with $|x_i|$ large, $(1 + \frac{\kappa_{m-1}}{x_i} + \cdots + \frac{\kappa_0}{x_i^m})$ cannot be zero since the κ_j are bounded. So x_i is bounded for each i and the map is proper as required. ◻

Proposition 3.1.4 *Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a finite map. Then the set of critical values (i.e. images of points where the Jacobian is degenerate) is closed, both in the Zariski and the standard topology.*

Proof Let $y_n \to y$ with y_n critical and let x_n be a corresponding critical point, so $|df(x_n)| = 0$. Let D be a compact neighbourhood of y: since f is proper (see previous result), $f^{-1}(D)$ is also compact. Hence the x_n have an accumulation point x, say. Now $|df(x)| = \lim_{n\to\infty} |df(x_n)| = 0$. So x is critical and $f(x) = \lim_{n\to\infty} f(x_n) = y$. Thus y is also a critical value and we find that the set of critical values is closed in the standard topology.

Now the set of critical points is Zariski closed (determined by the vanishing of a determinant) and hence its image is constructible. But by Lemma 3.1.5 below, we know that a closed, constructible set is Zariski closed. ◻

Lemma 3.1.5 *A closed, constructible set is Zariski closed.*

Proof Let $T \supset S$ be affine varieties in $\mathbf{C^n}$ with no components in common. We may write T as a union of its irreducible components

$$T = \bigcup_{i=1}^{r} T_i$$

and so

$$T - S = \bigcup_{i=1}^{r} (T_i - S_i) \tag{3.2}$$

where $S_i = S \cap T_i$. Now we may write $T_i - S_i = P(T_i) - (P(S_i) \cup (P(T_i) \cap H_\infty))$, where $P()$ represents the projective closure and $H_\infty \cong \mathbf{P}^{n-1}$ is the "hyperplane at infinity". Now $P(S_i)$ and $(P(T_i) \cap H_\infty)$ are both Zariski closed by definition, so their union must be too. Considering the subspace topology on $P(T_i)$, $(P(S_i) \cup$

$(P(T_i) \cap H_\infty))$ will also be Zariski closed in $P(T_i)$. Thus $T_i - S_i$ is Zariski open in $P(T_i)$ and non-empty, since T and S have no components in common. By [Mu, p39], the closure (in the classical topology on $\mathbf{P}^n$) of $T_i - S_i$ is $P(T_i)$. We will write this as

$$\overline{T_i - S_i}^{\mathbf{P}^n} = P(T_i).$$

Applying this to (3.2), we obtain

$$\overline{T - S}^{\mathbf{P}^n} = \bigcup_{i=1}^{r} P(T_i)$$

which in fact equals $P(T)$ for the following reason. The set $\bigcup_{i=1}^{r} P(T_i)$ is a projective variety containing T, but $P(T)$ is the smallest such variety by definition (see [CLO, p377]). So

$$P(T) \subseteq \bigcup_{i=1}^{r} P(T_i).$$

Suppose $P(T_j)$ is not a subset of $P(T)$. Consider the variety $P(T_j) \cap P(T)$. This is a variety containing T_j which is strictly smaller than $P(T_j)$. This contradicts the definition of $P(T_j)$, so $P(T_j) \subset P(T)$. The required equality follows.

Now since the classical topology on $\mathbf{C}^n$ is a subspace topology of that on $\mathbf{P}^n$, $\overline{T - S}^{\mathbf{C}^n}$ is the affine part of $P(T)$, i.e. T itself. Hence a (classically) closed set of the form $T - S$ must in fact be Zariski closed.

Any constructible set is a disjoint union of sets of the form $T - S$, so a closed constructible set is Zariski closed. □

The degree of a map $f : \mathbf{C}^n \to \mathbf{C}^n$ is defined to be the degree of the field extension $[\mathbf{C}(x) : f^*\mathbf{C}(y)]$. The number of inverse images of a point in the target is at most the degree of the map. The ramification locus is the set of points where this number falls below the degree. The following allows us to determine if a given point falls within this variety.

Proposition 3.1.6 *Again, let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a finite map. Then the ramification locus and the set of critical values are equal.*

Proof Let y be a non-ramified point and let $x \in f^{-1}(y)$. Then $df(x)$ is an isomorphism, so y is a regular (non-critical) value (see [Sh, p120]).

Now suppose y is a regular value, then there is a compact neighbourhood D of y of regular values. Since the ramification locus is closed and a proper subset of $\mathbf{C}^n$ (see [Sh, p117]), we can find points arbitrarily near y which are non-ramified. Choosing D small enough, we find $f : f^{-1}(D) \to D$ is a cover and the number of sheets is $|f^{-1}(y)|$ which is also the degree of f. So y is also a non-ramification point.

Thus we have shown inclusions in both directions, so the two sets are equal.□

We have already seen that for $f : \mathbf{C}^n \to \mathbf{C}^n$ finite, $\mathbf{C}[x_1, \ldots, x_n]$ is finitely generated as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module. The following gives further infomation concerning this module:

Proposition 3.1.7 *Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a finite map. Then $\mathbf{C}[x_1, \ldots, x_n]$ is a free $f^*\mathbf{C}[y_1, \ldots, y_n]$ module.*

Proof Since the source and target varieties are in fact affine spaces, they are certainly Cohen-Macaulay and smooth respectively. Thus by [Ma, p179], the map f (and hence $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module) is flat. Now by [C2, p242 Cor 6.4], a flat module which is finitely generated over a noetherian ring is projective. The ring $f^*\mathbf{C}[y_1, \ldots, y_n]$ is certainly noetherian, so $\mathbf{C}[x_1, \ldots, x_n]$ is a projective $f^*\mathbf{C}[y_1, \ldots, y_n]$ module. Finally, by [Q] a projective module over a polynomial ring is free, so we have the required result. □

3.2 Simple types of finite map

We begin by giving some simple types of finite map, which are easy to identify. In the following, given a set of weights $w = (w_1, \ldots, w_n) \in \mathbf{N}^n$ (each $w_i > 0$), we define the *w-degree* of a monomial by $\deg_w(x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}) = \sum_{i=1}^n w_i a_i$. Now a polynomial is said to be *quasihomogeneous* if all its monomials have the same w-degree.

Proposition 3.2.1 *Let $f = (f_1, \dots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map with each component f_i quasihomogeneous with respect to a set of weights $w = (w_1, \dots, w_n)$. Then*

$$dim_{\mathbf{C}} \frac{\mathbf{C}[x_1, \dots, x_n]}{<f_1, \dots, f_n>} \tag{3.3}$$

is finite if and only if the map f is finite. Moreover, if $\mathcal{B}$ is a monomial basis for the above vector space, then $\mathcal{B}$ generates $\mathbf{C}[x_1, \dots, x_n]$ as a module over $f^*\mathbf{C}[y_1, \dots, y_n]$.

Proof Let $\phi \in \mathbf{C}[x_1, \dots, x_n]$ be an arbitrary element. We wish to express ϕ in terms of $\mathcal{B}$ and $f^*\mathbf{C}[y_1, \dots, y_n]$ so that $\phi = \sum_{i=1}^m \delta_i b_i$ with $b_i \in \mathcal{B}$, $\delta_i \in f^*\mathbf{C}[y_1, \dots, y_n]$. (This shows that $\mathbf{C}[x_1, \dots, x_n]$ is a finitely generated module over $f^*\mathbf{C}[y_1, \dots, y_n]$ and hence that f is finite). Now we may write

$$\phi = \sum_{i=1}^m \lambda_i b_i + \sum_{i=1}^n \alpha_i f_i \tag{3.4}$$

where $\lambda_i \in \mathbf{C}$ and $\alpha_i \in \mathbf{C}[x_1, \dots, x_n]$ by consideration of ϕ as an element of the quotient in (3.3). In fact, by Lemma 3.2.2 and the fact that the f_i are non-constant, we also have that $\alpha_i = 0$ or $\deg_w \alpha_i < \deg_w \phi$. Now since $\lambda_i \in f^*\mathbf{C}[y_1, \dots, y_n]$, the first sum in (3.4) is of the required form. The second sum will also be of the correct form if the α_i lie in the $f^*\mathbf{C}[y_1, \dots, y_n]$-module generated by $\mathcal{B}$. Thus we now express each non-zero α_i as in (3.4) and continue the process. If all the polynomials are eventually reduced to zero, then we have shown, as required, that ϕ is an element of the module generated by $\mathcal{B}$. Now at each stage, the (non-negative) weighted degrees are strictly decreasing, or the resulting α_i are zero, so this will eventually occur.

For the converse, suppose that the dimension in (3.3) was infinite. Then, this would imply that 0 had an infinite number of preimages (see Proposition 1.2.5) and this would contradict the finiteness of f. The second result has already been proved. □

Lemma 3.2.2 *Let $f_1, \ldots, f_n$ be non-constant polynomials in $\mathbf{C}[x_1, \ldots, x_n]$, each of which is quasihomogeneous with respect to a set of weights $w = (w_1, \ldots, w_n) \in \mathbf{N}^n$. Then if $\phi \in <f_1, \ldots, f_n>$, $\phi \neq 0$, we may write*

$$\phi = \sum_{i=1}^{n} \alpha_i f_i$$

where each α_i satisfies $\deg_w \alpha_i f_i \leq \deg_w \phi$ and thus $\deg_w \alpha_i < \deg_w \phi$

Proof Since $\phi \in < f_1, \ldots, f_n >$, we may write $\phi = \sum_{i=1}^{n} \beta_i f_i$ for some $\beta_i \in \mathbf{C}[x_1, \ldots, x_n]$. Suppose there is a $\beta_j f_j$ such that $\deg_w \beta_j f_j > \deg_w \phi$, otherwise the proof is complete. Take the quasihomogeneous part of each $\beta_i f_i$ of this w-degree. Since the f_i are quasihomogeneous, these may be written as $\zeta_i f_i$ in each case. ($\zeta_i \in \mathbf{C}[x_1, \ldots, x_n]$ and possibly zero). But $\sum_{i=1}^{n} \zeta_i f_i = 0$, so

$$\phi = \sum_{i=1}^{n} \beta_i f_i - \sum_{i=1}^{n} \zeta_i f_i = \sum_{i=1}^{n} (\beta_i - \zeta_i) f_i.$$

In this manner, all the higher terms may be discarded. □

Since the proof of Proposition 3.2.1 is constructive, it is easy to give an example of how an arbitrary element of $\mathbf{C}[x_1, \ldots, x_n]$ may be rewritten as an element of the module.

Example 3.2.3 Let $f = (f_1, f_2) = (x^2 + xy^3 + y^6, y^3)$, which is quasihomogeneous with respect to the set of weights w=(3,1). Now we find that $\{1, x, y, xy, xy^2, y^2\}$ is an LT monomial basis for the quotient of $\mathbf{C}[x_1, \ldots, x_n]$ by the ideal $<x^2 + xy^3 + y^6, y^3>$ (See Proposition 1.2.5). So by Proposition 3.2.1, we know that f is finite. Now let us choose $\phi = x^3$ as our polynomial to reduce. This can be written as

$$\phi = 0 + (x) f_1 + (-x^2 - xy^3) f_2.$$

Now rewriting $-x^2 - xy^3$ gives

$$-x^2 - xy^3 = 0 + (-1)f_1 + (y^3)f_2$$

and y^3 can be written

$$y^3 = 0 + (0)f_1 + (1)f_2$$

which gives

$$\phi = x(f_1) + (-f_1 + f_2^2)f_2.$$

In other words,

$$\phi = 1 \cdot (f_2^3 - f_1 f_2) + x \cdot (f_1) + y \cdot 0 + xy \cdot 0 + y^2 \cdot 0 + xy^2 \cdot 0.$$

Thus x^3 has been expressed in terms of the given basis, with coefficients in $f^*\mathbf{C}[y_1, \dots, y_n]$.

3.3 Further finite maps

Having shown that a map with quasihomogeneous components can be tested for finiteness by looking at the dimension of a quotient space, we extend this result to a larger set of mappings.

Proposition 3.3.1 *Let $(f_1, \dots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a finite polynomial map with quasihomogeneous components with respect to a set of weights $w = (w_1, \dots, w_n)$. Let $(g_1, \dots, g_n) \in \mathbf{C}[x_1, \dots, x_n]^n$ be such that for each i, $\deg_w g_i < \deg_w f_i$. Then the map $(f+g) = (f_1 + g_1, \dots, f_n + g_n) : \mathbf{C}^n \to \mathbf{C}^n$ is also finite. Moreover, if $\mathcal{B}$ is a monomial basis for the vector space,*

$$\frac{\mathbf{C}[x_1, \dots, x_n]}{<f_1, \dots, f_n>}$$

then $\mathcal{B}$ generates $\mathbf{C}[x_1, \dots, x_n]$ as a module over $(f+g)^*\mathbf{C}[y_1, \dots, y_n]$

Proof Let $\phi \in \mathbf{C}[x_1, \ldots, x_n]$ be an arbitrary element. Then we may write ϕ as in Proposition 3.2.1 as

$$\phi = \sum_{i=1}^{m} \lambda_i b_i + \sum_{i=1}^{n} \alpha_i f_i$$

where $\lambda_i \in \mathbf{C}$, $b_i \in \mathcal{B}$, $\alpha_i \in \mathbf{C}[x_1, \ldots, x_n]$ and $\alpha_i = 0$ or $\deg_w \alpha_i < \deg_w \phi$. This gives

$$\phi = \sum_{i=1}^{m} \lambda_i b_i + \sum_{i=1}^{n} \alpha_i (f_i + g_i) - \sum_{i=1}^{n} \alpha_i g_i. \tag{3.5}$$

Now $\sum \alpha_i g_i = 0$ or $\deg_w(\sum \alpha_i g_i) < \deg_w \phi$, so rewrite $\sum \alpha_i g_i$ in the form of (3.5) and continue to reduce. Since the degree is strictly decreasing, the process will terminate, giving

$$\phi = \sum_{i=1}^{m} \kappa_i b_i + \sum_{i=1}^{n} \beta_i (f_i + g_i) \tag{3.6}$$

where $\kappa_i \in \mathbf{C}$, $\beta_i \in \mathbf{C}[x_1, \ldots, x_n]$ and $\deg_w \beta_i f_i \leq \deg_w \phi$. Now rewrite β_i as in (3.6) and continue this process inductively. Since at each stage $\deg_w \beta_i < \deg_w \phi$, this will also terminate, giving ϕ as a linear combination of elements of $\mathcal{B}$ with coefficients in $(f+g)^* \mathbf{C}[y_1, \ldots, y_n]$ as required. □

Since this proof uses two different inductive processes, it is more complicated to do an example by hand. (See below for a computer program which will write an arbitrary element in the required form). However, the second reduction step is almost identical to the reduction step of Proposition 3.2.1, so we just illustrate the first step of the proof.

Example 3.3.2 Again, take $f = (f_1, f_2) = (x^2 + xy^3 + y^6, y^3)$, a quasihomogeneous finite map, but this time, let $g = (g_1, g_2) = (x + 8, y)$ be the extra, lower order part. Take $\phi = x^2$ as our polynomial to reduce to the form of (3.6). Now we may write it as

$$\phi = f_1 + (-x - y^3) f_2.$$

Adding and subtracting the g_i gives

$$\phi = (f_1 + g_1) + (-x - y^3)(f_2 + g_2) - g_1 + (x + y^3) g_2$$

and hence

$$\phi = (f_1 + g_1) + (-x - y^3)(f_2 + g_2) - x - 8 + xy + y^4.$$

Now rewriting y^4 gives

$$y^4 = y(f_2) = y(f_2 + g_2) - y(g_2) = y(f_2 + g_2) - y^2.$$

So we obtain

$$\phi = -8 - x + xy - y^2 + (f_1 + g_1) + (-x - y^3 + y)(f_2 + g_2)$$

and so x^2 is expressed in the form required. Now ϕ is of w-degree 6 and the largest w-degree on the right is also 6, in $(f_1 + g_1)$ and $-y^3(f_2 + g_2)$, so the given inequality does hold.

This Maple routine will express an element of $\mathbf{C}[x_1, \ldots, x_n]$ as a member of a $\mathbf{C}[f_1 + g_1, \ldots, f_n + g_n]$-module where f_i,g_i are as in Proposition 3.3.1.

```
asmodule:=proc(h,F,G,v,order)
local i,Ans,rm,Sp,Trm,j;
Sp:=FGR(h,F,G,v,order);
Ans:=Sp[1];
rm:=[[Sp[i+1],F[i]+G[i]] $ i=1 ..  nops(F)];
while rm <> [] do
  if rm[1][1]=0 then rm:=[rm[i] $ i=2 ..  nops(rm)];
    else Sp:=FGR(rm[1][1],F,G,v,order);
    Ans:=Ans+(Sp[1]*rm[1][2]);
    Trm:=[rm[i] $ i=2 ..  nops(rm),
      [Sp[j+1],(F[j]+G[j])*rm[1][2]] $j=1 ..  nops(F)];
    rm:=eval(Trm);
  fi;
od;
```

```
RETURN(Ans);
end:
```

In the above, h is the element to be rewritten. F and G are $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ respectively, v is the list of variables and order is the monomial order, given as a list of vectors. (See Appendix C for the definition). As the program runs, Ans contains the answer as it builds up, rm contains the remainder, yet to be rewritten, in the form [[a_1,F_{i_1}+G_{i_1}],..., [a_s,F_{i_s}+G_{i_s}]], where the remainder is actually $\sum a_j(f_{i_j} + g_{i_j})$. Then a_1 is itself split up, some of which is added to Ans, the rest to rm. The following subroutine is also needed:

```
FGR:=proc(h,F,G,v,order)
local Ans,temp,i,Extra,Textra;
for i from 1 to nops(F)+1 do
  Ans[i]:=0;
od;
Extra:=h;
while Extra <>0 do
  temp:=remainder(Extra,F,v,order);
  Ans[1]:=Ans[1]+temp[1];
  Textra:=0;
  for i from 1 to nops(F) do
    Ans[i+1]:=Ans[i+1]+temp[2][i];
    Textra:=eval(Textra-temp[2][i]*G[i]);
  od;
  Extra:=eval(Textra);
od;
RETURN(convert(Ans,list));
end:
```

The subroutine FGR above uses the `remainder` function from Albert Lin's Gröbner basis package (See Appendix C). With `h,F,G,v,order` as in the routine `asmodule`, it gives `[r,a`$_1$`,...,a`$_n$`]` as output, where $h = r + \sum a_i(f_i + g_i)$ as in (3.6) above.

3.4 Projective geometry of finite maps

We may consider the components of a map as defining hypersurfaces in projective space. By studying these, especially their intersections, we may determine whether a map is finite, in a similar way to that of Proposition 3.2.1.

Proposition 3.4.1 *Let $f = (f_1, \dots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map. Let $f_1', \dots, f_n'$ be the leading homogeneous parts of the components of f. Given $y = (y_1, \dots, y_n) \in \mathbf{C}^n$, let $f_1^y, \dots, f_n^y$ be the homogenizations with respect to a variable z of $f_1 - y_1, \dots, f_n - y_n$. Then*

1. *If $f_1', \dots, f_n'$ have no common root in $\mathbf{P}^{n-1}$, then f is finite.*

2. *If for some y, the curves given by $f_1^y = 0, \dots, f_n^y = 0$ do not meet in $\mathbf{P}^n$ at the hyperplane given by $z = 0$, then f is finite.*

Proof Part 1: By the Projective Weak Nullstellensatz ([CLO, p370]), for each i there exists m_i such that $x_i^{m_i} \in <f_1', \dots, f_n'>$. Hence the quotient

$$\frac{\mathbf{C}[x_1, \dots, x_n]}{<f_1', \dots, f_n'>}$$

is a finite dimensional $\mathbf{C}$-vector space. Thus (see Proposition 3.2.1) $(f_1', \dots, f_n') : \mathbf{C}^n \to \mathbf{C}^n$ is finite. This in turn means that f itself is finite (see Proposition 3.3.1).

Part 2: Now $f_1^y = 0, \dots, f_n^y = 0$ meet in the hyperplane $z = 0$ if and only if $f_1', \dots, f_n'$ have a common root, since substituting $z = 0$ into f_i^y gives f_i'. □

Remark In the second part of the above, the points of intersections on $z = 0$ do not depend on y. If f is finite, but has intersections on $z = 0$, then the intersection multiplicity of $f_1^y = 0, \ldots, f_n^y = 0$ is also fixed. Suppose that there is a point on $z = 0$ which is also a point of intersection of $f_1^y = 0, \ldots, f_n^y = 0$. The only way the intersection multiplicity here can increase is if an intersection away from $z = 0$ moves out and joins it. Suppose we take $y = y_0$ and at this point a certain intersection lies away from $z = 0$, but as we move to $y = y_1$ this root moves to the hyperplane $z = 0$. Returning to the affine viewpoint, if we take a compact set containing a path between y_0 and y_1, its preimage will be unbounded, since it will contain all the positions of the root in question as it moves out to the "hyperplane at ∞". So the preimage will not be compact and this contradicts the properness of the finite map.

The description of finiteness in terms of projective varieties yields the following result.

Proposition 3.4.2 *In the space of all polynomial maps $f = (f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ such that $\deg f_i = d_i$ for fixed $d_1, \ldots, d_n$, the non-finite maps lie inside a proper (Zariski) closed subset.*

Proof From Proposition 3.4.1, we know that if the leading homogeneous parts $f_1', \ldots, f_n'$ have no common root in $\mathbf{P}^{n-1}$, then f itself is finite. We will show that the set where they do have a common root is a closed subset of the given space of mappings.

For a given degree d_i, we may represent the space of homogeneous polynomials of this degree in n variables by a projective space $\mathbf{P}^{N_i}$, where each homogeneous coordinate $y_{i,j}$ represents a monomial $\boldsymbol{x}^{i,j}$ in $x_1, \ldots, x_n$. So we have an isomorphism

$$\Psi : \sum_{j=0}^{N_i} \kappa_j \boldsymbol{x}^{i,j} \mapsto (\kappa_0 : \kappa_1 : \cdots : \kappa_{N_i}) \in \mathbf{P}^{N_i}$$

where $\kappa_j \in \mathbf{C}$. (NB All non-zero scalar multiples of a polynomial are represented by the same point of the space). Consider the following space:

$$\mathbf{P}^{n-1} \times \underbrace{\mathbf{P}^{N_1} \times \cdots \times \mathbf{P}^{N_n}}_{\substack{\parallel \\ M}}$$

and let $\Delta \subset \mathbf{P}^{n-1} \times M$ be the set

$$\Delta = \{(x, h_1, \ldots, h_n) : \Psi^{-1}(h_1)(x) = \cdots = \Psi^{-1}(h_n)(x) = 0\}.$$

Now a closed subset of $\mathbf{P}^{n-1} \times M$ is one defined by equations which are homogeneous in each set of variables (one set per projective space in the product). Can Δ be defined in this way? For each i, consider variables $x_1, \ldots, x_n$ on $\mathbf{P}^{n-1}$ and variables $y_{i,0}, \ldots, y_{i,N_i}$ on $\mathbf{P}^{N_i}$. Then $\Psi^{-1}(h_i)(x) = 0$ is equivalent to

$$\sum_{j=0}^{N_i} y_{i,j} \boldsymbol{x}^{i,j} = 0$$

which is a polynomial in the given variables, homogeneous of degree 1 in the y variables and degree d_i in the x variables. Thus the common locus of all such polynomials (i.e. Δ) is closed in $\mathbf{P}^{n-1} \times M$.

Now a product of projective spaces may be considered as a variety in a larger projective space via the Segre mapping (see [H, p25],[Sh, p41]). So we may consider $\mathbf{P}^{n-1} \times M$ as a product of varieties. Thus by ([Sh, p45 Thm 3]) the projection

$$\pi : \mathbf{P}^{n-1} \times M \to M$$

carries closed sets to closed sets. This means that $\pi(\Delta)$, (i.e. those n-tuples which do not have a common root) form a closed subset. Now $(x_1^{d_1}, \ldots, x_n^{d_n})$ lies outside $\pi(\Delta)$, so it is a proper closed subset. □

We now give some examples.

Example 3.4.3 The map $f = (x^3 + xy, y^4 - y) : \mathbf{C}^2 \to \mathbf{C}^2$ is finite, as we will show using the above proposition. We will choose the point in the target (called

y in the above) to be (0,0). Homogenizing f with respect to the variable z, we obtain $(x^3 + xyz, y^4 - yz^3)$. Now the curves $x^3 + xyz = 0$ and $y^4 - yz^3 = 0$ will meet at the hyperplane $z = 0$ when $x^3 = y^4 = 0$, which since we are working in projective space, will not occur. Thus f is finite by part 2 above.

Example 3.4.4 The map $f = (x^2y^2 + y^2, x^2 - xy) : \mathbf{C}^2 \to \mathbf{C}^2$ is in fact finite, although it fails the criterion given above. Let the chosen point in the target be (a_1, a_2). Now, homogenizing we obtain $(x^2y^2 + y^2z^2 - a_1z^4, x^2 - xy - a_2z^2)$. The curves $x^2y^2 + y^2z^2 - a_1z^4 = 0$ and $x^2 - xy - a_2z^2 = 0$ meet on the hyperplane $z = 0$ at the point $(0 : 1 : 0)$, so f does indeed fail the criterion. As noted above, the curves have intersection multiplicity 2 at this point, regardless of the values of a_1 and a_2.

3.5 Equality of leading term ideals

Since the ideal of leading terms can be used to determine if a map is finite or not, we now examine them a little a more closely.

Proposition 3.5.1 *Let* $(f_1, \dots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ *be a finite polynomial map with quasihomogeneous components with respect to a set of weights* $w = (w_1, \dots, w_n)$. *Let* $(g_1, \dots, g_n) \in \mathbf{C}[x_1, \dots, x_n]^n$ *be such that for each* i, $\deg_w g_i < \deg_w f_i$. *Let* $\mathcal{I}_f = <f_1, \dots, f_n>$ *and* $\mathcal{I}_{f+g} = <f_1 + g_1, \dots, f_n + g_n>$. *Then in an order graded with respect to the set of weights* w, $<LT(\mathcal{I}_f)> \subseteq <LT(\mathcal{I}_{f+g})>$.

Proof Suppose $\phi \in \mathcal{I}_f$. Then we may write it as

$$\phi = \sum_{i=1}^{n} \alpha_i f_i$$

where $\alpha_i \in \mathbf{C}[x_1, \dots, x_n]$ and $\deg_w \alpha_i f_i \le \deg_w \phi$, by Lemma 3.2.2. Now consider the polynomial

$$\tilde{\phi} = \sum_{i=1}^{n} \alpha_i (f_i + g_i).$$

This is in all respects the same as ϕ, except for the addition of some terms $\alpha_i g_i$ where $\deg_w \alpha_i g_i < \deg_w \phi$. Thus, $LT(\phi) = LT(\tilde{\phi})$. So $LT(\mathcal{I}_f) \subseteq LT(\mathcal{I}_{f+g})$ and hence the result. □

Now this result can be strengthened to equality, but we need the following proposition. It gives a strong condition on the syzygies among the f_i, relying on the property that $\mathbf{C}[x_1, \ldots, x_n]$ is free over $f^*\mathbf{C}[y_1, \ldots, y_n]$

Proposition 3.5.2 (The Syzygy Result) *For a finite map $f = (f_1, \ldots, f_n)$: $\mathbf{C}^n \to \mathbf{C}^n$, the syzygies on the components are generated by the trivial syzygies of the form $f_i e_j - f_j e_i$ where $\{e_1, \ldots, e_n\}$ is the basis of the module $\mathbf{C}[x_1, \ldots, x_n]^n$.*

Proof Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbf{C}[x_1, \ldots, x_n]^n$ be a syzygy on the f_i, so

$$\boldsymbol{\sigma} \cdot \mathbf{f} = \sum_{i=1}^{n} \sigma_i f_i = 0. \tag{3.7}$$

Let $b_1, \ldots, b_s$ be a free basis for $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module. We may write

$$\sigma_i = \sum_{j=1}^{s} \tau_{ij}(f) b_j$$

and so writing $\boldsymbol{\tau}_j = (\tau_{1j}, \ldots, \tau_{nj})$, we get

$$\boldsymbol{\sigma} = \sum_{j=1}^{s} \boldsymbol{\tau}_j(f) b_j. \tag{3.8}$$

Substituting this definition of $\boldsymbol{\sigma}$ into (3.7), we obtain

$$\sum_{j=1}^{s} (\boldsymbol{\tau}_j(f) \cdot \mathbf{f}) b_j = 0$$

and hence since the module is free, $(\boldsymbol{\tau}_j(f) \cdot \mathbf{f}) = 0$ for all j. Now this is another syzygy on the f_i, but observe that $\boldsymbol{\tau}_j(f) \in f^*\mathbf{C}[y_1, \ldots, y_n]^n$. Since $f^*\mathbf{C}[y_1, \ldots, y_n]$ is isomorphic to $\mathbf{C}[y_1, \ldots, y_n]$ via f^* (see Proposition 3.1.7), this is equivalent to the syzygy

$$(\boldsymbol{\tau}_j(y) \cdot \mathbf{y}) = 0$$

where $\mathbf{y} = (y_1, \ldots, y_n)$. Now the syzygies on the y_i are generated by the trivial ones $y_i e_j - y_j e_i$, so $\boldsymbol{\tau}_j(y)$ must also lie in this syzygy module. Hence via the map f^*, $\boldsymbol{\tau}_j(f)$ lies in the module generated by the trivial syzygies $f_i e_j - f_j e_i$. Now $\boldsymbol{\sigma}$ is just a linear combination of the $\boldsymbol{\tau}_j(f)$ (see (3.8)), so it also lies in this module.□

Using the syzygy result we obtain the following result, which shows why the finiteness of a quasihomogeneous map is unaffected if we add extra lower order terms.

Proposition 3.5.3 *With the same conditions as Proposition 3.5.1,* $<LT(\mathcal{I}_f)> = <LT(\mathcal{I}_{f+g})>$.

Proof Now Proposition 3.5.1 showed $<LT(\mathcal{I}_f)> \subseteq <LT(\mathcal{I}_{f+g})>$, so only the reverse inclusion needs proving.

Suppose a polynomial $\phi \in \mathbf{C}[x_1, \ldots, x_n]$ is written

$$\phi = \sum_{i=1}^{n} \alpha_i (f_i + g_i) + h$$

where $\alpha_i \in \mathbf{C}[x_1, \ldots, x_n]$ and $h \in \mathcal{I}_f \cap \mathcal{I}_{f+g}$.

Define $d(\phi) = \max\{\deg_w(\alpha_i(f_i + g_i)) : i = 1, \ldots, n\}$ if some $\alpha_i \neq 0$ and 0 otherwise. Thus $d(\phi)$ depends not only on ϕ, but also on how it is written.

We wish to show that $LT(\phi) \in <LT(\mathcal{I}_f)>$ or f may be rewritten such that $d(\phi)$ decreases.

There are 3 cases to consider:

- **Case 1:** $LM(h) > LM(\sum \alpha_i (f_i + g_i))$.
- **Case 2:** $LM(h) = LM(\sum \alpha_i (f_i + g_i))$.
- **Case 3:** $LM(h) < LM(\sum \alpha_i (f_i + g_i))$.

Case 1: Here, $LT(\phi) = LT(h) \in <LT(\mathcal{I}_f)>$ and so this is a trivial case.

Case 2: Now since $h \in \mathcal{I}_{f+g}$, write

$$h = \sum_{i=1}^{n} \zeta_i (f_i + g_i).$$

This can be split up, according to the w-degree of $\zeta_i f_i$ as follows:

$$h = \sum_{i=1}^{n} \eta_i (f_i + g_i) + \sum_{i=1}^{n} \theta_i (f_i + g_i) + \sum_j \sum_{i=1}^{n} \epsilon_{ji} (f_i + g_i).$$

The splitting is defined as follows: for each j, $\sum \epsilon_{ji}(f_i)$ is a quasihomogeneous part of $\sum \zeta_i(f_i)$ of a w-degree j strictly greater than $d(\phi)$. The element $\sum \eta_i(f_i)$ is the part of $\sum \zeta_i(f_i)$ of w-degree equal to $d(\phi)$. Now $\sum \theta_i(f_i)$ is the remainder, i.e. the part of $\sum \zeta_i(f_i)$ of w-degree strictly less than $d(\phi)$. We can split up $\sum \alpha_i(f_i)$ itself similarly:

$$\sum_{i=1}^{n} \alpha_i(f_i) = \sum_{i=1}^{n} \beta_i(f_i) + \sum_{i=1}^{n} \delta_i(f_i), \tag{3.9}$$

where $\sum \beta_i(f_i)$ is the part of w-degree $d(\phi)$ and $\sum \delta_i(f_i)$ is the remainder. Consider $\sum(\beta_i + \eta_i)(f_i)$. If this is non-zero, then $LT(\phi) = LT(\sum(\beta_i + \eta_i)(f_i))$ and hence $LT(\phi) \in <LT(\mathcal{I}_f)>$. However, if this sum is zero, then the Syzygy Result gives $\beta_i + \eta_i \in \mathcal{I}_f$ and so

$$\phi = \sum_{i=1}^{n} (\delta_i + \theta_i)(f_i + g_i) + \underbrace{\sum_{i=1}^{n} (\beta_i + \eta_i)(f_i + g_i) + \sum_j \sum_{i=1}^{n} \epsilon_{ji}(f_i + g_i)}_{\substack{\parallel \\ \tilde{h}}} \tag{3.10}$$

Now $\tilde{h} \in \mathcal{I}_f \cap \mathcal{I}_{fg}$ since the first sum certainly is, from above and $\epsilon_{ji} \in \mathcal{I}_f$ by the following argument. We know that for each j, $\deg_w LM(h) = \deg_w LM(\sum \alpha_i(f_i + g_i)) < d(\phi) < j$ and so $\sum \epsilon_{ji}(f_i) = 0$ otherwise there would be a monomial in h greater than $LM(h)$. Thus by the Syzygy Result, $\epsilon_{ji} \in \mathcal{I}_f$.

Thus, by consideration of (3.10), we see that ϕ has been rewritten in such a way that $d(\phi)$ has decreased.

Case 3: Splitting $\sum \alpha_i(f_i)$ into two sums as in (3.9), suppose $\sum \beta_i f_i$ is non-zero, then $LT(\phi) = LT(\sum \beta_i f_i) \in <LT(\mathcal{I}_f)>$ as required. If the sum is zero, then by the Syzygy Result, $\beta_i \in \mathcal{I}_f$ for all i. This gives

$$\phi = \sum_{i=1}^{n} \delta_i(f_i + g_i) + \underbrace{\sum_{i=1}^{n} \beta_i(f_i + g_i) + h}_{\substack{\| \\ \tilde{h}}} .$$

Now $\tilde{h} \in \mathcal{I}_f \cap \mathcal{I}_{fg}$, so again ϕ has been rewritten so that $d(\phi)$ has decreased.

Now given some ϕ with $h = 0$, i.e. an arbitrary element of $\mathcal{I}_{f+g}$, we have shown that either $LT(\phi) \in <LT(\mathcal{I}_f)>$ or $d(\phi)$ can be repeatedly reduced. Suppose the latter happens. Now $d(f) = 0$ can only occur if

$$\phi = \sum_{i=1}^{n} 0 \cdot (f_i + g_i) + h$$

in which case, $LT(\phi) = LT(h) \in <LT(\mathcal{I}_f)>$. Thus $<LT(\mathcal{I}_{f+g})> \subseteq <LT(\mathcal{I}_f)>$ and hence $<LT(\mathcal{I}_{f+g})>=<LT(\mathcal{I}_f)>$. □

3.6 Finiteness criterion

We can now give a criterion for determining if a general polynomial map $f : \mathbf{C}^n \to \mathbf{C}^n$ is finite, but first we have a preliminary lemma:

Lemma 3.6.1 *Suppose that $>_1$ and $>_2$ are product monomial orders on $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$ such that $x_i >_k z_j$ for $i, j = 1, \ldots, n$ and $k = 1, 2$. Let $f_1, \ldots, f_n$ be polynomials in $\mathbf{C}[x_1, \ldots, x_n]$ and let LT_k denote the leading term with respect to $>_k$. Then there exist $m_j \in \mathbf{N}$ for $j = 1, \ldots, n$ such that*

$$x_j^{m_j} \in LT_1(<f_1 - z_1, \ldots, f_n - z_n>)$$

if and only if there exist $r_j \in \mathbf{N}$ for $j = 1, \ldots, n$ such that

$$x_j^{r_j} \in LT_2(<f_1 - z_1, \ldots, f_n - z_n>)$$

Proof For the definition of product orders, see §1.1. Now the sets $\mathcal{B}_1$ and $\mathcal{B}_2$, where $\mathcal{B}_k = \{\mathbf{x}^\alpha : \mathbf{x}^\alpha \notin LT_k(<f_1 - z_1, \ldots, f_n - z_n>)\}$ form a basis for the $\mathbf{C}$ vector space

$$\frac{\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]}{<f_1 - z_1, \ldots, f_n - z_n, z_1, \ldots, z_n>}$$

Thus $|\mathcal{B}_1| = |\mathcal{B}_2|$ and so one is finite if and only if the other is. Hence the result follows. (Compare Proposition 1.2.5.) □

Proposition 3.6.2 (Finiteness Criterion) *Let* $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$ *have, as in Lemma 3.6.1, a product monomial order with* $x_i > z_j$ *for* $i, j = 1, \ldots, n$. *Then a polynomial map* $f = (f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ *is finite if and only if there exist* $m_j \in \mathbf{N}$ *for* $j = 1, \ldots, n$ *such that*

$$x_j^{m_j} \in LT(<f_1 - z_1, \ldots, f_n - z_n>) \tag{3.11}$$

Proof Suppose the condition (3.11) holds for the order $>$ and some polynomial map $f : \mathbf{C}^n \to \mathbf{C}^n$. Now by Lemma 3.6.1, it will also hold for the order $>_{lex(j)}$ for any given $j \in \{1, \ldots, n\}$, where $>_{lex(j)}$ is defined as lexicographic order with $x_1 > x_2 > \cdots > x_{j-1} > x_{j+1} > \cdots > x_n > x_j > z_1 > \cdots > z_n$. This is a product order since we may consider it as a product of lex on the x variables and lex on the z variables. Hence we have

$$x_j^{m_j} = LT_{lex(j)}\left(\sum_{i=1}^{n} \alpha_{i,j}(f_i - z_i)\right)$$

for some $\alpha_{i,j} \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$. By consideration of the monomial order, we may write

$$\sum_{i=1}^{n} \alpha_{i,j}(f_i - z_i) = \sum_{k=1}^{m_j} \beta_k(z) x_j^k$$

where β_k is a polynomial in $\mathbf{C}[z_1, \ldots, z_n]$, with $\beta_{m_j} = 1$. Now setting $z_i = f_i$ for all i we obtain

$$0 = \sum_{k=1}^{m_j} \beta_k(f) x_j^k$$

which is a monic polynomial in x_j with coefficients in $f^*\mathbf{C}[y_1, \ldots, y_n]$. Thus x_j is integral over $f^*\mathbf{C}[y_1, \ldots, y_n]$. This may be repeated for each j and hence $\mathbf{C}[x_1, \ldots, x_n]$ is integral over $f^*\mathbf{C}[y_1, \ldots, y_n]$, i.e. the map f is finite.

