EVOLUTES INVOLUTES AND THE COXETER GROUP H_4

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ABSTRACT

This is a study of the singularities of the evolutes and involutes of curves and hypersurfaces in R^n, suggested by the work of Arnold on Lagrangian and Legendrian singularities. A family of parallel surfaces is constructed, which, considered as a hypersurface in space-time, appears to be locally diffeomorphic to the discriminant $\Delta(H_4)$ of the Coxeter group H_4. This is the only Coxeter group, not so far linked with singularity theory. By considering the family of distance squared functions for these surfaces, one would be able to find a function 'of type H_4' and so complete the correspondence between Coxeter groups and singularities of functions, which was noted by Arnold.

Chapter One of this work contains algebraic material. A method is given for constructing a rational parametrisation of the discriminant of any Coxeter group. This is used to describe $\Delta(H_4)$. A set of generators for the ring of invariants of H_4 is calculated explicitly. The results of this lengthy calculation (performed by computer) are included as an appendix.

The second chapter concerns the focal sets of curves. The idea of a family of parallel hypersurfaces, all having the same focal set, is generalised to apply also to families of curves.

The focal set of a curve is always developable, and so its properties are encapsulated in those of its cuspidal edge, which we call the <u>space evolute</u> of the curve. Results are obtained relating the singularities of a curve to the singularities of its space evolute. A number of examples are calculated for curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. A new and simple proof is given of Shcherbak's result, that the big involute of a plane curve with an inflexion is the discriminant of H_3.

The third chapter concerns hypersurfaces in $\mathbb{R}^n$. The curves obtained by lifting the lines of curvature of a hypersurface M to the focal set of M (the <u>raised lines of curvature</u>, or RLCs) play an important role. It is proved that the RLCs are geodesics on the focal set, and the singularities of the RLCs at singular points of the focal set are described. It is suggested that H_4 points will be obtained by combining A_3 points (ribs) and H_3 points in a natural way. Therefore, the behaviour of the focal sets of surfaces with cuspidal edges is investigated, using elementary methods. Surfaces M and F are identified, such that the big involute of F, and the big wave front formed by the parallels to M, appear to give the discriminant $\Delta(H_4)$ (though this is not proved rigorously). In a final brief section, it is shown that if M is a submanifold of $\mathbb{R}^n$ which is neither a curve, nor a hypersurface, then a family of parallels to M does not exist in general.

CONTENTS

Contents of Chapter One

Contents of Chapter Two

Contents of Chapter Three

LIST OF ILLUSTRATIONS

PREFACE

It gives me great pleasure to acknowledge the help I have received from a number of people in the course of my stay in Liverpool. First and foremost, I wish to thank my supervisor, Dr.I.R.Porteous, for his guidance. Every conversation with him has been a source of fresh stimulation and encouragement. Secondly, I would like to express my gratitude to Professor C.T.C.Wall, with whom I have had a number of very helpful discussions, and my other colleagues at Liverpool University for their interest in this work. Thirdly, I thank the Science and Engineering Research Council for their financial support. Finally, I would like to thank Parry for her assistance with typing and photocopying, and my father for help with proof reading.

As a result of the research presented here, the following papers have arisen. Two articles have been submitted to the Mathematical Proceedings of the Cambridge Philosophical Society, entitled

'The Discriminant of the Coxeter Group H_4'

and 'Wave Fronts and the Coxeter Group H_4'.

Papers have also been read at the following meetings.

'Wave Fronts and the Coxeter Group H_4',

Splinter Group Talk at British Mathematical Colloquium, 1984

'The Hidden Coxeter Group Revealed',

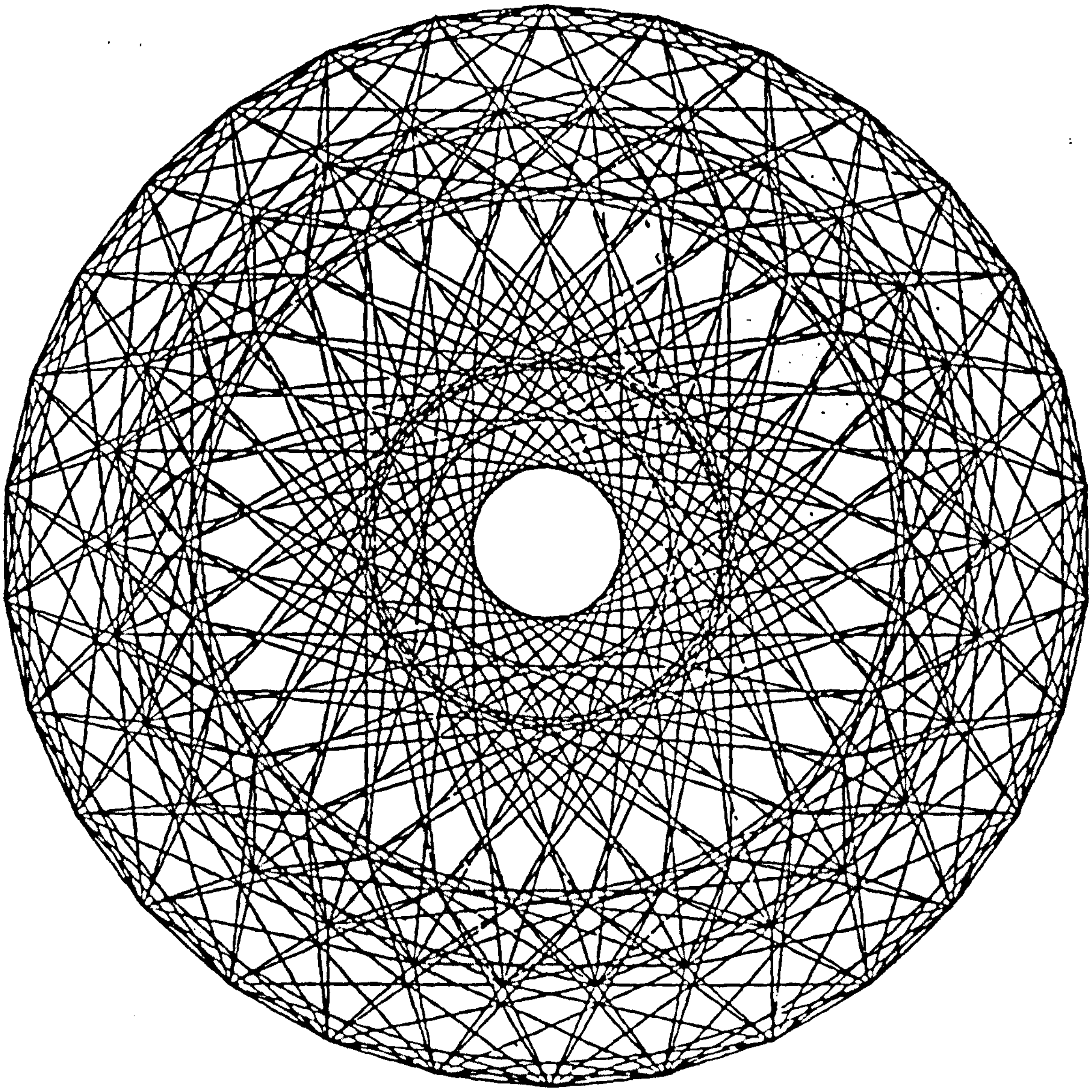
Splinter Group Talk at British Mathematical Colloquium, 1985

'Evolutes and Involutes of Singular Surfaces',

Singularity Days II, Southampton, July 1985

Figure 0.1 A Projection of The Polytope {3,3,5} in $\mathbb{R}^4$

Whose Symmetry Group is H_4 (Drawn by van Oss)



HISTORICAL INTRODUCTION

In this thesis we investigate the relationship between a k-dimensional submanifold M^k of $\mathbb{R}^n$, its parallels, and its <u>focal set F</u>. This study also involves ideas from two other areas of mathematics, namely Coxeter groups and singularity theory. Some use is made of ideas from singularity theory, to help study these geometric objects. Conversely, by looking at the geometric examples, a function is described we believe ought to be called 'of type H_4', and so from a geometric example, we derive a function which it is believed will be of interest to singularity theorists.

The focal set of a nonsingular submanifold $M^k \subset \mathbb{R}^n$ can be defined in three equivalent ways:

(1) the envelope of the normals to M or

(2) the locus of centres of hyperspheres having at least 3 point contact with M or

(3) the locus of cusps of parallels to M

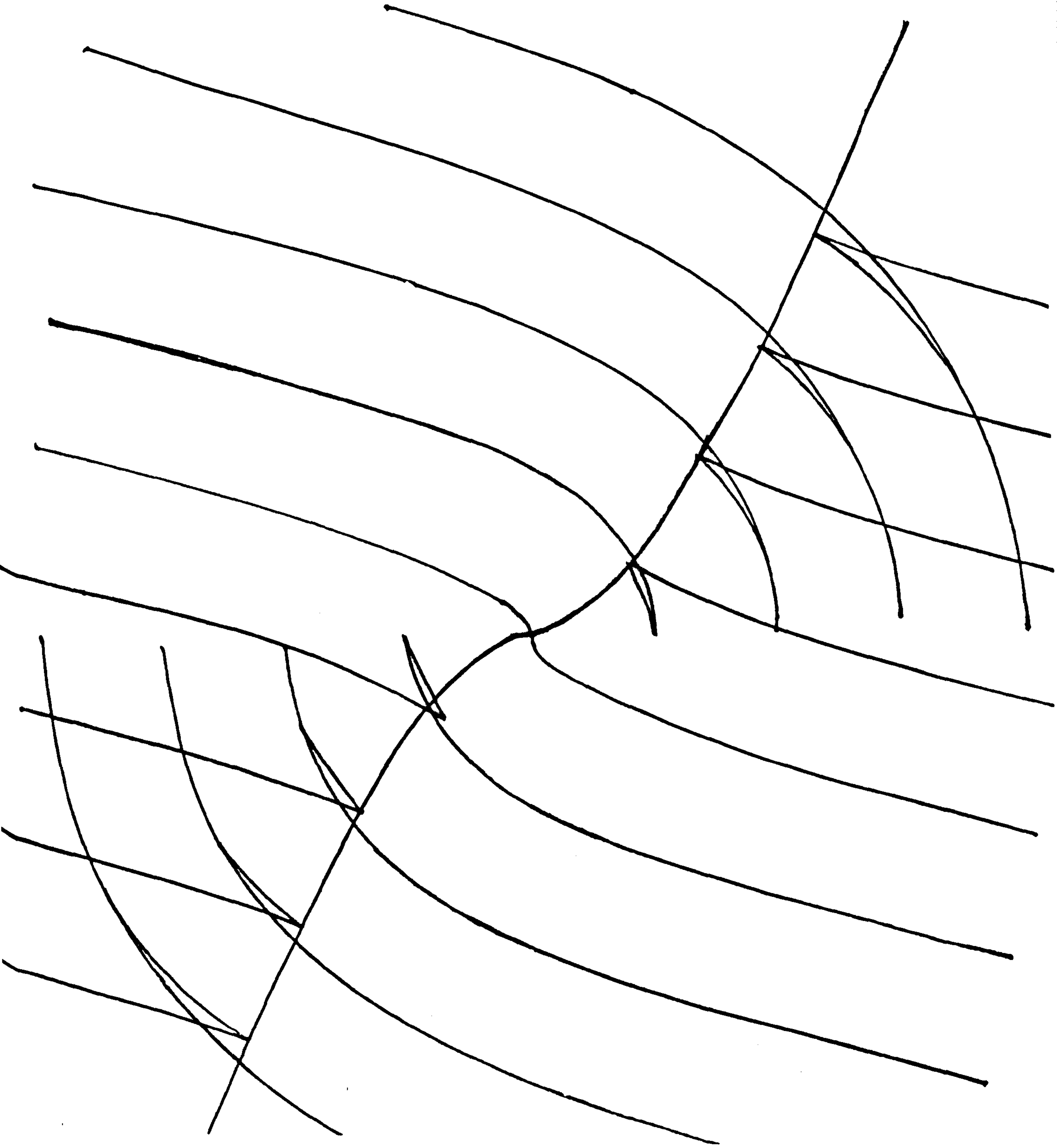
If $k = n-1$, the terms <u>focal set</u> and <u>evolute</u> are used interchangeably and M is said to be an <u>involute</u> of F. Unfortunately, the terms 'evolute' and 'involute' cannot be used for space curves, since the phrase 'evolute of a space curve' has been used classically in a way which is not synonymous with the term 'focal set'.

The focal set is one of the earliest objects to be studied in differential geometry. The evolute of a plane curve was first described by Huyghens in 1659 [17]. Huyghens also discovered the unwinding construction for the involute of a plane curve, and noted that this gave rise to a family of parallel curves, all having the same evolute. He also noted that, starting off with a smooth curve, cusps may start to appear on the parallels. The motivations for this work were the study of light caustics and the practical problem of constructing accurate pendulum clocks.

Further work on curves was done by Bernoulli and Tschirnhausen. In 1696 L'Hospital became the first person to distinguish between ordinary and rhamphoid cusps on a plane curve. He realised the possibility of unwinding any plane curve, including those curves which do not occur as evolutes of a nonsingular curve, and in [16] he described the family of involutes of a plane curve having an inflexion. This family of curves, shown in figure (0.2) recently became the focus of considerable attention, as will be described below.

The idea of the evolute in the case $n=3$ was introduced by the French mathematicians of the 18th and 19th centuries. The properties of the focal set of a curve and of a surface in $\mathbb{R}^3$ were described by Monge, Darboux and Dupin. The classical theory included the procedure for reconstructing a surface in $\mathbb{R}^3$ from

Figure 0.2 The family of Involutes of a Curve With an Inflexion



one of the two sheets of its focal set and an associated geodesic foliation, which is described in chapter 3 of this thesis. Subsequently, the definition of the focal set was extended so as to apply to k-manifolds in $\mathbb{R}^n$ for any pair (k,n) [19].

Another concept which is equally old as the focal set is the idea of parallels to a hypersurface. A hypersurface M^{n-1} in $\mathbb{R}^n$ has a one-parameter family of parallels. Since they describe the propagation of light or other wave phenomena, these parallels are also called wave fronts. As remarked by Huyghens, a wave front which is initially smooth can, after a certain time, develop singularities. Further historical references concerning the development of these differential geometric ideas are given in [5] and [26].

The second subject area involved in this thesis is Coxeter groups. A detailed history of Coxeter groups can be found in [6]. A Coxeter group is a subgroup of O(n) generated by reflections. They are named after Coxeter, since he was the first to enumerate all possible such groups [10] §11.5. In this study, we shall only be interested in finite Coxeter groups. For brevity, the word 'finite' will henceforth be omitted, but will always be implied when the term 'Coxeter group' is used. Coxeter groups were first defined as the sets of symmetries of certain regular and semi-regular polytopes in $\mathbb{R}^n$. Thus initially they were discrete objects which arose in a purely algebraic context. But they then started to reappear in various different contexts.

For example, Shephard and Todd showed [30] that Coxeter groups can also be characterised as groups whose ring of invariants is particularly simple. It was also discovered that most of the Coxeter groups (those which are crystallographic) occurred as the Weyl groups of semi-simple Lie algebras.

The third subject we will be considering is singularity theory. An excellent reference for this subject is [4]. When singularity theory was developed in the 1950s and 1960s, one of the first results was the classification of simple functions. This list brought to light a remarkable and unexpected relationship with Coxeter groups, for each simple singularity was found to correspond to a Coxeter group.

But singularity theory also proved to be a powerful tool in differential geometry, enabling one to describe what parallels and focal sets really look like. The connection with the differential geometric problems mentioned above is as follows. Given a smooth manifold M, one can study the parallels to M and the focal set of M by looking at the singularities of the family of distance-squared function on M. This idea is due to Thom. Conversely, given any family V of functions, one can construct a hypersurface in $\mathbb{R}^N$ (for some N) having V as its family of distance squared functions [4].

For $n < 10$, a list of normal forms for the germ of the distance squared function can be found in [4]. From this list, we can get a set of standard models for the behaviour of focal sets and wave fronts in $\mathbb{R}^n$. For almost all nonsingular hypersurfaces M [33], the big wave front generated by M and the focal set of M are locally diffeomorphic at each point to one of the resulting standard models. The first few normal forms for the distance squared functions are associated with the Coxeter groups A_k, D_k, and E_k [4]. These are the groups whose Coxeter Dynkin diagram consists entirely of unmarked branches, and are associated with function germs on a smooth manifold. In each case the standard model for the big wave front is the **discriminant** of the Coxeter group.

A few years later, it was discovered that the groups B_k and F_4 are also associated with simple function germs. The difference is that these are function germs defined on a smooth manifold with boundary (see [2] and [4]). In the wavefront interpretation, one is taking the parallels to (or the focal set of) a smooth hypersurface with boundary in $\mathbb{R}^n$. The half-space $H = \{x \in \mathbb{R}^n | x_1 \geq 0\}$ is the quotient space $\mathbb{R}^n/Z_2$ for the group Z_2 acting on $\mathbb{R}^n$ by reflection, so a function on H can equally well be regarded as a function on the whole of $\mathbb{R}^n$, which is invariant under reflection in some hyperplane. Thus it is possible to think of these functions as describing the behaviour of the parallels to (or focal set of) a manifold M with a line of symmetry [28]. The groups B_k and F_4 are distinguished from A_k, D_k and E_k by the presence in their Coxeter-Dynkin diagram of branches labelled with the integer 4.

The remaining Coxeter groups are H_3 H_4 and $I_2(k)$ $(k > 5)$. They are (apart from $I_2(6)=G_2$) noncrystallographic, since their Coxeter-Dynkin diagrams include branches labelled with the integers 5 (in the case of H_3 and H_4) and k (in the case of $I_2(k)$).

For ten years, it was thought that these groups did not correspond to singularities of functions. It was then observed that the family of parallel curves in figure (0.2), first considered by L'Hospital in the 1690s, gave rise to a big wave front diffeomorphic to the discriminant of H_3, which is the group of symmetries of the regular icosahedron. This led to the works [18] and [29], in which singularities of functions are described which are linked to the groups H_3 and $I_2(k)$. The distinguishing feature of these functions is that they are defined on a manifold with a singular boundary, e.g. the region of the plane bounded by the curve $x^2=y^5$. The wave front interpretation is that the groups H_3 and $I_2(n)$ should be associated to a family of parallel curves, each of which is singular. The distance squared function for such a curve is a function on the (cuspidal) curve which is obtained as the restriction of a function defined on the whole plane.

Consider the family of curves shown in figure (0.2), which, as mentioned above, are associated to the group H_3. Each of these curves is singular, but their common focal set is a smooth curve with an inflexion. Thus the H_3 family of curves can be constructed by taking the involutes of a smooth curve with an ordinary point of inflexion. As well as starting with a nonsingular hypersurface and looking at the possible singularities of its focal set, it is thus also profitable to investigate what kinds of singularities can arise on the invoilutes of a nonsingular hypersurface.

When the singularities of types H_3 and $I_2(k)$ were discovered, Arnold conjectured that a singularity would soon be discovered, which bore a similar relation to the one remaining Coxeter group H_4. This was one of the aims of the present work. The arguments used are mainly geometric. The idea is first to describe a type of wave-front, and then by looking at the distance squared function to find a function of type H_4. This is the opposite to the usual squence of argument, which is first to classify certain functions by analytic methods, and then to deduce geometric information as a corollary. The present arrangement approximates more closely to the order in which the ideas are usually developed, and so it is hoped that this will make it easier for the reader to follow.

Another aim of this work is to take a fresh look at the differential geometric problems raised above. Let M^k be a k-dimensional submanifold of $\mathbb{R}^n$. We may ask the following questions.

for a hypersurface in $\mathbb{R}^n$ to be the focal set of some curve (the hypersurface must be developable). The essential properties of a developable hypersurface are encapsulated in its cuspidal edge, which we call the <u>space evolute</u> of M. We compute a number of examples in the cases n=2 and n=3, and we provide a new and simple proof of Shcherbak's result that the big involute of a curve with an inflexion is $\Delta(H_3)$.

Chapter 3 is about families of parallel hypersurfaces in $\mathbb{R}^n$. The family of lines of curvature of M play an important part in the relationship between M and its focal set, and we prove some results describing the behaviour of <u>raised lines of curvature</u>. We explain why the group H_4 should arise in connection with a singular surface having a singular focal set, and then we look at the focal sets of some cuspidal surfaces. We describe a family of parallel surfaces in $\mathbb{R}^3$ which we believe give rise to a big involute of type H_4, and we describe the associated distance-squared function, which should have a singularity of type H_4. This chapter concludes with a brief look at focal sets and parallels for submanifolds of $\mathbb{R}^n$ with dimension and codimension both greater than 1. In this case, the differential geometric problems (3) to (5) are still open, even for the case when k=2 and n=4.

Before commencing reading the main text of the thesis, the reader's attention is drawn to appendix two, which explains most of the notation used.

Bourbaki's notation is different from that of Coxeter, as indicated in the following table.

Bourbaki's name	A_n	B_n	D_n	E_n	F_4	H_n	$I_2(k)$
Coxeter's name	A_n	C_n	B_n	E_n	F_4	G_n	D_2^k

CHAPTER 1 DISCRIMINANTS OF COXETER GROUPS

This chapter contains some results of an algebraic nature which arose from a study of the discriminant variety $\Delta(H_4)$. One of the aim of this project was to find a family of wave fronts diffeomorphic to $\Delta(H_4)$. Since no published description of $\Delta(H_4)$ was available, the first step was to describe $\Delta(H_4)$, so that when it was found, it would be recognized.

In this chapter, we define the discriminant of a Coxeter group and prove some results which provide a description not only of the discriminant of H_4, but also of the discriminants of other Coxeter groups.

1.1 Review of Standard Facts About Coxeter Groups

A **reflection** is a linear transformation in O(n) with one eigenvalue equal to -1 and all the other eigenvalues equal to +1. The set of points fixed by a particular reflection is called its **mirror**. A **Coxeter group** is a finite subgroup of O(n) which is generated by reflections. Any such group is a direct product of **irreducible Coxeter groups**. The irreducible Coxeter groups form four infinite series and six exceptional groups [6]. We will use the notation of Bourbaki, in which the infinite series are denoted by A_n $(n\geqslant 1)$, B_n $(n\geqslant 2)$, D_n $(n\geqslant 3)$, $I_2(k)$ $(k\geqslant 3)$, and the other six groups are E_6 E_7 E_8 F_4 H_3 and H_4. In each case, the subscript indicates the dimension of the Euclidean space on which the group acts, which will be called the <u>rank</u> of the group. Note that there is some duplication in this list for small values of the rank: $I_2(3)=A_2$ $I_2(4)=B_2$ and $A_3=D_3$.

Every Coxeter group has a <u>fundamental region</u>. This is a subset of $\mathbb{R}^n$ containing precisely one point from each orbit of the action of G on $\mathbb{R}^n$. We can, and will, assume that a fundamental region D has the following additional properties:

(1) D is a closed convex subset of $\mathbb{R}^n$ consisting of a cone on an (n-1) simplex, i.e. the union of the lines joining the points of an (n-1) simplex σ to a fixed point P not lying in the hyperplane of σ.

(2) The n hyperplanes which form the boundary of D are mirrors of G and the group G is generated by these n reflections.

(3) The interior of D does not meet any of the mirrors of G.

The group G acts not only on $\mathbb{R}^n$, but also on $\mathbb{C}^n$ and on the polynomial ring $R = \mathbb{C}[x_1,\ldots,x_n]$. The action on the ring R is given by $\sigma.f(\underline{x}) = f(\sigma(\underline{x}))$.

Those polynomials which are invariant under the action of G form a subring R(G) of R, and for any Coxeter group G the ring R(G) is itself isomorphic to R (see [30]). In other words, the ring R(G) is generated by n invariant polynomials $f_1,\ldots,f_n$ which are algebraically independent. These n polynomials may be chosen to be homogeneous, and if this is done, the degrees of the f_i must be as shown in Table 1.1 (see [30]). A set of n algebraically independent homogeneous generators for R(G) will be called a set of **basic invariant polynomials** for G. While the degrees of the polynomials in such a set are uniquely determined (see [30] or [32]), the polynomials themselves are not unique. For example, if f_2 and f_4 are basic invariants for B_2, where f_i has degree i, then for any $\lambda \in C$, the polynomials f_2 and $f_4 + \lambda f_2^2$ will form another set of basic invariants.

The reflections of any Coxeter group G form either one or two conjugacy classes in G. The number of reflections in each conjugacy class [12] is shown in Table 1.1.

Consider the quotient mapping $\pi_G:C^n \longrightarrow C^n/G$ in the category of algebraic varieties and regular mappings. The quotient π_G can be constructed by letting its components be a set of basic invariant polynomials for G. Since the polynomials in a basic set are algebraically independent, it follows that the image of π_G, which is the quotient space C^n/G, is the whole of C^n.

Furthermore, by the Malgrange Preparation Theorem, π_G is a quotient not only in the algebraic category but also in the category of smooth manifolds and smooth mappings between them (the argument in chapter 6 of [7] for A_k can be adapted for all groups).

Table 1.1 The List of Irreducible Coxeter Groups

Group	Order	Degrees of Basic Invariants	No.of Reflections in each Conjugacy Class
A_n $(n \geq 1)$	$(n+1)!$	$2,3,4,\ldots,(n+1)$	$n(n-1)/2$
B_n $(n \geq 2)$	$2^n.n!$	$2,4,\ldots,2n$	$n, n(n-1)$
D_n $(n \geq 3)$	$2^{n-1}.n!$	$2,4,\ldots,2n-2,n$	$n(n-1)$
E_6	$72.6!= 51840$	2,5,6,8,9,12	36
E_7	$8.9!= 2903400$	2,6,8,10,12,14,18	63
E_8	$192.10!= 696729600$	2,8,12,14,18,20,24,30	120
F_4	1152	2,6,8,12	12,12
H_3	120	2,6,10	15
H_4	14400	2,12,20,30	60
$I_2(k)$ $(k \geq 3)$	$2k$	$2,k$	k (k odd) $k/2, k/2$ (k even)

The pseudodiscriminant $\Delta(G)$ is the real part of the image of the mirrors under a quotient mapping π_G: $C^n \longrightarrow C^n/G = C^n$ whose components are homogeneous polynomials. As it stands, there is some ambiguity in this definition. Two quotient mappings π and π' for the same group G give rise to two pseudodiscriminants Δ and Δ'. By the general uniqueness properties of quotient mappings, there is an algebraic automorphism of C^n which maps the complexification of Δ onto the complexification of Δ'. But this automorphism may not preserve the real part of C^n, and as real algebraic subvarieties of $\mathbb{R}^n$, the pseudodiscriminants Δ and Δ' may be very different. To remove this ambiguity, we will make the following definition.

Definition 1.2

The discriminant of G is the pseudodiscriminant constructed from a quotient map whose components are homogeneous polynomials with real coefficients.

This assumption distinguishes one particular real form of $\Delta(G)$ for each group G. In § (1.5), we discuss whether there is any significant difference between the discriminant and other pseudodiscriminants.

The pseudodiscriminant $\Delta(G)$ is sometimes called the variety of non-regular orbits of G, since it consists precisely of those orbits containing strictly fewer than $|G|$ points. Yet another characterization of $\Delta(G)$ is as the set of critical values of π_G.

Proposition 1.3

Let G be a Coxeter group acting on $\mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, the set of linear transformations in G leaving x fixed, denoted by $Fix_G(x)$, is a Coxeter group. It is generated by reflections in all those mirrors of G which pass through x.

Proof See [6] V 3.3.

Each Coxeter group has an associated graph $\Gamma(G)$ called its Coxeter-Dynkin diagram. The vertices of this graph correspond to generators and the edges to relations in a particular presentation of G (for details see [10] §11·3). The Coxeter-Dynkin diagram of $Fix_G(x)$ is obtained from that of G by removing any vertex corresponding to a mirror not passing through x. Thus the list of possible stabiliser subgroups within G can easily be written down, by looking at all the possible subgraphs of $\Gamma(G)$.

Proposition (1.3) can be used to describe the local structure of $\Delta(G)$, as follows.

Proposition 1.4

Let $x \in \mathbb{C}^n$ with $Fix_G(x) = H$ where H is a Coxeter group of rank k. Then

(i) There is a linear space L of dimension n-k passing through x such that for all $y \in L$ sufficiently close to x, $Fix_G(y) = H$.

(ii) In some neighbourhood $\mathcal{N}$ of $\pi_G(x)$, the discriminant $\Delta(G)$ is diffeomorphic to $\Delta(H) \times \mathbb{R}^{n-k}$.

Proof

Let $L = \bigcap_1^N M_i(x)$ where $M_1(x),\dots,M_N(x)$ is the set of mirrors of G passing through x. Now (i) follows from (1.3) and (ii) follows from the uniqueness of quotients and the fact that the restriction of π_G to $\pi_G^{-1}(\mathcal{N}) = \bar{\mathcal{N}}$ must be a quotient in the smooth category for the action of H on $\bar{\mathcal{N}}$.

Corollary 1.5

$\Delta(G)$ is smooth at $\pi_G(x)$ if and only if $Fix_G(x) = A_1$.

1.2 Parametrizations of $\Delta(G)$

Let G be a Coxeter group in which all the reflections are conjugate. This covers the cases A_n D_n E_6 E_7 E_8 H_3 H_4 and $I_2(2k+1)$. Select arbitrarily one mirror M_1 and let the other mirrors be $M_2,\ldots,M_N$. Then there will be unit vectors

$$v_1,\ldots,v_n \in S^{n-1} \subset \mathbb{R}^n$$

such that $M_i = \{ x \mid x.v_i = 0 \}$. Let H be the stabiliser subgroup $\mathrm{Fix}_G(v_1)$.

By (1.3), the group H is a Coxeter group, and since each element of H preserves M_1, we can consider H as a group generated by reflections acting on M_1.

Let $\pi_H: M_1 \dashrightarrow M_1/H$ be the quotient map and let M_i' denote $M_1 \cap M_i$ for $i \geqslant 2$. Note that the M_i' will not all be distinct. There will usually be pairs (i,j) with $M_i \neq M_j$ but $M_i' = M_j'$. Since all the mirrors are conjugate, the discriminant $\Delta(G)$ will be equal to the real part of $\pi_G(M_1)$, where $\pi_G: C^n \dashrightarrow C^n/G$ is the quotient map. Since the map π_G is invariant under the action of H, π_G factors through π_H.

Let q be the map defined in commutative diagram (1.6) (see next page). We shall investigate the properties of the map q, which provides useful information about the discriminant $\Delta(G)$.

Examples of the map q for particular groups will be given in section (1.3).

Diagram 1.6

$$C^{n-1} = M_1/H \overset{\pi_H}{\longleftarrow} M_1 \subset C^n \overset{\pi_G}{\longrightarrow} C^n/G = C^n$$

$$q$$

We observe that the mirrors of the group H consist of some but not all of the M_i'. The situation is explained in the following easy lemma.

Lemma 1.7 The following conditions are equivalent.

(i) Reflections in M_1 and M_i commute.

(ii) The hyperplanes M_1 and M_i are perpendicular.

(iii) M_i' is a mirror for the action of H on M_1.

Now let x be a point of M_1 and let $y = \pi_G(x)$. By (1.3), the stabiliser group $Fix_G(x)$ is a group generated by reflections.

If $Fix_G(x) = A_1$, this means by (1.3) that x does not lie on any of the M_i for $i \neq 1$. In particular, no mirror of H passes through x, so the orbit Hx is regular and

$$|Hx| = |H| = |Fix_G(v_1)| = \frac{|G|}{|Gv_1|} =$$

$$= \frac{|G|}{|\{v_1,\ldots,v_N,-v_1,\ldots.-v_N\}|} = \frac{|G|}{2N}$$

Also $|Gx| = \dfrac{|G|}{|Fix_G(x)|} = \dfrac{|G|}{|A_1|} = \dfrac{|G|}{2}$

The $|G|/2$ points of Gx lie on the N mirrors of G, so there are $|G|/2N$ on each mirror. In particular, there are $|G|/2N = |Hx|$ on M_1. This means that a single orbit of H is mapped onto y by q.

Suppose now that $Fix_G(x) = I_2(2k+1)$ for some $k \geqslant 1$ (recall that $I_2(3) = A_2$). By (1.3), the point x lies on precisely $2k+1$ mirrors of G, one of which is M_1. Without loss of generality, $x \in M_i$ for $i = 1,\dots,2k+1$. Since no two mirrors of $I_2(2k+1)$ are perpendicular, none of the M_i is perpendicular to M_1 for $i = 2,3,\dots,2k+1$ and so no mirror of H passes through x.

Thus $|Hx| = |H| = \frac{|G|}{2N}$

Also $|Gx| = \frac{|G|}{|Fix_G(x)|} = \frac{|G|}{|I_2(2k+1)|} = \frac{|G|}{2(2k+1)}$

The $\frac{|G|}{2(2k+1)}$ points of Gx lie on the N mirrors of G, with each point lying on $(2k+1)$ mirrors, so each mirror contains $G/2N = |Hx|$ points of Gx. Once again, a single orbit of H is mapped onto y by q.

Suppose now that $Fix_G(x) = I_2(2k)$ for some $k \geqslant 1$ (where $I_2(2)$ denotes the direct product $A_1 \times A_1$). By (1.3), the point x lies on $2k$ mirrors of G, one of which is M_1. Precisely one of these $2k$ mirrors will be perpendicular to M_1, so that one mirror of H passes through x. Thus

$$|Hx| = \frac{|H|}{|Fix_H(x)|} = \frac{|H|}{|A_1|} = \frac{|H|}{2} = \frac{|G|}{4N}$$

Also $|Gx| = \dfrac{|G|}{|Fix_G(x)|} = \dfrac{|G|}{|I_2(2k)|} = \dfrac{|G|}{4k}$

The $|G|/4k$ points of Gx lie on the N mirrors of G, with each point lying on 2k mirrors, so each mirror contains $|G|/2N = 2|Hx|$ points. This means that two orbits of H, each with $|G|/4N$ points, are mapped onto y by q. The above results can be summarized as follows:

Proposition 1.8

(i) The real part of the image of q is $\Delta(G)$ and the components of q are quasihomogeneous polynomials.

(ii) Let $x \in M_1$ and let $y = \pi_G(x)$. Then the set $q^{-1}(y)$ is

a single point if $Fix_G(x) = A_1$ or A_2 or $I_2(2k+1)$

a pair of points if $Fix_G(x) = A_1 \times A_1$ or $I_2(2k)$

Remarks 1.9

(i) For almost all $x \in M_1$, the group $Fix_G(x)$ will be A_1. This is because the point x will not lie in any of the M_i' for $i \geqslant 2$. So for almost all $y \in \Delta(G)$, we have $|q^{-1}(y)| = 1$. This means that q is a parametrization of $\Delta(G)$ as an irreducible rational algebraic hypersurface in $\mathbb{R}^n$.

(ii) Unless G is actually a dihedral group $I_2(k)$, it follows from enumerating the subgraphs of the Coxeter-Dynkin diagram of G that the only possible stabilisers of rank two or less are

A_1, $A_1 x A_1 = I_2(2)$, $A_2 = I_2(3)$, $B_2 = I_2(4)$ and $I_2(5)$.

The case when G is a dihedral group is not of great interest since the discriminant varieties of the dihedral groups are well known.

(iii) The same counting argument used above can be used to find the number of pre-images of y for any point $y \in \Delta(G)$.

