QUADRATIC DIVERGENCES IN RENORMALISABLE FIELD THEORIES

BY

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THEISIS SUBMITTED IN ACCORDANCE WITH THE REQUIREMENTS OF THE UNIVERSITY OF LIVERPOOL FOR DEGREE OF DOCTOR OF PHILOSOPHY .

JUNE 1992

TEXT BOUND INTO

THE SPINE

DECLARATION

THE WORK PRESENTED IN THIS THESIS WAS CARRIED OUT IN THE DEPARTMENT OF APPLIED MATHEMATICS AND THEORTEICAL PHYSICS AT THE UNIVERSITY OF LIVERPOOL BETWEEN OCT. 1989 AND JUNE 1992 UNDER THE SUPERVISION OF DR. D.R.T.JONES .

THE MATERIAL PRESENTED IN THIS THESIS IS ORIGINAL UNLESS OTHERWISE STATED . THE WORK OF CHAPTERS ONE , FOUR AND FIVE HAVE BEEN PUBLISHED AS FOLLOWS :

M.S.AL-SARHI, I.JACK, D.R.T.JONES, NUCL.PHYS.B345,431 (1990)

M.S.AL-SARHI, I.JACK, D.R.T.JONES, Z.PHYS.C (IN PRESS).

ACKNOWLEDGEMENTS

All praise be to God alone.

I am extremely grateful to my supervisor Dr.D.R.T.Jones for his help, assistance and encouragement during this work . My great thanks to Dr.I.Jack for his valuable help . I would like to thank also Prof. C.Michael for his kind treatment and encouragement . Thanks also to all of the D.A.M.T.P. at liverpool . Thanks to Mrs R.Morris for her help in typing this work.

I am extremely grateful to my FATHER AND MOTHER for everything they have done for me and for the financial support and encouragement . Thanks also to my Brothers and Sisters specially brother Hussam Al-Sarhi for his financial support. Thanks to my Uncles and Aunts for their encouragement and support.

Table of Contents

CHAPTER I : INTRODUCTION

In the last twenty years Quantum Field Theory (QFT) has witnessed a spectacular progress.

First introduced to describe quantum electrodynamics (QED), QFT has become the framework for the discussion of all the fundamental interactions except gravity.

The concept of renormalisable field theory first emerged empirically in QED, where it led to predictions of extraordinary accuracy, and now forms the basis of a complete theory of strong, weak and electromagnetic interactions. Very early it was realized that in massless renormalisable field theories a renormalisation group could be associated with transformation properties under space dilations but only later was this property used to discuss the short distance structure of physical processes.

Quantum electrodynamics, as well as all more complete field theories in particle physics, is afflicted by a strange disease.In a straightforward calculation all physical quantities are infinite, due to the short distance singularities of the theory. A strange remedy to this disease has been found: one artificially modifies the theory at short distance, at a scale characterized by a short distance cut-off, and one then re-expresses all physical quantities in terms of a small number of physical constants, such as the physical charge.

After this change of parametrisation the cut-off is removed and , somewhat miraculously, when the theory is so-called renormalisable, all other physical quantities have a finite limit. Moreover this limit is independent of the precise form of the short distance modification. We can summarize this property by saying that renormalisable field theories are short distance insensitive, in the sense that they can be described in terms of a finite number of effective parameters relevant to the scale of observation without a detailed knowledge of the microscopic structure. The infinities , or divergences , that we meet when calculating physical processes show that the field theories we want to construct cannot be defined by a straightforward perturbative expansion without some modification. We shall modify the field theory at large momentum in such a way that the new Feynman diagrams become well-defined finite quantities, and such that when one control parameter approaches some limit (for example the cut-off is sent to infinity), we recover the original perturbation theory[1] . This procedure is called regularisation. It will allow us to isolate well-defined divergent parts of diagrams and deal with them with renormalisation. There are many regularisation methods but in any particular application there are some criteria which guide our choice of a regularisation method: in some theories, symmetries play a crucial role and it is helpful to find a regularisation which preserve the symmetry (for example in gauge theories).

Another criterion is that if we wish explicitly to calculate Feynman diagrams, we shall look for the regularisation which leads to the simplest practical calculation. In some theories some essential property apart from the symmetry is violated by the regularisation method, e.g. the antisymmetric tensor $\varepsilon_{\mu 1,\ldots\ldots\mu d}$ and γ_5 are specific to integer dimensions which causes problem with dimensional regularisation [2].

There are also non-perturbative regularisations the best known being lattice regularization for which the regularised functional integral can be calculated by non-perturbative methods, for example Monte Carlo calculations. It also preserves most global and local symmetries.

Despite the development of the machinery of renormalisation, there remained a widespread feeling that the divergences, although trained, were indicative of something unsatisfactory in our approach to QFT, if not in QFT itself. This attitude has changed over the last twenty years, and the modern consensus is that the divergent nature of the radiative corrections to the particle masses and coupling constants is a reflection of the existence of an energy scale at which new degrees of freedom become excited. From this viewpoint, the quadratic divergences in renormalisable field theories in four dimensions become important.

We can see this clearly in the standard model, which is a very successful theory which accurately describes weak and electromagnetic phenomena and quantum chromodynamics (QCD). The standard model is a gauge theory based on SU(3) x SU(2) x U(1). One of the main pieces of the puzzle is missing, namely the spin zero elementary Higgs boson needed by the Standard Model for spontaneous symmetry breaking (which is responsible for the masses of the $W^{\pm}$, Z and fermions). Although one could argue that it is only a matter of time until the Higgs boson will be discovered (depending on its mass which is not fixed by the theory), it is widely thought that deeper problems exist, connected with the Higgs boson, which suggest that it is necessary to look beyond the Standard Model to understand the Higgs sector of the theory. For this reason a great deal of interest has developed in super-symmetric extensions of the standard model (for reviews see e.g. ref. [2]).

There are three kinds of reasons why the standard model is incomplete.

First, it contains many arbitrary assumptions and parameters, e.g. Why are there three colours? Why are left-handed fermions in SU(2) doublets and right-handed ones in SU(2) singlets? etc.

Secondly, the Standard Model, like QED, is not asymptotically free, so ultimately, at some energy scale, its interactions must become strong. Even though this could be at very high energy, it suggests that in principle the Standard Model is the low energy effective theory of a more fundamental one.

The above two reasons do not necessary suggest that supersymmetry is a particularly good approach to going beyond the Standard Model, although it could be relevant to them. However, the third reason does .

If one calculates the radiative corrections to the mass of the Higgs boson of the Standard Model, e.g. from a fermion loop in the propagator, one has a loop integral of the form:

$$\int \frac{d^4k}{(\not{k} - m_f)\ ((\not{p}+\not{k}) - m_f)} \qquad (I.1)$$

for a Higgs of momentum p. This integral diverges quadratically for large k, so it gives a correction to the mass $\delta m^2 \sim \Lambda^2$, where Λ is a cut-off, a scale beyond which the low energy theory no long applies. Could the Higgs particle mass in fact be superheavy? For some Higgs' mass of the Order of a few Tev, the Higgs self-coupling gets too strong, and we should not be observing the apparently successful perturbation theory at low energies.

Since corrections larger than this mass scale would seem equally unphysical, we expect the new physics to give an effective cut-off scale below a few Tev. In fact, the Higgs vacuum expectation value, which determines m_W's and in principle, the fermion masses, is about 250 Gev. So far, there are three kinds of attempts which have emerged to try to deal with this problem[2].

One approach is to have quarks, leptons and gauge bosons as composite objects [3]. A second approach is to eliminate fundamental scalars from the theory by making them composites of new fermions - the Technicolor approach [4]. We are not in the position to justify either of the above approaches - each has its own problems.

The third approach is to use a higher symmetry to eliminate the quadratic divergences in the Higgs mass , which can be arranged in supersymmetric theories. In supersymmetric theories there is always a loop of superpartners accompanying the loop of normal particles; the extra minus sign that goes with any fermion loop, plus the supersymmetric relations between masses and couplings, guarantee that the coefficient of the divergence in zero. We give an example of one of the calculations.

We can demonstrate this at one-loop level in the supersymmetric model of Wess-Zumino [5].

$$L = \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} i \bar{\Psi} \not{\partial} \Psi - \frac{1}{2} m^2 (A^2 + B^2)$$
$$- \frac{1}{2} m \bar{\Psi} \Psi - g\, m\, A\, (A^2 + B^2) - \frac{1}{2} g^2 (A^2 + B^2)^2 - g \bar{\Psi} (A - iB \gamma_5) \psi \qquad (I.2)$$

where A and B are real scalar field and Ψ is a four-component Majorana spinor.

At one-loop, the only potential quadratic divergence can occur in the self-energy of the scalar. We will illustrate the cancellation of quadratic divergences in the one-loop graphs of the A field self-energy. All possible one-loop quadratically divergent graphs are given in Fig. (1). The necessary Feynman rules are given in Fig. (2). Now the sum of the two boson-loop graphs (a) and (b) is the quadratically divergent integral:

$$8 g^2 \int \frac{d^4k}{(k^2 - m^2)} \qquad (I.3)$$

The fermion-loop graph (c) is given by:

$$- 2 g^2 \,\mathrm{Tr} \int \frac{d^4k \quad (\not{k} + m)\ (\not{k} - \not{p} + m)}{(k^2 - m^2)\ [\ (k-p)^2 - m^2]} \tag{I.4}$$

The trace is

$$\mathrm{Tr}\ (\not{k} + m)\ (\not{k} - \not{p} + m) = 4\ (k^2 - k.p + m^2)$$
$$= 2\ [\ (k^2 - m^2) + ((k-p)^2 - m^2) - P^2 + 4m^2\ ] \tag{I.5}$$

Inserting (I.5) into (I.4) we obtain

$$- 4\ g^2 \int \frac{d^4k}{(k^2 - m^2)} - 4\ g^2 \int \frac{d^4k}{(k-p)^2 - m^2} + I(p,m) \tag{I.6}$$

where the integral I (p, m) is only logarithmically divergent. If we shift variable $k \longrightarrow k + p$ in the second term of (I.6) it then becomes clear that the first two terms of (I.6) exactly cancel the result given in (I.3). Thus the quadratic divergence has indeed cancelled.

As in the above example, the standard model's quadratic divergence problem can be resolved elegantly if the low energy theory is rendered supersymmetric. However , the lack of experimental evidence for the super-partners of the known particle is an embarrassment .

Because of that a conjecture that it might be the case that there exist non-supersymmtric theories free of quadratic divergences has been made. To ensure that a given theory is free from quadratic divergences we need to impose a cancellation condition at each loop level of the theory .

So, the question we set out to answer is as follows :whether we can understand the quadratic divergences cancellation conditions to all orders in terms of the scale dependence of the one loop condition .

We will give a simple example to illustrate :

We consider the Φ^4 theory in d = 4 with the Lagrangian :

$$L = \frac{1}{2} (\partial_\mu \Phi^a)^2 + U(\Phi) \qquad (I.7)$$

where a = 1,.....,N and $U(\Phi)$ is a polynomial in Φ^a of degree four. In dimensional regularisation , quadratic divergences manifest themselves at d = 2 (see chapter one). Denoting the coefficient of the quadratic divergences at L-loop by Δ_L, the one loop result is[6]:

$$\Delta_1 = U_{aa} \qquad (I.8)$$

where $U_a = \frac{\partial U}{\partial \Phi^a}$ etc. The normalisation of Δ_1 is arbitrary , we have suppressed overall numerical cofficients, these being irrelevant to our purpose here.

Therefore the condition for the absence of the quadratic divergnces at the one loop level is

$$\Delta_1 = 0 \qquad \text{(I.9)}$$

At the two loop level the quadratic divergences manifest themselves as a pole at d = 3 , and the two loop result is given by[6]

$$\Delta_2 = U_{abc}\, U_{abc} \qquad \text{(I.10)}$$

where the normalisation of Δ_2 is arbitrary .we have suppressed

overall numerical cofficients, these being irrelevant to our purpose here.

We now show how in fact the information in (I.10) is already present (in a sense) in (I.9) . To do that we recall that the renormalised couplings of a theory are functions of the renormalisation scale μ (running couplings) .

Then by differentiating (I.9) with respect to μ we obtain

$$\mu \frac{\partial}{\partial\mu} \Delta_1 = \mu \frac{\partial U}{\partial\mu} \frac{\partial}{\partial U} \Delta_1$$

$$= \beta_U^{(1)} \frac{\partial}{\partial U} \Delta_1 \qquad \text{(I.11)}$$

where $\beta_U = \mu \frac{\partial}{\partial\mu} U$.

At the lowest order , the right-hand side of (I.11) can be written as

$$a_{11} = \beta_U^{(1)} \frac{\partial}{\partial U} \Delta_1 - \Delta_1 \frac{\partial}{\partial U} \beta_U^{(1)} \qquad \text{(I.12)}$$

provided the condition (I.9) holds .
Now substituting for $\beta_U^{(1)}$ in (I.12) then it is straightforward to verify

$$\Delta_2 = a_{11} \qquad \text{(I.13)}$$

So we have obtained Δ_2 in term of the scale invariance of Δ_1.
This is also true for a general gauge theory at one loop (ref [7]).
In the light of this result we set up four chapters in this work to test this conjecture for different renormalisable field theories motivated by the hope of finding a non-supersymmetric theory free of quadratic divergences .
As it has been mentioned before, the standard model suffers from quadratic divergences,the one loop cancellation condition ,in fact, leads to a relationship between the top quark and the Higgs masses .
The one loop cancellation condition is

$$\lambda + \frac{3}{4} g^2 + \frac{1}{4} g''^2 - 2 h^2 = 0 \qquad \text{(I.14)}$$

where λ is the Higgs self coupling , g and g" are the gauge couplings and h is the Yukawa coupling .
This relationship can be translated to a relationship between the masses using the Higgs vacuum expectation value[8] :

$$H + 3 + \tan^2\theta_W - 4T = 0 \qquad (I.15)$$

where

$$H = m_H^2 / m_W^2 \quad , \quad T = m_t^2 / m_W^2$$

and θ_W is the weak mixing angle

The quadratic divergences of a given theory depend in a non trivial way on the regulator employed . This happens even at one loop level as demonstrated in the pioneering work of Veltman [8] in the case of the standard model .
Although the Veltman's formula in (I.15) originally derived in the context of regularisation by dimensional reduction it can be reproduced by any straightforward regularisation methods that does not involve continuation in dimension [9] , for example non-local regularisation (see Chapter 5) or point-splitting regularisation [10]
In this work we demonstrate an unexpected relationship between the β-function and the quadratic divergences in a renormalisation field theories at d = 4, d = 2, d = 6 and d = 3.

In Chapter 1 we investigate the structure of the quadratic divergences in d = 4 Φ^4 theory and ask whether the relationship which we call the cancellation condition conjecture persists at the four loop level.

In Chapters 2 and 3, we examine the renormalisable theories of Φ^3 in d = 6 and Φ^6 in d = 3 and calculate the quadratic divergences up to 2-loop level in the case of Φ^3 in d = 6 and up to 6-loop level in the case of Φ^6 in d = 3. We also test the cancellation condition conjecture for these theories.

In Chapter 4 we turn to d = 2 and examine non-linear sigma models. In the string context, the quadratic divergences in this case have a definite interpretation as a renormalisation of the tachyon background field; the fact that the tachyon can be decoupled from the spectrum in the superstring case relates to the fact that the corresponding sigma model is free of quadratic divergences. We analyse up to the four-loop level, and find interesting similarities with the d = 4 calculation of Chapter 1. In Section 2 of Chapter 4 we generalise the sigma model case with the inclusion of a torsion (antisymmetric tensor) term.

In Chapter 5 we considered the gauge theories and calculate the quadratic divergences using dimensional regularization and a new regularisation called non-local regularisation. We rederive Veltman formula and give a prediction (if the strong interaction term α_3 is ignored) that for m_t and m_H .

We end with a Conclusion in which we discuss the results we found.

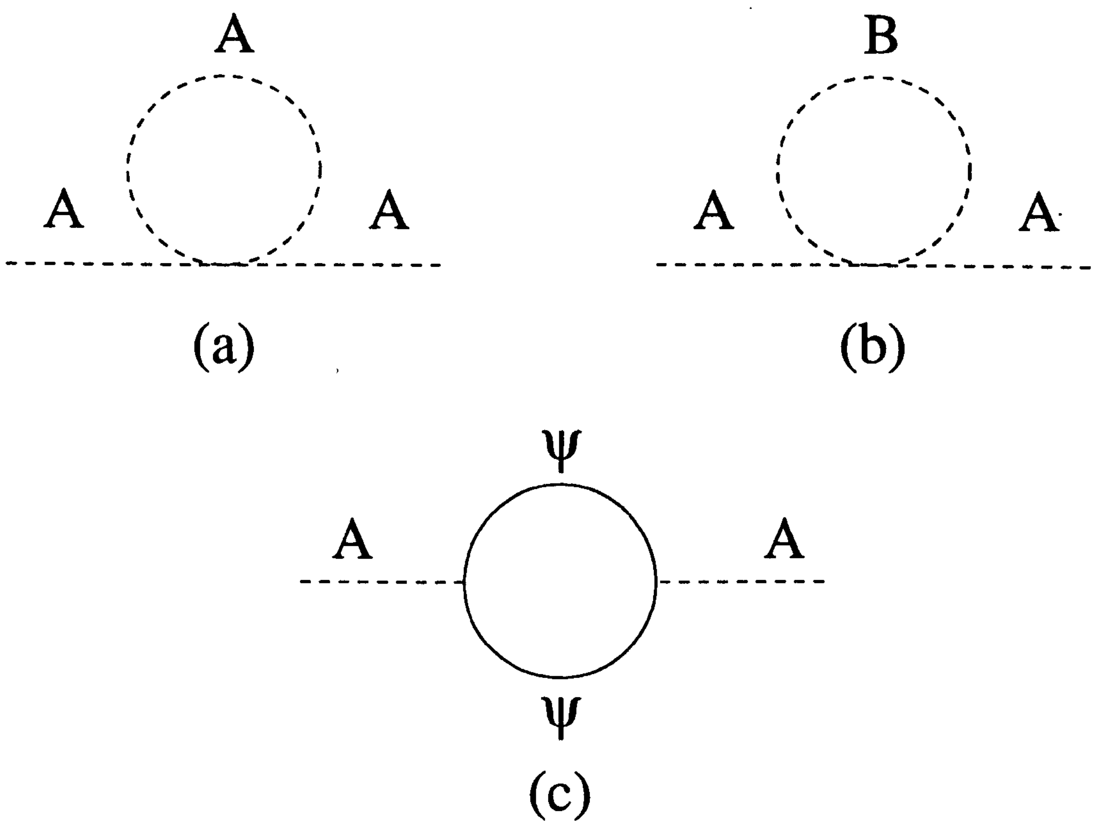


fig (1)

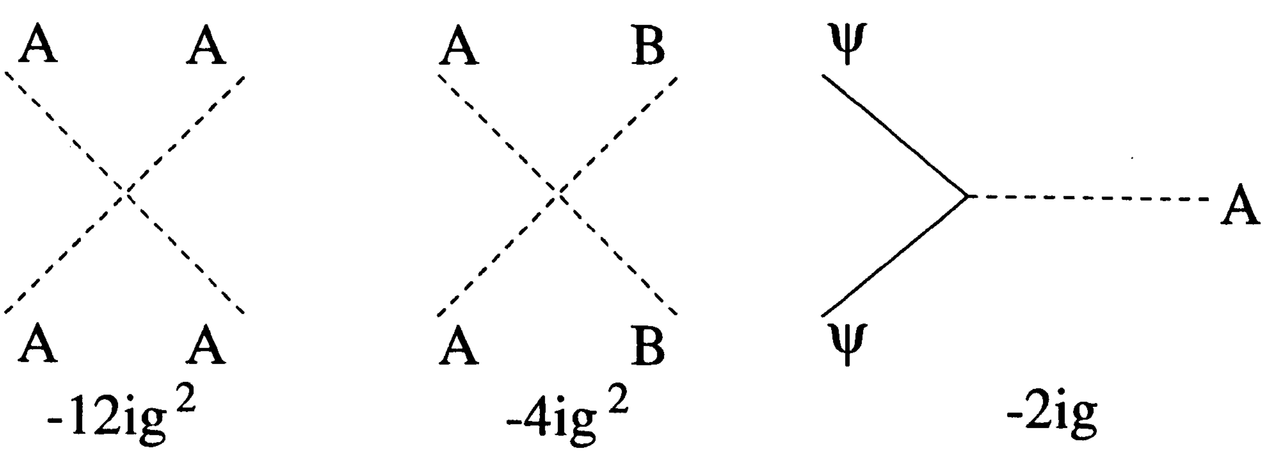


fig (2)

CHAPTER 1 : QUADRATIC DIVERGENCES IN Φ^4 THEORY IN d = 4 .

In this chapter we consider renormalisable theories of scalar fields in four space-time dimensions. The quadratic divergences manifest themselves in the context of dimensional regularisation (DREG)as a poles at unphysical values of d. This can be seen from elementary power counting. We consider a diagram of order V ,i.e. with V vertices, E external lines, I internal lines and L loops and with a space-time dimensions d . So the degree of divergences D is given by

$$D = dL - 2I \tag{1.1}$$

We want to express D in terms of E and V, so we want to eliminate I and L. There exists an identity which is true for any interaction in any dimension,

$$L = I - V + 1 \tag{1.2}$$

In ϕ^4 theory, each vertex gives 4 legs, so there are 4V legs,some external and some internal, however; the internal ones count twice because they are connected to two vertices, so

$$4V = E + 2I \tag{1.3}$$

Now using (1.2) and (1.3) we have

$$D = (d-4)L + 4 - E \tag{1.4}$$

For d = 4

$$D = 4 - E \tag{1.5}$$

and so when E = 2 we have a quadratically divergent graph. In dimensional regularisation the corresponding poles in the Green functions first occur when D = 0 i.e. when[6]

$$d = 4 - 2/L \tag{1.6}$$

by analogy to the usual $\varepsilon^{(L)} = 4 - (2/L) - d$ (1.7)

where d = 4 - 2/L is the leading divergences .In this chapter we will calculate the divergences to the four-loop level.

Quadratic divergences are of particular interest due to the naturalness problem. A theory is deemed unnatural if the radiative corrections to a physical observable have an intrinsic magnitude much greater than the observed value.[8] The discovery of supersymmetry had solved the problem of quadratic divergences, since generally supersymmetric theories are free of quadratic divergences. But because of the lack of the experimental evidence for supersymmetry any example of a non-supersymmetric natural theory would be most interesting. Our aims in this chapter are to study the structure of the quadratic divergences at four-loop level for Φ^4 theory and test a conjecture[7] that the quadratic divergence cancellation conditions can be understood to all orders in terms of the scale dependence of the one loop condition.

In ϕ^4 theory this conjecture has been successfully tested up to the 3-loop level in ref. [6]. But, as we shall see, at the 4-loop level some features occur that were not present in the lower order corrections in ref. [6].
We begin with the basic Lagrangian in Minkowski space

$$L = \frac{1}{2} (\partial_\mu \Phi^a)^2 - U(\Phi) \qquad (1.8)$$

where a = 1, ..., N and $U(\Phi)$ is polynomial in Φ of degree four.