For the converse, suppose that f is a finite map. Then for each $j \in \{1, \ldots, n\}$, there exist polynomials $\delta_i \in f^*\mathbf{C}[y_1, \ldots, y_n]$ such that

$$x_j^m + \delta_{m-1}(f)x_j^{m-1} + \cdots + \delta_0(f) = 0$$

since $\mathbf{C}[x_1, \ldots, x_n]$ is integral over $f^*\mathbf{C}[y_1, \ldots, y_n]$. Denote by $\mathcal{I}_{fz}$ the ideal $<f_1 - z_1, \ldots, f_n - z_n>$. Now $f_i = z_i$ modulo $\mathcal{I}_{fz}$ for $i = 1, \ldots, n$ and hence if ϕ is any polynomial, then $\phi(f_1, \ldots, f_n) = \phi(z_1, \ldots, z_n)$ modulo $\mathcal{I}_{fz}$. Thus we have

$$\sum_{k=1}^{m} \delta_k(z)x_j^k = \sum_{k=1}^{m} \delta_k(f)x_j^k = 0 \text{ modulo } \mathcal{I}_{fz}$$

where $\delta_k(z)$ is simply $\delta_k(f)$ with each f_i replaced by z_i. Hence

$$\sum_{k=1}^{m} \delta_k(z)x_j^k = \sum_{i=1}^{n} \epsilon_i(f_i - z_i)$$

with $\epsilon_i \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$. So $x_j^m = LT\left(\sum \epsilon_i(f_i - z_i)\right)$, in other words, $x_j^m \in LT(\mathcal{I}_{fz})$. This may be repeated for each j and hence the result follows. □

We can use Proposition 3.6.2 to determine a generating set for $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module as follows.

Proposition 3.6.3 *Let $(f_1, \ldots, f_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map. Suppose $\mathcal{B} = \{b_\lambda(\mathbf{x}, \mathbf{z}) \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]\}_{\lambda \in \Lambda}$ is a basis for the $\mathbf{C}$ vector space*

$$\frac{\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]}{<f_1 - z_1, \ldots, f_n - z_n>} \tag{3.12}$$

If $\mathcal{B}_m$ is the set of monomials of elements of $\mathcal{B}$ and Ψ is the map obtained by substituting $z_i = 1$ for all i, then $\Psi(\mathcal{B}_m)$ generates $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module.

Proof Let $\phi(\mathbf{x})$ be an arbitrary element of $\mathbf{C}[x_1, \ldots, x_n]$. We may write

$$\phi(\mathbf{x}) = \sum_{\lambda \in \Lambda} \mu_\lambda \cdot b_\lambda(\mathbf{x}, \mathbf{z}) + \sum_{i=1}^{n} \beta_i(\mathbf{x}, \mathbf{z})(f_i - z_i)$$

where $\mu_\lambda \in \mathbf{C}$ and $\beta_i \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$. Substitution of $z_i = f_i$ gives

$$\phi(\mathbf{x}) = \sum_{\lambda \in \Lambda} \mu_\lambda \cdot b_\lambda(\mathbf{x}, \mathbf{f}) = \sum_{\mathbf{x}^\alpha \in \Psi(\mathcal{B}_m)} \mathbf{x}^\alpha \cdot \delta_\alpha(\mathbf{f})$$

where $\delta_\alpha(\mathbf{f})$ is some element of $f^*\mathbf{C}[y_1, \ldots, y_n]$. Thus $\Psi(\mathcal{B}_m)$ is a generating set for $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module as required. □

Note that if we take f to be a finite map in the above, then the set $\mathcal{B}$ will of course be finite. The following corollaries give a particularly simple set of module generators and condition for this to be a free basis.

Corollary 3.6.4 *Let* $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$ *have a product monomial order with* $x_i > z_j$ *for* $i, j = 1, \ldots, n$ *and let* $\mathcal{L} = \{\mathbf{x}^\alpha : \mathbf{x}^\alpha \notin LT(\mathcal{I}_{fz})\}$. *Then* $\mathcal{L}$ *generates* $\mathbf{C}[x_1, \ldots, x_n]$ *as an* $f^*\mathbf{C}[y_1, \ldots, y_n]$ *module.*

Proof Now any element of the LT-monomial basis of $\mathcal{I}_{fz}$ is expressible as $\mathbf{x}^\alpha \mathbf{z}^{\alpha'}$ where $\mathbf{x}^\alpha \in \mathcal{L}$, so will lie in $\mathcal{L}$ under Ψ.□

Corollary 3.6.5 $\mathcal{L}$ *(defined as above) is a free basis for* $\mathbf{C}[x_1, \ldots, x_n]$ *as an* $f^*\mathbf{C}[y_1, \ldots, y_n]$ *module if and only if the LT-monomial basis of* $\mathcal{I}_{fz}$ *(using the same monomial order as above) is* $\{\mathbf{x}^\alpha \mathbf{z}^{\alpha'} : \mathbf{x}^\alpha \in \mathcal{L}, \text{ any } \mathbf{z}^{\alpha'}\}$ *(i.e. the ideal* $<LT(\mathcal{I}_{fz})>$ *is generated by monomials in* $\mathbf{C}[x_1, \ldots, x_n]$*).*

Proof Suppose we have

$$\sum_{l \in \mathcal{L}} l \cdot \epsilon_l(\mathbf{f}) = 0$$

for some $\epsilon_l(\mathbf{f}) \in f^*\mathbf{C}[y_1, \ldots, y_n]$. Let $\{\mathbf{f}^k\}_{k \in K}$ be the set of all monomials in $f_1, \ldots, f_n$. Then for each l we may write

$$\epsilon_l(\mathbf{f}) = \sum_{k \in K} \lambda_{l,k} \cdot \mathbf{f}^k$$

where $\lambda_{l,k} \in \mathbf{C}$. So

$$\sum_{l \in \mathcal{L}, k \in K} \lambda_{l,k}(l \cdot \mathbf{f}^k) = 0$$

Now modulo $\mathcal{I}_{fz}$, this is

$$\sum_{l \in \mathcal{L}, k \in K} \lambda_{l,k}(l \cdot \mathbf{z}^k) = 0$$

But the monomials $l \cdot \mathbf{z}^k$ form a basis of the vector space (3.12), hence $\lambda_{l,k} = 0$ for all l, k. Thus $\epsilon_l(\mathbf{f})$ is identically zero for all l, so $\mathcal{L}$ is a free basis as required.

Now suppose that $\mathcal{L}$ is a free basis, but there is a monomial $\mathbf{x}^\alpha \mathbf{z}^{\alpha'}$ with $\mathbf{x}^\alpha \in \mathcal{L}$ but which lies outside the LT monomial basis. We may suppose without loss of generality that every monomial which divides it lies inside the monomial basis. If we write G for a Gröbner basis of $\mathcal{I}_{fz}$ (as defined above) then there is some $\gamma \in G$ such that $\mathrm{LT}(\gamma) = \mathbf{x}^\alpha \mathbf{z}^{\alpha'}$. By consideration of the monomial ordering, we see that the x part of any monomial of γ must be less than $\mathbf{x}^\alpha$ and hence in $\mathcal{L}$. Using this, we may write

$$\gamma = \sum_{l \in \mathcal{L}} l \cdot \zeta_l(\mathbf{z}) \in \mathcal{I}_{fz}$$

where $\zeta_l \in \mathbf{C}[z_1, \ldots, z_n]$ and with zero constant term. Now replacing each z_i by f_i gives

$$\sum_{l \in \mathcal{L}} l \cdot \zeta_l(\mathbf{f}) = 0$$

which is an $f^*\mathbf{C}[y_1, \ldots, y_n]$ syzygy on $\mathcal{L}$, contradicting the fact that it is a free module basis. □

Suppose for some multi-index β, $\gamma \cdot \boldsymbol{x}^\beta$ still consisted of monomials whose x components lay in $\mathcal{L}$. This product would also form an $f^*\mathbf{C}[y_1, \ldots, y_n]$ syzygy on $\mathcal{L}$. We will call such a syzygy a *parallel syzygy*. The method given above yields the following:

Corollary 3.6.6 *Let G be a Gröbner basis for $\mathcal{I}_{fz}$ (as defined above) under the given product monomial order. Let G' be the set of elements of G whose leading monomials are such that the x component lies in $\mathcal{L}$. Now to each $\gamma \in G'$ we* *can associate (as shown in the proof of the previous corollary) an* $f^*\mathbf{C}[y_1, \ldots, y_n]$ *syzygy* $\sigma(\gamma)$ *on* $\mathcal{L}$. *Then* $\sigma(G')$ *and parallel syzygies generate the* $f^*\mathbf{C}[y_1, \ldots, y_n]$*-syzygy module on* $\mathcal{L}$.

Proof Let $\boldsymbol{\tau}(f)$ be an $f^*\mathbf{C}[y_1, \ldots, y_n]$ syzygy on $\mathcal{L}$. Writing $\boldsymbol{l}$ for the elements of $\mathcal{L}$ as a vector, this means that

$$\boldsymbol{\tau}(f) \cdot \boldsymbol{l} = 0 \tag{3.13}$$

thus if we replace each f_i by z_i, we obtain

$$\boldsymbol{\tau}(z) \cdot \boldsymbol{l} \in \mathcal{I}_{fz}.$$

So $\boldsymbol{\tau}(z) \cdot \boldsymbol{l}$ is a $\mathbf{C}[x_1, \ldots, x_n,\ z_1, \ldots, z_n]$-linear combination of elements of G. Note that each of the components of $\boldsymbol{\tau}$ has zero constant term. Otherwise, setting $f_i = 0$ for all i in (3.13) above, we find that the elements of $\mathcal{L}$ are linearly dependent. If we divide by the Gröbner basis G, we obtain quotients α_γ and $\alpha'_{\gamma'}$ such that

$$\boldsymbol{\tau}(z) \cdot \boldsymbol{l} = \sum_{\gamma \in G - G'} \alpha_\gamma(x, z)\gamma + \sum_{\gamma' \in G'} \alpha'_{\gamma'}(x, z)\gamma'$$

where $\alpha_\gamma, \alpha'_{\gamma'} \in \mathbf{C}[x_1, \ldots, x_n,\ z_1, \ldots, z_n]$. Now the left hand side of the above consists of monomials whose x component lies in $\mathcal{L}$. But every $\gamma \in G - G'$ has a leading monomial with x component outside $\mathcal{L}$ and so would not divide $\boldsymbol{\tau}(z) \cdot \boldsymbol{l}$. Thus α_γ must be identically zero for all $\gamma \in G - G'$. So we have

$$\boldsymbol{\tau}(z) \cdot \boldsymbol{l} = \sum_{\gamma' \in G'} \alpha'_{\gamma'}(x, z)\gamma'.$$

We may write

$$\gamma' = \sum_{l \in \mathcal{L}} l \cdot \zeta_l(z)$$

which becomes $\sigma(\gamma') \cdot \boldsymbol{l}$ on replacing z_i by f_i. So we obtain

$$\boldsymbol{\tau}(f) = \sum_{\gamma' \in G'} \alpha'_{\gamma'}(x, f)\sigma(\gamma')$$

Thus τ is a $f^*\mathbf{C}[y_1, \ldots, y_n]$-linear combination of the syzygies $\sigma(\gamma')$ and parallel syzygies as required. □

Note that it is possible to determine a free basis for $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$-module using an algorithm. See [LS] for an explanation and a description of a (rather impractical) method.

We now give some examples of the use of the criterion.

Example 3.6.7 We return to the example of the map $f = (x, xy^2 + y)$ which we claimed was quasifinite but not finite. Consider the points (x, y) in the set $f^{-1}(a_1, a_2)$. This implies that $x = a_1$ and y is a root of $a_1y^2 + y - a_2$. Thus $f^{-1}(a_1, a_2)$ is a finite set for all $(a_1, a_2) \in \mathbf{C}^2$, i.e. f is quasifinite. But is f also finite? Let us apply the criterion of Proposition 3.6.2. Calculating a Gröbner basis G for $<f_1 - z_1, \ldots, f_n - z_n>$ using lex order with $x > y > z_1 > z_2$ gives

$$G = \{x - z_1, z_1y^2 + y - z_2\}$$

and so $\mathrm{LT}(G) = \{x, z_1y^2\}$. Hence f fails the criterion since there is no power of y in this set. So f is, as claimed, not finite.

Example 3.6.8 Now consider the map $f = (x^2y^2 + y^2, x^2 - xy)$. This time, if we calculate a Gröbner basis G for $<f_1 - z_1, \ldots, f_n - z_n>$ using lex order with $x > y > z_1 > z_2$, we obtain

$$\begin{aligned}
G = \{ \ & x^2 - xy - z_2, \\
& xy^3 + y^2 - z_1 + y^2z_2, \\
& xy^2z_2 - y^3 + yz_1 + xy^2 - xz_1, \\
& xyz_1 + y^2 - z_1 + 2y^2z_2 - z_2z_1 + y^2z_2^2 + y^4 - y^2z_1, \\
& xz_1^2 + y^3 - yz_1 + 3y^3z_2 - 2yz_2z_1 + 3y^3z_2^2 + y^5 - yz_1^2 - yz_2^2z_1 + \\
& \quad + y^3z_2^3 + y^5z_2 - y^3z_2z_1, \\
& y^6 - 2y^2z_1 + z_1^2 - 2z_1y^2z_2 + y^4 + 2y^4z_2 + y^4z_2^2 - z_1y^4 \ \}
\end{aligned}$$

and so f is finite, since there is a power of each of x and y in the leading terms. We can also see that the monomials $\{1, x, y, xy, y^2, xy^2, y^3, y^4, y^5\}$ generate $\mathbf{C}[x_1, \ldots, x_n]$ as an $f^*\mathbf{C}[y_1, \ldots, y_n]$ module. This is not a free basis however, since there are leading terms of elements of G which contain z variables.

The following Maple routine implements the criterion, allowing for the simple checking of finite polynomial maps. Given a map F as a list of components and variables V, it calculates a Gröbner basis using lex order and extra variables _Z to check finiteness. If a third argument is given, this is assigned the list of leading terms of the basis calculated.

```
checkfinite:=proc(F,V)
  local i,j,VV,FF,G,LT,marker;
  if nops(V)<>nops(F) then ERROR(`incorrect No of components`);
  fi;
  n:=nops(V);
  _Z:=array(1..n);
  VV:=[op(V),_Z[i] $ i=1..n];
  FF:=[F[i]-_Z[i] $ i=1..n];
  G:=grobner[gbasis](FF,VV,plex);
  LT:=map(x->x[2],map(grobner[leadmon],G,VV,plex));
  if nargs=3 then assign(args[3],[LT]);
  fi;
  for i from 1 to n do
    marker:=0;
    for j from 1 to nops(LT) do
      pow:=coeffs(LT[j],V[i]);
      if type(pow,numeric) then
        marker:=1;
        break;
```

```
      fi;
    od;
    if marker=0 then RETURN(false);
    fi;
  od;
  RETURN(true);
end:
```

3.7 The degree of a finite map

Now that we can determine whether a polynomial map f is finite, we may consider the various degrees associated with it:

- $\dim_{\mathbf{C}}(\frac{\mathbf{C}[x_1,\ldots,x_n]}{<f_1-z_1,\ldots,f_n-z_n>})$ for any value of $(z_1,\ldots,z_n) \in \mathbf{C}^n$
- The dimension of $\mathbf{C}[x_1,\ldots,x_n]$ as a free module over $f^*\mathbf{C}[y_1,\ldots,y_n]$
- $\deg f = [\mathbf{C}(x_1,\ldots,x_n) : f^*\mathbf{C}(y_1,\ldots,y_n)]$

In fact, we will show that all of these numbers are equal:

Proposition 3.7.1 *If $\{b_1,\ldots,b_s\}$ is a (free module) basis for $\mathbf{C}[x_1,\ldots,x_n]$ over $f^*\mathbf{C}[y_1,\ldots,y_n]$, then it also forms a (vector space) basis for $\mathbf{C}(x_1,\ldots,x_n)$ over $f^*\mathbf{C}(y_1,\ldots,y_n)$.*

Proof Suppose there is linear relation among the elements $\{b_1,\ldots,b_s\}$ over the field $f^*\mathbf{C}(y_1,\ldots,y_n)$. So

$$\sum_{i=1}^{s} \beta_i(f)b_i = 0$$

where $\beta_i \in f^*\mathbf{C}(y_1,\ldots,y_n)$. Then multiplying up by denominators, we obtain

$$\sum_{i=1}^{s} \overline{\beta_i}(f)b_i = 0$$

where $\overline{\beta_i} \in f^*\mathbf{C}[y_1, \ldots, y_n]$. Thus, since $\{b_1, \ldots, b_s\}$ is a basis, we obtain $\overline{\beta_i}(f) = 0$ for all i. Hence $\beta_i(f) = 0$ for all i and so the b_i are linearly independent over $f^*\mathbf{C}(y_1, \ldots, y_n)$.

But are they a spanning set? Let $\phi \in \mathbf{C}(x_1, \ldots, x_n)$ be an arbitrary element. Now the extension of $\mathbf{C}(x_1, \ldots, x_n)$ over $f^*\mathbf{C}(y_1, \ldots, y_n)$ is finite and hence algebraic. Thus $\mathbf{C}(x_1, \ldots, x_n) = f^*\mathbf{C}(y_1, \ldots, y_n)[x_1, \ldots, x_n]$ (see [ZS, ChII §2 Thm2]). So we may write ϕ as a finite sum

$$\phi = \sum_\alpha \beta_\alpha(f)\mathbf{x}^\alpha$$

where $\beta_\alpha(f) \in f^*\mathbf{C}(y_1, \ldots, y_n)$ and α ranges through all the possible multi-indices. Now finding a common denominator $\delta(f)$, we obtain

$$\phi = \frac{1}{\delta(f)} \sum_\alpha \overline{\beta_\alpha}(f)\mathbf{x}^\alpha$$

where again $\overline{\beta_\alpha} \in f^*\mathbf{C}[y_1, \ldots, y_n]$. Since the sum in the above expression lies in $\mathbf{C}[x_1, \ldots, x_n]$, we can rewrite it as $\sum \epsilon_i(f)b_i$ where $\epsilon_i \in f^*\mathbf{C}[y_1, \ldots, y_n]$ and thus

$$\phi = \frac{1}{\delta(f)} \sum_{i=1}^{s} \epsilon_i(f)b_i$$

in other words, $\{b_1, \ldots, b_s\}$ is also a spanning set and hence a basis. □

Proposition 3.7.2 *The elements $b_1, \ldots, b_s$ form a (free module) basis for the ring $\mathbf{C}[x_1, \ldots, x_n]$ over $f^*\mathbf{C}[y_1, \ldots, y_n]$, if and only if they are a (vector space) basis for*

$$V_{\mathbf{z}} = \frac{\mathbf{C}[x_1, \ldots, x_n]}{<f_1 - z_1, \ldots, f_n - z_n>}$$

for all $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{C}^n$.

Proof We will begin by proving the 'only if' part of the proof. Let us suppose that $b_1, \ldots, b_s$ are a free basis for the module. Choose a point $\mathbf{z} \in \mathbf{C}^n$ and let

$\phi \in \mathbf{C}[x_1, \ldots, x_n]$ be arbitrary. Then

$$\phi = \sum_{i=1}^{s} \beta_i(f) b_i$$

with $\beta_i \in f^* \mathbf{C}[y_1, \ldots, y_n]$ and

$$\phi = \sum_{i=1}^{s} \beta_i(z) b_i \text{ modulo } <f_1 - z_1, \ldots, f_n - z_n> .$$

Now this is simply a linear combination of the b_i, so they span $V_{\mathbf{z}}$.

Are they linearly independent? Suppose

$$\sum_{i=1}^{s} \lambda_i b_i = \sum_{j=1}^{n} \zeta_j (f_j - z_j)$$

with $\lambda_j \in \mathbf{C}, \zeta_i \in \mathbf{C}[x_1, \ldots, x_n]$, so this linear combination is zero in $V_{\mathbf{z}}$. We may rewrite each ζ_j as an element of the module:

$$\zeta_j = \sum_{i=1}^{s} \delta_{ji}(f) b_i.$$

This yields

$$\sum_{i=1}^{s} \left(\lambda_i - \sum_{j=1}^{n} \delta_{ji}(f)(f_j - z_j) \right) b_i = 0$$

and since the module is free,

$$\lambda_i - \sum_{j=1}^{n} \delta_{ji}(f)(f_j - z_j) = 0$$

for each i. Now f is an epimorphism (see Proposition 3.1.1), so the inverse image of $\mathbf{z}$ is non-empty. If $\mathbf{x} = (x_1, \ldots, x_n)$ lies in $f^{-1}(z)$, evaluating at x we obtain $\lambda_i = 0$ for all i. Thus the b_i are linearly independent and so form a basis for $V_{\mathbf{z}}$.

Now for the reverse implication, suppose that the elements $b_1, \ldots, b_s$ are again a free basis for the module, while $b'_1, \ldots, b'_s$ form a basis for $V_{\mathbf{z}}$ for all $\mathbf{z} \in \mathbf{C}^n$. We may therefore write

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

where A is a matrix with entries in $f^*\mathbf{C}[y_1, \dots, y_n]$, just by writing the b'_i as elements of the module. Now, the b_i are also a basis for $V_{\mathbf{z}}$ for every $\mathbf{z} \in \mathbf{C}^n$. Thus we know that on setting $f_1 = z_1, \dots, f_n = z_n$, i.e. modulo $<f_1 - z_1, \dots, f_n - z_n>$, the matrix A is an invertible (change of basis) matrix and so has non-zero determinant. Now since the determinant is non-zero for all values of $\mathbf{z} \in \mathbf{C}^n$, it must in fact be identically a constant. Thus A is invertible as a matrix over the ring $f^*\mathbf{C}[y_1, \dots, y_n]$. In effect, A is a change of basis matrix in the free module, so $b'_1, \dots, b'_s$ are also a free basis. □

We have thus shown that the three definitions of degree given above coincide.

Chapter 4

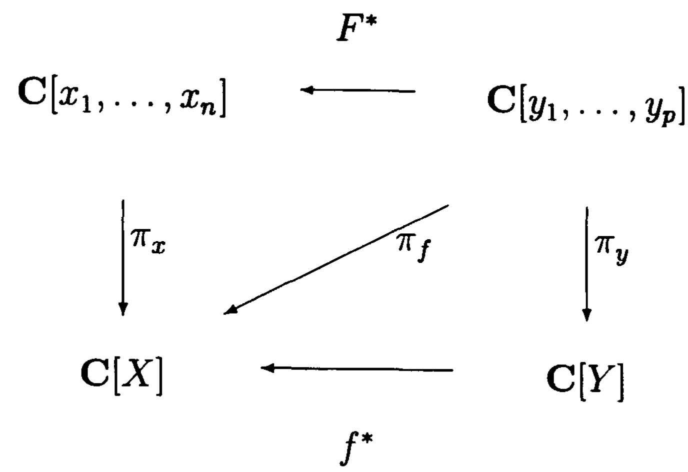
Generalising to varieties

In this section, we generalise our results on finite maps $\mathbf{C}^n \to \mathbf{C}^n$ to finite maps $X \to Y$ where X and Y are affine varieties of the same dimension.

4.1 Definitions

Firstly, suppose we are given a regular map $f : X \to Y$ where $X \subset \mathbf{C}^n$, $Y \subset \mathbf{C}^p$ and $Y = \overline{f(X)}$ (i.e. the Zariski closure of the image). Now f has components in $\mathbf{C}[X] = \frac{\mathbf{C}[x_1,\ldots,x_n]}{I_X}$ where I_X is the ideal of X. We can choose a 'representative' polynomial map $F : \mathbf{C}^n \to \mathbf{C}^p$ such that F_i lies in the coset given by f_i for each i. This gives

Lemma 4.1.1 *Given f and F as above and defining f^* and F^* to be the associated pullback maps, the following diagram commutes:*



where $\mathbf{C}[Y] = \frac{\mathbf{C}[y_1,\ldots,y_p]}{I_Y}$, *with* I_Y *being the ideal of* Y, π_x *and* π_y *being the natural maps to the quotients and* π_f *is the map obtained by substituting* f_i *for* y_i

Proof Let $k(y_1, \ldots, y_p) \in \mathbf{C}[y_1, \ldots, y_p]$ be an arbitrary element. Now

$$\pi_y(k) = k(y_1 + I_Y, \ldots, y_p + I_Y)$$

Mapping by f^* yields

$$\begin{aligned} f^* \circ \pi_y(k) &= k(f_1, \ldots, f_p) \\ &= \pi_f(k). \end{aligned}$$

However, following the other way round the diagram gives

$$F^*(k) = k(F_1, \ldots, F_p)$$

and thus

$$\begin{aligned} \pi_x \circ F^*(k) &= k(F_1 + I_X, \ldots, F_p + I_X) \\ &= k(f_1, \ldots, f_p) \\ &= \pi_f(k). \end{aligned}$$

Thus the result is independent of the route taken. □

This leads to a condition for finiteness using the 'representative' map F:

Corollary 4.1.2 *With f and F as above, the map f is finite if and only if for every $\phi \in \mathbf{C}[x_1, \dots, x_n]$ there exists some m and $\alpha_i \in F^*\mathbf{C}[y_1, \dots, y_p]$ for $i = 1, \dots, m$ such that*

$$\phi^m + \alpha_{m-1}\phi^{m-1} + \cdots + \alpha_0 \in I_X. \tag{4.1}$$

Proof Now f is finite if and only if for every $\psi \in \mathbf{C}[X]$ there exists some m and $\beta_i \in f^*\mathbf{C}[Y]$ such that

$$\psi^m + \beta_{m-1}\psi^{m-1} + \cdots + \beta_0 = 0 \tag{4.2}$$

Now considering the diagram in Lemma 4.1.1 above and since the map π_y is onto we obtain

$$f^*\mathbf{C}[Y] = \pi_x \circ F^*\mathbf{C}[y_1, \dots, y_p]$$

so we may rewrite each β_i as $\pi_x(\alpha_i)$ where $\alpha_i \in F^*\mathbf{C}[y_1, \dots, y_p]$. Also rewriting ψ as $\pi_x(\phi)$ where $\phi \in \mathbf{C}[x_1, \dots, x_n]$, (4.2) becomes:

$$\pi_x(\phi)^m + \pi_x(\alpha_{m-1})\pi_x(\phi)^{m-1} + \cdots + \pi_x(\alpha_0) = 0$$

and so

$$\pi_x(\phi^m + \alpha_{m-1}\phi^{m-1} + \cdots + \alpha_0) = 0$$

which gives

$$\phi^m + \alpha_{m-1}\phi^{m-1} + \cdots + \alpha_0 \in I_X.$$

□

4.2 New finiteness criterion

Using the previous result, we can extend the finiteness criterion to regular maps between varieties:

Proposition 4.2.1 *Let $f : X \to Y$ be a regular map, with X, Y varieties in $\mathbf{C}^n, \mathbf{C}^p$ respectively. Let F be a 'representative' polynomial map $\mathbf{C}^n \to \mathbf{C}^p$ for f as above. Let $<g_1, \ldots, g_r>$ be the ideal corresponding to X and let the polynomial ring $\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_p]$ have a product monomial order with $x_i > z_j$ for all i, j. Then f is finite if and only if there exist $m_j \in \mathbf{N}$ for $j = 1, \ldots, n$ such that*

$$x_j^{m_j} \in LT(<g_1, \ldots, g_r, F_1 - z_1, \ldots, F_p - z_p>). \tag{4.3}$$

Proof Suppose the condition (4.3) holds for the order $>$ and some polynomial map $F : \mathbf{C}^n \to \mathbf{C}^p$ which is a 'representative' for f. Now by Lemma 3.6.1, it will also hold for the order $>_{lex(j)}$ for any given $j \in \{1, \ldots, n\}$, where $>_{lex(j)}$ is defined as lexicographic order with $x_1 > x_2 > \cdots > x_{j-1} > x_{j+1} > \cdots > x_n > x_j > z_1 > \cdots > z_p$. This is a product order since we may consider it as a product of lex on the x variables and lex on the z variables. Hence we have

$$x_j^{m_j} = LT_{lex(j)}\left(\sum_{i=1}^{p} \alpha_{i,j}(F_i - z_i) + \sum_{i=1}^{r} \beta_{i,j} g_i\right)$$

for some $\alpha_{i,j}, \beta_{i,j} \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_p]$. By consideration of the monomial order, we may write

$$\sum_{k=1}^{m_j} \delta_k(\mathbf{z}) x_j^k = \sum_{i=1}^{p} \alpha_{i,j}(F_i - z_i) + \sum_{i=1}^{r} \beta_{i,j} g_i$$

where δ_k is a polynomial in $\mathbf{C}[z_1, \ldots, z_n]$, with $\delta_{m_j} = 1$. Now setting $z_i = F_i$ for all i we obtain

$$\sum_{k=1}^{m_j} \delta_k(\mathbf{F}) x_j^k = \sum_{i=1}^{r} \beta_{i,j}(\mathbf{x}, \mathbf{F}) g_i$$

the lefthand side of which is a monic polynomial in x_j with coefficients in the ring $F^*\mathbf{C}[y_1, \ldots, y_p]$. Thus x_j satisfies the equation (4.1). This may be repeated for each j and hence any $h \in \mathbf{C}[x_1, \ldots, x_n]$ will satisfy (4.1) i.e. the map f is finite.

For the converse, suppose that f is a finite map. Then for each $j \in \{1, \ldots, n\}$, there exist polynomials $\epsilon_i \in F^*\mathbf{C}[y_1, \ldots, y_p]$ such that

$$x_j^m + \epsilon_{m-1}(F) x_j^{m-1} + \cdots + \epsilon_0(F) = \sum_{i=1}^{r} \beta_{i,j} g_i$$

for some $\beta_{i,j} \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_p]$, by Corollary 4.1.2. Denote by $\mathcal{I}$ the ideal $<g_1, \ldots, g_r, F_1 - z_1, \ldots, F_n - z_n>$. Now $F_i = z_i$ modulo $\mathcal{I}$ for $i = 1, \ldots, n$ and hence if ϕ is any polynomial, then $\phi(f_1, \ldots, f_n) = \phi(z_1, \ldots, z_n)$ modulo $\mathcal{I}$. Thus we have

$$\sum_{k=1}^{m} \epsilon_k(z) x_j^k = \sum_{k=1}^{m} \epsilon_k(F) x_j^k = 0 \text{ modulo } \mathcal{I}$$

where $\epsilon_k(z)$ is simply $\epsilon_k(F)$ with each F_i replaced by z_i. Hence

$$\sum_{k=1}^{m} \epsilon_k(z) x_j^k = \sum_{i=1}^{p} \eta_{i,j}(F_i - z_i) + \sum_{i=1}^{r} \theta_{i,j} g_i$$

with $\eta_{i,j}, \theta_{i,j} \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_p]$. So

$$x_j^m = LT\left(\sum_{i=1}^{p} \eta_{i,j}(F_i - z_i) + \sum_{i=1}^{r} \theta_{i,j} g_i\right)$$

in other words, $x_j^m \in LT(\mathcal{I})$. This may be repeated for each j and hence the result follows. □

Corollary 4.2.2 *If we define* $\mathcal{I}_z = \mathcal{I} \cap \mathbf{C}[z_1, \ldots, z_p]$, *then* $\mathcal{I}_z \cong I_Y$ *by simply re-writing* y *variables as* z *variables and vice versa.*

Proof The points of $f(X)$ are given by the projection of the set

$$\mathcal{F} = \{(\mathbf{z}, \mathbf{x}) : F_1(\mathbf{x}) = z_1, \ldots, F_p(\mathbf{x}) = z_p, g_1(\mathbf{x}) = 0, \ldots, g_r(\mathbf{x}) = 0\}$$

onto the z variables. By Proposition 1.2.8 this means that Y is defined by the ideal $\mathcal{I}_z$ using z variables in the target. □

Now to actually calculate generators for the ideal I_Y, the method is as follows. First, calculate a Gröbner basis, G, for the ideal $\mathcal{I}$ under a product order as in Proposition 4.2.1 above. Then $G_k = G \cap \mathbf{C}[z_1, \ldots, z_p]$ is a Gröbner basis and hence a set of generators for $\mathcal{I}_k$ (see Proposition 1.2.7).