(iv) Every Coxeter group consists entirely of orthogonal transformations in $O(n)$. Therefore they all preserve the invariant quadratic form, which is given, if we take an orthonormal co-ordinate system, by

$$f_2(x) = x_1^2 + \ldots\ldots + x_n^2.$$

It follows that if π_G and π_H are suitably chosen, the map q will preserve the first co-ordinate.

Proposition 1.10

(i) If $x \in \mathbb{R}^{n-1}$ then $q(x) \in \mathbb{R}^n$.

(ii) Suppose $q(x) \in \Delta(G)$. Then either $x \in \mathbb{R}^n$ or $q(x)$ is a singular point of $\Delta(G)$.

Proof

(i) Since the components of π_G and π_H are assumed to be real polynomials, the components of q will also be real.

(ii) Suppose $q(x) \in \mathbb{R}^n$ with $x \notin \mathbb{R}^{n-1}$ and $q(x) = \pi_G(P)$. Then $q(x) = q(\bar{x})$ with $x \neq \bar{x}$ where the bar denotes complex conjugation. So by (1.8), $Fix_G(P)$ is either $I_2(2k)$ or a group of rank at least 3. Thus by Corollary (1.5) $q(x)$ is a singular point of $\Delta(G)$.

The set of singular points of $\Delta(G)$ has codimension 1 in $\Delta(G)$ and codimension 2 in $\mathbb{C}^n/G$. The above proposition therefore tells us that by looking at $q(\mathbb{R}^n)$ we get almost all of $\Delta(G)$.

<u>Table 1.11 Table of the Groups H</u>

G	H
A_n $(n \geq 3)$	A_{n-2}
D_4	$A_1 \times A_1 \times A_1$
D_n $(n \geq 5)$	$D_{n-2} \times A_1$
E_6	A_5
E_7	D_6
E_8	E_7
H_3	$A_1 \times A_1$
H_4	H_3
$I_2(2m+1)$ $(m \geq 1)$	1

The constructions described above can also be carried out for the groups B_k F_4 and $I_2(2m)$ containing two conjugacy classes of reflections. Let M_1 and M_2 be mirrors in different conjugacy classes, let $M_i = \{ x \mid x.v_i = 0 \}$, and let $H_i = Fix_G(v_i)$. Let q_i be the map defined in the commutative diagram below.

$$\mathbb{C}^{n-1} = M_i/H_i \xleftarrow{\pi_{H_i}} M_i \subset \mathbb{C}^n \xrightarrow{\pi_G} \mathbb{C}^n/G = \mathbb{C}^n$$

$$q_i$$

Then the image of each of the maps q_i is an irreducible algebraic hypersurface in C^n. The real parts of these two hypersurfaces together make up $\Delta(G)$.

The following theorem is the result analogous to (1.8) for these groups. The proof will not be given here since it is similar to that for (1.8).

<u>Proposition 1.12</u>

Let $x \in C^n$ with $\pi_G(x) = y$

(i) If $Fix_G(x) = A_1$ then x lies on a single mirror of G, $|q_i^{-1}(y)| = 1$ and $|q_j^{-1}(y)| = 0$ where $\{i,j\} = \{1,2\}$

(ii) If $Fix_G(x) = I_2(2k+1)$ then x lies on (2k+1) mirrors of G, all from the same conjugacy class,

$|q_i^{-1}(y)| = 1$ and $|q_j^{-1}(y)| = 0$ where $\{i,j\} = \{1,2\}$

(iii) If $Fix_G(x) = I_2(2k)$ then there are two possibilities. Either x lies on 2k mirrors, all from the same conjugacy class,

$|q_i^{-1}(y)| = 2$ and $|q_j^{-1}(y)| = 0$ where $\{i,j\} = \{1,2\}$

or x lies on 2k mirrors, k from each conjugacy class and

$$|q_1^{-1}(y)| = |q_2^{-1}(y)| = 1.$$

<u>Table 1.13 Table of the Groups H_1 and H_2</u>

G	H_1	H_2
B_3	B_2	$A_1 \times A_1$
B_n $(n \geqslant 4)$	B_{n-1}	$B_{n-2} \times A_1$
F_4	B_3	B_3
$I_2(2m)$ $(m \geqslant 2)$	A_1	A_1

1.3 Examples of Discriminants

In this section we give examples of the map q introduced above. In particular, we describe $\Delta(H_4)$.

Example 1.14 The Discriminant of H_3

This discriminant is also described in [3] and [18].

Let $G = H_3$ acting as the symmetry group of the icosahedron whose 12 vertices are

$(0 \ \pm\tau \ \pm1) \quad (\pm1 \ 0 \ \pm\tau) \quad (\pm\tau \ \pm1 \ 0)$

where we take all possible combinations of signs and

$\tau = (1 + \sqrt{5})/2.$

Then the 15 mirrors of H_3 are [10] §12·6

$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0 \qquad \tau^2 x_1 \pm x_2 \pm \tau x_3 = 0$

$\tau x_1 \pm \tau^2 x_2 \pm x_3 = 0 \qquad x_1 \pm \tau x_2 \pm \tau^2 x_3 = 0$

Let M_1 be the plane $x_3 = 0$. Then the fourteen lines $M_2',\ldots\ldots,M_{15}'$ are:

$x_1 \pm \tau x_2 = 0$ each counted four times

$\tau^2 x_1 \pm x_2 = 0$ each counted twice

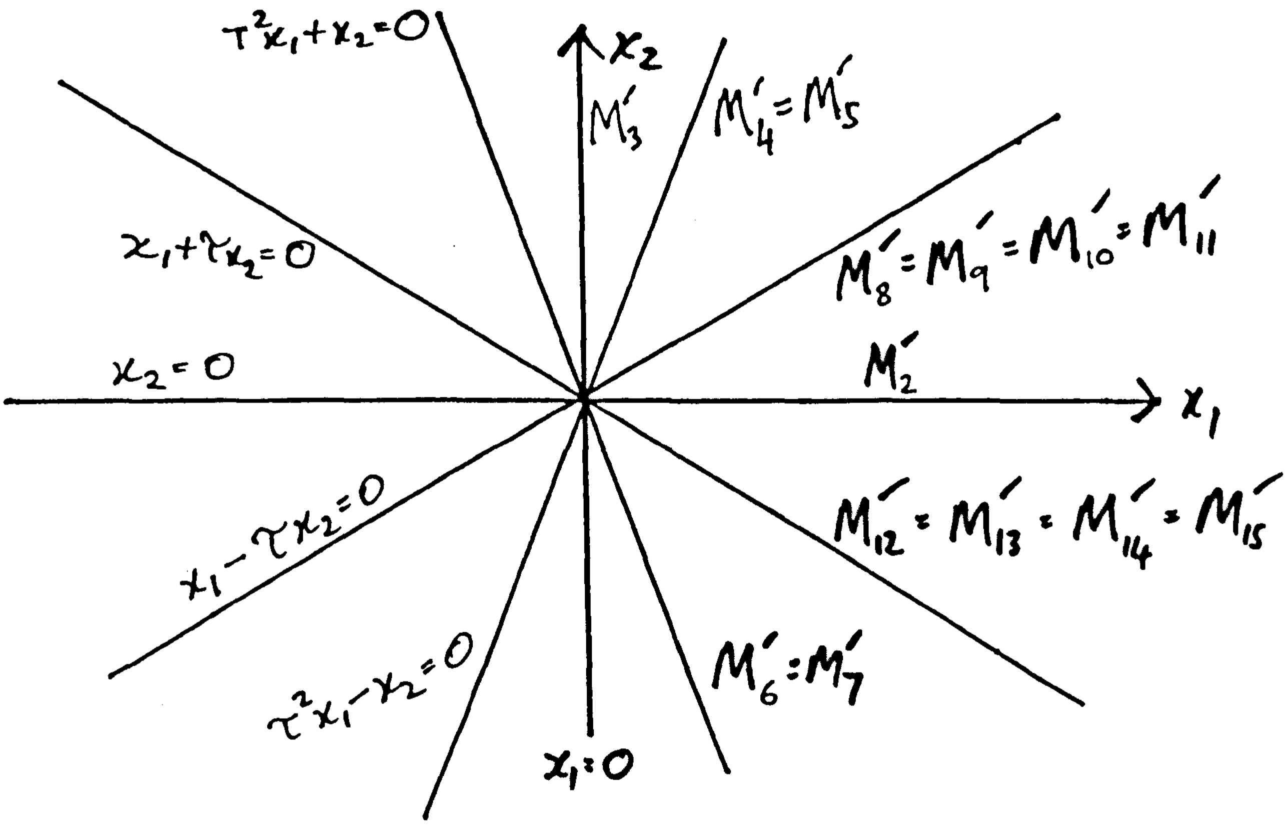
$x_1 = 0$ and $x_2 = 0$ each counted once

Thus, of the fourteen lines $M_2',\ldots\ldots,M_{15}'$, only six are distinct. The group H is $A_1 \times A_1$ generated by reflections in $x_1 = 0$ and $x_2 = 0$ and we can take

$\pi_H(x_1,x_2) = (x_1^2, x_2^2) = (y_1, y_2)$.

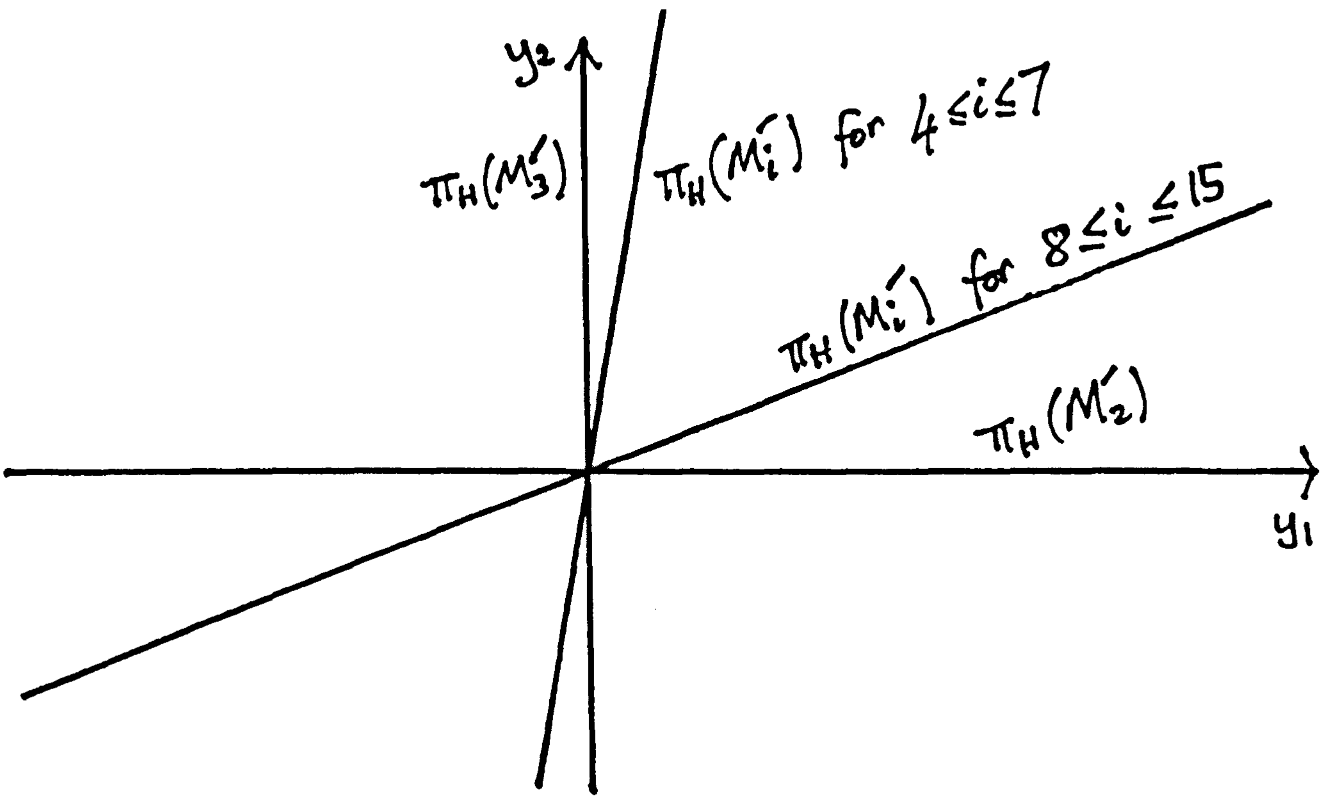
The plane M_1 and the six lines $M_2', \ldots\ldots, M_{15}'$ are shown in figure (1.15).

Figure 1.15 The plane M_1 and the Six Lines $M_2', \ldots\ldots, M_{15}'$



The plane $\pi_H(M_1)$ is shown in figure (1.16). The six lines $M_2', \ldots\ldots, M_{15}'$ are mapped by π_H to four lines. The real part of M_1 is mapped to the first quadrant of the plane $\pi_H(M_1)$.

Figure 1.16 The plane $\pi_H(M_1)$ for the Group H_3



By [18], π_G can be given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} i_2 \\ i_6 \\ i_{10} \end{pmatrix}$$

where

$$\left.\begin{aligned} i_2 &= x_1^2 + x_2^2 + x_3^2 \\ i_6 &= (2\tau - 3)(\tau x_1^2 - \tau^{-1} x_3^2)(\tau x_3^2 - \tau^{-1} x_2^2)(\tau x_2^2 - \tau^{-1} x_1^2) \\ i_{10} &= (2\tau-1)(-x_1^4 - x_2^4 - x_3^4 + 2x_1^2x_2^2 + 2x_1^2x_3^2 + 2x_2^2x_3^2)(\tau^2 x_1^2 - \tau^{-2} x_2^2)(\tau^2 x_2^2 - \tau^{-2} x_3^2)(\tau^2 x_3^2 - \tau^{-2} x_1^2) \\ \tau &= (1 + \sqrt{5})/2 \end{aligned}\right\} \quad (1.17)$$

The factors $(2\tau-3)$ and $(2\tau-1)$ are included to simplify subsequent calculations.

q is given by

$$q:\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} y_1 + y_2 \\ (2\tau-3)\, y_1 y_2(\tau y_2 - \tau^{-1} y_1) \\ (2\tau-1)\, y_1 y_2 (y_1 - y_2)^2(\tau^2 y_2 - \tau^{-2} y_1) \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad (1.18)$$

The discriminant $\Delta(H_3) = q(\mathbb{R}^2)$ is shown in figure (1.19).

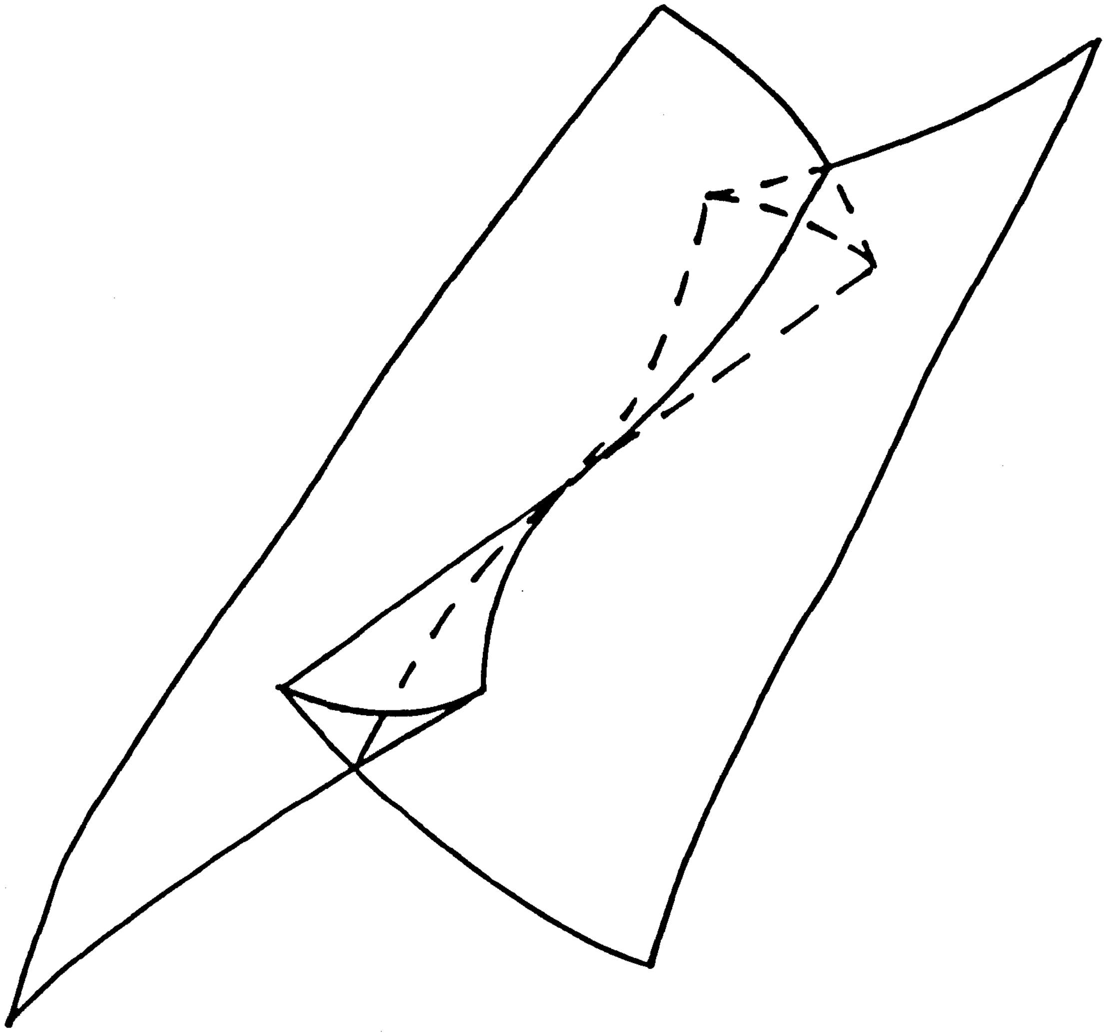


Figure 1.19 The discriminant variety $\Delta(H_3)$

Eliminating the variables y_1, y_2 from (1.18) gives the algebraic equation of the discriminant

$$0 = F(X,Y,Z) = -8640Y^5 + 3440Y^4X^3 + 3600XY^3Z - 455X^6Y^3$$
$$- 795X^4Y^2Z + 20X^9Y^2 - 325X^2YZ^2 + 40X^7YZ - 25Z^3 + 4X^5Z^2$$

where $Z = W\sqrt{5}$. Composing F with the map

$$\phi : \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ -32Y/45 + X^3/5 \\ 205Z/81 - 32YX^2/45 + X^5/25 \end{pmatrix} \qquad (1.20)$$

gives $0 = F(\phi(X,Y,Z)) = \lambda G(X,Y,Z)$ where λ is a non-zero number and

$$0 = G(X,Y,Z) = 7776Y^5 - 2000Z^3 - 16200XY^3Z + 9000X^2YZ^2$$
$$- 2025X^3Y^4 + 4050X^4Y^2Z - 2025X^5Z^2 \qquad (1.21)$$

is the algebraic equation of the tangent developable to the space curve γ given by

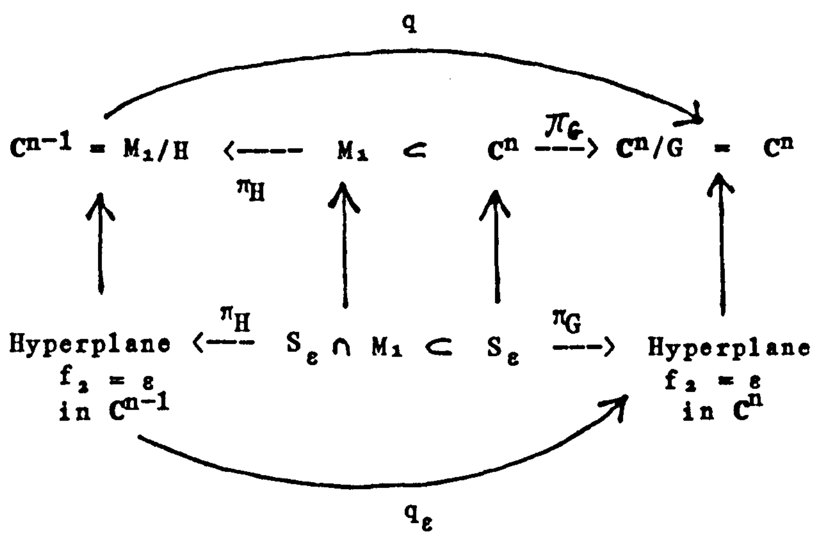
$$s \longrightarrow (X(s),Y(s),Z(s)) = (s, s^3/3, s^5/5)$$

We have thus shown that there is a real algebraic automorphism (given by (1.20)) mapping $\Delta(H_3)$ onto this tangent developable, as stated in [3].

<u>Example 1.22 The Discriminant of H_4</u>

We shall describe this variety by looking at hyperplane sections, for two reasons. Firstly, because it is easier to visualise a family of surfaces in $\mathbb{R}^3$ than a single hypersurface in $\mathbb{R}^4$, and secondly because this is what is needed for the application we have in mind (one-parameter families of surfaces in $\mathbb{R}^3$). The next four paragraphs actually apply to any Coxeter group G.

Let S_ε denote the invariant quadric in $\mathbb{C}^n$ with equation $\varepsilon = f_2(z_1,\dots,z_n) = z_1^2 + \dots + z_n^2$ and let q_ε denote the map defined in the following commutative diagram, in which all the vertical arrows are inclusions.



Let $\Delta(\varepsilon)$ denote the intersection of $\Delta(G)$ with the hyperplane $f_2 = \varepsilon$ in $\mathbb{C}^n$. Then $\Delta(\varepsilon)$ is the real part of the image of q_ε. Furthermore the sections $\Delta(\varepsilon)$ ($\varepsilon \in \mathbb{R}$) are a generic set of sections of $\Delta(G)$. A family of sections of G can be characterized as the level sets of a real-valued function defined on $\Delta(G)$. Almost all smooth functions on $\Delta(G)$ can be expressed as the composition $f_2\,\theta$ of f_2 with a self-diffeomorphism θ of $\mathbb{R}^n$ which preserves the discriminant $\Delta(G)$ (see [1]), and, if a function can be expressed in this way, its level sets will be essentially the same as the sections $\Delta(\varepsilon)$. Therefore, by describing the surfaces

$\Delta(\varepsilon)$, we are describing almost all families of sections of $\Delta(G)$.

The map $x \longrightarrow \lambda x$ ($\lambda \in \mathbb{C}$, $\lambda \neq 0$) commutes with every element of G and so induces an algebraic automorphism ϕ_λ of $\mathbb{C}^n/G$. Then if $\lambda \in \mathbb{R}$, this automorphism maps $\Delta(\varepsilon)$ to $\Delta(\lambda^2\varepsilon)$. The same is true for $\lambda = i$ provided that all the invariants of G have even degree. This condition is necessary to ensure that ϕ_λ maps real points of $\mathbb{C}^n/G$ to real points, and is satisfied by the groups F_4 and H_4. So if $\varepsilon \neq 0$, there is an algebraic automorphism of $\mathbb{R}^{n-1}$ which maps $\Delta(\varepsilon)$ onto $\Delta(1)$.

Let $\Delta_1(\varepsilon) = \Delta(\varepsilon) \cap \pi_G(\mathbb{R}^n)$ and

Let $\Delta_2(\varepsilon) = \Delta(\varepsilon) \cap q(\mathbb{R}^n)$

Then $\Delta_1(\varepsilon) \subset \Delta_2(\varepsilon) \subset \Delta(\varepsilon)$

The group H_4 is the symmetry group of the regular polytope $\{3,3,5\}$ in $\mathbb{R}^4$ which has 120 vertices and 600 tetrahedral cells ([10] §8·5). The dual polytope $\{5,3,3\}$ with 600 vertices and 120 dodecahedral cells also has H_4 as symmetry group.

A fundamental region for the group H_4 meets the hypersphere S_1 in a curvilinear tetrahedron T. This solid tetrahedron is mapped homeomorphically onto its image by π_G, and the real parts of the mirrors are mapped on to the boundary of the tetrahedron $\pi_G(T)$. Thus $\Delta_1(1)$ is homeomorphic to the boundary of a tetrahedron.

The tetrahedron T is shown (after projection and application of a suitable homeomorphism) in figure (1.23). The stabilisers

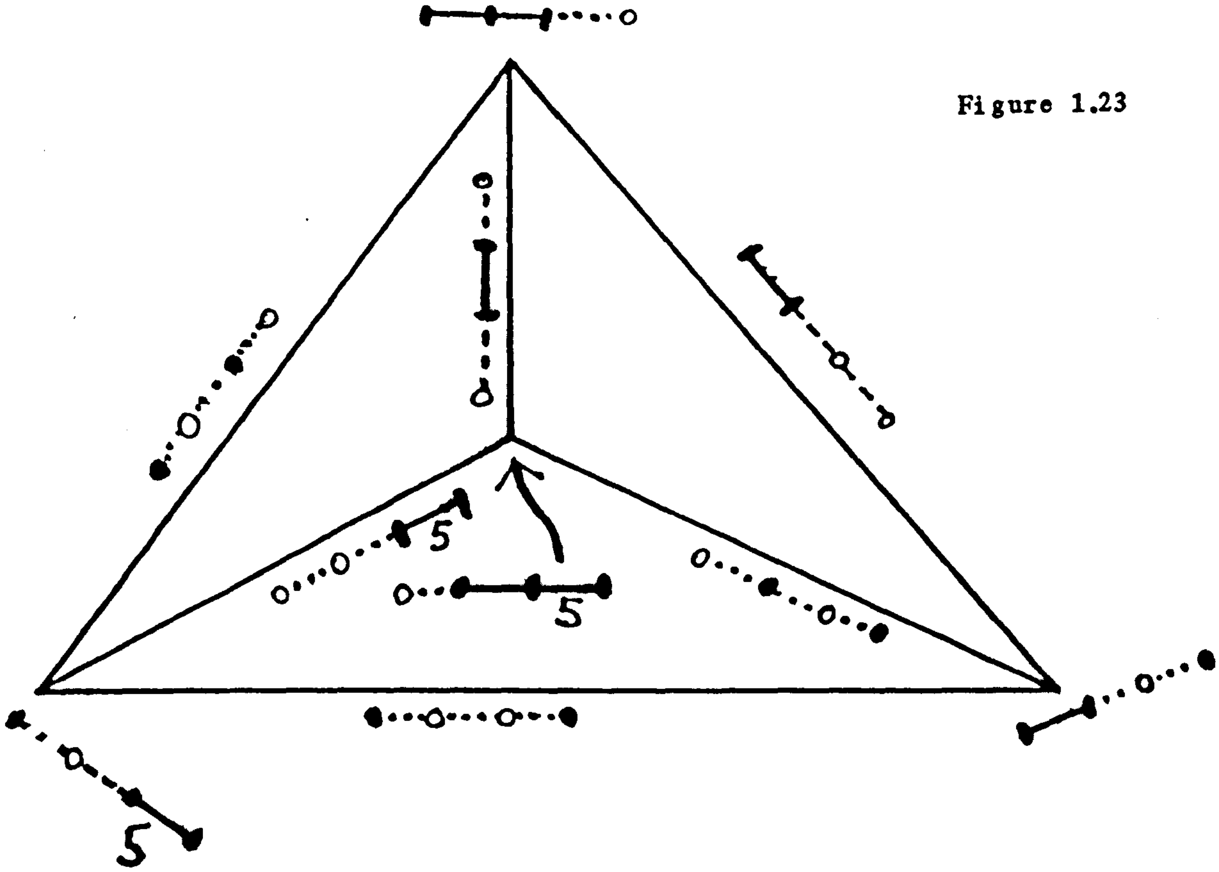


Figure 1.23

swallowtail

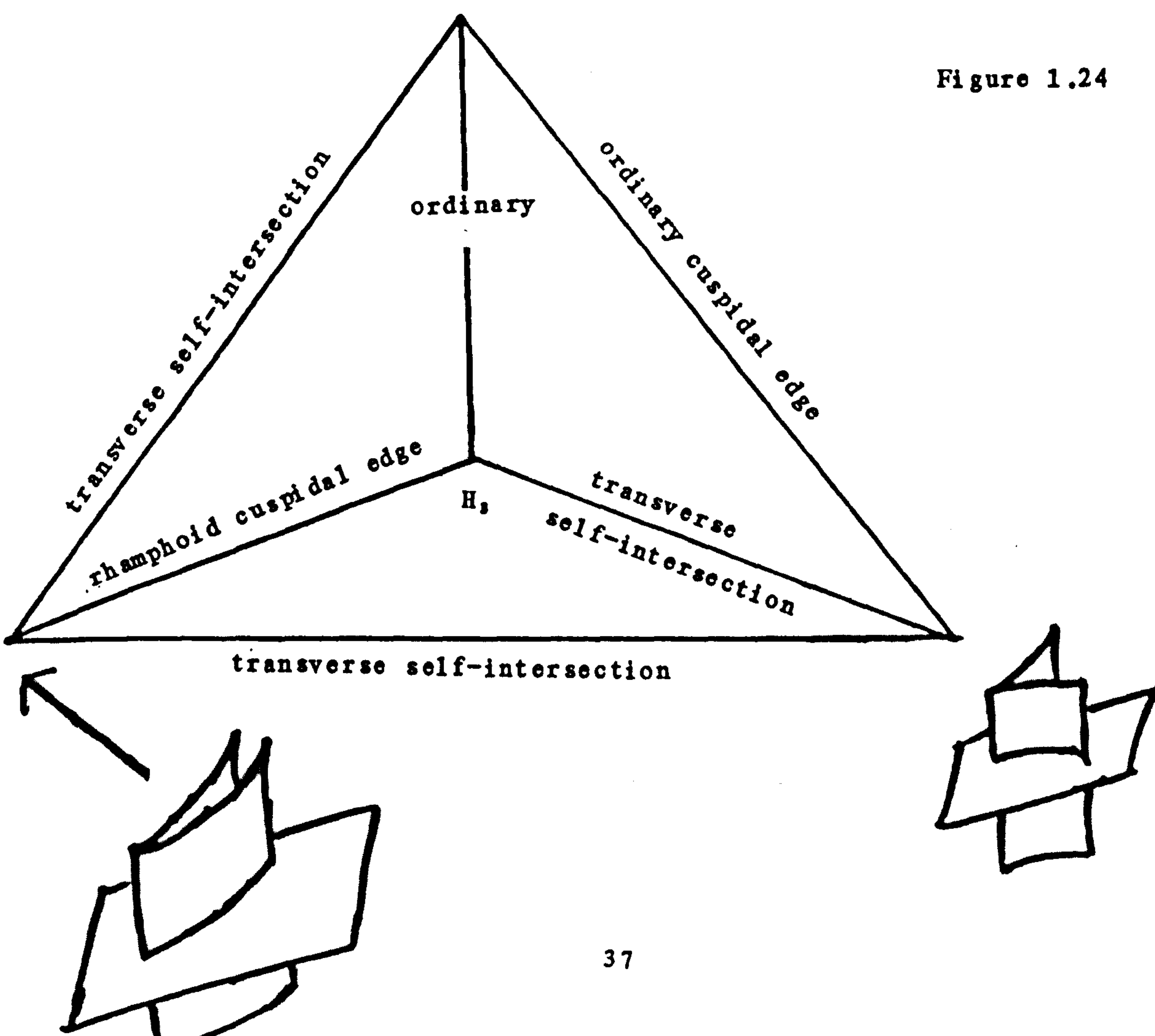


Figure 1.24

of vertices and points on the edges of T are shown in this drawing (by (1.3) the stabiliser of any other point is A_1). The differentiable structure of $\Delta_1(1)$ can now be deduced by applying (1.4): see figure (1.24).

The next stage in this discussion is to describe $\Delta_2(1)$. We can take ([10] §8.7) a co-ordinate system in which the 120 vertices of the polytope {3,3,5} are

$$(\pm 1\ 0\ 0\ 0)\quad (0\ \pm 1\ 0\ 0)\quad (0\ 0\ \pm 1\ 0)\quad (0\ 0\ 0\ \pm 1)$$

$$(\pm 1/2\ \pm 1/2\ \pm 1/2\ \pm 1/2)\quad (\pm\tau/2\ \ \pm 1/2\ \pm\tau^{-1}/2\ \ 0) \qquad (1.25)$$

(take all combinations of signs and all even permutations of the co-ordinates)

and the mirrors of H_4 are the sixty hyperplanes of the form

$$\{x \mid x.v = 0\}$$

where v is one of the vectors listed in (1.25).

Let $\pi_G(x) = \begin{pmatrix} p_2(x) \\ p_{12}(x) \\ p_{20}(x) \\ p_{30}(x) \end{pmatrix}$ where p_i is homogeneous of degree i and $p_2(x) = x_1^2+\ldots+x_4^2$.

Illustrations on Page 37

Figure 1.23 The tetrahedron T with the stabilisers of vertices and of points on the edges.

Figure 1.24 The differentiable structure of $\Delta_1(1) = \pi_G(T)$

Let M_1 be the hyperplane $x_4 = 0$. The polytope $\{3,3,5\}$ intersects M_1 in an icosidodecahedron: a semiregular polytope with symmetry group H_3 formed by the intersection of a regular icosahedron and a regular dodecahedron (**). The group H has order 120 and preserves this icosidodecahedron, so must be H_3.

Let Ω be an icosahedron lying in M_1 with symmetry group H and let Ψ be a dodecahedron lying in M_1 with symmetry group H.

By direct calculation, the 59 planes $M'_2,\ldots,M'_{60}$ are:

(A) The 15 planes $\{ x \in M_1 \mid x.v=0 \}$ where v is the position vector of the midpoint of an edge of the icosahedron Ω (or equivalently where v is the position vector of the midpoint of an edge of the dodecahedron Ψ). These are the 15 mirrors of H.

(B) The 10 planes $\{ x \in M_1 \mid x.v = 0 \}$ where v is the position vector of a vertex of the dodecahedron Ψ.

(C) The 6 planes $\{ x \in M_1 \mid x.v = 0 \}$ where v is the position vector of a vertex of the icosahedron Ω.

This list seems to consist of 31 planes rather than 59. But in the enumeration of planes $M'_2,\ldots,M'_{60}$, each plane in the set (A) occurs once, each plane in (B) twice and each plane in (C) four times. So taking multiplicities into account, the total number of planes becomes 59.

The quotient map π_H can be given [18] by equations (1.17). The images of the planes $M'_2,\ldots,M'_{60}$ under π_H are:

** See [10] § 2.3 and [10] Table $\overline{V}$(iii).

(A) The discriminant surface $\Delta\,(H_3)$ of the group $H = H_3$ shown in figure (1.19)

(B) The plane $i_{10}=0$ (1.26)

(C) The plane $i_6=0$

The plane $i_2=1$ in M_1/H is mapped by q to $\Delta(1)$ and the positions of the points (A) (B) and (C) of (1.26) are shown in figure (1.27). The map q can be thought of as a set of instructions on how to fold up an (elastic) sheet of paper to form the surface $\Delta_2(1)$.

The behaviour of the surface $\Delta_2(1)$ along the singular curves (A) (B) and (C) can be deduced by numerous applications of (1.4) (see figure 1.28). For example, $\Delta_2(1)$ has a curve of transverse self-intersections along q(A), an ordinary cuspidal edge along q(B) and a rhamphoid cuspidal edge along q(C).