It is straightforward to show from (1.4) that quadratic divergences occur in graph with E = 0, 1, 2 where E is the external line.

We apply the Background field method, i.e. we let

$$\Phi \longrightarrow \Phi + \Phi_q \qquad (1.9)$$

where Φ_q is the internal (quantum) field and ϕ is the external (classical) field Since we are only interested in the leading divergences (i.e. the pole in $\varepsilon^{(L)}$ at L loop) graphs with non-overlapping divergences or counterterm insertions can be ignored.

Now (1.4) becomes

$$L = \frac{1}{2} (\partial_\mu \Phi^a)^2 - U(\Phi) + \frac{1}{2} (\partial_\mu \Phi^a{}_q)^2$$
$$- \Phi^a{}_q U_a - \frac{1}{2!} \Phi^a{}_q \Phi^b{}_q U_{ab} - \frac{1}{3!} \Phi^a{}_q \Phi^b{}_q \Phi^c{}_q U_{abc}$$
$$- \frac{1}{4!} \Phi^a{}_q \Phi^b{}_q \Phi^c{}_q \Phi^d{}_q U_{abcd} \qquad (1.10)$$

where $U_a = \partial U / \partial \Phi^a$ etc.

Now we want to calculate the vacuum graphs which contribute to the quadratic divergences in the effective action up to the 4-loop level, displayed in figs.(1, 2, 3 and 4). The 2-point function graphs can be obtained by differentiating twice with respect to ϕ^a. If we denote the coefficient of the leading quadratic divergences at L loops as Δ_L then for L = 1, 2, 3 the results are[6]

$$\Delta_1 = U_{aa} \qquad (1.11)$$
$$\Delta_2 = U_{abc} U_{abc} \qquad (1.12)$$
$$\Delta_3 = U_{ab} U_{acde} U_{bcde} - \frac{9}{2} U_{abcd} U_{abe} U_{cde} \qquad (1.13)$$

where $U_a = \partial U / \partial \Phi^a$ etc. The normalisation of Δ_L is arbitrary for each value of L; that is , we have suppressed overall numerical cofficients , these being irrelevant to our purpose here.

1.1 Four Loop Calculations:

Essential details concerning our regularisation procedure (dimensional regularisation), signs, factors of i and other facets of our calculation are contained in Appendix One.

In this calculations, we have arranged matters in such a way as to ensure that the answers obtained are directly proportional to the quadratic divergences.

Now we proceed with the calculations:

From (1.6) the quadratic divergences at 4-loop level occur at $d = \frac{7}{2} - \varepsilon$.

In our calculation we are looking at the simple pole only which is an ultra-violet divergences.

The treatment of infra-red divergences is straightforward via the insertion of a regulator mass or routing of an external momentum p as appropriate.

For the first graph fig. (4a) we have :

$$A \int \frac{d^dk \;\; d^dq \;\; d^dr \;\; d^ds}{(k-p)^2 \; q^2 \; (k-q)^2 \; r^2 (k-r)^2 \; s^2 (k-s)^2} \tag{1.14}$$

where $A = \quad U_{abc} \; U_{bcde} \; U_{defg} \; U_{afg}$

but

$$\int \frac{d^dk}{(k-p)^2 \;\; k^2} \quad = \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(2-d/2) \;\; \Gamma((d/2)-1)^2}{\Gamma(d-2)} \quad (p^2)^{(d/2)-1}$$

$$= I \quad (p^2)^{(d/2)-1} \tag{1.15}$$

where

$$I = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\ \Gamma((d/2)-1)^2}{\Gamma(d-2)} \tag{1.16}$$

Therefore (1.14) can be written as

$$A\ I^3 \int \frac{d^d k}{(k^2)^{6-3d/2}\ (k-p)^2} \tag{1.17}$$

Substituting $d = \frac{7}{2} - \varepsilon$,then

$$\text{fig(4a)} = (2/3)\ A\ Y\ \Gamma(2\varepsilon) + \ldots\ldots \tag{1.18}$$

where $$Y = \frac{1}{(4\pi)^7} \frac{\Gamma(1/4)^3\ \Gamma(3/4)^5}{\Gamma(1/\ 2)^3} \tag{1.19}$$

Now the graph in fig(4b) gives

$$B \int \frac{d^d k\ d^d q\ d^d s\ d^d r}{k^2\ (q-k)^2\ (s-q)^2\ (s^2)^2\ (r-s)^2\ (p-r)^2} \tag{1.20}$$

where $$B = U_{abc}\ U_{bcd}\ U_{aefg}\ U_{defg} \tag{1.21}$$

Performing the k integral gives

$$\text{fig}(4b) = B\ I \int \frac{d^dq\ d^ds\ d^dr}{(q^2)^{2-(d/2)}\ (s-q)^2\ (s^2)^2\ (r-s)^2\ (p-r)^2} \tag{1.22}$$

where I is given in (1.16).

Now it is straightforward to do the integrals over q, s and r respectively.

Then (1.22) become

$$\text{fig}(4b) = B.I.J_1.J_2.J_3 \tag{1.23}$$

where

$$J_1 = (s^2)^{3-d} \int \frac{d^dq}{(q^2)^{2-d/2}\ (s-q)^2}$$

$$= \frac{i}{(4\pi)^{d/2}}\ \frac{\Gamma(3-d)\ \Gamma(d-2)\ \Gamma((d/2)-1)}{\Gamma(2-d/2)\ \Gamma((3d/2)-3)} \tag{1.24}$$

$$J_2 = (r^2)^{6-3d/2} \int \frac{d^ds}{(s^2)^{5-d}\ (r-s)^2}$$

$$= \frac{i}{(4\pi)^{d/2}}\ \frac{\Gamma(6-3d/2)\ \Gamma((3d/2)-5)\ \Gamma((d/2)-1)}{\Gamma(6-3d/2)\ \Gamma(2d-6)} \tag{1.25}$$

and $$J_3 = \int \frac{d^dr}{(r^2)^{6-3d/2}\ (p-r)^2}$$

$$J_3 = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(7-2d)\ \Gamma(2d-6)\ \Gamma((d/2)-1)}{\Gamma(6-3d/2)\ \Gamma((5d/2)-7)} (p^2)^{2d-7} \tag{1.26}$$

by substituting $d = \frac{7}{2} - \varepsilon$. The result is

$$\text{fig}(4b) = -\frac{32}{45}\ X\ B\ \Gamma(2\varepsilon) + \ldots\ldots. \tag{1.27}$$

where
$$X = \frac{1}{(4\pi)^7}\ \Gamma(3/4)^4 \tag{1.28}$$

The third graph contributing to the quadratic divergences at 4-loop level is graph fig(4c. This graph can be redrawn as in fig. (4ci). Thus we have

$$C\ Z \int \frac{d^dk\ d^dq}{(k^2)^{5-d}\ (k-q)^2\ (p-q)^2} \tag{1.29}$$

where Z is the graph in fig. (4cii)

and
$$C = U_{abc}\ U_{ade}\ U_{bdfg}\ U_{cefg} \tag{1.30}$$

Once again p is an arbitrary momentum routed to control the infra-red divergences.

Now doing the q integral:

$$C\ Z\ I \int \frac{d^dk}{(k^2)^{5-d}\ ((p-k)^2)^{2-d/2}} \tag{1.31}$$

where I is given in (1.16).

For Z given in graph fig. (4cii) we have

$$Z = \int \frac{d^d k \; d^d q}{q^2 \; k^2 \; (k+p)^2 \; (q+p)^2 \; (k-q)^2} \qquad (1.32)$$

To evaluate this integral we use an identity derived in ref. [11]

$$Z = \frac{2}{(d-4)} \; [\; (3-d).I^2 + (3d-10).I.M \;] \qquad (1.33)$$

where I is given in (1.16)

and

$$M = \int \frac{d^d k}{(k^2)^{3-d/2} \; (\; k+p)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(4-d) \; \Gamma(d-3\;) \; \Gamma((d/2)-1)}{\Gamma(3-d/2) \; \Gamma((3d/2)-4)} \; (p^2)^{d-4} \qquad (1.34)$$

After substituting $d = \frac{7}{2} - \varepsilon$. Then

$$Z = \frac{-\,8}{(4\pi)^{7/2}} \; \Gamma(3/4)^3 \left[\frac{\Gamma(3/4) \; \Gamma(1/4)^2}{\Gamma(1/2)^2} \; - \; \frac{8\,\Gamma(1/2)}{\Gamma(1/4)} \right] \qquad (1.35)$$

Now the result for graph fig. (4c) is

$$\text{fig}(4c) = \frac{c}{(4\pi)^7} \left[\frac{8}{3} \; \frac{\Gamma(3/4)^5 \; \Gamma(1/4)^3}{\Gamma(1/2\;)^3} \; - \; \frac{64}{3} \, \Gamma(3/4)^4 \right]$$

$$\text{fig}(4c) = C \left[\frac{8}{3} Y - \frac{64}{3} X \right] \Gamma(2\varepsilon) \qquad (1.36)$$

The graph in fig. (4d) gives:

$$D \int \frac{d^dk \; d^dq \; d^ds \; d^dr}{k^2 \; (q-k)^2 \; s^2 \; q^2 \; r^2 \; (s-r)^2 \; (p-s-q)^2} \qquad (1.37)$$

where

$$D = U_{abc} \; U_{bcde} \; U_{adfg} \; U_{efg} \qquad (1.38)$$

performing first the k and v integrals:

$$\text{fig}(4d) = D \; I^2 \int \frac{d^dq \; d^ds}{(q^2)^{3-d/2} \; (s^2)^{3-d/2} \; (p-s-q)^2}$$

$$= D.I^2. \; W_1.W_2 \qquad (1.39)$$

where I is given in (1.16) and

$$W_1 = ((p-q)^2)^{4-d} \int \frac{d^ds}{(s^2)^{3-d/2} \; (p-s-q)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(4-d) \; \Gamma(d-3\;) \; \Gamma((d/2)-1)}{\Gamma(3-d/2) \; \Gamma((3d/2)-4)} \qquad (1.40)$$

$$W_2 = \int \frac{d^dq}{(q^2)^{3-d/2} \; ((p-q)^2)^{4-d}}$$

$$= \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(7-2d) \; \Gamma(d-3\;) \; \Gamma((3d/2)-4)}{\Gamma(3\;-d/2) \; \Gamma(4-d) \; \Gamma((5d/2)-7)} \; (p^2)^{2d-7} \qquad (1.41)$$

Now substituting $d = \frac{7}{2} - \varepsilon$ into (1.40) and (1.41),

The result for graph fig. (4d) is

$$\text{fig}(4d) = \frac{32}{3} X D \Gamma(2\varepsilon) + \ldots\ldots \qquad (1.42)$$

where

$$X = \frac{1}{(4\pi)^7} \Gamma(3/4)^4 \qquad (1.43)$$

Finally we have the graph in fig(4e) which gives :

$$E \int \frac{d^dk\, d^dq\, d^ds\, d^dr}{(k^2+m^2)^2\, q^2\, s^2\, (s-q-k)^2\, r^2\, (r-q-k)^2} \qquad (1.44)$$

$$\text{where } E = U_{ab}\, U_{acde}\, U_{defg}\, U_{bcfg} \qquad (1.45)$$

Here we insert a mass m to control the infra-red divergences .

Now doing the s and r integrals we have

$$\text{fig}(4e) = E\, I^2 \int \frac{d^dk\ \ d^dq}{q^2\, [(q-k)^2]^{4-d}\, (k^2+m^2)^2} \qquad (1.46)$$

$$= E.I^2.N_1.N_2$$

where I is given by (1.16)

and

$$N_1 = (k^2)^{5-3d/2} \int \frac{d^dq}{q^2\, [(q-k)^2]^{4-d}}$$

$$N_1 = \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(5-3d/2)\ \Gamma((3d/2)-3\ )\ \Gamma((d/2)-1)}{\Gamma(4-d)\ \Gamma(2d-5)} \qquad (1.47)$$

and

$$N_2 = \int \frac{d^d k}{(k^2)^{5-3d/2}\ (k^2+m^2)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(7-2d)\ \Gamma(2d-5)}{\Gamma(2)\quad \Gamma(d/2)} \ (m^2)^{2d-7} \qquad (1.48)$$

Now substituting $d = \frac{7}{2} - \varepsilon$

Then the result for graph fig. (4e) is

$$\text{fig}(4e) = -\frac{2}{3}\ Y\ E\ \Gamma(2\varepsilon) + \ldots\ldots. \qquad (1.49)$$

where

$$Y = \frac{1}{(4\pi)^7} \ \frac{\Gamma(1/4)^3\ \Gamma(3/4)^5}{\Gamma(1/\ 2)^3} \qquad (1.50)$$

IF we denote the coefficient of the leading quadratic divergences at L loops as Δ_L then from (1.19), (1.27), (1.36), (1.42) and (1.49) we have the quadratic divergences at 4-loop level

$$\Delta_4 = \frac{2}{3}\ Y.A - \frac{32}{45}\ X.B + \left(\frac{8}{3}\ Y + \frac{64}{3}\ X \right) C + \frac{32}{3}\ X.D - \frac{2}{3}\ Y.E \qquad (1.51)$$

The essential new feature of the L = 4 calculation is the presence in the result of X and Y .

Note that

$$\delta = \frac{Y}{X} = \sqrt{\frac{2}{\pi}} \quad \Gamma(1/4)^2 \qquad (1.52)$$

This ratio cannot be expressed as a rational number, nor have we been able to find any relationship between it and $\zeta(3)$, which would have been interesting since $\zeta(3)$ occurs in β_3, as we shall see below[12].

1.2 The Cancellation Condition Conjecture:

Now we would like to test the Conjecture in ref. [6] that the quadratic divergences cancellation conditions can be understood to all orders in terms of the scale dependence of the one loop condition. In the case of Φ^4 theory the relationship up to the 3-loop level is

$$\Delta_2 = a_{11} \qquad (1.53)$$

$$\Delta_3 = - \frac{23}{36} a_{12} + \frac{1}{3} a_{21} \qquad (1.54)$$

where

$$a_{LM} = \beta_L \frac{\partial}{\partial U} \Delta_M - \Delta_M \frac{\partial}{\partial U} \beta_L \qquad (1.55)$$

and β_L is the L-loop contribution to β-functions where $\beta(U) = \mu \frac{\partial U}{\partial \mu}$

To verify equs. (1.53) and (1.54) we need β_1, β_2, which are given by[13][14]

$$\beta_1 = \frac{1}{2} U_{ab} U_{ab} \tag{1.56}$$

$$\beta_2 = -\frac{1}{2} U_{ab} U_{acd} U_{bcd} + \frac{1}{12} U_a U_{abcd} U_{ebcd} \Phi^e \tag{1.57}$$

Here we have suppressed a factor of $(16\pi^2)^{-L}$.

We will verify (1.54) as an example:

from (1.55) we have

$$a_{12} = \beta_1 \frac{\partial}{\partial U} \Delta_2 - \Delta_2 \frac{\partial}{\partial U} \beta_1 \tag{1.58}$$

Using (1.56) and (1.12)

$$a_{12} = \frac{1}{2} U_{ef} U_{ef} \frac{\partial}{\partial U} U_{abc} U_{abc} - \frac{1}{2} U_{abc} U_{abc} \frac{\partial}{\partial U} U_{ef} U_{ef}$$

$$= [U_{ab} U_{ab}]_{abc} U_{abc} - [U_{abc} U_{abc}]_{ef} U_{ef} \tag{1.59}$$

where $[U_{abc} U_{abc}]_d = \frac{\partial}{\partial \Phi^d} [U_{abc} U_{abc}]$ etc.

Then

$$a_{12} = 6 \ U_{abc} U_{efab} U_{efc} - 2 \ U_{ef} U_{abce} U_{abcf} \tag{1.60}$$

Now for the second term in (1.54)

$$a_{21} = \beta_2 \frac{\partial}{\partial U} \Delta_1 - \Delta_1 \frac{\partial}{\partial U} \beta_2 \tag{1.61}$$

Using (1.11) and (1.57) then

$$a_{21} = \frac{1}{2} U_{ab} U_{acd} U_{bcd} \frac{\partial}{\partial U} U_{kk}$$

$$+ \frac{1}{12} U_a U_{abcd} U_{ebcd} \Phi^e \frac{\partial}{\partial U} U_{kk}$$

$$- \frac{1}{12} U_{kk} \frac{\partial}{\partial U} U_a U_{abcd} U_{ebcd} \Phi^e$$

$$- \frac{1}{2} U_{kk} \frac{\partial}{\partial U} U_{ab} U_{acd} U_{bcd}$$

$$= - 2 U_{abk} U_{acdk} U_{bcd} - \frac{5}{6} U_{ab} U_{acdk} U_{bcdk} \qquad (1.62)$$

Now we would like to find α_1 and α_2 such that

$$\Delta_3 = \alpha_1 a_{12} + \alpha_2 a_{21} \qquad (1.63)$$

substituting (1.13),(1.61),(1.62) and solving for α_1,α_2 we have

$$\alpha_1 = -\frac{23}{36} \quad , \quad \alpha_2 = \frac{1}{3} \qquad (1.64)$$

It follows that the relationship in (1.55) holds at 3- loop level. Now the crucial question with regard to the relationship in (1.55), for the 4-loop level, is whether we can find $\alpha_1,\ldots\ldots,\alpha_3$ such that

$$\Delta_4 = \alpha_1 a_{13} + \alpha_2 a_{22} + \alpha_3 a_{31} \qquad (1.65)$$

To discuss this we need β_3 .

β_3 can be expressed as[14]

$$\beta_3 = \frac{1}{4} U_{abcd} U_{efcd} U_{ae} U_{bf} - \frac{3}{16} U_{abcd} U_{ebcd} U_{af} U_{ef}$$
$$+ 2 U_{abcd} U_{ae} U_{bcf} U_{def} - \frac{1}{4} U_{abcd} U_{ef} U_{abe} U_{cdf}$$
$$- \frac{1}{8} U_{abc} U_{dbc} U_{aef} U_{bef} + \frac{1}{2} \zeta(3) U_{abc} U_{ade} U_{dbf} U_{cef}$$
$$- \frac{1}{16} U_a U_{abcd} U_{ebfg} U_{cdfg} \Phi^e . \qquad (1.66)$$

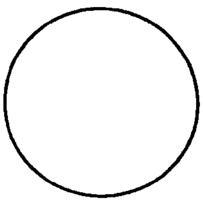
Using the results for Δ_L [eqns (1.11)-(1.13)] and β_L eqns (1.56), (1.57) and (1.66)] it is straightforward to show that

$$a_{13} = - \frac{27}{2} A + B - 9 C - 18 D + 15 E \qquad (1.67)$$

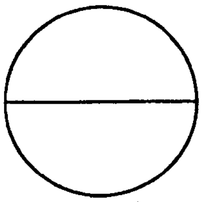
$$a_{22} = \frac{3}{2} B - 6 C - 6 D \qquad (1.68)$$

$$a_{31} = - \frac{3}{2} A - \frac{7}{8} B + \left(\frac{9}{2} + 6 \zeta(3) \right) C + \frac{7}{2} D + \frac{27}{8} E \qquad (1.69)$$

Unfortunately it is also straightforward to show that there exists no values of $(\alpha_1,\ldots,\alpha_3)$ such that eqn.(1.65) is true. It would thus appear that the previous conjecture, namely that absence of quadratic divergences at L-loop is equivalent to demanding scale invariance of the naturalness conditions for all L"< L is not true. Thus the precise relationship between scale invariance and quadratic divergences remains unclear.



fig(1)



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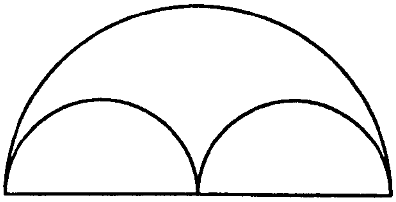
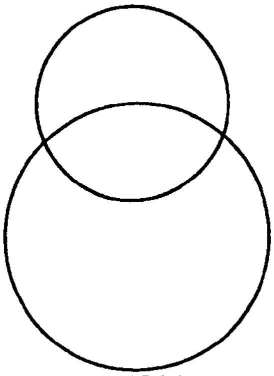
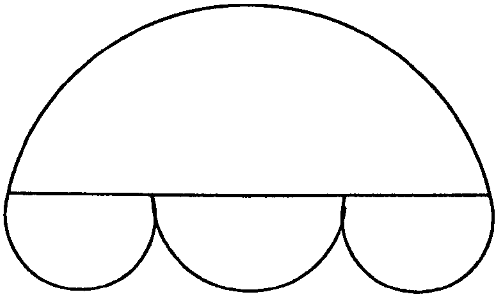


fig(3a)



fig(3b)



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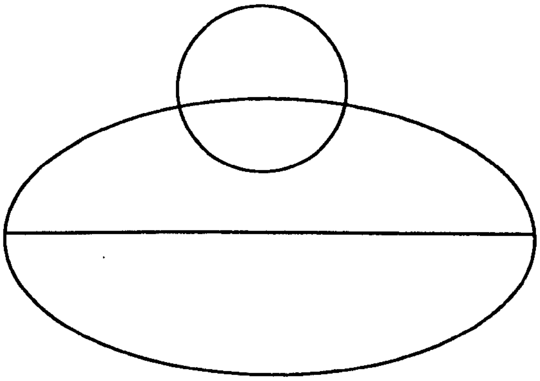
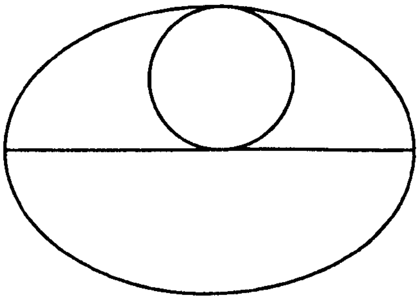
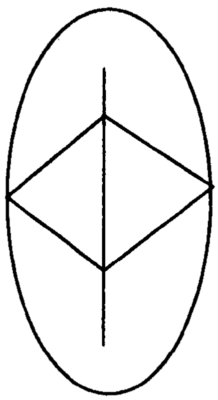


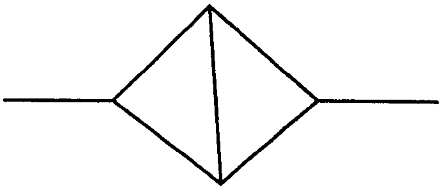
fig(4b)



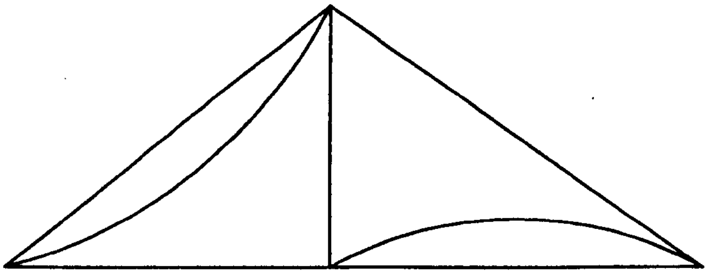
fig(4c)



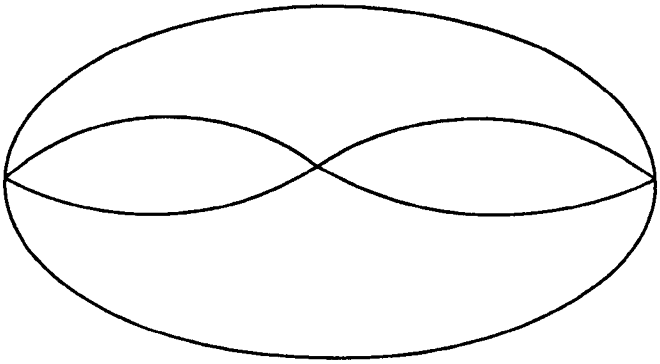
fig(4ci)



fig(4cii)



fig(4d)



fig(4e)

CHAPTER 2 : QUADRATIC DIVERGENCES IN Φ^3 THEORY IN d = 6 .