Corollary 4.2.3 *If we define* $\mathcal{L} = \{\mathbf{x}^\alpha : \mathbf{x}^\alpha \notin LT(\mathcal{I})\}$, *then* $\pi_x(\mathcal{L})$ *generates* $\mathbf{C}[X]$ *as a* $f^*\mathbf{C}[Y]$ *module.*

Proof If we define $\mathcal{L}'$ to be the LT-monomial basis of $\frac{\mathbf{C}[x_1,\ldots,x_n,z_1,\ldots,z_p]}{\mathcal{I}}$, then for an arbitrary element of $\phi \in \mathbf{C}[x_1,\ldots,x_n]$, we may write

$$\phi = \sum_{l\in\mathcal{L}'} \mu_l \cdot l(\mathbf{x},\mathbf{z}) + \sum_{i=1}^{r} \alpha_i(\mathbf{x},\mathbf{z})g_i + \sum_{j=1}^{p} \beta_j(\mathbf{x},\mathbf{z})(F_j - z_j)$$

where $\phi \in \mathbf{C}$, $\alpha_i, \beta_j \in \mathbf{C}[x_1,\ldots,x_n,z_1,\ldots,z_p]$. Now substituting F_j for z_j everywhere gives

$$\phi = \sum_{l\in\mathcal{L}'} \mu_l \cdot l(\mathbf{x},\mathbf{F}) + \sum_{i=1}^{r} \alpha_i(\mathbf{x},\mathbf{F})g_i$$

Now every monomial in $\mathcal{L}'$ is of the form $\mathbf{x}^\alpha\mathbf{z}^{\alpha'}$ where $\mathbf{x}^\alpha \in \mathcal{L}$ and since $\alpha_i(\mathbf{x},\mathbf{F}) \in \mathbf{C}[x_1,\ldots,x_n]$, this is simply

$$\phi = \sum_{l\in\mathcal{L}} \epsilon_l(\mathbf{F}) \cdot l(\mathbf{x}) + \sum_{i=1}^{r} \delta_i(\mathbf{x})g_i$$

where $\epsilon_l(\mathbf{F}) \in F^*\mathbf{C}[y_1,\ldots,y_p]$ and $\delta_i \in \mathbf{C}[x_1,\ldots,x_n]$. Now mapping under π_x gives

$$\begin{aligned}\pi_x(\phi) &= \sum_{l\in\mathcal{L}} \pi_x(\epsilon_l(\mathbf{F})) \cdot \pi_x(l(\mathbf{x}))\\ &= \sum_{l\in\mathcal{L}} \epsilon_l(\mathbf{f}) \cdot \pi_x(l(\mathbf{x}))\end{aligned}$$

where $\epsilon_l(\mathbf{f}) \in \mathbf{C}[Y]$ since $\pi_f = \pi_x \circ F^*$. Any element of $\mathbf{C}[X]$ can be written as $\pi_x(\phi)$ for some $\phi \in \mathbf{C}[x_1,\ldots,x_n]$, so $\pi_x(\mathcal{L})$ does generate the module as required. □

The following Maple routine implements the new criterion. Given a representative map F as a list of components, equations defining the source variety X and variables V, it calculates a Gröbner basis using lex order and extra variables _Z to check finiteness. If a fourth argument is given, this is assigned the list of leading terms of the basis calculated and the equations defining the target variety (given in terms of the variables _Z).

```
vcheckfinite:=proc(F,X,V)
  local i,j,VV,FF,G,LT,marker,Y,l;
  n:=nops(V);
  p:=nops(F);
  nox:=nops(X);
  _Z:=array(1..p);
  VV:=[op(V),_Z[i] $ i=1..p];
  FF:=[X[i] $ i=1..nox, F[i]-_Z[i] $ i=1..p];
  G:=grobner[gbasis](FF,VV,plex);
  Y:=[];
  for i from 1 to nops(G) do
    l:=grobner[leadmon](G[i],VV,plex)[2];
    if l=coeffs(l,V) then Y:=[op(Y),G[i]];
    fi;
  od;
  LT:=map(x->x[2],map(grobner[leadmon],G,VV,plex));
  if nargs=4 then assign(args[4],[LT,Y]);
  fi;
  for i from 1 to n do
    marker:=0;
    for j from 1 to nops(LT) do
      pow:=coeffs(LT[j],V[i]);
      if type(pow,numeric) then
        marker:=1;
        break;
```

```
      fi;
    od;
    if marker=0 then RETURN(false);
    fi;
  od;
  RETURN(true);
end:
```

Example 4.2.4 We now demonstrate the use of vcheckfinite. In each case, the source variety, X is given by $x_1^2 - 4x_2x_3 + 3x_2^2 + 2x_1 - 3x_3$ in $\mathbf{C}^3$.

```
> X:=[x[1]^2-4*x[2]*x[3]+3*x[2]^2+2*x[1]-3*x[3]]:
-----------------------------------------------------------
> F1:=[x[1],x[2]]:
-----------------------------------------------------------
> vcheckfinite(F1,X,[x[1],x[2],x[3]]);
                                      false
-----------------------------------------------------------
> F2:=[x[2],x[3]];
-----------------------------------------------------------
> vcheckfinite(F2,X,[x[1],x[2],x[3]]);
                                      true
-----------------------------------------------------------
> F3:=[x[1]^2,x[2]^2,x[3]]:
-----------------------------------------------------------
> vcheckfinite(F3,X,[x[1],x[2],x[3]],'B');
                                      true
-----------------------------------------------------------
> B[2];
```

$$- 288 \_Z[3]^{2} \_Z[2]^{3} + 576 \_Z[3]^{3} \_Z[2]^{2} - 288 \_Z[3]^{4} \_Z[2] +$$

$$256 \_Z[3]^{4} \_Z[2]^{2} - 32 \_Z[3]^{2} \_Z[2] \_Z[1]^{2} - 192 \_Z[3]^{2} \_Z[2]^{2} \_Z[1]$$

$$+ 192 \_Z[3]^{3} \_Z[2] \_Z[1] + 16 \_Z[1]^{2} - 8 \_Z[1]^{3}$$

$$+ 196 \_Z[1] \_Z[3]^{2} \_Z[2] + 81 \_Z[3]^{4} - 108 \_Z[1] \_Z[3]^{3}$$

$$+ 144 \_Z[1] \_Z[3] \_Z[2] + 54 \_Z[1]^{2} \_Z[3]^{2} - 12 \_Z[1]^{3} \_Z[3]$$

$$+ 54 \_Z[1]^{2} \_Z[2]^{2} + 12 \_Z[2] \_Z[1]^{3} + \_Z[1]^{4}$$

$$- 108 \_Z[1]^{2} \_Z[3] \_Z[2] - 324 \_Z[1] \_Z[2]^{2} \_Z[3]$$

$$- 324 \_Z[3]^{3} \_Z[2] - 72 \_Z[1] \_Z[3]^{2} - 48 \_Z[2] \_Z[1]^{2}$$

$$+ 48 \_Z[1]^{2} \_Z[3] - 72 \_Z[1] \_Z[2]^{2} + 81 \_Z[2]^{4}$$

$$+ 108 \_Z[2]^{3} \_Z[1] - 324 \_Z[2]^{3} \_Z[3] + 486 \_Z[3]^{2} \_Z[2]^{2}$$

Thus the map $X \to \mathbf{C}^2$ via projection onto the first two coordinates does not yield a finite map. However, if we project onto the last two, this does give us a finite map. The map $X \to \mathbf{C}^2$ by squaring the first two coordinates is, as we would expect, finite. The target variety Y is given above in terms of `_Z[1],_Z[2],_Z[3]`.

4.3 Further results

The following are further generalisations of our results on finite maps $\mathbf{C}^n \to \mathbf{C}^n$.

Proposition 4.3.1 *Let $f = (f_1, \ldots, f_p) : X \to Y$ be a map between varieties $X \subset \mathbf{C}^n$, $Y \subset \mathbf{C}^p$, with X defined by the ideal $< g_1, \ldots, g_r >$. Suppose $F = (F_1, \ldots, F_p) : \mathbf{C}^n \to \mathbf{C}^p$ is a representative map for f and suppose that $F'_1, \ldots, F'_p$ are the leading homogeneous parts of its components. Now let F' be the map given by the F'_i and $f' : X \to Y'$ the map given by the restriction of F' to X. (Where Y' is the Zariski closure of $f'(X)$). Then*

1. *f' is finite if and only if the quotient*

$$\frac{\mathbf{C}[x_1, \ldots, x_n]}{<g_1, \ldots, g_r, F'_1, \ldots, F'_p>} \tag{4.4}$$

   is finite dimensional as a $\mathbf{C}$ vector space.

2. *If f' is finite, then f is also finite.*

Proof Part 1: Assume (4.4) to be finite dimensional and let $b_1, \ldots, b_q$ be a basis for it. Take $\phi \in \mathbf{C}[x_1, \ldots, x_n]$ to be arbitrary. Then we may write

$$\phi = \sum_{j=1}^{q} \kappa_j b_j + \sum_{i=1}^{p} \alpha_i F'_i + \sum_{k=1}^{r} \beta_k g_k \tag{4.5}$$

where $\kappa_j \in \mathbf{C}$, $\alpha_i, \beta_k \in \mathbf{C}[x_1, \ldots, x_n]$ and $\deg \alpha_i F_i' \leq \deg \phi$ (via the homogeneity). Now inductively rewriting each α_i as in (4.5), and using the fact that the degree will decrease at each stage, enables us to write

$$\phi = \sum_{j=1}^{q} \delta_j(F')b_j + \sum_{k=1}^{r} \mu_k g_k$$

where $\mu \in \mathbf{C}$ and $\delta_j \in (F')^*\mathbf{C}[y_1, \ldots, y_p]$. Thus the quotient

$$\frac{\mathbf{C}[x_1, \ldots, x_n]}{<g_1, \ldots, g_r>}$$

is an $(F')^*\mathbf{C}[y_1, \ldots, y_p]$-module spanned by the b_i, so f' is finite.

For the reverse implication, f' finite implies that there is a LT-monomial basis for

$$\frac{\mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_p]}{<g_1, \ldots, g_r, F_1' - z_1, \ldots, F_p' - z_p>}$$

with a finite number of monomials in the x variables. Substituting $z_1 = \cdots = z_p = 0$ into this gives (4.4) as a finite dimensional $\mathbf{C}$ vector space.

Part 2: Let ϕ and $b_1, \ldots, b_q$ be as before, since we know that (4.4) is a finite dimensional $\mathbf{C}$ vector space. We may again write ϕ as in (4.5) above with F for F', but may rewrite this as

$$\phi = \sum_{j=1}^{q} \kappa_j b_j + \sum_{i=1}^{p} \alpha_i F_i - \underbrace{\sum_{i=1}^{p} \alpha_i (F_i - F_i')}_{(*)} + \sum_{k=1}^{r} \beta_k g_k$$

Now the expression labelled (*) is either zero or has degree strictly less than ϕ. Rewriting (*) as in (4.5) and continuing inductively yields

$$\phi = \sum_{j=1}^{q} \lambda_j b_j + \sum_{i=1}^{p} \zeta_i F_i + \sum_{k=1}^{r} \eta_k g_k$$

where $\lambda_j \in \mathbf{C}$, $\zeta_i, \eta_k \in \mathbf{C}[x_1, \ldots, x_n]$ and $\deg \zeta_i F_i \leq \deg \phi_i$. This is almost identical to (4.5) and the proof of finiteness finishes as in Part 1. □

Proposition 4.3.2 *If $f : X \to Y$ is a regular mapping of affine varieties and if every point $x \in Y$ has an affine neighbourhood U such that $f : V \to U$ is finite, then f itself is also finite.*

Proof The proof follows [Sh, p49]. For every point, we may take a neighbourhood U being a principal open set (i.e. a set of the form $D(\phi) = X - \mathbf{V}(\phi)$ for some polynomial ϕ). Let $D(g_\alpha)$ be a system of such open sets, which we may take to be finite in number. Then $Y = \bigcup D(g_\alpha)$, that is, the g_α generate an ideal equal to $\mathbf{C}[Y]$. In our case, $V_\alpha = f^{-1}(D(g_\alpha)) = D(f^*(g_\alpha))$, $\mathbf{C}[D(g_\alpha)] = \mathbf{C}[Y][1/g_\alpha]$, $\mathbf{C}[V_\alpha] = \mathbf{C}[X][1/g_\alpha]$. By hypothesis, $\mathbf{C}[X][1/g_\alpha]$ has a finite basis $\omega_{i,\alpha}$ over $\mathbf{C}[Y][1/g_\alpha]$. Here, we may assume $\omega_{i,\alpha} \in \mathbf{C}[X]$, for if the basis consisted of elements $\omega_{i,\alpha}/g_\alpha^{m_i}$, then the elements $\omega_{i,\alpha}$ would also be a basis. We consider the union of all the basis elements $\omega_{i,\alpha}$ and show that they form a basis of $\mathbf{C}[X]$ over $\mathbf{C}[Y]$. Every element $\phi \in \mathbf{C}[X]$ has a representation

$$\phi = \sum_i \frac{a_{i,\alpha}}{g_\alpha^{n_\alpha}} \omega_{i,\alpha}$$

for each α. Since the elements $g_\alpha^{n_\alpha}$ generate the unit ideal of $\mathbf{C}[Y]$, there exist $h_\alpha \in \mathbf{C}[Y]$ such that $\sum_\alpha g_\alpha^{n_\alpha} h_\alpha = 1$. Therefore

$$b = b\sum_\alpha g_\alpha^{n_\alpha} h_\alpha = \sum_i \sum_\alpha a_{i,\alpha} h_\alpha \omega_{i,\alpha},$$

which proves the theorem. □

In other words, if a regular map looks finite on affine neighbourhoods, then it is finite globally. This is in fact the definition for finiteness of maps between projective varieties. If the map is finite on restriction to affine neighbourhoods, then it is called finite (see [Sh, p49]).

Proposition 4.3.3 *Let $X \subset \mathbf{C}^n$ be an affine variety of dimension d and consider maps $f : X \to \mathbf{C}^r$ for some $r \geq d$. If we restrict our attention to those maps with a representative map F whose components have degree $\leq k$, then these are generically finite.*

Proof Firstly, consider the projective closure of our variety $P(X) \subset \mathbf{P}^n$. We can embed this in a larger projective space $\mathbf{P}^N$ by means of the Veronese mapping ν_k of degree k.

Given any map $f : X \to \mathbf{C}^r$ as specified, we can take its representative, homogenize (to degree k) with respect to a new variable x_0 and obtain

$$F^h|_{P(X)} = f^h : P(X) \to \mathbf{C}^r \subset \mathbf{P}^r.$$

Since the components of F^h are linear combinations of monomials of degree k, we can choose a projection $\pi : \mathbf{P}^N \to \mathbf{P}^r$ such that $\pi(\nu_k(P(X))) = F^h(P(X))$ so long as $\ker \pi \cap \nu_k(P(X)) = \emptyset$, (see [Sh, p40]). Now $\dim(\nu_k(P(X))) = d$ and $\dim \ker \pi = N - r - 1$ (determined by $r + 1$ equations). But $N - r - 1 + d < N$, thus $\ker \pi \cap \nu_k(P(X)) = \emptyset$ is a generic condition on π. Now the map ν_k is an embedding and hence finite. The projection π is also finite (see [Sh, p50 Thm7]). Thus we find that f^h must also be finite. Since this is the projective case, as mentioned above, this means that it looks finite in affine charts. A regular mapping g of quasiprojective varieties is called finite if each point in the target has an affine neighbourhood V such that the restriction of the map to $g^{-1}(V)$ is finite in the affine sense. In this case, if we restrict everything to our original affine spaces, we will recover the map f we began with. Now the condition of the map looking finite locally still holds and by Proposition 4.3.2, f is finite. □

Chapter 5

The trace bilinear form

5.1 Polynomials in one variable

The definition of the trace bilinear form (or TBF), in the one variable case, is encapsulated in the following proposition.

Proposition 5.1.1 *Let f,g be polynomials in $\mathbf{R}[x]$ with no common roots. Suppose that $f = 0$ has r real roots where $g > 0$ and s real roots where $g < 0$. Define a bilinear form on the vector space*

$$W = \frac{\mathbf{R}[x]}{<f(x)>} \tag{5.1}$$

as follows. Let $T_g(a, b) = trace(abg)$ where abg is considered as the endomorphism of W determined by multiplication. This form is known as the trace bilinear form *(or TBF)* weighted by *g. The rank of the quadratic form $T_g(a, a)$ is the total number of distinct roots of $f = 0$ (including complex solutions) and its signature is $r - s$.*

Proof We will suppose that f and g are monic and also assume initially that f has distinct roots $\alpha_1, \ldots, \alpha_m \in \mathbf{C}$. We claim that

$$T_g(a, a) = \sum_{j=1}^{m} g(\alpha_j)(a_1 + \alpha_j a_2 + \ldots + \alpha_j^{n-1} a_n)^2 \tag{5.2}$$

where $a = a_1 + a_2x + \cdots + a_nx^{n-1} \in W$. To see this, note that x as an endomorphism of W has m distinct eigenvalues $\alpha_1, \ldots, \alpha_m$ with eigenvectors $\frac{f(x)}{(x-\alpha_1)}, \ldots, \frac{f(x)}{(x-\alpha_m)}$. So the matrix of x^r with respect to this basis is

$$\begin{pmatrix} \alpha_1^r & 0 & \cdots & 0 \\ 0 & \alpha_2^r & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_m^r \end{pmatrix}.$$

The endomorphism corresponding to g, is however given by

$$\begin{pmatrix} g(\alpha_1) & 0 & \cdots & 0 \\ 0 & g(\alpha_2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g(\alpha_m) \end{pmatrix}$$

with respect to the same basis. For if we write

$$g(x)\frac{f(x)}{(x-\alpha_j)} = \sum_{i=1}^{m} \lambda_{i,j}\frac{f(x)}{(x-\alpha_i)} + \langle f(x) \rangle$$

and evaluate at α_i, the matrix is found to be as above. Thus, $x^r g$ as an endomorphism with respect to the same basis has the matrix

$$\begin{pmatrix} g(\alpha_1)\alpha_1^r & 0 & \cdots & 0 \\ 0 & g(\alpha_2)\alpha_2^r & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g(\alpha_m)\alpha_m^r \end{pmatrix}$$

and (5.2) follows. If the roots of f are not distinct, then a small perturbation makes them so and (5.2) holds by continuity.

Returning to the case when the α_j are distinct, we note that the vectors $(1, \alpha_j, \ldots, \alpha_j^{m-1})$ are linearly independent, as their determinant is the vandermonde matrix. If α_j is a real root, then $g(\alpha_j)(a_1 + \alpha_j a_2 + \cdots + \alpha_j^{n-1}a_n)^2$ contributes a real square with sign equal to that of $g(\alpha_j)$. If α_j is complex, then $\bar{\alpha}_j$

is also a root and

$$g(\alpha_j)(a_1 + \alpha_j a_2 + \cdots + \alpha_j^{n-1} a_n)^2 + g(\bar{\alpha_j})(a_1 + \bar{\alpha_j} a_2 + \cdots + \bar{\alpha_j}^{n-1} a_n)^2$$

$$= 2\Re(g(\alpha_j)(a_1 + \alpha_j a_2 + \cdots + \alpha_j^{n-1} a_n)^2)$$

If we write $g(\alpha_j) = \lambda + i\mu$, $(a_1 + \alpha_j a_2 + \cdots + \alpha_j^{n-1} a_n) = L + iM$ where L and M are real forms in the a_i, then L and M are linearly independent. Moreover we can rewrite the expression above as follows.

$$2\Re((\lambda + i\mu)(L^2 - M^2 + 2iLM))$$

$$= 2(\lambda(L^2 - M^2 - 2\mu LM)$$

$$= \begin{cases} 2(\lambda(L - \mu\lambda^{-1}M)^2 - \lambda^{-1}(\lambda^2 + \mu^2)M^2) & \text{if } \lambda \neq 0 \\ -2\mu LM & \text{if } \lambda = 0 \end{cases}$$

Either way, there is a net contribution of 2 to the rank and of 0 to the signature. The result follows.

Suppose now that f has some coincident roots. Then for each k-tuple root α, we have an expression $kg(\alpha)(a_1 + \alpha a_2 + \cdots + \alpha^{n-1} a_n)^2$ and the result now follows. □

Corollary 5.1.2 (Sylvester, see [C2]) *If $g = 1$, then the rank of T_g is the number of distinct roots of $f = 0$ and its signature is the number of real roots.*

Corollary 5.1.3 *If we allow f and g to have common roots, then the rank of T_g is the number of roots of $f = 0$ distinct from those of g and its signature is $r - s$ as before.*

5.2 Higher dimensions

We now consider the higher dimensional case, where $f : \mathbf{R}^n \to \mathbf{R}^n$ is a finite polynomial map. This is very much like the situation looked at in [PRS]. In this paper, the authors related the number of roots of a set of polynomials with a trace bilinear form. The space considered was the real space formed by taking the quotient of the polynomial ring by the ideal generated by the polynomials in question. Our approach will in some sense look at the preimages of all points at once, as opposed to just the preimages of one point.

The field $\mathbf{R}(\mathbf{x})$ is an $f^*\mathbf{R}(\mathbf{y})$ algebra. We define the trace bilinear form of this algebra in an analogous way to that in the one variable case. Given $a, b \in \mathbf{R}(\mathbf{x})$, we consider the endomorphism $\mathbf{R}(\mathbf{x}) \to \mathbf{R}(\mathbf{x})$ given by multiplication by ab. Then $T(a,b) = \text{trace}(ab)$ as before. Again, given $g \in \mathbf{R}(\mathbf{x})$ we can define T_g, the bilinear form weighted by g, by $T_g(a,b) = \text{trace}(abg)$.

Since we are dealing with fields of characteristic zero, the extension of $\mathbf{R}(\mathbf{x})$ over $f^*\mathbf{R}(\mathbf{y})$ is separable. It is also finite, hence it must be a simple extension (see [ZS p65,84]). Thus we can choose a primitive element σ with $\sigma \in \mathbf{R}[\mathbf{x}]$. Now since f is finite, the minimal polynomial of σ has coefficients in $f^*\mathbf{R}[\mathbf{y}]$, i.e. it is an element of $f^*\mathbf{R}[\mathbf{y}][z]$ say. (compare [Sh, p116]). In fact, we shall see that we may take σ to be a real linear form.

Now given a point $c \in \mathbf{R}^n$, the bilinear forms T, T_g above determine bilinear forms

$$T^c, T^c_g : \mathbf{R}(\mathbf{x}) \times \mathbf{R}(\mathbf{x}) \to \mathbf{R}$$

defined by $T^c(a,b) = T(a,b)|_{f=c}$ and $T^c_g(a,b) = T_g(a,b)|_{f=c}$.

We now prove some results related to these bilinear forms.

Proposition 5.2.1 *(i) Given σ, there is a proper subvariety $V \subset \mathbf{R}^n$ (target) with the property that the bilinear form T^c, for $c \notin V$ is equivalent to the trace bilinear form on*

$$\frac{\mathbf{R}[z]}{<G_\sigma(z)>} \tag{5.3}$$

where $G_\sigma(z)$ is the minimal polynomial of σ with the polynomial coefficients evaluated at $f = c$. More generally, given $g \in \mathbf{R}[\mathbf{x}]$ and writing $g = \sum g_i(f)\sigma^i(x)$, then T^c_g is equivalent to the trace bilinear form on (5.3) weighted by $\sum g_i(c)z^i$.

(ii) Given any $c \in \mathbf{R}^n$, there is a primitive element σ for the extension of $\mathbf{R}(\mathbf{x})$ over $f^*\mathbf{R}(\mathbf{y})$ with c not in the corresponding exceptional variety V (as defined in (i) above).

Proof (i) For V we consider the set of $c \in \mathbf{R}^n$ such that σ does not take distinct values on the set $f^{-1}(c)$. In other words, those c such that the number of distinct roots of $G_\sigma(z) = 0$ is smaller than the number of distinct points of $f^{-1}(c)$. It is not hard to see that V is closed; it is proper because it is contained in the ramification locus of f. (See [Sh, p117]).

Now suppose

$$G_\sigma(z) = z^m + a_1(f)z^{m-1} + \cdots + a_m(f)$$

with the $a_i(f) \in f^*\mathbf{R}[\mathbf{y}]$, so $1, \sigma(x), \ldots, \sigma^{m-1}(x)$ is an $f^*\mathbf{R}(\mathbf{y})$ basis for $\mathbf{R}(\mathbf{x})$. With this ordered basis, the $(i,j)^{th}$ entry in the symmetric matrix for T is obtained as follows. The element $\sigma^{i+j-2}(x)$ determines an endomorphism of $\mathbf{R}(\mathbf{x})$, taking $\sigma^k(x)$ to $\sigma^{i+j+k-2}(x)$, which, using the relation provided by G_σ, can be written as an $f^*\mathbf{R}[\mathbf{y}]$ combination of $1, \sigma(x), \ldots, \sigma^{m-1}(x)$. Setting $f = c$, we obtain the real $n \times n$ matrix corresponding to the endomorphism of (5.3) determined by multiplication by z^{i+j-2} with respect to the basis $1, z, \ldots, z^{m-1}$. The argument for the bilinear form T_g is exactly the same.

(ii) We only need to ensure that the primitive element $\sigma(x)$ takes distinct values on the points of $f^{-1}(c)$. Now since $[\mathbf{R}(\mathbf{x}) : f^*\mathbf{R}(\mathbf{y})] = m$, there are m distinct $f^*\mathbf{R}(\mathbf{y})$ isomorphisms $\tau_1, \ldots, \tau_m : \mathbf{R}(\mathbf{x}) \to \mathbf{R}(\mathbf{x})$. For $i \neq j$, the equations $\tau_i(L) - \tau_j(L) = 0$ for L a real linear form in $x_1, \ldots, x_n$ determine a proper subspace of the $\mathbf{R}$ vector space of all such forms. So any linear form L not in the union of these subspaces has $\tau_i(L)$ pairwise distinct. So the minimum polynomial of L, $G_L(z)$ has m distinct roots and L is a primitive element.

Let $c_{(1)}, \ldots, c_{(k)}$ be the distinct points of $f^{-1}(c)$. The conditions $L(c_{(i)}) = L(c_{(j)})$, $i \neq j$, determines a proper subspace of the space of $\mathbf{R}$ linear forms and if L is not in their union, then c does not lie in the corresponding exceptional variety. □

This leads to the following result.

Proposition 5.2.2 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a finite mapping, with $g \in \mathbf{R}[\mathbf{x}]$ a polynomial and $c \in \mathbf{R}^n$ (target space). Let T be the trace bilinear form of f. Then the rank of T^c is the number of points (real or complex) of $f^{-1}(c)$ and its signature is the number of these points which are real. If T_g is the form weighted by g, then the rank of T_g^c is the number of points of $f^{-1}(c)$ which do not lie on $g = 0$. The signature of this form is $r - s$, where r is the number of real points in the region $g > 0$ and s the number in the region $g < 0$.*

Proof Given $c \in \mathbf{R}^n$, choose a real primitive element σ which distinguishes the points of $f^{-1}(c)$. Then T^c is equivalent to the trace bilinear form on

$$\frac{\mathbf{R}[z]}{<G_\sigma(z)>} \tag{5.4}$$

where $G_\sigma(z)$ is the minimal polynomial of σ with the polynomial coefficients evaluated at $f = c$ by Proposition 5.2.1. The rank of T^c is, by Proposition 5.1.1, the number of roots of $G_\sigma(z) = 0$. But there is one such root for each point of $f^{-1}(c)$. The signature is the number of real roots, i.e. the number of points of $f^{-1}(c)$ at which σ takes real values. Clearly, σ takes real values at real points of $f^{-1}(c)$. Conversely, given c', with $f(c') = c$ and $\sigma(c') \in \mathbf{R}$, note that $f(\overline{c'}) = c$ and $\sigma(\overline{c'}) = \overline{\sigma(c')} = \sigma(c')$. Thus, by our choice of primitive element, $\overline{c'} = c'$ and $c' \in \mathbf{R}$.

The result in the general case is proved similarly. Here T_g^c is equivalent to the trace bilinear form on (5.4) weighted by $\sum g_i(c) z^i$. The rank of T_g^c is, by Proposition 5.1.1, the number of roots of $G_\sigma(z) = 0$ which are not roots of $\sum g_i(c) z^i$. This then is the number of points of $f^{-1}(c)$ which do not lie on $g(x) = 0$. The signature, on the other hand, gives the difference $r - s$, where r is the number of real roots of $G_\sigma(z) = 0$ with $\sum g_i(c)z^i > 0$ and s the number with $\sum g_i(c)z^i < 0$. But if $\lambda \in \mathbf{R}$ is a root, then we have a unique $c' \in \mathbf{R}^n$ with $f(c') = c$, $\sigma(c') = \lambda$ and $\sum g_i(c)\lambda^i = \sum g_i(c)\sigma^i(c') = g(c)$. The result now follows. □

5.3 Calculating the TBF

Consider the following spaces:

(F) $\mathbf{R}(\mathbf{x})$ as an $f^*\mathbf{R}(\mathbf{y})$ vector space.

(M) $\mathbf{R}[\mathbf{x}]$ as an $f^*\mathbf{R}[\mathbf{y}]$ module.

(I) $\frac{\mathbf{R}[\mathbf{x}]}{<f_1-c_1,\ldots,f_n-c_n>}$ as an $\mathbf{R}$ vector space

where $c = (c_1, \ldots, c_n) \in \mathbf{R}^n$. We wish to calculate the trace bilinear form on (F), weighted by some polynomial $g \in \mathbf{R}[\mathbf{x}]$ and evaluated at $f_1 = c_1, \ldots, f_n = c_n$. However, it is not easy to see how to obtain this directly. The following simplifies the problem:

Proposition 5.3.1 *The trace bilinear form on the vector space (I) above, weighted by $g \in \mathbf{R}[\mathbf{x}]$ is the same as that on (F), weighted by g and evaluated at $f_1 = c_1, \ldots, f_n = c_n$.*

Proof Let $\{b_1, \ldots, b_m\}$ be a free basis for (M). Then this is also a basis for each of (F) and (I) (see §3.7). Let TF_g^c and TI_g be the TBFs on (F) and (I) respectively. In order to show that these forms are equal, it is enough to show that $TF_g^c(b_i, b_j) = TI_g(b_i, b_j)$ for all $i, j \in \{1, \ldots, m\}$.

Now $TF_g^c(b_i, b_j) = tr(b_ib_jg)|_{f=c}$ where b_ib_jg is considered as a linear map on (F), acting by multiplication. For any k we may write

$$(b_ib_jg)b_k = \sum_{l=1}^{m} \alpha_{k,l}(f)b_l \tag{5.5}$$

where $\alpha_{k,l} \in f^*\mathbf{R}[\mathbf{y}]$, since $(b_ib_jg)b_k$ is a polynomial and the b_i are a free basis for (M). Now the matrix of b_ib_jg as a linear map is $(\alpha_{k,l}(f))$ and so its trace is $\sum_{l=1}^m \alpha_{l,l}(f)$. Evaluating at $f = c$, this gives

$$TF_g^c(b_i, b_j) = \sum_{l=1}^{m} \alpha_{l,l}(c). \tag{5.6}$$

Turning our attention to the TBF on (I), we know that (5.5), with $\alpha_{k,l} \in f^*\mathbf{R}[\mathbf{y}]$, still holds, as this is simply a polynomial identity. But we may rewrite this as

$$(b_ib_jg)b_k = \sum_{l=1}^{m} \alpha_{k,l}(c)b_l \bmod <f_1 - c_1, \ldots, f_n - c_n> .$$

So the matrix of b_ib_jg as a linear map on (I) is $(\alpha_{k,l}(c))$. Therefore,

$$TI_g(b_i, b_j) = \sum_{l=1}^{m} \alpha_{l,l}(c) = TF_g^c(b_i, b_j)$$

and so the two bilinear forms are indeed equal. □

With this result, since the TBF is independent of the basis chosen, if we wish to calculate TF_g^c, it is enough to find TI_g with respect to any basis.

The following Maple procedure will calculate TF_g^c using Proposition 5.3.1 above.

```
TBF:=proc(F,v,c,g)
local i,j,k,l,TRIPLES,PIECES,G,m,T,B,thistriple,
red,FF,thispiece,entry,Ans,place,comp;
  FF:=[];
  for i from 1 to nops(F) do
    FF:=[op(FF),F[i]-c[i]];
  od;
  B:=[op(quotbasis(FF,v))];
  m:=nops(B);
  G:=grobner[gbasis](FF,v);
```

```
##### This part calculates all the (weighted) triples

  TRIPLES:=[];
  for i from 1 to m do
    for j from i to m do
      for k from j to m do
        TRIPLES:=[op(TRIPLES),[g*B[i]*B[j]*B[k],i,j,k]];
      od;
    od;
  od;

##### This part reduces and calculates the required components

  PIECES:=[];
  T:=nops(TRIPLES);
  for i from 1 to T do
    thistriple:=TRIPLES[i][1];
    red:=grobner[normalf](thistriple,G,v);
    thispiece:=[];
    for j from 1 to 3 do
      thispiece:=[op(thispiece),coeff_of_mon(red,\
      B[TRIPLES[i][j+1]],v)];
    od;
    PIECES:=[op(PIECES),thispiece];
  od;

##### This part works out the matrix of the TBF
```

```
  Ans:=array(sparse, 1..m,1..m);
  for i from 1 to m do
    for j from i to m do
      entry:=0;
      for k from 1 to m do
        thistriple:=[g*B[i]*B[j]*B[k],op(sort([i,j,k]))];
        for l from 1 to T do
          if thistriple=TRIPLES[l] then
            member(k,[thistriple[2..4]],'comp');
            entry:=entry+PIECES[l][comp];
          fi;
        od;
      od;
      Ans[i,j]:=entry;
      Ans[j,i]:=entry;
    od;
  od;
  if nargs=5 then assign(args[5],B);
  fi;
RETURN(evalm(Ans));
end:
```

In the procedure, F is the list of components of the map and v is the list of variables used. c is the point at which the form is evaluated (given as a list) and g is the polynomial weight. The method used is first to calculate all the weighted triples of the form $b_i b_j b_k g$. Then these are reduced modulo the ideal and the relevant components recorded. Finally, these components are added and placed in a matrix. The reason for this method is that the weighted triple $b_1 b_2 b_3 g$ will be considered (and hence reduced) when calculating the entries at (1, 2), (1, 3) and (2, 3) in the matrix. All the reduction is thus done at the start to improve the efficiency of the algorithm. The routines `quotbasis`, `coeff_of_mon`, `mon`, `quotbasis` and `getmonos` are also needed and are to be found in Appendix B.

We now give some examples of the routine `TBF` in use.

Example 5.3.2 Firstly, consider the finite map $f = (x^2, y^2) : \mathbf{R}^2 \to \mathbf{R}^2$. We wish to calculate TF^c and also TF_g^c where $c = (2, 3)$ and $g = 4xy$ (the Jacobian of f).

```
> f:=[x^2,y^2];
                                          2    2
                                f := [x , y ]
---------------------------------------------------------
> c:=[2,3];
                                c := [2, 3]
---------------------------------------------------------
> TBF(f,[x,y],c,1,'B');
                              [ 4  0   0   0 ]
                              [              ]
                              [ 0  8   0   0 ]
                              [              ]
                              [ 0  0  12   0 ]
                              [              ]
                              [ 0  0   0  24 ]
---------------------------------------------------------
> B;
                              [1, x, y, x y]
---------------------------------------------------------
> g:=4*x*y;
```

```
                           g := 4 x y
-------------------------------------------------------
> TBF(f,[x,y],c,g);
                                 [ 0   0   0  96 ]
                                 [               ]
                                 [ 0   0  96   0 ]
                                 [               ]
                                 [ 0  96   0   0 ]
                                 [               ]
                                 [ 96  0   0   0 ]
-------------------------------------------------------
```

Thus, since the first matrix has rank 4 and signature 4, $f^{-1}(2,3)$ has 4 points, all of which are real. Now the second matrix has rank 4 and signature 0, so the real degree of f is 0.

Example 5.3.3 Next consider $f := (x^2y^2 + y^2, x^2 - xy) : \mathbf{R}^2 \to \mathbf{R}^2$. We wish to find TF^c, where c is the origin.

```
> f:=[x^2*y^2+y^2,x^2-x*y];
                             2  2    2   2
                      f := [x  y  + y , x  - x y]
-------------------------------------------------------
> M:=TBF(f,[x,y],[0,0],1);
                         [ 6  -2   0   0  -2   0 ]
                         [                       ]
                         [ -2  2   0   0   2   0 ]
                         [                       ]
                         [ 0   0  -2  -2   0   2 ]
                    M := [                       ]
```

```
                                    [ 0    0  -2  -2   0   2 ]
                                    [                        ]
                                    [ -2   2   0   0   2   0 ]
                                    [                        ]
                                    [ 0    0   2   2   0  -2 ]
```

```
> linalg[eigenvals](M);

                                    0, 0, 0, 2, 8, -6
```

Thus in this case, the rank is 3 and the signature is 1, so $f^{-1}(0,0)$ has 3 points, only one of which is real.

Example 5.3.4 The following is an indication of the size of the bilinear forms obtainable via TBF:

```
> F:=[2*y^2-2*y*z+z^2,y-x^2+x*y,y^2-2*y*z+z^2,9*x^2+6*x*w+w^2]:
```

```
> TBF(F,[x,y,z,w],[1,1,1,1],1);


[ 16,  0, 0,  0, 0, 16,  0, 48, 0, 0,  0,  0,-128, 0,  0, 0 ]
[                                                            ]
[  0,-16, 0,  0, 0,  0,  48, 0, 0, 0,  0, 48,  0,  0,-288,0 ]
[                                                            ]
[  0,  0, 0,  0, 0,  0,  0,  0, 0, 0,  0,  0,  0,  0,  0, 0 ]
[                                                            ]
[  0,  0, 0, 16, 0,  0,  0,  0, 0, 0, 48,  0,  0,-128, 0, 0 ]
[                                                            ]
[  0,  0, 0,  0,-16, 0,  0,  0, 0, 48, 0,  0,  0,  0, 0,-288]
[                                                            ]
```

```
[ 16,  0, 0,  0, 0, 16,  0, 48, 0, 0,  0,  0,-128, 0,  0, 0 ]
[                                                             ]
[  0, 48, 0,  0, 0,  0,-128, 0, 0, 0,  0,-128, 0,  0,448, 0 ]
[                                                             ]
[ 48,  0, 0,  0, 0, 48,  0,128, 0, 0,  0,  0,-288, 0,  0, 0 ]
[                                                             ]
[  0,  0, 0,  0, 0,  0,  0,  0, 0, 0,  0,  0,  0,  0,  0, 0 ]
[                                                             ]
[  0,  0, 0,  0, 48, 0,  0,  0, 0,-128,0,  0,  0,  0,  0,448]
[                                                             ]
[  0,  0, 0, 48, 0,  0,  0,  0, 0, 0,128,  0,  0,-288, 0, 0 ]
[                                                             ]
[  0, 48, 0,  0, 0,  0,-128, 0, 0, 0,  0, -128,0,  0, 448,0 ]
[                                                             ]
[-128, 0, 0,  0, 0, -128,0,-288,0, 0,  0,  0, 448, 0,  0, 0 ]
[                                                             ]
[  0,  0, 0,-128,0,  0,  0,  0, 0, 0,-288, 0,  0, 448, 0, 0 ]
[                                                             ]
[  0,-288,0,  0, 0,  0, 448, 0, 0, 0,  0,448,  0,  0,5632,0 ]
[                                                             ]
[  0,  0, 0, 0,-288, 0,  0,  0, 0,448, 0,  0,  0,  0, 0,5632]
```

5.4 Calculating the form TF_g

Suppose we wish to find TF_g, i.e. a form with coefficients in $f_1, \ldots, f_n$ such that $TF_g|_{f=c} = TF_g^c$ for all $c \in \mathbf{R}$. We can find this using a TBF defined on (M) as follows. Let $\{b_1, \ldots, b_m\}$ be a free basis for (M), as before. Now using (5.5), we can again write

$$(b_i b_j g) b_k = \sum_{l=1}^{m} \alpha_{k,l}(f) b_l$$

where $\alpha_{k,l} \in f^*\mathbf{R}[\mathbf{y}]$ and hence define

$$TM_g = \sum_{l=1}^{m} \alpha_{l,l}(f).$$

This is obviously equivalent to TF_g, for putting $f = c$, we obtain $\sum \alpha_{l,l}(c) = TF_g^c$ (see (5.6)). We now show that this definition is independent of the free basis chosen:

Lemma 5.4.1 *The bilinear form TM_g defined above, which has coefficients in $f^*\mathbf{R}[\mathbf{y}]$, is well-defined.*

Suppose $\{b'_1, \ldots, b'_m\}$ is another free basis for (M) and let P be the change of basis matrix such that

$$\begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix}$$

where P has entries in $f^*\mathbf{R}[\mathbf{y}]$ and is invertible. Let $A = (\alpha_{k,l})$ be the matrix representing multiplication by $b_i b_j g$ on the module (M) and let B the matrix with respect to the new basis, so $B = PAP^{-1}$. Since these are similar matrices, we will show that they have the same characteristic polynomial and hence the same trace. (Compare [C1, p336]). Now

$$zI - B = P(zI - A)P^{-1}$$

where z is some (scalar) indeterminate. Thus

$$\begin{aligned} \det(zI - B) &= \det(P(zI - A)P^{-1}) \\ &= \det P(\det(zI - A))(\det P)^{-1} \\ &= \det(zI - A). \end{aligned}$$

So A and B have the same trace, which means that TM_g is independent of the basis chosen. □

Suppose we are given a finite map f and a monomial order on $\mathbf{R}[\mathbf{x}, \mathbf{z}]$ with $x_i > z_i$ for all i. Let G be the Gröbner basis for the ideal $<f_1 - z_1, \ldots, f_n - z_n>$. We say that f is z-independent with respect to this order if $\mathrm{LT}(G)$ contains only monomials in $\mathbf{R}[\mathbf{x}]$.