Here an ordinary cuspidal edge is one which is locally diffeomorphic to the surface $x^2=y^3$ and a rhamphoid cuspidal edge is one which is locally diffeomorphic to $x^2=y^5$ (see (3.38) and (3.4)).

This completes the description of $\Delta_2(1)$ and we now describe

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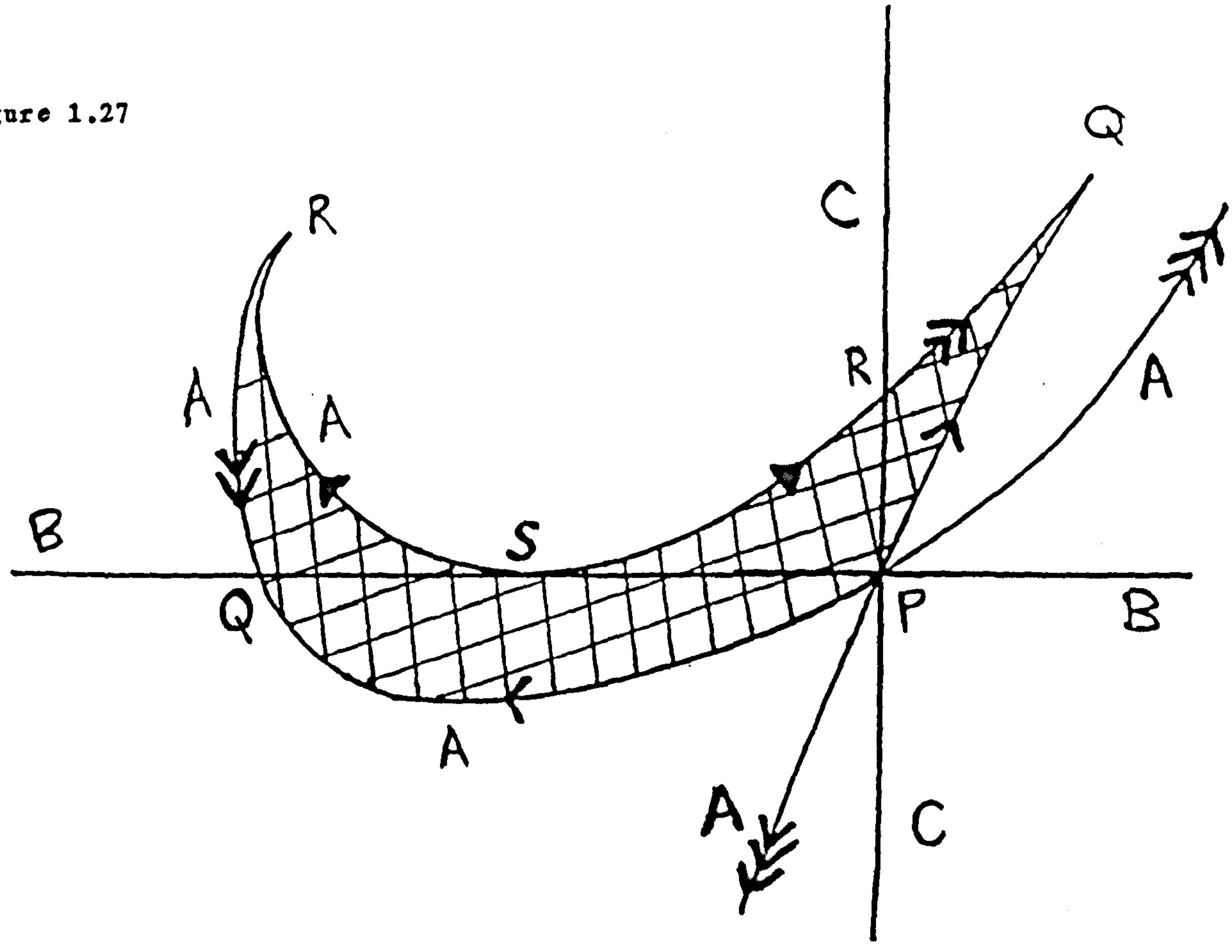
Illustrations on Page 41

Figure 1.27 The plane $i_2=1$ in M_1/H.
This plane is mapped by q_1 to the surface $\Delta_2(1)$.
The tetrahedron $\Delta_1(1)$ is the image of the shaded area and the portions of the curve (A) must be identified as shown.

Figure 1.28 Critical points of the map q_1.

In figure 1·27, (A), (B) and (C) denote the sets (A), (B) and (C) described at the top of page 40.

Figure 1.27



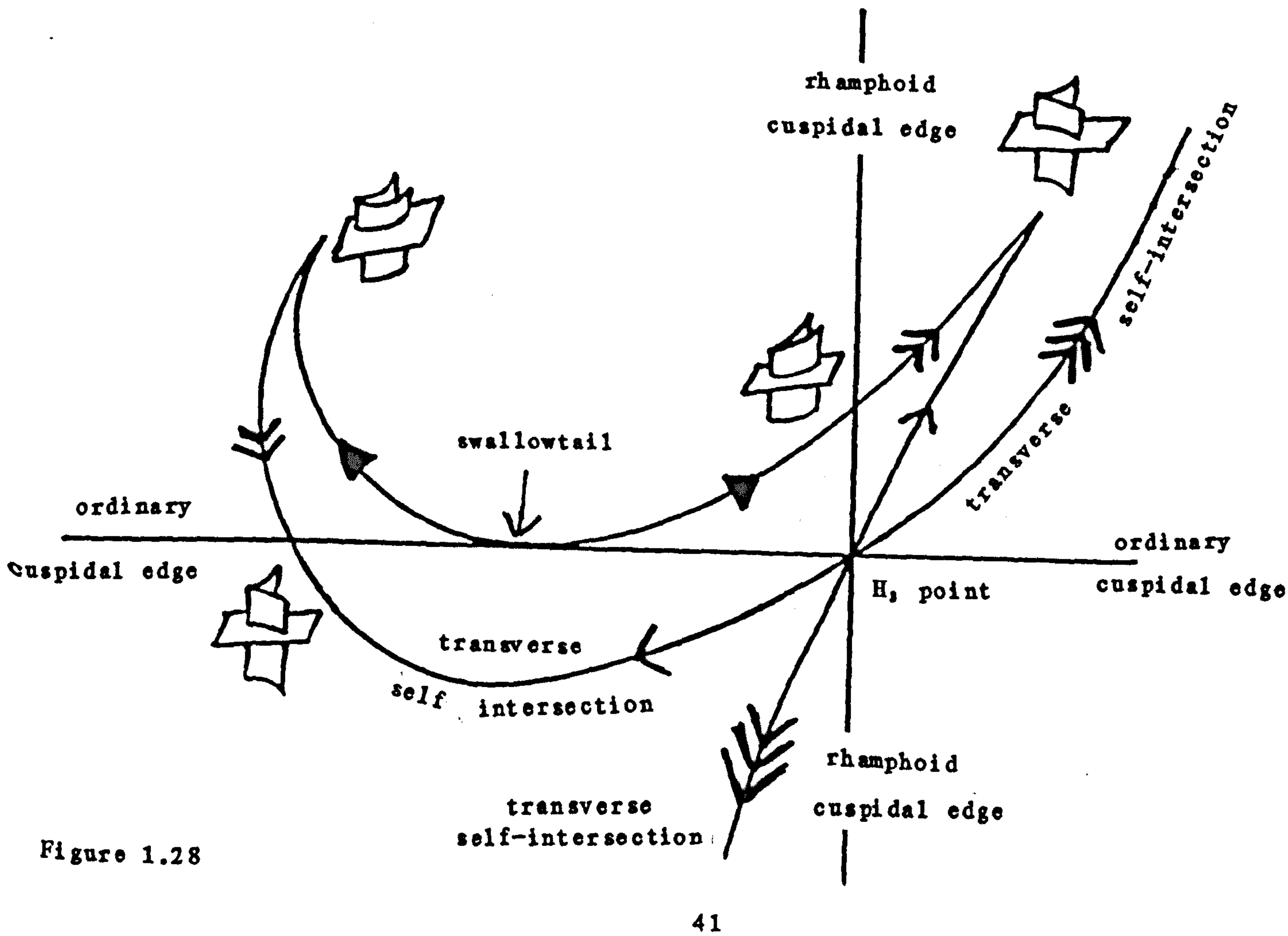


Figure 1.28

$\Delta(1)-\Delta_2(1)$. Suppose $\pi_G(x) \in \Delta(1)-\Delta_2(1)$, then by (1.4) and (1.10), $Fix_G(x)$ is either $A_1 \times A_1$ or $I_2(2m)$ $(m \geq 2)$ or a group of rank at least 3. The second possibility is ruled out because the Coxeter Dynkin diagram for $I_2(2m)$ is not a subgraph of that for H_4, and the third because if $Fix_G(x)$ has rank 3, then x lies on the complexification of one of the 4 edges of a fundamental region D for the action of H_4 on C^4. But each of these edges meet S_1 in 2 real antipodal points, which must be x and −x and so $\pi_G(x) \in \Delta_1(1)$. This is a contradiction.

Thus $Fix_G(x) = A_1 \times A_1$ and so x lies on one of the 15 mirrors of H_4 of type (A). Thus $\pi_H(x)$ lies on the rational curve $\pi_H(A)$ formed by the section $i_2 = 1$ of $\Delta(H_3)$. Hence $\pi_G(A) = q(\pi_H(A))$ is also rational.

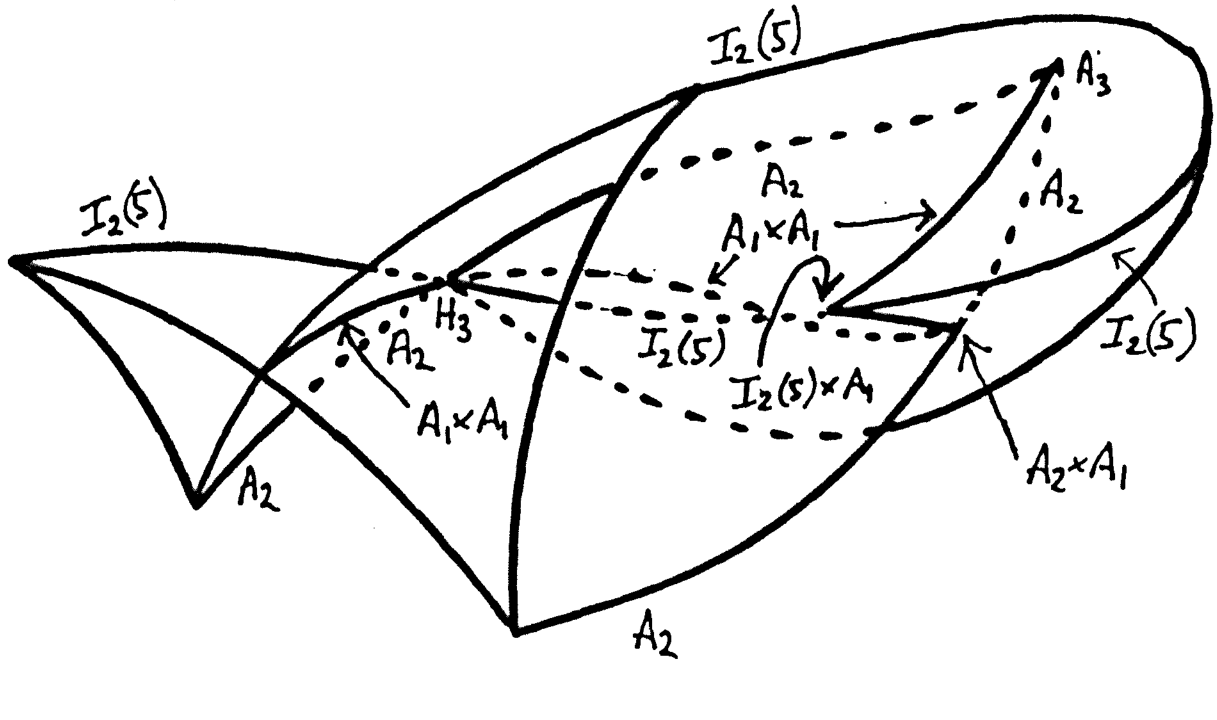


Figure (1.29) The surface $\Delta_2(1)$

For example, suppose $c = 0$. Then for arbitrary values of i_6 and i_{10}, the points (i_6, i_{10}) and $(i_6, -i_{10})$ will have the same image under q_0. This contradicts the fact that q_0 is injective almost everywhere.

The image under q of the real part of $\pi_H(A)$ is a portion of the curve coming from infinity, passing through cusps at q(Q) and q(R) (see figure (1.27)) and terminating at the swallowtail q(S). Consequently in $\Delta(1)-\Delta_2(1)$ this curve must continue smoothly away to infinity, as the real line of intersection of 2 complex conjugate sheets of the discriminant. This curve completes our picture of the section $\Delta(1)$ of $\Delta(H_4)$ (see figure 1.29).

Finally we describe $\Delta(0)$. By considering the degrees of the polynomials i_6, i_{10}, p_{12}, p_{20}, p_{30}, the restriction q_0 of the map q to the plane $i_2 = 0$ must be of the form

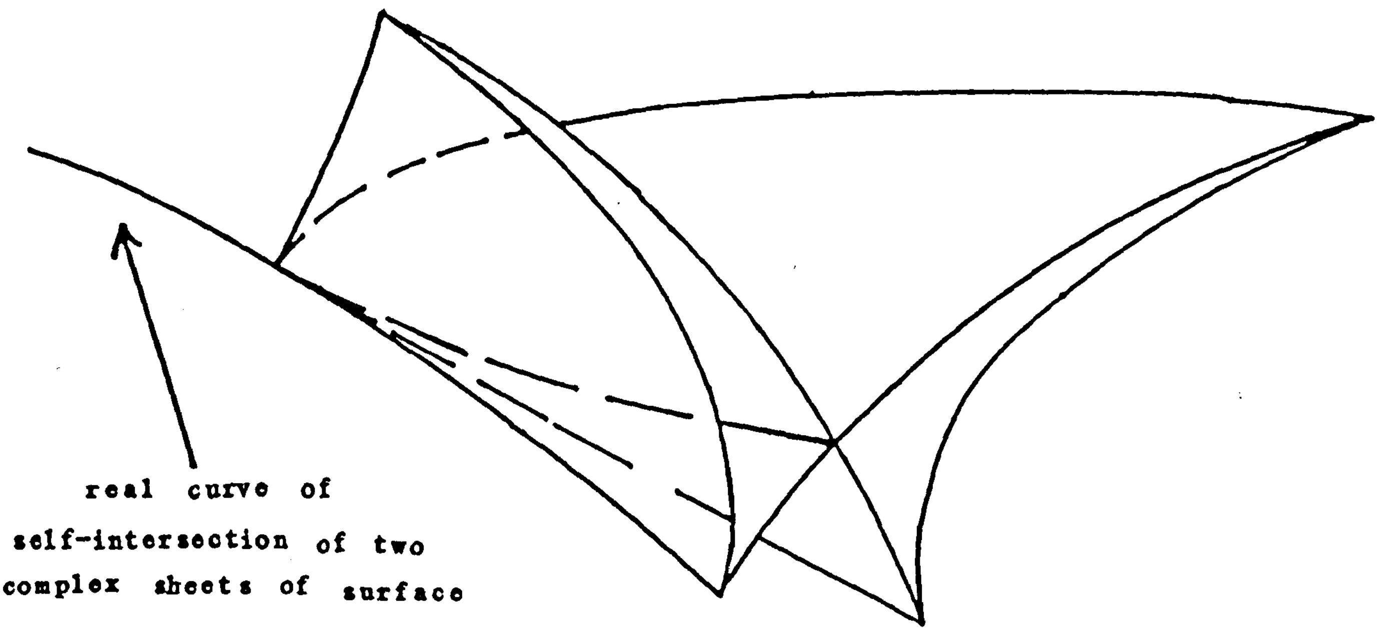
$$q_0 : (i_6, i_{10}) \longrightarrow (a i_6^2, b i_{10}^2, c i_{10}^3 + d i_6^5)$$

Since (by (1.7)) this map must be injective almost everywhere, the constants a, b, c, and d are all non-zero. By making linear changes of co-ordinates in source and target, we may arrange that a=b=c=1 and d=-1. Thus $\Delta(0)$ has the parametrisation

$$(u,v) \dashrightarrow (u^2, v^2, u^3 - v^5) \qquad (1.30)$$

This surface is shown in figure (1.31). In figure (1.32) the plane $i_2=0$ is shown with the critical points of q_0 indicated. This figure clearly shows how $\Delta(0)$ can be obtained from $\Delta(\varepsilon)$ for $\varepsilon \neq 0$ by letting the tetrahedron $\Delta_2(\varepsilon)$ shrink to zero size.

Figure 1.31 The Section Δ(0)



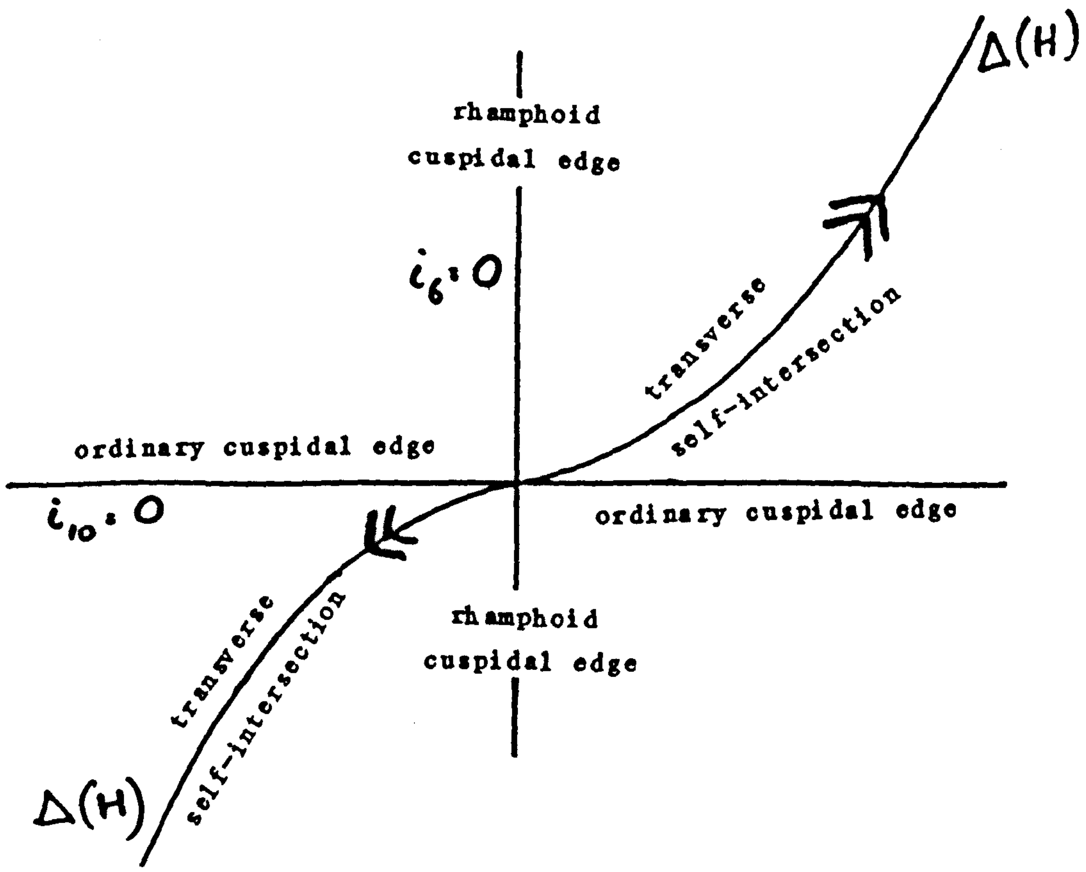


Figure 1.32 The plane $i_2 = 0$ with the Critical Points of qe

The results of the above work can be summarized as follows:

<u>Proposition 1.33</u>

(i) $\Delta(H_4)$ is an irreducible rational algebraic hypersurface in $\mathbb{R}^4$.

(ii) The section $\Delta(0)$ is given by equation (1.30) and illustrated in figure (1.31).

(iii) Any section $\Delta(\varepsilon)$ for $\varepsilon \neq 0$ is diffeomorphic to $\Delta(1)$. The surface $\Delta(1)$ is shown in figure (1.29). It has a parametrization q for which the critical points are as shown in figure (1.28).

1.4 Calculation of Invariant Polynomials

Explicit formulae for a basic set of invariant polynomials are given in [15] for the four infinite series, in [18] for the group H_3, and in [11] for E_6.

The purpose of this section is to obtain explicit formulae for a basic set of invariants for H_4. During this calculation, we discover some facts about the structure of the group H_4, which we believe are of interest in their own right.

The invariants of H_4 are polynomials in four variables of degrees 2,12,20 and 30. Because of the large number of terms involved, a computer had to be used. To minimize any possibility of error in the computer programs, the calculation was performed by two different methods. The same answer was obtained by both methods. In order to find the basic invariants of H_4, we first need to find those of F_4. This is done using the same two methods.

These two methods can be summarized as

1) averaging over all elements of the group and
2) using two subgroups that generate G

Let $\langle , \rangle$ denote the usual inner product on $\mathbb{R}^n$, and let

$$p^y{}_i(x) = \sum_{\sigma \in H_4} \langle x , \sigma(y) \rangle^i = \sum_{\sigma \in H_4} \langle \sigma(x) , y \rangle^i$$

As polynomials in x, the p^y_i are clearly invariant. Let $d(1),\ldots,d(n)$ be the degrees of the basic invariants of G.

Proposition 1.34

(i) For almost all $y \in \mathbb{R}^n$, the polynomials $p^y{}_{d(1)},\ldots,p^y{}_{d(n)}$ are algebraically independent, and hence form a set of basic invariants for G.

(ii) The group F_4 with order $1152 = 2^7.3^2$ is the symmetry group of the self-dual regular polytope {3,4,3} in $\mathbb{R}^4$ (see [10]).

For $G = F_4$, y may be chosen to be any of the following points:

1) a vertex of {3,4,3}

2) the mid-point of {3,4,3}

3) the centroid of a triangular face of {3,4,3}

4) the centroid of an octahedral cell of {3,4,3}

(iii) For $G = H_4$, y may be chosen to be either

1) a vertex of {3,3,5}

or 2) a vertex of {5,3,3}

<u>Proof</u>

The result (i) is due to Flatto and Weiner. In [13], they show that $p^y{}_{d(1)},\ldots,p^y{}_{d(n)}$ are algebraically independent provided that y does not lie on a certain algebraic subvariety of $\mathbb{R}^n$. To verify the suitability of the particular choices of y given in (ii) and (iii), we calculate the Jacobian

$$\frac{\partial(p^y{}_{d(1)},\ldots\ldots p^y{}_{d(n)})}{\partial(x_1,\ldots\ldots\ldots x_n)}$$

This is a lengthy calculation which will not be reproduced here. QED.

The second method of finding the invariants is to choose subgroups K_1 and K_2 such that $G = Gp\langle K_1,K_2\rangle$. Then a polynomial will be invariant under G if and only if it is invariant under the action of both K_1 and K_2. Therefore $R(G) = R(K_1) \cap R(K_2)$. If the invariants of K_1 and K_2 are known, we can calculate those of G by finding the intersection of the two known rings. The aim is to choose K_1 and K_2 as large as possible, so that the rings

$R(K_1)$ and $R(K_2)$ will be fairly small.

We can choose (see [10]) a rectangular Cartesian co-ordinate system for $\mathbb{R}^4$ in which the 24 vertices of {3,4,3} are:

$$(\pm 1\ 0\ 0\ 0)\quad (0\ \pm 1\ 0\ 0)\quad (0\ 0\ \pm 1\ 0)$$

$$(0\ 0\ 0\ \pm 1)\quad (\pm 1/2\ \pm 1/2\ \pm 1/2\ \pm 1/2) \qquad (1.35)$$

(take any combinations of signs)

We can partition the 24 vertices of {3,4,3} into three sets of eight in such a way that each set of eight forms the vertices of a cross-polytope $\{3,3,4\} = \beta_4$ ([10] §8.2). The symmetry group of each β_4 will be a copy of B_4 which forms a subgroup of index 3 in F_4. Thus F_4 contains three copies of B_4. Let K_1 be the symmetry group of the β_4 with vertices

$$(\pm 1\ 0\ 0\ 0)\quad (0\ \pm 1\ 0\ 0)\quad (0\ 0\ \pm 1\ 0)\quad (0\ 0\ 0\ \pm 1) \qquad (1.36)$$

and let K_2 be the symmetry group of the β_4 whose vertices have co-ordinates

$$(\pm 1/2\ \pm 1/2\ \pm 1/2\ \pm 1/2) \qquad (1.37)$$

(take an even number of + signs and an even number of − signs)

Then $R(K_1) = C[d_2,d_4,d_6,d_8]$, where $d_{2i} = \sigma_i(x_1^2,\dots,x_4^2)$ and σ_i are the elementary symmetric polynomials.

The ring $R(K_2)$ is $C[e_2,e_4,e_6,e_8]$, where

$$e_2 = d_2 = x_1^2 + \dots + x_4^2$$

$$e_4 = d_4 + 6s_4 \qquad (1.38)$$

$$e_6 = d_6 + d_2 s_4$$

$$e_8 = 8d_8 + 4d_4 s_4 - d_2^2 s_4$$

$$s_4 = x_1 x_2 x_3 x_4$$

Since $|G:K_1| = 3$ and $K_1 \neq K_2$, we have $G = \mathrm{Gp}\langle K_1,K_2\rangle$.

To find a basic invariant f_6 of degree 6 for F_4, it is sufficient to find λ_i such that

$$f_6 = \lambda_1.d_6 + \lambda_2.d_4d_2 + \lambda_3.d_2^3 = \lambda_4.e_6 + \lambda_5.e_4e_2 + \lambda_6.e_2^3$$

This is done by substituting suitable values for $(x_1,\dots,x_4)$ to give simultaneous linear equations for the λ_i. A similar procedure will give basic invariants f_8 and f_{12} having degrees 8 and 12.

Proposition 1.39

The following polynomials form a set of basic invariants for F_4.

$$f_2 = e_2 = d_2$$

$$f_6 = d_2^3 + 48d_6 - 8d_4d_2 = 48e_6 - 8e_4e_2 + e_2^3$$

$$f_8 = 3d_2^4 + 160d_4^2 + 1920d_8 - 40d_4d_2^2 - 240d_6d_2$$
$$= -480e_8 - 240e_6e_2 + 160e_4^2 - 40e_4e_2^2 + 3e_2^4$$

$$f_{12} = d_2{}^6 - 28d_4d_2^4 + 240d_4^2d_2^2 - 640d_4^3 - 192d_6d_2^3 + 1440d_6d_4d_2$$
$$- 4320d_6^2 - 7200d_8d_2^2 + 23040d_8d_4$$
$$= 2880e_8e_4 - 360e_8e_2^2 - 4320e_6^2 + 1440e_6e_4e_2 - 192e_6e_2^3$$
$$- 640e_4^3 + 240e_4^2e_2^2 - 28e_4e_2^2 + e_2^6.$$

Proof

By solving the simultaneous linear equations described above using a computer. QED.

Now we repeat this procedure for H_4, which we recall from §(1.3) is the symmetry group of the regular polytope {3,3,5} or of its reciprocal {5,3,3}.

We can choose ([10] §8.5) 24 of the 120 vertices of {3,3,5} in such a way that they form the vertices of an inscribed polytope {3,4,3}. The intersection of the symmetry groups of these two polytopes is a subgroup of index 25 in H_4 and index 2 in F_4 which will be denoted by hF_4. This stands for half of F_4. Although it is of no significance for the remainder of the argument, we note here that there are 25 different subsets of the vertices of a given {3,3,5} which form an inscribed {3,4,3}.

For computational purposes let F_4 be the symmetry group of the polytope {3,4,3} whose 24 vertices are given by (1.35).

$$(\pm 1\ 0\ 0\ 0)\quad (0\ \pm 1\ 0\ 0)\quad (0\ 0\ \pm 1\ 0)$$
$$(0\ 0\ 0\ \pm 1)\quad (\pm 1/2\ \pm 1/2\ \pm 1/2\ \pm 1/2) \qquad (1.35)$$

(take any combinations of signs)

Let the remaining 96 vertices of {3,3,5} be

$$(\pm\tau/2\quad \pm 1/2\quad \pm\tau^{-1}/2\quad 0) \qquad (1.40)$$

(take all combinations of signs and all even permutations of the co-ordinates)

Let $K_1 = hF_4$ and let K_2 be the stabiliser of the vertex (0,0,0,1). By (1.3), $K_2 = H_3$.

By proposition 4.7 of [32], the ring of invariants of hF_4 is
$R(K_1) = (1+j_{12})C[f_2,f_6,f_8,f_{12}]$
where j_{12} is the product of 12 linear factors vanishing on those mirrors of F_4 which are not in hF_4, and the f_i are the invariants of F_4 given in (1.39).

By [18] $R(K_2) = C[i_2,i_6,i_{10},x_4]$ where

$$i_2 = x_1^2 + x_2^2 + x_3^2$$

$$i_6 = (2\tau - 3)(\tau x_1^2 - \tau^{-1}x_3^2)(\tau x_3^2 - \tau^{-1}x_2^2)(\tau x_2^2 - \tau^{-1}x_1^2) \qquad (1.41)$$

$$i_{10} = (2\tau-1)(-x_1^4-x_2^4-x_3^4+2x_1^2x_2^2+2x_1^2x_3^2+2x_2^2x_3^2)(\tau^2x_1^2-\tau^{-2}x_2^2)(\tau^2x_2^2-\tau^{-2}x_3^2)(\tau^2x_3^2-\tau^{-2}x_1^2)$$

$$j_{12} = (2\tau-1)\prod_{i<j}(x_i^2 - x_j^2)$$

$$j_{12}^2 \in C[f_2, f_6, f_8, f_{12}]$$

$$\tau = (1 + \sqrt{5})/2$$

Provided that K_1 and K_2 generate H_4, the procedure used for F_4 will give the invariants of H_4. Before proving that $H_4 = Gp\langle K_1, K_2\rangle$, we state the following result.

Proposition 1.42

The polynomials $p_2, p_{12}, p_{20}, p_{30}$ given in Appendix 1 form a basic set of invariants for H_4.

Proof Similar to the proof of (1.39).

We now give the proof that K_1 and K_2 generate H_4.

Proposition 1.43 Let $H = Gp\langle K_1, K_2\rangle$. Then $H = H_4$.

Proof

$LCM(|K_1|, |K_2|) = LCM(576, 120) = 2880$ so either $H = H_4$ or $|H_4 : H| = 5$. So it is sufficient to prove that H_4 has no subgroup of index 5.

Let G° denote the set of linear transformations of determinant +1 in G. Then $G^\circ \triangleleft G$ with $|G:G^\circ| = 2$.
Let π_n : Spin(n)—> SO(n) be the standard double covering map and let $\tilde{G}^\circ = \pi_n^{-1}(G^\circ)$.

We will now identify all normal subgroups of H_3° $\tilde{H}_3^\circ$ $\tilde{H}_4^\circ$ H_4° and H_4.

The group $\tilde{H}_3^0$ is the well known simple group of order 60. If $N \lhd \tilde{H}_3^0$ then $\pi_3(N) \lhd H_3^0$ so it follows that N is either 1 or $\tilde{H}_3^0$ or the group $\pi_3^{-1}(1)$ of order 2.

We now need the following result.

<u>**Proposition 1.44**</u> $\tilde{H}_4^0 = \tilde{H}_3^0 \times \tilde{H}_3^0$

<u>**Proof**</u>

If q is a quaternion then the <u>**real part**</u> of q is Re q = $1/2(q+\bar{q})$ and the <u>**pure part**</u> of q is Pu q = $1/2(q-\bar{q})$. We identify $\mathbb{R}^4$ with the space of quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ where the summand $\mathbb{R}$ is the reals and the summand $\mathbb{R}^3$ is the set of pure quaternions. The group Spin(3) = S^3 is the set of quaternions of modulus 1, and the double covering π_3: Spin(3) —> SO(3) is given by

$$q \longrightarrow \rho(q) \mid \mathbb{R}^3$$

where $\rho(q)(x) = qx\bar{q} = qxq^{-1}$.

In fact the map $\rho(q)$ is an element of SO(4) fixing the plane of 1 and pu q and rotating the perpendicular plane through an angle of $2 \cos^{-1}$ (Re q).**

Let $I \subset S^3 \subset \mathbb{R}^4$ be the set of 120 vertices of {3,3,5} in the position given by (1.35) and (1.40). We claim that I is a group under quaternion multiplication . By identifying $\mathbb{R}^4$ with $\mathbb{H}$ we may consider $\pi_3(I)$. For each vertex z of {3,3,5}, we can

** This can be seen by expressing q as the product of two pure quaternions $q = p_1p_2$ in such a way that $|p_i| = 1$ and p_i is perpendicular to pu q for each i (this is possible by [25] proposition 10.17). Now $\rho(q) = \rho(p_1)\rho(p_2)$ and for a pure quaterion p the map $(-\rho(p))$ acts on the space of pure quaternions $\mathbb{R}^3$ as a reflection (see [25] proposition 10.21). Thus $\rho(q) \mid \mathbb{R}^3$ has been expressed as the product of two reflections.

check that $\rho(z)$ permutes the vertices of $\{3,3,5\}$. Thus $\rho(z) \in H_4^\circ$ and, since $\rho(z)(1)=1$ for all $z \in S^3$, $\rho(z) \in \mathrm{Fix}_{H_4}(1)$.

But $|\mathrm{Fix}_{H_4^\circ}(1)| = 60 = |\pi_3(I)| = |H_3^\circ|$
Thus $\pi_3(I) = H_3^\circ$ and $I = \tilde{H}_3^\circ$ is indeed a subgroup of Spin(3).

Now $\mathrm{Spin}(4) = S^3 \times S^3$ where $(q,r) \in \mathrm{Spin}(4)$ acts on the quaternion $x \in \mathbb{R}^4$ by quaternion multiplication according to the formula

$$\pi_4\,(q,r)(x) = qx\bar{r}$$

Consider the subgroup $I \times I$ of Spin(4). Since it permutes the 120 vertices of $\{3,3,5\}$, it must be a subgroup of $\tilde{H}_4^\circ$.
But $|I \times I| = 14400 = |\tilde{H}_4^\circ|$. Thus $\tilde{H}_4^\circ = I \times I = \tilde{H}_3^\circ \times \tilde{H}_3^\circ$.

This completes the proof of (1.44). We now continue with the enumeration of normal subgroups. Suppose that $N \lhd \tilde{H}_4^\circ$, then N must be $N_1 \times N_2$ for some $N_i \lhd \tilde{H}_3^\circ$. In other words each N_i is either 1 or $\pi_3^{-1}(1)$ or $\tilde{H}_3^\circ$.

Now suppose that $N \lhd H_4^\circ$. Then $\pi_4^{-1}(N) \lhd \tilde{H}_4^\circ$ so there are three non-trivial cases

(1) $N=Z(H_4)$ the centre of H_4 which has order 2.

(2) $N=I_L$ a copy of I acting by multiplication on the left.

(3) $N=I_R$ a copy of I acting by multiplication on the right.

Finally suppose $N \lhd H_4$. Then either $N \lhd H_4^\circ$ or N has a subgroup N° of index 2 with $N^\circ \lhd H_4^\circ$ and $N^\circ \lhd H_4$.