As we shall see in this chapter and the next two chapters we are going to inquire whether the intriguing features which we have discovered for quadratic divergences in four space-time dimensional theories are also displayed by theories in other space-time dimensions. In this chapter we choose to consider Φ^3 theory in six space-time dimensions. Our purpose will be once again to calculate the quadratic divergences and test the cancellation condition conjecture ,this time at the leading order . The first question to be asked is at what dimension the quadratic divergences manifest themselves in this theory ?

Using (1.1) , (1.2) and (1.3) we have

$$D = (d - 6)L - 2E + 6 \qquad (2.1)$$

where D is the degree of divergence , L is the number of loops , E is the external lines , and d is the space - time dimension.

For d = 6

$$D = 6 - 2E \qquad (2.2)$$

and so when E = 2 we have a quadratically divergent graph.

In dimensional regularization the corresponding poles in the Green functions occur when D = 0 i.e when

$$d = 6 - 2 / L \qquad (2.3)$$

So at L-loop level the quadratic divergences occur at d = 6 - 2/L .

We begin with the basic Lagrangian in Minkowski space

$$L = \frac{1}{2} (\partial_\mu \Phi^a)^2 - U(\Phi) \qquad (2.4)$$

where a = 1,......,N and $U(\Phi)$ is a polynomial in Φ^a of degree three. It is straightforward in d = 6 to show from (2.1) that the quadratic divergences occur in graphs with E = 2 where E is the external lines.

Now we apply the background field method , i.e. we let

$$\Phi \longrightarrow \Phi + \Phi_q \qquad (2.5)$$

where Φ_q is the internal (quantum) lines and Φ is the external (classical) lines .

Since we are only interested in the leading divergences, graphs with non-overlapping divergences can be ignored. So (2.4) becomes

$$\begin{aligned} L = & \frac{1}{2}(\partial_\mu \Phi^a)^2 - U(\Phi) - \frac{1}{2} (\partial_\mu \Phi_q)^2 \\ & - \Phi_q^{\ a} U_a - \frac{1}{2!} \Phi_q^{\ a} \Phi_q^{\ b} U_{ab} \\ & - \frac{1}{3!} \Phi_q^{\ a} \Phi_q^{\ b} \Phi_q^{\ c} U_{abc} \end{aligned} \qquad (2.6)$$

Where $U_a = \partial U / \partial \Phi^a$ etc.

We would like to calculate the vacuum graphs which contribute to the quadratic divergences effective action up to the 2-loop level which displayed in figs.(1) ,(2), and (3). The two point function graphs can be obtained by differentiating twice with respect to the field Φ . According to (2.1) we are interested in the graphs with two external lines which are quadratically divergent.

2.1 One and Two loop calculations :

Essential details concerning our regularization procedure (dimensional regularization) , signs , and factors are contained in appendix one .In this calculations , we have arranged matters in such a way as to ensure that the answers obtained below contribute directly to the quadratic divergences.

We now proceed with the calculations:

From (2.3) the dimension where the quadratic divergences occur at the one-loop level is

$$d = 6 - 2/L = 4 \ .$$

For the graph in fig(1) which gives

$$\frac{1}{2} A \int \frac{d^d k}{k^2 \ (p - k)^2} \tag{2.7}$$

where $A = U_{ab} U_{ab}$ (2.8)

and p is an arbitrary momentum .

$$\text{fig}(1) = \frac{i}{(4\pi)^{d/2}} \ A \frac{\Gamma(2-d/2) \ \Gamma(d/2-1)^2}{\Gamma(d-2)} \tag{2.9}$$

substituting $d = 4 - \varepsilon$ we have

$$\text{fig}(1) = \frac{i}{4(16\pi^2)} \ A \ \Gamma(\varepsilon/2) + \ldots . \tag{2.10}$$

where A is given in (2.8) .

Since we are looking for graphs with two external lines , the graph in fig(1) is the only graph which contributes to the quadratic divergences at the one-loop level . The other graph which can be drawn at the one loop level has three external lines . Both graphs contribute to the logarithmic divergences and hence to the one-loop level β-function as we shall see later in this chapter .

Now the graphs which contribute to the quadratic divergences at two loop level are displayed in fig(2) , and fig(3)

The graph in fig(2)gives

$$\frac{1}{2} \quad B \int \frac{d^d k \quad d^d q}{q^2 \; (k - q)^2 \; (k^2)^2 \; (p - k)^2} \tag{2.11}$$

Where
$$B = U_{ab} \; U_{adc} \; U_{dce} \; U_{eb} \tag{2.12}$$

After doing the q integral we have

$$\text{fig}(2) = \frac{i}{(4\pi)^{d/2}} \quad \frac{B \; \Gamma(2-d/2) \; \Gamma(d/2-1)^2}{\Gamma(d-2)} \; U \tag{2.13}$$

Where

$$U = \int \frac{d^d k}{(k^2)^{4-d/2} \; (p - k)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(5-d) \; \Gamma(d-4) \; \Gamma(d/2-1)}{\Gamma(4-d/2) \; \Gamma(3d/2 - 5)} \quad (p^2)^{d-5} \tag{2.14}$$

substituting $d = 5 - \varepsilon$ we have

$$\text{Fig}(2) = \frac{1}{6(4\pi)^5} \; \Gamma(1/2)^2 \; \Gamma(\varepsilon) \tag{2.15}$$

For the graph in fig(3) we have

$$C \int \frac{d^d k \quad d^d q}{k^2 \; (q - p)^2 \; q^2 \; (p - q)^2 \; (p - k)^2} \tag{2.16}$$

where $C = U_{ab} \, U_{acd} \, U_{de} \, U_{bce}$ (2.17)

and p is an arbitrary momentum

Now

$$\text{fig}(3) = \frac{1}{2} C \int \frac{d^d k \; d^d q}{(k^2)^2 \; (q - k)^2 \; (q^2)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \; C \; \frac{\Gamma(3-d/2) \; \Gamma(d/2-1) \; \Gamma(d/2-2)}{\Gamma(d-3)} \quad M \tag{2.18}$$

where

$$M = \int \frac{d^d q}{(q^2)^{4-d/2} \; (p - q)^2} \tag{2.19}$$

Then

$$M = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(5-d) \; \Gamma(d-4) \; \Gamma(d/2-1)}{\Gamma(4-d/2) \; \Gamma(3d/2 - 5)} \quad (p^2)^{d-5} \tag{2.20}$$

Now $d = 5 - \varepsilon$ we have

$$\text{fig}(3) = -\frac{1}{3(4\pi)^5} \; \Gamma(1/2)^2 \; \Gamma(\varepsilon) \tag{2.21}$$

If we denote the coefficient of the leading quadratic divergences at L loops as Δ_L then for Φ^3 theory in d = 6 we have

$$\Delta_1 = \frac{i}{4(16\pi^2)} A \qquad (2.22)$$

$$\Delta_2 = \left(\frac{B}{6(4\pi)^5} + \frac{C}{3(4\pi)^5} \right) \Gamma(1/2)^2 \qquad (2.23)$$

Where

$$\Gamma(1/2) = \sqrt{\pi}$$

$$A = U_{ab}\, U_{ab}$$

$$B = U_{ab}\, U_{adc}\, U_{dce}\, U_{eb}$$

$$C = U_{ab}\, U_{cad}\, U_{bce}\, U_{de}$$

As in (2.22) and (2.23) we have calculated the quadratic divergences for the one and two loop level in Φ^3 theory in d = 6 using dimensional regularization . In the next section we will produce the result in (2.23) from (2.22) and the β-function .

2.2 The cancellation condition conjecture :

In this section we will be testing the conjecture in ref.[7] for Φ^3theory in d = 6 . According to this conjecture we will be able to produce the two-loop quadratic divergences in (2.23) given the one-loop β-function and the one-loop quadratic divergences . We have calculated the one-loop quadratic divergences in the previous section. Now we want to calculate the β-function at one-loop level for the theory.

One-loop β-function

We start with the renormalisable Lagrangian

$$L^{ren} = L + L_{c.t} \qquad (2.24)$$

where L is the original lagrangian for Φ^3 theory

$$L = \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{\lambda}{3!} \Phi^3 \qquad (2.25)$$

and $L_{c.t}$ is the counterterm Lagrangian

$$L_{c.t.} = \frac{1}{2} A (\partial_\mu \Phi)^2 - \lambda \mu^{\varepsilon/2} B \Phi^3 \qquad (2.26)$$

As we can see from (2.25) and (2.26) that $L_{c.t.}$ is exactly of the same form as L , but with A and B so that the Green functions generated by L^{ren} are finite as $\varepsilon \longrightarrow 0$.

we can rewrite L^{ren} as

$$L^{ren} = \frac{1}{2} (\partial_\mu \Phi_B)^2 - \lambda_B \Phi_B^3 \qquad (2.27)$$

Where

$$\Phi_B = (1+A)^{1/2} \Phi = Z_\Phi^{1/2} \Phi \qquad (2.28)$$

$$\lambda_B = \lambda \mu^{\varepsilon/2} (1+B)/(1+A)^2 = \mu^{\varepsilon/2} Z_\lambda \lambda \qquad (2.29)$$

and Z is the wave function renormalisation constant, and Φ_B , λ_B are called the bare field and coupling constant respectively. we can see that L^{ren} looks the same as L except L^{ren} leads to a finite theory but L does not.

For the 1PI Green functions

$$\Gamma_B^{(n)}(p_1,\ldots,p_n;\lambda_B,m_B,\varepsilon) = Z_\Phi^{-n/2}\,\Gamma^{(n)}(p_1,\ldots,p_n;\lambda,m,\mu,\varepsilon). \tag{2.30}$$

In this equation we can see that 1PI function $\Gamma^{(n)}$ depends on μ through the dependence of Z_Φ on μ but $\Gamma_B^{(n)}$ does not . Therefore by differentiating the above equation with respect to μ we obtain a differential equation that summarizes the magic of renormalisation.

$$\left[\mu \frac{\partial}{\partial\mu} + \mu \frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + \mu \frac{\partial m}{\partial\mu}\frac{\partial}{\partial m} - \frac{n}{2}\,\mu \frac{\partial}{\partial\mu}\ln Z_\Phi\right]\Gamma^{(n)} = 0 \tag{2.31}$$

We define

$$\gamma(g) = 1/2\ \ \mu \frac{\partial}{\partial\mu}\ln Z_\Phi \tag{2.32}$$

$$\gamma_m = 1/2\ \ \mu \frac{\partial}{\partial\mu}\ln m^2 \tag{2.33}$$

$$\beta(\lambda) = \mu \frac{\partial\lambda}{\partial\mu} \tag{2.34}$$

each one of these coefficients is of particular interest but our interest is in the β-function in (2.34) . We need to calculate the β-function for Φ^3 theory in d = 6 .

Now

$$\begin{aligned} L &= \frac{1}{2}(\partial_\mu\Phi)^2 Z_\Phi - \lambda\,\Phi^3 Z_\lambda Z_\Phi^{3/2}\mu^{\varepsilon/2} \\ &= \frac{1}{2}(\partial_\mu\Phi)^2 Z_\Phi - \lambda\,\Phi^3 Z_1\,\mu^{\varepsilon/2} \end{aligned} \tag{2.35}$$

where

$$Z_1 = Z_\phi^{3/2} Z_\lambda \quad . \tag{2.36}$$

The graphs which contribute to the one-loop β-function in Φ^3 theory in d = 6 are in fig(4) and fig(5)

For the graph in fig(4) we have

$$\int \frac{d^dk}{k^2\ (p-k)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\ \Gamma(d/2-1)^2}{\Gamma(d-2)} \quad (p^2)^{d/2-2} \tag{2.37}$$

where p is an arbitrary momentum .

In d = 6 - ε

$$\begin{aligned} \text{fig}(4) &= \frac{i}{(4\pi)^3}(-1/2)\ (1/6)\ \Gamma(\varepsilon/2)\ p^2 \\ &= -\frac{i}{(4\pi)^3} 1/12\ (2/\varepsilon)\ \ p^2 \\ &= -\frac{i}{(4\pi)^3}(1/6)\ p^2/\varepsilon \ + \ldots\ldots \end{aligned} \tag{2.38}$$

where 1/2 is the symmetry factor for the graph in fig(4).

The other contribution comes from the graph in fig(5) which gives :

$$\int \frac{d^dk}{(k^2)^2 (p-k)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\ \Gamma(d/2-2)\ \Gamma(d/2-1)}{\Gamma(2)\ \Gamma(d-3)} \quad (p^2)^{d/2-3} \tag{2.39}$$

In d = 6 - ε

$$\text{fig}(5) = \frac{i}{(4\pi)^3} \ \frac{1}{\varepsilon} \ + \ldots\ldots \tag{2.40}$$

Now from (2.28) and (2.29) we can calculate Z_Φ and Z_1 hence Z_λ which will determine the β-function for the one-loop level.

$$Z_\Phi = 1 + \frac{1}{(4\pi)^3}(1/6\varepsilon)\lambda^2 \tag{2.41}$$

$$Z_1 = 1 + \frac{1}{(4\pi)^3}(1/\varepsilon)\lambda^2 \tag{2.42}$$

So

$$Z_\lambda = 1 + \frac{1}{(4\pi)^3}(3/4\varepsilon)\lambda^2 \tag{2.43}$$

From (2.29) and the definition of β-function in (2.34) we have the β-function for the one loop level in Φ^3 theory[15] :

$$\beta_1 = - \frac{1}{(4\pi)^3}(3/4)\ \lambda^3 \tag{2.44}$$

In the general case we have

$$\beta_1 = a\ (\ U_{de}\ U_{ef}\ U_{fd}) + b\ (\ \Phi_e\ U_{efg}\ U_{fgh}\ U_h) \tag{2.45}$$

where a and b are calculable constants.

The question now is whether we can produce the two -loop result for the quadratic divergences in (2.23) using the information of the one-loop quadratic divergences in (2.22) and the β-function in (2.40). Now

$$\Delta_2 = a_{11} \tag{2.46}$$

where

$$a_{11} = \beta_1\ (\ \frac{\partial}{\partial U}\Delta_1\ ) + \Delta_1(\ \frac{\partial}{\partial U}\beta_1\ )$$

$$= a\ (\ U_{de}\ U_{ef}\ U_{fd})\ \ \frac{\partial}{\partial U}(U_{ab}U_{ab})$$

$$+ b\ (\ \Phi_e\ U_{efg}\ U_{fgh}\ U_h)\ \ \frac{\partial}{\partial U}(U_{ab}U_{ab}\ ) - a(\ U_{ab}U_{ab})\ \ \frac{\partial}{\partial U}(U_{de}\ U_{ef}\ U_{fd})$$

$$- b(\ U_{ab}U_{ab})\ \ \frac{\partial}{\partial U}\ (\Phi_e\ U_{efg}\ U_{fgh}\ U_h)$$

$$a_{11} = 2a\ (U_{de}\ U_{ef}\ U_{fd}\ )_{ab}\ U_{ab}$$

$$+ 2b\ (\Phi_e\ U_{efg}\ U_{fgh}\ U_h\ )_{ab}\ U_{ab}$$

$$- 3a\ (U_{ab}U_{ab}\ )_{de}\ U_{ef}\ U_{fd}$$

$$- b\ \Phi_e\ (U_{ab}U_{ab}\ )_h\ U_{efg}\ U_{fgh}$$

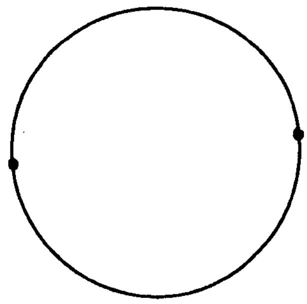
finally

$$a_{11} = a[\ 12\ U_{dea}\ U_{efb}\ U_{fd}\ U_{ab}$$

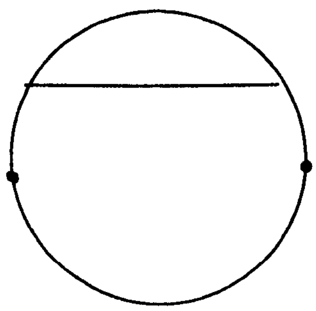
$$- 6\ U_{abd}\ U_{abe}\ U_{ef}\ U_{fd}\ ]$$

$$+ b[\ 4\ U_{efg}\ U_{fgh}\ U_{hb}\ U_{ab}] \qquad (2.47)$$

Unfortunately it is straightforward to see that (2.23) does not agree with (2.47) so the proposed conjecture , that the absence of quadratic divergences at L-loop is equivalent to demanding scale invariance is not true in this theory also .Thus the relationship between β-function and the quadratic divergences remains unclear in this theory .



fig(1)



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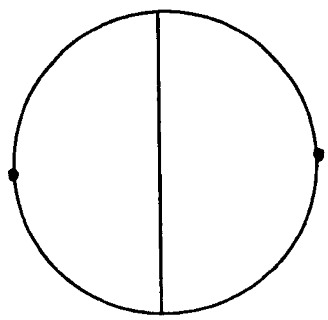

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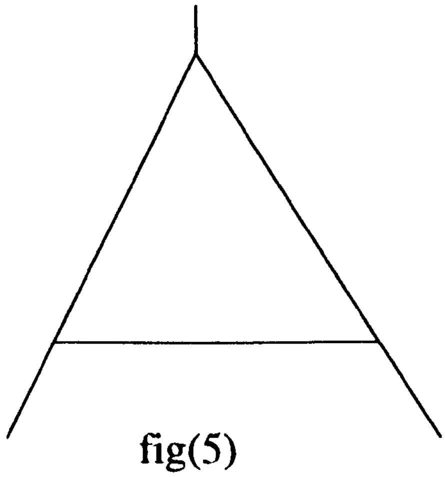
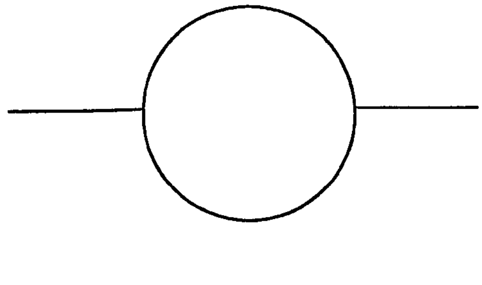


fig(5)



fig(4)

CHAPTER 3 : QUADRATIC DIVERGENCES IN Φ^6 IN $d = 3$:

In this chapter we examine another renormalisable Φ^r type theory . We will follow the same procedure as in the previous two chapters . First we have to determine the dimension where the quadratic divergences manifest themselves . Using (1.1)and (1.2) we have :

$$rV = 2I + E \tag{3.1}$$

where V is the number of vertices , I is the number of the internal lines (propagators) , E is the number of the external lines and r in this case is equal to 6 . Now

$$D = (d-3)L - \frac{1}{2}E + 3 \tag{3.2}$$

where D is the degree of divergence , L is the number of loops and d is the space-time dimension .

For d = 3 we have

$$D = 3 - \frac{1}{2}E \quad . \tag{3.3}$$

It is clear from (3.3) that when E = 2 we have quadratically divergent graphs . In dimensional regularization the corresponding poles in the Green function occur when D = 0 i.e. when

$$d = 3 - \frac{2}{L} \tag{3.4}$$

where d is the space time dimensions where the quadratic divergences occur in this theory.

We begin with the basic Lagrangian for this theory in Minkowski space :

$$L = \frac{1}{2} (\partial_\mu \Phi^a)^2 - U (\Phi) \qquad (3.5)$$

Where a = 1,.....,N and $U(\Phi)$ is a polynomial in Φ^a of degree six. The effective action for the 1PI vacuum graphs can be calculated by applying the background field method i.e. we let :

$$\Phi \longrightarrow \Phi + \Phi_q \qquad (3.6)$$

where Φ_q is the internal (quantum) lines and Φ is the external (classical) lines .

Applying the transformation (3.6) to the Lagrangian (3.5) we have :

$$\begin{aligned} L = & \frac{1}{2} (\partial_\mu \Phi^a)^2 - U(\Phi) - \frac{1}{2} (\partial_\mu \Phi_q{}^a)^2 \\ & - \Phi_q{}^a U_a - \frac{1}{2} \Phi_q{}^a \Phi_q{}^b U_{ab} \\ & - \frac{1}{3!} \Phi_q{}^a \Phi_q{}^b \Phi_q{}^c U_{abc} - \frac{1}{4!} \Phi_q{}^a \Phi_q{}^b \Phi_q{}^c \Phi_q{}^d U_{abcd} \\ & - \frac{1}{5!} \Phi_q{}^a \Phi_q{}^b \Phi_q{}^c \Phi_q{}^d \Phi_q{}^e U_{abcde} \\ & - \frac{1}{6!} \Phi_q{}^a \Phi_q{}^b \Phi_q{}^c \Phi_q{}^d \Phi_q{}^e \Phi_q{}^f U_{abcdef} \end{aligned} \qquad (3.7)$$

where $U_a = \frac{\partial U}{\partial \Phi^a}$ etc.

Since we are only interested in the leading divergences , graphs with non-overlapping divergences can be ignored.

The linear term in (3.7) does not contribute to the calculation since we are looking for 1PI graphs only .

From (3.3) the graphs which contribute to the quadratic divergences are the graphs with two external lines These graphs are displayed in fig(1) , fig(2) and fig(3) .

In the next section we will be calculating the vacuum graphs which contribute to the quadratic divergences in the effective action up to the six-loop level .

3.1 The loop calculations

In this theory the leading order quadratic divergences occur at the four-loop level . In this section we will calculate the quadratic divergences at four and six loop level . Essential detail concerning the regularization procedure (dimensional regularization) , signs , and factors are contained in appendix one . In this calculations , we have arranged matters in such a way as to ensure that the answers obtained below contribute directly to the quadratic divergences.