The following routine, which is very similar to that in §5.3, will calculate TF_g, but only if f is z-independent with respect to the lex order $x_1 > \cdots > x_n > z_1 > \cdots > z_n$.

```
TBFZ:=proc(F,v,Z,g)
local i,j,k,l,TRIPLES,PIECES,G,m,T,B,thistriple,red,FF,Gv,\
thispiece,entry,Ans,place,comp,thismon;
  vv:=v;
  for i from 1 to nops(F) do
    vv:=[op(vv),Z[i]];
  od;
  FF:=[];
  for i from 1 to nops(F) do
    FF:=[op(FF),F[i]-Z[i]];
  od;

##### This part calculates the free module basis

  G:=grobner[gbasis](FF,vv,plex);
  Gv:=[];
  for i from 1 to nops(G) do
```

```
    thismon:=grobner[leadmon](G[i],vv,plex)[2];
    for j from 1 to nops(v) do
      if divide(thismon,Z[j])
         then ERROR(`Map not z-independent`);
      fi;
    od;
    Gv:=[op(Gv),thismon];
  od;
  B:=getmonos(Gv,v);
  m:=nops(B);

##### This part calculates all the (weighted) triples

  TRIPLES:=[];
  for i from 1 to m do
    for j from i to m do
      for k from j to m do
        TRIPLES:=[op(TRIPLES),[g*B[i]*B[j]*B[k],i,j,k]];
      od;
    od;
  od;

##### This part reduces and calculates the required components

  PIECES:=[];
  T:=nops(TRIPLES);
  for i from 1 to T do
    thistriple:=TRIPLES[i][1];
    red:=grobner[normalf](thistriple,G,v);
```

```
    thispiece:=[];
    for j from 1 to 3 do
      thispiece:=[op(thispiece),coeff_of_mon(red,\
      B[TRIPLES[i][j+1]],v)];
    od;
    PIECES:=[op(PIECES),thispiece];
  od;

##### This part works out the matrix of the TBF

  Ans:=array(sparse, 1..m,1..m);
  for i from 1 to m do
    for j from i to m do
      entry:=0;
      for k from 1 to m do
        thistriple:=[g*B[i]*B[j]*B[k],op(sort([i,j,k]))];
        for l from 1 to T do
          if thistriple=TRIPLES[l] then
            member(k,[thistriple[2..4]],'comp');
            entry:=entry+PIECES[l][comp];
          fi;
        od;
      od;
      Ans[i,j]:=entry;
      Ans[j,i]:=entry;
    od;
  od;
  if nargs=5 then assign(args[5],B);
  fi;
```

```
RETURN(evalm(Ans));
end:
```

As before, `F` is the list of components of the map and `v` is the list of variables used. `Z` is the set of variables to be used in the matrix entries and `g` is the polynomial weight. The method used is as before, except that the weighted triples are reduced modulo the ideal generated by `F[1]-Z[1]`,..., `F[n]-Z[n]`. The routines `coeff_of_mon`, `mon`, `quotbasis` and `getmonos` from Appendix B are also required.

Example 5.4.2 We now show the routine `TBFZ` in use.

```
> f:=[x^2,y^2];
                                  2   2
                         f := [x , y ]
------------------------------------------------------------
> TBFZ(f,[x,y],Z,1,'B');
                    [ 4     0        0          0        ]
                    [                                    ]
                    [ 0   4 Z[1]     0          0        ]
                    [                                    ]
                    [ 0     0      4 Z[2]       0        ]
                    [                                    ]
                    [ 0     0        0     4 Z[1] Z[2]   ]
------------------------------------------------------------
> B;
                          [1, x, y, x y]
------------------------------------------------------------
> g:=4*x*y;
```

```
                                g := 4 x y
-----------------------------------------------------------------

> TBFZ(f,[x,y],z,g,'B');
                           [  0    0    0   %1 ]
                           [                   ]
                           [  0    0   %1    0 ]
                           [                   ]
                           [  0   %1    0    0 ]
                           [                   ]
                           [ %1    0    0    0 ]

%1 :=                          16 z[1] z[2]
```

These results agree with those found in Example 5.3.2.

Chapter 6

G-variant map germs

In the second half of this thesis, we seek to generalise the results of Damon ([D]), Gusein-Zade ([GZ]) and Roberts ([R]), regarding map germs which are equivariant with respect to the action of a group.

Roberts ([R2]) considers function germs, f, which are invariant with respect to the action of a finite group G. He then defines the associated "Equivariant Milnor number". This is given by the character of the representation of the action of G on the quotient of the ring of germs by the Jacobian ideal of f. The Equivariant Milnor number at a given group element, g, is shown to be equal to the (usual) Milnor number of the restriction of f to the fixed space of g. Roberts then examines Morse approximations of f and the critical points of such an approximation. The Equivariant Milnor number of f is related to the permutation representation derived from the action of G on these critical points. He then considers the lattice of fixed spaces associated with the subgroups of G and shows that this is a stronger invariant than the Equivariant Milnor number. This is in a similar vein to his thesis, ([R]) in which he considers map-germs which are equivariant with respect to the action of a compact Lie group on the source and target spaces. The determinacy of these map germs and conditions for the germ to be stable are considered.

In [D], Damon is mainly concerned with the action of a finite group G on the branches of a curve defined by a germ $\mathbf{R}^{n+1},0 \to \mathbf{R}^n,0$. This paper also looks at G-equivariant map-germs $F : \mathbf{R}^n,0 \to \mathbf{R}^n,0$ and defines the G-degree of f as a virtual modular character, i.e. the character of an element of the representation ring over the field $\mathbf{F}_2$ of two elements. This is then related to the G-signature of the local algebra of the map.

Gusein-Zade ([GZ]) looks at real analytic germs $F : \mathbf{R}^n,0 \to \mathbf{R}^n,0$ which are G-equivariant with respect to some representation T of a finite group G. The G-equivariant degree is defined by consideration of a G-invariant quadratic form on the local algebra of the map, to give an element of the representation ring. The action of the group on the preimages of zero under a suitable perturbation of the map is considered. It is shown that the G-equivariant degree may be obtained from the permutation representation associated with this action.

At first sight, Roberts' paper might seem fundamentally different from the other two in that it concerns functions and not maps. However, the "Equivariant Milnor number" described here for an invariant function f will be the same as the G-equivariant degree described by Damon and Gusein-Zade applied to the (G-equivariant) map grad f. Note that in [R], the term "equivariant" is used even when the action of the given group is different on the source and target spaces, but in [D] and [GZ] the action is always identical on the two spaces. We have opted to use the terminology "variant" to include the possibility of the two actions being different, reserving "equivariant" for the case when they are the same. Roberts' thesis also considers equivariance under the action of a compact Lie group, whereas we will be concerned solely with finite groups. We have three principal extensions of the results of Damon and Gusein-Zade. Firstly, we give a careful treatment of complex degree for arbitrary G-variant maps. Secondly, we show that the results hold in the real case for "matched" representations and finally we obtain results when the deformation of the germ is not "regular". (Such deformations do not always exist.)

6.1 Group Representations

Before we begin discussing map germs and the results from [AVG], we shall state some definitions and results from the theory of group representations. (See [JL] for example.)

Definition A *representation* of a group G over a field K is a homomorphism

$$R : G \to GL(n, K).$$

So $R(xy) = R(x)R(y)$ and $R(1_G) = I_n$.

We refer to n as the dimension of R and we say that R is a representation on K^n, since the matrices act on this space. We sometimes ignore the distinction between a space and the pair consisting of the space and the matrices which act upon it.

Definition Two representations R and S are *equivalent* if there exists an invertible matrix T such that

$$TR(g) = S(g)T$$

for all $g \in G$.

When a group G acts on a finite set $\{x_1, \ldots, x_r\}$, it gives rise to a natural homomorphism $\phi : G \to S_r$. We define the associated permutation representation

$R(g)$ as follows. Suppose $\phi : g \mapsto \sigma \in S_r$, then $R(g)$ is the matrix with exactly one non-zero entry (of value 1) in each row and column such that we have

$$\left(\quad R(g) \quad \right) \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(r)} \end{pmatrix}$$

Definition Given any representation R of a group G, the character χ_R of R is the map

$$\chi_R : G \to K$$

defined by $\chi_R(g) = \text{ trace } (R(g))$. Characters are functions which are constant on conjugacy classes.

Proposition 6.1.1 *If two representations R and S are such that $\chi_R = \chi_S$ then R and S are equivalent.* □

We can form a sum of representations by defining

$$(R \oplus S)(g) = \begin{pmatrix} R(g) & 0 \\ 0 & S(g) \end{pmatrix}$$

and a product by

$$(R \otimes S)(g) = \begin{pmatrix} R(g)_{11}S(g) & \cdots & R(g)_{1n}S(g) \\ \vdots & \ddots & \vdots \\ R(g)_{n1}S(g) & \cdots & R(g)_{nn}S(g) \end{pmatrix}.$$

These operations can be extended to form a ring by considering the set of pairs of representations, with one considered "positive" and the other "negative", written $R - S$. We define ring operations on this set by

$$(R - S) + (R' - S') = (R \oplus R') - (S \oplus S')$$
$$(R - S)(R' - S') = ((R \otimes R') \oplus (S \otimes S')) - ((R \otimes S') \oplus (S \otimes R'))$$

and by factoring out the elements of the form $R - R$. This is called the *representation ring* of G and we define (virtual) characters on the elements by

$$\chi_{R-S}(g) = \chi_R(g) - \chi_S(g).$$

6.2 Definitions

We shall recall some results from [AVG], beginning with some terminology.

Let $a \in \mathbf{C}^n$. The ring of holomorphic functions $\mathbf{C}^n, a \to \mathbf{C}$ is denoted by $\mathcal{O}_n(a)$; when $a = 0$ we simply write $\mathcal{O}_n$. The maximal ideal of functions vanishing at a is denoted by $\mathcal{M}_n(a)$, again simplified to $\mathcal{M}_n$ when $a = 0$. If $g : \mathbf{C}^n, a \to \mathbf{C}^n, b$ is a holomorphic map then I_g denotes the ideal in $\mathcal{O}_n(a)$ generated by the pullback of $\mathcal{M}_n(a)$ by g. The local algebra of g denoted by Q_g or $Q_g(a)$ is the quotient $\mathcal{O}_n(a)/I_g$.

Let U be an open set in $\mathbf{C}^n$. Then $A(U)$ denotes the algebra of holomorphic functions on U. If $g : U \to \mathbf{C}^n$ is holomorphic, then we denote by $I_g(U)$ the ideal of functions generated by the components of g. The quotient algebra $Q_g(U) = A(U)/I_g(U)$ is the algebra of the map g on the domain U. The polynomial subalgebra $Q_g[U]$ of the map g on the domain U is the image of the subalgebra of polynomials of $A(U)$ in the algebra $Q_g(U)$.

Definition A map-germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ is said to be finite if f is analytic and locally the inverse image of 0 is simply 0.

Lemma 6.2.1 (See [B, ch13], for example) *For an analytic map-germ f, the following are equivalent:*

1. *f is finite.*
2. *The local algebra Q_f is finite dimensional as a $\mathbf{C}$-vector space.*

3. The ideal I_f contains some power $\mathcal{M}_n^k$ of the maximal ideal.

Proof If f is finite, then the ideal $I_f = \mathcal{O}_n <f_1, \dots, f_n>$ determines the set $\{0\}$, so by the local Nullstellensatz, $I_f \supset \mathcal{M}_n^k$ for some k. So $\dim_{\mathbf{c}} Q_f \leq \dim_{\mathbf{c}}(\mathcal{O}_n/\mathcal{M}_n^k) < \infty$ and we have proved $1 \Rightarrow 2$.

If $\dim_{\mathbf{c}} Q_f < \infty$, consider the inclusions

$$I_f + \mathcal{M}_n^k \supset I_f + \mathcal{M}_n^{k+1} \supset \cdots$$

Since

$$\infty > \dim Q_f = \dim \frac{\mathcal{O}_n}{I_f} \geq \dim \frac{\mathcal{O}_n}{I_f + \mathcal{M}_n} \geq \cdots$$
$$\cdots \geq \dim \frac{\mathcal{O}_n}{I_f + \mathcal{M}_n^k} \geq \dim \frac{\mathcal{O}_n}{I_f + \mathcal{M}_n^{k+1}} \geq \cdots$$

this list of inclusions cannot be strict. So

$$I_f + \mathcal{M}_n^k = I_f + \mathcal{M}_n^{k+1}$$

for some k. Applying Nakayama's lemma shows that $I_f \subset \mathcal{M}_n^k$ and so $2 \Rightarrow 3$. In particular, $x_j^k \in I_f$ for each j, so $f^{-1}(0) = 0$ and thus $3 \Rightarrow 1$. □

We shall consider a deformation of a finite map-germ f, that is a mapping $F : \mathbf{C}^n \times \mathbf{C}^k, (0,0) \to \mathbf{C}^n, 0$ with $F(x, 0) = f(x)$. Given $t \in \mathbf{C}^k$ we denote $F(-, t)$ by F_t. We shall label ideals and algebras with a subscript t rather than F_t.

Now let U be a sufficiently small neighbourhood of 0 in $\mathbf{C}^n$. If $F_t^{-1}(0) \subset U$ is a finite set of points $\{a_1, \dots, a_r\}$ then the multilocal algebra of F_t, denoted by $\Lambda_t(U)$, is defined to be $\bigoplus Q_t(a_i)$ where $Q_t(a_i)$ is the local algebra of $F_t : \mathbf{C}^n, a_i \to \mathbf{C}^n, 0$.

We can now state the results we shall need from Arnold:

Proposition 6.2.2 ([AVG, p99]) *Let f as above be finite with deformation F_t and let L be a $\mathbf{C}$-linear space, spanned by functions $e_1, \dots, e_\mu$ whose germs at* *zero are a basis for Q_f. There is a neighbourhood of 0 in $\mathbf{C}^n$, say U, and a neighbourhood of 0 in $\mathbf{C}^k$, say V such that for any $t \in V$ the following holds.*

(i) The natural projections $\pi_t : L \to \Lambda_t(U)$ are isomorphisms of linear spaces.

(ii) Each polynomial P in the algebra $A(U)$ is equivalent modulo the ideal $I_t(U)$ to a unique element of the space L and this element depends analytically on t. □

We also need the following

Lemma 6.2.3 ([AVG, p98]) *Let $p_t : \mathbf{C}[x_1, \ldots, x_n] \to \Lambda_t(U)$ denote the natural projections. Then*

(i) p_t is in fact a surjection.

(ii) The kernel of p_t is $I_t(U) \cap \mathbf{C}[x_1, \ldots, x_n]$.

Proof The proof of (i) is to be found in [AVG]. For (ii), let P be a polynomial with $p_t(P) = 0$. We know that P is equivalent modulo the ideal $I_t(U)$ to a unique element h in L. But $\pi_t(h) = p_t(P) = 0$ and so since π_t is an isomorphism we deduce that $h = 0$, in other words $P \in I_t(U)$. If, conversely we are given a polynomial $P \in I_t(U)$, then we find $p_t(P) = 0$ immediately. □

We now introduce the action of a group G, which we will assume throughout to be finite. We will also assume that G acts on both source and target, via representations R_S and R_T respectively. We will call these representations *matched* when $\det(R_S(g)) = \det(R_T(g))$ for all $g \in G$.

Definition A G-variant map is a map-germ f which 'commutes' with the group, in other words

$$f(R_S(g)(x)) = R_T(g)(f(x)).$$

for all x and all $g \in G$.

We will often dispense with the explicit reference to the representations and just write the action of a group element g as $g \cdot x$. In this notation, the above condition for G-variance would be written as

$$f(g \cdot x) = g \cdot f(x)$$

but note that the dot on either side of the equality can mean quite different things. The class of G-variant maps includes both G-invariant and G-equivariant maps. The former is obtained by taking R_T to be the trivial representation, while the latter by setting $R_S = R_T$. In general, we will be interested in finite G-variant maps.

Before the next proposition, we need the following lemma:

Lemma 6.2.4 ([JL]) *Let V be a finite dimensional G-vector space (so G acts on it via a representation) with $U \subset V$ a G-invariant subspace. Then U has a G-invariant complement $U^\perp$ such that $V = U \oplus U^\perp$.*

Proof Suppose V has $v_1, \ldots v_n$ as a basis. Then define two complex inner products (non-degenerate Hermitian forms) $(,)$ and $[,]$ as follows:

$$\begin{aligned} \left(\sum_{i=1}^{n} \lambda_i v_i, \sum_{j=1}^{n} \mu_j v_j \right) &= \sum_{i=1}^{n} \lambda_i \overline{\mu_i} \\ [u, v] &= \sum_{g \in G} (g \cdot u, g \cdot v) \end{aligned}$$

Now the second of these is also G-invariant, since

$$\begin{aligned} [g' \cdot u, g' \cdot v] &= \sum_{g \in G} ((gg') \cdot u, (gg') \cdot v) \\ &= \sum_{g \in G} (g \cdot u, g \cdot v) \\ &= [u, v] \end{aligned}$$

Now taking

$$U^\perp = \{v \in V : [u, v] = 0 \text{ for all } u \in U\}$$

we find that this is indeed G-invariant for

$$\begin{aligned} & [u,v] = 0 \text{ for all } u \in U \\ \Rightarrow \quad & [g\cdot u, g\cdot v] = 0 \text{ for all } u \in U \\ \Rightarrow \quad & [u, g\cdot v] = 0 \text{ for all } u \in U \end{aligned}$$

and so $v \in U^\perp$ implies $g \cdot v \in U^\perp$. Finally, the fact that $V = U \oplus U^\perp$ is a standard property of complex inner products (see [C1] for example). □

Note that the components $f_1, \dots, f_n$ of a G-variant map f are such that the ideal they generate, I_f, is G-invariant. For we have for all $g \in G$,

$$\begin{pmatrix} f_1(g\cdot x) \\ \vdots \\ f_n(g\cdot x) \end{pmatrix} = \begin{pmatrix} & & \\ & R_T(g) & \\ & & \end{pmatrix} \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$$

So after applying an element of the group, we simply have linear combinations of the original elements. Since the matrix $R_T(g)$ is invertible, these new elements will generate the same ideal.

Proposition 6.2.5 *(i) If $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ is a finite G-variant map-germ, and it has a representative (also denoted by f) defined on a neighbourhood U, then we can choose a G-invariant neighbourhood $V \subset U$ of 0 (on which of course f is defined).*

(ii) Let $f_1, \dots, f_n$ be germs of functions $\mathbf{C}^n, 0 \to \mathbf{C}$ with the property that the ideal they generate $\mathcal{O}_n\langle f_1, \dots, f_n\rangle = I_f$ is G-invariant, and of finite codimension. Then we can find a complement to I_f in $\mathcal{O}_n$ of minimal dimension which is also G-invariant.

Proof (i) We simply let V denote the intersection of the open sets $g \cdot U$ as g varies over all elements of G.

(ii) Since I_f is of finite codimension it contains some power of the maximal ideal $\mathcal{M}_n^{k+1}$ (see Lemma 6.2.1). As a consequence it is enough to work in the finite-dimensional quotient space $\mathcal{O}_n/\mathcal{M}_n^{k+1}$ of polynomials of degree at most k. (Since G is acting linearly the group action preserves the filtration by degree.) We now apply Lemma 6.2.4 to $I_f/\mathcal{M}_n^{k+1}$ as a subspace of $\mathcal{O}_n/\mathcal{M}_n^{k+1}$. We will denote its G-invariant complement by L. □

Definition Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be finite and G-variant, so that I_f is invariant. Since G acts linearly on the ring $\mathcal{O}_n$ as a vector space, we see that it must also act linearly on the local algebra Q_f. The action of $g \in G$ on Q_f as a vector space can therefore be expressed as a matrix. We thus have a representation of G on Q_f, which we will call the *G-variant degree* of the complex map f.

Of course, since L (as in the proof of 6.2.5) is G-invariant there is a natural action of G on L. Suppose now that F is a finite G-variant deformation of f, so $F_t(g \cdot x) = g \cdot (F_t(x))$ for all $x \in U$ and t near $0 \in \mathbf{R}^k$. Our next task is to identify the group's action on the algebras $\Lambda_t(U)$. First note that the natural projections $p_t : A(U) \to \Lambda_t(U)$ are surjections. We have an action of the group G on $A(U)$ (recall that U is G-invariant).

Proposition 6.2.6 *(i) The group action on $A(U)$ gives a well defined group action on $\Lambda_t(U)$.*

(ii) If 0 is a regular value of F_t then $F_t^{-1}(0)$ consists of μ points where $\mu = \dim Q_f$. Also, $\Lambda_t(U)$ is isomorphic to the direct sum of μ copies of $\mathbf{C}$, one for each point of $F_t^{-1}(0)$. In this case the group action on $\Lambda_t(U)$ yields the permutation representation of G on the points of $F_t^{-1}(0)$.

Proof (i) If $F_t^{-1}(0) = \{a_1, \ldots, a_r\}$ then let $h^* = (h_1, \ldots, h_r) \in \bigoplus Q_t(a_i)$. We know that there is a polynomial function $h \in A(U)$ which projects to h^*. So we define $g \cdot h^*$ to be $p_t(g \cdot h)$. We need to show that this is well-defined; once we have established that it will be clear that we have a group action. So let H be another polynomial function with $p_t(h) = p_t(H) = h^*$. It follows from Lemma 6.2.3 that $h - H$ lies in the ideal $I_t(U)$. But $I_t(U)$ is G-invariant, so $g \cdot H - g \cdot h \in I_t$ and consequently $p_t(g \cdot H) = p_t(g \cdot h)$ as required.

(ii) If a is a regular point of F_t then the local algebra $Q_t(a)$ is simply $\mathbf{C}$, and the first result follows. It is also clear that the map $A(U) \to \Lambda_t(U)$ simply evaluates the function at the μ points. The rest is now straightforward. □

Now we can mimic the proof given by Arnold et al [AVG, p99].

Proposition 6.2.7 *Let f as above be finite and G-variant. There is a G-invariant neighbourhood of 0 in $\mathbf{C}^n$, say U, and a neighbourhood of 0 in $\mathbf{C}^k$, say V, such that for any $t \in V$, the natural projections $\pi_t : L \to \Lambda_t(U)$ are G-isomorphisms of linear spaces.*

Proof We wish to apply Proposition 6.2.2 above (by Proposition 6.2.5 we can choose U to be a G-invariant neighbourhood of 0 in $\mathbf{C}^n$). We start by choosing a G-invariant complement L to I_f in $\mathcal{O}_n$. This shows that for all $t \in V$ the map $\pi_t : L \to \Lambda_t(U)$ is an isomorphism. We need to show that it preserves the group action, in other words that $\pi_t(g \cdot h) = g \cdot (\pi_t(h))$ for all $g \in G$ and $h \in L$. But this is immediate. □

Corollary 6.2.8 *Suppose that with the given deformation of f we have 0 a regular value of F_t for some t in V. Then the permutation representation of G on the set of points $F_t^{-1}(0)$ is equivalent to the representation of the action of G on the local algebra Q_f.*

Proof The algebra $\Lambda_0(U)$ is G-isomorphic to the local algebra Q_f. On the other hand $\Lambda_t(U)$ is isomorphic to μ copies of $\mathbf{C}$ (the local algebra of a regular point is simply $\mathbf{C}$). The group action is then simply given by the permutation of these μ points. □

6.3 Bilinear forms

We start with an easy Lemma. Let G be a finite group acting on our source and target spaces via matched representations and let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be a finite G-variant map-germ.

Lemma 6.3.1 *The Jacobian determinant of f, denoted by J, is G-invariant.*

Proof We consider the representations R_S and R_T as linear maps $\mathbf{C}^n \to \mathbf{C}^n$. Since f is G-variant, we have $R_T(g)(f(x)) = f(R_S(g)(x))$. Taking derivatives and determinants we find that

$$(\det R_T(g))(\det df(x)) = (\det df(R_S(g)(x)))(\det R_S(g)),$$

and so $\det df(g \cdot x) = \det df(x)$ as required. □

Suppose now that $\alpha : Q_f \to \mathbf{C}$ is linear and G-invariant, in the sense that $\alpha(g \cdot h) = \alpha(h)$ for all $h \in Q_f$ and $g \in G$. We define a bilinear form B_α (sometimes just denoted by B) on Q_f by

$$B_\alpha(h_1, h_2) = \alpha(h_1 \cdot h_2).$$

Proposition 6.3.2 *The bilinear form B_α is G-invariant and is non-degenerate if and only if $\alpha(J) \neq 0$.*

Proof We have only to prove that the bilinear form is G-invariant; the other result follows from the classical case ([AVG, p100]). But $\alpha((g.h_1) \cdot (g.h_2)) = \alpha(g.(h_1 \cdot h_2)) = \alpha(h_1 \cdot h_2)$ whence the result. □

By [R2, Prop 3.3] we have the following result:

Proposition 6.3.3 *Let V be a complex representation of the finite group G. Then the following are equivalent:*

- *There is a G-invariant non-degenerate quadratic form defined on V.*
- *V is a real representation.* □

A complex representation is called *real* if it is the complexification of a representation on a real vector space. We have shown that there is a G-invariant non-degenerate quadratic form on Q_f, derived from the linear form α which takes the value 1 on J and 0 elsewhere. This means that in our case the representation of the action of G on Q_f is in essence just a representation on a real space. Here is an example to show how starting with an action of G on $\mathbf{C}^n$ given by a complex, non-real representation, we obtain a real representation on Q_f.

Example 6.3.4 Let $G = \mathbf{Z}_4$, generated by g, act on $\mathbf{C}$ by $[(i)]$, where we express a representation R by giving $R(g)$ enclosed in square brackets. The map $f : \mathbf{C} \to \mathbf{C}$ given by $x \mapsto x^5$ is finite and G-equivariant. Now Q_f is isomorphic to $\mathbf{C}$-span $\{1, x, x^2, x^3, x^4\}$ with G acting by

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right].$$

But this is in fact a real representation, for if we rewrite the basis for Q_f as $\{1, x^2, x^4, x - x^3, ix + ix^3\}$, we obtain the representation

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}\right].$$

On the other hand, see Example 7.1.1 in which non-matched representations give a representation on Q_f which is not real.

Remark Clearly there are plenty of invariant linear forms α with $\alpha(J) \neq 0$. Indeed we can proceed as follows. In the construction of the G-invariant complement to the ideal I_f above we can first choose a G-invariant complement L_1 to the sum of I_f and the invariant functions. We may then extend this by adding in a relevant subspace L_2 of the invariant functions; we have shown that J lies in the latter. Choose a basis for L_2 containing J. We now define α to be 1 on J, 0 on the other basis vectors of L_2, and 0 on L_1.

We wish to push the constructions of Arnold et al a little further, even in the classical case. Using the standard notation above we shall start with a linear form $\alpha : Q_f(U) \to \mathbf{C}$ with $\alpha(J) \neq 0$. In fact assuming that f is polynomial we need only consider a linear form $\alpha : \mathbf{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle \to \mathbf{C}$ with $\alpha(J) \neq 0$ (it is not hard to show that one can change co-ordinates so that f is polynomial and this quotient is finite dimensional). See [W], Wall's survey paper.

Proposition 6.3.5 *Let f , U and V be as before. For each $t \in V$ we define a bilinear form B_t on $\Lambda_t(U)$ as follows. Given $h_1,\ h_2 \in \Lambda_t(U)$ choose polynomials $H_1,\ H_2 \in A(U)$ with $p_t(H_i) = h_i$. Define $B_t(h_1, h_2)$ by considering their product in $Q_f(U)$ and then applying α.*

(i) The bilinear forms B_t are well defined and non-degenerate.

(ii) If $F_t^{-1}(0) = \{a_1, \dots, a_r\}$ then α determines a linear map $\alpha_i : \mathcal{O}(a_i) \to \mathbf{C}$. and these in turn determine a bilinear form $B(a_i)$ on the local algebras $Q_t(a_i)$. The form B_t is the direct sum of the $B(a_i)$.

Proof (i) First, we show that α gives a well-defined linear form, α_t on $\Lambda_t(U)$. Suppose $h = p_t(H) = p_t(H')$, then $H - H' \in I_f(U)$, so $\alpha(H) = \alpha(H')$. This means that $\alpha_t(h)$ is well defined. Now since α_t is well defined, this means that the bilinear form it defines, B_t is also well-defined. The form B_t is non-degenerate since B_0 is non-degenerate. (See the note below.)

(ii) If $p_t(H) = (h_1, \dots, h_r)$ and $p_t(H') = (h'_1, \dots, h'_r)$, then $p_t(H \cdot H') = (h_1 \cdot h'_1, \dots, h_r \cdot h'_r)$. So p_t carries the algebra structure of $A(U)$ into $\Lambda_t(U)$, with pointwise multiplication. Now $\alpha_i(h_i) = \alpha_t((0, \dots, 0, h_i, 0, \dots, 0))$ by definition. Thus

$$\alpha_t((h_1, \dots, h_r)) = \sum_{i=1}^{r} \alpha_i(h_i)$$

and so

$$\begin{aligned} B_t(h, h') &= \alpha(h \cdot h') \\ &= \alpha_t((h_1 \cdot h'_1, \dots, h_r \cdot h'_r)) \\ &= \sum_{i=1}^{r} \alpha_i(h_i \cdot h'_i) \\ &= \sum_{i=1}^{r} B(a_i)(h_i, h'_i). \end{aligned}$$

□

Remark One can prove the non-degeneracy of the bilinear forms in a *slightly* different way to that in Arnold et al (it still uses Hartog's Theorem). Using the result above we obtain a family of bilinear forms B_t on the space L the entries of which depend analytically on t. We can prove non-degeneracy by induction on $\mu = \dim Q_f$ as follows. Consider the set X of values of t with $F_t^{-1}(0)$ consisting of a single point. It turns out that we can choose a family F_t with the property that this has codimension > 2 unless f is a fold. The result in the case of the fold is trivial anyway. So we may assume that the bilinear forms B_t have maximal rank off this set by induction using (ii) above. Since they have maximal rank off a set of codimension 2 by Hartog's Theorem they have maximal rank everywhere. (The inverse matrices B_t^{-1} exist off this set, and the entries are holomorphic in t; by Hartog's Theorem they extend to holomorphic functions for all values of t. Since $B_t \cdot B_t^{-1} = I$ holds on a dense set it holds everywhere.)

Now suppose that we have a group G acting. We suppose once more that f is polynomial. Again, it is not difficult to find a G-invariant linear form $\alpha : \mathbf{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n\rangle \to \mathbf{C}$ which is non-zero on J as required. The same proof as above then shows that the following holds:

Proposition 6.3.6 *Let f , U and V be as above. For each $t \in V$ we define a bilinear form B_t on $\Lambda_t(U)$ as follows. Given h_1, $h_2 \in \Lambda_t(U)$ choose polynomials H_1, $H_2 \in A(U)$ with $p_t(H_i) = h_i$. Define $B_t(h_1, h_2)$ by considering the product in $Q_f(U)$ and then applying α.*

(i) The bilinear forms B_t are well defined G-invariant and non-degenerate.

(ii) If $F_t^{-1}(0) = \{a_1, \ldots, a_r\}$ then α determines a linear map $\alpha_i : \mathcal{O}(a_i) \to \mathbf{C}$, and these in turn determine a G-invariant bilinear form $B(a_i)$ on the local algebras $Q_t(a_i)$. The form B_t is the direct sum of the $B(a_i)$.

Proof This is now trivial. □

6.4 The real case

An analytic map-germ $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is said to be finite if its complexification $f_{\mathbf{c}} : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ is finite. Let $\mathcal{O}_n$ now denote the ring of analytic functions $\mathbf{R}^n, 0 \to \mathbf{R}$. We define Q_f to be $\mathcal{O}_n/I_f$ similarly to the complex case. If we write $Q_f^{\mathbf{R}}$ and $Q_f^{\mathbf{C}}$ for the real and complex local algebras respectively, then one can check that $Q_f^{\mathbf{C}} \cong Q_f^{\mathbf{R}} \otimes \mathbf{C}$ and thus f is finite if and only if $Q_f^{\mathbf{R}}$ is finite dimensional as a $\mathbf{R}$-space. Again, if $\alpha : Q_f \to \mathbf{R}$ is a linear form, define B_α on Q_f by $B_\alpha(h_1, h_2) = \alpha(h_1 \cdot h_2)$.

A complex function, ϕ on a set with involution τ is said to be τ-real if

$$\phi(\tau(a)) = \overline{\phi(a)}.$$

(NB A polynomial with real coefficients is τ-real when τ is the involution of complex conjugation.) The τ-real functions on a set of μ points $\{a_1, \ldots, a_\mu\}$ form an $\mathbf{R}$-algebra R of $\mathbf{R}$ dimension μ. For each function $\phi \in R$ (with $\phi(a_i) \neq 0$ for all a_i) we may define a bilinear form B_ϕ on R by

$$B_\phi(h_1, h_2) = \sum_{i=1}^{\mu} \phi(a_i) h_1(a_i) h_2(a_i).$$

We have the following proposition from Arnold et al.

Proposition 6.4.1 ([AVG, p103]) *Given R and B_ϕ as above,*

(i) the values of the form B_ϕ are real.

(ii) B_ϕ is non-degenerate.

(iii) The signature of the form B_ϕ is $\phi^+ - \phi^-$ where ϕ^+ is the number of fixed points of the involution on which $\phi > 0$ and ϕ^- is the number on which $\phi < 0$.

Proof Under τ, the set decomposes into invariant 1 and 2 point subsets. It is therefore sufficient to prove the proposition for these sets.

One Point Case: The τ-real functions at a point fixed under τ are isomorphic to a single copy of $\mathbf{R}$ and so have $\mathbf{R}$ dimension 1. This proves (i) and (ii) in this case. The signature is simply $+1$, -1 or 0 depending on the sign of the function at the point. This proves (iii).

Two Point Case: In this case, we have distinct points a_1, a_2 such that $\tau(a_1) = a_2$ and thus if ϕ is a τ-real function on the pair, $\phi(a_1) = \overline{\phi(a_2)}$. So by determining the value of a function on one point, it is determined on both. Thus the functions have $\mathbf{R}$ dimension 2. Now

$$\begin{aligned} B_\phi(h_1, h_2) &= \phi(a_1)h_1(a_1)h_2(a_1) + \phi(a_2)h_1(a_2)h_2(a_2) \\ &= \phi(a_1)h_1(a_1)h_2(a_1) + \overline{\phi(a_1)h_1(a_1)h_2(a_1)} \\ &\in \mathbf{R} \end{aligned}$$

so we have shown (i) in this case. Suppose $\phi(a_1) = \phi_1 + i\phi_2$ and $h(a_1) = h_1 + ih_2$ where $\phi_1, \phi_2, h_1, h_2 \in \mathbf{R}$, then

$$\begin{aligned} B_\phi(h, h) &= \phi(a_1)h(a_1)^2 + \overline{\phi(a_1)h(a_1)^2} \\ &= 2(\phi_1 h_1^2 - 2\phi_2 h_1 h_2 - \phi_1 h_2^2) \\ &= \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 2\phi_1 & -2\phi_2 \\ -2\phi_2 & -2\phi_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{aligned}$$

Now the matrix $\begin{pmatrix} 2\phi_1 & -2\phi_2 \\ -2\phi_2 & -2\phi_1 \end{pmatrix}$ has rank 2 and signature 0 for any $(\phi_1, \phi_2) \in \mathbf{R}^2 \backslash \{(0,0)\}$, since there exists a change of basis to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This completes the proof. □

When we have a group G acting on a real vector space, V, with a G-invariant bilinear form B, we define the G-signature of B by

$$\mathrm{sig}_G(B) = V^+ - V^-$$

in the ring of representations of G. Here V^+ and V^- are the G-spaces where B is positive and negative definite respectively.