We rule out any involvement of the groups I_L and I_R by noting that they are conjugate in H_4. If c is the map $q \longrightarrow \bar{q}$, then $cI_Lc^{-1} = I_R$. Thus neither I_L nor I_R is a normal subgroup of H_4, and so the only nontrivial normal subgroups of H_4 are as follows

$Z(H_4)$ (with order 2) and H_4^o (with order 7200 and index two).

We are now ready to complete the proof of (1.43). Let $N < H_4$ with $k = |H_4:N| \geqslant 3$. Let S be the set of right cosets of N in H_4. Then H_4 acts transitively on S by $(Ng)g' = Ngg'$. This action gives a homomorphism σ from H_4 into the symmetric group Sym(k). The kernel of σ must be a normal subgroup of H_4, and (since the action is transitive) the kernel is either 1 or $Z(H_4)$. But

$14400 = |H_4| = |\mathrm{Im}\ \sigma|\ |\mathrm{Ker}\ \sigma| \leqslant 2|\mathrm{Im}\ \sigma| \leqslant 2|\mathrm{Sym}(k)| = 2(k!)$

So $k \geqslant 8$.

We have thus proved the following proposition.

<u>Proposition 1.45</u>

H_4 has no subgroups of index less than eight apart from H_4^o. The only non-trivial normal subgroups of H_4 are H_4^o and $\pi_4^{-1}(1) = Z(H_4)$.

This of course completes the proof of (1.43). We might hope that an invariant polynomial f contains some geometric information. For example, the algebraic variety f=0 might have some geometric significance. The group theoretic information above gives the following result.

<u>Proposition 1.46</u>

Let p_2 p_{12} p_{20} p_{30} be any set of basic invariants for H_4 and let f_2 f_6 f_8 f_{12} be any basic set of invariants for F_4. Then each of the polynomials f_i and p_i is irreducible over $\mathbb{C}$.

<u>Proof</u>

This is a case-by-case argument. Since the discussion is similar in each case, we only give details for p_{30}. Suppose that p_{30} is reducible. It must then factorise into k irreducible factors, each having degree j, such that jk=30 and such that H_4 permutes these k factors.

The k factors are the equations of d distinct irreducible surfaces in CP^3, where d is a factor of k and H_4 permutes these d surfaces. We thus have homomorphisms

$$\sigma_1:H_4 \dashrightarrow Sym(k) \text{ and } \sigma_2:H_4 \dashrightarrow Sym(d).$$

As in the proof of (1.43), we must have $k\leq 2$ or $k\geq 8$ and $d\leq 2$ or $d\geq 8$

The only possibilities are therefore

j = 15	k = 2	d = 1 or 2
j = 3	k = 10	d = 1 2 or 10
j = 2	k = 15	d = 1 or 15
j = 1	k = 30	d = 1 2 10 15 or 30

Suppose j=15 and k=2. Then the two irreducible factors of p_{30} are each invariant under H_4^0. But by proposition (4.7) of [32] the ring $R(H_4^0)$ is

$$(1+p_{60})C[p_2\ p_{12}\ p_{20}\ p_{30}]$$

where p_{60} is the product of 60 linear factors vanishing on the mirrors of H_4. Thus $R(H_4^0)$ contains no polynomials of odd degree, which contradicts our assumption.

Suppose j=1 and k=30. Then p_{30} is the equation of the union of d planes in CP^3. Let S_1 be the invariant quadric $p_2 = 1$ and let P be the set

P = {x ∈ CP^3 | x is the polar with respect to the quadric S_1 of one of the d planes given by $p_{30} = 0$}.

Then the set P contains d points and must be the union of

The result (1.46) means that the invariants of H_4 cannot be interpreted geometrically in a way analogous to those of H_3. A basic set of invariants exists for H_3 (given by 1.17) such that two of the invariants, i_6 and i_{10}, factorise into linear factors. These factors are the equations of planes which are perpendicular to the lines joining opposite vertices of a dodecahedron or icosahedron.

Any basic set of invariants for H_4 will describe four irreducible hypersurfaces in $\mathbb{C}^4$. One (given by p_2) has only one real point. The geometric connection between the other hypersurfaces and the polytopes $\{3, 3, 5\}$ and $\{5, 3, 3\}$ with H_4 as symmetry group is not at all clear.

orbits of the action of H_4 on CP^3. (This action can be obtained from the action of H_4 on C^4 by considering CP^3 as the set of lines passing through the origin in C^4.) But the smallest orbit of this action is the orbit of the vertices of {3,3,5} which contains 60 points. We thus have a contradiction.

Suppose d=1 and k=10 or 15. The k factors of p_{30} must be of the form $p, p\eta, \ldots, p\eta^{k-1}$ where η is a primitive k^{th} root of unity. Any element of H_4 must act on these factors by multiplying all of them by η^i for some i. Thus the image of $\sigma_1: H_4 \dashrightarrow Sym(k)$ is cyclic of order k and so H_4 has a normal subgroup of index k. This contradicts (1.45).

It remains to consider the cases in which j=2 or 3 and d≠1.

To eliminate the remaining cases let $P_1, \ldots, P_d$ be the d surfaces given by $p_{30}=0$ and let $Q_{ij} = P_i \cap P_j \cap S_0$ where S_0 is the invariant quadric cone $p_2 = 0$. The group H_4 permutes the points Q_{ij} in CP^3. But each set Q_{ij} has $2j^2$ points and there are $d(d-1)/2$ such sets. Thus the total number of points is $d(d-1)j^2$. But the orbits of the action of H_4 on CP^3 have sizes 720, 1200, 1800, 3600 and 7200 and there is no way of combining these numbers to get $d(d-1)j^2$. QED.

1.5 The Uniqueness of the Discriminant Variety

In this section, we investigate the extent to which the variety $\Delta(H_4)$ described in detail above is in fact unique. More generally, we look at the uniqueness of $\Delta(G)$ for all groups G. In order to make sure that our object of study was well defined, we assumed that the components of π_G were real homogeneous polynomials. One can weaken these assumptions by looking at pseudodiscriminants formed from quotient maps π_G whose components are complex homogeneous polynomials, non-homogeneous polynomials (real or complex) or analytic invariant maps. In this section we shall only look at those constructed from quotient maps whose components are complex homogeneous polynomials.

Definition 1.47

A hypersurface in $\mathbb{R}^n$ is *real* if it is mapped into itself by the complex conjugation involution of $\mathbb{C}^n$.

It can happen that there is a quotient map π_1 with complex components such that the image of the mirrors is a real hypersurface Δ_1 which is of course a pseudodiscriminant for G. Let π_2 be a real quotient map for G and let Δ_2 be the associated discriminant, discussed in § 1.2 to § 1.4. Each of the two quotient maps π_1 and π_2 factors through the other, and so there is always a complex algebraic automorphism ϕ of the quotient space mapping Δ_1 to Δ_2. We are considering Δ_1 and Δ_2 as real subvarieties of $\mathbb{R}^n$, and so we want to know whether there is an algebraic automorphism of $\mathbb{R}^n$ mapping Δ_1 to Δ_2 (or equivalently, whether the map ϕ preserves the real part of $\mathbb{C}^n$).

Example 1.48 Dihedral Groups

Any plane curve which is diffeomorphic over C to $\Delta(I_2(k))$ must be the set of zeros of a function $\mathcal{A}$ equivalent to $x^2 - y^k$. Therefore if k is even there are two forms of pseudodiscriminant, given by

$$x^2 - y^k = 0 \text{ and } x^2 + y^k = 0.$$

If k is odd, the pseudodiscriminant is unique and is given by

$$x^2 - y^k = 0.$$

Example 1.49 The Discriminant $\Delta(D_{2k})$

The mirrors are $x_i = \pm x_j$ for $i \neq j$.

Let σ_i be the i^{th} elementary symmetric polynomial in 2k variables and let

$$\lambda_{2i} = \sigma_i(x_1^2,\ldots,x_{2k}^2) \quad \text{for } 1 \leqslant i \leqslant 2k-1$$

$$\mu_{2k} = x_1 x_2 \ldots x_{2k} \qquad (1.50)$$

$$p_{2k}^{\pm} = (\lambda_{2k} \pm i\mu_{2k})$$

Then the maps

$$\pi_1 : x \longrightarrow (\lambda_2,\ldots,\lambda_{4k-2},\mu_{2k}) \qquad (1.51)$$

$$\pi_2 : x \longrightarrow (\lambda_2,\ldots,\widehat{\lambda_{2k}},\ldots,\lambda_{4k-2},p_{2k}^{+},\ p_{2k}^{-})$$

are both quotients for D_{2k}, and the hat denotes that a term has been omitted.

The two pseudodiscriminants Δ_1 and Δ_2 are

$$\Delta_1 = \{(\lambda_2,\ldots,\lambda_{4k-2},\mu_{2k}) \text{ such that } 0=\mathrm{Disc}(x^{2k}-\lambda_2 x^{2k-2}+\ldots-\lambda_{4k-2}x+\mu_{2k}^2)=0\ \} \qquad (1.52)$$

$$\Delta_2 = \{(\lambda_2,\ldots,\widehat{\lambda_{2k}},\ldots,\lambda_{4k-2},p_{2k}^{+},p_{2k}^{-}) \text{ such that } 0=\mathrm{Disc}(x^{2k}-\lambda_2 x^{2k-2}+\widehat{\lambda_{2k}x^k}\ldots-\lambda_{4k-2}x+(p_{2k}+p_{2k})^2+(p_{2k}-p_{2k})^2)\} \qquad (1.53)$$

If $k \geqslant 3$, consider the two lines $l = \lambda \begin{pmatrix} \pm 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

These lines have the following properties.

(i) Each line is an edge of a fundamental region for D_{2k}.

(ii) A point $x \in C^n$ lies on one of these lines if and only if $Fix_G(x) = B_{2k-3} \times A_1 \times A_1$.

Under π_1, these two lines are mapped to two real curves, and under π_2 they are mapped into two complex conjugate curves.

Since these curves form the set of singular points of type $B_{2k-3} \times A_1 \times A_1$ on the pseudodiscriminant, they must be preserved by any automorphism of the quotient space which maps Δ_1 to Δ_2. Hence any such automorphism cannot be real.

If k=2, consider the images in the quotient spaces of the three lines $\lambda\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\lambda\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}$, and $\lambda\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

These curves are the locus of singularities of type $A_1 \times A_1 \times A_1$. Under π_1 all three lines are mapped to real curves, while under π_2 they are mapped to one real and two complex conjugate curves.

<u>Proposition 1.54</u>

Let π_1 be a quotient map for H_4 with complex homogeneous polynomials as components, such that the image of the mirrors under π_1 is a real hypersurface. Let Δ_1 be the pseudodiscriminant constructed from π_1, and let Δ_2 be the discriminant of H_4 described in (1.33). Then there is a real algebraic automorphism of $\mathbb{R}^4$ mapping Δ_2 into Δ_1.

<u>Proof</u>

Let π_2 be the quotient map described in § 1.3. Let G_c denote the set of algebraic automorphisms of C^n of the form

$$\begin{pmatrix} p_2 \\ p_{12} \\ p_{20} \\ p_{30} \end{pmatrix} \rightarrow \begin{pmatrix} ap_2 \\ bp_{12} + cp_2^6 \\ dp_{20} + ep_{12}p_2^4 + fp_2^{10} \\ gp_{30} + hp_{12}^2p_2^3 + ip_{20}p_2^5 + jp_{12}p_2^9 + kp_2^{15} \end{pmatrix} \qquad (1.55)$$

where a,....,k are complex numbers, and let G_r denote the subgroup of G_c consisting of those automorphisms for which a,...,k are all real. Then

$$\theta(\Delta_2) = \Delta_1 \qquad (1.56)$$

for some $\theta \in G_c$. The aim is to prove that (1.56) holds for some $\theta \in G_r$, using the fact that Δ_1 is a real hypersurface.

For any complex number η of modulus 1, the map

$$\phi_\eta : \begin{pmatrix} p_2 \\ p_{12} \\ p_{20} \\ p_{30} \end{pmatrix} \longrightarrow \begin{pmatrix} \eta p_2 \\ \eta^6 p_{12} \\ \eta^{10} p_{20} \\ \eta^{15} p_{30} \end{pmatrix}.$$

is a member of G_c which preserves Δ_2. We may therefore assume that $\theta(\Delta_2) = \Delta_1$, where θ is an element of G_c given by (1.55) with $a \in \mathbb{R}$.

We may replace Δ_2 by $\phi(\Delta_2)$ for any map $\phi \in G_r$, without affecting the truth of the result we are trying to prove. We shall now do this. The discriminant of H_4 includes four distinguished curves: the curve of H_3 points, the curve of points of type $I_2(5)xA_1$, the curve of swallowtails, and the curve of points of type A_3xA_1. There is a map $\phi \in G_r$ such that on $\phi(\Delta_2)$ these four curves take the particularly simple forms

$$\begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} t \\ t^6 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} t \\ \alpha t^6 \\ t^{10} \\ 0 \end{pmatrix} \quad \begin{pmatrix} t \\ \beta t^6 \\ \gamma t^{10} \\ t^{15} \end{pmatrix}$$

for some real numbers α, β, γ which are probably unique.

The images of these four curves under a map $\theta \in G_c$ are

$$\begin{pmatrix} at \\ ct^6 \\ ft^{10} \\ kt^{15} \end{pmatrix} \begin{pmatrix} at \\ (b+c)t^6 \\ (e+f)t^{10} \\ (h+j+k)t^{15} \end{pmatrix} \begin{pmatrix} at \\ (\alpha b+c)t^6 \\ (d+\alpha e+f)t^{10} \\ (\alpha^2 h+i+\alpha j+k)t^{15} \end{pmatrix} \begin{pmatrix} at \\ (\beta b+c)t^6 \\ (\gamma d+\beta e+f)t^{10} \\ (g+\beta^2 h+\gamma i+\beta j+k)t^{15} \end{pmatrix}$$

By assumption, a is real. If $\theta(\Delta_2) = \Delta_1$, the four curves (1.57) will all be real and so b, c, d, e, f, k, (h+j), $(\alpha^2 h+i+\alpha j)$, and $(g+\beta^2 h+\gamma i+\beta j)$ are all real. To complete the proof of (1.54), it only remains to prove that g, h, i, and j are real.

This follows from considering the surface of $I_2(5)$ points. This is the set of points at which the pseudodiscriminant is locally diffeomorphic to $\Delta(I_2(5))\times\mathbb{R}^2$. This set is a two-dimensional algebraic subvariety of the pseudodiscriminant, and it must be real in both Δ_1 and Δ_2, since otherwise a second surface of $I_2(5)$ points could be obtained by complex conjugation.

Let $\mathcal{J}$ be the set of $I_2(5)$ points in Δ_2. The surface $\mathcal{J}$ is the image under q of the plane $i_6=0$. Since $\mathcal{J}$ contains the curves $\begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} t \\ t^6 \\ 0 \\ 0 \end{pmatrix}$, it must be the image of a map

$$\sigma: \begin{pmatrix} i_2 \\ i_{10} \end{pmatrix} \longrightarrow \begin{pmatrix} i_2 \\ (\lambda_1 i_{10} + \lambda_2 i_2)i_2 \\ (\lambda_1 i_{10} + \lambda_2 i_2)(\lambda_3 i_{10} + \lambda_4 i_2) \\ (\lambda_1 i_{10} + \lambda_2 i_2)(\lambda_3 i_{10} + \lambda_4 i_2)(\lambda_5 i_{10} + \lambda_6 i_2) \end{pmatrix}$$

It follows by computing first the values of the constants α, β, γ, $\lambda_1,\ldots,\lambda_6$ and then the composite map $\theta\circ\sigma$ (whose image must be a real surface) that g, h, i, and j are all real. QED.

CHAPTER 2 INVOLUTES OF CURVES

2.1 Summary

In this chapter we investigate the following questions.

(1) Does a curve have parallels?

(2) Given a hypersurface F in $\mathbb{R}^n$, is there a curve M having F as its focal set?

We find that a curve always has an (n−1) parameter family of parallels. For the hypersurface F to be a focal set, it must be a developable hypersurface. All the classical developable hypersurfaces are focal sets. We show how to construct an (n−1) parameter family of parallel curves, all having F as focal set. If F has certain special properties, all curves in this (n−1) parameter family will be singular. We look at examples of such families in the cases n=2 and n=3.

2.2 Some Definitions

The term 'manifold' will be used in a nonstandard way, to mean what is usually called a manifold with singularities.

Definition 2.1

A **k-manifold** or **k-dimensional manifold** is a subset M of $\mathbb{R}^n$ such that the following conditions hold:

(2.1.1) For each point m ∈ M, there exists a neighborhood N(m) of m and a smooth map $\mathbb{R}^k \longrightarrow \mathbb{R}^n$ whose image is $N(m) \cap M$ and which is injective at almost all points of its image. Such a map will be called a **local parametrisation** of M.

(2.1.2) For almost all points m of M, there is a neighbourhood N'(m) of m and a diffeomorphism

N.B. The word 'locally' will sometimes be omitted.

$$\phi:(N'(m),N'(m) \cap M) \longrightarrow (\mathbb{R}^n,\mathbb{R}^k \times 0).$$

Points where (2.1.2) fails are called <u>singular points</u> of M and the set of such points is denoted by ΣM.

Throughout chapters 2 and 3, M will denote a k-dimensional manifold with local parametrisation $r:(\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^n,m)$. The dimension of a manifold will be denoted, if necessary, by a superscript. We will only be looking at the local properties of manifolds, in other words at germs of manifolds and germs of maps between them. We will not always distinguish between a germ, which is an equivalence class of manifolds (or maps), and a representative of such an equivalence class.

We define two equivalence relations on the set of all manifolds germs. Let $g_i:(\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^n,m_i)$ for i=1,2 be local parametrisations of two manifolds M_1 and M_2. Then the manifold germs (M_1,m_1) and (M_2,m_2) are <u>locally diffeomorphic</u> or <u>equivalent</u> if g_1 and g_2 are $\mathcal{A}$-equivalent, that is if $g_1 = \phi_n \circ g_2 \circ \phi_k$ where ϕ_j is a germ of a self-diffeomorphism of $\mathbb{R}^j$.

If it is possible to take ϕ_n to be an affine isometry of $\mathbb{R}^n$, we will say that M_1 and M_2 are <u>locally congruent</u>. In this case the focal sets of M_1 and M_2 will be locally diffeomorphic. (Focal sets are defined below in (2.4.2).)

<u>Definition 2.2</u>

Let M_1 and M_2 be two submanifolds of $\mathbb{R}^n$. Then M_1 is <u>parallel</u> to M_2 if there exists a continuous bijection $\tau:M_1 \longrightarrow M_2$ (which will be called a <u>parallel map</u>) with the following properties.

(2.2.1) The affine tangent spaces to M_1 and M_2 at corresponding points are parallel (i.e. $T_m M_1$ is parallel to $T_{\tau(m)} M_2$.

(2.2.2) The affine normal spaces to M_1 and M_2 at corresponding points coincide (i.e. $N_m M_1 = N_{\tau(m)} M_2$).

(2.2.3) If M_1 is smooth at m and M_2 is smooth at $\tau(m)$ then τ is a diffeomorphism in some neighbourhood of m.

Remarks

(1) This definition agrees with the usual definition of parallels to a hypersurface.

(2) For two manifolds to be parallel, it is a necessary condition that they should be of the same dimension.

(3) Parallelism is an equivalence relation.

The three conditions in the above definition can be summarised in a single phrase by saying that M_1 and M_2 are parallel if the bundles NM_1 and NM_2 consist of the same set of linear subspaces of $\mathbb{R}^n$.

Definition 2.3

The **contact** between two manifolds M_1 and M_2 at a point $m \in M_1 \cap M_2$ at which both M_1 and M_2 are nonsingular is measured by the $\mathcal{K}$-equivalence class of an appropriate map-germ (see [23] and [34] for the definition of $\mathcal{K}$-equivalence and other relevant singularity theory). If $g:(\mathbb{R}^i,0) \to (M_1,m)$ is a local parametrisation of M_1, immersive at 0, and M_2 is locally the zero set of a submersive map germ $F: (\mathbb{R}^n,m) \to (\mathbb{R}^j,0)$, the map germ concerned is the composite $F \circ g$.

We now recall the various definitions of the focal set of a smooth manifold $M^k \subset \mathbb{R}^n$. The **normal bundle** NM is naturally embedded in $M \times \mathbb{R}^n$ as

$$\{(m,y) \in M \times \mathbb{R}^n \mid (y-m) \text{ is perpendicular to } T_mM \}$$

The projection maps onto the first and second factors will be denoted by π_1 and π_2 respectively.

Definition 2.4.1

The **focal set** F is the set of critical points of the projection $\pi_2: NM \longrightarrow \mathbb{R}^n$. The map π_2 is sometimes called the **end-point map**, because it maps each vector in NM to its endpoint.

The **distance-squared function** $V^y: \mathbb{R}^k \longrightarrow \mathbb{R}$ is defined by

$$2V^y(x) = \|y - r(x)\|^2.$$

Definition 2.4.2

The **focal set** F is

$$\{(r(x),y) \in M \times \mathbb{R}^n \mid V^y \text{ has a degenerate critical point (a singularity of at least type } A_2) \text{ at } x\} =$$

$$= \{(m,y) \in M \times \mathbb{R}^n \mid \text{there is a hypersphere with centre } y \text{ making at least } A_2 \text{ contact (3 point contact) with } M \text{ at } m \}$$

The equivalence of these two definitions of the focal set is proved in [19] and [26].

Remarks

(i) The focal set of M^k has dimension (n-1) and is the **singularity set** of the family of distance squared functions . It is usually identified with its image under π_2, the hypersurface $\pi_2(F)$ in $\mathbb{R}^n$, which is the **bifurcation set** of the family of distance-squared functions..

(ii) As well as the projections π_1 and π_2 a third projection π can be defined locally by the commutative diagram

$$\begin{array}{ccccc} & F & \subset & M \times \mathbb{R}^n & \\ \pi_1 \swarrow & \searrow \pi_2 & & \downarrow & \pi_2 \\ M & \xleftarrow{\ \pi\ } \pi_2(F) & \subset & \mathbb{R}^n & \end{array}$$

(iii) For a hypersurface, the terms <u>focal set</u> and <u>evolute</u> are used interchangeably, and if F is the evolute of M, then M is said to be an <u>involute</u> of F.

(iv) Suppose that M_1 and M_2 are parallel. Then since the bundles NM_1 and NM_2 comprise the same set of affine subspaces of $\mathbb{R}^n$, the corresponding maps π_2 have the same critical values, and so the focal sets of M_1 and M_2, map under π_2 to the same hypersurface in $\mathbb{R}^n$. If we identify the focal set with its image under π_2, we can say that parallel manifolds have the same focal set.

There is a natural way [33] to include points at infinity in the focal set of M. Let [c,s] be homogeneous co-ordinates in $\mathbb{R}P^n$ with $c \in \mathbb{R}^n$ and $s \in \mathbb{R}$, where the hyperplane at infinity is given by s=0.

<u>**Definition 2.5**</u>

The <u>extended focal set</u> F of a smooth manifold M is

$$\{(r(x),[c,s]) \mid V^{[c,s]} \text{ has a degenerate critical point at } x\}$$

$$\text{where } 2V^{[c,s]}(x) = s||r(x)||^2 - 2\langle c,r(x)\rangle$$
$$= 2s||V^{c/s}(x)||^2 - (1/s)||c||^2$$

If s=0, the function $V^{[c,s]}$ measures contact between M and a hypersphere centred at c/s. For points [c,0] on the hyperplane at infinity, $V^{[c,0]}$ measures contact between M and a hyperplane which can be thought of as a hypersphere of infinite radius.

Until now, it has been assumed in discussing focal sets that M is non-singular. This restriction will now be removed.

Definition 2.6

The (*extended*) *focal set* of a manifold M with singularities is the closure of the focal set of $M - \Sigma M$.

We shall mainly be interested in the focal sets of manifolds which are in some sense typical, and we shall most of the time want to ignore pathological examples. To make these ideas precise, we need some definitions.

Definition 2.7

A curve $M^1 \subset \mathbb{R}^n$ is of *type* $(i_1,\ldots,i_n)$ *at* $r(t)$ if the derivatives $r_{i_1}(t),\ldots,r_{i_n}(t)$ are linearly independent, and each of the integers $i_1 < \ldots < i_n$ is least possible satisfying this condition.

Definition 2.8

A manifold M^k in $\mathbb{R}^n$ is *distance-generic* if it satisfies certain conditions given in [33]. Let $V:\mathbb{R}^k \times \mathbb{R}P^n \to \mathbb{R}$ be defined by $V(x,[c,s])=V^{[c,s]}(x)$. Roughly speaking, a distance-generic manifold is one for which, for a suitable integer N, the N-jet extension $j^N V$ of the map V is transversal to a finite set of submanifolds of the jet space $j^N(\mathbb{R}^n \times \mathbb{R}P^n,\mathbb{R})$ and for which similar transversality conditions also hold for the restriction of V to the subspace s=0. As a consequence of the transversality conditions, any singularities of the functions $V^{[c,s]}$ for a distance-generic manifold must have $\mathcal{K}$-codimension at most n+1 (if $s\neq 0$) and n (if s=0).

Example 2.9

For a distance-generic curve in $\mathbb{R}^n$, given by $t \dashrightarrow r(t)$, $V^{[c,s]}$ is a function of one variable, and so only has singularities of types $A_1,\ldots\ldots,A_{n+1}$. In particular, the curve is of type $(1,2,\ldots,n)$ everywhere except at isolated points, and at these points it is of type $(1,\ldots,(n-1),(n+1))$.

It is a result of Looijenga (see [33]) that in the set $Imm(\mathbb{R}^k,\mathbb{R}^n)$ of smooth immersions of $\mathbb{R}^k$ in $\mathbb{R}^n$, those whose image is distance-generic form a dense subset.

2.3 The Family of Parallels to a Curve

Any parallel to M must be an integral manifold of a distribution P(M) on NM that will be defined below.

Let Z be the zero-section of NM, which is naturally isomorphic to M since $Z = \{(m,m) \mid m \in M\}$. To each point z of Z will be associated the k-dimensional space $T_zZ \subset T_z(NM)$. This k-dimensional distribution defined on Z can be extended to a k-dimensional distribution on the whole of NM by parallel translation of the k-dimensional spaces along the fibres of the projection $\pi_1:NM \longrightarrow M$.

A submanifold N^k of $\mathbb{R}^n$ is parallel to M if and only if it is the projection under $\pi_2:NM \longrightarrow \mathbb{R}^n$ of an integral manifold of the k-dimensional distribution P(M) defined above. It follows that for any $m \in M$ there is at most one k-manifold germ parallel to M passing through each point of N_mM.

If $k=1$, any one-dimensional distribution is integrable. This means that there is a k-manifold germ parallel to M passing through any point of N_mM. If however $k \geqslant 2$, a general k-dimensional distribution is not integrable.

<u>**Theorem 2.10**</u>

Let m be an arbitrary point of M^k.

(i) If $k=1$, there is an (n-1) parameter family of germs of curves parallel to M, one passing through each point of N_mM.

(ii) If $k=(n-1)$, there is a 1-parameter family of germs of hypersurfaces parallel to M, one passing through each point of N_mM.

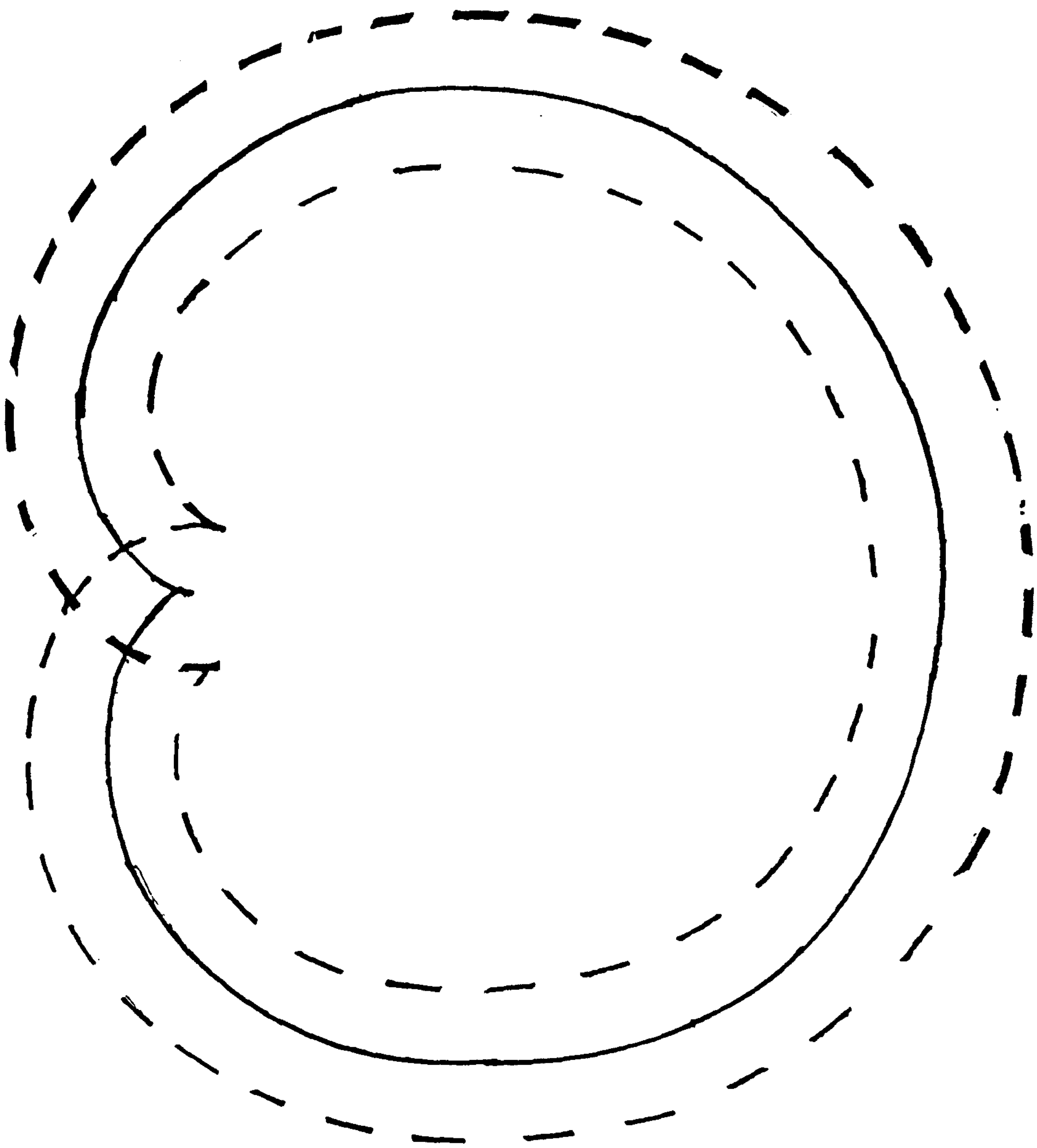
<u>**Remark**</u> The global form of this proposition is false. To illustrate the complexities that can arise in the global case, consider the cardioid M^1 with polar equation $r = 1 + \cos\theta$ in $\mathbb{R}^2$. Since a continuous globally defined choice of unit normal is not possible on M^1, the natural definition of a parallel to M^1 is a curve P which double-covers M^1 (see figure (2.11)).

In the case $k = (n-1)$, this theorem picks out a particular property of the distribution P(M) which is not shared by arbitrary distributions of dimension (n-1) on NM: this is the property of being integrable.

<u>**Proof of Theorem 2.10**</u>

(i) The parallel curves can be obtained from the integral curves of the line field P(M) by projection from NM to $\mathbb{R}^n$ by means of the map π_2.

Figure 2.11 A parallel to a cardioid



(ii) Recall that $r:(\mathbb{R}^k,0) \longrightarrow (M,m)$ is a local parametrisation of M. Let $n:\mathbb{R}^k \longrightarrow NM$ be a smoothly varying local choice of unit normal, defined by $n = \sigma \circ r$, where σ is a suitable section of the bundle NM. Then if c is a constant, the equation

$$\tau(r(x)) = x + cn(x)$$

gives a mapping τ which (near the point m) satisfies the conditions of definition (2.2). QED.

Thus if M has dimension 1 or codimension 1, the part of $\mathbb{R}^n$ lying close to M is foliated by parallels to M. For higher dimensions and codimensions, this is not the case: parallels can only rarely be found. In chapter 3, we shall construct a manifold M^2 in $\mathbb{R}^4$ such that the only 2-manifold germ parallel to M is M itself. In fact, we shall see that for purely local reasons, almost all two-dimensional submanifolds of $\mathbb{R}^4$ have this property.

This chapter is about curves, so we shall now concentrate on the case $k=1$. Let the curve

$$\gamma : t \longrightarrow r(t) + c(t)n(t) \qquad (2.12)$$

be parallel to M. Then the vectors r_1 and $r_1 + c_1 n + c n_1$ are parallel. Therefore $0 = n.(r_1 + c_1 n + c n_1) = c_1$ so that c must be constant. Therefore the curve γ lies on the <u>**tubular surface of radius c with core M**</u>, which is defined below.

<u>**Definition 2.13**</u>

Let $M^k \subset \mathbb{R}^n$. Then the <u>**tubular hypersurface $Tub_c(M)$ of radius c with core M**</u> is the union $\bigcup S_m$ of a set of $(n-k-1)$ dimensional spheres in $\mathbb{R}^n$, one for each point m of M, where the sphere S_m has centre m and radius c and lies in $N_m M$. The hypersurface $Tub_c(M)$ can also be defined as the set of points lying at distance c from M and is sometimes called a <u>**canal surface**</u>. For example, if M is a circle of radius r in $\mathbb{R}^3$ and $c < r$, then $Tub_c(M)$ is a torus.

The parallels to a curve M in $\mathbb{R}^n$ have the following geometric interpretation.

Theorem 2.14

Each of the parallels to the curve M^1 in $\mathbb{R}^n$ is a line of curvature on $Tub_c(M)$.

To prove this result, we investigate the focal set of $Tub_c(M)$. We temporarily remove the restriction on the dimension of M.

Proposition 2.15

Let $M^k \subset \mathbb{R}^n$ where $1 \leqslant k \leqslant (n-2)$.
Then the focal set of $Tub_c(M)$ consists of M itself (with each point of M counted (n-k-1) times) and the focal set of M (with each point counted once).

Proof of proposition 2.15

The projection $p:Tub_c(M) \longrightarrow M$ induces a projection mapping $p_N: NTub_c(M) \longrightarrow NM$ which is a diffeomorphism almost everywhere. The fibre of p_N above a point $(m,m) \in Z \subset NM$ is a sphere of dimension (n-k-1), while the fibre above any other point is a single point. The result now follows from (2.4.1). QED.

An analytic proof appears in [9].

Proof of theorem 2.14

Let $M^1 \subset \mathbb{R}^n$. Of the (n-1) principal curvature functions on $Tub_c(M)$, (n-2) of them take the constant value 1/c everywhere on $Tub_c(M)$. These are the principal curvatures which correspond to the (n-2) sheets of focal set that collapse, by 2.15, to the curve M. At each point $y \in S_m \subset Tub_c(M)$, the (n-2) dimensional space L_y of principal directions corresponding to these principal curvatures is simply T_yS_m, which is the tangent space to S_m at y. Then the principal direction at y corresponding to the one remaining principal curvature must be perpendicular to T_yS_m and so must lie in the direction of T_mM. We have therefore shown that the lines of curvature of $Tub_c(M)$ corresponding to the non-constant principal curvature function are parallel to M. QED.