Now we start with the four loop calculation :

The only graph which contributes at this loop level is the graph in fig(1) which gives

$$(1/5!)\ A \int \frac{d^d k\ d^d q\ d^d r\ d^d s}{k^2\ (\ q-k\ )^2\ (\ r-q\ )^2\ (\ s-r\ )^2\ \ (\ p-s\ )^2} \qquad (3.8)$$

Where $$A = U_{abcde}\ U_{abcde} \qquad (3.9)$$

and we have routed the arbitrary momentum p to control the infrared divergences. By doing the k,q,and r integrals then

$$\text{fig (1)} = (1/5!)\ A\ G_1\ G_2\ G_3 \int \frac{d^d s}{(s^2)^{4-3d/2}\ (\ p-s\ )^2} \tag{3.10}$$

where

$$G_1 = (\ q^2)^{2-d/2} \int \frac{d^d k}{(k^2)\ (\ q-k\ )^2} \tag{3.11}$$

$$= \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(2-d/2)\ \Gamma((d/2)-1)^2}{\Gamma(d-2)}$$

$$G_2 = (\ r^2\ )^{3-d} \int \frac{d^d q}{(q^2)^{2-d/2}\ (\ r-q\ )^2} \tag{3.12}$$

$$= \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(3-d)\ \Gamma(d-2)\ \Gamma((d/2)-1)}{\Gamma(2-d/2)\ \Gamma((3d/2)-3)}$$

$$G_3 = (\ s^2\ )^{4-3d/2} \int \frac{d^d r}{(r^2)^{3-d}\ (\ s-r\ )^2} \tag{3.13}$$

$$= \frac{i}{(4\pi)^{d/2}} \quad \frac{\Gamma(4-3d/2)\ \Gamma((3d/2)-3)\ \Gamma((d/2)-1)}{\Gamma(3-d)\ \Gamma(2d-4)}$$

Now we do the s integral in (3.10) which gives

$$\int \frac{d^d s}{(s^2)^{4-3d/2}\,(\,p-s\,)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(5-2d)\;\Gamma(2d-4)\;\Gamma((d/2)-1)}{\Gamma(4-3d/2)\;\Gamma((5d/2)\,-\,5)} \quad (p^2)^{2d-5} \tag{3.14}$$

Substituting $d = (5/2) - \varepsilon$ in (3.11), (3.12), (3.13) and (3.14) we have

$$\mathrm{fig}(1) = \frac{1}{(4\pi)^5} \quad (1/30)\; A\; \Gamma(1/4)^4\; \Gamma(2\varepsilon) + \ldots\ldots\ldots \tag{3.15}$$

The next order in the perturbation series where the quadratic divergences occur is at six-loop level and the graph which contribute to it are in fig(2) and fig(3).

For the graph in fig(2) we have :

$$\frac{5}{72} \quad B\, H^2 \int \frac{d^d k\; d^d q}{k^2\;(\,q-k\,)^2\;((\,p-q\,)^2)^{6-2d}} \tag{3.16}$$

where $B = U_{abcde}\; U_{cdefgh}\; U_{abfgh}$

and H comes from the sub graph in fig(2a) :

$$H = \quad ((\,p-q)^2)^{3-d} \int \frac{d^d m\; d^d n}{m^2\;(\,n-m\,)^2\;(\,p-n-q\,)^2}$$

$$= \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(3-d)\quad \Gamma((d/2)-1)^3}{\Gamma((3d/2)\,-3)} \tag{3.17}$$

Now it is straightforward by doing the k and q integral in (3.16)and substituting $d = (8/3) - \varepsilon$ we have

$$fig(2) = -\frac{15}{72\ (4\pi)^8}\ B\ \Gamma(1/3)^9\ \Gamma(3\varepsilon) + \ldots\ldots. \qquad (3.18)$$

For the other graph at six loop level which is in fig(3) :

$$fig(3) = \frac{5C}{72}\ H \int \frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ ((q-k)^2)^{3-d}\ r^2\ (q-r)^2\ s^2\ (q-s)^2} \qquad (3.19)$$

Where $C = U_{abcdef}\ U_{fehg}\ U_{abcdhg}$ and H is given in (3.17)

Then by doing the k , s and r integrals we have :

$$\frac{5C}{72}\ H\ Z_1\ Z_2\ Z_3 \int \frac{d^dq}{(q^2)^{7-5d/2}\ (q^2-m^2)} \qquad (3.20)$$

As we can see from (3.19) that the s integral is equal to the r integral then

$$Z_3 = Z_2 = (q^2)^{2-d/2} \int \frac{d^ds}{s^2\ (q-s)^2}$$

$$= \frac{i}{(4\pi)^{d/2}}\ \frac{\Gamma(2-d/2)\ \Gamma((d/2)-1)^2}{\Gamma(d-2)} \qquad (3.21)$$

and

$$Z_1 = (q^2)^{4-3d/2} \int \frac{d^dk}{k^2\ ((q-k)^2)^{3-d}}$$

$$= \frac{i}{(4\pi)^{d/2}}\ \frac{\Gamma(4-3d/2)\ \Gamma((3d/2)-3)\ \Gamma((d/2)-1)}{\Gamma(3-d)\ \Gamma(2d-4)} \qquad (3.22)$$

Now performing the integration in (3.20) and substituting for $d = 8/3 - \varepsilon$ we have

$$\text{fig}(3) = \frac{45}{72(4\pi)^8} \, C \, \Gamma(1/3)^6 \, \Gamma(3\varepsilon/2) \, \Gamma(3\varepsilon) + \ldots\ldots \quad (3.23)$$

As we can see from the above equation an interesting feature has been displayed ; the double pole . This double pole is due to a divergent sub graph in the original six-loop graph ,which can be isolated , and it is displayed in fig(3a) .The graph in fig(3a) has four external lines this feature makes it linearly divergent according to equation in (3.3) and this linear divergence at three loop level occurs at the same dimension where the quadratic divergences occur at six loop level .

It is obvious that the quadratic divergences in fig(3) are different to the quadratic divergences in fig(2) because of the double pole in fig(3) , we denote the coefficient of the quadratic divergences with a simple pole by Δ_L and the coefficient of quadratic divergences with double pole by $\hat{\Delta}_L$, then from (3.15) , (3.18) and (3.23) we have

$$\Delta_4 = \frac{1}{(4\pi)^5} \, (1/30) \, A \, \Gamma(1/4)^4 \quad (3.24)$$

$$\Delta_6 = \frac{-15}{72(4\pi)^8} \, B \, \Gamma(1/3)^9 \quad (3.25)$$

$$\hat{\Delta}_6 = \frac{-45}{72(4\pi)^8} \, C \, \Gamma(1/3)^6 \quad (3.26)$$

Where

$$A = U_{abcde} U_{abcde}$$

$$B = U_{abcde} U_{cdefgh} U_{abfgh}$$

and

$$C = U_{abcdef} U_{fehg} U_{abcdhg}$$

The next stage of our calculations is to test the cancellation condition conjecture for Φ^6 theory in $d = 3$.

3.2 The cancellation condition conjecture

The question to ask now is can we produce the coefficient of the quadratic divergences at six loop level as in (3.25) and (3.26) from the coefficient of the quadratic divergences at four loop level , and the β-function ?

From the conjecture we have :

$$\Delta_6 = \alpha_1 a_{15} + \alpha_2 a_{24} + \alpha_3 a_{33} + \alpha_4 a_{42} + \alpha_5 a_{51} \qquad (3.27)$$

where

$$a_{ML} = \beta_M \frac{\partial \Delta_L}{\partial U} - \Delta_L \frac{\partial \beta_M}{\partial U}$$

but the first order of the quadratic divergences in this theory occur at four-loop level and the next order occur at six-loop level as there is no contribution from one , two , three , and five-loop so $a_{15} = a_{33} = a_{42} = a_{51} = 0$ this reduces (3.27) to

$$\Delta_6 = \alpha \, a_{24} \qquad (3.28)$$

where

$$a_{24} = \beta_2 \frac{\partial}{\partial U} \Delta_4 - \Delta_4 \frac{\partial}{\partial U} \beta_2 \qquad (3.29)$$

To calculate Δ_6 in (3.28) we need to know the two-loop β-function . The only contribution to the two-loop β-function comes from the vertex graph in fig(4) and in general we have [15]:

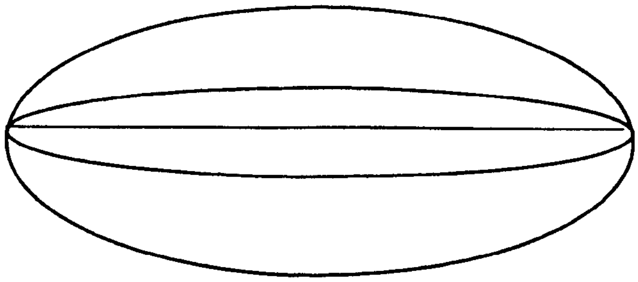
$$\beta_2 = c\, U_{mnk} U_{mnk} \qquad (3.30)$$

where c is a constant .

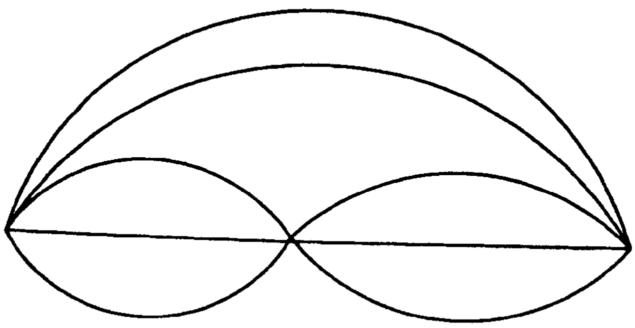
Using (3.24) and (3.30) then (3.29) gives:

$$\begin{aligned} a_{24} &= c\,(U_{mnk} U_{mnk}) \frac{\partial}{\partial U} (U_{abcd} U_{abcd}) \\ &\quad - c(U_{abcd} U_{abcd}) \frac{\partial}{\partial U} (U_{mnk} U_{mnk}) \\ &= 16\, c\, U_{mnkabc} U_{mnkd} U_{abcd} \\ &\quad + 12\, c\, U_{mnkab} U_{mnkcd} U_{abcd} \\ &\quad - 12\, c\, U_{abcdmn} U_{abcdk} U_{mnk} \end{aligned}$$

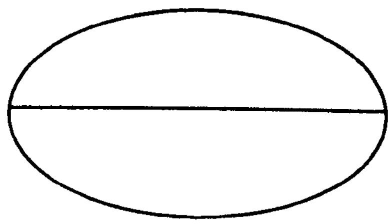
From this result it is clear that the proposed conjecture failed to produce Δ_6 and $\hat{\Delta}_6$ .Thus once again the relationship between scale invariance and the quadratic divergences remains unclear.



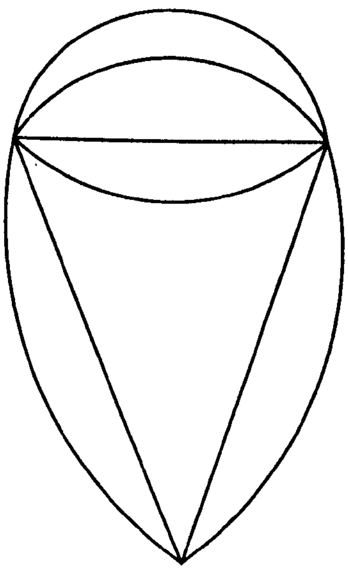
fig(1)



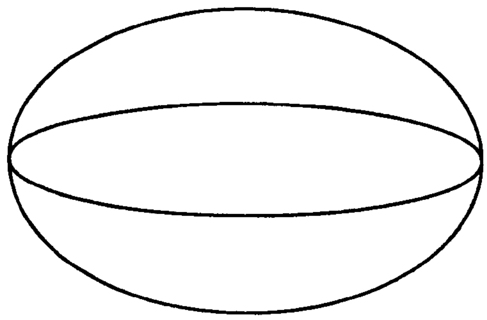
fig(2)



fig(2a)



fig(3)



fig(3a)

CHAPTER 4: QUADRATIC DIVERGENCES IN 2-DIMENSIONAL SIGMA MODELS

Two dimensional sigma models are currently being extensively investigated, primarily because of their close relationship to string theory. They are also interesting field theories in their own right. In this chapter , we shall investigate the structure of the quadratic divergences in this theory and inquire whether the intriguing features which we have discovered for quadratic divergences in theories in other dimensions in the previous chapters are also displayed here. This chapter will be in two sections; the first section will be devoted to examining the quadratic divergences for torsion-free two-dimensional non-linear sigma models up to four-loop order, and the other section will be devoted to the models with torsion.

4.1 The torsion free two dimensional sigma-model.

We start with the action:

$$S = \int d^dx \quad \frac{1}{2} g_{ij} (\Phi) \; \partial_\mu \Phi^i \; \partial^\mu \Phi^j \tag{4.1}$$

where the scalar field $\Phi^i(x)$, $i = 1 \ldots, D$, may be regarded as a map from the d-dimensional world-sheet to a D-dimensional target space, and, $g_{ij}(\Phi)$ represents a metric on the target space.

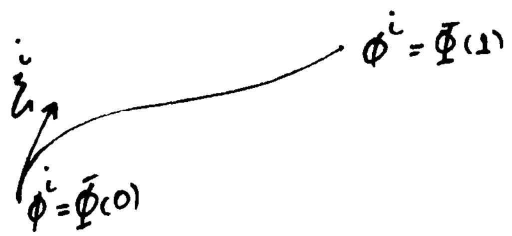
Quantization of Eq. (4.1) is most conveniently discussed using the background field method.

However, the usual background/quantum field split gives a quantum field which does not transform covariantly on the target space, making calculations cumbersome. We can obtain a quantum field which does transform as a contravariant target space vector as follows.

We first split the field $\Phi^i(x)$ into a background piece $\phi^i(x)$ and quantum piece $\pi^i(\phi,\zeta)$ [16]
i.e. we let

$$\Phi^i \longrightarrow \phi^i + \pi^i(\phi,\zeta) \qquad (4.2)$$

Now we can think of $\phi^i, \phi^i + \pi^i$ as the beginning and end points of a geodesic $\phi^i(S)$ in the target space. $S \in [0, 1]$:



and we define our quantum field, ζ^i to be the tangent vector to the geodesic, i.e.

$$\zeta^i = \left[\frac{d\phi^i}{dS} \right]_{S=0} \qquad (4.3)$$

giving

$$\pi^i = \zeta^i - \Gamma^i{}_{jk}\, \zeta^i \zeta^j + \ldots\ldots \qquad (4.4)$$

So the scalar Lagrangian can be expanded as a Taylor series. Then the action in (4.1) under (4.2) gives[17]

$$
\begin{aligned}
S^{\sigma}(\phi,\zeta) = S^{\sigma}(\phi) + \int d^{d}x \, [\, & g_{ij}\, \partial_{\mu}\Phi^{i}\, \partial^{\mu}\zeta^{j} \\
& + \tfrac{1}{2} R_{iklj}\, \partial_{\mu}\Phi^{i}\, \partial^{\mu}\Phi^{j}\; \zeta^{k}\zeta^{l} \\
& + \tfrac{1}{2}\, g_{ij}\, \partial_{\mu}\zeta^{i}\, \partial^{\mu}\zeta^{j} \\
& + \tfrac{1}{6}\, R_{iklj;m}\, \partial_{\mu}\Phi^{i}\, \partial^{\mu}\Phi^{j}\; \zeta^{k}\zeta^{l}\zeta^{m} \\
& + \tfrac{2}{3} R_{iklj}\, \partial_{\mu}\Phi^{i}\, \partial^{\mu}\zeta^{j}\; \zeta^{k}\zeta^{l} \\
& + \tfrac{1}{24} \left(R_{iklj;mn} + 4\, R^{p}{}_{kli}\, R_{pmnj} \right) \\
& \quad \times \partial_{\mu}\Phi^{i}\, \partial^{\mu}\Phi^{j}\; \zeta^{k}\zeta^{l}\zeta^{m}\zeta^{n} \\
& + \tfrac{1}{4} R_{iklj;m}\, \partial_{\mu}\Phi^{i}\, \partial^{\mu}\zeta^{j}\; \zeta^{k}\zeta^{l}\zeta^{m} \\
& + \tfrac{1}{6} R_{iklj}\, \partial_{\mu}\zeta^{i}\, \partial^{\mu}\zeta^{j}\; \zeta^{k}\zeta^{l} \\
& + \tfrac{1}{120} \left(R_{iklj;mnp} + 14\, R^{q}{}_{kli;m}\, R_{qnpj} \right) \\
& \quad \times \partial_{\mu}\Phi^{i}\, \partial^{\mu}\Phi^{j}\; \zeta^{k}\zeta^{l}\zeta^{m}\zeta^{n}\zeta^{p} \\
& + \tfrac{1}{15} \left(R_{iklj;mn} + 2\, R^{p}{}_{kli}\, R_{pmnj} \right) \\
& \quad \times \partial_{\mu}\Phi^{i}\, \partial^{\mu}\zeta^{j}\; \zeta^{k}\zeta^{l}\zeta^{m}\zeta^{n} \\
& + \tfrac{1}{12} R_{iklj;m}\, \partial_{\mu}\zeta^{i}\, \partial^{\mu}\zeta^{j}\; \zeta^{k}\zeta^{l}\zeta^{m} + \ldots
\end{aligned}
\tag{4.5}
$$

Now, by simple dimensional analysis, we can determine at what dimension the poles corresponding to quadratic divergences occur in this model.

Every vertex has two powers of momentum, so the degree of divergence of an arbitrary graph is given by

$$D = 2V - 2P + d\,L \qquad (4.6)$$

where V is the number of vertexes, P is the number of propagators and L is the number of loops. Using

$$L = P - V + 1 \qquad (4.7)$$

we find

$$D = 2 + (d - 2)\,L. \qquad (4.8)$$

We can see why d = 2 is special for sigma models. In d = 4, for example, the degree of divergence D increases with the number of Loops L.

Returning to d = 2, in dimensional regularisation the corresponding poles in the Green function occur when D = 0 i.e. when

$$d = 2 - 2/L \qquad (4.9)$$

(4.9) gives us the space-time dimension where the quadratic divergences appear as a poles in sigma model.

Now the field Φ^i is it self dimensionless; hence graph with two external $\partial_\mu\phi$ lines are logarithmically divergent, and provide correction to the metric term in equ. (4.1). While graphs with no external $\partial_\mu\phi$ lines are quadratically divergent, and generate corrections to the tachyon term in the action. (Such graphs may also have logarithmic divergences contributing to the metric corrections) so the action in (4.1) can be written as

$$S^\sigma = \int d^dx \left[\frac{1}{2} g_{ij} (\Phi) \partial_\mu \Phi^i \partial^\mu \Phi^j + T(\Phi) \right] \qquad (4.10)$$

where $T(\phi)$ will be necessary to absorb quadratic divergences; from the point of view of string theory it represents a background tachyon field.

For our purposes (calculating the quadratic divergences), as explained above, we may omit the parts of the normal coordinate expansion involving background $\partial_\mu\phi$ terms, and so we can reduce (4.5) to

$$\begin{aligned} S^\sigma (\phi,\zeta) = S^\sigma(\phi) + \int d^dx \, [\, & g_{ij} \partial_\mu\zeta^i \partial^\mu\zeta^j \\ & + \frac{1}{6} R_{iklj} \partial_\mu\zeta^i \partial^\mu\zeta^j \zeta^k \zeta^l \\ & + \frac{1}{12} R_{iklj;m} \partial_\mu\zeta^i \partial^\mu\zeta^j \zeta^k \zeta^l \zeta^m + .. \end{aligned} \qquad (4.11)$$

4.1.1 The Loop Calculation :

In this section we will be calculating the quadratic divergences in sigma models without torsion up to 4-loop level. Essential details concerning the regularisation procedure (dimensional regularisation), signs, and factors are contained in appendix one. In this calculation we have arranged matters in such a way as to ensure that the answers obtained below contribute directly to the quadratic divergences. The potentially quadratically divergent graphs up to 4-loop level are depicted in fig. (1), fig. (2), fig. (3) and fig. (4). The background-dependent vertices are given by the quartic and quintic terms in (4.11), while the propagator is derived from the quadratic term.

For the first graph in fig. (1) it is straightforward to see that this two loop diagram ,in fact, have no pole at d = 1 and hence does not produce a leading quadratic divergence according to the dimensional regularisation scheme.

Now for the graph in fig. (2), which is the first non-zero quadratic divergences to occur in sigma model without torsion, we have:

$$\frac{1}{48}\int \frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ q^2\ r^2\ s^2}\ \delta(s+k+q+r)\ [\ 6\ A_{ij:kl}\ A_{ij:kl}\ (k.q)^2$$
$$+\ 6\ A_{ij:kl}\ A_{kl:ij}\ (k.q)(r.s)$$
$$+24\ A_{ij:kl}\ A_{ik:jl}\ (k.q)(k.s)\,] \tag{4.12}$$

where the tensor A is defined by

$$A_{ij:kl} = \frac{1}{3} [R_{kijl} + R_{kjil}] \qquad (4.13)$$

The three integrals arising from this we call $I_1(\sim (k.q)^2)$, $I_2 = (\sim (k.q)(r.s))$ and I_3 $(\sim (k.q)(k.s))$. What we shall now show is that they can all be equated with I_2, thereby simplifying matters. We shall suppress integral signs. Using $[k + q + r + s = 0]$ we have

$$\begin{aligned} I_1 &= k.q\ k.q \\ &= k.q\ k.[-k-r-s] \\ &= -2I_3 \end{aligned} \qquad (4.14)$$

Next

$$\begin{aligned} I_2 &= k.q\ r.s \\ &= k.q\ r.[-k-q-r] \\ &= -2I_3 \end{aligned} \qquad (4.15)$$

So

$$I_1 = I_2 = -2I_3 \qquad (4.16)$$

We have chosen to evaluate I_2 (we could equally well have chosen either of $I_{1,3}$)

Now I_2 gives

$$- \int \frac{d^d k \; d^d q \; d^d r \quad k.(q-k) \;\; (r-q).r}{k^2 \;\; (q-k)^2 \;\; (r-q)^2 \;\; r^2} \qquad (4.17)$$

Performing the k and r integral, we find

$$I_2 = \left[- \frac{i}{(4\pi)^{d/2}} \; \frac{\Gamma(1-d/2) \quad \Gamma(d/2)}{\Gamma(d-1)} \right]^2 \times \int \frac{d^d q}{(q^2)^{1-d/2} \;\; [(p-q)^2]^{1-d/2}} \qquad (4.18)$$

Carrying out the q integral and substituting $d = \frac{4}{3} - \varepsilon$ according to (4.9) we have

$$I_2 = \frac{-\, i}{(4\pi)^2} \; \Gamma(2/3)^3 \; \Gamma((3/2)\varepsilon) + \ldots\ldots \qquad (4.19)$$

For the final result of the graph in fig. (2) we have to evaluate $A_{ij:kl} \, A_{ij:kl}$ and $A_{ij:kl} \;\; A_{ik:jl}$ in (4.12). From (4.13) we have

$$A_{ij:kl} \, A_{ij:kl} = \frac{1}{9} \, [\; 2 \; (R_{kijl})^2 + 2 \; R_{kijl} \; R_{kjil} \;]$$

$$= \frac{1}{3} \; (\; R_{klmn} \;)^2 \qquad (4.20)$$

and

$$A_{ij:kl}\, A_{ik:jl} = \frac{1}{9} [R_{kijl} + R_{kijl}] \quad [R_{jikl} + R_{jkil}]$$

$$= - \frac{1}{6} (R_{klmn})^2 \qquad (4.21)$$

combining this with the result from I_2 integral and simplifying gives us our result:

$$\text{fig}(2) = \frac{1}{12} \frac{i}{(4\pi)^2} \quad \Gamma(2/3)^3\, \Gamma((3/2)\varepsilon) \quad R_{klmn}\, R_{klmn} \quad + \ldots \qquad (4.22)$$

from which we find a correction to the tachyons

$$\Delta T^{(3)} = -i \frac{\mu^2}{\varepsilon^{(3)}} \quad \frac{\Gamma(2/3)^3}{12\ (4\pi)^2} \quad R_{klmn}\, R_{klmn} \qquad (4.23)$$

The first four-loop graph in fig. (3) gives

$$\frac{1}{48} \quad \nabla_k\, R_{lmnp}\, \nabla^k \quad R^{lmnp} \int \frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ (q-k)^2\ (s-q)^2\ (r-s)^2\ (p-r)^2}$$

$$\times \left[(k.q)\ (k.q) + (k.q)(s.r) -2(k.q)(q.s) \right] \qquad (4.24)$$

The three integrals arising from this graph we call $B_1(\sim (k.q)^2)$, $B_2(\sim (k.q)(s.r))$ and $B_3 = (\sim (k.q)(q.s))$.