Now, let G be a group acting via matched representations on source and target spaces isomorphic to $\mathbf{R}^n$. Let f_t be a finite G-variant deformation of f such that 0 is a regular value. Let $f_t^{-1}(0) = \{a_1, \dots, a_\mu\} \in \mathbf{C}^n$. Take τ to be the involution of complex conjugation acting on the roots. As before, we let R be the $\mathbf{R}$-algebra of τ-real functions on the points a_i. Now G acts on R in the obvious way, by simply permuting the points. For a G-invariant $\phi \in R$, with $\phi(a_i) \neq 0$ for all a_i, define a bilinear form B_ϕ on R as before,

$$B_\phi(h_1, h_2) = \sum_{i=1}^{\mu} \phi(a_i) h_1(a_i) h_2(a_i).$$

Now we may decompose $\{a_i\}$ as follows

$$\begin{aligned}
A_1 &= \{a_i : \tau(a_i) = a_i\} \\
A_2 &= \{a_i : \tau(a_i) \neq a_i, \tau(a_i) = g \cdot a_i \text{ some } g \in G\} \\
A_3 &= \{a_i : \tau(a_i) \neq a_i, \tau(a_i) \neq g \cdot a_i \text{ any } g \in G\}.
\end{aligned}$$

This leads to a decomposition of R as

$$R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$$

where

$$\begin{aligned}
R_1 &= \{\tau\text{- real functions } h \text{ on } A_1\} \\
R_2 &= \{\tau\text{- real functions } h \text{ on } A_2 : h(a_i) = h(\tau(a_i)) \in \mathbf{R}\} \\
R_3 &= \{\tau\text{- real functions } h \text{ on } A_2 : h(a_i) = -h(\tau(a_i)) \in i\mathbf{R}\} \\
R_4 &= \{\tau\text{- real functions } h \text{ on } A_3\}.
\end{aligned}$$

Then we have the following analogue of the above proposition:

Proposition 6.4.2 *With G, $\{a_i\}$, τ, R as above, Then*

$$sig_G B_\phi = (R_1^+ + R_2^+ + R_3^-) - (R_1^- + R_2^- + R_3^+)$$

in the representation ring of G, where the '+' and '−' superscripts denote the G-spaces where $\phi > 0$ and $\phi < 0$ respectively.

Proof We consider each of the spaces R_1, R_2, R_3 and R_4:

R_1: The contribution here follows as in the original case, since the matrix for B_ϕ here simply has the values of ϕ at the given point a_i down the diagonal. Thus this contributes $R_1^+ - R_1^-$ to the G-signature.

R_2 and R_3: Consider a pair of points $a_i, a_j \in A_2$ such that $\tau(a_i) = a_j$ and $\phi(a_i) = \phi(a_j) = \lambda \in \mathbf{R}$. The matrix for B_ϕ here is of the form $\begin{pmatrix} 2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}$, where the first basis vector corresponds to real functions on the two points and the second to purely imaginary ones. (Compare the proof of the original proposition.) Thus when $\lambda > 0$, the real functions contribute positively to the G-signature and the imaginary ones negatively. When $\lambda < 0$, the opposite occurs. Thus, overall, $(R_2^+ + R_3^-) - (R_2^- + R_3^+)$ is contributed to the G-signature. Note that the actions of G on R_2 and R_3 are in general different. A real function will be unaffected by a group element which has the same effect as τ, whereas a purely imaginary function will be multiplied by -1.

R_4: Suppose we consider an orbit in in R_4, where $\phi = \lambda + i\mu$. This will come with a 'τ mirror-image' orbit where $\phi = \lambda - i\mu$, since τ commutes with G. The matrix of B_ϕ on these two orbits will look like

$$\begin{pmatrix} \begin{matrix} 2\lambda & -2\mu \\ -2\mu & -2\lambda \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 2\lambda & -2\mu \\ -2\mu & -2\lambda \end{matrix} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \begin{matrix} 2\lambda & -2\mu \\ -2\mu & -2\lambda \end{matrix} \end{pmatrix}$$

where G acts by permuting the pairs of basis vectors. Now by an identical change of basis on each pair of vectors, we obtain

$$\begin{pmatrix} \begin{matrix}1 & 0\\ 0 & -1\end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix}1 & 0\\ 0 & -1\end{matrix} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \begin{matrix}1 & 0\\ 0 & -1\end{matrix} \end{pmatrix}$$

with G acting as before. This is the same as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}$$

with an identical G action on each summand. Thus there is no contribution to the G signature. □

Corollary 6.4.3 *If G is of odd order then the above result reduces to*

$$sig_G B_\phi = R_1^+ - R_1^-$$

which is analogous to the complex case, in the sense that this is the action of G on the preimages of zero counted up to the sign of ϕ.

Proof Since G is of odd order, the points a_i must also form orbits of odd order (since their order must divide $|G|$). Suppose A_2 is non-empty for otherwise the result is trivial. Choose an orbit in A_2. We know that if a_i lies in the orbit, then so does $\tau(a_i)$. Thus the orbit has an even number of elements. This is a contradiction, so we must have $A_2 = \emptyset$. □

In the case above, the G-signature is given by the permutation representation of G on the real roots of $f_t = 0$ counted with the sign of ϕ at that point. Here R_1^+ is given by the action of G on τ-real functions on A_1 (real roots of $f_t = 0$) where $\phi > 0$. This corresponds to just the action on the points. A similar result holds for R_1^-. Note that we need matched representations to have an action on $\{f_t = 0\} \cap \{\phi > 0\}$.

Definition We will call $R_1^+ - R_1^-$, with $\phi = 1/J$, the *G-index* of f at the point 0. Note that it will be dependent of the choice of deformation of f.

Now let $e_1, \ldots, e_\mu$ be a $\mathbf{R}$-basis for $L_\mathbf{R}$, which is a G-invariant complement to I_f in $\mathcal{O}_n$ as in the complex case. Note that this will also form a $\mathbf{C}$-basis for L, defined as previously. There is a natural map $\pi_R : L_\mathbf{R} \to R$ given by evaluation at the μ points. Now obviously $\pi_R = \pi_t \mid_{L_R}$ and since π_t is an isomorphism, this means that $L_\mathbf{R}$ is isomorphic to some subspace of R. But both these spaces have $\mathbf{R}$ dimension μ, so $L_\mathbf{R} \cong R$ via π_R. Since the group action is real, this also carries over from the complex case, so $L_\mathbf{R}$ and R are in fact isomorphic as G-spaces.

Now let us define a bilinear form B^t on $L_\mathbf{R}$ by

$$B^t(h_1, h_2) = \sum_{i=1}^{\mu} \frac{1}{J(a_i)} h_1(a_i) h_2(a_i).$$

By the G-isomorphism described above, this form has G-signature equal to that of $\mathrm{sig}_G B_\phi$ in Proposition 6.4.2 above, with $\phi = 1/J$. Let us denote this element of the representation ring by σ. Note that J is G-invariant by Lemma 6.3.1 and hence $1/J$ is a G-invariant function on the points a_i.

To continue further, we require the following results from Arnold et al.

Proposition 6.4.4 *For any holomorphic function h at 0, set*

$$l^t(h) = \sum h(a_i)/J(a_i)$$

for the points $\{a_1, \ldots a_\mu\} = f_t^{-1}(0)$. Then:

(i) As $t \to 0$, $l^t(h)$ tends to a finite value denoted $[h/f]$.

(ii) The linear form $\alpha_0(-) = [-/f]$ is zero on the ideal I_f and thus determines a linear form on the local algebra Q_f.

(iii) The bilinear form $B = B_{\alpha_0}$ on the local algebra, (constructed from the linear form α_0 in the usual way) is non-degenerate. □

Using this result, we find that as $t \to 0$, $B^t \to B$ on the local algebra, corresponding to the special linear form α_0. Since B is non-degenerate, it must also have G-signature equal to σ. Now join α_0 to an arbitrary α, which also has $\alpha(J) > 0$, by a line segment. Since all the points along this line correspond to non-degenerate forms, they all have G-signatures equal to σ. Thus we have proved

Proposition 6.4.5 *The G-signature of a bilinear form B_α on Q_f, derived from a linear form $\alpha : Q_f \to \mathbf{R}$ with $\alpha(J) > 0$ is independent of the choice of α.* □

We will call this G-signature the *G-variant degree* of the real map f. Notice that it is in general different from the G-variant degree of f considered as a complex map. If we take Corollary 6.4.3 into account, then we obtain

Corollary 6.4.6 *For G of odd order, the G-index of f at the point 0 is independent of the deformation chosen and the G-signature of a bilinear form B_α on Q_f, derived from a linear form $\alpha : Q_f \to \mathbf{R}$ with $\alpha(J) > 0$ is equal to this G-index.* □

6.5 Examples

Example 6.5.1 Let the group $\mathbf{Z}_2$ act on the source space $\mathbf{R}^2$ by the representation

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

and on the target space $\mathbf{R}^2$ by

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Then the map $f = (x^2 - y^2, x^2 + y^2) : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ is finite and G-variant with respect to these actions. Note that both these matrices have determinant -1, so they match. The local algebra Q_f is given by the $\mathbf{R}$-span of $\{1, x, y, xy\}$ and the action of the group (inherited from the action on the source) is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right].$$

Now the Jacobian determinant of f is $8xy$, so let us define $\alpha : Q_f \longrightarrow \mathbf{R}$ by $\alpha(xy) = 1$, $\alpha = 0$ elsewhere. We diagonalize the associated bilinear form B_α by taking $\{1 - xy, 1 + xy, x + y, x - y\}$ as a basis for Q_f. The quadratic form associated to B_α is positive on the second and third basis elements and negative on the first and fourth. The action of the group with respect to this new basis is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right].$$

and thus we obtain

$$\begin{aligned} \mathrm{sig}_G B_\alpha &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &= [(1)] - [(-1)]. \end{aligned}$$

Here is a simple example where the representations are not matched.

Example 6.5.2 The group $\mathbf{Z}_2$ acts on the source space $\mathbf{R}$ by $[(-1)]$ and on the target space $\mathbf{R}$ by $[(1)]$. Then $f = x^2 : \mathbf{R}, 0 \to \mathbf{R}, 0$ is finite and G-variant with respect to these actions. However, the determinants of the representations do not match. The local algebra Q_f is simply the $\mathbf{R}$-span of $\{1, x\}$ and the action of the group is given by

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

Now the spaces spanned by 1 and x respectively are G-invariant and this is the only possible decomposition of Q_f into such spaces, by consideration of eigenvalues. We define the linear form $\alpha : Q_f \to \mathbf{R}$ by $\alpha(x) = 1$, $\alpha(1) = 0$, since the Jacobian determinant of f is $2x$. The related bilinear form, B_α has the following matrix on this basis:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can see that B_α is not G-invariant and thus we cannot calculate $\mathrm{sig}_G B_\alpha$. So having matched representations is necessary when we are considering bilinear forms.

Consider the complexification of the above, with $\mathbf{Z}_2$ acting on $\mathbf{C}$ by $[(-1)]$ and $[(1)]$ on the two spaces and $f = x^2 : \mathbf{R}, 0 \to \mathbf{R}, 0$ as the G-variant map in question. The local algebra is given by $\mathbf{C}$-span $\{1, x\}$ and G acts on it exactly as in the real case. Let us take $f_t = x^2 - t^2$ as a finite G-variant deformation of f. This gives $f_t^{-1}(0) = \{t, -t\}$ and G acts on the τ real functions via the representation

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

which is equivalent to the representation above. Thus although we cannot calculate the real G-variant degree since the representations are not matched, we can calculate the complex G-variant degree as this does not require matched representations.

Example 6.5.3 Let $G = \mathbf{Z}_4$ with generator g and let G act on $\mathbf{R}^4$ under the standard permutation representation. We will, as before, denote a representation R by giving $R(g)$ enclosed in square brackets, so this representation is given by

$$\left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Let $f = (x_1^2, x_2^2, x_3^2, x_4^2) : \mathbf{R}^4, 0 \to \mathbf{R}^4, 0$, which is finite and G-equivariant. Consider the following finite G-equivariant deformations of it

$$\begin{aligned} f_t' &= (x_1^2 + t^2, x_2^2 + t^2, x_3^2 + t^2, x_4^2 + t^2) \\ f_t'' &= (x_1^2 - tx_1, x_2^2 - tx_2, x_3^2 - tx_3, x_4^2 - tx_4) \end{aligned}$$

where $t \in \mathbf{R}$.

Let us consider the first deformation and looking at its complexification, study $(f_t')^{-1}(0)$. This consists of the points of the form $(\pm it, \pm it, \pm it, \pm it)$, sixteen in total. The Jacobian determinant of f_t', $J = 16x_1x_2x_3x_4$, so we see that 0 is a regular value of f_t'. Looking at the action of G on the points we get the following orbits:

Isotropy group	**Order of orbits**	**Number of orbits**	**Representatives of orbits**
$\mathbf{Z}_4$	1	2	$\pm(it, it, it, it)$
$\mathbf{Z}_2$	2	1	$(it, -it, it, -it)$
$\mathbf{1}$	4	3	$\pm(-it, it, it, it), (it, it, -it, -it)$

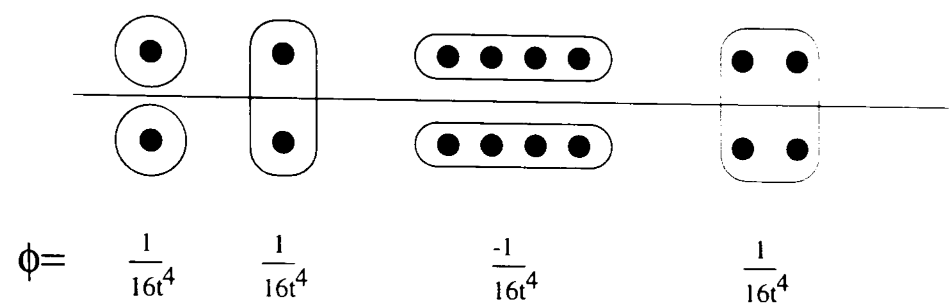


Figure 6.1: The $\mathbf{Z}_4$ orbits of $(f_t')^{-1}(0)$.

We can represent the orbits as shown in Figure 6.1, with orbits ringed and reflection in the line corresponding to complex conjugation (τ). In this figure, ϕ gives the value of $\phi = \frac{1}{J}$ on the given orbits.

We can now calculate $\mathrm{sig}_G B_\phi$. The points in the orbits of order 1 may be ignored, since they lie in A_3. For the orbit of order 2, G acts on the real valued τ-real functions as $[(1)]$. The action on the purely imaginary functions is $[(-1)]$. Thus the contribution here is $[(1)] - [(-1)]$. The first two order 4 orbits may also be ignored, but the points of the third lie in A_2, so must be considered. G acts on the real valued τ-real functions as $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$ and on the imaginary ones as $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$. This yields

$$\begin{aligned} \mathrm{sig}_G B_\phi &= [(1)] - [(-1)] + \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right]. \end{aligned}$$

Let us call this element of the representation ring R' and its (virtual) character ρ'.

Now let us consider the second deformation and $(f_t'')^{-1}(0)$ as before. This consists

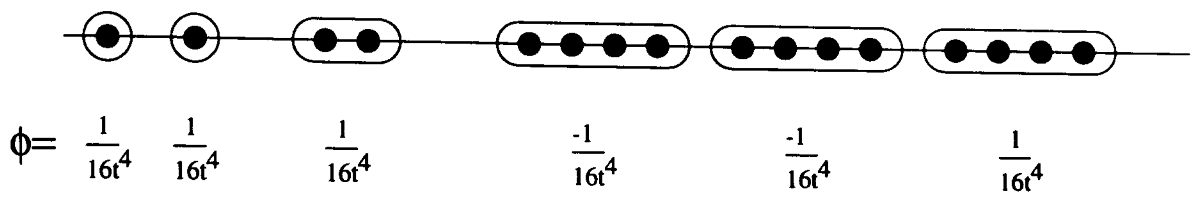


Figure 6.2: The $\mathbf{Z}_4$ orbits of $(f_t'')^{-1}(0)$.

of the points with each component having a value of 0 or t, again giving sixteen points. The Jacobian determinant of f_t'', $J = (2x_1 - t)(2x_2 - t)(2x_3 - t)(2x_4 - t)$, so we see that 0 is a regular value of f_t''. Looking at the action of G on these points we get the following table of orbits.

Isotropy group	Order of orbits	Number of orbits	Representatives of orbits
$\mathbf{Z}_4$	1	2	$(0,0,0,0),(t,t,t,t)$
$\mathbf{Z}_2$	2	1	$(0,t,0,t)$
$\mathbf{1}$	4	3	$(0,t,t,t),(t,0,0,0),(t,t,0,0)$

This can be represented by Figure 6.2, where again ϕ gives the value of $\phi = \frac{1}{J}$ on the given orbits.

These points all lie in A_1, so $\mathrm{sig}_G B_\phi$ in this case is just the permutation representation of the action of G on the points, taken with the sign of ϕ. So we obtain

$$\mathrm{sig}_G B_\phi = 2[(1)] + \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right].$$

Let us call this element of the representation ring R'' and its (virtual) character ρ''. Now we expect R' and R'' to be equal and if we compare the values of ρ' and ρ'' on the elements of the group, we obtain

	1	g	g^2	g^3
ρ'	0	2	4	2
ρ''	0	2	4	2

and hence R' and R'' are equivalent elements of the representation ring.

Now let us look at Q_f, which in this case is spanned by $\{x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4} : i_j = 0 \text{ or } 1\}$. Take $\alpha : Q_f \to \mathbf{C}$ by $\alpha(x_1x_2x_3x_4) = 1$ and 0 elsewhere, since this gives $\alpha(J) = \alpha(16x_1x_2x_3x_4) > 0$. Let B_α be the associated bilinear form. We can diagonalize it by taking the following as a basis:

$$\begin{array}{ll} e_1 = 1 + x_1x_2x_3x_4 & e_2 = 1 - x_1x_2x_3x_4 \\ e_3 = x_1x_3 + x_2x_4 & e_4 = x_1x_3 - x_2x_4 \\ e_5 = x_1 + x_2x_3x_4 & e_6 = x_1 - x_2x_3x_4 \\ \vdots & \vdots \\ e_{11} = x_4 + x_1x_2x_3 & e_{12} = x_4 - x_1x_2x_3 \\ e_{13} = x_1x_2 + x_3x_4 & e_{14} = x_1x_2 - x_3x_4 \\ e_{15} = x_1x_4 + x_2x_3 & e_{16} = x_1x_4 - x_2x_3 \end{array}$$

So B_α is positive definite on the 'odd' basis vectors and negative definite on the 'even' ones. The e_i break up into orbits as $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4\}$, $\{e_5, e_7, e_9, e_{11}\}$, $\{e_6, e_8, e_{10}, e_{12}\}$, $\{e_{13}, e_{15}\}$, $\{e_{14}, e_{16}\}$. Now the actions of G on $\{e_1\}$ and $\{e_2\}$ are equal and B_α is positive on one and negative on the other. This also occurs with the orbits of size 4. These can therefore be ignored. The action on $\{e_3\}$ is $[(1)]$

and on $\{e_4\}$ it is $[(-1)]$. On $\{e_{13}, e_{15}\}$ and $\{e_{14}, e_{16}\}$, we obtain $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$ and $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$ respectively. This yields

$$\mathrm{sig}_G B_\alpha = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right]$$

which gives the same element of the representation ring as previously calculated.

Example 6.5.4 Let $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ act on $\mathbf{R}^4$ by the standard permutation representation. In this example, this is written as the representation of $(0,1)$ and $(1,0)$ respectively, enclosed by square brackets:

$$\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right]$$

As in Example 6.5.3 above, consider $f = (x_1^2, x_2^2, x_3^2, x_4^2) : \mathbf{R}^4, 0 \to \mathbf{R}^4, 0$, which is also finite and G-equivariant in this situation. As before, consider the following finite G-equivariant deformation of it

$$f_t' = (x_1^2 + t^2, x_2^2 + t^2, x_3^2 + t^2, x_4^2 + t^2)$$

Again, looking at its complexification, we consider $(f_t')^{-1}(0)$. This consists of the sixteen points of the form $(\pm it, \pm it, \pm it, \pm it)$. Since the Jacobian determinant is the same as before, 0 is a regular value of f_t'. Looking at the action of G on the points we get the following orbits:

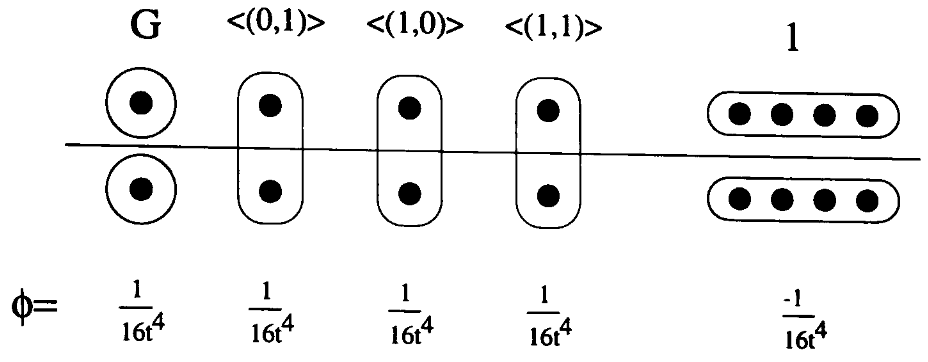


Figure 6.3: The $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbits of $(f_t')^{-1}(0)$.

Isotropy group	**Order of orbits**	**Number of orbits**	**Representatives of orbits**
$\mathbf{Z}_2 \times \mathbf{Z}_2$	1	2	$\pm(it, it, it, it)$
$<(0,1)>$	2	1	$(it, it, -it, -it)$
$<(1,0)>$	2	1	$(it, -it, it, -it)$
$<(1,1)>$	2	1	$(it, -it, -it, it)$
1	4	2	$\pm(-it, it, it, it)$

We can represent the orbits as shown in Figure 6.3, with orbits ringed and reflection in the line corresponding to complex conjugation (τ). In this figure, ϕ gives the value of $\phi = \frac{1}{j}$ on the given orbits and the isotropy subgroup is given above.

We can now calculate $\mathrm{sig}_G B_\phi$. This time, we are only concerned with the orbits of order 2, as the other points lie in A_3. For the orbit with isotropy group $<(0,1)>$, G acts on the real functions as $[(1),(1)]$ and on the imaginary ones as $[(1),(-1)]$, so the contribution here is $[(1),(1)] - [(1),(-1)]$. Similarly, the other two orbits give contributions of $[(1),(1)] - [(-1),(1)]$ and $[(1),(1)] - [(-1),(-1)]$. Thus

$$\mathrm{sig}_G B_\phi = 3[(1),(1)] - [(1),(-1)] - [(-1),(1)] - [(-1),(-1)]$$

which can be written as

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right].$$

If this has (virtual) character ρ then we obtain this table of values.

	(0,0)	(0,1)	(1,0)	(1,1)
ρ	0	4	4	4

Now let us look at Q_f, which is the same as in Example 6.5.3. Take $\alpha : Q_f \to \mathbf{C}$ by $\alpha(x_1x_2x_3x_4) = 1$ and 0 elsewhere. Let B_α be the associated bilinear form. We diagonalize it by taking the same basis $\{e_1, \ldots, e_{16}\}$ as before, so B_α is positive definite on the 'odd' basis vectors and negative definite on the 'even' ones. The only non-singleton orbits formed by the e_i are the orbits $\{e_5, e_7, e_9, e_{11}\}$ and $\{e_6, e_8, e_{10}, e_{12}\}$. The actions on each of these is identical and since one is positive and one negative, we may ignore them. The same is true of $\{e_1\}$ and $\{e_2\}$. The action on each of $\{e_3\}$, $\{e_{13}\}$ and $\{e_{15}\}$ is $[(1),(1)]$. Now $\{e_4\}$ gives $[(-1),(1)]$, $\{e_{14}\}$ gives $[(1),(-1)]$ and $\{e_{16}\}$ yields $[(-1),(-1)]$. So

$$\mathrm{sig}_G B_\alpha = 3[(1),(1)] - [(1),(-1)] - [(-1),(1)] - [(-1),(-1)]$$

which is the same element of the representation ring obtained above.

Example 6.5.5 Let us consider the symmetric group $G = S_3$ acting on $\mathbf{R}^3$ in the usual way. Since S_3 is generated by the permutations (12) and (123), we will describe a representation of S_3 by giving the matrices corresponding to these elements, enclosed in square brackets. So S_3 acts by

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

Spanning Vectors	**Action of G**	**Spanning Vectors**	**Action of G**
$1+x^2y^2z^2$	[(1),(1)]	$1+x^2y^2z^2$	[(1),(1)]
$x+xy^2z^2$ $y+x^2yz^2$ $z+x^2y^2z$	R_0	$x-xy^2z^2$ $y-x^2yz^2$ $z-x^2y^2z$	R_0
$x^2+y^2z^2$ $y^2+x^2z^2$ $z^2+x^2y^2$	R_0	$x^2-y^2z^2$ $y^2-x^2z^2$ $z^2-x^2y^2$	R_0
$yz+x^2yz$ $xz+xy^2z$ $xy+xyz^2$	R_0	$yz-x^2yz$ $xz-xy^2z$ $xy-xyz^2$	R_0
xy^2+xz^2 yz^2+x^2y x^2z+y^2z	R_0	xy^2-xz^2 yz^2-x^2y x^2z-y^2z	$\hat{R}_0$
xyz	[(1),(1)]		

Table 6.1: Q_f for $f=(x^2,y^2,z^2)$ under S_3

which we will call R_0. Now the map $f=(x^3,y^3,z^3):\mathbf{R}^3,0\to\mathbf{R}^3,0$ is finite and G-equivariant with respect to this action. The local algebra, Q_f, is given by the R-span of the monomials $\{x^{i_1}y^{i_2}z^{i_3} : i_j\in\{0,1,2\}\}$. Now the Jacobian determinant J of the map f is given by $27x^2y^2z^2$, so let us define the bilinear form B_α via the linear form α which takes the value 1 at $x^2y^2z^2$ and 0 elsewhere. Diagonalizing B_α we obtain the subspaces of Q_f shown in Table 6.1, where the spaces on the left-hand side are such that B_α is negative definite and those on the right, where it is positive. The representation $\hat{R}_0$ is given by

$$\left[\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\right].$$

Looking at the table, we obtain

$$\begin{aligned}
\mathrm{sig}_G B_\alpha &= R_0 + [(1),(1)] - \hat{R}_0 \\
&= \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right].
\end{aligned}$$

The characters of R_0, $\hat{R}_0$ and the trivial representation $[(1),(1)]$ are denoted ρ_0, $\hat{\rho}_0$ and ι. They are given by

	(1)	(12)	(13)	(23)	(123)	(132)
ρ_0	3	1	1	1	0	0
$\hat{\rho}_0$	3	-1	-1	-1	0	0
ι	3	3	3	3	3	3

and so we obtain

	(1)	(12)	(13)	(23)	(123)	(132)
$\rho_0 + \iota - \hat{\rho}_0$	3	5	5	5	3	3

which is the character of $\mathrm{sig}_G B_\alpha$.

Chapter 7

Further results

7.1 Non-regular deformations

We now consider the results when 0 is not a regular value of the deformation f_t. Indeed, it is often the case that f will have no deformation f_t with 0 a regular value (away from $t = 0$) as the following shows.

Example 7.1.1 Let our group G be $\mathbf{Z}_4$ acting on source and target spaces (both isomorphic to $\mathbf{C}$) via the representations

$$[(i)] \text{ and } [(-i)]$$

respectively. (Again, we give the matrix of the generator as a description of the whole representation.) Then $f(x) = x^3$ is finite and G-variant, but there is no "good" deformation. Indeed, the representation on Q_f is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}\right]$$

and since the trace of this matrix is i, it cannot be equivalent to a permutation matrix.

The first proposition is an alternative description of the action of G on Λ_t, which contains Proposition 6.2.6(ii) as a special case. First, though, we need the following lemma.

Lemma 7.1.2 *The local algebras associated to two points in the same point orbit are isomorphic.*

Proof Suppose we consider points a_i and a_j with local algebras $Q(a_i)$ and $Q(a_j)$ respectively, such that $g(a_i) = a_j$. Then $g : Q(a_i) \to Q(a_j)$ via a linear map with inverse g^{-1}. □

Proposition 7.1.3 *Let G act linearly on $\mathbf{C}^n$ and let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be finite G-variant, with f_t a finite G-variant deformation. Let $f_t^{-1}(0) = \{a_1, \ldots, a_r\}$ and let Λ_t be the multilocal algebra of the deformation. Now G acts on the points, so let O_l be the permutation representation of the action of G on the l^{th} point orbit. Each point has an associated local algebra and all the points in a given orbit have isomorphic local algebras (see Lemma 7.1.2 above). Denote by Q_l the local algebra (considered as a G-space) associated with the points of the l^{th} orbit. Then*

$$L \cong \bigoplus_l (O_l \otimes Q_l)$$

where this is an isomorphism of G-spaces.

Proof We know that $L \cong \Lambda_t$ by Proposition 6.2.7, so it is enough to show that Λ_t has the given form. Let us consider the 1^{th} orbit, consisting of points $\{a_1, \ldots a_s\}$ and suppose Q_l has $\{\epsilon_1, \ldots, \epsilon_p\}$ as a basis. The part of Λ_t corresponding to this orbit is spanned by $\{a_i \otimes \epsilon_j : i = 1 \ldots s, j = 1 \ldots p\}$, where $a_i \otimes \epsilon_j$ is the basis element ϵ_j in the algebra associated to a_i. Now $g \in G$ acts by $g(a_i \otimes \epsilon_j) = g(a_i) \otimes g(\epsilon_j)$, which is exactly what is given by $O_l \otimes Q_l$. Taking a sum over the orbits gives the required result. □

Note that each map-germ $f_t : \mathbf{C}^n, a_i \to \mathbf{C}^n, 0$ is variant with respect to G_{a_i}, the subgroup of G which fixes a_i. The G_{a_i}-variant degree is by definition the representation of the action of G_{a_i} on the local algebra at that point. It is therefore given by the restriction of the appropriate Q_l (as given in the Proposition above) to G_{a_i}.

Example 7.1.4 Let $G = \mathbf{Z}_2$ (generated by g) act on $\mathbf{C}^2$ by

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

where we describe a representation R by giving $R(g)$. Now $f = (x^3, y^3) : \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ is finite and G-equivariant and $f_t = (x^3 - t^2x, y^3)$ is a finite equivariant deformation of f. Notice that 0 is not a regular value of f_t and that $f_t^{-1}(0) = \{(0,0), (t,0), (-t,0)\}$. The local algebra at each of the three points is isomorphic to $\mathbf{C}$-span$\{1, y, y^2\}$. The action of G on this algebra is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right].$$

The points break up into orbits as $\{(0,0)\}$ and $\{(t,0), (-t,0)\}$ with G acting by $[(1)]$ and $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$ respectively. Thus the action of G on Λ_t is in this case given by

$$\left[\left((1) \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \oplus \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\right]$$

which has a character of 9 on the identity and 1 on g. In contrast, Q_f as a vector space is given by $\mathbf{C}$-span$\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2\}$ upon which G acts by $[\mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1, 1)]$, so this has the same character, as predicted.

We now consider the real case.

Let G act linearly on $\mathbf{R}^n$ (both source and target) via matched representations and let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be finite G-variant, with f_t a finite G-variant deformation. We take $L_{\boldsymbol{R}}$ be the $\mathbf{R}$-span of functions $e_1, \ldots, e_\mu$ whose germs at 0 span Q_f. If we now consider f_t as a complex map, let $f_t^{-1}(0) = \{a_1, \ldots, a_r\} \subset \mathbf{C}^n$ and let Λ_t be the (complex) multilocal algebra of the deformation. Let R be the image of $L_{\boldsymbol{R}}$ under $\pi : Q_{f,\boldsymbol{C}} \to \Lambda_t$. Now let us split the points $\{a_i\}$ into sets A_1, A_2 and A_3 as in §6.4.

Let $O_{1,l}$ be the permutation representation of the action of G on the lth A_1 point orbit.

Considering the action of G on points in A_2, we can look at the pairs consisting of points with their complex conjugate. Let $O_{2,l}$ be the permutation representation of G on the conjugate pairs of the lth A_2 point orbit. Similarly, let $O_{3,l}$ be the permutation representation of G on these pairs, but with a sign to determine the 'orientation' of the pair. Suppose we have an orbit consisting of $a_i, \overline{a_i}, a_j, \overline{a_j}$. If some $g \in G$ acts by

$$\begin{array}{ccc} a_i & \mapsto & \overline{a_j} \\ \overline{a_i} & \mapsto & a_j \\ a_j & \mapsto & a_i \\ \overline{a_i} & \mapsto & \overline{a_j} \end{array}$$

then the unsigned and signed permutations representations would look like

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The pairs have merely been exchanged, hence the first matrix, but the pair $\{a_i, \overline{a_i}\}$ is mapped to $\{a_j, \overline{a_j}\}$ with the reverse orientation, giving the -1 entry in the signed permutation representation.

The orbits of points in A_3 come in pairs, so here take $O_{4,l}$ to be the permutation representation of G on one orbit of the l^{th} pair of A_3 point orbits.

Each point a_j, in any orbit, has an associated (complex) local algebra $Q(a_j)$. The direct sum of these over all orbits forms Λ_t. Now an element of R can be expressed as element of Λ_t, having components in each local algebra. The real algebra of possible component values in the algebra $Q(a_j)$ will be denoted $R(a_j)$ and is really the projection of R onto this component. For the l^{th} A_1 orbit, define $R_{1,l}$ to be $R(a_j)$ for any point a_j in the orbit. For the l^{th} A_2 orbit, define $R_{2,l}$ and $R_{3,l}$ to be respectively the spaces of real- and purely imaginary-valued functions in $R(a_j)$ for any point a_j in the orbit. Finally, for the l^{th} pair of A_3 orbits, define $R_{4,l}$ to be $R(a_j)$ for any point a_j in one of the orbits.

Proposition 7.1.5 *If we are given $O_{k,l}$,$R_{k,l}$, $k = 1,2,3,4$ as above, then*

$$L_{\mathbf{R}} \cong \bigoplus_{\substack{k=1 \\ l\in I_k}}^{4} (O_{k,l} \otimes R_{k,l})$$

where this is an isomorphism of G-spaces and I_k indexes the orbits.

Proof We look at each of the three types of orbit:

$\boldsymbol{A_1}$: For orbits in A_1 the proof is identical to that in the complex case, but over $\mathbf{R}$.

$\boldsymbol{A_2}$:Suppose we have the l^{th} orbit in A_2, with points $\{a_1, \ldots a_s\}$. Given some point a_j in this orbit, we know that $\overline{a_j}$ is also in the orbit. Now $R(a_j)$ and $R(\overline{a_j})$ are isomorphic via complex conjugation. An element of $R(a_j)$ which is real-valued will therefore be unaffected by a change in orientation of the pairs, just by the permutation of them. Thus the representation of the action of G on the real-valued functions is given by $O_{2,l} \otimes R_{2,l}$. However, an element of $R(a_j)$ which is purely imaginary-valued will have a reversal of sign under an orientation change in a conjugate pair. Thus the representation here is given by $O_{3,l} \otimes R_{3,l}$

$\boldsymbol{A_3}$: Orbits in A_3 come in pairs, so suppose we have the l^{th} pair of orbits in A_3, with one orbit consisting of points $\{a_1, \dots a_s\}$, the other of the conjugate points. Let a_j be a point in our chosen orbit. We know that the algebras $R(a_j)$ and $R(\overline{a_j})$ are isomorphic via complex conjugation. In fact, the image of a point of $L_{\mathbf{R}}$ has conjugate components on these two spaces. So it is enough to know the component of some element of R on one of each pair of conjugate algebras. So the representation of the action of G on these orbits is given by $O_{4,l} \otimes R_{4,l}$ as required. □

Corollary 7.1.6 *Let $\alpha : Q_f \to \mathbf{R}$ be a G-invariant linear form with $\alpha(J) > 0$ and let $\alpha_{k,l}$ be the linear form it induces on $R_{k,l}$. These in turn give bilinear forms B_α and $B_{\alpha_{k,l}}$ on their respective spaces and*

$$sig_G B_\alpha = \sum_{\substack{k=1 \\ l \in I_k}}^{3} (O_{k,l} \cdot sig_G B_{\alpha_{k,l}})$$

in the representation ring of G.