This theorem provides another proof of the existence of the parallels to a curve.

If M is a hypersurface, it is usual to regard the parallels to M as forming part of a single hypersurface in **space-time** called the **big wave front generated by M** or simply a **big front**. The factor $\mathbb{R}^n$ is to be thought of as **space** and the factor $\mathbb{R}$ as **time**. This definition can be extended to the case where M is a curve as follows.

Definition 2.16

The **family of parallels to M** is the subset $\tilde{M}$ of $N_mM \times \mathbb{R}^n$ given by

$$\tilde{M} = \bigcup_{x \in N_mM} \{x\} \times M_x$$

where M_x is the parallel to M passing through the point x.

2.4 The Focal Set of a Curve

Until now, we have been looking at the parallels to curves. We now move on to study their focal sets. Given a hypersurface F in $\mathbb{R}^n$, what conditions must be imposed on F for it to be the focal set of some curve M. If one such curve exists, then by (2.10), there is an (n-1) parameter family of such curves. In order to answer these questions, we shall need some definitions.

Definition 2.17

A hypersurface $M \subset \mathbb{R}^n$ is **developable** if it is foliated by a 1-parameter family of affine subspaces of $\mathbb{R}^n$ of codimension 2, and if, in addition, the normal to M is constant along each leaf of this foliation.

It follows from this definition that (n-2) of the principal curvature functions of M are identically zero for a developable hypersurface.

Theorem 2.18

The focal set F of a curve M in $\mathbb{R}^n$ is a developable hypersurface in $\mathbb{R}^n$.

First Proof

The focal set is foliated by the fibres of the projection π_1. We claim that this foliation satisfies the conditions of definition (2.16). The fibre $\pi_1^{-1}(m)$ is the set of y such that

$$0 = V_1^y = (y - r(0)).r_1(0) \quad \text{and}$$
$$0 = V_2^y = (y - r(0)).r_2(0) - r_1(0).r_1(0) \qquad (2.19)$$

If $r_1(0)$ and $r_2(0)$ are linearly independent, the set of y satisfying equations (2.19) is an affine (n-2) dimensional subspace of $\mathbb{R}^n$. If, on the other hand, $r_1(0)$ and $r_2(0)$ are linearly dependent, the fibre $\pi_1^{-1}(m)$ is empty. (In this case, the fibre $\pi_1^{-1}(m)$ in the extended focal set is a projective (n-2) dimensional subspace of $\mathbb{RP}^n$ which lies entirely at infinity.)

Let $\quad t \longrightarrow (r(c(t)),e(t)) \qquad (2.20)$

be a curve lying on the focal set of M such that

$\gamma(0) = (m,y)$. Thus $c(t) \in \mathbb{R}$ and $e(t) \in \mathbb{R}^n$. Then

$$0 = \nabla \varrho^{(t)}(c(t)) = (e - \tilde{r}).\tilde{r}_1 \qquad (2.21)$$

where $\tilde{r} = r \circ c$, so differentiating (2.21) with respect to the variable t, and, for the sake of clarity, omitting the argument t of the functions e and $\tilde{r}$, gives

$$\begin{aligned} 0 &= (e - \tilde{r}).\tilde{r}_2 + (e_1 - \tilde{r}_1).\tilde{r}_1 \\ &= (e - \tilde{r}).\tilde{r}_2 - \tilde{r}_1.\tilde{r}_1 + e_1.\tilde{r}_1 \\ &= e_1.\tilde{r}_1 \end{aligned}$$

since the first two terms on the right hand side vanish by (2.19).

Letting γ vary over all possible curves on the focal set passing through (m,y) we find that for any curve (2.20) $e_1(0)$ lies in the hyperplane N_mM so $T_{(m,y)}F = N_mM$ depends only on m. QED.

__Second Proof__

The focal set F is the envelope of a one-parameter family of hyperplanes in $\mathbb{R}^n$ (the normal hyperplanes to M). Any envelope of a one-parameter family of hyperplanes is a developable hypersurface. QED.

We now recall briefly the classical examples of developable hypersurfaces in $\mathbb{R}^n$. These are reamers (see definition 2.26 below) cones, and cylinders. A __developable cone__ M is defined as the envelope of a one-parameter family of hyperplanes all passing through the same point v, the __vertex__ of M. The intersection of M with a hypersphere S^{n-1} centred at v is an (n-2) dimensional developable submanifold of S^{n-1} (the definition of such an object is very similar to (2.17) and should be clear to the reader).

A **developable cylinder** M in $\mathbb{R}^n$ is the envelope of a one-parameter family of hyperplanes all perpendicular to a fixed vector $v \in \mathbb{R}^n$. Thus a cylinder is a cone whose vertex is at infinity. Its intersection with any hyperplane H containing the vector v is a developable (n-2) dimensional submanifold of H.

We are mainly going to be interested in those curves whose focal sets are reamers. However, we first prove a result which describes those curves whose focal sets are cones and cylinders.

Proposition 2.22

(i) The following conditions are equivalent.

(A) The curve M lies on a hypersphere with centre v.

(B) The normal hyperplanes all pass through the point v.

(C) The focal set of M is a cone with vertex v.

(ii) The following conditions are equivalent.

(A) The curve M lies on a hyperplane normal to the vector v.

(B) The normal hyperplanes all contain the vector v.

(C) The focal set of M is a cylinder with vertex [v,0] in $\mathbb{R}P^n$.

Proof

In each case the implications (A)->(B)->(C)->(B) are obvious. To show that (B)->(A), suppose that the point $[c_0, s_0] \in \mathbb{R}P^n$ lies in all the normal hyperplanes to M. The normal hyperplane at r(t) is $\{ y \mid V_1^y(t)=0 \}$.

$$\text{Thus} \qquad 0 = V_1^{[c_0, s_0]}(t) \quad \text{for all } t \qquad (2.23)$$

and integrating (2.23) with respect to t, we find that

$$V^{[c_0, s_0]}(t) \text{ is constant, independent of } t \qquad (2.24)$$

But if $s_0 \neq 0$ (2.24) is the equation of a hypersphere and if $s_0 = 0$ it is the equation of a hyperplane. QED.

We now start looking at curves whose focal sets are reamers.

Proposition 2.25

Let M be a curve which is of type $(i_1, \ldots\ldots, i_n)$ at $r(0)$. Then, for each $j<n$, there is a unique affine space of codimension j, and a unique hypersphere (which may possibly be a hypersphere of infinite radius, i.e. a hyperplane) having best possible contact with the curve at $r(0)$. These are called the **osculating (n-j)-space** and **osculating hypersphere**.

If M is singular at $r(0)$, that is if $i_1 \neq 1$, this statement may have to be interpreted to mean that there is an osculating hypersphere of zero radius, i.e.the function $V^{r(0)}$ has a higher codimension singularity than any V^y for $y \neq r(0)$.

Proof

Let $(r(x),[c,s])$ lie on the extended focal set. The contact at $r(x)$ between M and a hypersphere of suitable radius with centre $[c,s]$ is measured by the $\mathcal{K}$ equivalence class of the map germ $V^{[c,s]}: \mathbb{R} \dashrightarrow \mathbb{R}$. This germ is of type A_k if and only if the first k derivatives $V^{[c,s]}{}_1(0), \ldots\ldots, V^{[c,s]}{}_k(0)$ vanish and $V^{[c,s]}{}_{k+1}(0) \neq 0$.

Each of these derivatives gives a linear condition on $[c,s]$. The set of simultaneous solutions $[c,s]$ of the M equations

$V^{[c,s]}{}_1(0) = \ldots\ldots = V^{[c,s]}{}_M(0) = 0$

is therefore a projective subspace of $\mathbb{R}P^n$. We claim that for sufficiently large M, this subspace is just a single point $[c_0, s_0]$ But this follows from the fact that $r_{i_1}(0), \ldots\ldots, r_{i_n}(0)$

span $\mathbb{R}^n$. The point $[c_0, s_0]$ will be the centre of the osculating hypersphere.

Let L be the affine subspace of $\mathbb{R}^n$ given by
$L = \{y \mid y.v_t = c_t \text{ for } t = 1,\ldots,j\}$
where $v_t \in \mathbb{R}^n$ and $c_t \in \mathbb{R}$. (Here the subscripts t do not denote differentiation.) The contact between M and L is measured by the $\mathcal{K}$-equivalence class of the map germ

$$t \longrightarrow \begin{pmatrix} v_1.r(t) - c_1 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ v_j.r(t) - c_j \end{pmatrix}$$

This germ is of type A_s if its first s derivatives vanish (but not the $(s+1)^{st}$). To make as many of these derivatives vanish as possible, $\mathbb{R}\{c_1,\ldots,c_j\}$ must be an orthogonal complement to the space spanned by $r_{i_1}(0),\ldots,r_{i_{n-j}}(0)$. So the osculating (n-j) space at r(0) is the space spanned by $r_{i_1}(0),\ldots,r_{i_{n-j}}(0)$, the first (n-j) linearly independent derivatives of r. QED.

The restriction that M should be of type $(i_1,\ldots,i_n)$ for some $i_1,\ldots,i_n$ is not a great restriction on the curve M since it holds for all distance-generic curves.

An example of a curve for which this result does not hold is a straight line in $\mathbb{R}^3$, for which the osculating planes and osculating spheres are not well defined.

<u>Definition 2.26</u>

The <u>reamer</u> of a curve M in $\mathbb{R}^n$ is the union of the osculating (n-2) spaces.

For example, the reamer of a curve in $\mathbb{R}^3$ is its tangent developable.

The locus of centres of osculating hyperspheres of a curve in $\mathbb{R}^n$ is called its **space evolute**. If E is the space evolute of M, then M will be called a **space involute** of E. The map $e:\mathbb{R} \to E$ will be defined by letting $e(t)$ be the centre of the osculating hypersphere at $r(t)$.

For plane curves (n=2), the space evolute is the same as the usual evolute, and the space involute is the same as the usual involute or **evolvent**. However for space curves (n=3) the space evolute defined here is what is classically called the **locus of centres of spherical curvature**. The term evolute is used classically to mean something slightly different, and the classical involutes of a space curve are also not the same as the space involutes defined above. For some space curves, the space evolute is very badly behaved.

Example 2.28

Let S and S' be two spheres in $\mathbb{R}^3$, with centres x and y respectively, intersecting transversally in a circle γ. Then there exists a smooth curve M given by $t \to r(t)$ such that, for $t<0$, $r(t)$ lies on S, and for $t>0$, $r(t)$ lies on S'. The point $r(0)$ lies on γ, and all the derivatives $r_i(0)$ are tangent to γ. For this curve,

$$e(t) = \begin{cases} x & \text{when } t<0 \\ y & \text{when } t>0 \end{cases}$$

and so the function $t \to e(t)$ is not even continuous.

For the curve of example (2.28), the functions V^x and V^y

have singularities of infinite codimension at 0. The next result shows that there is a very large class of curves, whose space evolutes are genuine curves.

Theorem 2.29

If $M^1 \subset \mathbb{R}^n$ is of type $(i_1,\ldots,i_n)$ at $r(t)$, the map $e:t \to e(t)$ is smooth. Thus the space evolute is a curve (with singularities).

Examples

The hypothesis of this theorem holds whenever M is a curve for which the germ of the map V (see (2.8)) is of finite $\mathcal{K}$ codimension at every point. It also holds if M is any algebraic curve in $\mathbb{R}^n$, other than a circle, lying on a hypersphere. So those curves M not covered by this theorem (such as (2.28)) are certainly exceptional cases.

Proof of theorem 2.29

The proof consists of three stages.

(i) Show that for all sufficiently small non-zero t, the curve M is of type $(1,2,\ldots,n)$ at $r(t)$.

(ii) Construct a smooth map $t \to \eta(t)$ such that $\eta(t)=e(t)$ for all sufficiently small non-zero t.

(iii) Show that $e(0)=\eta(0)$

(i) Without loss of generality, $r(t) = (a_1t^{i_1}+\ldots,\ldots,a_nt^{i_n}+\ldots)$ where $a_1\ldots a_n \neq 0$.

Let $j_k = i_k-k$ and consider the determinant of the matrix D whose rows are $r_1, r_2,\ldots,r_n$. The terms of lowest order in t in this determinant are

$$a_1 \ldots a_n \begin{vmatrix} i_1 t^{i_1-1} & \cdots & i_n t^{i_n-1} \\ \cdots & \cdots & \cdots \\ i_1(i_1-1)\ldots(i_1-n+1)t^{i_1-n} & \cdots & i_n(i_n-1)\ldots(i_n-n+1)t^{i_n-n} \end{vmatrix}$$

$$= a_1 \ldots a_n i_1 \ldots i_n t^{j_1+\ldots j_n} \begin{vmatrix} 1 & \cdots & 1 \\ i_1-1 & & i_n-1 \\ \cdots & \cdots & \cdots \\ (i_1-1)\ldots(i_1-n+1) & \cdots & (i_n-1)\ldots(i_n-n+1) \end{vmatrix}$$

$$= \text{after performing row operations} \quad a_1 \ldots a_n i_1 \ldots i_n t^{j_1+\ldots j_n} \begin{vmatrix} 1 & \cdots & 1 \\ i_1 & \cdots & i_n \\ \cdots & \cdots & \cdots \\ i_1^{n-1} & \cdots & i_n^{n-1} \end{vmatrix}$$

Thus the coefficient of $t^{j_1+\ldots+j_n}$ in $|D|$ is non-zero and so $|D|$ has an isolated zero at $t=0$. But where $|D| \neq 0$, $r_1, \ldots, r_n$ are linearly independent.

(ii) It follows from (i) that for all sufficiently small non-zero t, the point $e(t)$ is given by

$$V_1^{e(t)}(t) = \ldots = V_n^{e(t)}(t) = 0$$

Therefore

$$D(e-r) = \begin{pmatrix} 0 \\ r_1 . r_1 \\ 3r_1 . r_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} -V_1^{r(t)}(t) \\ -V_2^{r(t)}(t) \\ \vdots \\ -V_n^{r(t)}(t) \end{pmatrix}$$

where D is the matrix defined in section (i) of this proof. Thus for all sufficiently small non-zero t, e(t) is given by

$$e(t) = r(t) + D^{-1} \begin{pmatrix} -V_1^{r(t)}(t) \\ -V_2^{r(t)}(t) \\ \vdots \\ -V_n^{r(t)}(t) \end{pmatrix}$$

Now define $\eta(t)$ by

$$\eta(t) = r(t) + D^{-1} \begin{pmatrix} -V_1^{r(t)}(t) \\ -V_2^{r(t)}(t) \\ \\ -V_n^{r(t)}(t) \end{pmatrix}$$

The map η is well defined, provided that it is allowed to take values in RP^n. This follows from the fact proved above, that the determinant of the matrix D is $t^{j_1+\dots+j_n}F(t)$ for some function F with $F(0)\neq 0$.

(iii) It was shown during the proof of (2.25) that e(0) is given by the equations

$$0 = V_{i_1}^{e(0)}(0) = \dots = V_{i_n}^{e(0)}(0) \qquad (2.30)$$

To complete the proof of (2.28) it willtherefore suffice to show that $0 = V_{i_1}^{\eta(0)}(0) = \dots = V_{i_n}^{\eta(0)}(0)$

But $0 = V_1^{\eta(t)}(t) = \dots = V_n^{\eta(t)}(t)$ (2.31)

for all non-zero t, and hence by continuity for all t.

In particular

$$V_1^{\eta(t)}(t) = (\eta - r).r_1 \qquad (2.32)$$

So differentiating (2.32) (n-1) times with respect to t,

$0=(\eta-r).r_2 + (\eta_1-r_1).r_1 = V_2 + \eta_1.r_1$

$0=(\eta-r).r_3+2(\eta_1-r_1).r_2+(\eta_2-r_2).r_1 = V_3+2\eta_1.r_2+\eta_2.r_1$

$$0=V_n+\binom{n-1}{1}\eta_1.r_{n-1}+\ldots+\binom{n-1}{1}\eta_{n-2}.r_2+\eta_{n-1}.r_1$$

Similarly by differentiating $0 = V\eta^{(t)}(r(t))$ (n-2) times with respect to t, we get

$0 = V_3 + \eta_1.r_2$

$0 = V_4 + 2e_1.r_3 + \eta_2.r_2$

etc.

Assembling derivatives of all the equalities in (2.31) gives

Table 2.33

$0 = V_1$

$0 = V_2 = V_2+\eta_1.r_1$

$0 = V_3 = V_3+\eta_1.r_2 = V_3+2\eta_1.r_2+\eta_2.r_1$

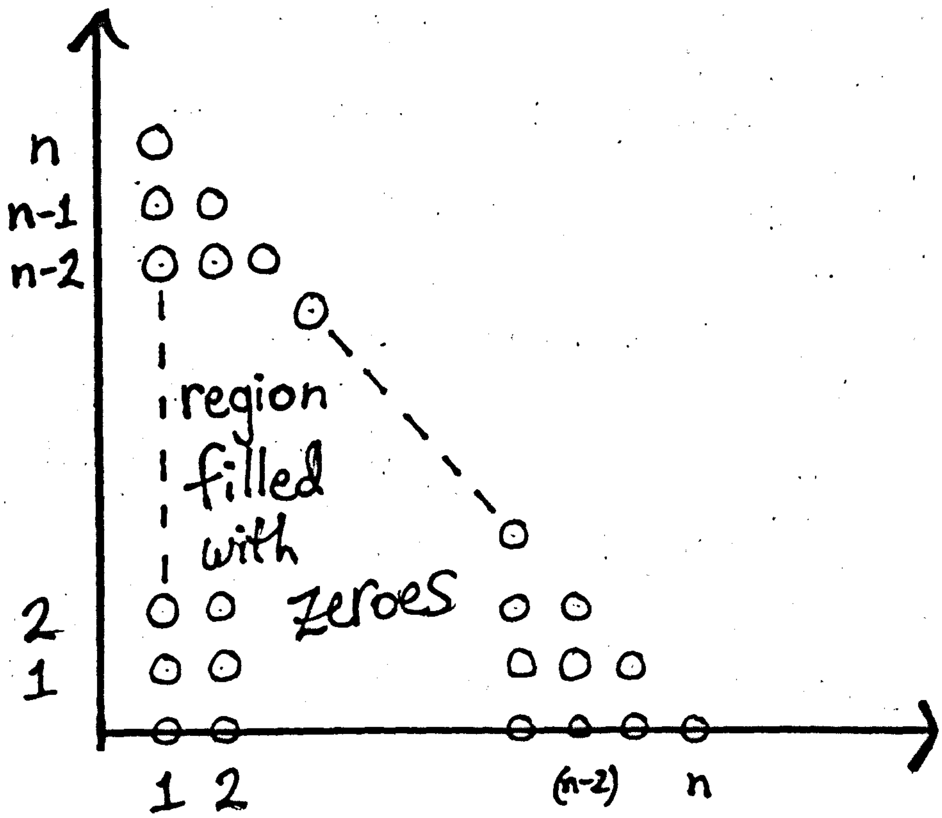
$0 = V_n = V_n+\eta_1.r_{n-1} = V_n+2\eta_1.r_{n-1}+\eta_2.r_{n-2}$

$$= V_n+\binom{n-1}{1}\eta_1.r_{n-1}+\ldots+\binom{n-1}{1}\eta_{n-2}.r_2+\eta_{n-1}.r_1$$

It follows that each summand of each expression in table (2.33) is equal to zero. This information can also be expressed graphically by placing a zero at the point (i,j) of the integer lattice to indicate the equality $r_i.\eta_j = 0$ and a cross to indicate $r_i.\eta_j \neq 0$. Furthermore the point (i,0) will be marked with a zero if $V_i = 0$ and a cross if $V_i \neq 0$. This results in figure (2.34)

Figure 2.34 A Graphical Representation of Table 2.33:

A Triangular Region Filled With Zeroes



Proposition 2.35

Figure (2.34) is completed as follows. Either the entire first quadrant is filled with zeros or the region of zeros is bounded by a staircase with (n-1) steps as shown in figure 2.36.

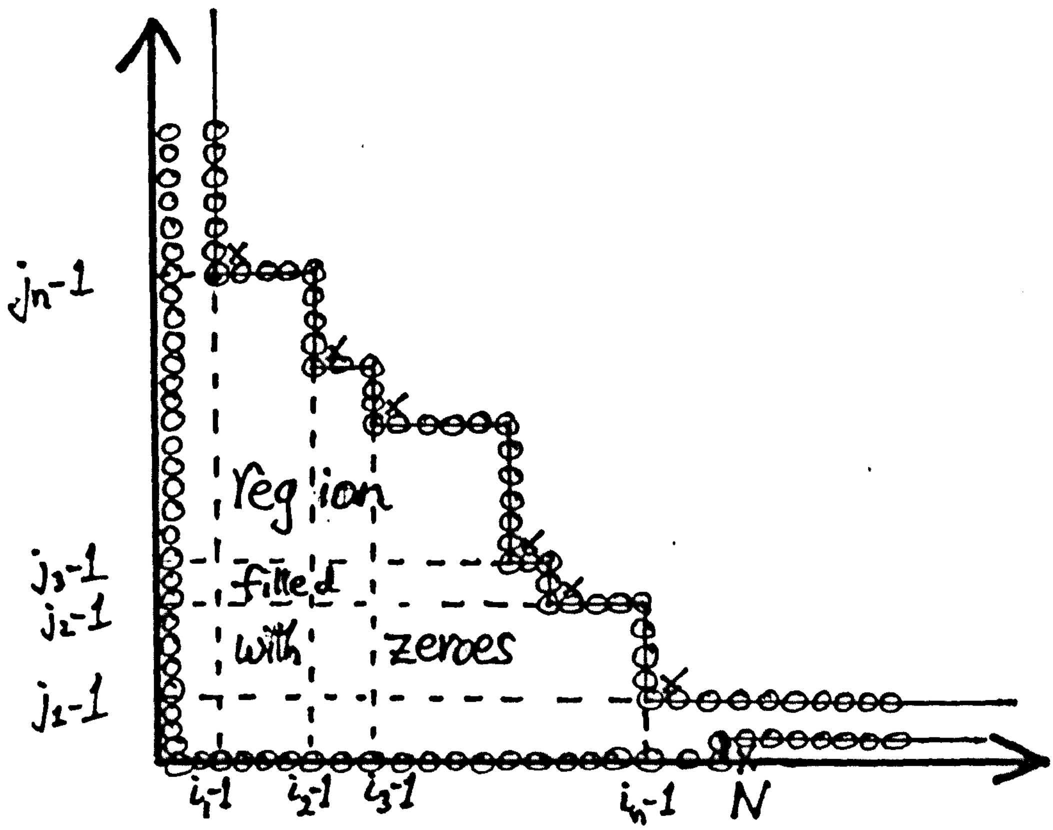
Proof

Let L_k denote the diagonal $i+j = k$ in the plane of figure 2.36.

Suppose that $r_i(0).\eta_j(0) \neq 0$ for some pair (i,j). Let L_N be the first diagonal containing a cross. Let $r(t)$ be a point of type $(i_1,\ldots,i_n)$. Whenever $i<i_t$, $r_i(0) \in \mathbb{R}\{r_{i_1}(0),\ldots r_{i_{t-1}}(0)\}$ and so if $r_{i_1}.\eta_j = \ldots = r_{i_{t-1}}.\eta_j = 0$ then $r_j.\eta_j = 0$.

So the triangular region of zeros in figure 2.34 should be extended as in figure (2.36).

Figure 2.36 Graphical Representation of some Equalities That Follow From Table 2.33: a Staircase Region Filled With Zeroes



Now all but n+1 of the positions on the diagonal L_N have been filled with zeros. But by hypothesis the diagonal L_N contains at least one non-zero term, and hence must contain at least n+1 non-zero terms. For if there were n or fewer non zero terms, these would be related by n equalities obtained by differentiating the bottom line of (2.33) N-n times. It would then follow that these terms were in fact all zero. Since we assumed that some were non zero, this would be a contradiction.

Therefore the (n+1) remaining places on the diagonal L_N must be filled with crosses, and we have obtained the information shown in figure (2.36). This completes the proof of (2.35).

Furthermore, the first (N-1) positions on the x axis of figure (2.36) are filled with zeros. This means

$$0 = V_1\eta^{(0)}(0) = \ldots = V_{N-1}\eta^{(0)}(0) \qquad (2.37)$$

and since $N = i_n + j_1 > i_n$ we certainly have as a special case of (2.37)

$$0 = V_{i_1}\eta^{(0)}(0) = \ldots = V_{i_n}\eta^{(0)}(0)$$

and so $\eta(0) = e(0)$ which also completes the proof of (2.28).

Since it has now been shown that $\eta(t) = e(t)$, it is now possible to replace η by e in all the above equations.

Proposition 2.38

Suppose M is of type $(i_1,\ldots i_n)$ at r(0) and E is its space evolute. Then

either $e_j(0)=0$ for all j and the function $V^{e(0)}$ is flat at 0

or E is of type $(j_1,\ldots,j_n)$ where $i_t+j_{n+1-t}=N$

and the function $V^{e(0)}$ has an A_{N-1} singularity at 0.

Remarks

(1) A flat function is one all of whose derivatives vanish.

(2) If $r_1(0) \neq 0$, there is A_{N-1} contact between M and its osculating hypersphere at r(0).

Proof

These results can be read off from figure 2.36. QED.

Proposition 2.39

Suppose M is of type $(i_1,\ldots i_n)$ at r(0) and E is of type $(j_1,\ldots j_n)$ at e(0) for some $i_1<\ldots<i_n$ and $j_1<\ldots<j_n$. Then

(i) The focal set of M is the reamer of E

(ii) The bundle of osculating hyperplanes to E is the bundle of normal planes to M.

Proof

(i) The osculating (n-2) space to E at e(0) is spanned by $e_{i_1}(0) \ldots e_{i_{n-2}}(0)$.

By figure (2.36) this space is perpendicular to $r_1(0)$ and $r_2(0)$ and hence is parallel to the intersection of the focal set with $N_{r(0)}M$. Since e(0) lies in both of these parallel affine subspaces of $\mathbb{R}^n$ they coincide.

(ii) The osculating hyperplane to E at e(0) is spanned by $e_{i_1}(0) \ldots e_{i_{n-1}}(0)$. By figure 2.36 each of these vectors is perpendicular to $r_{j_1}(0)$ and hence lies in $N_{r(0)}M$. QED.

Theorem 2.40

Let E be a curve of type $(j_1, \ldots j_n)$. There is an (n-1) parameter family of germs of curves M (with singularities) which are space involutes of E . Any two space involutes of E are parallel.

Proof

Suppose a space involute M exists. Then by (2.39) the normal bundle of M is the bundle ΩE of osculating hyperplanes of E. So any two space involutes of E have the same normal bundle, and hence are parallel.

Furthermore, this construction produces a smooth line field on $\Omega E \subset E \times \mathbb{R}^n$. The integral curves of this line field, project under the map $E \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ onto the space involutes. QED.

Proposition 2.41

Suppose E is of type $(j_1,\ldots,j_n)$ at e(0). Then either $r_i(0)=0$ for all i and the function $V^{e(0)}$ is flat at 0 or M is of type $(i_1,\ldots,i_n)$ where $i_t + j_{n+1-t}=N$ and the function $V^{e(0)}$ has an A_{N-1} singularity at r(0).

Proof

These results can be read off from figure 2.36. QED.

We have thus solved the problem of finding curves M whose focal set is an arbitrary hypersurface F in $\mathbb{R}^n$ in virtually all cases. By (2.12), such curves can only exist if F is developable. If F is a developable cone or cylinder, then by (2.18) any such curve M lies on a hypersphere or hyperplane of $\mathbb{R}^n$. Let H be this hypersphere or hyperplane. The focal set of M regarded as a curve in H is F $\cap$ H, so we should now apply the construction of (2.40) in H. We have thus reduced the dimension of the ambient space by one. Repeating this last step as many times as necessary, we may assume without loss of generality that F is not a cone or cylinder. In almost all cases of practical interest, F is then the reamer of a curve E lying in some sphere or Euclidean space. Provided this curve is of type $(i_1<\ldots<i_n)$ for some $i_1<\ldots<i_n$, an (n-1) parameter family of curves M with the desired property is given by (2.40) or an analogous result for curves on spheres.

2.6 Examples in R^2 and R^3

In this section $E \subset R^n$ is a curve with parametrisation $e:(R,0) \longrightarrow (R^n,0)$ for some $n > 1$, and M is a space involute of E where e(t) is the centre of the osculating hypersphere of M at r(t). Let Ω be the osculating hyperplane of E at e(0). Then by (2.40), the family of space involutes of E can be parametrised by the points of Ω. The following definition allows us to treat the whole family as a single entity.

Definition 2.42

The **big involute** of E is the subset $\tilde{M}$ of $\Omega \times R^n$ given by

$$\tilde{M} = \bigcup_{x \in \Omega} \{x\} \times M_x$$

where M_x is the space involute of E passing through the point $x \in \Omega$. This is an extension of Arnold's definition of the big involute of a hypersurface. If M is one particular space involute of E, then $\tilde{M}$ is the family of parallels to M.

If $x, x' \in \Omega$ with $x \neq x'$ local co-ordinates $(r, \theta_1, .. \theta_{n-2})$ can be chosen for Ω near x' in such a way that r(P) measures the distance of the point P from x. For example if n = 4, (r, θ_1, θ_2) could be spherical polar co-ordinates on Ω.

Proposition 2.43

Let $(x', y) \in \tilde{M}$ with $x' \neq x$. In the neighbourhood of the point (x', y), the projection $p:(r, \theta_1, .. \theta_{n-2}, z) \longrightarrow (r, z)$ maps the big involute into

$$\bigcup_{r \in R^+} \{r\} \times Tub_r(M_x) \qquad (2.44)$$

This is a one-parameter family of parallel tubular hypersurfaces whose core is the curve M_x, the space involute of E passing through the point $x \in \Omega$.

If for some $x \in \Omega$, M_x is distance-generic, then the hypersurface (2.44) is the image of a generic Legendrian mapping. Using classification theorems for such mappings, it follows that the hypersurface (2.44) is locally diffeomorphic at any point to one of the standard models described in [4]. The first few standard models turn out to be of the form $\Delta(G) \times \mathbb{R}^i$ where G is one of the Coxeter groups A_k, D_k, or E_k [4].

Next we define two terms which will be used later. Let $r:(\mathbb{R},0) \longrightarrow (\mathbb{R}^n,m)$ be a parametrisation of M.

<u>Definition 2.45</u>

The curve M has an <u>ordinary cusp</u> at m if the germ of the map r at zero is $\mathcal{A}$-equivalent to the map germ

$$t \longrightarrow \begin{pmatrix} t^2 \\ t^3 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (2.46)$$

The curve M has a <u>rhamphoid cusp</u> at m if the germ of r at zero is $\mathcal{A}$-equivalent to the germ

$$t \longrightarrow \begin{pmatrix} t^2 \\ t^5 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (2.47)$$

Proposition 2.48

(i)Let $n = 2$ or 3.

Then the curve $M \subset \mathbb{R}^n$ has an ordinary cusp at m <=>

$r_1(0) = 0$ and $r_2(0), r_3(0)$, are linearly independent.

(ii) The curve $M \subset \mathbb{R}^2$ has a rhamphoid cusp at m <=>

$r_1(0)=0$, $r_2(0)\neq 0$, $r_2(0)$ and $r_3(0)$ are linearly dependent and

$$0 \neq \begin{vmatrix} r_3(0) & 0 \\ r_4(0) & 3r_3(0).r_2(0) \\ r_5(0) & 10r_2(0).r_3(0) \end{vmatrix}$$

(iii) The curve $M \subset \mathbb{R}^3$ has a rhamphoid cusp at m <=>

$r_1(0)=0$, $r_2(0)\neq 0$, $r_2(0)$ and $r_3(0)$ are linearly dependent and

$$3r_2(0) \times r_5(0) \neq 10\, r_3(0) \times r_4(0)$$

Proof

There are three stages.

(i) Verify that the normal forms (2.46) and (2.47) satisfy the conditions of (2.48).

(ii) Check that if the map r satisfies the conditions of (2.48), then so do $r^\circ \phi$ and $\theta^\circ r$ where

$$\phi:(\mathbb{R},0) \to (\mathbb{R},0) \text{ and } \theta:(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$$

are diffeomorphisms.

(iii) Check that if r satisfies the conditions listed above, it is $\mathcal{A}$-equivalent to (2.46) or (2.47). Let k=3 for an ordinary cusp and let k=5 for a rhamphoid cusp. There is clearly an $\mathcal{A}$-equivalence that will put r into the form

$$t \longrightarrow \begin{pmatrix} t^2 \\ t^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mathcal{O}(k+1)$$

But the maps (2.46) and (2.47) are k-$\mathcal{A}$-determined (by [34] corollary (4.2.1)) and the result follows. QED.

Let $E^1 \subset \mathbb{R}^2$ be a plane curve given by $e:\mathbb{R} \longrightarrow \mathbb{R}^2$. Then the individual involutes of E will be curves of the form

$$t \longrightarrow r(t) = e(t) + \frac{(\lambda - s(t))}{s_1(t)} e_1(t) \qquad (2.49)$$

where $s:(\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ is the arc length function on E. The big involute of E is the surface given by

$$\begin{pmatrix} \lambda \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda \\ e(t) + \dfrac{(\lambda - s(t))}{s_1(t)} e_1(t) \end{pmatrix} \qquad (2.50)$$

The map germ (2.50) is an unfolding of the map germ

$$t \longrightarrow e(t) - \frac{s(t)}{s_1(t)} e_1(t) \qquad (2.51)$$

which is a parametrisation of the involute passing through e(0).

<u>Example 2.52</u>

If $e_1(0) \neq 0$ and $e_1(0), e_2(0)$ are linearly independent, then there will be some neighbourhood of e(0) in which every point of E is of type (1,2). Let $t \longrightarrow r(t)$ be an involute of E, where e(t) is the centre of a sphere which osculates at r(t). By (2.41), the involute is of type (N,N+1) at r(t) and the distance squared function $V^{e(t)}$ has an A_{N+1} singularity at t. The integer N is of course a function of t. But now the equations

$$0=V^{e(t)}_N(t) = (e(t)-r(t)).r_N(t) +\dots$$

$$0=V^{e(t)}_{N+1}(t) = (e(t)-r(t)).r_{N+1}(t) +\dots.$$

$$0 \neq V^{e(t)}_{N+2}(t) = (e(t)-r(t)).e_{N+2}(t) +\dots$$

show that $N(t) = 1$ when $r(t) \neq e(t)$

$N(t) = 2$ when $r(t) = e(t)$

and $N(t) \geqslant 3$ leads to a contradiction.

By (2.41) an involute M is of type (2,3) precisely when $s(t) = \lambda$ (since this is the condition for r(t) and e(t) to be equal). At these points, the singularities of the involutes are ordinary cusps (by (2.48)). Let $M_{e(0)}$ be the involute passing through e(0). Then $M_{e(0)}$ is locally diffeomorphic to the curve

$$t \longrightarrow \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \tag{2.53}$$

An $\mathcal{A}$-versal deformation of (2.53) is

$$\begin{pmatrix} a \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} t^2 \\ t^3+at \end{pmatrix} \tag{2.54}$$

Therefore (2.50) is $\mathcal{A}$-equivalent to

$$\begin{pmatrix} \lambda \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda \\ t^2 \\ t^3+a(\lambda)t \end{pmatrix} \quad \text{for some function } a(\lambda).$$

But for sufficiently small λ, there is a value of t satisfying $s(t)=\lambda$ (the germ of the function s at zero is invertible), and so for sufficiently small λ, the involute (2.40) has an ordinary cusp.