Now what we shall show is that they can all be equated with B_2, thereby simplifying matters. We shall suppress integral signs. Then we have

$$\begin{aligned} B_1 &= k.q\ k.q \\ &= k.q\ k.(-k-r-s-n) \\ &= -3B_3 \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} B_3 &= k.q\ s.q \\ &= k.q\ s.(-s-r-n-k) \\ &= -\ B_2 \end{aligned} \tag{4.26}$$

Now we shall calculate B_2 which gives

$$B_2 = \int \frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ (q-k)^2\ (s-q)^2\ (r-s)^2\ (p-r)^2} [k.(k-q)\ \ (s-q).(r-s)] \tag{4.27}$$

By making shift of the origin $s = s - q$ we have

$$B_2 = \int \frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ (q-k)^2\ (s)^2\ (r-s)^2\ (p-r)^2} [k.(k-q)\ \ s.(r-s)] \tag{4.28}$$

Performing the k, and s integrals, we find

$$B_2 = \left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)\quad \Gamma(d/2)^2}{\Gamma(d-1)} \right]^2 \int \frac{d^dq\ d^dr}{(q^2)^{1-d/2}\ r^2\ [(r-q)^2]^{1-d/2}}$$

$$B_2 = \left[\frac{-i}{(4\pi)^{3d/2}} \frac{\Gamma(1-d/2)\ \Gamma(d/2)^2\ \Gamma(2-d)\ \Gamma((d/2)-1)}{\Gamma((3d/2)-2)} \right] \int \frac{d^d q}{(q^2)^{3-3d/2}} \tag{4.29}$$

Doing the q integral and substituting for $d = \frac{3}{2} - \varepsilon$ we have

$$B_2 = -\frac{1}{(4\pi)^3}\ 4\ \Gamma(3/4)^4\ \Gamma(2\varepsilon) \tag{4.30}$$

substituting the result of B_2 into (4.24) using (4.25) and (4.26) and simplifying gives us our result for the graph in fig. (3):

$$\text{fig}(3) = \frac{-1}{2(4\pi)^3} \left(\nabla_k R_{lmnp} \nabla^k R^{lmnp} \right) \Gamma(3/4)^4\ \Gamma(2\varepsilon) + \ldots \tag{4.31}$$

There is another graph which is potentially quadratically divergent at 4-loop level, depicted in fig. (4):

$$\frac{1}{6} \int \frac{d^d k\ d^d q\ d^d s\ d^d u}{k^2\ (q-k)^2\ u^2\ (q-u)^2\ s^2\ (q-s)^2}$$

$$\times\ [\ R_{klmn}\ R^{k\ m}_{\ p\ q}\ R^{lpnq}\ (\ k.(q-k)\ u.(q-u)\ s.(q-s)\ )$$

$$+\ R_{klmn}\ R^{mnpq}\ R_{pq}^{\ \ kl}\ (\ (k.s)\ (k.u)\ (u.s)\ )\ \ ] \tag{4.32}$$

So we have two kinds of integral arising from the graph in fig. (4).

Writing fig(4) = F + M, we have

$$F = \frac{1}{6} \int \frac{d^dk\ d^dq\ d^ds\ d^du}{k^2\ (q-k)^2\ u^2\ (q-u)^2\ s^2\ (q-s)^2} k.(q-k)\ u.(q-u)\ s.(q-s)\ R_1 \tag{4.33}$$

where $$R_1 = R_{klmn}\ R^{k\ m}_{\ p\ q}\ R^{lpnq} \tag{4.34}$$

but

$$\int d^dk \frac{k.(q-k)}{k^2\ (q-k)^2} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)\ \ \Gamma(d/2)^2}{\Gamma(d-1)} (q^2)^{(d/2)-1} \tag{4.35}$$

therefore

$$F = \frac{1}{6} R_1 \left[\frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)\ \ \Gamma(d/2)^2}{\Gamma(d-1)} \right]^3 \int \frac{d^dq}{(q^2)^{3-3d/2}}$$

$$= \frac{1}{6} \frac{1}{(4\pi)^{2d}} R_1 \left[\frac{-\Gamma(1-d/2)\ \ \Gamma(d/2)^2}{\Gamma(d-1)} \right]^3 \left[- \frac{\Gamma(3-2d)\ \Gamma(2d-2)\ \Gamma((d/2)-1)}{\Gamma(2-3d/2)\ \Gamma((5d/2)-3)} \right] \tag{4.36}$$

Substituting $d = \frac{3}{2} - \varepsilon$, we have

$$F = \frac{1}{6} \frac{1}{(4\pi)^3} R_1 \frac{\Gamma(1/4)^3\ \Gamma(3/4)^5}{\Gamma(1/2)^3} \Gamma(2\varepsilon) + \ldots\ldots \tag{4.37}$$

We also have

$$M = \frac{1}{6} R_2 \int \frac{d^dk\ d^dq\ d^ds\ d^du}{k^2\ (q-k)^2\ u^2\ (q-u)^2\ s^2\ (q-s)^2} (\ k.s \quad k.u \quad u.s\ ) \qquad (4.38)$$

where $R_2 = R_{klmn}\ R^{mnpq}\ R_{pq}{}^{kl}$ (4.39)

but we have

$$\int \frac{d^dk}{k^2\ (q-k)^2}\ k_\mu\ k_\sigma = \frac{I}{4(d-1)} [\ d\ q_\mu\ q_\sigma - g_{\mu\sigma} q^2\ ] \qquad (4.40)$$

where

$$I = \int \frac{d^dk}{k^2\ (k-p)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\ \ \Gamma((d/2)-1)^2}{\Gamma(d-2)} (p^2)^{(d/2)-1} \qquad (4.41)$$

therefore we have

$$M = \frac{1}{6} R_2 \frac{I^3}{4^3 (d-1)^3} [\ d\ q_\mu q_\sigma - g_{\mu\sigma}\ ][\ d\ q_\sigma q_\beta - g_{\sigma\beta}\ ][\ d\ q_\mu q_\beta - g_{\mu\beta}\ ]$$

$$\times\ g_{\mu\sigma}\ g_{\sigma\rho}\ g_{\alpha\beta} \int \frac{d^dq}{(q^2)^{3-3d/2}} \qquad (4.42)$$

Performing the q integral and substituting for $d = \frac{3}{2} - \varepsilon$ we have:

$$M = \frac{1}{16} \frac{1}{(4\pi)^3} R_2 \frac{\Gamma(1/4)^3\ \Gamma(3/4)^5}{\Gamma(1/2)^3} \Gamma(2\varepsilon) + \ldots\ldots \qquad (4.43)$$

Therefore our result for the graph in fig. (4) is

$$\text{fig}(4) = \frac{1}{6} \frac{1}{(4\pi)^3} R_1 \, Y \, \Gamma(2\varepsilon) + \frac{1}{16} \frac{1}{(4\pi)^3} R_2 \, Y \, \Gamma(2\varepsilon) + \ldots\ldots \tag{4.44}$$

where R_1 and R_2 are in (4.34) and (4.39) respectively and

$$Y = \frac{\Gamma(1/4)^3 \, \Gamma(3/4)^5}{\Gamma(1/2)^3} \tag{4.45}$$

So, at 4-loop level, the quadratic divergences give a correction to the tachyon,

$$\Delta T^{(4)} = \frac{\mu^2}{(4\pi)^3 \, \varepsilon^{(4)}} \left[-\frac{1}{4} X \nabla_k R_{lmnp} \nabla^k R^{lmnp} + \frac{1}{32} Y R_{klmn} R^{mnpq} R_{pq}{}^{kl} + \frac{1}{12} Y R_{klmn} R^{k}{}_{p}{}^{m}{}_{q} R^{lpnq} \right] \tag{4.46}$$

where Y is in (4.45) and

$$X = \Gamma(3/4)^4 \tag{4.47}$$

It is remarkable that the same Γ-function combinations arise at four loops in both four dimensions (see Chapter One) and two dimensions.

4.1.2 Cancellation Condition Conjecture :

We now wish to enquire whether there is any relation between the quadratic divergences at different loop order in this model. The natural analogue of the relationship described in chapters one, two and three is

$$C_{ij} = \beta^{g(i)} \frac{\partial}{\partial g} \Delta T^{(j)} - \Delta T^{(j)} \frac{\partial}{\partial T} \beta^{T(i)} \qquad (4.48)$$

where $\beta^{g(i)}$ is the ith order β-function for the metric g and $\beta^{T(i)}$ is the ith order β-function for the tachyon.

The lowest order result for β and β^T are in ref [18]

$$\beta^{g(1)}_{ij} = R_{ij} \qquad (4.49)$$

$$\beta^{T(1)} = -\frac{1}{2} \nabla^2 T \qquad (4.50)$$

Now substituting (4.23), (4.49) and (4.50) in (4.48) we have as the lowest order C_{ij},

$$C_{13} = \text{const.} \left[\nabla_k R_{lmnp} \nabla^k R^{lmnp} - R_{klmn} R^{mn}{}_{pq} R^{pqkl} + 4 R_{klmn} R^{k}{}_{p}{}^{m}{}_{q} R^{lpnq} \right] \qquad (4.51)$$

where we used the identity

$$H \frac{\partial}{\partial g} R^k{}_{lmn} = \frac{1}{2} \Big[H^k{}_{l;mn} + H^k{}_{m;ln} - H_{lm;}{}^k{}_n - H^k{}_{l;nm} + H^k{}_{n;lm} + H_{ln;}{}^k{}_m \Big] \qquad (4.52)$$

for any tensor H_{ij} .

As we can see from (4.51), the potential terms involving the Ricci tensor cancel leaving only terms which also appear in $\Delta T^{(4)}$ in equ. (4.46). However, as in chapter one, the fact that Y/X is transcendental clearly rules out any relation between C_{13} and $\Delta T^{(4)}$.

Of course the lack of any relationship between the four loop leading quadratic divergences in four dimensions and the lower-order result are a disappointing result which should probably have led us to be pessimistic in this case also.

4.2. Quadratic divergences in 2-dimensional sigma model with torsion

In the previous case, i.e. a 2-dimensional sigma model without torsion, the problem was the lack of any non-trivial one- and two-loop result which would enable us to test for relationships at lower orders than L = 4. We can partially remedy this by adding a torsion term to the sigma model action, which generates non-trivial two-loop quadratic divergences. We add a term containing an antisymmetric tensor field to the sigma model action. This may be regarded as representing torsion on the target manifold. The action now becomes:

$$S^{\sigma} = \int d^dx \, [\; \frac{1}{2} g_{ij}(\Phi) \, \partial_{\mu}\Phi^i \, \partial^{\mu}\Phi^j \; + \frac{1}{2} b_{ij}(\Phi) \, \varepsilon^{\mu\nu} \, \partial_{\mu}\Phi^i \, \partial_{\nu}\Phi^j + T(\Phi) \;] \tag{4.53}$$

where b_{ij} is antisymmetric and where $\varepsilon^{\mu\nu}$ is the 2-dimensional alternating symbol, appropriately extended to d-dimensions. It is well known that the definition of $\varepsilon^{\mu\nu}$ away from two-dimensional is difficult, and some care is needed[19-22] to obtain correct results for the renormalisation group β-functions when dimensional regularisation is used in the presence of torsion. The most satisfactory prescription [21-23] seems to be to define $\varepsilon^{\mu\nu}$as an almost complex structure on the d-dimensional world-sheet, so that it is required to satisfy

$$\varepsilon_\mu{}^\rho \, \varepsilon_\rho{}^\nu = - \delta_\mu{}^\nu ,$$
$$\varepsilon_{\mu\nu} = - \varepsilon_{\nu\mu} \qquad (4.54)$$

Nevertheless consistent results for the β-function can also be obtained by taking in general [24]

$$\varepsilon_\mu{}^\rho \, \varepsilon_\rho{}^\nu = - (1+ c \, \varepsilon \,) \, \delta_\mu{}^\nu \qquad (4.55)$$

where $\varepsilon = 2 - d$, and c is arbitrary. The β-functions for different values of c are then related by field redefinitions [23]. Thus the treatment of $\varepsilon^{\mu\nu}$ is reasonably well understood. In the context of quadratic divergences, we have the additional complication that the relevant expansion parameter becomes $\varepsilon^{(L)}$. Therefore ε as defined above is not "small". We will present our result in the form which avoids any assumption concerning properties of $\varepsilon^{\mu\nu}$ (other than antisymmetry), and afterwards consider the effect of employing a specific natural generalisation of equ. (4.55).

As we have done with the torsion free model, the background field expansion of the action in equ. (4.53) is given by [17]

$$S^{\sigma}(\phi,\zeta) = S^{\sigma}(\phi) + \int d^dx \, [\, \tfrac{1}{2} g_{ij} \, \partial_\mu \zeta^i \, \partial^\mu \zeta^j \\ + \tfrac{1}{6} R_{iklj} \, \partial_\mu \zeta^i \, \partial^\mu \zeta^j \, \zeta^k \, \zeta^l \\ + \tfrac{1}{3} H_{ijk} \, \varepsilon^{\mu\nu} \, \partial_\mu \zeta^i \, \partial_\nu \zeta^j \, \zeta^k \\ + \tfrac{1}{4} \nabla_l H_{ijk} \, \varepsilon^{\mu\nu} \, \partial_\mu \zeta^i \, \partial_\nu \zeta^j \, \zeta^k \, \zeta^l + .. \qquad (4.56)$$

where $H_{ijk} = 3 \, \nabla_{[i} \, b_{jk]}$.
Again we omit terms involving $\partial_\mu \Phi$ which do not contribute to quadratic divergence. We shall try, in this section, to find a relationship between the three-loop and the two-loop quadratic divergences for this case. The two and three-loop graphs contributing leading quadratic divergences constructed from the action in equ. (4.46) are displayed in figs. (5), (6), (7), (8) and (9) and (10).

4.2.1 The Loop Calculations :

As explained before, in this case the quadratic divergence start at two-loop level due to the torsion term in the action. Essential detail concerning the regularisation procedure (dimensional regularisation), signs, and factors are contained in appendix one. In this calculations, we have arranged matters in such a way as to ensure that the answers obtained below contribute directly to the quadratic divergences.

We start with the two-loop calculation. At this loop level there is only one graph contributing to the quadratic divergences which is in fig. (5):

$$\frac{1}{9}\,\varepsilon_1\, H_{klm}\, H^{klm} \int \frac{d^dk\; d^dq\; d^dr\; \delta(k+q+r)}{k^2\; q^2\; r^2}\; [\; k.q\; k.q \; - \; 2\; k.q\; k.r\;] \tag{4.57}$$

where $\varepsilon_1 = \varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$ and the two integrals arising from this graph are $C_1 \sim (\; k.q\; k.q\;)$ and $C_2 \sim (\; k.q\; k.r\;)$ which could be related to each other as follows

$$\begin{aligned} C_1 &= k.q\; k.q \\ &= k.q\;\; k.(\; -k \; - \; r\;) \\ &= -\; C_2 \; + \; \ldots \end{aligned} \tag{4.58}$$

So, equ. (4.57) can be written as

$$\text{fig}(5) = -\,\frac{1}{3}\,\varepsilon_1\, H_{klm}\, H^{klm} \int \frac{d^dk\; d^dq\; d^dr\; \delta(k+q+r)}{k^2\; q^2\; r^2}\; [\; k.q\; k.r\;] \tag{4.59}$$

$$= -\,\frac{1}{3}\,\varepsilon_1\, H_{klm}\, H^{klm}\, g_{\mu\nu} g_{\sigma\rho} \int \frac{d^dk\; d^dq\; d^dr\; \delta(k+q+r)}{k^2\; q^2\; r^2}\, k_\mu q_\nu\; k_\sigma r_\rho \tag{4.60}$$

$$\text{fig}(5) = -\frac{1}{3}\,\varepsilon_1\, H_{klm}\, H^{klm}\, g_{\mu\nu} g_{\sigma\rho} \int \frac{d^d k\; d^d q\; k_\mu (q-k)_\nu\; k_\sigma (p-q)_\rho}{k^2\;(q-k)^2\;(p-q)^2} \tag{4.61}$$

$$= -\frac{1}{3}\,\varepsilon_1\, H_{klm}\, H^{klm}\, g_{\mu\nu} g_{\sigma\rho} \int \frac{d^d k\; d^d q\; [\; k_\mu k_\sigma q_\nu - k_\mu k_\sigma k_\nu]\;(p-q)_\rho}{k^2\;(q-k)^2\;(p-q)^2} \tag{4.62}$$

But we have

$$\int \frac{d^d k}{k^2\;(q-k)^2}\; k^\mu\, k^\nu\, k^\rho = I \left[\frac{d+2}{8(d-1)}\, q^\mu q^\nu q^\rho - \frac{(\, g^{\mu\nu} q^\rho + g^{\nu\rho} q^\mu + g^{\mu\rho} q^\nu \,)}{8(d-1)} \right] \tag{4.63}$$

where I as in (4.41) .

Now using (4.63) and (4.40) in (4.62), and simplifying, the algebra gives us: (where p = 0 is understood)

$$\text{fig}(5) = \Gamma(1/2)^3\, g_{\mu\nu} g_{\sigma\rho} \int d^d q \left[\frac{1}{4}\, \frac{q_\mu q_\sigma q_\nu q_\rho}{(q^2)^{3-d/2}} + \frac{1}{2}\, g_{\mu\sigma}\, \frac{q_\nu q_\rho}{(q^2)^{2-d/2}} - \frac{1}{8}\, \frac{[\; g_{\mu\sigma}\; q_\nu q_\rho + g_{\sigma\nu}\; q_\mu q_\rho + g_{\mu\nu}\; q_\sigma q_\rho]}{(q^2)^{3-d/2}} \right] \tag{4.64}$$

Now evaluating the q integrals in (4.62) (where the formula for the integrals are given in appendix one) then we have:

$$\text{fig}(5) = -\frac{1}{6}\frac{1}{(4\pi)}\varepsilon_{\mu\nu}\varepsilon^{\mu\nu}H_{klm}H^{klm}\ \Gamma(1/2)^2\,\Gamma(\varepsilon) + \ldots \qquad (4.65)$$

So, the two-loop quadratic divergences can be cancelled by the following correction to the tachyon field

$$\Delta T^{(2)} = -\mu^2\frac{\Gamma(1/2)^2}{4\pi\ \varepsilon^{(2)}}\ \frac{1}{6}\varepsilon_1 H_{klm}H^{klm} \qquad (4.66)$$

where $\varepsilon_1 = \varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$. (4.67)

The fact that there is a quadratic divergence proportional to H^2, but no corresponding term proportional to the Ricci Scalar is reminiscent of the dilaton β-function, where the same thing happens.