Proof We wish to calculate the G-signature on some $O_{k,l} \otimes R_{k,l}$. The bilinear form $B_{\alpha_{k,l}}$ induces a bilinear form $B_{k,l}$ on $\bigoplus_j R_{k,l}(a_j)$ where the a_j are the points in the orbit (or pairs in the A_2 case). This is given by

$$B_{k,l}(h_1, h_2) = \sum_j B_{\alpha_{k,l}}(h_{1,j}, h_{2,j})$$

where $h_1, h_2 \in \bigoplus_j R_{k,l}(a_j)$ with components $h_{1,j}, h_{2,j}$ in the j^{th} place. Considering $B_{k,l}(h, h)$ in order to diagonalize the bilinear form, we see that a permutation of the a_j will have no effect on the value of $B_{k,l}$. Also, multiplication by a signed permutation will also not alter the form, since $B_{k,l}(h,h) = B_{k,l}(-h,-h)$. In other words, we can write

$$O_{k,l} \otimes R_{k,l} = (O_{k,l} \otimes R^+_{k,l}) \oplus (O_{k,l} \otimes R^-_{k,l})$$

where $B_{k,l}$ is positive definite on the first summand and negative on the second. Looking at all the orbits gives

$$\operatorname{sig}_G B_\alpha = \sum_{\substack{k=1\\ l\in I_k}}^{4} (O_{k,l} \cdot \operatorname{sig}_G B_{\alpha_{k,l}}).$$

It remains to show that $\operatorname{sig}_G B_{\alpha_{4,l}} = 0$ for all $l \in I_4$. Let us first write

$$R_{4,l} = R'_{4,l} \oplus R''_{4,l}$$

where $R'_{4,l}$ is the space of real-valued functions and $R''_{4,l}$ is the space of purely imaginary-valued ones. Let $\epsilon_1, \dots \epsilon_s$ be a basis for $R'_{4,l}$ which diagonalizes $B_{4,l}$ on the space. This means that $i\epsilon_1, \dots i\epsilon_s$ will be a basis for $R''_{4,l}$ which diagonalizes the form on that space. Since the group action is real, the actions on $R'_{4,l}$ and $R''_{4,l}$ will be the same. But $B_{k,l}(i\epsilon_j, i\epsilon_j) = -B_{k,l}(\epsilon_j, \epsilon_j)$ by definition, so the two spaces are isomorphic as G-spaces, but $B_{4,i}$ has an opposite sign on each, thus $\operatorname{sig}_G B_{\alpha_{4,l}} = 0$. □

Example 7.1.7 We look at the same case as the previous example, but over $\mathbf{R}$. Thus $G = \mathbf{Z}_2$ (generated by g) acts on $\mathbf{R}^2$ by

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

as before. We consider $f = (x^3, y^3) : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$, which is finite and G-equivariant and $f_t = (x^3 - t^2 x, y^3)$, a finite equivariant deformation of it. As before, $f_t^{-1}(0) = \{(0,0), (t,0), (-t,0)\}$. Notice that all three points are of type

A_1. The local algebra at each of the three points is isomorphic to $\mathbf{R}$-span$\{1, y, y^2\}$. The action of G on this algebra is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right].$$

In this case, we look at $\alpha : Q_f \to \mathbf{R}$ given by $\alpha = 1$ on x^2y^2, 0 elsewhere. What are the induced linear forms on the three local algebras? Taking a basis of $\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2\}$ for Q_f, then the isomorphism $\pi : L_{\mathbf{R}} \to R$ is given by

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9) \longmapsto ((\lambda_1, \lambda_4, \lambda_7),$$
$$(\lambda_1 + t\lambda_2 + t^2\lambda_3, \lambda_4 + t\lambda_5 + t^2\lambda_6, \lambda_7 + t\lambda_8 + t^2\lambda_9),$$
$$(\lambda_1 - t\lambda_2 + t^2\lambda_3, \lambda_4 - t\lambda_5 + t^2\lambda_6, \lambda_7 - t\lambda_8 + t^2\lambda_9))$$

where $R = R((0,0)) \oplus R((t,0)) \oplus R((-t,0))$ and each has $\mathbf{R}$-basis $\{1, y, y^2\}$. This is obtained by rewriting a general element of $L_{\mathbf{R}}$ as a germ at the appropriate point and factoring out by the ideal generated by the components of f_t. Now α is equivalent to projection onto the final basis vector in $L_{\mathbf{R}}$. The induced map in R is therefore given by projection onto $((0,0,-2),(0,0,1),(0,0,1))$, since $-2(\lambda_7) + 1(\lambda_7 + t\lambda_8 + t^2\lambda_9) + 1(\lambda_7 - t\lambda_8 + t^2\lambda_9) = 2t^2\lambda_9$.

On the real algebra at $(0,0)$, which we shall call R_1, the induced linear form, α_1 acts by

$$\alpha_1 : \begin{array}{ccc} 1 & \mapsto & 0 \\ y & \mapsto & 0 \\ y^2 & \mapsto & -2 \end{array}$$

We can diagonalize B_{α_1} by taking a basis $\{1-y^2, 1+y^2, y\}$, which gives the action

$$\left[(1) \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

with B_{α_1} positive definite on the first summand, negative on the second. The G-signature here is thus $-[(-1)]$.

On the other orbit, the algebra R_2 has the induced form α_2 on it given by

$$\alpha_2 : \begin{array}{rcl} 1 & \mapsto & 0 \\ y & \mapsto & 0 \\ y^2 & \mapsto & 1 \end{array}$$

Diagonalizing with the same basis as above gives the action

$$\left[(1) \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

with B_{α_2} negative definite on the first summand, positive on the second. The G-signature here is thus $[(-1)]$.

The action of G on the two orbits is given by $[(1)]$ and $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$ as before. We therefore obtain

$$\begin{aligned} \mathrm{sig}_G B_\alpha &= [(1)] \cdot -[(-1)] + \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \cdot [(-1)] \\ &= -[(-1)] + \left[\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right] \\ &= -[(-1)] + \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &= [(1)]. \end{aligned}$$

Now considering Q_f, we can diagonalize the bilinear form B_α by taking the basis $\{1+x^2y^2, 1-x^2y^2, x+xy^2, x-xy^2, x^2+y^2, x^2-y^2, x^2y+y, x^2y-y, xy\}$. B_α is positive definite on the 'odd' basis vectors $\{1+x^2y^2, x+xy^2, \ldots\}$ and negative on the 'even' ones $\{1-x^2y^2, x-xy^2, \ldots\}$ and G acts by $[\mathrm{diag}(1,1,-1,-1,1,1,-1,-1,1)]$.

Thus we get

$$\begin{aligned}\mathrm{sig}_G B_\alpha &= \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right] \\ &= [(1)]\end{aligned}$$

as expected.

7.2 Two group actions

Let G_1 (respectively G_2) be a finite group with two representations on $\mathbf{C}^{n_1}$ (respectively $\mathbf{C}^{n_2}$) one on the source and one on the target space. Then $G = G_1 \times G_2$ acts on $\mathbf{C}^{n_1} \times \mathbf{C}^{n_2}$ in the obvious way. Let $f_1 : \mathbf{C}^{n_1}, 0 \to \mathbf{C}^{n_1}, 0$ and $f_2 : \mathbf{C}^{n_2}, 0 \to \mathbf{C}^{n_2}, 0$ be finite and G_1- and G_2-variant respectively. Then $f = (f_1, f_2) : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ is finite G-variant ($n = n_1 + n_2$). Now $Q_{f_1} \otimes Q_{f_2} \cong Q_f$ by $(h_1 \otimes h_2)(x, y) = h_1(x) \cdot h_2(y)$ where the x coordinates are in $\mathbf{C}^{n_1}$ and the y in $\mathbf{C}^{n_2}$. The action of G on Q_f is given by

$$(g_1, g_2) \cdot \sum_{i,j} \lambda_{i,j}(\alpha_i \otimes \beta_j) = \sum_{i,j} \lambda_{i,j}(g_1 \cdot \alpha_i \otimes g_2 \cdot \beta_j)$$

where $\alpha_i \in Q_{f_1}$, $\beta_j \in Q_{f_2}$, $g_k \in G_k$ and $\lambda_{i,j} \in \mathbf{C}$. This action is related to those on Q_{f_1} and Q_{f_2} by the following

Proposition 7.2.1 *Let G_1 and G_2 act on Q_{f_1} and Q_{f_2} via representations R_1 and R_2. Then G acts on Q_f via the representation*

$$\begin{aligned} R((g_1, g_2)) &= R_1(g_1) \otimes R_2(g_2) \\ &= R_1|^G((g_1, g_2)) \otimes R_2|^G((g_1, g_2)) \end{aligned}$$

where '$|^G$' denotes the induced representation on G.

Proof Let $\{\alpha_j\}$ and $\{\beta_l\}$ be $\mathbf{C}$-space bases for Q_{f_1} and Q_{f_2}. Suppose that

$$g_1 \cdot \alpha_i = \sum_j r_{i,j}\alpha_j$$

and

$$g_2 \cdot \beta_k = \sum_l s_{k,l}\beta_l,$$

so $r_{i,j}$ and $s_{k,l}$ are the entries of $R_1(g_1)$ and $R_2(g_2)$ respectively. Then

$$\begin{aligned}(g_1, g_2) \cdot (\alpha_i \otimes \beta_k) &= \sum_j r_{i,j}\alpha_j \otimes \sum_l s_{k,l}\beta_l \\ &= \sum_{j,l} r_{i,j}s_{k,l}(\alpha_j \otimes \beta_l)\end{aligned}$$

But $r_{i,j}s_{k,l}$ is exactly the coefficient of $\alpha_j \otimes \beta_l$ in $R_1(g_1) \otimes R_2(g_2)$. It is easy to see that since G_1 and G_2 are normal in $G_1 \times G_2$, then $R_1|^G((g_1, g_2)) = R_1(g_1)$ and $R_2|^G((g_1, g_2)) = R_2(g_2)$. □

This carries over to the real case and G-signatures as follows. Now let G_1 and G_2 act on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, with matched pairs of representations. Let $f_1 : \mathbf{R}^{n_1}, 0 \to \mathbf{R}^{n_1}, 0$ and $f_2 : \mathbf{R}^{n_2}, 0 \to \mathbf{R}^{n_2}, 0$ again be finite and G_1- and G_2-variant respectively. Suppose we have a non-degenerate, G_1-invariant bilinear form B_1 on Q_{f_1} and a similarly defined form B_2 on Q_{f_2}. We can define a form B on $Q_f \cong Q_{f_1} \otimes Q_{f_2}$ by

$$B(\alpha_i \otimes \beta_k) = B_1(\alpha_i) \cdot B_2(\beta_k)$$

which is non-degenerate and G-invariant by construction. Now express Q_{f_1} as $R_1^+ \oplus R_1^-$, a sum of G_1-space with B_1 positive definite on R_1^+ and negative definite on R_1^- and similarly for Q_{f_2}. Then Q_f can be expressed as

$$(P_1^+ \otimes P_2^+) \oplus (P_1^- \otimes P_2^-) \oplus (P_1^+ \otimes P_2^-) \oplus (P_1^- \otimes P_2^+)$$

where $P_1^+ = R_1^+|^G$ etc. Now B is positive definite on the first two summands and negative on the last two. In the representation ring we have

$$(P_1^+ - P_1^-) \cdot (P_2^+ - P_2^-) = (P_1^+ \cdot P_2^+ + P_1^- \cdot P_2^-) - (P_1^+ \cdot P_2^- + P_1^- \cdot P_2^+),$$

so we have proved

Corollary 7.2.2 *With G, G_1, G_2, B, B_1, B_2 as above,*

$$sig_G B = (sig_{G_1} B_1)|^G \cdot (sig_{G_2} B_2)|^G.$$

□

7.3 Modular representations

A representation V in characteristic 0 has a corresponding modular representation $V_{(p)}$ in characteristic p, as an element of the Grothendieck ring of representations. (See [DP], or [D] for the case $p = 2$.) We are mainly concerned with the case when $p = 2$ for the following reason. In the description of the G-variant degree in the real case (Proposition 6.4.2), the complications lie mainly in the points of type A_2, which are symmetric under the involution of complex conjugation. We are also only really interested in the action of G on the real points of $f_t^{-1}(0)$. By considering modular representations with $p = 2$, we can eliminate the contribution from the A_2 points.

Given a G-signature of some G-invariant bilinear form B on a real vector space V as

$$\mathrm{sig}_G(B) = V^+ - V^-$$

we can define the modular G-signature of B to be

$$\mathrm{sig}_{(2)G}(B) = V^+_{(2)} - V^-_{(2)},$$

where $V^+_{(2)}$ and $V^-_{(2)}$ are the modular versions of V^+ and V^-. Now if the representation in characteristic 0 is defined over the integers, then the modular representation over $\mathbf{F}_2$ (the field of two elements) is obtained by reducing the coefficients mod 2 (Again, see [D], [DP].)

We now consider the results on G-signature from §6.4, but using modular representations. We let G act on $\mathbf{R}^n$ and take f to be a finite G-variant map with f_t a finite G-variant deformation such that 0 is a regular value (assuming here that such a deformation exists). Let $f_t^{-1}(0) = \{a_1, \ldots, a_\mu\} \in \mathbf{C}^n$. Take τ to be the involution of complex conjugation acting on the roots. As before, we let R be the $\mathbf{R}$-algebra of τ-real functions on the points a_i. We decompose R as

$$R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$$

where R_1, R_2, R_3, R_4 are as before. We then have the following result, which is due to Damon ([D]) in the equivariant case.

Proposition 7.3.1 *With G, $\{a_i\}$, τ, R as above, then*

$$sig_{(2)G} B_\phi = R_1^+ - R_1^-$$

in the modular representation ring of G, where the '+' and '−' superscripts denote the G-spaces where $\phi > 0$ and $\phi < 0$ respectively, as before.

Proof As before, we look at each of the spaces R_1, R_2, R_3 and R_4:

R_1: As in the original proof.

R_2 and R_3: Consider a pair of points $a_i, a_j \in A_2$ such that $\tau(a_i) = a_j$ and $\phi(a_i) = \phi(a_j) = \lambda \in \mathbf{R}$. The matrix for B_ϕ here is of the form $\begin{pmatrix} 2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}$, where the first basis vector corresponds to real functions on the two points and the second to purely imaginary ones. Thus when $\lambda > 0$, the real functions contribute positively to the G-signature and the imaginary ones negatively. When $\lambda < 0$, the opposite occurs. Thus, overall, $(R_{(2)2}^+ + R_{(2)3}^-) - (R_{(2)2}^- + R_{(2)3}^+)$ is contributed to the G-signature. Note that the actions of G on R_2 and R_3 are identical in the modular case, as the multiplication by -1 which occurred in the original case when a conjugate pair were swapped has no effect. Thus $R^+_{(2)2} = R^+_{(2)3}$ and $R^-_{(2)2} = R^-_{(2)3}$ and so there is no contribution to the G-signature.

R_4: As in the original proof.

We have so far shown that $\mathrm{sig}_{(2)G} B_\phi = (R^+_{(2)1}) - (R^-_{(2)1})$, but since R^+_1 and R^-_1 are just permutation matrices, reduction to the modular case makes no difference to the matrices in question. □

Note that although R^+_1 and R^-_1, considered as modular representations, appear identical to the original (characteristic 0) representations, the element of the modular representation ring, $R^+_1 - R^-_1$, may be very different from the original version. A non-zero element of the representation ring in the original case could give rise to a zero element in the modular case. For example, the following element of the representation ring of $\mathbf{Z}_2$,

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

is non-zero in the characteristic 0 case, but zero when reduced to the modular case. The following result is derived identically to Proposition 6.4.5, with representations assumed to be modular.

Proposition 7.3.2 *The modular G-signature of a bilinear form B_α on Q_f, derived from a linear form $\alpha : Q_f \to \mathbf{R}$ with $\alpha(J) > 0$ is independent of the choice of α.* □

Proposition 7.3.1 is very similar to Corollary 6.4.3. Again, when we take $\phi = 1/J$, we will call the obtained (modular) G-signature the (modular) G-index of f. This allows us to state a result analogous to Corollary 6.4.6:

Corollary 7.3.3 *The modular G-signature of a bilinear form B_α on Q_f, derived from a linear form $\alpha : Q_f \to \mathbf{R}$ with $\alpha(J) > 0$ is equal to the modular G-index of the singular point 0.* □

As in the original case, the G-signature is equal to the G-index for any given deformation and the G-signature is independent of the deformation. Thus, using modular representations, the G-index is independent of the deformation chosen.

We can also look at the case when we do not have a 'good' deformation of f and obtain the following analogue of Corollary 7.1.6.

Proposition 7.3.4 *Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be finite and G-variant, with f_t a finite G-variant deformation. Suppose $f_t^{-1}(0) = \{a_1, \ldots, a_r\}$ with $O_{k,l}$, $R_{k,l}$ as defined in §7.1. Let $\alpha : Q_f \to \mathbf{R}$ be a G-invariant linear form with $\alpha(J) > 0$ and let $\alpha_{k,l}$ be the linear form it induces on $R_{k,l}$. These in turn give bilinear forms B_α and $B_{\alpha_{k,l}}$ on their respective spaces and*

$$sig_{(2)G} B_\alpha = \sum_{l \in I_1} (O_{1,l} \cdot sig_{(2)G} B_{\alpha_{1,l}})$$

in the (modular) representation ring of G. Here I_1 indexes the orbits of A_1 points.

Proof Now by reducing everything to the modular case, we certainly have

$$\mathrm{sig}_{(2)G} B_\alpha = \sum_{\substack{k=1 \\ l \in I_k}}^{3} (O_{(2)k,l} \cdot \mathrm{sig}_{(2)G} B_{\alpha_{k,l}}).$$

Now since we are working in modular representations, $O_{(2)2,l}$ and $O_{(2)3,l}$ will be isomorphic as G-spaces, since a signed and an unsigned permutation representation are equivalent. Now consider the subspace of R of (complex) functions on the l^{th} A_2 point orbit (i.e. $R_{2,l} \oplus R_{3,l}$). The linear form α will induce a bilinear form B on this space which restricted to $R_{2,l}$ or $R_{3,l}$ will give $B_{\alpha 2,l}$ and $B_{\alpha 3,l}$ respectively.

We now proceed as in the proof of Corollary 7.1.6. Let $\epsilon_1, \ldots \epsilon_s$ be a basis for $R_{2,l}$ which diagonalizes B on the space. This means that $i\epsilon_1, \ldots i\epsilon_s$ will be a basis for $R_{3,l}$ which diagonalizes the form on that space. Since the group action is real, the actions on $R_{2,l}$ and $R_{3,l}$ will be the same. But $B(i\epsilon_j, i\epsilon_j) = -B(\epsilon_j, \epsilon_j)$ by definition, so the two spaces are isomorphic as G-spaces, but B has an opposite sign on each, thus

$$O_{(2)2,l} \cdot \mathrm{sig}_{(2)G} B_{\alpha_{2,l}} + O_{(2)3,l} \cdot \mathrm{sig}_{(2)G} B_{\alpha_{3,l}} = 0$$

in the modular representation ring. Thus there is no contribution to $\mathrm{sig}_{(2)G}$ from A_2 points and the result follows. Note that since $O_{1,l}$ is a permutation representation it is already a modular representation, so we may dispense with the '(2)' subscript. □

7.4 A stronger invariant

The isomorphism class (or equivalently the character) of a permutation representation does not determine the isomorphism class of the associated G-set. This means that if we obtain (the character of) the G-variant degree of a complex map, then this is not enough information to determine the action of G on the preimages of 0 under some deformation. We therefore need a stronger invariant. First, however, we give an example of two non-isomorphic G-sets with equivalent permutation representations.

Example 7.4.1 Let our group G be $\mathbf{Z}_2 \times \mathbf{Z}_2$ again and consider the following permutation representations of G:

$$\begin{aligned} R_1 \ : \ & \{(0,0),(0,1),(1,0),(1,1)\} \to \\ & \{(1),(1),(1),(1)\} \end{aligned}$$

R_2 : $\{(0,0),(0,1),(1,0),(1,1)\} \to$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

R_3 : $\{(0,0),(0,1),(1,0),(1,1)\} \to$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

R_4 : $\{(0,0),(0,1),(1,0),(1,1)\} \to$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

R_5 : $\{(0,0),(0,1),(1,0),(1,1)\} \to$

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right\}$$

Now consider the permutation representations $2R_1 \oplus R_5$ and $R_2 \oplus R_3 \oplus R_4$. These clearly correspond to different G-sets, since the first has one orbit of order 4 and two of order 1, whereas the second has three of order 2. Let us denote the characters of these representations by σ_1 and σ_2 respectively. These give us the values

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
σ_1	6	2	2	2
σ_2	6	2	2	2

showing that these are in fact equivalent representations.

In [R2], Roberts considers the lattice of subspaces fixed by subgroups of G and the Milnor number of the invariant function f restricted to these subspaces. We modify the results to our own situation, but adopt some of his notation. We will denote the source and target spaces (still isomorphic to $\mathbf{C}^n$) by V and W respectively. We take $f : V, 0 \to W, 0$ to be finite and G-variant and let f_t be a finite G-variant deformation such that 0 is a regular value when $t \neq 0$ (again, assuming that such a deformation exists). We set $f_t^{-1}(0) = \{a_1, \ldots, a_\mu\}$. As in [R2], we let L be the lattice of vector subspaces of the source of the form

$$V(H) = \{v \in V : h \cdot v = v \text{ for all } h \in H\}$$

where $H \leq G$. The partial order on L is inherited from the order given by inclusion on the subgroups of G. Let $f_H = f|_{V(H)}$ and define

$$\begin{array}{rccl} \rho: & L & \rightarrow & \mathbf{N} \\ & V(H) & \mapsto & \deg(f_H) = |\{a_i : a_i \in V(H)\}|. \end{array}$$

We also need the following definition.

Definition (see [K]). The Möbius function of a lattice $m : L \times L \rightarrow \mathbf{Z}$ is defined by

$$m(x,x) = 1 \text{ and } m(x,y) = -\sum_{x \geq z > y} m(x,z).$$

If we have a function $f : L \rightarrow \mathbf{R}$ and another function $f^* : L \rightarrow \mathbf{R}$ given by

$$f^*(z) = \sum_{x \geq z} f(x)$$

then the Möbius inversion formula

$$f(x) = \sum_{y \geq x} m(y,x) f^*(y)$$

holds. These results also hold in the dual case, when the partial order is reversed throughout.

We may now state the result we require

Proposition 7.4.2 *Let f, f_t,$\{a_i\}$, L and ρ be as above, then given L and ρ, we are able to determine the isomorphism class of the G-set $\{a_1, \ldots, a_\mu\}$.*

Proof First we define

$$\begin{array}{rccl} \rho^*: & L & \rightarrow & \mathbf{N} \\ & V(H) & \mapsto & |\{a_i : a_i \in V(H), a_i \notin V(H') \text{ for } H' > H\}|. \end{array}$$

Now ρ gives the degree of f restricted to $V(H)$, which is equivalent to the number of the a_i which lie in $V(H)$. The function ρ^*, however, gives the number of the a_i which lie in $V(H)$ but no greater element of L. We can therefore write

$$\rho(V(H)) = \sum_{H' \geq H} \rho^*(V(H')).$$

Now if we let $m : L \times L \rightarrow \mathbf{Z}$ be the Möbius function of L (see below), then we obtain

$$\rho^*(V(H)) = \sum_{H' \geq H} m(V(H'), V(H)) \cdot \rho(V(H'))$$

by Möbius inversion. If we denote by $c(H)$ the number of orbits whose elements have isotropy group H, or a conjugate of H, then

$$\rho^*(V(H)) = \left|\frac{N(H)}{H}\right| \cdot c(H)$$

where $N(H)$ is the normalizer of H. We can then write

$$c(H) = \frac{|H|}{|N(H)|} \sum_{H' \geq H} m(V(H'), V(H)) \cdot \rho(V(H')) \qquad (7.1)$$

and so $c(H)$ is determined by L (which gives m) and ρ. Now $c(H)$ is the number of orbits of 'type G/H' and this gives sufficient information to recover the isomorphism class of our G-set. □

Example 7.4.3 We now look at an actual example of the use of the lattice invariant in practice, by considering a case similar to Example 6.5.4. Let $f = (x_1^2, x_2^2, x_3^2, x_4^2) : \mathbf{C}^4, 0 \rightarrow \mathbf{C}^4, 0$ and $f_t = (x_1^2 - t^2, x_2^2 - t^2, x_3^2 - t^2, x_4^2 - t^2)$. This gives us $f_t^{-1}(0) = \{(\pm t, \pm t, \pm t, \pm t)\}$, which contains 16 points. Let $G = \mathbf{Z}_2 \times \mathbf{Z}_2$

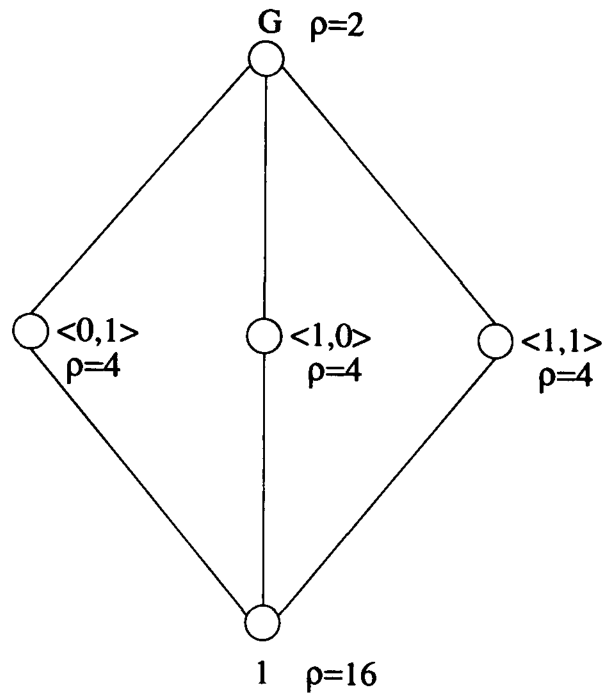


Figure 7.1: The lattice of subgroups of $\mathbf{Z}_2 \times \mathbf{Z}_2$ with values of ρ given for f.

which acts on both source and target via the action

$$\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right]$$

where the first matrix is the representation of $(0,1)$ and the second of $(1,0)$. The map f is finite and G-equivariant with respect to this action. By consideration of the various orbits of the points in $\{(\pm t, \pm t, \pm t, \pm t)\}$, we obtain the following table.

Representatives of orbits	**Order of orbits**	**Group action**	**Number of orbits**
$\pm(t,t,t,t)$	1	R_1	2
$(t,t,-t,-t)$	2	R_2	1
$(t,-t,t,-t)$	2	R_3	1
$(t,-t,-t,t)$	2	R_4	1
$\pm(-t,t,t,t)$	4	R_5	2

This means that the permutation representation of the action on the points is given by $2R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus 2R_5$. Its character, σ, has values as follows.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
σ	16	4	4	4

We showed in Example 7.4.1 that the permutation representations $2R_1 \oplus R_5$ and $R_2 \oplus R_3 \oplus R_4$ have identical characters, so given σ above, we cannot determine the isomorphism class of the G-set formed by the points $\{(\pm t, \pm t, \pm t, \pm t)\}$. From σ, we can obtain the following equations.

$$\begin{aligned} 4c(1) + 2c(<0,1>) + 2c(<1,0>) + 2c(<1,1>) + c(G) &= 16 \\ 2c(<0,1>) + c(G) &= 4 \\ 2c(<1,0>) + c(G) &= 4 \\ 2c(<1,1>) + c(G) &= 4 \end{aligned}$$

This is obviously not enough information to recover all the values of $c(H)$ for $H \leq G$. The lattice of (isotropy) subgroups is shown in Figure 7.1, with the value of ρ given at each point. The Möbius function is given by the following table.

$m(x,y)$	1	$<0,1>$	$<1,0>$	$<1,1>$	G
1	1	-1	-1	-1	2
$<0,1>$	0	1	0	0	-1
$<1,0>$	0	0	1	0	-1
$<1,1>$	0	0	0	1	-1
G	0	0	0	0	1

From this, using (7.1) we have

$$\begin{aligned} c(G) &= \frac{|G|}{|G|}\left(m(G,G)\cdot\rho(G)\right) \\ &= (1\cdot 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} c(<0,1>) &= \frac{|<0,1>|}{|G|}(m(<0,1>,<0,1>)\cdot\rho(<0,1>)+ \\ &\qquad +m(G,<0,1>)\cdot\rho(G)) \\ &= \frac{1}{2}(1\cdot 4+(-1)\cdot 2) \\ &= 1 \end{aligned}$$

$$\begin{aligned} c(1) &= \frac{|1|}{|G|}(m(1,1)\cdot\rho(1)+m(<0,1>,1)\cdot\rho(<0,1>)+ \\ &\qquad +m(<1,0>,1)\cdot\rho(<1,0>)+m(<1,1>,1)\cdot\rho(<1,1>)+ \\ &\qquad +m(G,1)\cdot\rho(G)) \\ &= \frac{1}{4}(1\cdot 16+(-1)\cdot 4+(-1)\cdot 4+(-1)\cdot 4+2\cdot 2) \\ &= 2 \end{aligned}$$

and similarly for $c(<1,0>)$ and $c(<1,1>)$. This gives us exactly the G-set isomorphism class expected.

Now we turn to the real case. We will define a signed G-set to be a G-set with each orbit designated to be either positive or negative. Two signed G-sets will be said to be isomorphic if they are isomorphic in the usual way, but also with preservation of sign. A 'cancelling pair of orbits' is a pair of isomorphic orbits of opposite sign. Suppose we have $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$, which is finite and G-variant and f_t, a finite G-variant deformation such that 0 is a regular value when $t \neq 0$. We set $f_t^{-1}(0) = \{a_1, \ldots, a_\mu\}$ as before. Now given the G-index of f with this deformation, we have insufficient information to recover the signed G-set formed by the points $\{a_1, \ldots, a_\mu\}$ (with sign given by the Jacobian). However, we can mimic the results in the complex case as follows.

Let L be the lattice of subspaces of the source of the form

$$V(H) = \{v \in \mathbf{R}^n : h \cdot v = v \text{ for all } h \in H\}$$

where $H \leq G$, much as before. Let $f_H = f|_{V(H)}$ and define

$$\begin{array}{rccl} \rho : & L & \to & \mathbf{N} \\ & V(H) & \mapsto & deg(f_H) = \sum_{a_i \in V(H)} \mathrm{sign}(J(a_i)) \end{array}$$

We may now state the real analogue of the previous result:

Proposition 7.4.4 *Let f, f_t,$\{a_i\}$, L and ρ be as above, then given L and ρ, we are able to determine the isomorphism class of the signed G-set formed by $\{a_1, \ldots, a_\mu\}$, up to cancelling pairs.*

Proof We define

$$\begin{array}{rccl} \rho^* : & L & \to & \mathbf{N} \\ & V(H) & \mapsto & \sum_{a_i \in A_H} \mathrm{sign}(J(a_i)) \end{array}$$

where $A_H = \{a_i \in V(H) : a_i \notin V(H') \text{ for } H' \geq H\}$. Then the argument of Proposition 7.4.2 gives us

$$c^+(H) - c^-(H) = \frac{|H|}{|N(H)|} \sum_{H' \geq H} m(V(H'), V(H)) \cdot \rho(V(H'))$$

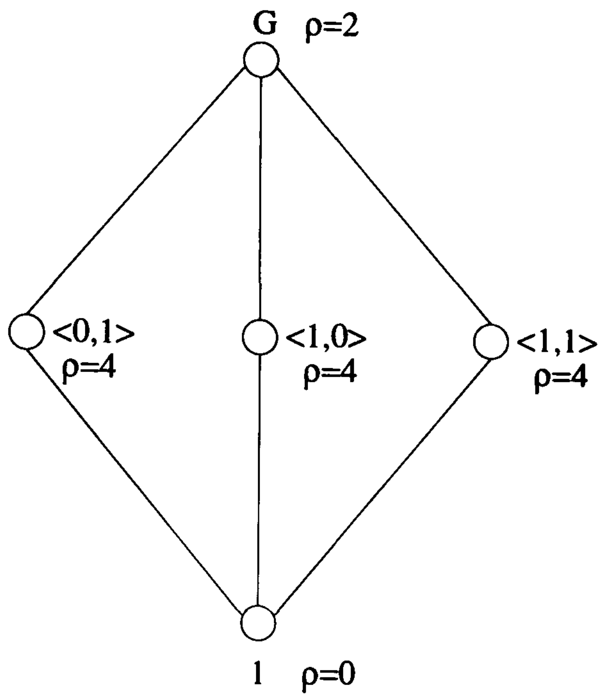


Figure 7.2: The lattice of subgroups of $\mathbf{Z}_2 \times \mathbf{Z}_2$, with ρ given for f in the real case.

where $c^+(H)$ (resp. $c^-(H)$) is the number of orbits with isotropy group H or a conjugate, with positive (resp. negative) sign. □

Example 7.4.5 Let us look at the real version of Example 7.4.3, taking $f = (x_1^2, x_2^2, x_3^2, x_4^2) : \mathbf{R}^4, 0 \to \mathbf{R}^4, 0$ and $f_t = (x_1^2 - t^2, x_2^2 - t^2, x_3^2 - t^2, x_4^2 - t^2)$. This gives us $f_t^{-1}(0) = \{(\pm t, \pm t, \pm t, \pm t)\}$ as before. We take $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ acting as before, so f and f_t are both finite and G-equivariant. Now the Jacobian determinant, $J = 16x_1x_2x_3x_4$ and so we obtain the following list of orbits.

Representatives of orbits	**Order of orbits**	**Group action**	**Number of orbits**	**Sign of Jacobian**
$\pm(t,t,t,t)$	1	R_1	2	+
$(t,t,-t,-t)$	2	R_2	1	+
$(t,-t,t,-t)$	2	R_3	1	+
$(t,-t,-t,t)$	2	R_4	1	+
$\pm(-t,t,t,t)$	4	R_5	2	−

This means that the G-variant degree of the map is given by $(2R_1+R_2+R_3+R_4)-(2R_5)$ in the representation ring of G. Now the associated lattice of subgroups is shown in Figure 7.4. Using this and the Möbius function (which is the same as before) we obtain

$$\begin{aligned} c^+(G) - c^-(G) &= \frac{|G|}{|G|}(m(G,G)\cdot\rho(G)) \\ &= (1\cdot 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} c^+(<0,1>) - c^-(<0,1>) &= \frac{|<0,1>|}{|G|}(m(<0,1>,<0,1>)\cdot\rho(<0,1>) + \\ &\quad +m(G,<0,1>)\cdot\rho(G)) \\ &= \frac{1}{2}(1\cdot 4 + (-1)\cdot 2) \\ &= 1 \end{aligned}$$

$$\begin{aligned} c^+(1) - c^-(1) &= \frac{|1|}{|G|}(m(1,1)\cdot\rho(1) + m(<0,1>,1)\cdot\rho(<0,1>) + \\ &\quad +m(<1,0>,1)\cdot\rho(<1,0>) + \\ &\quad +m(<1,1>,1)\cdot\rho(<1,1>) + \\ &\quad +m(G,1)\cdot\rho(G)) \\ &= \frac{1}{4}(1\cdot 0 + (-1)\cdot 4 + (-1)\cdot 4 + (-1)\cdot 4 + 2\cdot 2) \end{aligned}$$

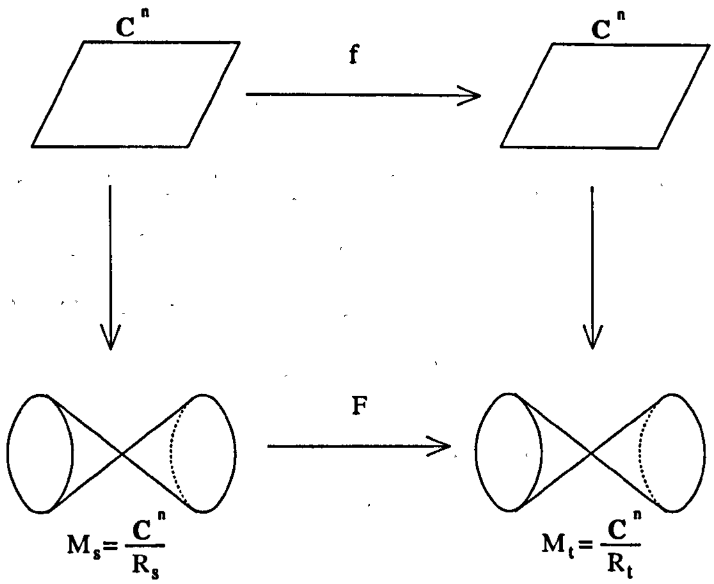


Figure 7.3: The map F derived from the G-variant map f

$$= \quad -2$$

and similarly for <1, 0> and <1, 1>. This gives us the signed G-set isomorphism class expected, up to cancelling pairs (in fact there are none in this case).

7.5 Quotient spaces

We now consider the spaces formed by taking the quotient by the action of the group on each of the target and and quotient spaces. We will denote the quotient by the actions R_S and R_T by M_S and M_T respectively. We will use simply M when we wish to refer to some general quotient space. The finite G-variant map $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ gives a new map $F : M_S \to M_T$ as shown in Figure 7.3. The (complex) G-variant degree of f can sometimes be calculated from F as the following example shows. (We will not pursue this approach.)