Since the curve (2.54) is nonsingular for $a \neq 0$, the function $a(\lambda)$ must be identically zero. Thus the big involute is locally diffeomorphic to the surface given by

$$\begin{pmatrix} \lambda \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda \\ t^2 \\ t^3 \end{pmatrix}$$

In other words the big involute has an ordinary cuspidal edge (see (3.38) below).

<u>Example 2.55</u>

It is well known that the evolute of a nonsingular plane curve cannot have a point of inflexion. This can be deduced from (2.34) as follows. Let M be a plane curve. Then either $e_i(0) = 0$ for all i, or $r_1(0)$ and $r_2(0)$ are linearly dependent, in which case $e(0)$ is at infinity, or M is of type (1,2) at $r(0)$ and its evolute E is of type (N,N+1) at $e(0)$. In all three cases E is never of type (1,k) ($k \geqslant 3$) at $e(0)$.

There is nothing to stop us from considering the family of involutes of a plane curve with an inflexion.

Let $E^1 \subset \mathbb{R}^2$ be a plane curve given by $e:\mathbb{R} \longrightarrow \mathbb{R}^2$ with an ordinary inflexion at $e(0)$. By (2.41), the involute $M_{e(0)}$ passing through $e(0)$ is of type (3,5). Since the map germ

$$t \longrightarrow \begin{pmatrix} t^3 \\ t^5 \end{pmatrix} \qquad (2.56)$$

is 5-$\mathcal{A}$-determined, the curve $M_{e(0)}$ is locally diffeomorphic to the curve (2.56).

An $\mathcal{A}$versal deformation of (2.56) is

$$\begin{pmatrix} a \\ b \\ c \\ d \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} t^3+at \\ t^5+bt^4+ct^2+dt \end{pmatrix}$$

Thus the map (2.50) which parametrises the big involute is A-equivalent to

$$\begin{pmatrix} \lambda \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda \\ t^3+a(\lambda)t \\ t^5+b(\lambda)t^4+c(\lambda)t^2+d(\lambda)t \end{pmatrix}$$

for some functions $a(\lambda),\ldots,d(\lambda)$. For small non-zero values of λ, the individual involute given by (2.49) must have an ordinary cusp at the point with parameter t such that $s(t)=\lambda$, and a rhamphoid cusp at the point with parameter 0. Furthermore the difference between the values of the parameter at the two cuspidal points is

$$s^{-1}(\lambda)-s^{-1}(0) = s^{-1}(\lambda)$$

Note that, for all sufficiently small values of λ, $S^{-1}(\lambda)$ is a nonsingular function of λ.

Here s^{-1} is the inverse of the function s, so that

$$s^{-1}(s(t)) = s(s^{-1}(t)) = t$$

For the curve

$$t \longrightarrow \begin{pmatrix} t^3+at \\ t^5+bt^4+ct^2+dt \end{pmatrix}$$

to have one rhamphoid cusp and one ordinary cusp, the following equations must be satisfied

$a = -12b^2/25 \qquad c = -8b^3/25 \qquad d = -16b^4/125$

The ordinary cusp occurs at $t = 2b/5$ and the rhamphoid cusp at $t = -2b/5$. It follows that the big involute of E is locally diffeomorphic to

$$\begin{pmatrix} \lambda \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda \\ t^3 - 12\lambda^2 t/25 \\ t^5 + \lambda t^4 - 8\lambda^3 t^2/25 - 16\lambda^4 t/125 \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad (2.57)$$

We now claim that this is the surface $\Delta(H_3)$. Eliminating the variables λ, t from (2.57) gives

$$\begin{aligned} 0 = H(X,Y,Z) &= 11264X^{12}Y - 42240X^{10}Z + 368000X^9Y^2 - 3600000X^7YZ \\ &+ 750000X^6Y^3 + 7500000X^5Z^2 - 28125000X^4Y^2Z - 21484375X^3Y^4 \\ &+ 117187500X^2YZ^2 + 146484375XY^3Z + 48828125Y^5 - 48828125Z^3 \end{aligned}$$

Composing H with the map

$$\phi : \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ -96Y/125 + 16X^3/125 \\ -256Z/625 + 48X^5/3125 \end{pmatrix}$$

gives $0 = H(\phi(X,Y,Z)) = \lambda G(X,Y,Z)$ where λ is a non-zero number and $G(X,Y,Z)$ is the equation of $\Delta(H_3)$ as given by (1.21). We have thus provided a new proof of the result of Shcherbak [29], that the big involute of a curve with an inflexion is $\Delta(H_3)$.

<u>Example 2.58</u>

Let E be a curve in $\mathbb{R}^3$ which is of type (1,2,3) at e(0). Then by (2.41) any space involute M is of type (N,N+1,N+2) and the distance squared function $V^{e(0)}$ for M has an A_{N+2} singularity at 0. But now the equations

$$0 = V^{e(0)}_N(0) = (e(0)-r(0)).r_N(0)$$
$$0 = V^{e(0)}_{N+1}(0) = (e(0)-r(0)).r_{N+1}(0)+\ldots\ldots\ldots$$
$$0 = V^{e(0)}_{N+2}(0) = (e(0)-r(0)).r_{N+2}(0)+\ldots\ldots\ldots$$
$$0 = V^{e(0)}_{N+3}(0) = (e(0)-r(0)).r_{N+3}(0)+\ldots\ldots\ldots$$

show that N=1 when r(0)is not on the tangent line to E at e(0)

N=2 when r(0) lies on the tangent line $T_{e(0)}E$
but $r(0) \neq e(0)$

N=3 when r(0) = e(0)

and $N \geqslant 4$ leads to a contradiction.

It follows that M is locally congruent near t = 0 to one of the following standard forms near t = 0.

$$\text{(A)} \quad (t^3, at^4+\ldots bt^5+\ldots) \qquad ab \neq 0$$
$$\text{(B)} \quad (t^2, at^3+\ldots, bt^4+ct^5+\ldots) \qquad abc \neq 0 \qquad (2.59)$$
$$\text{(C)} \quad (t, at^2+bt^3+\ldots, et^3+ft^4+\ldots)$$

where in case (C) $a^2e+2bf \neq ce$ and $ae \neq 0$.

Let T denote the tangent developable to E, which by (2.39) is the focal set of M. Then (see figure (2.60))

Case (A) occurs at points where the curve M meets E

Case (B) occurs at points where M meets T-E

and Case (C) occurs everywhere else.

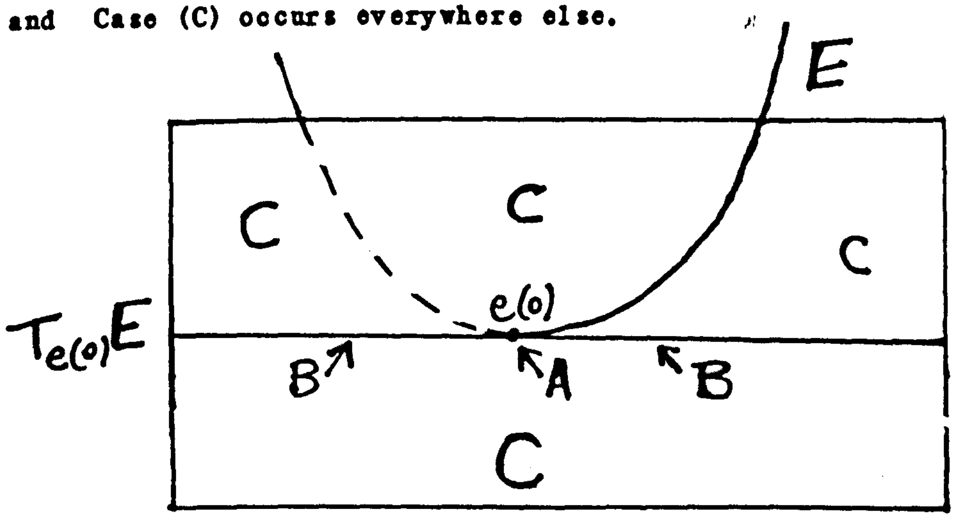


Figure 2.60 Normal Forms for Involutes of a Curve With a Point of Type (1,2,3)

The conditions $c \neq 0$ in case (B) and $a^2e+2bf \neq 0$ in case (C) arise from the inequalities $V_5^{e(0)} \neq 0$ and $V_4^{e(0)}(0) \neq 0$ respectively. The reparametrisation used to obtain equations (2.59) means that for $t \neq 0$, points on M and E with the same value of the parameter will no longer correspond.

The big involute $\tilde{M}$ is a 3-dimensional submanifold of the 5-dimensional space $\Omega \times R^3$ where $\Omega = T_{e(0)}E$.

For any individual space involute M, there is, by (2.43), a projection

$$\Omega \times R^3 \longrightarrow R \times R^3$$

which maps $\tilde{M}$ onto the hypersurface

$$D = \bigcup_{c \in \mathbb{R}^+} \{c\} \times \mathrm{Tub}_c(M)$$

in $\mathbb{R} \times \mathbb{R}^3$. Since the curve (2.59C) is distance generic near zero, D will be locally diffeomorphic at any point to one of the following three standard models

At points (2.59A) $\Delta(A_3) \times \mathbb{R}$

At points (2.59B) $\Delta(A_2) \times \mathbb{R}^2$

At points (2.59C) $\Delta(A_1) \times \mathbb{R}^3 = \mathbb{R}^3$ (a smooth hypersurface).

Conjecture 2.61

The big involute $\tilde{M} \subset \mathbb{R}^6$ is locally diffeomorphic at any point to one of the following three standard models:

At points (2.59A) S x $\mathbb{R}$ where S is an open swallowtail in $\mathbb{R}^4$.

At points (2.59B) $\Delta(A_3) \times \mathbb{R}^2$

At points (2.59C) $\Delta(A_1) \times \mathbb{R}^3 = \mathbb{R}^3$

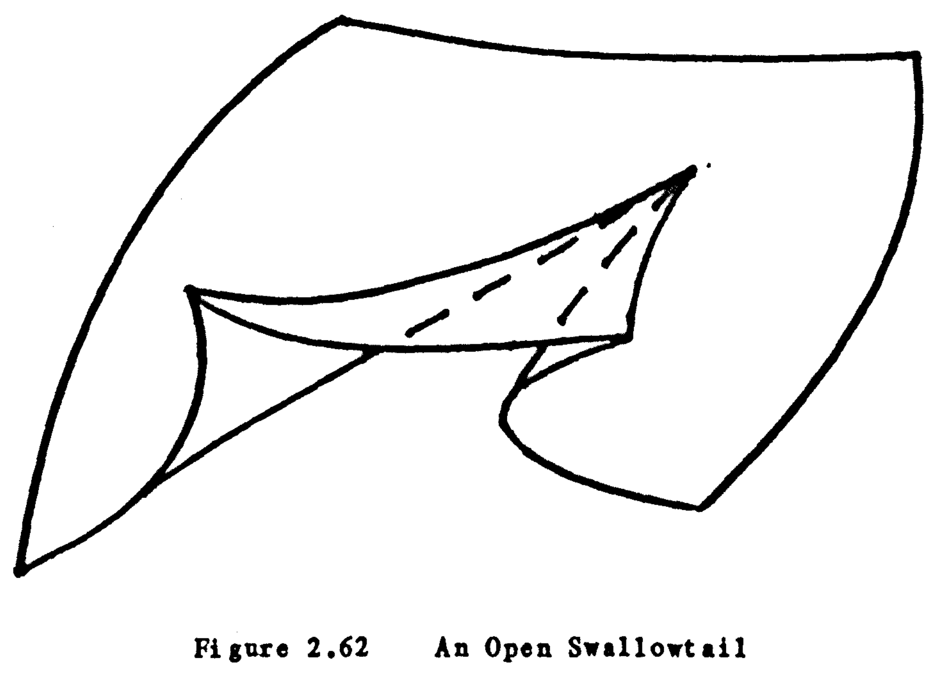


Figure 2.62 An Open Swallowtail

Note The open swallowtail is the set of monic quintic polynomials whose roots add up to zero and which have a root of multiplicity at least three [14]. A sketch of an open swallowtail in $\mathbb{R}^3$ is given in figure (2.62).

Example 2.63

Let E be a space curve of type (2,3,4) at e(0). Then by (2.41) M is of type (N,N+1,N+2) at r(0) for some N and the distance squared function $V^{e(0)}$ for M has an A_{N+3} singularity at 0. From the equations

$V^{e(0)}_N(0) = \dots = V^{e(0)}_{N+3}(0) = 0$ and $V^{e(0)}_{N+4} \neq 0.$

it follows that N=1 when e(0) is not on $T_{e(0)}E$

N=2 when r(0) lies on $T_{e(0)}E$ but $r(0) \neq e(0)$

N=4 when r(0) = e(0)

and other values of N lead to a contradiction.

The curve M is therefore locally congruent to one of the normal forms

(A) $(t^4, at^5+\dots, bt^6+\dots)$ $ab \neq 0$

(B) $(t^2, at^3+\dots, bt^4+ct^6+\dots)$ $ab \neq 0$, $a^2b \neq c$ (2.64)

(C) $(t, at^2+bt^3+ct^4+dt^5+\dots, et^3+ft^4+gt^5+\dots)$

In case (C) the constants a,b,....,g must satisfy $ae \neq 0$, $V_\xi^{(0)} = 0$ and $V_\xi^{(0)} \neq 0$

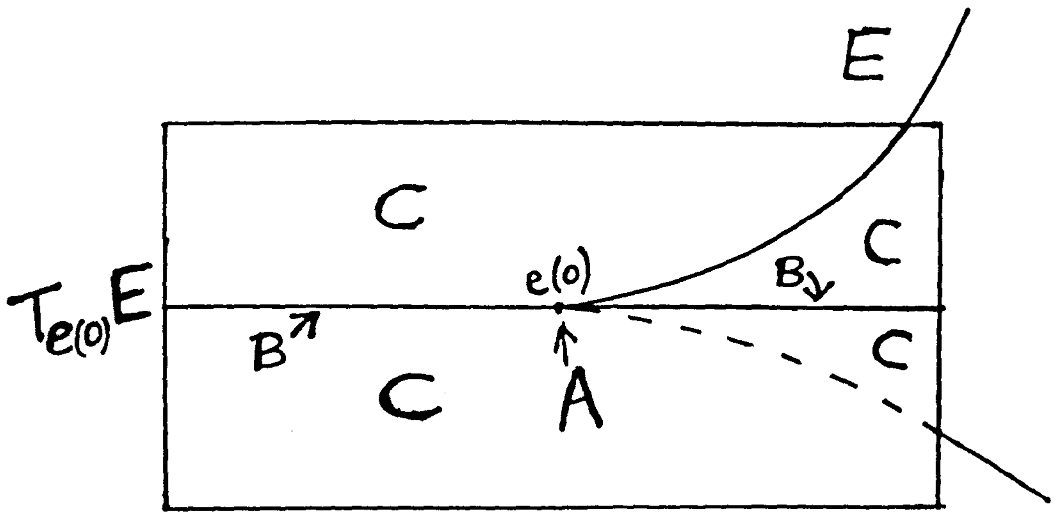
Let T denote the tangent developable to E, which by (2.39) is the focal set of M. Then (see figure (2.65))

Case (A) occurs at points where the curve M meets E

Case (B) occurs at points where M meets T−E

and Case (C) occurs everywhere else.

Figure 2.65 Normal Forms for Involutes of a Curve With a Point of Type (2,3,4)



The big involute $\tilde{M}$ is a 3-manifold in $\mathbb{R}^5$ which can be projected to a hypersurface that is locally diffeomorphic

at the point (2.64A) to $\Delta(A_4)$

at points (2.64B) to $\Delta(A_3) \times \mathbb{R}^2$

and at points (2.64C) to $\Delta(A_1) \times \mathbb{R}^3 = \mathbb{R}^3$

<u>Conjecture 2.66</u>

The big involute $\tilde{M}$ is locally diffeomorphic

at the point (2.64A) to an open butterfly in $\mathbb{R}^5$

at points (2.64B) to $\Delta(A_3) \times \mathbb{R}^2$

and at points (2.64C) to $\Delta(A_1) \times \mathbb{R}^3 = \mathbb{R}^3$

<u>Note</u> The open butterfly is the set of monic polynomials of degree 7 having a root of multiplicity at least 4 such that the sum of the roots is zero [14].

Example 2.67

Let E be of type (1,2,4) at e(0) and let M be a space involute of E. Then by (2.41) M is of type (N,N+2,N+3) where $V^{e(0)}$ has an A_{N+3} singularity at 0. As in the previous examples,

(A) N=4 when r(0)=e(0).

(B) N=3 when r(0) lies on $T_{e(0)}E$ and r(0) ≠ e(0).

(C) N=2 when r(0) does not lie on $T_{e(0)}E$.

The curve M is thus locally congruent to one of the following normal forms. The letters (A), (B) and (C) correspond to the three cases listed above.

(A) $(t^4, at^6+.., bt^7+)$ $\quad ab \neq 0$

(B) $(t^3, at^5+.., bt^6+ct^7+...)$ $\quad abc \neq 0$

(C) $(t^2, at^4+.., bt^5+..)$ $\quad V_6^{e(0)} \neq 0$

The focal set of M is the tangent developable to E. By [22], this is a surface with a cuspidal cross cap (this term is defined in (3.41)) as illustrated in figure (2.69).

Example 2.68

Let E be of type (1,3,4) at e(0) and let M be a space involute of E.

By (2.41) M is of type (N,N+1,N+3) where

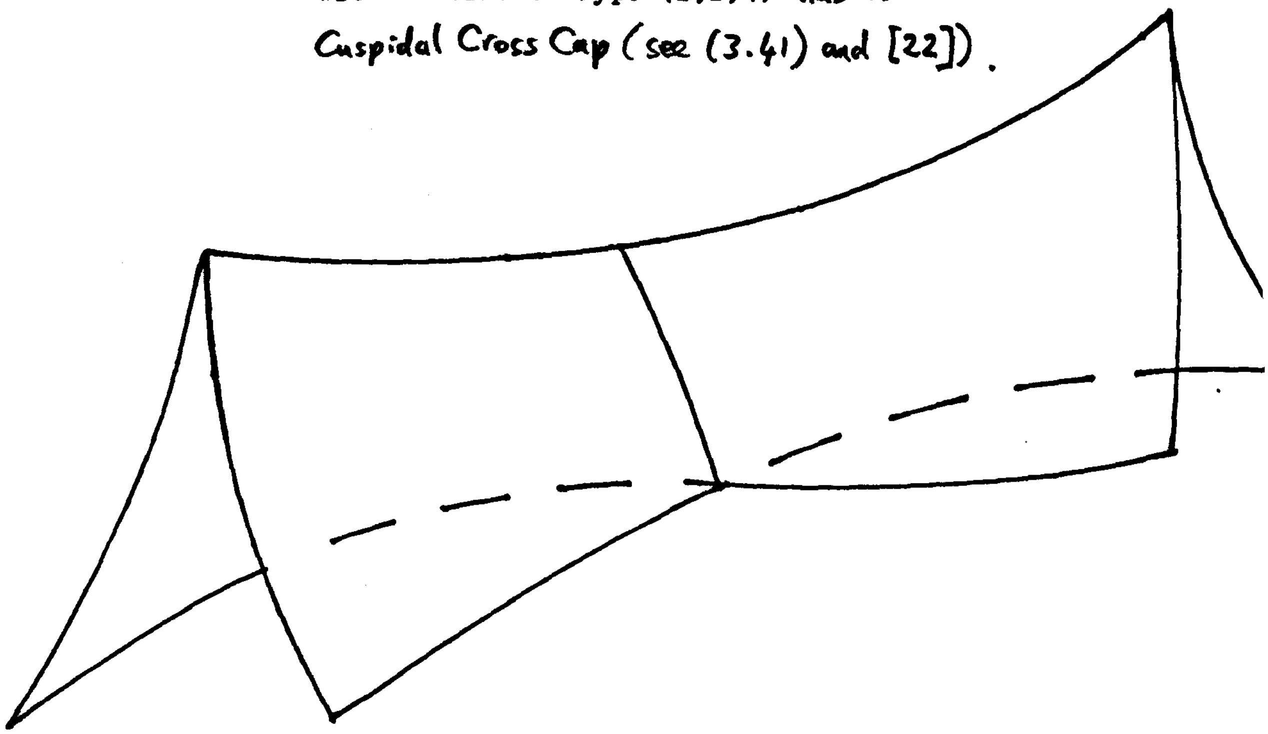
(A) N=4 when r(0)=e(0)

(B) N=3 when r(0) lies on $T_{e(0)}E$ but r(0) ≠ e(0)

(C) N=1 when r(0) does not lies on $T_{e(0)}E$

The general point of zero torsion described opposite cannot occur on a space involute. This is because its space evolute is at infinity. Therefore any point of zero torsion on a space involute must be in some sense 'atypical'.

Figure 2.69 The Tangent Developable to any Curve E With a Point of Type (1,2,4) Has a Cuspidal Cross Cap (see (3.41) and [22]).



By smooth changes of co-ordinates in $\mathbb{R}$ and an affine isometry in $\mathbb{R}^3$, the following normal forms for r are obtained

$$\begin{array}{lll}
\text{(A)} & (t^4, at^5+\ldots, bt^7+\ldots) & ab \neq 0 \\
\text{(B)} & (t^3, at^4+\ldots, bt^6+ct^7+\ldots) & abc \neq 0 \\
\text{(C)} & (t, at^2+bt^4+ct^5+\ldots, dt^4+et^5+\ldots) & 3cd+a^2e-3be \neq 0 \text{ and } ad \neq 0
\end{array} \qquad (2.70)$$

Equation (2.70C) is a parametrisation of a nonsingular space curve with a point of zero torsion at r(0). A point of zero torsion at space curve is one at which the osculating plane has A_3 contact with the curve (rather than A_2 contact, as is usual). In general a point of zero torsion occurs when the unique sphere making A_3 contact with the curve (the osculating sphere) is in fact a plane.

This is not the case for the curve given by (2.70C). The parametrisation (2.70C) gives a curve M for which there is a whole pencil of spheres (whose centres lie on a line C) making A_3 contact with M at the origin. There will be one sphere in this pencil which has infinite radius (and so is a plane).

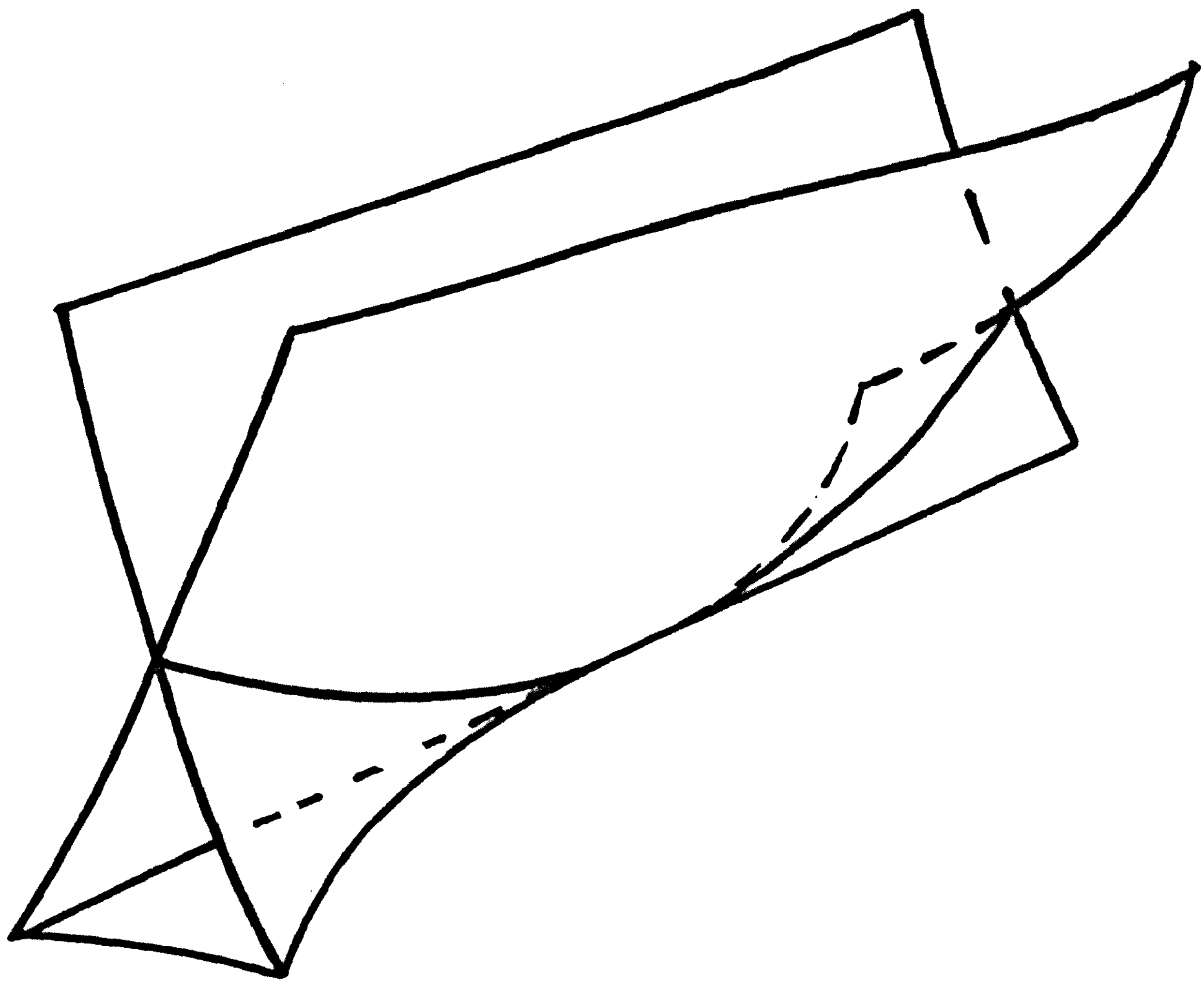


Figure 2.71 The Tangent Developable to any Curve With a Point of Type (1,3,4) is Locally Diffeomorphic to this Standard Model [22]

There will be precisely one sphere in the pencil (the osculating sphere) making A_4 contact with the curve: its centre will be the point $C \cap E$.

The focal set of M, which is the tangent developable to E will be the surface with two ordinary cuspidal edges illustrated in figure (2.71). The points on the two edges are the centres of A_3 spheres: one edge is the curve E and the other is the line C which is inflexional tangent to E. This surface is discussed in the survey [22] of possible singularities on the tangent developable of a curve in $\mathbb{R}^3$.

CHAPTER 3 INVOLUTES OF HYPERSURFACES

3.1 Introduction

The focal set F of a surface $M \subset \mathbb{R}^3$ consists of two sheets F_1 and F_2, which will be regarded as separate surfaces. The surface M will be called an **involute** of F_1 or of F_2. We examine some properties of the focal set and describe the classical procedure for reconstructing a surface M from either of the F_i. We then discuss certain configurations which cannot occur on the focal set of a smooth surface. By performing the classical construction on such a surface S, we can construct one-parameter families of surface germs which are all involutes of S, and all have singularities. This has already been done by Shcherbak [29] to obtain the Coxeter group H_3. In this chapter we will describe how to obtain the group H_4 in a similar way.

Recall from (2.10) that a hypersurface M always has a one-parameter family of parallels, and that it is usual to regard all these parallels as forming constant time-sections of a single hypersurface $\tilde{M}$ in space-time $\mathbb{R} \times \mathbb{R}^n$. The hypersurface $\tilde{M}$ is also called the **big wave front generated by M**, or simply a **big wave front**. Individual parallels are called **wave fronts** and are to be thought of as describing the propagation of a disturbance, such as light, through an isotropic space $\mathbb{R}^n$.

3.2 Behaviour of lines of curvature

In this section, we examine some of the properties of the focal set of a hypersurface in $\mathbb{R}^n$. Throughout this chapter, $r:(\mathbb{R}^{n-1},0) \to (M,m)$ will be a local parametrisation of the hypersurface $M^{n-1} \subset \mathbb{R}^n$.

The map $n: \mathbb{R}^{n-1} \to NM$ will be a smooth choice of unit normal obtained by composing r with a suitable section of NM.

The fibre of $\pi: F \to M$ above a non-singular point $m \in M$ consists of the (n-1) points

$$m + (1/k)n(0) \qquad (3.1)$$

for which k is a solution of the equations.

$$e = m + (1/k)n(0)$$

$$0 = V_2^e(0)(u) = (e-r(0)).r_2(0)(u) - r_1(0)(u).r_1(0) \qquad (3.2)$$

$$= (1/k)n(0).r_2(0)(u) - r_1(0)(u).r_1(0)$$

for some $u \neq 0$.

These conditions can be expressed in the form

$$0 = \det(n(0).r_2(0) - kr_1(0).r_1(0)) \qquad (3.3)$$

where det is a function on the space $S^2\mathbb{R}^{n-1}$ of quadratic forms in (n-1) variables for which $\det^{-1}(0)$ is the space of degenerate quadratic forms. If M is smooth, there are (n-1) solutions $k_1,\dots,k_{n-1}$ to (3.3) which are all real, but not necessarily distinct. If $k_i = 0$ then (3.1) is to be interpreted as giving a point at infinity in $\mathbb{R}P^n$. The function $x \to k_i(x)$ is the i^{th} principal curvature function on M and its reciprocal is called the i^{th} principal radius of curvature.

We shall only consider the case when the principal curvatures k_i are all distinct. If the curvature k_i has multiplicity λ at m, then V^x has a singularity of corank λ at 0. Thus if m is a distance generic hypersurface (see 2.8), the set of points of M at which two or more of the principal curvatures coincide is of codimension 2 in M. Thus our discussion covers the behaviour of almost all hypersurfaces at almost all points.

A vector u satisfying (3.2) is called a **principal direction**, and an integral curve of one of the (n-1) fields of principal directions on M is called a **line of curvature**. There are (n-1) systems of lines of curvature on M and each system is associated via (3.2) with a definite sheet of the focal set. Given any point $m \in M$, there is exactly one curve of each system passing through M.

Proposition 3.4

A curve γ on M is a line of curvature if and only if the normals at points of γ form a **developable** surface, i.e. a ruled surface whose tangent plane is constant along the straight lines of the ruling. (Equivalently a developable surface is a ruled surface with Gaussian curvature everywhere identically zero).

Proof

This is a consequence of Rodriguez's formula (see [31] vol.3 chap.4). For properties of developable surfaces see [31] vol.3,chap.5. QED.

Let F_i be the i^{th} sheet of the focal set. Then considered as a submanifold of $M \times \mathbb{R}^n$, F_i is given by

$$x \longrightarrow (r(x), e(x)) = (r(x), r(x) + n(x)/k_i(x)) \qquad (3.5)$$

Considered as a subset of $M \times \mathbb{R}^n$, the focal set F_i is singular only at places where two of the principal curvatures are equal (since anywhere else k_i is a smooth function of x and so equation (3.5) gives a smooth parametrisation). However, when F_i is projected to $\mathbb{R}^n$ by π_2, it can acquire extra singularities, such as cuspidal edges.

Theorem 3.6

The inverse image of a line of curvature of the i^{th} system under $\pi : F_i \longrightarrow M$ is a geodesic on the hypersurface $F_i \subset \mathbb{R}^n$

Definition 3.7

Such a curve will be called a **raised line of curvature**, abbreviated to RLC.

Let $t \longrightarrow \phi(t)$ and $t \longrightarrow \theta(t)$ be germs of curves such that $r \circ \phi$ and $r \circ \theta$ are lines of curvature of the i^{th} and j^{th} systems respectively. Assume that $k_i \neq 0$, and let $e : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be given by (3.5). Then $e \circ \phi$ is an RLC and we shall prove that $e \circ \phi$ is a geodesic on F_i.

Let $\rho = 1/k_i$ and let tilde denote composition with ϕ, so that, for example $\tilde{e} = e \circ \phi$, $\tilde{r} = r \circ \phi$.

Similarly, let bar denote composition with θ, so that for example $\bar{e} = e \circ \theta$.

A curve on a hypersurface is a geodesic if and only if each osculating plane of the curve contains the normal to the hypersurface. We shall prove that $\tilde{e}$ has this property. In fact, we shall prove the following more general result about the behaviour of the RLCs on the surface F_i.

Theorem 3.8

Let M be a nonsingular hypersurface with $k_i(0) \neq k_{i'}(0)$ for all $i \neq i'$ and $k_i(0) \neq 0$. Use the notation introduced in the previous two paragraphs. Then the vector $\tilde{r}_1(0)$ is normal to F_i at e(0) and either there is an integer $j \geqslant 1$ such that the conditions (i) to (vii) below are satisfied, or the conditions (i)' to (v)' below are satisfied.

(i) $V^{e(0)}$ has an A_{j+1} singularity, in other words the tangent hypersphere with centre e(0) tangent to M at r(0) has A_{j+1} contact with M.

(ii) $(V^{e(0)} \circ \phi)$ has an A_{j+1} singularity at the origin.

(iii) At the origin, the first (j+1) derivatives of $(V^{\varrho(0)} \circ \phi)$ vanish, but its (j+2)nd derivative is non-zero.

(iv) $\tilde{\rho}$ has an A_{j-1} singularity at 0.

(v) $\tilde{e}_1(0) = \ldots\ldots = \tilde{e}_{j-1}(0) = 0$

(vi) $\tilde{e}_j(0)$ and $\tilde{e}_{j+1}(0)$ are linearly independent

(vii) $\tilde{e}_j(0).r_1(0) = 0$ and $\tilde{e}_{j+1}(0)$ lies in the plane spanned by $\tilde{e}_j(0)$ and $\tilde{r}_1(0)$.

(i)' $V^{e(0)}$ has a singularity of corank 1 and infinite codimension at 0.

(ii)' $(V^{e(0)} \circ \phi)$ is a flat function at 0 (all its derivatives vanish there).

(iii)' $(V^{\varrho(0)} \circ \phi)$ is a flat map at 0 (all its derivatives vanish there).

(iv)' $\tilde{\rho}$ is a flat function at 0.

(v)' $\tilde{e}$ is a flat map at 0.

The hypothesis that $k_i \neq 0$ ensures that the sheet of the focal set under consideration is not at infinity.

Remarks about Notation

In this proof V with no superscript will mean $V^{e(o)}$ and the tilde operation will be performed before differentiation, so that, for example, $\tilde{r}_j$ denotes $(\tilde{r})_j$. The equations (3.9) to (3.11) and (3.14) to (3.23) describe the behaviour of points on the line of curvature given by ϕ, but to make the formulae more concise, the arguments $\phi(t)$ and t have been omitted. Thus, for example,

ρ_j denotes $\rho_j(\phi(t))$

$\tilde{e}_j$ denotes $\tilde{e}_j(t)$

and V^e denotes $V^{e\ \phi(t)}(\phi(t)) = V^{e(t)}(\phi(t))$.