At three loops we have the graphs displayed in fig. (6) - (11). The first graph is the graph which comes from torsion free part of the model. So, the graph in fig. (6) is the same as in equ. (4.17). For the graph in fig. (7) we have

$$\frac{1}{24}\varepsilon_1 \nabla_k H_{lmn}\nabla^k H^{lmn} g_{\mu\nu}g_{\sigma\rho}\int\frac{d^dk\ d^dq\ d^dr\ d^ds}{k^2\ q^2\ s^2\ r^2}\ \delta(k+q+s+r)$$

$$\times\left[-4(k_\mu q_\nu\ q_\sigma s_\rho) + 2\,(k_\mu q_\nu\ k_\sigma q_\rho)\right] \qquad (4.68)$$

where ε_1 is given in (4.67)

We call the integrals

$$Q_1 \sim k_\mu q_\nu \; q_\sigma s_\rho \qquad\qquad Q_2 \sim k_\mu q_\nu \; k_\sigma q_\rho$$

and $\quad Q_3 \sim k_\mu q_\nu \; s_\sigma r_\rho$

It is easy to show that

$$Q_2 = -2\,Q_1 = Q_3 \tag{4.69}$$

therefore (4.68) can be written as

$$\text{fig}(7) = \frac{1}{24}\,\varepsilon_1\,\nabla_k H_{lmn}\,\nabla^k H^{lmn}\,g_{\mu\nu}g_{\sigma\rho}\int \frac{d^dk\;d^dq\;d^dr\;d^ds}{k^2\;q^2\;s^2\;r^2}\,\delta(k+q+s+r)\,(4Q_3) \tag{4.70}$$

$$\text{fig}(7) = \frac{1}{24}\,\varepsilon_1\,\nabla_k H_{lmn}\,\nabla^k H^{lmn}\,g_{\mu\nu}g_{\sigma\rho}$$

$$\times \int \frac{d^dk\;d^dq\;d^dr\;d^ds}{k^2\;(q-k)^2\;s^2(q-s)^2}\,k_\mu(q-k)_\nu\;s_\sigma(q-s)_\rho \tag{4.71}$$

Using (4.40) and

$$\int \frac{d^dk}{k^2\;(q-k)^2}\;k_\mu = \frac{1}{2}\,I\,q_\mu \tag{4.72}$$

where I is given in (4.41) we find

$$\text{fig}(7) = \frac{1}{24}\, \varepsilon_1\, \nabla_k\, H_{lmn}\, \nabla^k\, H^{lmn} \quad g_{\mu\nu} g_{\sigma\rho}$$

$$\times \int d^d q \,\frac{I^2}{16(d-1)} \left[\frac{1}{2}(1-2d)\, q_\mu q_\nu - g_{\mu\nu} q^2\right]\left[\frac{1}{2}(1-2d)\, q_\sigma q_\rho - g_{\sigma\rho} q^2\right] (q^2)^{d-4} \tag{4.73}$$

carrying out the q integrals and substituting for $d = \frac{4}{3} - \varepsilon$

we have

$$\text{fig}(7) = \frac{3}{4}\, \frac{-i}{(4\pi)^2} \Gamma(2/3)^3\, \Gamma(3\varepsilon/2)\, \varepsilon_1\, \nabla_k\, H_{lmn}\, \nabla^k\, H^{lmn} + \ldots \tag{4.74}$$

The third graph at three loop is the graph in fig. (8) which gives:

$$- \frac{4}{3}\, \varepsilon_1\, R_{klmn}\, H_p{}^{kl}\, H^{pmn} \quad g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta}$$

$$\int \frac{d^d k\, d^d q\, d^d s}{k^2\, (q-k)^2\, s^2 (q-s)^2\, q^2}\, k_\mu\, s_\nu\, s_\sigma\, q_\rho\, k_\alpha\, q_\beta \tag{4.75}$$

$$= - \frac{4}{3}\, \varepsilon_1\, R_{klmn}\, H_p{}^{kl}\, H^{pmn}\, g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta}$$

$$\int \frac{d^d k\, d^d q\, d^d s}{k^2\, (q-k)^2\, s^2 (q-s)^2\, q^2}\, k_\mu\, k_\alpha\, s_\sigma\, s_\nu\, q_\rho\, q_\beta \tag{4.76}$$

Now using the formulas in equ. (4.40) we have

$$\text{fig}(8) = -\frac{4}{3}\,\varepsilon_1\, R_{klmn}\, H_p{}^{kl}\, H^{pmn} g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta} \int d^d q\, \frac{I^2}{16(d-1)^2}$$

$$\times [\, d\, q_\mu q_\nu - g_{\mu\nu} q^2]\, [\, d\, q_\nu q_\sigma - g_{\nu\sigma} q^2]\, \frac{q_\rho\, q_\beta}{(q^2)^{3-d}} \qquad (4.77)$$

carrying out the q integrals in (4.66) and substituting for

$d = \frac{4}{3} - \varepsilon$ we have

$$\text{fig}(8) = \frac{-i}{(4\pi)^2}\,\frac{27}{16}\,\varepsilon_1\, R_{klmn}\, H_p{}^{kl}\, H^{pmn}\, \Gamma(2/3)^3\, \Gamma(3\varepsilon/2) + \ldots \qquad (4.78)$$

The other two graphs are 4 vertex graphs. The first in fig. (9) gives:

$$\varepsilon_2\, H_{kmn}\, H^{lmn}\, H^{npq}\, H_{lpq}\, g_{\mu\nu}\, g_{\sigma\rho}\, g_{\alpha\beta}\, g_{\gamma\delta}$$

$$\int \frac{d^d k\, d^d q\, d^d s \qquad k_\mu\, q_\nu\, q_\sigma\, s_\rho\, s_\alpha\, q_\beta\, q_\gamma\, k_\delta}{k^2\, (q-k)^2\, s^2 (q-s)^2\, q^2} \qquad (4.79)$$

$$\text{where} \quad \varepsilon_2 = \varepsilon_1^2 + 2\, \varepsilon^{\mu\nu}\, \varepsilon_{\mu\sigma}\, \varepsilon^{\rho\sigma}\, \varepsilon_{\rho\nu} \qquad (4.80)$$

Now the integral is straightforward to evaluate, being very similar to those we have already considered.

$$fig(9) = \varepsilon_2 \; H_{kmn} \; H^{lmn} \; H^{npq} \; H_{lpq} \; g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta} g_{\gamma\delta} \int d^dk \; d^dq \; d^ds$$

$$\times \; \frac{k_\mu k_\delta}{k^2 (q-k)^2} \quad \frac{s_\sigma s_\alpha}{s^2 (q-s)^2} \; \frac{q_\nu q_\sigma q_\beta q_\gamma}{(q^2 \;)^2} \qquad (4.81)$$

$$fig(9) = \varepsilon_2 \; H_{kmn} \; H^{lmn} \; H^{npq} \; H_{lpq} \; g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta} g_{\gamma\delta} \int d^dk \; d^dq \; d^ds \; \frac{I^2}{16 \; (d-1)}$$

$$[\; d \; q_\mu q_\delta - \; g_{\mu\delta} q^2] [\; d \; q_\rho q_\alpha - \; g_{\rho\alpha} q^2] \; \frac{q_\nu q_\sigma q_\beta q_\gamma}{(q^2)^{2-d} \; (q^2)^2} \qquad (4.82)$$

where we used equ. (4.60) and the formulae from appendix one. The last step is to perform the q integrals using the functions in appendix one and substitute $d = \frac{4}{3} - \varepsilon$. So, the graph in fig(9) gives

$$fig(9) = \frac{81}{160} \; \frac{-i}{(4\pi)^2} \; \varepsilon_2 \; H_{kmn} \; H^{lmn} \; H^{npq} \; H_{lpq} \; \Gamma(2/3)^3 \; \Gamma(3\varepsilon/2) \; + \; .. (4.83)$$

The final quadratically divergent graph is in fig. (10). This is

$$\varepsilon_2 \; G \; g_{\mu\nu} g_{\sigma\rho} g_{\alpha\beta} g_{\gamma\delta} \int \frac{d^dk \; d^dq \; d^dr \quad k_\mu \; k_\sigma \; k_\gamma \; q_\nu \; q_\beta \; q_\delta \; r_\rho \; r_\alpha}{k^2 \; (k-q)^2 \; (r+k)^2 \; q^2 \; (q-r)^2 \; r^2} \qquad (4.84)$$

where $G = H_{klm} \; H^{kpq} \; H^l{}_{pn} \; H_q{}^{nm}$

Now we can write

$$\int \frac{d^dk\ d^dq\ d^dr \quad k_\mu\ k_\sigma\ k_\gamma\ q_\nu\ q_\beta\ q_\delta\ r_\rho\ r_\alpha}{k^2\ (k-q)^2\ (r+k)^2\ q^2\ (q-r)^2\ r^2} \tag{4.84}$$

$$= A\ F_1 + B\ F_2 + C\ F_3 + D\ F_4 + E\ F_5 + K\ F_6 \tag{4.85}$$

where $F_1, \ldots\ldots, F_6$ are combinations of g-functions e.g.

$$\begin{aligned} F_1 = [\ & g_{\mu\sigma}\ g_{\gamma\nu}\ g_{\beta\delta}\ g_{\rho\alpha} + g_{\mu\gamma}\ g_{\sigma\nu}\ g_{\beta\delta}\ g_{\rho\alpha} \\ & + g_{\gamma\sigma}\ g_{\mu\nu}\ g_{\beta\delta}\ g_{\rho\alpha} + g_{\mu\sigma}\ g_{\gamma\beta}\ g_{\nu\delta}\ g_{\rho\alpha} \\ & + g_{\mu\sigma}\ g_{\gamma\delta}\ g_{\beta\nu}\ g_{\rho\alpha} + g_{\mu\gamma}\ g_{\sigma\beta}\ g_{\nu\delta}\ g_{\rho\alpha} \\ & + g_{\mu\gamma}\ g_{\sigma\delta}\ g_{\beta\nu}\ g_{\rho\alpha} + g_{\gamma\sigma}\ g_{\mu\beta}\ g_{\nu\delta}\ g_{\rho\alpha} \\ & + g_{\gamma\sigma}\ g_{\mu\delta}\ g_{\nu\beta}\ g_{\rho\alpha}\ ] \end{aligned} \tag{4.86}$$

$$\begin{aligned} F_2 = [\ g_{\rho\alpha}\ (\ & g_{\mu\nu}\ g_{\sigma\beta}\ g_{\gamma\delta} + g_{\mu\beta}\ g_{\sigma\nu}\ g_{\gamma\delta} \\ & g_{\mu\delta}\ g_{\sigma\beta}\ g_{\gamma\nu} + g_{\mu\nu}\ g_{\sigma\delta}\ g_{\gamma\beta} \\ & g_{\mu\beta}\ g_{\sigma\delta}\ g_{\gamma\nu} + g_{\mu\delta}\ g_{\sigma\nu}\ g_{\gamma\beta}\ )\ ] \end{aligned}$$

$$F_3 = [\ g_{\mu\rho}g_{\alpha\sigma}\ (\ g_{\gamma\nu}g_{\beta\delta} + g_{\gamma\delta}g_{\beta\nu} + g_{\gamma\beta}g_{\varphi\delta}\ )$$

$$+\ g_{\mu\alpha}g_{\sigma\rho}\ (\ g_{\gamma\nu}g_{\beta\delta} + g_{\gamma\delta}g_{\beta\nu} + g_{\gamma\beta}g_{\varphi\delta}\ )$$

$$+\ g_{\mu\rho}g_{\alpha\gamma}\ (\ g_{\sigma\nu}g_{\beta\delta} + g_{\sigma\delta}g_{\beta\nu} + g_{\sigma\beta}g_{\varphi\delta}\ )$$

$$+\ g_{\mu\alpha}g_{\rho\gamma}\ (\ g_{\sigma\nu}g_{\beta\delta} + g_{\sigma\delta}g_{\beta\nu} + g_{\sigma\beta}g_{\varphi\delta}\ )$$

$$+\ g_{\sigma\rho}g_{\gamma\alpha}\ (\ g_{\mu\nu}g_{\beta\delta} + g_{\mu\beta}g_{\nu\delta} + g_{\mu\delta}g_{\beta\nu}\ )$$

$$+\ g_{\sigma\alpha}g_{\gamma\rho}\ (\ g_{\mu\nu}g_{\beta\delta} + g_{\mu\beta}g_{\nu\delta} + g_{\mu\delta}g_{\beta\nu})$$

$$F_4 = [\ g_{\mu\sigma}g_{\nu\beta}\ (\ g_{\gamma\rho}g_{\delta\alpha} + g_{\gamma\alpha}g_{\delta\rho}\ )$$

$$+\ g_{\mu\sigma}g_{\beta\delta}\ (\ g_{\gamma\rho}g_{\nu\alpha} + g_{\gamma\alpha}g_{\nu\rho}\ )$$

$$+\ g_{\mu\sigma}g_{\nu\delta}\ (\ g_{\gamma\rho}g_{\beta\alpha} + g_{\gamma\alpha}g_{\beta\rho}\ )$$

$$+\ g_{\sigma\alpha}g_{\nu\beta}\ (\ g_{\mu\rho}g_{\delta\alpha} + g_{\mu\alpha}g_{\delta\rho}\ )$$

$$+\ g_{\sigma\gamma}g_{\beta\delta}\ (\ g_{\mu\rho}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\rho}\ )$$

$$+\ g_{\sigma\gamma}g_{\nu\delta}\ (\ g_{\mu\rho}g_{\beta\alpha} + g_{\mu\alpha}g_{\beta\rho}\ )$$

$$+\ g_{\mu\gamma}g_{\nu\beta}\ (\ g_{\sigma\rho}g_{\delta\alpha} + g_{\sigma\alpha}g_{\delta\rho}\ )$$

$$+\ g_{\mu\gamma}g_{\beta\delta}\ (\ g_{\sigma\rho}g_{\nu\alpha} + g_{\sigma\alpha}g_{\nu\rho}\ )$$

$$+\ g_{\mu\gamma}g_{\nu\delta}\ (\ g_{\sigma\rho}g_{\beta\alpha} + g_{\sigma\alpha}g_{\beta\rho}\ )$$

$$F_5 = [\ (\ g_{\sigma\beta}g_{\gamma\delta} + g_{\sigma\delta}g_{\gamma\beta}\ )\ (\ g_{\mu\rho}g_{\nu\alpha} + g_{\mu\alpha}g_{\nu\rho}\ )$$

$$+\ (\ g_{\sigma\nu}g_{\gamma\delta} + g_{\sigma\delta}g_{\gamma\nu}\ )\ (\ g_{\mu\rho}g_{\beta\alpha} + g_{\mu\alpha}g_{\beta\rho}\ )$$

$$+\ (\ g_{\sigma\nu}g_{\gamma\delta} + g_{\sigma\beta}g_{\gamma\nu}\ )\ (\ g_{\mu\rho}g_{\delta\alpha} + g_{\mu\alpha}g_{\delta\rho}\ )$$

$$+\ (\ g_{\sigma\rho}g_{\nu\alpha} + g_{\sigma\alpha}g_{\nu\rho}\ )\ (\ g_{\mu\beta}g_{\gamma\delta} + g_{\mu\delta}g_{\gamma\beta}\ )$$

$$+\ (\ g_{\sigma\rho}g_{\beta\alpha} + g_{\sigma\alpha}g_{\beta\rho}\ )\ (\ g_{\mu\nu}g_{\gamma\delta} + g_{\mu\delta}g_{\gamma\nu}\ )$$

$$+\ (\ g_{\sigma\rho}g_{\delta\alpha} + g_{\sigma\alpha}g_{\delta\rho}\ )\ (\ g_{\mu\nu}g_{\gamma\beta} + g_{\mu\beta}g_{\gamma\nu}\ )$$

$$+ (g_{\gamma\rho}g_{\nu\alpha} + g_{\gamma\alpha}g_{\nu\rho}) (g_{\mu\beta}g_{\sigma\delta} + g_{\mu\delta}g_{\sigma\beta})$$
$$+ (g_{\gamma\rho}g_{\beta\alpha} + g_{\gamma\alpha}g_{\beta\rho}) (g_{\mu\nu}g_{\sigma\delta} + g_{\mu\delta}g_{\sigma\nu})$$
$$+ (g_{\gamma\rho}g_{\delta\alpha} + g_{\gamma\alpha}g_{\delta\rho}) (g_{\mu\nu}g_{\sigma\beta} + g_{\mu\beta}g_{\sigma\nu})$$

$$F_6 = [(g_{\nu\rho}g_{\beta\alpha} + g_{\nu\alpha}g_{\beta\rho}) (g_{\mu\delta}g_{\sigma\gamma} + g_{\sigma\delta}g_{\mu\gamma} + g_{\gamma\delta}g_{\mu\sigma})$$
$$+ (g_{\nu\rho}g_{\delta\alpha} + g_{\nu\alpha}g_{\delta\rho}) (g_{\mu\beta}g_{\sigma\gamma} + g_{\sigma\beta}g_{\mu\gamma} + g_{\gamma\beta}g_{\mu\sigma})$$
$$+ (g_{\beta\rho}g_{\delta\alpha} + g_{\beta\alpha}g_{\delta\rho}) (g_{\mu\nu}g_{\sigma\gamma} + g_{\sigma\nu}g_{\mu\gamma} + g_{\gamma\nu}g_{\mu\sigma})$$

Contracting both side with one of the combination from each F's e.g for F_1 we contract with $g_{\mu\sigma}\ g_{\gamma\nu}\ g_{\beta\delta}\ g_{\rho\alpha}$, then we have on the left hand side a momentum integral and in the right hand side a functions of d , then we solve for A,B,C,D,E and find the result for fig(10) is

$$\frac{81}{1120} \frac{-i}{(4\pi)^2} \Gamma(2/3)^3\ \Gamma((2/3)\varepsilon)\ H_{klm}\ H^{kpq}\ H^{l}{}_{pn}\ H_{q}{}^{nm}\ \varepsilon_2 + \ldots \qquad (4.87)$$

where ε_2 is given in (4.80).

Now from (4.17), (4.74), (4.78) (4.83) and (4.87) we have the three loop correction to the tachyon field:

$$\Delta T^{(3)} = - i \, \frac{2}{3} \, \frac{\mu^2 \, \Gamma(2/3)^2}{(4\pi)^2 \, \varepsilon^{(3)}} \Big[\frac{1}{8} R_{klmn} R^{klmn} + \frac{3}{8} \varepsilon_1 \nabla_k H_{lmn} \nabla^k H^{lmn}$$

$$- \frac{27}{16} \varepsilon_1 R_{klmn} H_p{}^{kl} H^{pmn}$$

$$+ \frac{81}{160} \varepsilon_2 H_{kmn} H^{lmn} H^{npq} H_{lpq}$$

$$+ \frac{81}{1120} \varepsilon_2 H_{klm} H^{kpq} H^l{}_{pn} H_q{}^{nm} \Big] \quad (4.88)$$

where ε_1 and ε_2 are given in (4.67) and (4.80) respectively

4.2.2 Cancellation Condition Conjecture:

We wish to investigate whether $\Delta T^{(3)}$ can be derived from $\Delta T^{(2)}$. The quantity C_{ij} which we defined in equ. (4.48) for the torsion free model now become in the presence of torsion:

$$C_{ij} = \Big[\beta^{g(i)} \frac{\partial}{\partial g} + \beta^{b(i)} \frac{\partial}{\partial b} \Big] \Delta T^{(j)} - \Delta T^{(j)} \frac{\partial}{\partial T} \beta^{T(i)} \quad (4.89)$$

where $\beta^{g(i)}$ is the ith order β-function for the metric g, $\beta^{b(i)}$ in the ith order β-function for the antisymmetric tensor $b_{ij}(\phi)$, and $\beta^{T(i)}$ is the ith order β-function for the tachyon.

The lowest order results for β^g , β^b and β^T are:

$$\beta^{g(1)} = \frac{1}{16\ \pi^2} (R_{ij} - H_{ikl} H_j{}^{kl}) \qquad (4.90)$$

$$\beta^{b(1)} = - \frac{1}{16\ \pi^2} \nabla^k H_{kij} \qquad (4.91)$$

$$\beta^{T(1)} = - \frac{1}{2} \nabla^2 T \qquad (4.92)$$

Substituting (4.90), (4.91) and (4.92) in (4.89) we find the lowest order of c_{ij}

$$c_{12} = \text{const.} [3 H_{kpq} H^{lpq} H^{krs} H_{lrs} - 2 R_{pqkl} H^{klm} H_m{}^{pq} + \nabla_k H_{lmn} \nabla^k H^{lmn}] \qquad (4.93)$$

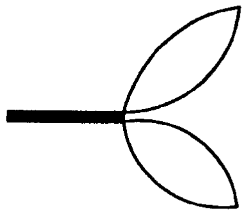
where we used the identity:

$$H \frac{\partial}{\partial g} R^k{}_{lmn} = \frac{1}{2} \Big[H^k{}_{l;mn} + H^k{}_{m;ln} - H_{lm;}{}^k{}_n - H^k{}_{l;nm} + H^k{}_{n;lm} + H_{ln;}{}^k{}_m \Big]$$

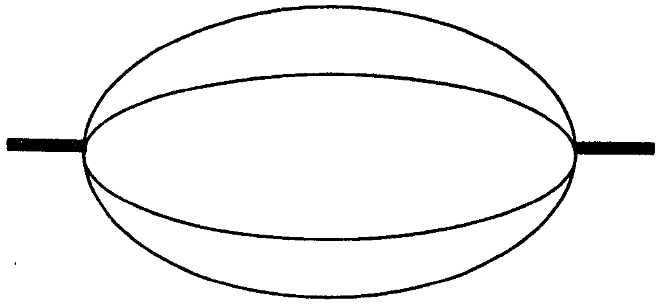
for any tensor H_{ij} .

As in Section (4.1), it is intriguing that terms involving R_{ij} and $\nabla^k H_{kij}$ cancel out, leaving only a subset of the terms present in equ. (4.88). However it is immediately apparent that c_{12} bears no relation to $\Delta T^{(3)}$, for any value of $\lambda^{(3)}$, since c_{12} does not contain $R_{klmn} R^{klmn}$.

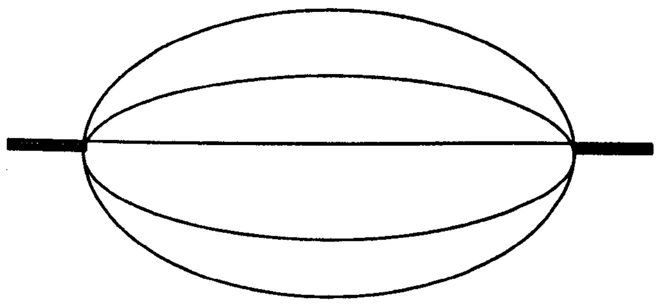
In the last 3 chapters and this chapter we have calculated the leading quadratic divergences as manifested in dimensional regularisation. Our main purpose was to explore the apparent connection between quadratic divergences and logarithmic divergences which has been discovered in ref. [7]. The original goal in this investigation was to demonstrate that there might exist non-supersymmetric theories free of quadratic divergences. While this question remains open with regard to the class of theories proposed in ref. [7], it appears that vanishing of the quadratic divergence at L loops is not guaranteed by requiring scale invariance (to L loops) of the quadratic divergences condition for $L'' < L$.



fig(1)



fig(2)



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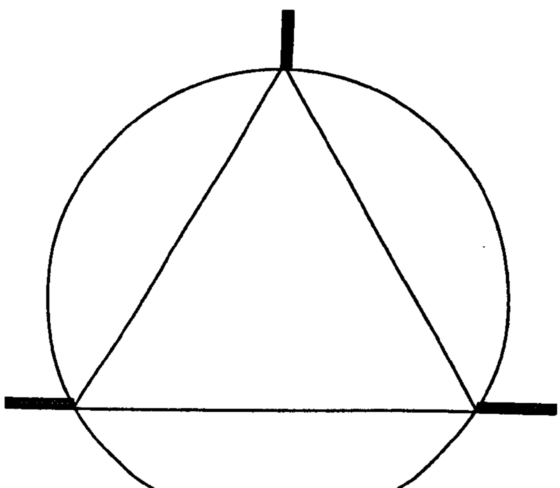

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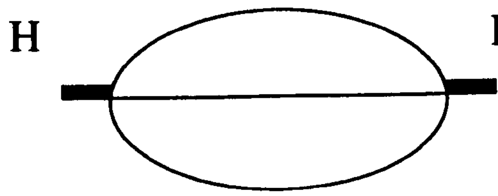
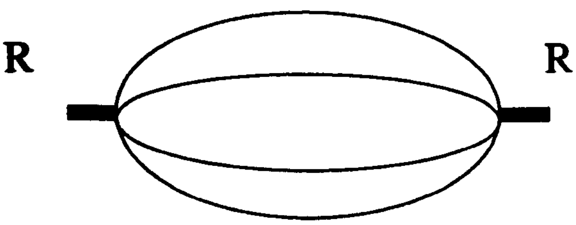


fig(5)



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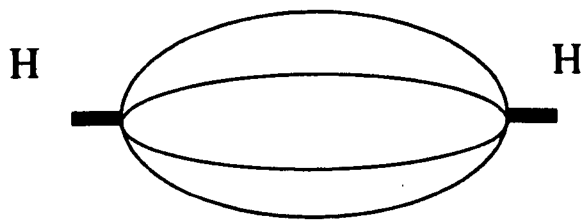
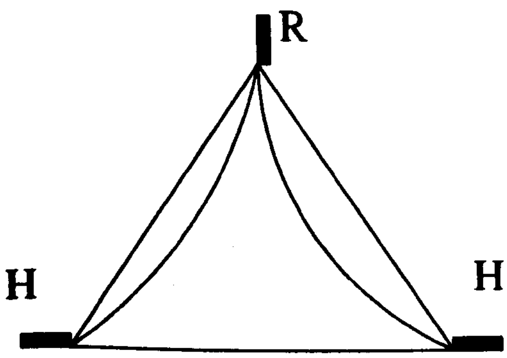
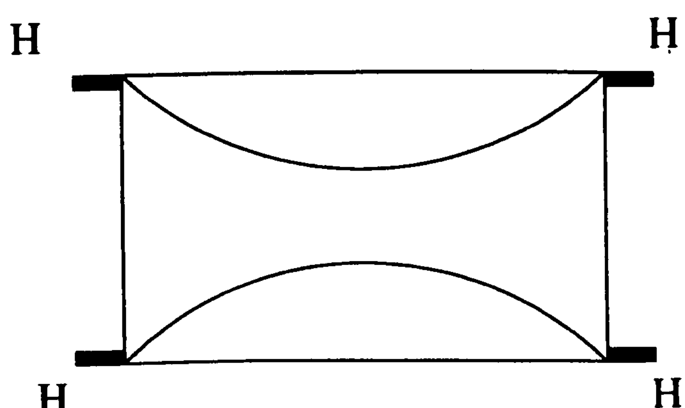


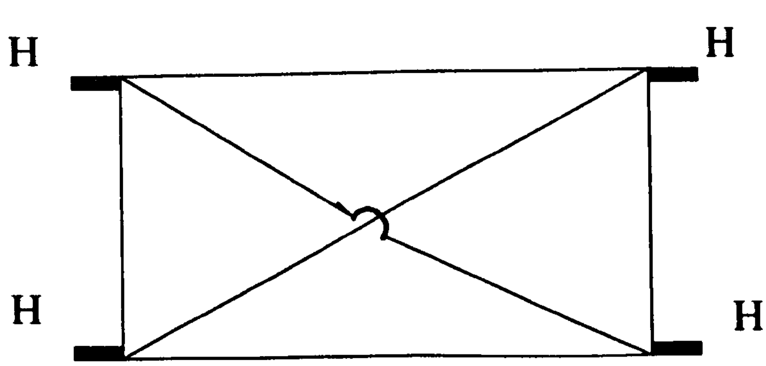
fig(7)



fig(8))



fig(9)



fig(10)

CHAPTER 5: QUADRATIC DIVERGENCES AND GAUGE THEORIES

In this chapter we will explore the structure of quadratic divergences in the standard model and rederive Veltman's formula using the recently proposed non-local regularisation. We will also conjecture that with this regulator the mismatch between the two-loop constraint and scale invariance of the one loop constraint would disappear.