Example 7.5.1 Take $G = \mathbf{Z}_2$ acting on $\mathbf{C}^2$ (both source and target) via the representation

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

Then we have $M_S \cong M_T \cong \{(x^2, xy, y^2) : (x,y) \in \mathbf{C}^2\} \subset \mathbf{C}^3$. This is the variety $\{(u,v,w) : v^2 = uw\} \subset \mathbf{C}^3$. Take $f = (x^3, y^3) : \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ to be our finite G-(equi)variant map. Then this gives

$$\begin{aligned} F : M_S &\to M_T \\ (x^2, xy, y^2) &\mapsto (x^6, x^3y^3, y^6) \\ (u,v,w) &\mapsto (u^3, v^3, w^3). \end{aligned}$$

Let us take $f_t = (x^3 - t^2x, y^3 - t^2y)$ to be our finite G-variant deformation. This gives a deformation of F given by

$$\begin{aligned} F_t : M_S &\to M_T \\ (x^2, xy, y^2) &\mapsto (x^6 - 2t^2x^4 + t^4x^2, x^3y^3 - t^2x^3y - t^2xy^3 + t^4xy, \\ &\qquad y^6 - 2t^2y^4 + t^4y^2) \\ (u,v,w) &\mapsto (u^3 - 2t^2u^2 + t^4, v^3 - t^2uv - t^2vw + t^4v, w^3 - 2t^2w^2 + t^4w). \end{aligned}$$

Now if we consider the preimages of zero under the map F_t we obtain

$$\begin{aligned} F_t^{-1}(0) &= \{(0,0,0), (t^2,0,0), (0,0,t^2), \\ &\qquad (t^2,t^2,t^2), (t^2,-t^2,t^2)\}. \end{aligned}$$

Now since we are in a quotient space, each of these points corresponds to an orbit in the original affine space. Now $(0,0,0)$ is the only possible trivial orbit and the group must act via the representation

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

on the rest. Thus the (complex) G-variant degree is given by the direct sum of one copy of the trivial action $[(1)]$ and four copies of the representation above.

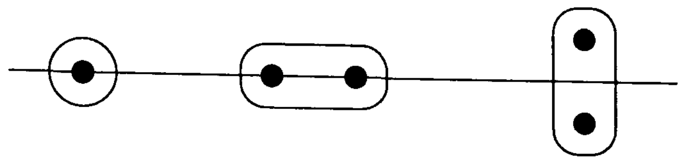


Figure 7.4: The three types of orbit in $M_{S,\mathbf{R}}$

Now let us look at the real case. Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a finite G-variant map which we can consider as a map $\mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ to give a new map F as before. Now complex conjugation on the original space maps M to itself with a set of invariant points $M_{\mathbf{R}}$. Now this will include the image of $\mathbf{R}^n$, but will in general be larger. To calculate the real G-variant degree, we are only interested in orbits which are mapped to themselves under complex conjugation (A_1 and A_2 points). This means that it is possible to calculate the real G-variant degree of f by consideration of F and $M_{S,\mathbf{R}}$ as the following example shows.

Example 7.5.2 We take $G = \mathbf{Z}_2$ acting on $\mathbf{R}^2$ (both source and target) via the same representation as the previous example. Now $M_{S,\mathbf{R}}$ is given by the image of the set

$$\{(x^2, xy, y^2) : x, y \in \mathbf{R}\} \cup \{(x^2, xy, y^2) : x, y \in i\mathbf{R}\}.$$

The three types of orbit these points can represent are shown in Figure 7.4. Now only the point $(0, 0, 0)$ corresponds to an A_1 fixed point, while the A_1 unfixed orbit and A_2 orbit correspond to the sets

$$\{(x^2, xy, y^2) : x, y \in \mathbf{R}\} - \{(0, 0, 0)\}$$

and

$$\{(x^2, xy, y^2) : x, y \in i\mathbf{R}\} - \{(0, 0, 0)\}$$

respectively. These two sets are given by points in the upper and lower halves of the cone shown in Figure 7.5. Now if we take $f = (x^3, y^3) : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ to be

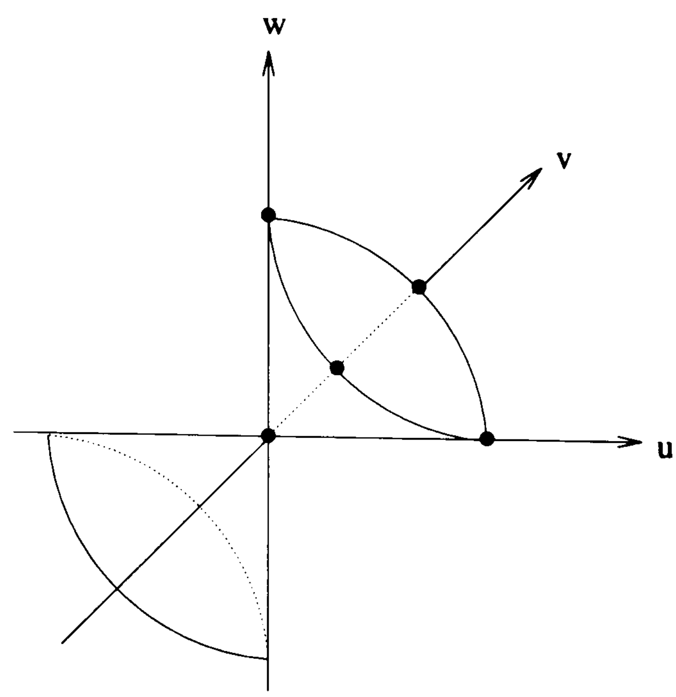


Figure 7.5: The preimages of 0 in $M_{S,\mathbf{R}}$

our finite G-variant map as before, then we also get $F : (u, v, w) \mapsto (u^3, v^3, w^3)$ as previously. Let us take $f_t = (x^3 - t^2x, y^3 - t^2y)$, $t \in \mathbf{R}$ to be our finite G-variant deformation. This gives a deformation of F given by $F_t : (u, v, w) \mapsto (u^3 - 2t^2u^2 + t^4, v^3 - t^2uv - t^2vw + t^4v, w^3 - 2t^2w^2 + t^4w)$. As before, we have

$$\begin{aligned} F_t^{-1}(0) &= \{(0,0,0), (t^2,0,0), (0,0,t^2), \\ &\quad (t^2,t^2,t^2), (t^2,-t^2,t^2)\} \end{aligned}$$

(see Figure 7.5). Now the Jacobian determinant J of f_t is given by $(3x^2 - t^2)(3y^2 - t^2)$ and so $\phi = 1/J$ has the following values.

Point	ϕ
$(0,0,0)$	$1/t^4$
$(t^2,0,0)$	$-1/2t^4$
$(0,0,t^2)$	$-1/2t^4$
(t^2,t^2,t^2)	$1/4t^4$
$(t^2,-t^2,t^2)$	$1/4t^4$

Since t is real, all these points are in A_1, with only $(0,0,0)$ being a fixed point. This means that the last four points cancel each other out, since they are all of the same type, but with two each positive and negative. So we obtain the trivial representation $[(1)]$ as the real G-variant degree of f. (Compare Example 7.1.7).

Chapter 8

The class of G-variant mappings

In this chapter, we examine the G-variant mappings which are associated to a given pair $\{R_S, R_T\}$ of representations of a finite group G on $\mathbf{C}^n$. We will adopt some of the notation from [R2] (also used in §7.4) and often denote the source and target spaces by V and W respectively. This is useful mainly for writing $V(H)$ and $W(H)$ to distinguish the space fixed by the subgroup H in the source from that in the target space.

8.1 Basic results

Lemma 8.1.1 *Let G be a finite group acting via representations R_S and R_T on the source and target spaces respectively, each isomorphic to $\mathbf{C}^n$. Then the set M of G-variant maps $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ is a module over the ring of invariant functions $h : \mathbf{C}^n, 0 \to \mathbf{C}$ in the source, denoted $\mathcal{O}_n^{G(S)}$.*

Proof If $g \in G$, $f_1, f_2 : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ are variant and $h \in \mathcal{O}_n^{G(S)}$, then we have

$$\begin{aligned}(f_1 + f_2)(g(x)) &= f_1(g(x)) + f_2(g(x)) \\ &= g \cdot f_1(x) + g \cdot f_2(x) \\ &= g \cdot (f_1 + f_2)(x)\end{aligned}$$

and also

$$\begin{aligned}(hf_1)(g(x)) &= h(g(x))f_1(g(x)) \\ &= h(x)(g \cdot f_1(x)) \\ &= g \cdot (hf_1)(x)\end{aligned}$$

thus giving the desired result. □

Corollary 8.1.2 *With R_S and R_T as above, the set $M^k = M \cap (\mathcal{M}_n^k \mathcal{O}(n,n))$ (i.e. the variant maps with components in $\mathcal{M}_n^k$) is a module over $\mathcal{O}_n^{G(S)}$.*

Proof As above, but with $f_1, f_2 \in M^k$. □

Definition Let M_p denote $M \cap \mathbf{C}[x]^G$, the set of G-variant maps with polynomial components and let M_p^k be those consisting of monomials of degree at least k.

We also require the following, related to subgroups of the group G.

Definition Given a point x let G_x be the isotropy subgroup of this point, $\{g \in G : g \cdot x = x\}$. For a subgroup $H \leq G$ acting on a space V, define $V(H) = \{x \in V : h(x) = x \text{ all } h \in H\}$.

The following is a useful lemma regarding invariant polynomials.

Lemma 8.1.3 *Suppose a finite group G acts on the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ by acting linearly on the coordinates. If the ring of G-invariant polynomials is non-empty, then for any non-zero x there is a G-invariant polynomial f such that $f(x) \neq 0$.*

Proof Suppose the ring of G-invariant polynomials is generated by $f_1, \ldots, f_N$. We can define $F : \mathbf{C}^n \to \mathbf{C}^N$ to be the map which takes these polynomials as components. Then $F^{-1}(0) = 0$ since $F(x_1) = F(x_2)$ if and only if x_1 and x_2 lie in the same orbit. Thus for every non-zero point x, there is a polynomial f such that $f(x) \neq 0$. □

In what follows, we will be most interested in those G-variant maps which are also finite. However, it is possible that the given pair of representations might force all G-variant maps to be non-finite. The following two results gives conditions which must hold for any finite G-variant maps to exist.

Proposition 8.1.4 *Suppose G acts via representations R_S and R_T on the source and target spaces respectively. If the set of finite G-variant maps is non-empty then the representations must satisfy order($R_S(g)$) is divisible by order($R_T(g)$) for all $g \in G$.*

Proof Let f be a finite G-variant map and g some element of G. Suppose that there is some $m \in \mathbf{N}$ such that $R_S(g^m) = I$, $R_T(g^m) \neq I$. Then we have

$$f(x) = R_T(g^m)(f(x))$$

for all x, which means that for each i we can write

$$f_i = \alpha_{i,1} f_1 + \cdots + \alpha_{i,j} f_j + \cdots + \alpha_{i,n} f_n$$

for some $\alpha_{i,j} \in \mathbf{C}$ or alternatively,

$$0 = \alpha_{i,1} f_1 + \cdots + (\alpha_{i,j} - 1) f_j + \cdots + \alpha_{i,n} f_n. \tag{8.1}$$

Now as $R_T(g^m) \neq I$, there is some i such that the coefficients in (8.1) above are not all zero. In other words, the components of the map f are linearly dependent and the ideal I_f is generated by $n - 1$ elements, say $f_1, \ldots, f_{n-1}$. Now if we are given r germs $f_1, \dots f_r$, then the (germ of the) analytic variety they give has codimension at most r. (See [GR, p93].) So if $r = n-1$ then this codimension will be at most $n-1$. But $I_f \subset \mathcal{M}_n^k$ for some k, since f is finite (see Lemma 6.2.1) and so $\mathbf{V}(f_1, \dots f_{n-1}) = \{0\}$ which has codimension n. We have therefore reached a contradiction, showing that order($R_S(g)$) is divisible by order($R_T(g)$). □

Proposition 8.1.5 *Let V, W be equidimensional vector spaces which have a representation of G acting on them. If there is a finite, G-variant mapping $f : V, 0 \to W, 0$, then for all $H < G$, we must have dim $V(H) \leq$ dim $W(H)$.*

Proof If $x \in V(H)$ then $h \cdot x = x$ for all $h \in H$. So $h \cdot f(x) = f(h \cdot x) = f(x)$ and thus $h \in G_{f(x)}$, i.e. $H < G_{f(x)}$. But then $W(H) \supset W(G_{f(x)})$ so $f(x) \in W(H)$ i.e. $f(V(H)) \subset W(H)$ and as f is finite, dim $V(H) \leq$ dim $W(H)$. □

It is possible that there are no finite G-variant maps associated with a given pair of representations on source and target as the following demonstrates.

Example 8.1.6 We take our group G to be $\mathbf{Z}_2$, generated by g. We let this act on the source space, $V = \mathbf{C}^2$ by $g \cdot (x, y) = (x, -y)$ and on the target space, $W = \mathbf{C}^2$ by $g \cdot (x, y) = (-x, -y)$. Then a map-germ $f : V, 0 \to W, 0$ is G-variant if and only if

$$f_j(x, -y) = -f_j(x, y)$$

for $j = 1, 2$. So this means that $f_j(x, y) = y g_j(x, y^2)$ for some function g_j and that $f^{-1}(0) \supset \mathbf{C} \times \{0\}$ i.e. f is non-finite.

We see that for any point $x = (a, 0)$, we have $G_x = \mathbf{Z}_2$. But for any point x and any G-variant map f, $G_{f(x)} \geq G_x$, so $G_{f(x)} = \mathbf{Z}_2$. But $W(\mathbf{Z}_2) = \{0\}$, i.e. $f(x) = 0$. So the condition dim $V(H) \leq$ dim $W(H)$ does not hold for each isotropy group H.

Definition Let us consider M_p, the $\mathbf{C}[x]^G$-module of polynomial G-variant mappings $f : V, 0 \to W, 0$. There is a map $E : V(H) \times M_p \to W(H)$ defined by $E_x(f) = E(x, f) = f(x)$. Let $V(H)^* = \{x \in V(H) : G_x = H\}$. We say that the pair of representations is *full* if for all $x \in V(H)^*$ sufficiently close to the origin, dim im $E_x \geq$ dim $V(H)$, for all isotropy subgroups $H \leq G$.

Before we give the first proposition regarding full representations, we need the following results.

Lemma 8.1.7 *Let X, Y be affine varieties with $\pi : X \to Y$ a regular mapping. If dim $\pi^{-1}(y) \geq r$ for all y on some open subset, then dim $X \geq$ dim $Y + r$.*

Proof Let $X_1, \ldots, X_r$ be the irreducible components of X. The hypotheses hold for (at least) one of the restrictions $\pi : X_i \to Y$. Now on an open subset of Y we have dim $\pi^{-1}(y)$=dim X_i- dim Y by [Sh, p60]. This means dim X_i- dim $Y \geq r$, i.e. dim $X_i \geq$ dim $Y + r$. But dim X = max { dim X_j} $\geq$ dim X_i and the result follows. □

Lemma 8.1.8 *Let A,B and C be smooth affine spaces and suppose $f : A \times B \to C$ is a polynomial map with rank $df(a, b) \geq$ dim A for all $(a, b) \in f^{-1}(C)$. Then, for all $b \in B$ off an algebraic subset of codimension at least 1, we have $f_b^{-1}(c)$ finite.*

Proof We claim that $f^{-1}(c)$ is an algebraic set of dimension at most dim B. For suppose dim $f^{-1}(c) >$ dim B. We can choose a smooth point $(a, b) \in f^{-1}(c)$ where the rank of $df(a, b)$ is maximal (and at least dim A). But $df(a, b)$ annihilates a subspace of dimension dim $f^{-1}(c)$. Thus dim ker $df(a, b) \geq$ dim $f^{-1}(c) >$ dim B and the rank-nullity theorem yields a contradiction.

Now consider the projection $\pi : f^{-1}(c) \subset A \times B \to B$. For each $b \in B$, we have $\pi^{-1}(b) = f_b^{-1}(c)$ and if dim $\pi^{-1}(b) \geq 1$ on an open subset of B, we would have dim $f^{-1}(c) > \dim B$ by Lemma 8.1.7 above. □

Proposition 8.1.9 *If the pair of representations is full, then almost all mappings (in the strong sense) $f : V, 0 \to W, 0$ which are G-variant are finite.*

Proof When we say "almost all in the strong sense" we mean that the set of elements for which the property fails to hold has infinite codimension. Given $f : V, 0 \to W, 0$, a polynomial, G-variant map of degree at most k, we wish to find some polynomial map ϕ of degree at least $k+1$ with $f + \phi : V, 0 \to W, 0$ both G-variant and finite. (Note that by degree in this case we mean the maximum degree of the polynomial components.) In this way, if we look at the maps of a certain maximum degree, as we allow this degree to increase, the codimension of the set of 'bad' maps will also increase. Thus in the limit, we have a set of infinite codimension. (See [B, ch 13] for the full definitions and results.)

We claim that we can find $\phi_1, \ldots, \phi_N \in M_p$ of degree $\geq k + 1$ such that for all $x \in V(H)^*$ (and each isotropy group H) $\phi_1(x), \ldots, \phi_N(x)$ span a space of dimension at least dim $V(H)$. First, let us fix our isotropy group $H \leq G$ and $x \in V(H)^*$. We can now certainly find $\psi_1, \ldots, \psi_r$ such that $\psi_1(x), \ldots, \psi_r(x)$ span a space of dimension at least dim $V(H)$, as the representation pair is full. We can also find some polynomial $\alpha \in \mathbf{C}[x]^G$ such that $\alpha(0) = 0$, but $\alpha(x) \neq 0$. Then $\alpha^k\psi_1(x), \ldots, \alpha^k\psi_r(x)$ span a space of the required dimension and are of degree $\geq k+1$. Now suppose that $\phi_1, \ldots, \phi_N$ generate M_p^k as a $\mathbf{C}[x]^G$-module. Then we see that

$$\text{span}\{\phi_1(x), \ldots, \phi_N(x)\} = \text{span}\{f(x) : f \in M_p^k\}.$$

Now this space must contain a set of the form $\alpha^k\psi_1, \ldots, \alpha^k\psi_r$ (as constructed above) for each x and thus must span a space of dimension of at least dim $V(H)$ for each such x.

Now we choose a representative map for the germ f, which we will also call f. Consider the map

$$F : V \times \mathbf{C}^N, 0 \to W, 0$$

defined by

$$F(x, \lambda) = f(x) + \sum_{i=1}^{N} \lambda_i \phi_i(x).$$

For each isotropy group H, we have the restriction

$$F_H : V(H)^* \times \mathbf{C}^N, 0 \to W(H), 0.$$

We now claim that $dF_H(x, \lambda)$ has rank $\geq \dim V(H)^*$ for each (x, λ). At (x, λ) consider

$$\begin{aligned} dF_H(e_j) &= lim_{t \to 0} \frac{F_H(x, \lambda + te_j) - F_H(x, \lambda)}{t} \\ &= lim_{t \to 0} \frac{F_H(x, \lambda) + t\phi_j(x) - F_H(x, \lambda)}{t} \\ &= \phi_j(x) \end{aligned}$$

where e_j is the j^{th} unit vector in $\mathbf{C}^N$. Now $x \neq 0$, so

$$\begin{aligned} \text{Im } dF_H(x, \lambda) &\supset \text{span}\{dF_H(x, \lambda)e_j\} \\ &\supset \text{span}\{\phi_j(x)\} \end{aligned}$$

where dim span$\{\phi_j\} \geq$ dim $V(H)$. Thus for almost all λ, we have $F_H(-, \lambda)^{-1}(0)$ finite by Lemma 8.1.8. Choosing a value for λ which satisfies this condition for each isotropy group H, we consider $f_\lambda = F(-, \lambda)$. This is G-variant and $f_\lambda^{-1}(0) \cap V(H)^*$ is finite for each H. But $\bigcup_{H \leq G} V(H)^* = V$ and we just need to take the germ of f_λ at the origin and we are done. □

We are interested in conditions for our map f to have a 'good' deformation f_t. This means we are seeking a family of G-variant maps such that O is a regular value for all t close to 0. We begin by showing that this is always the case when considering the invariant situation.

Proposition 8.1.10 *Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a finite G-invariant map (so the action of G on the target is trivial). Then there exists a G-invariant deformation f_t of f such that 0 is a regular value for all t close to 0.*

Proof Let y_0 be a regular value of f and define $F = f - y_0$. Now

$$F(g \cdot x) = f(g \cdot x) - y_0 = f(x) - y_0 = F(x)$$

so F is G-variant and

$$F(x_0) = f(x_0) - y_0 = y_0 - y_0 = 0$$

for any $x_0 \in f^{-1}(y_0)$, so 0 is a regular value of F. To define the deformation, we then join 0 to y_0 by a path $\gamma(t)$ and set $f_t = f + \gamma(t)$. See Corollary 8.1.12 below for a detailed description in a more general case. □

We now consider another class of G-variant maps.

Proposition 8.1.11 *Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be G-equivariant and finite and suppose there exists an invertible linear map L which is also G-equivariant. Then if the multiplicity of f is r, we can find $t_1, \ldots, t_r \in \mathbf{C}$ arbitrarily small and $u_{(1)}, \ldots, u_{(r)} \in \mathbf{C}^n$ arbitrarily close to 0 such that the map*

$$f(x) + \sum_{i=1}^{r} t_i L(x - u_{(i)})$$

has 0 as a regular value and is also G-equivariant.

Proof We construct the new map by an inductive process, which is similar to one mentioned, but not given explicitly in [GZ]. Firstly, we look at the polynomial in t given by det $(df(0) + tL)$. We can choose an arbitrarily small value t_1 which is not a root of this equation. Now $F_1 = f + t_1 L$ is G-equivariant and 0 is a regular point of this map with value 0. Note that for any small deformation of F_1 there will be a regular point close to 0 which maps to 0. In other words, it will remain regular provided subsequent deformations are sufficiently small. Let us denote the other (i.e. not 0) points of $F_1^{-1}(0)$ by $x_{(1)}, \ldots, x_{(s)}$. Note that $s < r$ and that by choosing t_1 small we can get the $x_{(i)}$ arbitrarily close to 0. since F_1 is G-equivariant, these points form a union of orbits under the action of G. Suppose $x_{(1)}, \ldots, x_{(m)}$ is an orbit of non-regular points. We apply the same argument as above to the points $x_{(i)}$. Let us define

$$F_{2,t}(x) = F_1(x) + t \sum_{i=1}^{m} L(x - x_{(i)}).$$

This new map is in fact G-equivariant, for

$$\begin{aligned} g \cdot F_{2,t} &= g \cdot F_1(x) + t \sum g \cdot L(x - x_{(i)}) \\ &= F_1(g \cdot x) + t \sum L(g \cdot x - g \cdot x_{(i)}) \\ &= F_1(g \cdot x) + t \sum L(g \cdot x - x_{(i)}) \\ &= F_{2,t}(g \cdot x). \end{aligned}$$

Also, $F_{2,t}(x_{(i)}) = F_1(x_{(i)}) = 0$. If we look at the derivative, we obtain

$$dF_{2,t}(x) = df(x) + t_1 L + tL.$$

We can clearly find t_2 arbitrarily small with $dF_{2,t_2}(x_{(i)})$ invertible for $1 \leq i \leq m$. We have now have $m+1$ regular points, with $m+1 \geq 2$. We now continue inductively, by setting $F_2 = F_{2,t_2}$, considering non-regular points of $F_2^{-1}(0)$, selecting an orbit and defining F_3 and so on. We ensure at each stage that all regular points remain so under small deformations. The process must terminate as the map has finite multiplicity and the result follows. □

Note that the G-equivariance (as opposed to G-variance) of f is not used explicitly in the above. However, the hypothesis that an invertible linear G-variant map exists forces the two actions to be equal.

Corollary 8.1.12 *Suppose f is as given above. Then there is an analytic map $F : \mathbf{C}^n \times \mathbf{C}, 0 \to \mathbf{C}^n$, with each $F_t = F(-,t)$ G-variant and for all t sufficiently close to 0 we have 0 a regular value of F_t.*

Proof Let us define

$$F(x, \lambda, u) = f(x) + \sum_{i=1}^{r} \lambda_i L(x - u_{(i)}).$$

The condition on (λ, u) for $F(-, \lambda, u)$ to be G-variant is that

$$\sum_{i=1}^{r} \lambda_i L(g \cdot u_{(i)} - u_{(i)}) = 0$$

for all $g \in G$. The condition above gives finitely many polynomial equations in (λ, u) space.

Now choose a neighbourhood $\mathcal{U}$ of 0 with $f^{-1}(0) \cap \bar{\mathcal{U}} = \{0\}$. We can choose a neighbourhood $\mathcal{V}$ of $(0,0)$ in (λ, u) space such that for $(\lambda, u) \in \mathcal{V}$, we have

$$F(-, \lambda, u)^{-1}(0) \cap \partial\mathcal{U} = \emptyset.$$

So if f has multiplicity r, then $F(-, \lambda, u)^{-1}(0) \cap \mathcal{U}$ consists of r points when counted with multiplicity. Consider the set

$$\Sigma = \{(x, \lambda, u) : F(x, \lambda, u) = 0, \det(dF_{\lambda,u}(x)) = 0\}$$

This is an analytic subset of $\mathcal{U} \times \mathcal{V}$. If we look at the projection $\pi : \Sigma \to \mathcal{V}$, we find that this projection is finite to one. So by [GR, p83], the image $\pi(\Sigma)$ is analytic. If $X \subset \mathcal{V}$ is the set of G-variant maps then $0 \in X \cap \pi(\Sigma)$ and we know from the previous result that 0 lies in the closure of $X - \pi(\Sigma)$. So dim $X \cap \pi(\Sigma) <$ dim X at 0 and by [GR, p104] there is a line through 0 meeting $\pi(\Sigma)$ in an isolated point at 0. The result follows. □

Example 8.1.13 We now consider an example. We take $G = \mathbf{Z}_2$, and set

$$R_S = R_T = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

Now the G-variant map (x, y) is linear and invertible, so by Proposition 8.1.11 every finite G-variant map must have a "good" deformation. For example, the map $f = (x^3, y^3)$ is G-variant and finite and has the deformation $f_t = (x^3 - t^2x, y^3 - t^2y)$. The Jacobian of f_t is $(3x^2 - t^2)(3y^2 - t^2)$ which is non-zero at $x, y = 0, t, -t$ and so 0 is a regular value.

We now take a close look at the action of the symmetric group S_n acting naturally by permutation of the basis vectors. In what follows, our group $G = S_n$ will act on both target and source spaces in this way. Thus we are interested in equivariant maps.

Proposition 8.1.14 *Suppose $G = S_n$ acts as above, then f is G-equivariant if and only if we have*

$$f = (\sum_j x_1^j g_j(x_2, x_3, \dots, x_n), \sum_j x_2^j g_j(x_1, x_3, \dots, x_n), \dots, \sum_j x_n^j g_j(x_1, x_2, \dots, x_{n-1}))$$

for some $g_j(u_1, \dots, u_{n-1})$ which are S_{n-1} invariant in the obvious sense.

Proof Suppose $(f_1, \dots, f_n)$ is G-equivariant, then if σ_{ij} is the transposition (i, j), we have

$$\begin{aligned} \sigma_{ij} \cdot (f_1, \dots, f_n) &= (f_1, \dots, f_{i-1}, f_j, f_{i+1}, \dots, f_{j-1}, f_i, f_{j+1}, \dots, f_n) \\ (f_1, \dots, f_n) \cdot \sigma_{ij} &= (f_1(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_n), \dots) \end{aligned}$$

So we see that $f_k \cdot \sigma_{ij} = f_k$ if $i, j \neq k$. Thus if we write

$$f_k = \sum_l x_k^l g_{l,k}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n),$$

each of the $g_{l,k}$ must be symmetric in its $n-1$ variables (i.e. S_{n-1} invariant). We also have that $f_i \cdot \sigma_{ij} = f_j$, so

$$\sum_l x_j^l g_{l,i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sum_l x_j^l g_{l,j}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and thus $g_{l,i} = g_{l,j}$ for all l. Finally, the map $(f_1, \ldots, f_n)$ defined by

$$f_k = \sum_l x_k^l g_l(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),$$

is clearly equivariant. □

Proposition 8.1.15 *Given the action of S_n as above, on both our target and source spaces, this pair of actions is full.*

Proof Let us denote the fixed point hyperplane of a transposition σ by $H(\sigma)$. If we write $R = \{(ij) : i \leq j\}$ then the fixed point set (of any isotropy subgroup) can be written

$$H(I) = \bigcap_{\sigma \in I} H(\sigma)$$

with $I \subset R$, as it must be generated by transpositions. We have

$$H(I)^* = H(I) - \bigcup_{J \subset I, J \neq I} H(J).$$

If we have a point of $H(I)^*$ we can reorder the variables so that it is of the form

$$(\underbrace{u_1, \ldots, u_1}_{m_1}, \underbrace{u_2, \ldots, u_2}_{m_2}, \ldots, \underbrace{u_s, \ldots, u_s}_{m_s})$$

where $\sum_{j=1}^{s} m_j = n$ and $u_i \neq u_j$ for $i \neq j$. The m_j form a partition of n and each such partition corresponds to an orbit type. We wish to find G-equivariant maps $F_1, \ldots, F_s$ with $F_1(x), \ldots, F_s(x)$ linearly independent for each $x \in H(I)^*$. For, given such F_i, if $x \in H(I)^*$, then if g fixes x, then

$$g \cdot F_i(x) = F_i(g \cdot x) = F_i(x)$$

and so $F_i(x) \in H(I)$. If we define

$$F_j(x) = (x_1^j, \dots, x_n^j)$$

then this is clearly a G-equivariant mapping. Also,

$$\begin{array}{ccc}
(1,\dots,1, & \dots & ,1,\dots,1) \\
(x_1,\dots,x_1, & \dots & ,x_s,\dots,x_s) \\
(x_1^2,\dots,x_1^2, & \dots & ,x_s^2,\dots,x_s^2) \\
 & \vdots & \\
(x_1^{s-1},\dots,x_1^{s-1}, & \dots & ,x_s^{s-1},\dots,x_s^{s-1})
\end{array}$$

are linearly independent since the x_i are distinct (Vandermonde matrix). □

We now restrict our attention to the hyperplane $L = \{x_1 + \cdots + x_n = 0\}$. Clearly, S_n acts on L and we wish to prove that this action on source and target yields a full pair of representations. The u_i must satisfy $\sum_{j=1}^{s} m_j u_j = 0$ and the F_i we require must be G-equivariant and also preserve L.

Lemma 8.1.16 *Let $\pi : \mathbf{C}^n \to L$ be the projection*

$$\pi(x) = x - \frac{<x,(1,\dots,1)>}{n}(1,\dots,1)$$

where $<,>$ denotes the standard inner product. Then for all $g \in G$ $(= S_n)$ we have $\pi(g \cdot x) = g \cdot \pi(x)$.

Proof Now we have

$$g \cdot \pi(x) = g \cdot x - \frac{<x,(1,\dots,1)>}{n} g \cdot (1,\dots,1),$$

but $g \cdot (1,\dots,1) = (1,\dots,1)$ and $<x,(1,\dots,1)>=<(g \cdot x),(1,\dots,1)>$ and the result follows. □

Lemma 8.1.17 *If $F : \mathbf{C}^n \to \mathbf{C}^n$ is G-equivariant, then so is $\pi \circ F : L \to L$. Conversely, if $f : L \to L$ is G-equivariant then there is an equivariant $F : \mathbf{C}^n \to \mathbf{C}^n$ with $f = \pi \circ F$*

Proof In the first case, we have

$$\begin{aligned} g \cdot (\pi \circ F(x)) &= \pi(g \cdot F(x)) \\ &= \pi(F(g \cdot x)) \\ &= \pi \circ F(g \cdot x) \end{aligned}$$

and so $\pi \circ F$ is equivariant. Now, if $f : L \to L$ is G-equivariant, define F by $F = f \circ \pi$, then this gives

$$\begin{aligned} g \cdot F(x) &= g \cdot (f \circ \pi(x)) \\ &= g \cdot f(\pi(x)) \\ &= f(g \cdot \pi(x)) \\ &= f(\pi(g \cdot x)) \\ &= F(g \cdot x) \end{aligned}$$

and we have shown that F is equivariant. □

Proposition 8.1.18 *The pair of actions given by the restriction of the natural action of S_n to the hyperplane L is full.*

Proof If $F_r(x_1, \ldots, x_n) = (x_1^r, \ldots, x_n^r)$, then $f_r = \pi_r \circ F_r$ is G-equivariant. If $x \in H(I)^* \cap L$, where

$$x = (\underbrace{u_1, \ldots, u_1}_{m_1}, \underbrace{u_2, \ldots, u_2}_{m_2}, \ldots, \underbrace{u_s, \ldots, u_s}_{m_s})$$

as before, then $F_r(x) = (u_1^r, \ldots u_1^r, \ldots, u_s^r, \ldots, u_s^r)$, so

$$f_r(x) = (u_1^r, \ldots u_1^r, \ldots, u_s^r, \ldots, u_s^r) - \frac{\sum_{j=1}^{s} m_j u_j^r}{n}(1, \ldots, 1)$$

Now $f_1(x), \dots, f_{s-1}(x)$ must be linearly independent, for if they were dependent, this would mean that $(1, \dots, 1), F_1(x), \dots, F_{s-1}(x)$ would be too, but we know these to be independent. $\square$

8.2 Finding generators for M

We can find generators for M_p using a method analogous to that for finding ring generators for the ring of invariant polynomials $\mathbf{C}[x]^G$ (see [CLO]). But once we have generators for M_p as a $\mathbf{C}[x]^G$ module, these will in fact generate M itself as an $\mathcal{O}_n^{G(S)}$-module. We calculate generators for M_p by using the following notion, restricted to the polynomial case.

Definition The *module Reynolds operator*, $\mathcal{R}$ is a map from $\mathcal{O}(n,n)$ to itself given by

$$\mathcal{R}(f) = \frac{1}{|G|} \sum_{g \in G} R_T(g^{-1}) \cdot f(R_S(g)(x)).$$

Lemma 8.2.1 *The Reynolds operator $\mathcal{R}$ has the following properties.*

1. *$\mathcal{R}$ is $\mathbf{C}$-linear and for any $f \in \mathcal{O}(n,n)$, $\mathcal{R}(f)$ is G-variant.*

2. *$\mathcal{R}(f) = f$ iff f is G-variant.*

3. *$\mathcal{R}$ preserves degrees.*

4. *If $f \in M_p$, $\alpha \in \mathbf{C}[x]$, then $\mathcal{R}(\alpha f) = \overline{\mathcal{R}}(\alpha) f$, where $\overline{\mathcal{R}}$ is the ring Reynolds operator in the source.*

Proof 1. For the $\mathbf{C}$-linearity,

$$\begin{aligned} \mathcal{R}(\alpha_1 f_1 + \alpha_2 f_2) &= \frac{1}{|G|} \sum_{g \in G} R_T(g^{-1}) \cdot (\alpha_1 f_1 + \alpha_2 f_2)(R_S(g)(x)) \\ &= \frac{1}{|G|} \sum_{g \in G} R_T(g^{-1}) \cdot ((\alpha_1 f_1)(R_S(g)(x)) + (\alpha_2 f_2)(R_S(g)(x))) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|}\sum_{g\in G}\alpha_1 R_T(g^{-1})f_1(R_S(g)(x)) + \alpha_2 R_T(g^{-1})f_2(R_S(g)(x)) \\
&= \alpha_1\mathcal{R}f_1 + \alpha_2\mathcal{R}f_2
\end{aligned}
$$

If we let $h \in G$ be an arbitrary element of G, then

$$
\begin{aligned}
R_T(h)\cdot\mathcal{R}(f) &= R_T(h)\cdot(\frac{1}{|G|}\sum_{g\in G}R_T(g^{-1})\cdot f(R_S(g)(x))) \\
&= \frac{1}{|G|}\sum_{g\in G}R_T(hg^{-1})\cdot f(R_S(g)(x)) \\
&= \frac{1}{|G|}\sum_{g_1\in G}R_T(g_1^{-1})\cdot f(R_S(g_1h)(x)) \\
&= \mathcal{R}(f)\cdot R_S(h)
\end{aligned}
$$

and thus $\mathcal{R}(f)$ is G-variant.

2. If f is G-variant, then clearly $\mathcal{R}(f) = f$. If $\mathcal{R}(f) = f$, then by 1. above, $\mathcal{R}(f)$, hence f itself is G-variant.

3. Trivial, since the group acts linearly.

4. If $\alpha \in \mathbf{C}[x]$, $f \in M_p$, then

$$
\begin{aligned}
\mathcal{R}(\alpha f) &= \frac{1}{|G|}\sum_{g\in G}R_T(g^{-1})\cdot(\alpha f)(R_S(g)(x)) \\
&= \frac{1}{|G|}\sum_{g\in G}R_T(g^{-1})\cdot(\alpha(R_S(g)(x))f(R_S(g)(x))) \\
&= \frac{1}{|G|}\sum_{g\in G}\alpha(R_S(g)(x))R_T(g^{-1})\cdot f(R_S(g)(x)) \\
&= \frac{1}{|G|}\sum_{g\in G}\alpha(R_S(g)(x))f(x) \\
&= \overline{\mathcal{R}}(\alpha)f.
\end{aligned}
$$

□

We are now in a position to prove that M_p is a finitely generated module.