Proof of theorem 3.8

On the line of curvature given by ϕ,

$$0 = V_1^e = (e-r).r_1 \qquad (3.9)$$

$$0 = V_1^e\ \phi_1 = (e-r).r_2\phi_1 - r_1\phi_1.r_1 \qquad (3.10)$$

[Reminder: In unsimplified notation, (3.9) would read

$$0 = V_1{}^{e(t)}(\phi(t)) = (\tilde{e}(t) - \tilde{r}(t)).r_1(\phi(t))]$$

Differentiating (3.9) with respect to t and evaluating at t=0, we get

$$0 = (e - r).r_2\phi_1 + (\tilde{e}_1 - \tilde{r}_1).r_1 = V_1^e\ \phi_1 + \tilde{e}_1.r_1 = \tilde{e}_1.r_1 \qquad (3.11)$$

As well as (3.9), we also have

$$0 = V_1^{\rho(t)}(\theta(t)) = (\bar{e}(t) - \bar{r}(t)).(r_1(\theta(t)))$$

Differentiating this equation with respect to t gives

$$0 = (\bar{e} - \bar{r}).(r_2\circ\theta)\theta_1 + (\bar{e}_1 - \bar{r}_1).(r_1\circ\theta) =$$

$$V_1^{\rho(t)}\ \theta_1 + (e_1\circ\theta)\theta_1.(r_1\circ\theta) \qquad (3.12)$$

So substituting t=0 in (3.12) and applying the operator (3.12) to $\phi_1(0)$ we get

$$0 = e_1(0)\theta_1(0).\tilde{r}_1(0) = \bar{e}_1(0).\tilde{r}_1(0) \qquad (3.13)$$

Equations (3.11) and (3.13) now tell us that

$$e_1(0).\tilde{r}_1(0) = 0 \qquad (3.14)$$

so that the vector $\tilde{r}_1(0)$ is perpendicular to the hypersurface $F_1 \subset R^n$.

We note that $\rho\tilde{n}_1 + \tilde{r}_1 = 0$ (Rodriguez's formula: see [31] vol.3, chap.4) so differentiating $e = r + \rho n$ in the ϕ_1 direction gives

$$\tilde{e}_1 = \tilde{r}_1 + \tilde{\rho}_1 n + \rho\tilde{n}_1 = \tilde{\rho}_1 n \qquad (3.15)$$

Consider now the following set of conditions:

$$\begin{aligned} 0 &= (V_1{}^e\theta)_{j+1} \\ 0 &= \tilde{e}_j.r_2\phi_1 \\ 0 &= \tilde{e}_{j+1}.r_1 \\ 0 &= \tilde{\rho}_j \\ 0 &= \tilde{e}_j \\ 0 &= (V_0\theta)_{j+2} \\ 0 &= \tilde{e}_j.\tilde{r}_2 \\ 0 &= \tilde{e}_{j+1}.\tilde{r}_1 \end{aligned} \qquad (3.16)_j$$

The proof of (3.8) will follow easily once the following technical result has been proved.

<u>Assertion 3.17</u>

Suppose that the conditions $(3.16)_1, \ldots\ldots, (3.16)_{j-1}$ all hold at the point t=0 of the line of curvature ϕ. (This hypothesis is vacuous if j=1.) Then the conditions $(3.16)_j$ are all equivalent: either they all hold or they all fail.

We now justify this assertion. A special case of (3.9) is

$$0 = V_1{}^e\phi_1 \qquad (3.18)$$

Differentiating (3.18) (j+1) times and (3.10) j times with respect to t and evaluating at t=0 gives

$$0 = (V_0\theta)_{j+2} + (j+1)\tilde{\sigma}_j.\tilde{r}_2 + \tilde{\sigma}_{j+1}.\tilde{r}_1 \text{ at } t=0 \qquad (3.19)$$

and $$0 = (V_1 \circ \theta)_{j+1} + \tilde{\sigma}_j.r_2\phi_1 \qquad \text{at } t=0 \qquad (3.20)$$

(since by hypothesis $\tilde{\sigma}_1(0) = \ldots\ldots = \tilde{\sigma}_{j-1}(0) = 0$)

Differentiating (3.11) j times with respect to t and evaluating at t=0 gives

$$0 = \tilde{\sigma}_{j+1}.r_1 + j\tilde{\sigma}_j.r_2\phi_1 \qquad \text{at } t=0 \qquad (3.21)$$

(since by hypothesis $\tilde{\sigma}_1(0) = \ldots\ldots = \tilde{\sigma}_{j-1}(0) = 0$)

Differentiating (3.15) (j-1) times with respect to t and evaluating at t=0 gives

$$\tilde{\sigma}_j = \tilde{\rho}_j n \qquad \text{at } t=0 \qquad (3.22)$$

(since by hypothesis $\tilde{\rho}_1(0) = \ldots\ldots = \tilde{\rho}_{j-1}(0) = 0$)

We now assemble all these equations.

$(V_1\circ\theta)_{j+1}$ = (by (3.20)) $- \tilde{\sigma}_j.r_2\phi_1$ = (by (3.21))

$(1/j)\tilde{\sigma}_{j+1}.r_1$ = (by (3.22)) $- \tilde{\rho}_j n.r_2\phi_1$ = (by (3.10))

$$- (\tilde{\rho}_j/\rho)\tilde{r}_1.r_1 \qquad \text{at} \quad t=0 \qquad (3.23)$$

Equation (3.23) gives a cotangent vector L_j to R^{n-1} at the origin. For L_j to be identically zero, it is necessary and sufficient that it be zero when applied to the vector $\phi_1(0)$, since it is of the form $\lambda\tilde{r}_1(0).r_1(0)$. Thus $L_j = 0$ iff

$0 = L_j\phi_1 = - \tilde{\sigma}_j.\tilde{r}_2$ = (by (3.19) and (3.21)) $(V_0\theta)_{j+2}$.

This proves assertion (3.17). We now conclude the proof of (3.8). It follows from (3.17) that either there is an j such that $(3.16)_1,\ldots\ldots,(3.16)_{j-1}$ hold and $(3.16)_j$ fails, and so (3.8.ii) to (3.8.vi) hold, or else $(3.16)_k$ is true for all k and so (3.8.ii)' to (3.8.v)' are satisfied.

In the first case, the operator L_j of (3.23) gives zero when applied to $\theta_1(0)$, and so $0 = \tilde{e}_{j+1}.\bar{r}(0)$. This proves (3.8.vii).

It remains to prove that conditions (3.8.i)and (3.8.i)' hold. This is done using a result from singularity theory. Let $F:(\mathbb{R}^{n-1},0) \longrightarrow (\mathbb{R},0)$ be a function germ with a corank 1 singularity at the origin. Then F has an A_k singularity if and only if there is a curve germ $\phi:(\mathbb{R},0) \longrightarrow (\mathbb{R}^{n-1},0)$ such that the composites

composites $F_0\phi:(\mathbb{R},0) \longrightarrow (\mathbb{R},0)$

and $F_1 \circ \phi:(\mathbb{R},0) \longrightarrow ((\mathbb{R}^{n-1})^*,0)$

both have A_k singularities at the origin (see [27])
This completes the proof of (3.8)

In the remainder of this section, some classical results about the evolutes of surfaces in $\mathbb{R}^3$ are generalised to the case of hypersurfaces in $\mathbb{R}^n$. Two of these results concern the function ρ which associates to each point x the i^{th} principal radius of curvature at x. This function can be considered as having either F_i or M as its domain.

<u>Proposition 3.24</u>

Using the notation above, wherever F_i is smooth, the vectors $e_1\phi_1$ and $e_1\theta_1$ are conjugate on F_i (i.e. $S(e_1\phi_1,e_1\theta_1) = 0$ where S is the second fundamental form of the surface F_i).

<u>Proof</u>

We must show that $N.e_1\phi_1\theta_1 = 0$ where N is a vector normal to the surface F_i. We may take $N = \tilde{r}_1$ since by (3.14), $e_1.\tilde{r}_1 = 0$.

By (3.11), $\tilde{e}_1.r_1 = 0$. We want to differentiate (3.11) in the θ_1 direction. In order to be able to do this, the domain of ϕ must be extended. Let $\Omega(s,t) = \phi^s(t)$ where $t \to r(\phi^s(t))$ is the line of curvature of the same system as $r(\phi^0(t))$ passing through $r(\theta(s))$, in particular $\phi^0 = \phi$. Now

$$0 = \tilde{e}_1.r_2\theta_1 + e_1\eta.r_1 + e_2\phi_1\theta_1.r_1 \qquad (3.25)$$

$$\text{where } \eta = \left.\frac{\partial^2\Omega}{\partial s\partial t}\right|_{(s,t)=(0,0)}$$

So applying (3.25) to ϕ_1 gives

$$0 = \tilde{e}_1.r_2\phi_1\theta_1 + e_1\eta.\tilde{r}_1 + e_2\phi_1\theta_1.\tilde{r}_1 = e_2\phi_1\theta_1.\tilde{r}_1 \qquad (3.26)$$

since $e_1\eta.\tilde{r}_1 = 0$ by (3.14)

and $\tilde{e}_1.r_2\phi_1\theta_1 =$ (by (3.22)) $\tilde{\rho}n.r_2\phi_1\theta_1 =$

(by (3.10)) $(\tilde{\rho}_1/\rho)r_1\phi_1.r_1\theta_1 = 0.$

<u>Proposition 3.27</u>

Take a region in which F_i is smooth and consider ρ as a function on F_i. Then ρ: $F_i \to \mathbb{R}$ is a submersion and the RLCs are orthogonal to the level sets of the function ρ.

<u>Proof</u>

If F_i is nonsingular at a point y, then any geodesic on F_i must be nonsingular, so conditions $(3.16)_1$ do not hold at y, and so $\tilde{\rho}_1 \neq 0$ at y. It follows that the function ρ is submersive.

We want to show that if $\eta \in T_0\mathbb{R}^{n-1}$ is such that $e_1\eta$ is perpendicular to $\tilde{r}_1$ then the derivative of ρ in the direction of η is zero, i.e. $\rho_1\eta = 0$.

But $\rho^2 = (e-r).(e-r)$ so

$\rho(\rho_1\eta) = (e-r).(e_1\eta - r_1\eta) = \rho n.(e_1\eta - r_1.\eta)$ and so

$$\rho_1\eta = n.e_1\eta = (1/\tilde{\rho}_1)\tilde{e}_1.e_1\eta \qquad (3.28)$$

As was mentioned in the previous paragraph, the hypothesis that F_i is smooth ensures that $\tilde{\rho}_1 \neq 0$. The desired result follows from (3.28). QED.

Suppose that M is a non-singular hypersurface. The next result characterises those points of M which are critical points for the function ρ. A necessary condition for a point to be critical was found during the proof of (3.27): any critical point of ρ must be a point at which (in the notation of (3.8)) $\tilde{\rho}_1 = 0$, and so must be the point of contact of an A_k sphere for some $k \geq 3$.

<u>Proposition 3.29</u>

Use the notation of (3.8). Suppose that M is non-singular, and that conditions (i) to (vii) of (3.8) hold for some $j \geq 2$. Let y a point on F_i, with $\pi(y) = m$. Then the function $\rho : M \to \mathbb{R}$ has a critical point at $m \in M$ if and only if the RLC passing through y is perpendicular to Im e_1 at y.

<u>Proof</u>

As in (3.8), the RLC passing through y is parametrised by the map $t \to \tilde{e}(t)$, where $\tilde{e}(0) = y$. By hypothesis $\tilde{e}_j(0) \neq 0$ and $\tilde{e}_1(0) = \ldots = \tilde{e}_{j-1}(0) = 0$. We wish to prove that $e_1(0)\eta . \tilde{e}_j(0) = 0$ for all vectors $\eta \in \mathbb{R}^{n-1}$ if and only if $\rho_1(0)\eta = 0$ for all vectors $\eta \in \mathbb{R}^{n-1}$.

But $\tilde{e}_j(0) = \tilde{\rho}_j n$ (by (3.22)). So

$e_1(0)\eta . \tilde{e}_j(0) = \tilde{\rho}_j n . e_1(0)\eta = (\tilde{\rho}_j/\rho)(e(0)-r(0)).e_1(0)\eta =$

$(\tilde{\rho}_j/\rho)(e(0)-r(0)).e_1(0)\eta =$

$(\tilde{\rho}_j/\rho)(e(0)-r(0)).(e_1(0)\eta - r_1(0)\eta) =$

$(\tilde{\rho}_j/2\rho)[(e(0)-r(0))^2]_1 \eta = (\tilde{\rho}_j/2\rho)(\rho^2)_1 \eta = \tilde{\rho}_j(\rho_1 \eta)$

The result now follows. QED .

Proposition (3.27) is geometrically obvious, because as γ_1 is normal to M, ρ must change fastest if one travels in the γ_1 direction and ρ will vary least if one travels in a direction perpendicular to γ_1.

However, (3.27) has significant implications in the case of hypersurfaces in R^n for $n \geq 4$. For (3.27) means that the geodesic foliation on F_i formed by the RLCs is everywhere perpendicular to a one-parameter family of surfaces. An arbitrary geodesic foliation on a k-manifold ($k \geq 3$) does not have this property (see [35] § 100).

__Definition 3.30__

Let R^N be a Riemannian manifold with a geodesic foliation. A distribution can be constructed on R by associating to each point the (N-1)-dimensional space perpendicular to the geodesic passing through that point. If this distribution is integrable, the foliation will be called __orthogonally integrable__.

To say that the distribution is integrable, means that the geodesics are everywhere perpendicular to a family of hypersurfaces in R. If one integral manifold exists, then there is a one-parameter family of integral manifolds. It cannot happen that integral manifolds exist passing through some points but not through others. Note that any geodesic foliation on a surface is orthogonally integrable.

The results (3.6) and (3.27) can be combined as follows:

__Theorem 3.31__

The RLCs on F_i form a orthogonally integrable geodesic foliation.

We now demonstrate how the results discussed so far in this chapter apply to surfaces in R^3. If M is a non-singular distance generic surface, then the only possible singularities of the distance squared functions for M are A_1, A_2, A_3, A_4, and D_4. The focal set F is the set of points y for which the function V^y has singularities of types A_2 A_3 A_4 and D_4.

The classification of Lagrangian and Legendrian singularities [4] provides normal forms for the focal set of M and the big wave front generated by M. If the function V^y for the surface M has an A_k(or D_4) singularity then the big wave front is locally diffeomorphic at (τ, y) to $\Delta(A_k) \times R^{4-k}$ (or one of the two forms of $\Delta(D_4)$). When the big involute is locally diffeomorphic to A_4 or D_4, the individual involutes are locally diffeomorphic to the standard family of sections of the discriminant, given by f_2 = constant, described in chapter one.

The A_2 points on F are those at which F is non-singular. By (3.6) the RLCs passing through such points are geodesics. Furthermore, by (3.8) an RLC can never have an inflexion (a point of zero curvature) at a nonsingular (A_2) point of F.

The A_3 points form curves on F called <u>**ribs**</u>, which are ordinary cuspidal edges (see (3.38) below). The ribs project under π:F —> M to nonsingular curves called <u>**ridges**</u>.

By (3.8) and (2.48) an RLC passing through a rib-point has an ordinary cusp there (ordinary cusps are defined in (2.46)).

Although there is a singular normal form $\Delta(A_3) \times \mathbb{R}$ for the big wave front, there may be many ways of taking sections to obtain the individual parallels. There turn out to be essentially two different ways in which the parallels can evolve (see [8]). Either each parallel is locally diffeomorphic to $\Delta(A_3)$ and the big involute has a product structure, or a pair of swallowtails is born as the wave front passes through the rib point on F. These two different configurations can be related to the RLCs crossing the cuspidal edge as follows.

Proposition 3.32

(i) The cuspidal tangent to an RLC at an A_3 point is not tangent to the rib.

(ii) The following conditions at a rib point y are equivalent.

(A) The cuspidal tangent to the RLC at the point y is perpendicular to the rib.

(B) $\pi(y)$ is a critical point of the principal curvature function on M

(C) Birth of a pair of swallowtails occurs in the family of parallels to M at y.

Proof of proposition 3.32

Use the notation of (3.8).

(i) By (3.11) and (3.12) $e_{1,r_1} = -V_2 \neq 0$

but by (3.8 vii) $\tilde{e}_{2,r_1} = 0$

Since the plane Im r_1 is perpendicular to $\tilde{e}_2$ but not to

$R\{\bar{\sigma}_1\}$ = Im e_1, it follows that $\tilde{c}_2$ (the cuspidal tangent to the RLC) and $\bar{\sigma}_1$ (the tangent to the rib) are not tangent to each other.

(ii) This is a restatement of (3.29). The equivalence of conditions (B) and (C) is also discussed in [8] and [23]. QED.

The A_4 and D_4 points are isolated points on F which for a distance-generic surface always lie on ribs. At an A_4 point y. F has a swallowtail and there is one rib passing through y which has an ordinary cusp. By (3.8) the RLC passing through y is congruent to a curve of the form $(t^3, at^4+\ldots, bt^5+\ldots)$ with $a \neq 0$. By an argument similar to (3.33) the singular tangent to the RLC and the cuspidal tangent to the rib are never tangent to each other. They are perpendicular if and only if the A_4 point projects under π to a critical point of the principal curvature ρ. (This means, in the notation of (3.8) that $\tilde{\rho}_1 = \tilde{\rho}_2 = 0$ and $\rho_1=0$. This is something that does not occur at the A_4 points of a generic surface.)

3.3 The Construction for Involutes of a Hypersurface

It was shown in the previous section that if M is a hypersurface, the RLCs on each sheet of its focal set form a orthogonally integrable geodesic foliation.

The original hypersurface can be recovered from any one of the F_i if we are also given the associated geodesic foliation. This classical procedure (see [35] § 81) is briefly described below. We need to introduce the idea of __unwinding__ a curve M in R^n. This is done (physically) by taking a piece of string stretched tightly along the curve. We choose an arbitrary point P of the curve (the __starting point__), cut the string at the point P, and attach two pieces of chalk to the two free ends. Now unwind the string from the curve, keeping it taut it taut at all times. The curve traced out by the chalk will be called an __unwinding__ of M. If $\gamma: t \to \gamma(t)$ is an arc-length parametrisation of M, the unwinding of M with starting point $\gamma(c)$ is given by

$$t \to \gamma(t) + (c-t)\gamma_1(t) \qquad (3.33)$$

Let F_1 be any hypersurface equipped with a orthogonally integrable geodesic foliation. Let σ be an orthogonal trajectory to these geodesics. Let M be the hypersurface formed by taking the union of the curves obtained by unwinding each geodesic, starting from the point at which it meets σ. The hypersurface F_i forms one of the (n-1) sheets of the focal set of M and the RLCs on F_1 are the geodesics we started with. The hypersurface M is therefore called an __involute__ of F_1 and σ is called the __starting set__ (if n=3, the __starting line__). Any hypersurface parallel to M is also called an involute of M: some,but not necessarily all of these parallel hypersurfaces can be constructed directly from M by changing the starting set.

The hypersurface M and its all its parallels are the constant-time sections of a big wave-front in space-time. This big front will be called the <u>big involute</u> of the foliated hypersurface F_1.

We have constructed a one-parameter family of hypersurfaces whose (n-1) sheets of focal set are F_1 and some other hypersurfaces $F_2,\ldots,F_{n-1}$. The (n-2) hypersurfaces $F_2,\ldots,F_{n-1}$ are called the <u>complementary hypersurfaces</u> to the foliated hypersurface F_1. A different choice of geodesic foliation on F_1 would give a different one-parameter family of parallel surfaces, whose focal set consists of F_1 and (n-2) other sheets $G_2,\ldots,G_{n-1}$. Thus two hypersurfaces M and M' can have the same focal set (in the sense that one sheet of the focal set of M coincides with one sheet of the focal set of M') without M and M' being parallel. This is in contrast to the behaviour of curves. If two curves have the same focal set, they are necessarily parallel.

The problem of describing the singularities of the big involute of a hypersurface F has been called by Arnold the problem of avoiding an obstacle. This name arises from considering F as forming the boundary of an obstacle in $\mathbb{R}^n$, and looking for paths in $\mathbb{R}^n$ which minimize length, subject to the constraint of not passing through the obstacle. Such paths clearly consist of partially unwound geodesics, i.e. once differentiable piecewise smooth curves consisting of portions of geodesics on F and portions of straight lines.

There are a number of well known results describing the behaviour of involutes. Let $F \subset \mathbb{R}^3$ be an arbitrary nonsingular surface, equipped with a geodesic foliation. Let G be the complementary surface, let M be an involute with starting line σ, and let $\pi: F \to M$ be the natural projection. If none of the geodesics have inflexions, then M is nonsingular, except for ordinary cuspidal edges (these are defined in (3.38) below) along the starting line (where M meets F) and along the locus where M meets G. Take an arbitrary point $x \in F$, and let l be the tangent line to the geodesic passing through x. Then each involute, will contain one point of l. Precisely two of the involutes of F have singularities along the line l. These are the involutes M_x and M_y containing the points x and y respectively, where y is the point of intersection of l with the complementary surface G. If M is a third involute of F, then M, M_x and M_y are all parallel and so by (2.2) there are parallel maps $\tau: M_x \to M$ and $\tau': M_y \to M$ such that the images of the singular points x and y are nonsingular. So these singularities (the ordinary cuspidal edges) can be eliminated by applying a parallel map.

Let the surface F have an ordinary cuspidal edge γ and suppose that the geodesics of the foliation cross the edge with ordinary cusps, in such a way that the cuspidal tangents are neither tangent nor perpendicular to the edge. The presence of the cuspidal edge does not alter the fact that the involutes of this foliation are only singular where they meet F and G. As in the previous example, each individual involute M has an ordinary cuspidal edge along the starting line $\sigma = \pi(\sigma)$. The new feature is that the starting line σ may meet the cuspidal edge at a point P. If this occurs, the starting line will have an ordinary cusp at the point P, where it crosses over from one side of the edge γ to the other. The involute will then have a swallowtail at P. The curve $\pi(\gamma)$ will be non-singular and will be a ridge on the involute, passing through the swallowtail point P. This curve can only be located by examining the metric properties of the M. Up to diffeomorphism, none of the points of $\pi(\gamma)$, except for P itself, can be distinguished from the other non-singular points of the involute.

If one of the geodesics on F has an inflexion, then by (3.8) this particular foliation cannot be the system of RLCs arising a non-singular surface M. It is still possible to use the classical construction to produce a one-parameter family of involutes of the foliated surface F. Each of these involutes will have a singularity at the point where it intersects the inflexional tangent l. These singularities will persist under the application of a parallel map, since any such map preserves the line l. Involutes of such foliations were first considered in [2] [24] and [29]. In [29] the following results were obtained.

If one geodesic has a <u>non-degenerate point of zero curvature</u>

(a point of type (1,3,4)) then each nearby geodesic has also a point of type (1,3,4), and these points form a smooth curve γ on the surface F.

Let M be any involute and let $\pi : F \to M$ be the projection. Then M has a rhamphoid cuspidal edge (see (3.39)) along the curve $\pi(\gamma)$. There will also be an ordinary cuspidal edge (see (3.38)) along the starting line. Suppose the starting line σ of the involute crosses the curve γ at a point P in such a way that the curves σ and γ are neither perpendicular nor tangent. Then the point P = π(P) on the involute M will be the point of intersection of an ordinary cuspidal edge and a rhamphoid cuspidal edge. It was shown by Shcherbak [29] that M is locally diffeomorphic at P to $\Delta(H_3)$.

Any involute sufficiently close to M will also have a $\Delta(H_3)$ singularity and the big involute will be locally diffeomorphic to $\Delta(H_3) \times \mathbb{R}$.

As explained above, the behaviour of the involutes depends on whether the geodesics have zero curvature. The following lemma shows when this occurs.

<u>Lemma 3.34</u>

Let γ be a geodesic on a non-singular surface F. Then γ has an inflexion at $x \in F$ if and only if the tangent direction to γ at x is an asymptotic direction for the surface F. Thus geodesics with inflexions can occur only in the hyperbolic regions of F (where the two asymptotic directions are real: in the notation of Arnold [3], in $\Pi_{3,1}$ regions), and not in the elliptic regions (where the two asymptotic directions are complex: in the notation of Arnold, in Π_3 regions).

It was anticipated by Arnold that the group H_4 would occur in a similar way to the other Coxeter groups, with the big involute of a suitable foliated surface F being locally diffeomorphic to $\Delta(H_4)$, and individual involutes being locally diffeomorphic to the sections $\Delta(\varepsilon)$ described in chapter one. To say that there is a surface whose involute is $\Delta(0)$ is not a statement of any significance, since the involute of the evolute of $\Delta(0)$ is bound to be $\Delta(0)$. We are looking for more than that: we expect each involute of F to be locally diffeomorphic to one of the sections $\Delta(\varepsilon)$ of $\Delta(H_4)$. We now make some informal remarks to suggest what such a surface F might be like. The purpose of these comments is to provide some motivation for studying the surfaces with cuspidal edges which will be examined later.

The H_3 points described above arise when a rhamphoid cuspidal edge meets an ordinary cuspidal edge in the most natural way. By analogy with the way in which H_3 was obtained, we look for the most natural way to combine big involutes of types A_3 and H_3. The groups A_3 and H_3 are chosen because, in the discriminant $\Delta(H_4)$ described in chapter 1, there is a curve consisting of swallowtails (A_3 points) and a curve of $\Delta(H_3)$ singularities, intersecting at the H_4 point.

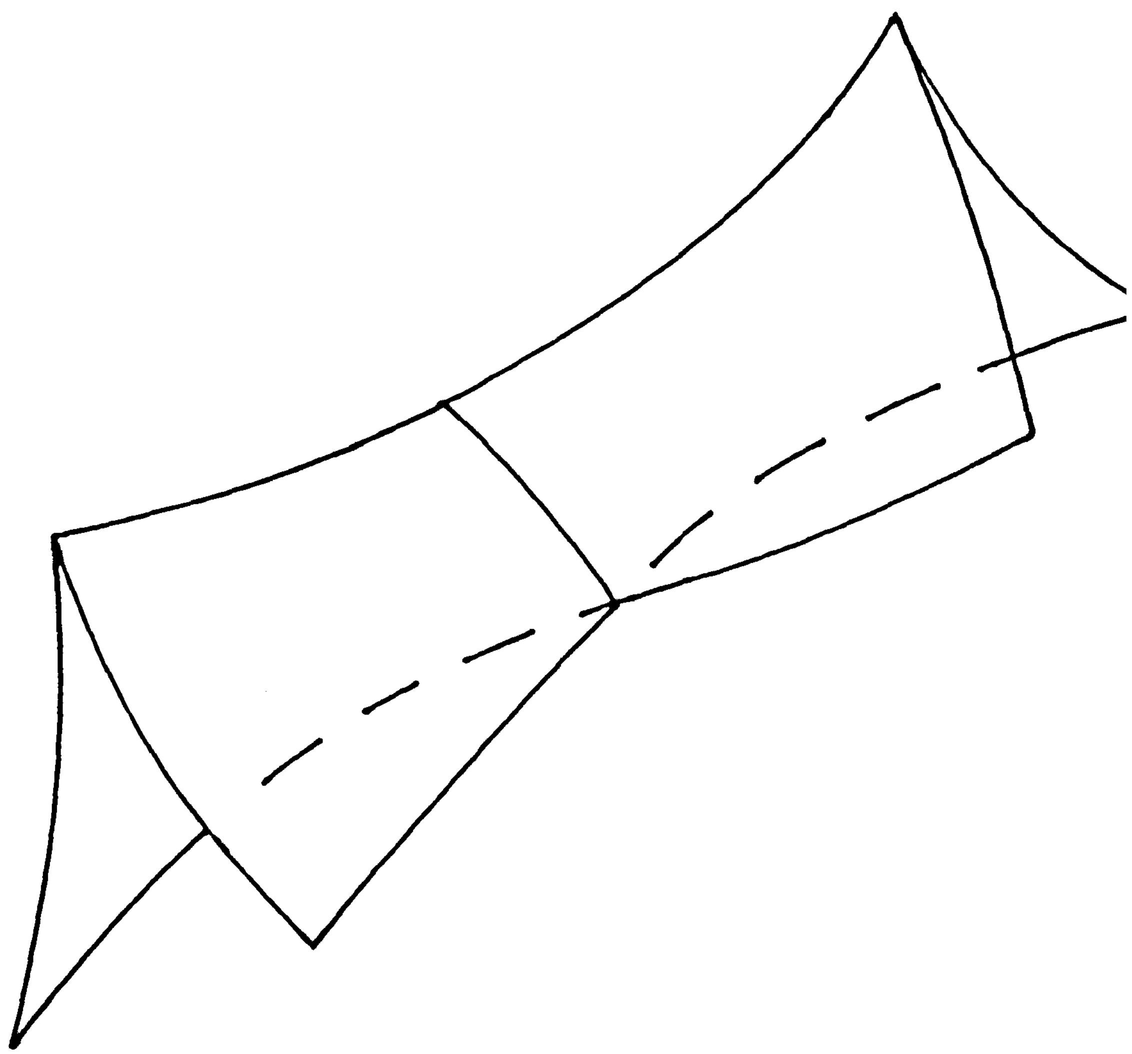
To get A_3 points, one takes involutes of a cuspidal surface F with a family of geodesics, each of which has an ordinary cusp. To get H_3 points, one takes a family of singular involutes of a singular surface F with a family of inflexional geodesics. These features can be combined by taking a family $\mathcal{F}$ of curves in $\mathbb{R}^3$ each having an ordinary cusp and an inflexion, such that the inflexions lie on one nonsingular curve γ_i and the cusps on other nonsingular curve γ_c. The curves γ_c and γ_i should meet at a point P (this will be the H_4 point) and the family $\mathcal{F}$ should form a geodesic foliation on a surface F. Such a family of curves is shown in figure (3.35). To form this illustration, the curves have been projected from $\mathbb{R}^3$ to $\mathbb{R}^2$ and this projection has introduced self-intersections that were not present in $\mathbb{R}^3$.

The surface F must necessarily have γ_c as a cuspidal edge, since otherwise the geodesics on F could not have cusps. Therefore in the next section we study surfaces with cuspidal edges.

To prove that surfaces exist with such a geodesic foliation, we give an example. This example is admittedly rather special since the surface F is developable. Let F be the tangent developable to a curve having a point P of type (1,2,4) (a point of zero torsion) and let l be the tangent at the point P. For such a surface, l is a line of flat umbilics. Therefore by (3.34) any geodesic has an inflexion where it crosses l. In addition any geodesic which is not tangent to the curve γ has a cusp where it crosses γ, and this cusp is rhamphoid (if it occurs at P) or ordinary (if it occurs at any other point of γ). Therefore almost all geodesic foliations on F will be of the desired form. The surface F is depicted in figure (3.36).

Figure 3.35 A Family of Geodesics Having Both Cusps and Inflexions

Figure 3.36 A Surface F on Which Can Be Found a Family of Geodesics Like That Illustrated in Figure 3.35



3.4 Surfaces With Cuspidal Edges

In this section we investigate the properties of certain types of surface having a cuspidal edge. We calculate the evolutes of some of these surfaces. Let $M \subset \mathbb{R}^3$ be a surface with parametrisation

$$r:(\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^3,m)$$

Definition 3.37

The surface M has an <u>cuspidal edge</u> at m (and the map germ r has a <u>cuspidal edge</u> at 0) if there is a non-singular curve γ passing through (0,0) such that r_1 has rank one at points of γ and rank two elsewhere.

Such map germs have been investigated in [20]. Here we will only consider three of the simplest types of point that can occur on a cuspidal edge.

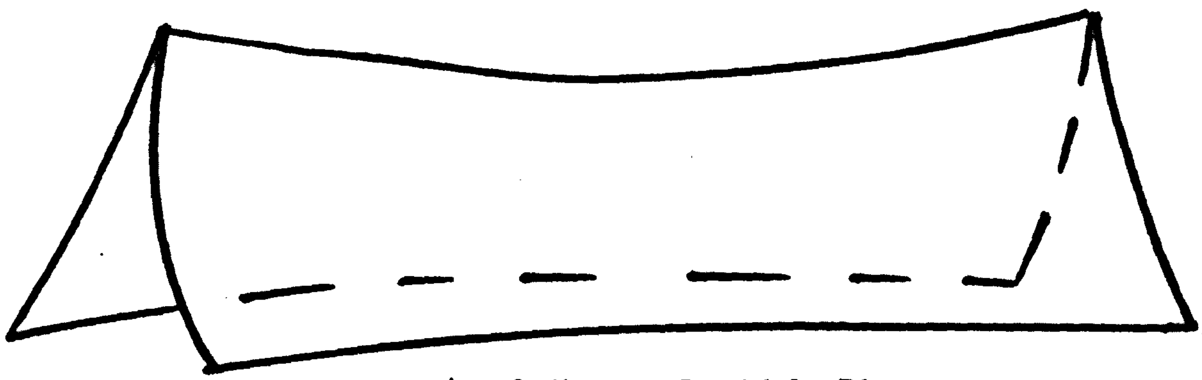
Definition 3.38

The surface M has an <u>ordinary cuspidal edge</u> (or <u>cuspidal edge of type 3/2</u>) at m if the germ of r at zero is $\mathcal{A}$ equivalent to the germ at zero of the map $(u,v) \longrightarrow (u,v^2,v^3)$.

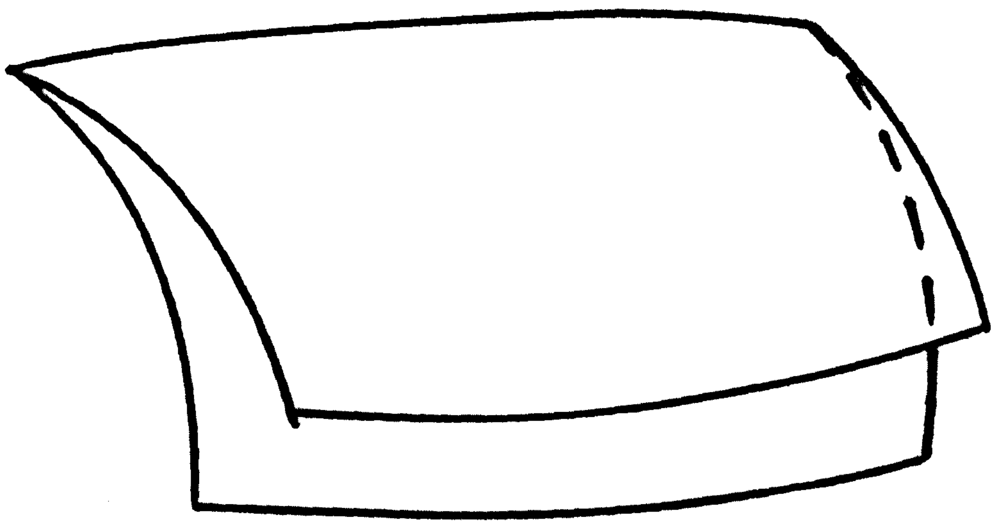
Definition 3.39

The surface M has a <u>rhamphoid cuspidal edge</u> (or <u>cuspidal edge of type 5/2</u>) at m if the germ of r at zero is $\mathcal{A}$ equivalent to the germ at zero of the map $(u,v) \longrightarrow (u,v^2,v^5)$.