5.1. Gauge Theories:

In this section we review the calculation of the quadratic divergences in general renormalisable Gauge Theories and demonstrate the mismatch referred to above .We consider a general renormalisable gauge theory, with the Lagrangian:

$$L = -\frac{1}{4}\left(G^{A}_{\mu\nu}\right)^2 + i\,\Psi\,\gamma.D\,\bar{\Psi} + \frac{1}{2}\left(D_\mu\Phi^{a}\right)^2 - U(\phi) - \left[\,\Psi\, M\,\bar{\Psi} + C.C\,\right] \qquad (5.1)$$

where

$$G^{A}_{\mu\nu} = \partial_\mu W^{A}_{\nu} - \partial_\nu W^{A}_{\mu} - g\, f^{ABC}\, W^{B}_{\mu}\, W^{C}_{\nu} \qquad (5.2)$$

$$D_\mu\phi^{a} = \partial_\mu\phi^{a} + ig\,(\theta^{A})^{ab}\, W^{A}_{\mu}\,\phi^{b} \qquad (5.3)$$

$$D_\mu \Psi^i = \partial_\mu \Psi^i + ig\,(t^A)^{ij}\, W_\mu^A\, \Psi^j \qquad (5.4)$$

and ϕ ,Ψ transform according to gauge group representations θ^A, t^A respectively:-

$$[\,\theta^A, \theta^B] = i\, f^{ABC}\, \theta^C \qquad (5.5)$$

$$[\,t^A, t^B] = i\, f^{ABC}\, t^C \qquad (5.6)$$

$U(\phi)$ is a gauge invariant polynomial, which for renormalisablity we restrict to be quartic . $M(\phi)$ is given by

$$M = m_{ij} + Y_{aij}\, \phi^a \qquad (5.7)$$

Now within dimensional regularisation and using the simple dimensional analysis of Chapter One, the pole in d-dimension that characterises quadratic divergences in renormalisable theories in four dimensions occurs at[8] $d = 4 - 2/L$ where L is the number of loops.

The one loop result for the quadratic divergences in ref.[7] is

$$\Delta_1 = (\, d-1\,)\, g^2\, \phi^T\, \theta^A\, \theta^A\, \phi + U_{aa} - 2\,\mathrm{Tr}\,(\, M\, M^*) \qquad (5.8)$$

where $U_a = \dfrac{\partial U}{\partial \phi^a}$,etc.

and g is the gauge coupling.

In (5.8), d stands for the trace of the metric tensor and arises from the contribution to the scalar self-energy and consequently is the dimensionality of the μ -index of the vector field W_μ.

The question arises now as whether it should be set equal to 4 or to 2. This problem was addressed by Veltman [8] in his original discussion of quadratic divergences in the context of the standard model. He concluded that although conventional dimensional regularisation [26] leads to d = 2, the appropriate choice was d = 4. This preserves the number of gauge degrees of freedom and hence respects supersymmetry. Supersymmetric theories (which are quite free from quadratic divergences) indeed satisfy the equation $\Delta_1 = 0$, with d = 4 (not d = 2). The following issue arises : imagine that supersymmetry were yet to be discovered. Could one have chanced on it by seeking solutions to the equation $\Delta_1 = 0$? Evidently only with the choice of d = 4; that is, by use of a regularisation that preserves the undiscovered symmetry! The use of dimensional reduction in a non-supersymmetric context leads to a problem [27]. Consider the infinitesimal gauge transformation on the vector field:

$$\delta W_\mu = \partial_\mu \alpha + \alpha \times W_\mu \qquad (5.9)$$

With dimensional reduction we can separate W_μ into d dimensional components $W_{\hat{\mu}}$ and ε = 4-d components W_σ . Then equ. (5.9) becomes

$$\delta W_{\hat{\mu}} = \partial_{\hat{\mu}}\alpha + \alpha \times W_{\hat{\mu}} \qquad (5.10)$$

$$\delta W_\sigma = \alpha \times W_\sigma . \qquad (5.11)$$

The W_σ transform as scalars (and are sometimes called ε-scalars)[28]. This has the effect that the "gauge" coupling constant of W_σ to matter fields renormalises differently from the true gauge coupling constant g, and the tree W_σ^4 interaction is not form-invariant under renormalisation. In the present context, it means that the components of the W_μ self-energy corresponding to the ε-scalars are not protected by gauge invariance from quadratic divergences.

Now for the two loop leading pole residue we have[7]

$$\begin{aligned}\Delta_2 = & -12 \, \mathrm{tr} \, (Y_a Y_a^* M M^*) - 6 \, [\, \mathrm{tr} \, (Y_a M^* Y_a M^*) + \mathrm{C.C} \,] \\ & + 6 \, U_{ab} \, \mathrm{tr} \, (Y_a Y_a^*) + U_{abc} U_{abc} - 12 g^2 U_{ab} (\theta^2)_{ab} \\ & + (12n - 18) g^4 \phi^T \theta^2 \theta^2 \phi + 24 g^2 \, \mathrm{tr} \, (t^2 M M^*) \\ & + 6 (n - 1) g^2 \, \mathrm{tr} \, (Y_a Y_b^*) (\theta^A \phi)_a (\theta^A \phi)_b \\ & + g^4 [3T(\theta) + 6(n-2) T(t) - 3(n+7) C_2(G)] \phi^T \theta^2 \phi \end{aligned} \qquad (5.12)$$

where $T(R) \delta^{AB} = \mathrm{Tr}(R^A R^B)$ and $C_2(G) \delta^{AB} = f^{ACD} f^{BCD}$.

where n = 3 in dimensional regularisation and n = 4 in dimensional reduction.

The relationship between the scale invariance and the quadratic divergences in this case defined by:

$$A_{11} = \left[\beta_U^1 \frac{\partial}{\partial U} + \beta_M^1 \frac{\partial}{\partial M} + \beta_g^1 \frac{\partial}{\partial g} \right] \Delta_1 - \Delta_1 \frac{\partial}{\partial U} \beta_U^1 \qquad (5.13)$$

where we have the one-loop β- functions[29]

$$16\,\pi^2\,\beta_g^1 = [\, \tfrac{1}{6}\, T(\theta) + \tfrac{2}{3}\, T(t) - \tfrac{11}{3}\, C_2(G)\,]\, g^3 \qquad (5.14)$$

$$\begin{aligned} 16\,\pi^2\,\beta_U^1 = {} & \tfrac{1}{2}\, U_{ab}\, U_{ab} + \tfrac{3}{2}\, g^4 (\, \phi^T\, \theta^A\, \theta^B\, \phi\,)^2 \\ & - tr\, (M\, M^*\, M\, M^*\,) - 3\, U_a\, (\, \theta^2\, \phi\,)_a\, g^2 \\ & + \tfrac{1}{2}\, [\, U_a\, tr\, (\, Y_a\, Y_b^*\,)\, \phi_b + C.C\,] \end{aligned} \qquad (5.15)$$

$$\begin{aligned} 16\,\pi^2\,\beta_M^1 = {} & -\, 3\, \{\, t^2,\, M\, \}\, g^2 + 2\, Y_a^*\, M\, Y_a^* \\ & + \tfrac{1}{2}\, (\, Y_a\, Y_a^*\, M + M\, Y_a^*\, Y_a\,) \\ & + \tfrac{1}{2}\, Y_a\, \phi_b\, tr\, (\, Y_a\, Y_b^* + Y_a^*\, Y_b\,) \end{aligned} \qquad (5.16)$$

Substituting (5.8) and the above β -functions into equ. (5.13) we have:

$$A_{11} = \Delta_2 + g^4 \,[\, 2(d-2)\, T(\theta) + 2(8-3d)\, T(t) + 2d\, C_2(G)]\, \phi^T \theta^2 \phi$$
$$- 2\, (d-4)\, [\, g^2\, U_{ab}(\theta^2)_{ab} + g^4\, \phi^T \theta^2 \theta^2 \phi$$
$$+ 2\, g^2\, \mathrm{tr}\, (Y_a\, Y_b^*)\, (\, \theta^A \phi\,)_a (\, \theta^A\, \phi\,)_b\,] \qquad (5.17)$$

In non-gauge theories in 4-dimension as in Chapter One we have $\Delta_2 = A_{11}$. The mismatch here in gauge theories , however, is a direct consequence of the ε-scalar problem discussed above.

Indeed if we set d = 4 then $\Delta_2 - A_{11}$ we find it proportional to the ε-scalar component of the vector boson self energy. So the absence of the full 4-dimensional gauge invariance caused by regularisation by dimensional reduction is responsible for the breakdown in the relationship between quadratic divergences and scale invariance.

5.2. The Standard Model .

In this section we take the standard model as an example. We believe that it is unlikely that the standard model could be rendered free of quadratic divergences to all orders by imposing constraints among a finite number of parameters. Nevertheless this topic has generated a certain amount of interest. In the standard model quadratic divergences arise only in the Higgs self-energy. The resulting pole residues are functions of the dimensionless couplings :

$$\Delta_1 = \lambda + \frac{3}{4} g^2 + \frac{1}{4} g''^2 - 2 h^2 . \qquad (5.18)$$

where g , g" , h , λ are the SU(2), U(1), top Yukawa and quartic Higgs couplings respectively.

The normalisations are the conventional ones such that:

$$m_W^2 = \frac{1}{4} g^2 v^2 \qquad (5.19)$$

$$m_Z^2 = \frac{1}{4} (g^2 + g''^2) v^2 \qquad (5.20)$$

$$m_H^2 = \lambda v^2 \qquad (5.21)$$

$$m_t^2 = \frac{1}{2} h^2 v^2 \qquad (5.22)$$

where v is the Higgs vacuum expectation value. Now using the above equs., (5.18) can be written as relationships between particle masses.

The result for Δ_1 is[8]

$$\Delta_1 = H + 3 + \tan^2\theta_W - 4T \qquad (5.23)$$

where

$$H = m_H^2 / m_W^2 \qquad (5.24)$$

$$T = m_t^2 / m_W^2 \qquad (5.25)$$

and θ_W is the weak mixing angle, where we ignored the contribution from other quarks and Leptons.

Values of m_t and m_H can be obtained by setting $\Delta_1 = 0$, as envisaged originally by Veltman.

As a special case from (5.12) we have the two-loop quadratic divergences for the standard model Δ_2 where we have used the equs. (5.19)-(5.22). Thus ,

$$\Delta_2 = \frac{9}{2} H^2 + 27 H T - 54 T^2 - 9H (3 + \tan^2 \theta_W) - T (27 - 7 \tan^2 \theta_W - S) + \frac{189}{2} + 45 \tan^2 \theta_W + \frac{261}{2} \tan^4 \theta_W \quad . \tag{5.26}$$

where $S = 192 \alpha_3 \sin^2\theta_W/\alpha$, α ,α_3 are the fine-structure and the strong coupling respectivly.
where H and T are given in equ. (5.24) and (5.25) respectively,

Now using the β-functions given in equs. (5.14)-(5.16) and equ. (5.13) we have

$$A_{11} = \frac{9}{4} H^2 + 27 H T - 54 T^2 - 9H (3 + \tan^2 \theta_W) - T (27 - 7 \tan^2 \theta_W - S) + \frac{21}{2} + 45 \tan^2 \theta_W + \frac{109}{2} \tan^4 \theta_W \quad . \tag{5.27}$$

Now if we substitute $\Delta_1 = 0$ we obtain

$$\Delta_2 = 126 T^2 - T (324 - 92 \tan^2 \theta_W - S) + 216 + 126 \tan^2 \theta_W + 144 \tan^4 \theta_W \quad . \tag{5.28}$$

and

$$A_{11} = 126\ T^2 - T\ (\ 324 - 92 \tan^2 \theta_W - S\ ) + 132$$
$$+ 126 \tan^2 \theta_W + 68 \tan^4 \theta_W\ . \qquad (5.29)$$

Neither $\Delta_2 = 0$ nor $A_{11} = 0$ can be achieved for any value of T. If we permit ourselves to delete the contribution involving α_3 , then one still cannot achieve $\Delta_2 = 0$, but we can achieve $A_{11} = 0$ for $m_t \cong 115$ GeV and correspondingly from equ. (5.23) $m_H \cong 180$ GeV. Thus requiring simply that the one-loop condition $\Delta_1 = 0$ be scale invariant leads (if the α_3 term is ignored) to unique predictions for both top and Higgs masses! How seriously should we take this? We hope to resolve the mismatch between Δ_2 and A_{11} by the use of a more suitable regulator. The opportunistic neglect of the α_3 terms is harder to justify. It would be interesting, of course, if m_t and m_H (when known) happen to satisfy $\Delta_1 = 0$; new physics might then enter to ensure $A_{11} = 0$.

5.3. Non-local regularisation and quadratic divergences:

We have seen in the previous sections the problems which arise when dimensional regularisation is used, whether in dimensional regularisation or dimensional reduction form, and its unsatisfactory treatment of the quadratic divergence in gauge theories.

So, we turn to a new regularisation procedure that has been advocated and applied in a recent series of papers [30]. In this section we will use this regularisation procedure to calculate the quadratic divergences in the standard model then rederive the Veltman formula for the top quark and Higgs masses , equ. (5.23).

We start first with a review of this method. The main idea emerges when the fact that the finiteness of string theory would follow trivially from the non-locality of its interactions [31]. This was central to the argument which the inventor of the non-local regularisation relied upon . The method can be described, simply, in two stages. Stage one involves the introduction of non-local convergence factor into the interaction terms (but not the quadratic term)in the Lagrangian, so that the loop integrals become infinite. By introducing the convergence factor only in the interaction terms the method overcomes the problems of the higher-derivative method which fails to regulate the one-loop graphs. The factor which we will use to calculate the quadratic divergences is

$$\xi_m = \exp\left[\frac{\partial^2 - m^2}{2\Lambda^2}\right] \qquad (5.30)$$

where Λ is the cut-off.

This factor is not unique, but we will use this because it makes explicit loop calculations simple [31]. There is evidently a price to pay here over gauge invariance. This will be recovered in the second stage of this regularisation procedure by the addition of suitable finite (but non-local) counter terms.

Now we would like to apply this method to regulate the quadratic divergences in the standard model. As has been mentioned before the quadratic divergence in the standard model arise only in the Higgs self-energy.

The Higgs sector of the Lagrangian:

$$L = - \left| \partial_\mu \phi - \frac{ig}{2} \tau . W \phi - i Y_\phi \frac{g''}{2} \phi B_\mu \right|^2 - \frac{\lambda}{2} (\phi^+ \phi - v^2)^2 - h (\bar{L} \phi R + \bar{R} \phi^+ L) \qquad (5.31)$$

where W^μ , B^μ are the gauge field of SU(2) and U(1) respectively g ,g'' are the coupling constant of SU(2) and U(1) respectively, h is the Yukawa coupling and ϕ is the Higgs field.

This Lagrangian is invariant under the local gauge group SU(2) x U(1):

$$\delta\phi = i \left(\frac{g}{2} \alpha(x) . \tau - \frac{g''}{2} \theta(x) \right) \phi \qquad (5.32)$$

$$\delta W_\mu = - \partial_\mu \alpha(x) - g(\alpha \times W_\mu) \qquad (5.33)$$

$$\delta B_\mu = - \partial_\mu \theta(x) \qquad (5.34)$$

Now using the method of non-local regularisation described above we write

$$L \longrightarrow L^{\text{nonlocal}} \qquad (5.35)$$

where L^{nonlocal} is L with the non local factor in equ. (5.30) inserted into its interaction terms i.e. equ. (5.31) under (5.35) transformation becomes:

$$\begin{aligned} L^{\text{nonlocal}} = &- \partial_\mu \phi \, \partial^\mu \phi^+ + \frac{ig}{2} [\partial_\mu \hat{\phi}^+ \tau . \hat{W}^\mu \hat{\phi} - \hat{\phi}^+ \tau . \hat{W}^\mu \partial_\mu \hat{\phi}] \\ &+ i \, Y_\phi \frac{g''^2}{2} [\partial_\mu \hat{\phi}^+ \hat{B}^\mu \hat{\phi} - \hat{\phi}^+ \hat{B}^\mu \partial_\mu \hat{\phi}] \\ &+ \frac{g^2}{4} [\hat{\phi}^+ (\tau . \hat{W})^2 \hat{\phi}] - \frac{g''^2}{4} Y_\phi^2 [\hat{\phi}^+ (\hat{B}_\mu)^2 \hat{\phi}] \\ &- \frac{\lambda}{2} (\hat{\phi}^+ \hat{\phi} - v^2)^2 \\ &- h (\hat{\bar{L}} \hat{\phi} \hat{R} + \hat{\bar{R}} \hat{\phi}^+ \hat{L}) \end{aligned} \qquad (5.36)$$

where

$$\hat{\phi} = \xi_m \phi \qquad (5.37)$$

$$\hat{W} = \xi_o W \qquad (5.38)$$

$$\hat{B} = \xi_o B \qquad (5.39)$$

and

$$\xi_m = \exp\left[\frac{\partial^2 - m^2}{2\Lambda^2}\right] \qquad (5.40)$$

$$\xi_o = \exp\left[\frac{\partial^2}{2\Lambda^2}\right] \qquad (5.41)$$

It is now clear that (5.36) is no longer gauge invariant under the local gauge transformation in equs. (5.32)-(5.34). But since current conservation at orders g, g" depend only upon the (unchanged) free theory there must be an associated symmetry at this order. One finds it by nonlocalising the transformation law to become:

$$\delta\,\phi = i\,\left(\frac{g}{2}\,\xi_m\,\hat{\alpha}(x).\tau\ -\ \frac{g''}{2}\,\xi_m\,\hat{\theta}(x)\,\right)\hat{\phi} \qquad (5.42)$$

$$\delta\,W_\mu = -\,\xi_m\,\partial_\mu\hat{\alpha}(x)\ -\ g\,\xi_m(\,\hat{\alpha}\times\hat{W}_\mu) \qquad (5.43)$$

$$\delta\,B_\mu = -\,\xi_m\,\partial_\mu\hat{\theta}(x) \qquad (5.44)$$

where $\hat{\alpha} = \xi_o\,\alpha$ and $\hat{\theta} = \xi_o\,\theta$.

Now at order g^2 , g''^2 the theory's invariance is violated in a physically meaningful way by the breakdown of current conservation and the loss of decoupling of longitudinal particle .

Here the second stage of this regularisation scheme is needed to postpone this violation to the next order, g^3 , g''^3 , by adding a new interaction term to the Lagrangian.