Proposition 8.2.2 *The module of G-variant polynomial mappings M_p is finitely generated over the ring $\mathbf{C}[x]^G$ of invariant polynomials in the source.*

Proof The ring $\mathbf{C}[x]$ is Noetherian and $\mathbf{C}[x]^p$ is a finitely generated $\mathbf{C}[x]$-module, so M_p is Noetherian (see [E2, p28]) i.e. every submodule is finitely generated. Let $\tilde{M}_p$ be the $\mathbf{C}[x]$-module generated by all homogeneous G-variant maps of positive total degree. Clearly, due to the Noetherian condition, $\tilde{M}_p$ is finitely generated, i.e. $\tilde{M}_p = \mathbf{C}[x]\{F_1, \ldots, F_N\}$ for some F_i. We could replace the F_i by their homogeneous parts and thus we may assume without loss of generality that they are homogeneous. We claim that $M_p = \mathbf{C}[x]^G\{F_1, \ldots, F_N\}$. Otherwise, we can find $F \in M_p - \mathbf{C}[x]^G\{F_1, \ldots, F_N\}$ and again we may suppose that F is homogeneous. Let us choose such F of minimal degree, k. Now $F \in \tilde{M}_p$, so

$$F = \sum_{i=1}^{N} h_i F_i$$

for $h_1, \ldots, h_N \in R$. We may suppose that each $h_i F_i$ is homogeneous either of degree k or degree 0. Applying $\mathcal{R}$, we obtain

$$\mathcal{R}(F) = F = \sum_{i=1}^{N} \overline{\mathcal{R}}(h_i) F_i$$

by Lemma 8.2.1. Again, $\overline{\mathcal{R}}(h_i) F_i$ homogeneous of degree k or 0. But note that $\overline{\mathcal{R}}(h_i) \in \mathbf{C}[x]^G$ and we have a contradiction. □

We can also use the Reynolds operator to obtain the following results.

Lemma 8.2.3 *If $f = f_0 + f_1 + \cdots + f_k$ is a decomposition of f into its homogeneous parts, then f is variant if and only if the f_i are variant.*

Proof Clearly f is variant if the f_i are. But conversely, if f is variant, we have

$$\mathcal{R}(f) = \mathcal{R}(\sum_i f_i) = \sum_i \mathcal{R}(f_i)$$

and since $\mathcal{R}$ preserves degrees, $\mathcal{R}(f_i) = f_i$ for each i. □

Lemma 8.2.4 *Any G-variant polynomial map f is a linear combination of elements of the form $\mathcal{R}(x^\alpha e_j)$.*

Proof Any G-variant f can be written as a sum $\sum c_{j,\alpha} x^\alpha e_j$, where $c_{j,\alpha} \in \mathbf{C}$. Then

$$f = \mathcal{R}(f) = \mathcal{R}(\sum c_{j,\alpha} x^\alpha e_j) = \sum c_{j,\alpha} \mathcal{R}(x^\alpha e_j)$$

and we have the required result. □

We can now give the result analogous to (and derived from) Noether's Theorem.

Proposition 8.2.5 *The set of G-variant polynomial maps, M_p, is generated as a $\mathbf{C}[x]^G$-module by the elements $\{\mathcal{R}(x^\alpha e_j) : |\alpha| \leq |G| - 1, 1 \leq j \leq p\}$.*

Proof Define the map

$$\pi : \mathbf{C}[x]^p \to \mathbf{C}[x, y]$$

by

$$\sum_{i=1}^{p} f_i(x) e_i \mapsto \sum_{i=1}^{p} f_i(x) y_i.$$

This is $\mathbf{C}[x]$-linear and injective, so $\mathbf{C}[x]^p$ is isomorphic to im π as a $\mathbf{C}[x]$-module. We can define an action of G on $\mathbf{C}[x, y]$ via a representation R by setting

$$\begin{aligned} R(g)(y_i) &= \pi(R_T(g^{-1})(e_i)) \\ R(g)(x_i) &= R_S(g)(x_i) \\ R(g)(x^\alpha y^\beta) &= R(g)(x^\alpha) R(g)(y^\beta). \end{aligned}$$

If $f = \sum f_i(x)e_i$ is G-variant, then

$$f = \sum_{i=1}^{p} R_T(g^{-1})(e_i) \cdot f_i(R_S(g)(x))$$

so

$$\begin{aligned}
\pi(f) &= \sum_{i=1}^{p} \pi(R_T(g^{-1})(e_i)) \cdot f_i(R_S(g)(x)) \text{ by } \mathbf{C}[x]\text{-linearity} \\
&= \sum_{i=1}^{p} R(g)(y_i) \cdot f_i(R(g)(x)) \\
&= R(g) \sum_{i=1}^{p} f_i(x) y_i \\
&= R(g)\pi(f)
\end{aligned}$$

Thus the image of the variant maps lies in $\mathbf{C}[x,y]^G$. Letting $\mathcal{R}_\pi$ be the Reynolds operator on $\mathbf{C}[x,y]$, by Noether's Theorem we have

$$\mathbf{C}[x,y]^G = \mathbf{C}[\mathcal{R}_\pi(x^\alpha y^\beta) : |\alpha| + |\beta| \le |G|]$$

so if F is the image of some variant map under π, we can write

$$F = \sum_{\gamma \in \mathbf{N}^N} c_\gamma r^\gamma = \sum_{\gamma \in \mathbf{N}^N} c_\gamma \prod_{i=1}^{N} r_i^{\gamma_i}$$

where $\{r_1, \ldots, r_N\} = \{\mathcal{R}_\pi(x^\alpha y^\beta : |\alpha| + |\beta| \le |G|\}$ and $c_\gamma \in \mathbf{C}$. Now let us write $r_i = r_{i,0} + \cdot + r_{i,|G|}$, where $r_{i,j}$ is homogeneous of degree j in the y variables. Since the action on $\mathbf{C}[x,y]$ is induced from that on $\mathbf{C}[x]$ and $\mathbf{C}[y]$, $\mathcal{R}_\pi(x^\alpha y^\beta)$ will be of the form $\sum d_j x^{\alpha(j)} y^{\beta(j)}$ where $|\alpha(j)| = |\alpha|$ and $|\beta(j)| = |\beta|$. Thus the $r_{i,j}$ are invariant. Now since F is homogeneous of degree 1 in the y variables, the only r_i which can appear in

$$\prod_{i=1}^{N} r_i^{\gamma_i}$$

are homogeneous of degree 0 or 1 in the y variables. The only multi-indices γ which can appear are those which have index 1 on exactly one of the r_i which is a 1-form and index 0 on the others.

Thus F is a $\mathbf{C}[x]^G$-linear combination of elements of the form $r_j = r_{j,1}$ since $r_{i,0} \in \mathbf{C}[x]^G$ for any i. We know that $r_j = \mathcal{R}_\pi(x^\alpha y_j)$ where $|\alpha| + 1 \leq |G|$ as $\mathcal{R}_\pi$ maintains the total x and y degrees.

Thus the image of the variant maps lies within the $\mathbf{C}[x]^G$-module generated by $\{\mathcal{R}_\pi(x^\alpha y_j) : |\alpha| + 1 \leq |G|\}$. This is isomorphic (via π^{-1}) to the $\mathbf{C}[x]^G$-module $\{\mathcal{R}(x^\alpha e_j) : |\alpha| + 1 \leq |G|\}$ which contains the variant maps. □

Suppose we have found generators for $\mathbf{C}[x]^G$ and for M_p as a $\mathbf{C}[x]^G$-module. We can determine if a given polynomial map lies in the module and if so, express it as an element of the module by using the method given in §2.1. The following example shows that the bound on $|\alpha|$ in the above cannot be sharpened.

Example 8.2.6 Let G be the cyclic group $\mathbf{Z}_l$ and suppose it acts on the spaces specified as source and target (each isomorphic to $\mathbf{C}^n$) via

$$\left[\begin{pmatrix} \omega & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega \end{pmatrix}\right]$$

and

$$\left[\begin{pmatrix} \omega^{-1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{-1} \end{pmatrix}\right]$$

respectively, where $\omega = e^{2\pi i/l}$. Suppose $f = (f_1, \cdots f_n)$ is a G-variant map. This is equivalent to the condition

$$\omega^{-1} \cdot f_i(x_1, \ldots, x_n) = f_i(\omega \cdot x_1, \ldots, \omega \cdot x_n)$$

for all i. Now let $f_{i,k}$ be the homogeneous part of f_i of degree k. We obtain from the above

$$\begin{aligned} f_{i,k}(x_1, \ldots, x_n) &= \omega \cdot f_{i,k}(\omega \cdot x_1, \ldots, \omega \cdot x_n) \\ &= \omega \cdot \omega^k f_{i,k}(x_1, \ldots, x_n). \end{aligned}$$

Thus $k+1$ is a multiple of l. So the only monomials which can appear in a component of a G-variant map are of the degree $m \cdot l - 1$, $m \in \mathbf{N}$. Since the Reynolds operator preserves degree and considering Proposition 8.2.5 above, we find that the G-variant maps must be generated by elements of the form $\mathcal{R}(x^\alpha e_j)$ where $|\alpha| = |G| - 1$.

We can now extend Proposition 8.2.5 to the analytic case. First we need the following result.

Proposition 8.2.7 ([Ca]) *Let $\{q_i\}$ be a finite set of homogeneous polynomials generating the algebra $\mathbf{C}[x_1, \ldots, x_n]^G$. Then any element of $\mathcal{O}_n^G$ can be expressed as an analytic function in the q_i.* □

Using this we can prove.

Proposition 8.2.8 *The set of G-variant maps, M, is generated as a $\mathcal{O}_n^{G(S)}$-module by the elements $\{\mathcal{R}(x^\alpha e_j) : |\alpha| \leq |G| - 1, 1 \leq j \leq p\}$.*

Proof This proof follows almost exactly that of Proposition 8.2.5. We begin by defining the map

$$\pi : \mathcal{O}(n, n) \to \mathcal{O}_{n+p}$$

by

$$\sum_{i=1}^{p} f_i(x) e_i \mapsto \sum_{i=1}^{p} f_i(x) y_i.$$

which is simply the extension to the analytic case of the map used before. We then define an action of G on $\mathcal{O}_{n+p}$ exactly as before, via a representation R. We then find that

$$\pi(f) = R(g)\pi(f)$$

and so the image of the variant maps lies in $\mathcal{O}_{n+p}^G$. Letting $\mathcal{R}_\pi$ be the Reynolds operator on $\mathcal{O}_{n+p}$, by Noether's Theorem and Proposition 8.2.7 above, we have

$$\mathbf{C}[x,y]^G = \mathbf{C}\{\mathcal{R}_\pi(x^\alpha y^\beta) : |\alpha| + |\beta| \leq |G|\},$$

where $\mathbf{C}\{\cdots\}$ denotes a complex analytic function in the given variables. Thus if F is the image of some variant map under π, we can write

$$F = \sum_{\gamma \in \mathbf{N}^N} c_\gamma r^\gamma = \sum_{\gamma \in \mathbf{N}^N} c_\gamma \prod_{i=1}^{N} r_i^{\gamma_i} \tag{8.2}$$

where $\{r_1, \ldots, r_N\} = \{\mathcal{R}_\pi(x^\alpha y^\beta : |\alpha| + |\beta| \leq |G|\}$ and $c_\gamma \in \mathbf{C}$. Now, by the same argument as before, in each product of the form

$$\prod_{i=1}^{N} r_i^{\gamma_i}$$

we must have exactly one r_i which is a linear form in the y variables. The other r_i which appear will be elements of $\mathbf{C}[x]^G$. We see that F is a linear combination of elements r_i which are 1-forms in the y-variables. However, since the sum (8.2) is possibly infinite in this case, the coefficients in this linear combination are (possibly infinite) sums of elements in $\mathbf{C}[x]^G$, ie elements of $\mathcal{O}_n^{G(S)}$. Thus via the isomorphism π^{-1} we see that M itself is generated by the given elements as a $\mathcal{O}_n^{G(S)}$-module. □

We can also obtain a version of Molien's theorem for variant maps as follows.

Proposition 8.2.9 *The Hilbert series of M_p is given by*

$$H_G(z) = \frac{1}{|G|} \sum_{g \in G} \frac{tr(R_T(g))}{det(I - zR_S(g))}$$

Proof We again use the map $\pi : \mathbf{C}[x]^p \to \mathbf{C}[x, y]$ defined in Proposition 8.2.5. But we consider the decomposition

$$\text{im } \pi = \bigoplus_{d=1}^{\infty} V_d$$

where V_d consists of those elements of im π which are homogeneous of degree d in the x variables. We allow G to act on $\mathbf{C}[x, y]$ via the representation R, also defined in Proposition 8.2.5. Since this action preserves both x and y degrees, R is in fact the direct sum of representations on the V_d. Let the representation on V_d be given by $R^{(d)}$. Now by Lemma 2.2.2 in [St], we find that the dimension of V_d (which is what we require to construct the Hilbert series) is given by the average of trace$(R^d(g))$ as g varies over G. We can identify V_1 with $\mathbf{C}^n \otimes \mathbf{C}^p$. Consider $R^{(1)}(g)$, letting $l_{g,1}, \ldots, l_{g,n}$ and $m_{g,1}, \ldots, m_{g,n}$ be the eigenvectors of the action of $R^{(1)}(g)$ on $\mathbf{C}^n$ and $\mathbf{C}^p$ respectively. (These matrices are diagonalisable as they are of finite order.) Let $\lambda_{g,1}, \ldots, \lambda_{g,n}$ and $\mu_{g,1}, \ldots, \mu_{g,n}$ be the associated eigenvalues. Then we see that the eigenvectors of $R^{(d)}(g)$ are the elements of the form

$$(l_{g,1}^{\alpha_1} \cdots l_{g,n}^{\alpha_n}) \otimes m_{g,j}$$

where $|\alpha| = d$, $1 \leq j \leq p$, with eigenvalue $\lambda_{g,1}^{d_1} \cdots \lambda_{g,n}^{d_n} \mu_{g,j}$.

Now since the trace of a linear transformation equals the sum of the eigenvalues, we have the equation

$$\begin{aligned} \text{tr } R^{(d)}(g) &= \sum_{|\alpha|=d} \sum_{j=1}^{p} \lambda_{g,1}^{\alpha_1} \cdots \lambda_{g,n}^{\alpha_n} \mu_{g,j} \\ &= \left(\sum_{|\alpha|=d} \lambda_{g,1}^{\alpha_1} \cdots \lambda_{g,n}^{\alpha_n} \right) \text{tr } (R_T(g)) \end{aligned}$$

Now using the Lemma in [St] we obtain

$$H_G(z) = \sum_{d=0}^{\infty} \frac{1}{|G|} \sum_{g \in G} \left(\left(\sum_{|\alpha|=d} \lambda_{g,1}^{\alpha_1} \cdots \lambda_{g,n}^{\alpha_n} \right) \text{tr } (R_T(g)) \right) z^d$$

$$
\begin{aligned}
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\alpha}\lambda_{g,1}^{\alpha_1}\cdots\lambda_{g,n}^{\alpha_n}\cdot \text{tr }(R_T(g))z^{\alpha_1+\cdots+\alpha_n}\\
&= \frac{1}{|G|}\sum_{g\in G}\frac{\text{tr }(R_T(g))}{(1-z\lambda_{g,1})\cdots(1-z\lambda_{g,n})}\\
&= \frac{1}{|G|}\sum_{g\in G}\frac{\text{tr }(R_T(g))}{\det\,(I-zR_S(g))}
\end{aligned}
$$

giving the required result. □

Example 8.2.10 Let our group G be $\mathbf{Z}_2$ and let both R_S and R_T be given by

$$
\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].
$$

So we are considering the equivariant situation. Now the invariant polynomials (in the source) are given by $\mathbf{C}[x,y]^G = \mathbf{C}[x+y, xy]$. We now begin applying the Reynolds operator to maps of the form $x^\alpha e_j$.

$$
\begin{aligned}
\mathcal{R}(e_1) &= 1/2((1,0)+(0,1))\\
&= 1/2(1,1)\\
\mathcal{R}(e_2) &= 1/2((0,1)+(1,0))\\
&= 1/2(1,1)\\
\mathcal{R}(xe_1) &= 1/2((x,0)+(0,y))\\
&= 1/2(x,y)\\
\mathcal{R}(ye_1) &= 1/2((y,0)+(0,x))\\
&= 1/2(y,x)\\
\mathcal{R}(xe_2) &= 1/2((0,x)+(y,0))\\
&= 1/2(y,x)\\
\mathcal{R}(ye_2) &= 1/2((0,y)+(x,0))\\
&= 1/2(x,y)
\end{aligned}
$$

These are all the maps of this form, with $|\alpha| \leq |G| - 1$ and so, by Proposition 8.2.5, M_p is generated by $\{(1,1),(x,y),(y,x)\}$ as a $\mathbf{C}[x+y,xy]$-module.

Example 8.2.11 The usefulness of Proposition 8.2.9 is demonstrated by the following example. Let our group G be $\mathbf{Z}_2$ again, but let both R_S and R_T be given by

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

Suppose we wish to find generators for M_p. We begin by applying Proposition 8.2.9, to obtain the following:

$$\begin{aligned} H_G(z) &= \frac{1}{2}\left(\frac{\operatorname{trace}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1-z & 0 \\ 0 & 1-z \end{pmatrix}} + \frac{\operatorname{trace}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{\det\begin{pmatrix} 1+z & 0 \\ 0 & 1+z \end{pmatrix}}\right) \\ &= \frac{1}{2}\left(\frac{2}{(1-z)^2} + \frac{-2}{(1+z)^2}\right) \\ &= (1+2z+3z^2+4z^3+\cdots) + (-1+2z-3z^2+4z^3+\cdots) \\ &= 4z+8z^3+12z^5+16z^7+\cdots \end{aligned}$$

The invariant polynomials (in the source) are given this time by $\mathbf{C}[x,y]^G = \mathbf{C}[x^2,xy,y^2]$. Now that we know the Hilbert series for M_p, we begin to apply the Reynolds operator to elements of the form $x^\alpha e_j$, as in Proposition 8.2.5.

$$\begin{aligned} \mathcal{R}(e_1) &= 1/2((1,0)+(-1,0)) \\ &= 0 \\ \mathcal{R}(e_2) &= 1/2((0,1)+(0,-1)) \\ &= 0 \\ \mathcal{R}(xe_1) &= 1/2((x,0)+(x,0)) \end{aligned}$$

$$
\begin{aligned}
&= (x,0)\\
\mathcal{R}(ye_1) &= 1/2((y,0)+(y,0))\\
&= (y,0)\\
\mathcal{R}(xe_2) &= 1/2((0,x)+(0,x))\\
&= (0,x)\\
\mathcal{R}(ye_2) &= 1/2((0,y)+(0,y))\\
&= (0,y)
\end{aligned}
$$

Since we know that the Hilbert series has no z^2 term, we know that M_p contains no elements of degree 2 and thus we have no need to consider elements of the form $x^\alpha e_j$ with $|\alpha| = 2$. So, by Proposition 8.2.5 we have found all the generators of M_p. Calculating the Hilbert series thus saved us applying the Reynolds operator 6 times.

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Appendix A

Quadratic form routines

Since maple provides no routines for handling quadratic forms, the following routines are useful.

The first routine, **qfmat**, takes as input a quadratic form, p and its variables, v and returns its associated matrix.

```
qfmat:=proc(p,v)
local i,j,M;
  M:=array(1..nops(v), 1..nops(v));
  for i from 1 to nops(v) do
    for j from i to nops(v) do
      if i=j then M[i,j]:=coeff(p,v[i]^2);
        else M[i,j]:=coeff(coeff(p,v[i]),v[j])/2;
        M[j,i]:=M[i,j];
      fi;
    od;
  od;
RETURN(evalm(M));
end:
```

The next routine, matqf is essentially the reverse of qfmat. Given a matrix M and variables v it returns the associated quadratic form.

```
matqf:=proc(M,v)
RETURN(evalm(linalg[transpose](v)&*M&*v));
end:
```

Finally we have the routine qfdiag. This takes as input a quadratic form p, variables v and new variables Z. It then diagonalizes the given form, expressing it in terms of the new variables. If a fourth argument is given, this is assigned a list consisting of the definition of the new variables in terms of the old, the rank and the signature of the form.

```
qfdiag:=proc(p,v,Z)
local i,j,pp,flag,nice,nicev,h,k,lump,lumpv,X,rk,sig;
  rk:=0;
  sig:=0;
  pp:=p;
  nice:=0;
  nicev:=array(v);
  for i from 1 to nops(v) do
    pp:=subs(v[i]=Z[i],pp);
  od;
  do
    flag:=0;
    for i from 1 to nops(v) do
      if coeff(pp,Z[i]^2)<>0
      then
        flag:=i;
        break;
```

```
  fi;
od;
if flag=0
then
  h:=0;
  k:=0;
  for i from 1 to nops(v) do
    for j from i+1 to nops(v) do
      if coeff(coeff(pp,Z[i]),Z[j])<>0 then
        h:=i;
        k:=j;
        break;
      fi;
    od;
    if h<>0 then break;
    fi;
  od;
  if h=0 then
    for i from 1 to nops(v) do
      flag:=coeff(nice,Z[i]^2);
      if flag<>0 then rk:=eval(rk+1);
        sig:=sig+sign(flag);
      fi;
    od;
    if nargs=4 then assign(args[4],\
[convert(nicev,list),rk,sig]);
    fi;
    RETURN(nice);
  fi;
```

```
      pp:=expand((subs(Z[k]=Z[k]+Z[h],pp)));
      nicev[k]:=nicev[k]-nicev[h];
    else
      nice:=nice+coeff(pp,Z[flag]^2)*Z[flag]^2;
      lump:=0;
      lumpv:=0;
      for i from 1 to nops(v) do
        if i<>flag then
          lump:=lump+(coeff(coeff(pp,Z[i]),\
Z[flag])*Z[i])/(2*coeff(pp,Z[flag]^2));
          lumpv:=lumpv+(coeff(coeff(pp,Z[i]),\
Z[flag])*nicev[i])/(2*coeff(pp,Z[flag]^2));
        fi;
      od;
      pp:=expand(subs(Z[flag]=X-lump , pp))\
-coeff(pp,Z[flag]^2)*X^2;
      nicev[flag]:=nicev[flag]+lumpv;
    fi;
  od;
end:
```

We now give an example of these routines in use:

```
> p:=12*x*y+24*y*z+6*z*w+2*x*w;
                        p := 12 x y + 24 y z + 6 z w + 2 x w
----------------------------------------------------------
> v:=[x,y,z,w]:
----------------------------------------------------------
> M:=qfmat(p,v);
                                  [ 0   6   0  1 ]
```

```
              [                ]
              [ 6   0  12   0 ]
       M :=   [                ]
              [ 0  12   0   3 ]
              [                ]
              [ 1   0   3   0 ]
```

```
> matqf(M,v);
                  12 x y + 24 y z + 6 z w + 2 x w
```

```
> pp:=qfdiag(p,v,Z,'S');
                       2           2           2            2
        pp := 12 Z[1]  - 3 Z[2]  + 2 Z[3]  - 1/2 Z[4]
```

```
> S;
 [[1/2 x + 1/2 y + z + 1/12 w, y - x - 2 z + 1/6 w,

 1/2 z + 1/2 w, w - z], 4, 0]
```

Thus we have shown that the form $12xy + 24yz + 6zw + 2xw$ can be rewritten as $12z_1^2 - 3z_2^2 + 2z_3^2 - 1/2z_4^2$ where

$$
\begin{aligned}
z_1 &= 1/2x + 1/2y + z + 1/12w \\
z_2 &= y - x - 2z + 1/6w \\
z_3 &= 1/2z + 1/2w \\
z_4 &= w - z
\end{aligned}
$$

and that it has rank 4, signature 0.

Appendix B

Further computer routines

These two routines are required for mgbasis and subring_syz. In effect they map between a module and a ring, replacing something of the form [0,... ,0,1,0,... ,0] with Z[i] and vice versa.

```
into_ring:=proc(f,v,Z,s)
local Ans,i;
  Ans:=0;
  for i from 1 to s do
    Ans:=Ans+f[i]*Z[i];
  od;
RETURN(Ans);
end:
```

```
 from_ring:=proc(f,Z)
local i,xx,zz,temp,Ans,place;
xx:=[coeffs(f,Z,'zz')];
zz:=[zz];
Ans:=table([0 $ k=1..nops(Z)]);
```

```
for i from 1 to nops(xx) do
  if degree(zz[i],Z)<>1 then RETURN(0) fi;
  member(zz[i],Z,'place');
  Ans[place]:=Ans[place]+xx[i];
od;
RETURN(convert(Ans,list))
end:
```

This routine gives a set of generators for a syzygy module. Here F is the list of polynomials for which the syzygies are calculated. v is the list of variables to be used. The generators are returned as the rows of a matrix. It uses [E, Thm 6.10 p155] to determine syzygies on the extension of F to a Gröbner basis, then Proposition 2.3.1 to find generators for the syzygies on F itself. NB this routine does not give a minimal generating set and may even return some zero vectors or duplicates in its output.

```
syz:=proc(F,v)
local n,i,ii,j,G,Ext,e,Pairs,Ans,pair,L,S,qu,Conv;
  G:=matrixgrobner(F,v,grevlex(nops(v)));
  Ext:=F;
  Conv:=array(1..nops(F),1..nops(F), identity);
  for i from 1 to nops(G[2]) do
    for ii from 1 to nops(F) do
      if G[2][i]=F[ii] then break;
      elif ii=nops(F) then Ext:=[op(Ext),G[2][i]];
        Conv:=linalg[stack](Conv,linalg[row](G[1],i));
      fi;
    od;
  od;
```

```
    n:=nops(Ext);
    for i from 1 to n do
      e[i]:=linalg[vector]([0 $ j=1..i-1,1,0 $ j=i+1..n]);
    od;
    Pairs:=[];
    Ans:=[];
    for i from 1 to n-1 do
      for ii from i+1 to n do
        Pairs:=[op(Pairs),[i,ii]];
      od;
    od;
    while Pairs<>[] do
      pair:=Pairs[1];
      Pairs:=[Pairs[j] $ j=2..nops(Pairs)];
      for i from 1 to 2 do
        L[i]:=product(grobner[leadmon](Ext[pair[i]],v)[j],j=1..2);
      od;
      gcd(L[1],L[2],'S[1]','S[2]');
      chunk:=S[1]*Ext[pair[2]]-S[2]*Ext[pair[1]];
      qu:=linalg[vector](remainder(chunk,Ext,v,grevlex(nops(v)))[2]);
      Ans:=[op(Ans),evalm((S[1]*e[pair[2]])-(S[2]*e[pair[1]])-qu)];
    od;
   RETURN(evalm(linalg[matrix](convert(Ans,listlist)) &* Conv));
end:
```

```
coeff_of_mon:=proc(poly,mon,v)
local i,collected,C,M;
```

```
  collected:=collect(poly,v,'distributed');
  C:=[coeffs(poly,v,'M')];
  M:=[M];
  for i from 1 to nops(M) do
    if M[i]=mon then RETURN(C[i]);
    fi;
  od;
RETURN(0);
end:
```

```
getmonos:=proc(lterms,vars)
local i,j,bas,marker,pow,current,monos,mono,redmono,n;
  n:=nops(vars);
  marker:=array(1..n);
  for i from 1 to n do
    marker[i]:=0;
    for j from 1 to nops(lterms) do
      pow:=coeffs(lterms[j],vars[i]);
      if type(pow,constant) then
        marker[i]:=simplify(ln(lterms[j])/ln(vars[i]));
        break;
      fi;
    od;
    if marker[i]=0 then RETURN(0);
    fi;
  od;
  current:=array(1..n,sparse);
```

```
monos:=[];
do
  mono:=mon(current,vars);
  redmono:=grobner[normalf](mono,lterms,vars);
  if redmono<>0 then monos:=[op(monos),redmono];
    else current[1]:=marker[1];
  fi;
  current[1]:=eval(current[1])+1;
  for i from 1 to n-1 do
    if current[i]>=marker[i] then
      current[i]:=0;
      current[i+1]:=eval(current[i+1])+1;
    fi;
  od;
  if current[n]>=marker[n] then RETURN (monos);
  fi;
 od:
end:
```

```
quotbasis:=proc(polys,vars)
local gbas,lterms,second;
  second:=(x) -> x[2];
  gbas:=grobner[gbasis](polys,vars);
  lterms:=map(second,map(grobner[leadmon],gbas,vars));
RETURN(getmonos(lterms,vars));
end:
```

```
mon:=proc(current,vars)
local i,answer;
  answer:=1;
  for i from 1 to nops(vars) do
    answer:=eval(answer)*vars[i]^current[i];
  od;
RETURN(answer);
end:
```

Appendix C

Albert Lin's Gröbner basis commands

This section details the collection of Gröbner basis commands written by Albert Lin. The monomial orders are defined by a list of vectors as follows. Given two multi-indices α and β and a list of vectors $v_1, \ldots, v_r$, then $\alpha > \beta$ if for some k, $v_i \cdot \alpha = v_i \cdot \beta$ for all $i < k$ and $v_k \cdot \alpha > v_k \cdot \beta$.

```
#
#                          THE THREE MAJOR COMMANDS
#
#       This package has three major commands:
#
#       1) matrixgrobner, which computes a reduced grobner basis
#          together with a matrix telling you how to transform the
#          original basis into the grobner basis.
#       2) grobnerbasis, which computes a grobner basis which in
#          general is neither minimal nor reduced.
#       3) remainder, which computes the remainders AND quotients for
#          the division algorithm.
#
#       These commands use a monomial order which is specified by the
#       user as a list of vectors.  The names of the predefined Maple
#       term orders (plex and tdeg) should not be used.  However,
```

```
#       three commands (lex, grlex, grevlex) are provided that make it
#       easier to use some of the more common term orders.

matrixgrobner:=proc(F,V,termorder)
        local grob, mingrob, redgrob;
        grob:=grobnerbasis(F,V,termorder);
        mingrob:=_minimalgb(grob,V,termorder);
        redgrob:=_reducegrobner(mingrob,V,termorder);
        redgrob;
end:


with(grobner,leadmon);

_leadingterm:=proc(f,V,U)
        local f1,h,mono,t,a,i,j,k,l,m,n,P,r;
                f1:=expand(f);
                if type(f1,monomial) then
                        leadmon(f1,V,plex);
                else
                        n:=nops(U);
                        _Wa:=convert(U,array);
                        m:=nops(V);
                        P:=array(1..m,1..nops(f1));
                        r:=array(1..n,1..nops(f1));

                        for i to nops(f1) do
                                t[i]:=op(i,f1);
                        od;

                        for j to nops(f1) do
                                for k to m do
                                        P[k,j]:=degree(t[j],V[k]);
                                od;
                        od;

                        r:=evalm(&*(_Wa,P));
                        a:=1;
                        for l from 2 to nops(f1) do
                                for h to n do
                                        if r[h,l]<r[h,a] then
                                                break;
                                        fi;
```

```
                                    if r[h,l]>r[h,a] then
                                        a:=l;
                                        break;
                                    fi;
                                od;
                            od;
                            mono:=leadmon(simplify(op(a,f1)),V,plex);
                            mono;
                    fi;
end:



with(grobner,leadmon);

remainder:=proc(g,Set,V,termorder)
        local h,i,a,v,f,lmf,lmg,b,c,d;
        a := array(1..nops(Set));
        for b to nops(Set) do
                f:=Set[b];
                a[b]:=0;
                lmf[b]:=_leadingterm(f,V,termorder);
        od;
        v:=0;
        h:=g;
        while h<>0 do
                lmg := _leadingterm(h,V,termorder);
                for i  to nops(Set) do
                        f:=Set[i];
                        d:=degree(denom(simplify(lmg[2]/lmf[i][2])));
                        if d=0 then
                                a[i]:=simplify(a[i]+lmg[1]*lmg[2]/
                                        (lmf[i][1]*lmf[i][2]));
                                h:=simplify(h-f*lmg[1]*lmg[2]/
                                        (lmf[i][1]*lmf[i][2]));
                                if h<>0 then i:=0 fi;
                                lmg:=_leadingterm(h,V,termorder);
                        fi;
                od;
                v:=simplify(v+lmg[1]*lmg[2]);
                h:=simplify(h-lmg[1]*lmg[2]);
        od;
        a:=convert(a,list);
```

```
        c:=[v,a];
        c;
end:


with(linalg,submatrix);
with(linalg,swaprow);

_minimalgb:=proc(G,V,termorder)
        local y,i,j,n,H,Ga,Gb;
        n:=nops(G);
        H:=G;
        i:=1;
        while i<=n do;
                j:=1;
                while j<=n do;
                        if i<>j then;
                                if divide(_leadingterm(H[i],V,termorder)[2],
                                        _leadingterm(H[j],V,termorder)[2])
                                then
                                        Ga:=[H[1..i-1]];
                                        Gb:=[H[i+1..n]];
                                        for y from i to n-1 do
                                                _grobM:=swaprow(_grobM,y,y+1);
                                        od;
                                        _grobM:=submatrix(_grobM,1..n-1,
                                                1.._Ralph);
                                        n:=n-1;
                                        if j<i then
                                                i:=i-1;
                                        fi;
                                        j:=0;

                                        H:=[op(Ga),op(Gb)];
                                fi;
                        fi;
                        j:=j+1;
                od;
                i:=i+1;
        od;
        H;
end:
```

```
with(grobner,leadmon);

grobnerbasis:=proc(H,V1,ord)
        local x,y,setofpairs,z,p,l,F,r,i,n,lmf,lmg,ernie,u,T,Q,
                index1,index2,index,subscript;
        F:=[];
        _grobM:=table(sparse);
        for i to nops(H) do
                _grobM[i,i]:=1;
        od;
        F:=H;

        setofpairs:=[];
        for index1 to nops(H)-1 do
                for index2 from index1+1 to nops(H) do
                        setofpairs:=[op(setofpairs),[index1,index2]];
                od;
        od;

        while nops(setofpairs) <>0 do
                _Ralph:=nops(H);
                subscript:=setofpairs[1];
                setofpairs:=[op(2..nops(setofpairs),setofpairs)];
                lmf:=_leadingterm(F[subscript[1]],V1,ord);
                lmg:=_leadingterm(F[subscript[2]],V1,ord);
                if gcd(lmf[2],lmg[2])<>1 then
                        ernie:=0;
                        for u to subscript[1]-1 do
                                if divide(lcm(lmf[2],lmg[2]),
                                        _leadingterm(F[u],V1,ord)[2]) then
                                        ernie:=1;
                                        break;
                                fi;
                        od;
                        if ernie = 0 then

                        p:=lcm(lmf[2],lmg[2]);
                        T:=p*F[subscript[1]]/(lmf[1]*lmf[2])-
                                p*F[subscript[2]]/(lmg[1]*lmg[2]);
                        Q:=remainder(simplify(T),F,V1,ord);
                        r:=Q[1];
                        if r<>0 then
                                for index to nops(F) do
```

```
                                        setofpairs:=[op(setofpairs),
                                        [index,nops(F)+1]];
                                od;
                                F:=[op(F),r];
                                n:=nops(F);
                                for l to n-1 do
                                        _grobM[n,l]:=-op(l,Q[2]);
                                od;
                                _grobM[n,subscript[1]]:=simplify(_grobM[n,
                                        subscript[1]]+p/(lmf[1]*lmf[2]));
                                _grobM[n,subscript[2]]:=simplify(_grobM[n,
                                        subscript[2]]-p/(lmg[1]*lmg[2]));
                        fi;
                        fi;
                fi;
        od;
        if nops(F) > nops(H) then
                for x from _Ralph+2 to n do
                        for y to _Ralph do
                                for z from _Ralph+1 to x-1 do
                                        _grobM[x,y]:=simplify(_grobM[x,y]+
                                        _grobM[x,z]*_grobM[z,y]);
                                od;
                        od;
                od;
        fi;
        _grobM:=convert(_grobM,array,sparse);
        _Ralph1:=_Ralph;
        _grobM:=submatrix(_grobM,1..nops(F),1.._Ralph);
        F;
end:


_reducegrobner:=proc(G,V,termorder)
        local x,Ca,t,i,j,k,l,n,a,o,Da,Db,r,J;
        J:=[];
        J:=G;
        n:=nops(J);
        i:=1;
        Ca:=table(sparse);
        if n<>1 then
                while i<= n do
                        Da:=[op(1..i-1,J)];
```

```
                        Db:=[op(i+1..n,J)];
                        r:= remainder(J[i],[op(Da),op(Db)],V,termorder);
                        J:=[op(Da),r[1],op(Db)];
                        for j to n do
                                for k to i-1 do
                                        Ca[i,j]:=simplify(Ca[i,j]-
                                                r[2][k]*Ca[k,j]);
                                od;
                        od;
                        for l from i+1 to n do
                                Ca[i,l]:=simplify(Ca[i,l]-r[2][l-1]);
                        od;
                        Ca[i,i]:=simplify(Ca[i,i]+1);
                        i:=i+1;
                od;
        else
                Ca[1,1]:=1;
        fi;
        for a to n do
                x:=_leadingterm(J[a],V,termorder)[1];
                o:=J[a]/x;
                for t to n do
                        Ca[a,t]:=Ca[a,t]/x;
                od;
                J:=[op(1..a-1,J),o,op(a+1..n,J)];
        od;
        Ca:=convert(Ca,array,sparse);
        _superM:=evalm(&*(Ca,_grobM));
        [evalm(_superM),J];
end:

lex := proc(n)
        local u,i,j;
        if n=1 then [[1]]; else
                u := [0 $ i=1..j-1,1,0 $ i=j+1..n];
                [[1,0 $ i=1..n-1],u $ j=2..n-1,[0 $ i=1..n-1,1]] fi;
end:

grlex := proc(n)
        local u,i,j;
        if n=1 then [[1]]; else
                u := [0 $ i=1..j-1,1,0 $ i=j+1..n];
                [[1 $ i=1..n],u $ j=1..n-1] fi;
```

```
end:

grevlex := proc(n)
        local u,i,j;
        if n=1 then [[1]]; else
                u := [0 $ i=1..n-j-1,-1,0 $ i=1..j];
                [[1 $ i=1..n],[0 $ i=1..n-1,-1], u $ j=1..n-2] fi;
end:
```