The loss of gauge invariance at g, g" order signals no physical problem since the current conservation continues to hold because the current conservation comes from the free unchanged part of the Lagrangian. The physical problem is the loss of decoupling which occurs at order g^2, g''^2 . This can be seen in the (Compton like) tree amplitude:

$$F_{W\phi} = g^2 \varepsilon''^{\mu} \varepsilon^{\nu} \left(\frac{p''_{\mu} p_{\nu}}{s - m^2} \exp\left[\frac{s^2 - m^2}{\Lambda^2}\right] + \frac{p''_{\nu} p_{\mu}}{u - m^2} \exp\left[\frac{u^2 - m^2}{\Lambda^2}\right] - \frac{1}{2} \delta_{\mu\nu} \right) \qquad (5.45)$$

where s and u are the Mandelstam parameters. Suppose now we let the first W particle be longitudinal, $\varepsilon^{\nu} = k^{\nu}$. Then

$$F_{W\phi} = g^2 \varepsilon''^{\mu} \left(\frac{1}{2} p''_{\mu} \exp\left[\frac{s^2 - m^2}{\Lambda^2}\right] + p_{\mu} \exp\left[\frac{u^2 - m^2}{\Lambda^2}\right] - \frac{1}{2} k_{\mu} \right) \qquad (5.46)$$

Hence longitudinal particles couple to physical particles. To cancel this failure of decoupling we need a suitable interaction and to find that we go backwards from the longitudinal particle coupling we wish to cancel. From (5.46) and $k_\mu = (p''-p)_\mu$ we have

Now

$$F_{W\phi} = \frac{1}{2} g^2 \varepsilon''^{\mu} \left(p''_\mu \; [\exp \left[\frac{s - m^2}{\Lambda^2} \right] - 1] - p_\mu \; [\exp \left[\frac{u - m^2}{\Lambda^2} \right] - 1] \right) \quad (5.47)$$

$$= \frac{1}{2} g^2 \varepsilon''^{\mu} (2\, k.p)\, p''_\mu \left(\frac{[\exp \left[\frac{s - m^2}{\Lambda^2} \right] - 1]}{s - m^2} \right)$$

$$+ \frac{1}{2} g^2 \varepsilon''^{\mu} (2\, p''.k)\, p_\mu \left(\frac{[\exp \left[\frac{u - m^2}{\Lambda^2} \right] - 1]}{u - m^2} \right) \quad (5.48)$$

Now $k^\nu = \varepsilon^\nu$

$$F_{W\phi} = g^2 \varepsilon''^{\mu} \varepsilon^{\nu} p_\mu\, p''_\nu \left(\frac{[\exp \left[\frac{s - m^2}{\Lambda^2} \right] - 1]}{s - m^2} \right)$$

$$+ g^2 \varepsilon''^{\mu} \varepsilon^{\nu}\, p''_\mu\, p_\nu \left(\frac{[\exp \left[\frac{u - m^2}{\Lambda^2} \right] - 1]}{u - m^2} \right) \quad (5.49)$$

This last expression would come from the following (momentum space) interaction:

$$g^2 \hat{W}_\mu \partial^\nu \hat{\phi}^+ \left(\frac{[\exp\left[\frac{\partial^2 - m^2}{\Lambda^2}\right] - 1]}{\partial^2 - m^2} \right) \hat{W}_\nu \partial^\mu \hat{\phi} \qquad (5.50)$$

and similarly for B_μ

$$g''^2 \hat{B}_\mu \partial^\nu \hat{\phi}^+ \left(\frac{[\exp\left[\frac{\partial^2 - m^2}{\Lambda^2}\right] - 1]}{\partial^2 - m^2} \right) \hat{B}_\nu \partial^\mu \hat{\phi} \qquad (5.51)$$

By adding these two terms (5.50) and (5.51) to the Lagrangian we cancel the failure of decoupling at one loop level. Now we want to calculate the one loop quadratic divergences in the standard model. The method of non-local regularisation only regulates the Euclidean loop integrals so our calculations in this section will be in Euclidean space. The graphs which contribute to the quadratic divergences at one loop level are displayed in fig. (1) to (6):

For the graph in fig. (1) we have:

$$\frac{g''^2}{2} Y_\phi^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2p+k)_\mu (2p+k)_\nu}{k^2 ((p+k)^2 + m^2)} \left[\delta_{\mu\nu} + (\alpha-1) \frac{k_\mu k_\nu}{k^2} \right]$$

$$\times \exp\left[\frac{-p^2 - m^2}{\Lambda^2} - \frac{(p+k)^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \qquad (5.52)$$

Now we set the external momentum p = 0 and promote the propagator to a Schwinger integral and then perform the momentum integral.

$$\mathrm{fig}(1) = \frac{\alpha\, g''^2}{2(2\pi)^4}\, Y_\phi^2 \;\exp\left[\frac{-\, m^2}{\Lambda^2}\right] \times \int \frac{d^4k}{k^2+m^2} \;\exp\left[\frac{-k^2-m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2}\right] \quad (5.53)$$

$$= \frac{\alpha\, g''^2}{2(2\pi)^4}\, Y_\phi^2 \;\exp\left[\frac{-\, m^2}{\Lambda^2}\right] \int_1^\infty \frac{d\tau}{\Lambda^2} \int d^4k \;\exp\left[\tau\,\frac{-k^2-m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2}\right] \quad (5.54)$$

performing the k integrals we have

$$\mathrm{fig}(1) = \frac{\alpha\, g''^2}{2(2\pi)^4}\, Y_\phi^2 \;\exp\left[\frac{-\, m^2}{\Lambda^2}\right] \pi^2 \Lambda^2 \int_1^\infty \frac{d\tau}{(\tau+1)^2} \;\exp\left(-\tau\,\frac{m^2}{\Lambda^2}\right) \quad (5.55)$$

$$= \frac{\alpha\, g''^2}{64\,\pi^2}\, Y_\phi^2 \;\exp\left[\frac{-\, m^2}{\Lambda^2}\right] \Lambda^2 + \frac{\alpha\, g''^2}{64\,\pi^2}\, Y_\phi^2 \; m^2 \ln \frac{2m^2}{\Lambda^2} + \ldots \quad (5.56)$$

It is interesting to note that this method gives the quadratic divergences and the logarithmic divergences which would be expected by power counting.

Next the graph of fig. (2) gives:

$$\frac{g^2}{2} (\tau^a \tau^a)_{ij} \int \frac{d^4k}{(2\pi)^4} \frac{(p+k)_\mu (p+k)_\nu}{k^2((p+k)^2 + m^2)} \left[\delta_{\mu\nu} + (\alpha-1) \frac{k_\mu k_\nu}{k^2} \right]$$

$$\times \exp \left[\frac{-p^2 - m^2}{\Lambda^2} - \frac{(p+k)^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \quad (5.57)$$

As we have done before for the graph in fig.(2) we set p = 0 and promote the propagator to Schwinger integrals. Then

$$\text{fig}(2) = \frac{\alpha g^2}{2(2\pi)^4} (\tau^a \tau^a)_{ij} \exp \left[\frac{- m^2}{\Lambda^2}\right]$$
$$\times \int \frac{d^4k}{k^2+ m^2} \exp \left[\frac{-k^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \quad (5.58)$$

$$= \frac{\alpha g^2}{2(2\pi)^4} (\tau^a \tau^a)_{ij} \exp \left[\frac{- m^2}{\Lambda^2}\right]$$
$$\times \int_1^\infty \frac{d\tau}{\Lambda^2} \int d^4k \exp \left[\tau \frac{-k^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \quad (5.59)$$

Now we perform the k integral

$$\text{fig}(2) = \frac{\alpha g^2}{64 \pi^2} (\tau^a \tau^a)_{ij} \exp \left[\frac{- m^2}{\Lambda^2} \right] \Lambda^2$$

$$\times \int_1^\infty \frac{d\tau}{(\tau+1)^2} \exp (-\tau \frac{m^2}{\Lambda^2}) \tag{5.60}$$

$$= 3 \frac{\alpha g^2}{64 \pi^2} \exp \left[\frac{- m^2}{\Lambda^2} \right] \Lambda^2$$

$$+ 3 \frac{\alpha g^2}{64 \pi^2} m^2 \ln \frac{2m^2}{\Lambda^2} + \ldots . \tag{5.61}$$

The third graph in fig. (3) gives:

$$- \frac{g''^2}{4} Y_\phi^2 \int \frac{d^4k}{(2\pi)^4} \frac{\delta_{\mu\nu}}{k^2} \left[\delta_{\mu\nu} + (\alpha-1) \frac{k_\mu k_\nu}{k^2} \right]$$

$$\times \exp \left[\frac{-p^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \tag{5.62}$$

We promote the propagator to a Schwinger integral , obtaining

$$\text{fig}(3) = \frac{- g''^2}{4(2\pi)^4} Y_\phi^2 (\alpha+3) \exp \left[\frac{- m^2}{\Lambda^2} \right]$$

$$\int_1^\infty \frac{d\tau}{\Lambda^2} \int d^4k \exp \left[\tau - \frac{k^2}{\Lambda^2} \right] \tag{5.63}$$

Performing the k integral gives

$$\text{fig}(3) = \frac{-g''^2}{4(2\pi)^4} Y_\phi^2 (\alpha+3) \exp\left[\frac{-m^2}{\Lambda^2}\right] \int_1^\infty \frac{d\tau}{\Lambda^2} \frac{\pi^2 \Lambda^2}{\tau^2} \quad (5.64)$$

$$= \frac{-g''^2}{64\pi^2} Y_\phi^2 (\alpha+3) \exp\left[\frac{-m^2}{\Lambda^2}\right] \Lambda^2 \,. \quad (5.65)$$

Similarly for the gauge field in the graph fig. (4) we have

$$-\frac{g^2}{4} (\tau^a \tau^a)_{ij} \int \frac{d^4k}{(2\pi)^4} \frac{\delta_{\mu\nu}}{k^2} \left[\delta_{\mu\nu} + (\alpha-1) \frac{k_\mu k_\nu}{k^2}\right] \times \exp\left[\frac{-p^2 - m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2}\right] \quad (5.66)$$

$$\text{fig}(4) = -\frac{3 g^2}{64\pi^2} (\alpha+3) \exp\left[\frac{-m^2}{\Lambda^2}\right] \Lambda^2 \,. \quad (5.67)$$

For the one loop self interacting Higgs in graph fig. (5) gives

$$\frac{-3\lambda}{(2\pi)^2} \delta_{ij} \int \frac{d^4k}{k^2 + m^2} \exp\left[\frac{-p^2 - m^2}{\Lambda^2} - \frac{k^2 - m^2}{\Lambda^2}\right] \quad (5.68)$$

$$= \frac{-3\lambda}{(2\pi)^2}\delta_{ij} \int_1^\infty \frac{d\tau}{\Lambda^2} \int d^4k \, \exp\left[\frac{-m^2}{\Lambda^2} - \tau\frac{(k^2+m^2)}{\Lambda^2} \right] \quad (5.69)$$

Performing the k integral we have

$$\text{fig}(5) = \frac{-3\lambda}{16\pi^2}\delta_{ij} \exp\left[\frac{-m^2}{\Lambda^2}\right] \Lambda^2 \times \int_1^\infty \frac{d\tau}{\tau^2} \exp\left(-\tau\frac{m^2}{\Lambda^2}\right) \quad (5.70)$$

$$= \frac{-3\lambda}{16\pi^2}\delta_{ij} \exp\left[\frac{-2m^2}{\Lambda^2}\right] \Lambda^2 + \ldots\ldots \quad (5.71)$$

Finally the contribution from the Yukawa coupling which is in the graph in fig. (6) gives:

$$\sum_{\text{fermions}} (-1)\, h^2 \int \frac{d^4k}{(2\pi)^4} \frac{(i\,\gamma_\mu k^\mu)^2}{k^2} \times \exp\left[\frac{-m^2}{\Lambda^2} \; \frac{-k^2}{\Lambda^2} - \frac{k^2}{\Lambda^2}\right] \quad (5.72)$$

where h is Yukawa coupling constant.

$$\text{fig}(6) = \sum_{\text{fermions}} (-1)\, h^2 \frac{4}{(2\pi)^4} \exp\left[\frac{-m^2}{\Lambda^2}\right] \int_1^\infty \frac{d\tau}{\Lambda^2} \int d^4k \, \exp\left[-\tau\frac{k^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \quad (5.73)$$

Performing the k-integral

$$\text{fig}(6) = \sum_{\text{fermions}} (-1)\ h^2 \frac{4}{(2\pi)^4} \exp\left[\frac{-\ m^2}{\Lambda^2}\right] \int_1^\infty \frac{d\tau}{(\tau+1)^2}\ \pi^2\ \Lambda^2 \tag{5.74}$$

$$\text{fig}(6) = \sum_{\text{fermions}} \frac{2\ h^2}{16\pi^4} \exp\left[\frac{-\ m^2}{\Lambda^2}\right] \Lambda^2 \ . \tag{5.75}$$

As we can see the results are similar to that from using dimensional regularization.

Now we consider the terms in (5.50) and (5.51) which we add to cancel the failure of decoupling. These terms will not contribute to the quadratic divergence. This can be seen by calculating the graphs arising from them, and find they are finite.

One can now write the result of the quadratic divergence defined above in terms of masses using the relationships in (5.19) - (5.22) So we have:

$$\Delta_1 = H + 3 + \tan^2 \theta_W - 4T \tag{5.76}$$

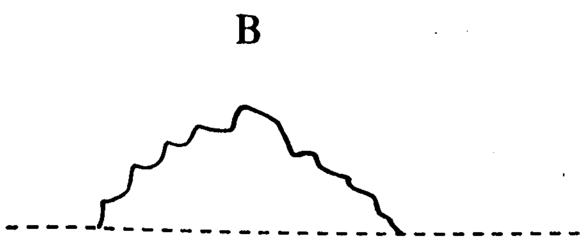
where

$$H = m_H^2 \ / \ m_W^2 \tag{5.77}$$

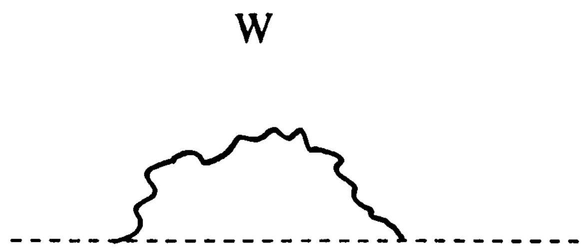
and

$$T = m_t^2 \ / \ m_W^2 \tag{5.78}$$

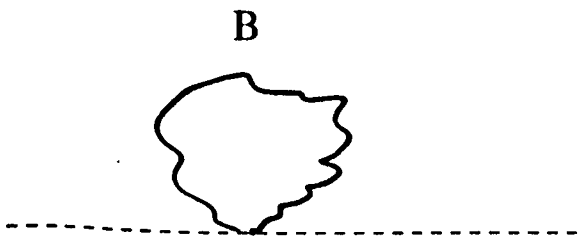
As we can see we have been able to rederive Veltman's formula for the Higgs and the top masses via non-local regularisation .as we have seen in section 5.1 the mismatch between Δ_2 and $A_{11}$is proportional to the ε-scalar components of the vector boson , equ.(5.17) , which is the consequence of using dimensional reduction regularisation . In this case since we remained in four dimension, the ε-scalar problem does not arise here. We conjecture that with this method , the conditions $\Delta_1 = 0$ will suffice to ensure absence of quadratic divergence through two loops.



fig(1)



fig(2)



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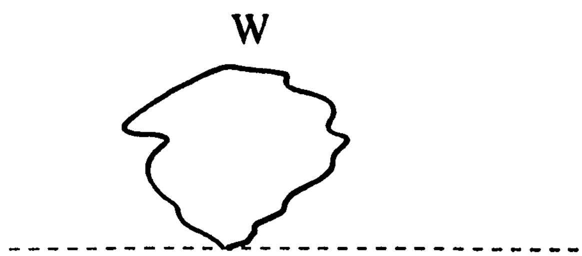
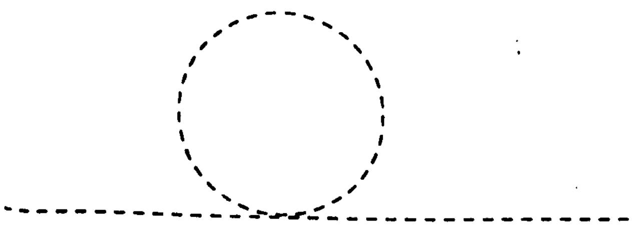
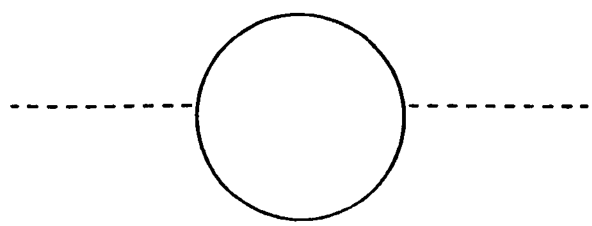


fig(4)



fig(5)



fig(6)

CONCLUSION :

In this work we have calculated the quadratic divergences in renormalisable field theories in d = 4 , d = 2 , d = 3 and d = 6 theories ,namely Φ^4 theory in d = 4 , Φ^3 theory in d = 6 , Φ^6 theory in d = 3 , d = 2 sigma model and a gauge theory (the standard model) .
Our main purpose in non-gauge theories was to explore the apparant connection between quadratic divergences and the logarithmic divergences . This connection , which succeeded at one,two ,and three loop level in Φ^4 theory in d = 4 , appears to fail at the four-loop level . A similar disappointment was reached in the case of Φ^3 theory in d = 6 and Φ^6 theory in d = 3 at the leading order .
In d = 2 sigma model ,where the ambiguities with $\varepsilon^{\mu\nu}$ lead to farther problem , the connection was not successful also.
The original goal of this investigation was to demonstrate that there might exist non-supersymmetric theories free of quadratic divergences . While this question remains open with regard to the class of theories proposed in ref[7] , it now appears that vanishing of the quadratic divergences at L- loop is not guaranteed by requiring scale invariance (to L -loops) of the quadratic divergences conditions for L" < L. This would suggest that there is in general more than one condition that must be satisfied to ensure quadratic divergences cancellation at a given loop order.

Quadratic divergences are themselves sensitive to the choice of regulator , as in the case of the gauge theories .

In chapter 5 we suggested that some other regulator apart from dimensional regularisation might be more appropriate in the gauge theories case and we tested the recently proposed non-local regularisation .

As anticipated the one loop result is identical to that obtained by dimensional regularisation , but because we remain in four dimensions , the ε-scalar problem does not arise.

In conclusion , we feel that the most interesting aspect of this work remains the relationship between the quadratic divergences and the scale dependence . Further study of the quadratic divergences may lead to a better understanding of their structure and role in determining the sensitivity of the theories to new physics.

APPENDIX ONE: DIMENSIONAL REGULARISATION.

In this Appendix we will give a brief review of our regularisation technique, dimensional regularisation [ ]. The features of dimensional regularisation is available in the literature, so we will give a short summary of some features, and a listing of the integral formula used throughout this work.

It is well known that direct calculation of a typical Feynman integral gives divergent quantities. It is important to distinguish between two types of potential divergences: ultraviolet divergences and infrared divergences. The first type arise from taking the momenta in the integral to infinity, and relevant to our calculation and the calculation of the β-functions, whereas the second type correspond to the zero momentum limits.

In quantum field theory calculation, the regularisation of the divergences is necessary, that is, rendering them finite so as to permit their mathematical manipulation in a meaningful manner. This is usually done by cutting off the momentum integrations at some value Λ , or by altering the dimension of the space-time. In the first case the divergences manifest themselves as logarithms of the cut-off parameter Λ or as powers of Λ , while in the second case -dimensional regularisation - they appear as poles in the small parameter measuring the difference between the real dimension and that to which we have continued.

So, in dimensional regularisation we shall regulate their divergences by going to

$$d = 4 - \varepsilon$$

for 4-dimensional theories, where ε is the small parameter. According to the above discussion, these divergences will appear as poles in ε.

Now, the renormalisation process consists of adding counterterms to the couplings in such a way that the amplitudes become finite. Within the theories which we considered in the first four chapters these counterterms are simply the regulated divergences arising in the perturbation theory but with a different sign.

The ε-series generally receives contributions from both infrared and ultraviolet divergences. The quadratic divergences are ultraviolet divergences, so we need to filter out the infrared divergences. The standard way to do that is to add a mass of terms to all propagators, which removes any zero-momentum infinities, in our calculation this corresponds to adding an invariant mass term.

$$+ m^2 \zeta_i \zeta^i$$

to the background field expansion.

The problem with this approach is that not all propagators need to be regulated. An alternative procedures is to add the mass terms only to these propagators, that need them, this is what we use in our calculation [32].

Now let us proceed with listing those integrals of use to us in our calculation. We have worked throughout in momentum space, and d-dimension. The integrals given here are evaluated in Minkowski space-time [16].

$$\int \frac{d^dk}{(k^2)^\alpha \;[(p-k)^2]^\beta} = i\,\pi^{d/2}\,\frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\;\Gamma(\beta)}\,B[(d/2)-\alpha,(d/2)-\beta] \times(-1)^{\alpha+B}\,(p^2)^{(d/2)-\alpha-\beta}$$

$$\int \frac{d^dk\;\;k^\mu}{(k^2)^\alpha \;[(p-k)^2]^\beta} = i\,\pi^{d/2}\,\frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\;\Gamma(\beta)}\,B[(d/2)-\alpha+1,(d/2)-\beta]\;p^\mu \times(-1)^{\alpha+B}\,(p^2)^{(d/2)-\alpha-\beta}$$

$$\int \frac{d^dk\;\;k^\mu\;k^\nu}{(k^2)^\alpha \;[(p-k)^2]^\beta} = \frac{i\,\pi^{d/2}}{\Gamma(\alpha)\;\Gamma(\beta)}\Big[\,\Gamma(\alpha+\beta-d/2)\;B[(d/2)-\alpha+2,(d/2)-\beta]\;p^\mu p^\nu + \frac{1}{2}\,g^{\mu\nu}p^2\;\Gamma(\alpha+\beta-1-d/2)\;B[(d/2)+1-\alpha,(d/2)+1-\beta]\,\Big] \times(-1)^{\alpha+B}\,(p^2)^{(d/2)-\alpha-\beta}$$

$$\int \frac{d^dk \; k^\mu \; k^\nu k^\lambda}{(k^2)^\alpha \; [(p-k)^2]^\beta} = \frac{i \; \pi^{d/2}}{\Gamma(\alpha) \; \Gamma(\beta)} \Big[\Gamma(\alpha+\beta-d/2) \; B[(d/2)-\alpha+3,(d/2)-\beta] \; p^\mu p^\nu p^\lambda$$
$$+\frac{1}{2} \, p^2 \; \Gamma(\alpha+\beta-d/2-1) \; B[(d/2)-\alpha+2,(d/2)-\beta+1] \; \{ \; g^{\mu\nu} p^\lambda + \ldots \} \Big]$$
$$\times \; (-1)^{\alpha+B} \; (p^2)^{(d/2) \; -\alpha-\beta}$$

$$\int \frac{d^dk \; k^\mu \; k^\nu k^\lambda k^\rho}{(k^2)^\alpha \; [(p-k)^2]^\beta} = \frac{i \; \pi^{d/2}}{\Gamma(\alpha)\Gamma(\beta)} \Big[\Gamma(\alpha+\beta-d/2) \; B[(d/2)-\alpha+4,(d/2)-\beta] \; p^\mu p^\nu p^\lambda p^\rho$$
$$+\frac{1}{2} \; \Gamma(\alpha+\beta-d/2-1) \; B[(d/2)-\alpha+3,(d/2)-\beta+1] \; \{ \; g^{\lambda\rho} p^\nu p^\mu + \ldots \}$$
$$+\frac{1}{4} \; \Gamma(\alpha+\beta-d/2-2) \; B[(d/2)-\alpha+2,(d/2)-\beta+2] \; \{ \; g^{\mu\nu} g^{\lambda\rho} + \ldots\ldots \} \Big]$$
$$\times \; (-1)^{\alpha+B} \; (p^2)^{(d/2) \; -\alpha-\beta}$$

Certain integrals can be more simply dealt with by use of the following formulas :

$$\int \frac{d^dk}{(k^2) \; (p-k)^2} = I(p^2) = \frac{\Gamma(2-d/2)}{\Gamma(d-2 \;)} \quad \Gamma((d/2)-1)^2 .$$

$$\int \frac{d^dk \quad k^\mu}{(k^2) \; (p-k)^2} = \frac{1}{2} \, I \; p^\mu \; .$$

$$\int \frac{d^dk \;\; k^\mu \; k^\nu}{(k^2) \; (p-k)^2} = \frac{I}{4(d-1)} \quad [\; d.p^\mu \; p^\nu - g^{\mu\nu} \; p^2 \;] \qquad .$$

For diagrams with infrared divergence we have needed only one integral:

$$\int \frac{d^d k}{(k^2)^{\alpha} \, [(k^2+ m^2)]^{\beta}} = i \, \pi^{d/2} \frac{\Gamma(\alpha+\beta - d/2) \, \Gamma((d/2)- \alpha)}{\Gamma(d/2) \, \Gamma(\beta)} (m^2)^{(d/2) -\alpha-\beta}$$

The Gamma function $\Gamma(x)$ has the short distance expansion

$$\Gamma(x) = \frac{1}{x} - \gamma_E + O(x) \ .$$

for small x where γ_E is the Eulers constant .

Also F(x) obey the identity,

$$\Gamma(x) = (x-1) \, \Gamma(x-1)$$

For integer x, $\Gamma(x) = (x-1)!$

B(a, b) is the Beta function and is given by

$$B(x,y) = \frac{\Gamma(x) \, \Gamma(y)}{\Gamma(x+y)} \qquad .$$

when calculating a Feynman integral we generally find not only poles in ε, but also finite parts whether or not we include these pieces in the definition of our counterterms is a matter of preference, properly, of renormalisation schemes. The simplest option is to exclude them, and remove only the divergent parts of the counterterms. This procedure is known as minimal subtraction.

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