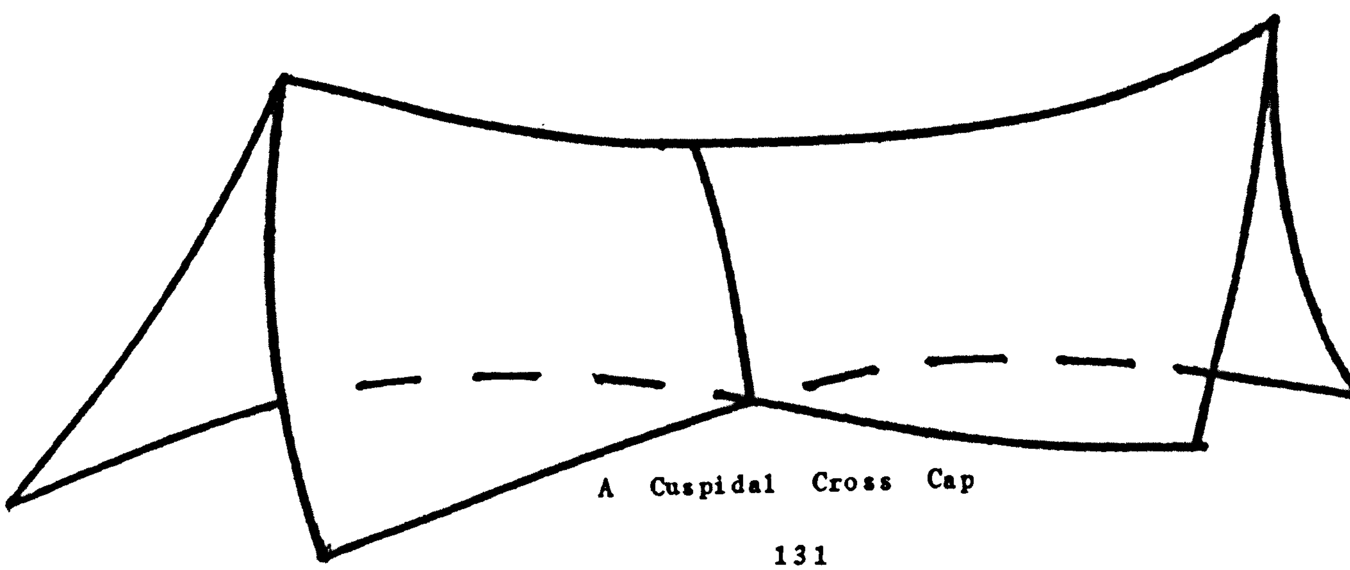
Figure 3.40 Surfaces With Cuspidal Edges



An Ordinary Cuspidal Edge



A Rhamphoid Cuspidal Edge



A Cuspidal Cross Cap

Definition 3.41

The surface M has a **cuspidal cross cap** (or **cuspidal pinch point**) at m if the germ r is $\mathcal{A}$ equivalent to the germ at zero of the map $(u,v) \to (u,v^2,uv^3)$.

Examples

The surface $\Delta(A_2) \times \mathbb{R}$ and $\Delta(I_2(5)) \times \mathbb{R}$ have cuspidal edges of types 3/2 and 5/2 respectively. The cuspidal cross-cap has already been mentioned in (2.69) and (3.36).

Surfaces having ordinary and rhamphoid cuspidal edges and cuspidal cross caps are illustrated in figure (3.40).

Proposition 3.42

Let r be any map germ with a cuspidal edge, such that the restriction of r to the edge is an immersion. Then by affine isometries in the target and smooth co-ordinate changes in the source, the map-germ r can be reduced to the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u \\ u^2 f(u)+v^2 \\ u^2 g(u)+uv^2 h(u)+v^3 S(u,v) \end{pmatrix} \qquad (3.43)$$

where $S(0,0) \neq 0$ for the ordinary cuspidal edge.

$S = uk(u)+vT(u,v)$ with $k(0) \neq 0$ for the cuspidal cross cap

$S = vH(u,v)$ with $\left.\frac{\partial H}{\partial v}\right|_{(0,0)} \neq 0$ for the rhamphoid edge.

<u>Notes</u>

(i) The letters f,g,h,k,H, and S denote arbitrary functions. Lower case letters will be used for functions of one variable and capitals for functions of two variables. In future we shall try to adhere to this convention.

(ii) We shall want to study the focal sets of these singular surfaces and the type of co-ordinate changes described above do not affect the behaviour of the focal set.

<u>Proof of proposition 3.42</u>

Let γ be the smooth curve in $\mathbb{R}^2$ at which r_1 has rank 1. Define two functions u and v on some neighbourhood of the origin in $\mathbb{R}^2$ as follows. Let $v:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be a submersion such that $\gamma = v^{-1}(0)$ and let $u:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by the composite $\pi \circ r$ where $\pi:\mathbb{R}^3 \longrightarrow \mathbb{R}$ is orthogonal projection onto the tangent line to the cuspidal edge at m. This can be done in such a way that $u(0,0) = v(0,0) = 0$. Then (u,v) form a local co-ordinate system on $\mathbb{R}^2$ and after rotations and translations in the target, r becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2 f(u) + v^2 N(u,v) \\ u^2 g(u) + uv^2 h(u) + v^3 S(u,v) \end{pmatrix}$$

where $S(u,v) = P(u,v^2) + vQ(u,v^2)$ and $N(0,0) > 0$.

Without loss of generality, we may assume that $N(u,v) = 1$ (since this can be achieved by a co-ordinate change in the source which replaces v by $v\sqrt{N(u,v)}$). The map r is now in the form $\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2 f(u) + v^2 \\ u^2 g(u) + uv^2 h(u) + v^3 S(u,v) \end{pmatrix}$ (3.44)

where $S(u,v) = P(u,v^2) + vQ(u,v^2)$ and $N(0,0) > 0$.

So far in this proof the only properties of the map r that have been used are that there is a smooth curve γ of points where r_1 has rank 1 and that $r|\gamma$ is an immersion at 0.

$$R(u,v^2) = \begin{cases} 1 & \text{for an ordinary cuspidal edge} \\ v^2 & \text{for a rhamphoid edge} \\ u & \text{for a cuspidal cross cap} \end{cases}$$

Now let $R(u,v^2) =$ as above.

From (3.44) $\quad r \underset{\mathcal{A}}{\sim} \begin{pmatrix} u \\ v^2 \\ v^3 P(u,v^2) \end{pmatrix}$

But by hypothesis $\quad r \underset{\mathcal{A}}{\sim} \begin{pmatrix} u \\ v^2 \\ v^3 R(u,v^2) \end{pmatrix}$

Hence by a result of Mond [20] $P(u,v^2) \sim R(u,v^2)$

This gives the conditions

$S(0,0) \neq 0$

$S = uk + vT$

$S = vH$ with $\left.\dfrac{\partial H}{\partial v}\right|_{(0,0)} \neq 0$

for the ordinary edge, cuspidal cross cap and rhamphoid edge respectively. QED.

We now present some criteria, which can be used to identify different types of cuspidal edges.

<u>**Proposition 3.45**</u>

Let r be a map germ with a cuspidal edge at m, such that the restriction of r to the edge γ is an immersion at m. Let M be the image of r.

(i) If the 3-jet of r at the origin is $\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ v^3 \end{pmatrix}$

then M has an ordinary cuspidal edge at m.

(ii) If the 4-jet of r at the origin is $\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ uv^3 \end{pmatrix}$

then M has a cuspidal cross cap at m.

(iii) If the 5-jet of r at the origin is $\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ v^5 \end{pmatrix}$

and if there is some neighbourhood of m containing no ordinary cuspidal edge points of M, then M has a rhamphoid cuspidal edge at m.

(iv) Let $k \in T_0\mathbb{R}^2$ be a non-zero vector such that $r_1(0)k=0$ (k stands for kernel).

If the image of $r_1(0)$ and the two vectors $r_2(0)k^2$ and $r_3(0)k^3$ span $\mathbb{R}^3$, then M has an ordinary cuspidal edge at m.

Proof

(i) and (ii) See [22] corollary 2.6.

(iii) The map r is given by $r\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + O(6) \\ v^2 + O(6) \\ v^5 + O(6) \end{pmatrix}$

As in the proof of (3.42),

$$r \underset{\mathcal{A}}{\sim} \begin{pmatrix} u \\ v^2 N(u,v) \\ v^5 + Y(u,v) \end{pmatrix} \text{ where } Y \in \mathcal{M}^6, \text{ and } N(0,0) \neq 0.$$

Since r has rank one along the curve v=0, there must be functions z, A, and B such that

$$z(u) + v^2 A(u,v^2) + v^3 B(u,v^2) = Y(u,v)$$

As in the proof of (3.42) we may assume without loss of generality that N(u,v) = 1 (since otherwise we can make a change of co-ordinates in the source which replaces v by v N(u,v)). Now, by an obvious co-ordinate transformation,

$$r \underset{\mathcal{A}}{\sim} \begin{pmatrix} u \\ v^2 \\ v^5 + v^3 B(u,v^2) \end{pmatrix}.$$

Because of the particular form of the 5-jet of the original map r, we must have $B \in \mathcal{M}^3$. If $B(u,0) \neq 0$, then by (i) the map r has an ordinary cuspidal edge at (u,0). But by hypothesis, there are no points near the origin for which this occurs. So B(u,0) = 0 for all suffiiciently small u, which means that $B(u,v^2) = v^2 C(u,v^2)$ for some function C. Now by [20], two map germs with cuspidal edges given by

$$\begin{pmatrix} u \\ v^2 \\ v^3 S(u,v^2) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u \\ v^2 \\ v^3 S'(u,v^2) \end{pmatrix}$$

are $\mathcal{A}$-equivalent if and only if the function germs S and S' are $\mathcal{K}$-equivalent. Applying this result to the germs $S(u,v^2) = v^2$ and $S'(u,v^2) = v^2(1 + C(u,v^2))$ we find that

$$r \underset{\mathcal{A}}{\sim} \begin{pmatrix} u \\ v^2 \\ v^5 \end{pmatrix} . \quad \text{QED.}$$

(iv) By (i), it is sufficient to show that r is $\mathcal{A}$-equivalent to a map germ g with 3-jet

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ v^3 \end{pmatrix} .$$

But this was shown in the proof of (3.42).

QED.

There is an important difference between, on the one hand, the ordinary cuspidal edge, and, on the other hand, the singularities (3.39) and (3.41). The ordinary cuspidal edge (3.38) can occur on the parallels to a nonsingular surface, while the rhamphoid edge and cuspidal pinch point cannot. Consider the Legendrian Submanifold M^* of $PT^*\mathbb{R}^3$ determined by the surface M, which is defined by

$M^* = \{(\tau, m) \in PT^*\mathbb{R}^3 \mid m \in M$

and τ is the tangent plane to M at m $\}$.

If M is a surface, possibly singular, which has a nonsingular parallel, then M^* is necessarily nonsingular.

<u>Proposition 3.46</u>

The surface M^*

(i) is nonsingular at (τ, m) if M has an ordinary cuspidal edge at m.

(ii) is locally diffeomorphic at (τ, m) to the surface

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ uv \\ v^2 \\ v^3 \\ 0 \end{pmatrix} \qquad \text{if } M \text{ has a cuspidal cross-cap at } m.$$

(iii) is locally diffeomorphic at (τ, m) to the surface

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ v^3 \\ 0 \\ 0 \end{pmatrix} \qquad \text{if } m \text{ has a rhamphoid cuspidal edge at } m.$$

<u>Proof</u>

By (3.42) M can be parametrised by the map

$$r: \begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2 f + v^2 \\ u^2 g + uv^2 h + v^3 S \end{pmatrix}$$

Then the direction normal to the surface M is given by the vector product $(1/v)\ r_u(0) \times r_v(0) =$

$$\begin{pmatrix} 1 \\ u^2 f_1 + 2uf \\ u^2 g_1 + 2ug + v^2 h + uv^2 h_1 + v^3 S_u \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 2uh + 3vS + v^2 S_v \end{pmatrix} =$$

$$= \begin{pmatrix} A \\ -2uh - 3vS - v^2 S_v \\ 2 \end{pmatrix} \qquad (3.47)$$

where $A = -4ug + 4u^2 fh + 6uvfS - 2u^2 g_1 - 2v^2 h + 2uv^2 fS_v +$
$+ 2u^3 f_1 h + 3u^2 v f_1 S - 2uv^2 h_1 - 2v^3 S_u + u^2 v^2 f_1 S_v$

So a local parametrisation of M^* near (τ,m) is given by

$$r^* \begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2 f + v^2 \\ u^2 g + uv^2 h + v^3 S \\ A \\ -2uh - 3vS - v^2 S_v \end{pmatrix} \qquad (3.48)$$

For an ordinary cuspidal edge, $S(0,0) \neq 0$ and so by inspection the map r^* of (3.48) is an immersion at (0,0).

In the other two cases the result follows by applying a sequence of co-ordinate changes to (3.48), after having made one of the substitutions

In theorem (3.49) $k : M \to \mathbb{R}$ is the function which assigns to each point x the curvature of the geodesic passing through x. The function k was essentially considered in [35]. The second fundamental form of a surface is a matrix-valued function $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ and if a

suitable co-ordinate system is used, our function k coincides with Weatherburn's function L ([35] §41 and [35] §56).

$S = uk+vT$ with $k(0) \neq 0$ (for a cuspidal cross cap)

$$S = vH \quad \text{with} \quad \left.\frac{\partial H}{\partial v}\right|_{(0,0)} \neq 0 \quad \text{(for a rhamphoid edge).}$$

In the case of the cuspidal cross cap, it is only necessary to work with the 3-jet of (3.48). This is because

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ uv \\ v^2 \\ v^3 \\ 0 \end{pmatrix}$$

is a 3-$\mathcal{A}$ determined map germ whose image is a surface with an isolated singular point [34]. QED.

We shall now see how surfaces with cuspidal edges arise on the involutes of non-singular surfaces.

<u>Theorem 3.49</u>

Let F be a non-singular surface with a geodesic foliation, and let M be an involute with starting line σ. Let $\pi: M \longrightarrow F$ be the projection map and let x be an arbitrary point of F.

(i) Suppose that $k(x) \neq 0$ and that the starting line σ passes through x. Then M has an ordinary cuspidal edge along σ. The lines of curvature of M of one system (obtained by unwinding the geodesics on F) have ordinary cusps where they cross the edge, while the lines of curvature of the other system are non-singular where they meet σ.

(ii) Suppose that $k(x)=0$, and that the geodesic passing through x has a point of type (1,3,N) there. Then the set of points of F at which $k=0$ is a nonsingular curve η passing through x. Suppose that the starting line σ <u>does not</u> pass through x. Then in general, M has a rhamphoid cuspidal edge along $\pi(\eta)$, and the lines of curvature of M of both systems have cusps where they cross the edge.

Proof

Recall that the function $k:M \longrightarrow \mathbb{R}$ assigns to each point x, the curvature of the geodesic passing through x. Usually, it is not possible to give a sign to the curvature of a space curve in a meaningful way. However, if the curve is a geodesic on a surface, it is possible to use a local orientation of F to determine a sign for k in a locally consistent manner, and k is then a smooth function on the surface which can take both positive and negative values. If the geodesic γ passing through x is of type (1,3,N) there, for any $N \geqslant 4$, then $k(x) = 0$, and the derivative of k along γ is nonzero at x. Therefore $k: M \longrightarrow \mathbb{R}$ is a submersion at x and so the set of points at which $k = 0$ is a smooth curve $\eta \subset M$ passing through x. At each point point of η sufficiently close to x, the geodesic there will be of type (1,3,N') for some $N' \leqslant N$.

The function $\tau:M \longrightarrow \mathbb{R}$, which assigns to each point x the torsion of the geodesic passing through x, is also smooth.

(i) Let $(u,v) \longrightarrow e(u,v)$ be a parametrisation of F such that u measures arc length along the geodesics (which are given by v=constant), the curves u = constant are orthogonal trajectories to the geodesics, and $e(0,0) = x$. Let $i:M \longrightarrow \mathbb{R}^3$ be the inclusion map and let γ be the geodesic passing through x. By (3.33), M is given by

$$r: (u,v) \longrightarrow e(u,v) - ue_u(u,v) \qquad (3.50)$$

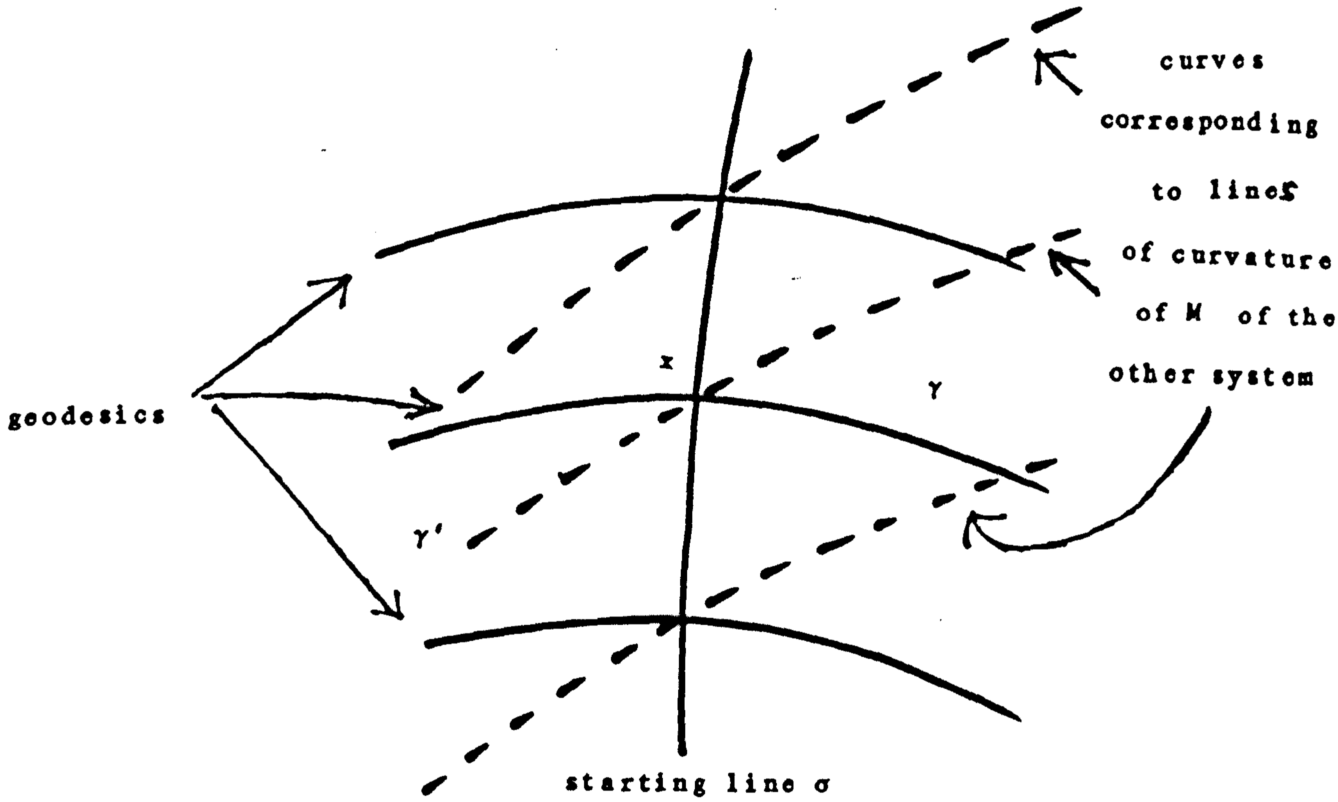
By direct calculation from (3.50), the derivatives $r_{uu}(0,0)$ and $r_{uuu}(0,0)$ are linearly independent, i.e. the unwinding of γ with starting point x has an ordinary cusp. This calculation also shows that the tangent to the starting line at x does not lie in the osculating plane to the unwound curve at the cusp point x. So, by (3.41.iv), the composite $i \circ r$ is $\mathcal{A}$ equivalent to the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v^2 \\ v^3 \end{pmatrix} \qquad (3.51)$$

Let $\pi(\gamma')$ be the line of curvature of M of the other system passing through $x = \pi(x)$. By (3.24) the tangent directions to γ and γ' at x are conjugate on F. Since γ does not have an inflexion at x, its tangent direction is not asymptotic, so is not self-conjugate, and so the curves γ and γ' on F intersect transversely at x. From the standard form (3.51) it now follows that $\pi(\gamma')$ is nonsingular at x. Some of the geodesics on F are shown in figure (3.52).

Figure 3.52 A Family of Geodesics Without Inflexions

Ready for Unwinding



(ii) Let (u,v) --> e(u,v) be a parametrisation of F such that u measures arc length along the geodesics (which are given by v=constant), the curve η has equation u = 0, and e(0,0) = x. Let $i: M \longrightarrow \mathbb{R}^3$ be the inclusion map and let γ be the geodesic passing through x. Let γ, e and r be as in the proof of (i). Direct calculation shows that an unwinding of γ whose starting point is not x has an ordinary cusp at x. In contrast to the situation in case (i), the tangent to the starting line at x does lie in the osculating plane to the unwound curve at the cusp point x, and so (3.41.iv) does not apply.

By (3.33), M is given by

$$r: (u,v) \longrightarrow e(u,v)+(c(v)-u)e_u(u,v) \qquad (3.54)$$

for some function c. Define three smooth fields of unit vectors $\underline{t}$, $\underline{n}$, and $\underline{b}$ on M such that $\underline{t}=e_u$ is tangent to the geodesic passing through x, $\underline{n}(x)$ is normal to F, and $\underline{b}(x)=\underline{t}(x)\times\underline{n}(x)$. Then the Serret–Frenet formulae hold:

$$\underline{t}_u = e_{uu} = k\underline{n} \qquad \underline{n}_u = \tau\underline{b} - k\underline{t} \qquad \text{and} \qquad \underline{b}_u = -\tau\underline{n}$$

Differentiating (3.54) and evaluating at (0,0),

$$r_u(0,0) = (c-u)e_{uu} = ck\underline{n} = 0$$

$$r_v(0,0) = e_v(0,0) + ce_{uv}(0,0) + c_1(0)\underline{t}$$

$$r_{uu}(0,0) = ck_u\underline{n}$$

Since $e_u.e_u = 1$ for all (u,v), $e_u.e_{uv} = 0$ and so $e_{uv}(0,0) = \alpha\underline{n} + \beta\underline{b}$ for some constants α and β. In addition, the co-ordinate system was chosen so that $e_v = \delta\underline{b} + \epsilon\underline{t}$ for some δ, ϵ.

We now assume that the involute M is such that $c \neq -\delta/\beta$. This is the meaning of the proviso 'in general' in the statement of the theorem. The particular involute with $c = -\delta/\beta$ meets the complementary surface G at r(0,0) and has a more complicated singularity there which we will not analyse now (see remarks in (3.63)).

Under the assumption that $c \neq -\delta/\beta$, the vectors $r_v(0,0)$ and $r_{uu}(0,0)$ are linearly independent. Take co-ordinates in $\mathbb{R}^3$ with the origin at r(0,0) and the vectors $r_v(0,0)$, $\underline{n}(x)$ and $\underline{t}(x)$ pointing along the X, Y, and Z axes respectively.
(N.B. These axes are not orthogonal.)

Then

$$r\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v + O(2) \\ u^2 N(u,v) + f(v) \\ O(2) \end{pmatrix} \text{ for some functions } f, N \text{ with } N(0,0) \neq 0$$

Let $\bar{r}$ denote r followed by the projection $(X,Y,Z) \mapsto (Y,Z)$.

A straightforward calculation using the Serret Frenet formulae shows that

$$\begin{vmatrix} \bar{r}_2(0) & 0 \\ \bar{r}_4(0) & 3\bar{r}_2(0).\bar{r}_2(0) \\ \bar{r}_5(0) & 10\bar{r}_2(0).\bar{r}_3(0) \end{vmatrix} = 24c^3 k_1^5 \neq 0$$

After a few more co-ordinate changes, we can apply (3.45.iii) to conclude that M does indeed have a rhamphoid cuspidal edge. QED.

In the case of a geodesic foliation with inflexions, (3.49) provides a clue as to the behaviour of the complementary surface to F. Since both families of lines of curvature on M behave similarly (they are cuspidal), we predict that the two sheets of the focal set of M will also behave similarly. Therefore we expect the complementary surface G also to be nonsingular, and the RLCs on G also to have inflexions. We shall return to this question later, in (3.63).

We now want to try to establish a converse to (3.49). Let M be a surface with a cuspidal edge (ordinary or rhamphoid). Consider the focal set of M. By (3.49), it is possible that one sheet of the focal set could be non-singular, with the geodesic foliation of RLCs consisting of curves without inflexions (if the edge is ordinary) and with inflexions (if the edge is rhamphoid). Is this necessarily the case, or are there other possibilities for the focal set?

We mention explicitly two phenomena that might occur at a point m on a cuspidal edge. There might be more than one line of curvature of each system passing through the point m (this is the case at a generic umbilic [26]). In this case we shall say that m is **source or sink for lines of curvature**. Also the limiting position of the pair of points $\pi^{-1}(m)$, as one approaches m along some curve in M, many depend on the direction in which one approches. If this occurs, then the focal set F will include the whole of the line $N_m M$, rather than just two points of it, and we shall say that the focal set **blows up** at m. This can happen at a nongeneric umbilic.

Proposition 3.55

Let M be a surface with an ordinary cuspidal edge γ, given as the image of the map r of (3.43) and let $m = r(0) \in \gamma$. Then, in the neighbourhood of M, the focal set behaves as follows.

(i) The focal set of M does not blow up at m and m is not a source or sink for lines of curvature (these terms are defined in the previous paragraph). The lines of curvature of M are the images under r of two systems of non-singular curves in $\mathbb{R}^2$, which intersect transversally.

(ii) One sheet of the focal set is nonsingular. The associated principal radius of curvature is identically zero along γ and the associated lines of curvature cross the edge with ordinary cusps. Lifting these lines of curvature to the focal set gives a family of nonsingular RLCs without inflexions.

(iii) The lines of curvature of the other system are non-singular where they meet the edge .

<u>**Proof**</u>

The surface M is given as the image of the map (3.43)

$$r : \begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2 f + v^2 \\ u^2 g + uv^2 h + v^3 S \end{pmatrix}$$

In this parametrisation, the curve γ is given by $v = 0$. Then by (3.47) a vector normal to M (not necessarily of unit length) is given by

$$N\begin{pmatrix} u \\ v \end{pmatrix} = (1/v)\ r_u \times r_v = \begin{pmatrix} A \\ -2uh-3vS-v^2S_v \\ 2 \end{pmatrix}$$

where $A = -4ug+4u^2fh+6uvfS-2u^3g_1-2v^2h+2uv^2fS_v+$
$+2u^3f_1h+3u^2vf_1S-2uv^2h_1-2v^3S_u+u^2v^2f_1S_v$

By (3.2) and (3.3) the focal set of M

is parametrised by $$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow r\begin{pmatrix} u \\ v \end{pmatrix} + \lambda\begin{pmatrix} u \\ v \end{pmatrix} N\begin{pmatrix} u \\ v \end{pmatrix} \qquad (3.56)$$

where $0 = \det(\lambda N.r_2 - r_1.r_1) =$

$$= \left| \lambda \begin{pmatrix} L & M \\ M & N \end{pmatrix} - \begin{pmatrix} E & F \\ F & G \end{pmatrix} \right| =$$

$$= \lambda^2(6vSL + v^2P) - \lambda(6vSE+v^2Q) + v^2R \qquad (3.57)$$

where the functions E,F,G,L,N,P,Q,R can be expressed in terms of u,v,f,g,h and S.

Explicitly,

$$E = 1 + 4u^2f^2 + 4u^2g^2 + 4u^3ff_1 + 4u^3gg_1 + 4uv^2gh + O(4)$$
$$F = 2uvf + u^2v(4gh + 2f_1) + 6uv^2gS_v + vO(3) \qquad (3.58)$$
$$G = 4v^2 + 4u^2v^2h^2 + 12uv^3hS_v + 9v^4S_v^2 + v^4O(1)$$
$$L = 4g + 4ug_1-4ufh-6vfS + 2u^2g_1+4v^2h_1-2v^2fS_v-4^2f_1h-6uvf_1S+O(3)$$
$$M = 4vh + 4uvh_1 + 6v^2S_u + 2v^3S_{uv}$$
$$N = 6vS + 10v^2S_v + 2v^3S_{vv}$$

Consider the behaviour of the two functions $\lambda_1,\lambda_2 : \mathbb{R}^2-\{v=0\} \longrightarrow \mathbb{R}$ defined by (3.57) as one approaches the point (0,0).

The equation (3.57) may be divided by v without affecting the values of its roots λ_1 and λ_2. It now becomes

$$\lambda^2(6SL+vP) - \lambda(6SE+vQ) + vR = 0 \qquad (3.59)$$

Substituting $(u,v)=(0,0)$ in (3.59) gives

$$0 = (\lambda^2 4g(0) - \lambda)6S(0,0)$$

which is a quadratic equation for λ with two distinct roots in $\mathbb{R}P^1$ (if $g(0)=0$ one of the roots for λ is infinity). Consequently (3.59) defines two smooth function germs

$$\lambda_1 : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}P^1,0)$$
$$\lambda_2 : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}P^1,1/4g(0))$$

which are extensions of the functions λ_1,λ_2 defined at the immersive points of $\mathbb{R}^2$ by (3.57). It follows that the focal set of M does not blow up at m, but contains two well-defined points of the normal line $N_m M$.

We now identify the lines of curvature of M. Let $\mathcal{S}$ denote the set of 2 x 2 matrices of rank 1. Then the map $\mathcal{S} \longrightarrow \mathbb{R}P^1$ given by $A \longrightarrow \ker A$ is smooth. Consider the two line fields ξ_1 and ξ_2 on $\mathbb{R}^2$ given by

$$0 = \left[\lambda_i \begin{pmatrix} L & M \\ M/v & N/v \end{pmatrix} - \begin{pmatrix} E & F \\ F/v & G/v \end{pmatrix}\right] \xi_i(u,v)$$

Since the matrix in square brackets has rank 1 throughout some neighbourhood of the origin, and is a smooth function of u and v, these line fields will integrate to give two non-singular systems of curves in $\mathbb{R}^2$. Whenever $v \neq 0$, these line fields agree with the principal directions for M, in the sense that $r_1(\xi_i(u,v))$ is a principal direction at $r(u,v)$. We have thus identified the lines of curvature.

By direct calculation,

$$\xi_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \xi_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3S(0,0) \\ 2h(0) \end{pmatrix} \tag{3.60}$$

Hence the integral curves of these two line fields cross transversally at (0,0).

(ii) and(iii) It follows from (3.60) that the images under r of the integral curves of ξ_1 will have ordinary cusps and the images of the integral curves of ξ_2 will be non-singular where they meet the edge γ. From (3.59), the function λ_1 is clearly identically zero along the line v = 0. It only remains to prove that the sheet of the focal set corresponding to λ_1 and ξ_1 is nonsingular near r(0,0). This is done by showing that the map e given by

$$e:\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow r\begin{pmatrix} u \\ v \end{pmatrix} + \lambda_1\begin{pmatrix} u \\ v \end{pmatrix} N\begin{pmatrix} u \\ v \end{pmatrix} \tag{3.61}$$

is an immersion at (0,0). Calculating those terms in E, L, P, Q, and R which are linear in u and v, and solving the quadratic equation (3.59), it can be shown that

$$\lambda_1 = 2v/3S(0,0) + O(2)$$

It follows that

$$e\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ 0 \\ 4v/3S(0,0) \end{pmatrix} + O(2)$$ which is an immersion. QED.

Corollary 3.62

If M has an ordinary cuspidal edge at m, then all except one of the parallels to M are nonsingular where they meet $N_m M$.

Proof

By (3.55), one sheet of the focal set of M is a nonsingular surface on which the RLCs do not have inflexions. Any parallel to M must be an involute of this foliated surface. Look at the involutes of this surface. The result follows. QED.

This corollary means that any surface with an ordinary cuspidal edge arises both as a parallel to some nonsingular surface, and as an involute of some nonsingular surface equipped with a foliation without inflexions.

Having investigated the focal sets of surfaces with ordinary cuspidal edges, we now consider the more interesting question of describing the focal sets of surfaces with rhamphoid cuspidal edges.

Theorem 3.63

Let M be a surface given by (3.42) with a rhamphoid cuspidal edge at m=r(0). The behaviour of the focal set in the neighbourhood of m is as follows.

(i) The focal set does not blow up at m. There are two points of the focal set on the line $N_m M$, neither of which is m.
They are distinct $\Longleftrightarrow$ $g(0) - H(0,0)$ and $h(0)$ are not both zero.
If these two points are distinct, and if in addition $h(0) \neq 0$,

Remark

In case (3.63.i), by [29], one expects two of the parallels to M to have an H_3 singularity on $N_m M$ (the two parallels which meet the two sheets of focal set). Let us express this in another way. Suppose we start with a nonsingular surface F, a geodesic foliation with a smooth curve γ of inflexions, and a distinguished point P on γ. Then, of the involutes, we expect all to have rhamphoid cuspidal edges at $\pi(P)$, except for two, which will have H_3 singularities at $\pi(P)$. This answers a point raised during the proof of (3.49).

The above discussion assumes that the involutes of F are of type (3.63.i). If they are of type (3.63.ii), then the situation at the complementary surface to F is more complicated.

then m is not a source or sink of lines of curvature. Both sheets of the focal set are nonsingular near m, and the two lines of curvature passing through m do so cuspidally. The corresponding RLCs on both sheets are nonsingular curves with inflexions.

(ii) If $h(0) = 0$ and $g(0) \neq H(0,0)$, then one sheet of the focal set is singular and the other sheet is nonsingular, and the lines of curvature corresponding to the nonsingular sheet form a foliation of M.

Proof

Proceed as in the proof of (3.55) to derive the equation (3.59). Since $S = vH$, it is possible to divide (3.59) by v giving

$$\lambda^2(6HL + P) - \lambda(6HE + Q) + R = 0 \qquad (3.64)$$

Substituting $(u,v)=(0,0)$ in (3.64) gives

$$0 = \lambda^2(6H(0,0)4g(0)+P(0,0)) - \lambda(6H(0,0)+Q(0,0)) + 4$$

$$= \lambda^2(64g(0)H(0,0)-16h(0)^2) - \lambda(16H(0,0)+16g(0)) + 4 \qquad (3.65)$$

The discriminant of the quadratic (3.65) is 256Δ, where

$$\Delta = (H(0,0)-g(0))^2 + h(0)^2 \qquad (3.66)$$

So if $H(0,0)-g(0)$ and $h(0)$ are not both zero, equation (3.64) defines two smooth function germs

$$\lambda_1:(\mathbb{R}^2,0) \longrightarrow (\mathbb{RP}^1,a_1)$$
and
$$\lambda_2:(\mathbb{R}^2,0) \longrightarrow (\mathbb{RP}^1,a_2) \qquad (3.67)$$

where a_1 and a_2 are the two (non-zero) values of λ satisfying (3.65). By direct calculation, $1/a_i = 2(g(0)+H(0,0)\pm\sqrt{\Delta})$. The functions (3.67) are extensions of the functions λ_1 and λ_2

defined by (3.57), and hence there are two well defined points of F on the line $N_m M$, given by $r(0,0) + a_i N(0,0)$ for $i = 1,2$. This proves that the focal set of M does not blow up at m.

We now investigate the lines of curvature of M. For convenience, instead of working directly with λ_i, we will work with the functions $\mu_i = 1/\lambda_i$. Consider the smooth vector fields ξ_i on $\mathbb{R}^2$, defined by

$$\xi_i = \begin{pmatrix} \mu_i F - M \\ L - \mu_i E \end{pmatrix}$$

They have the property that at each point $r_1(\xi_i(x))$ must be either zero or a principal direction on M. But

$L - \mu_i E = 4g(0) - \mu_i(0) + O(1)$ and for this to vanish at the origin, we must have $4g(0) = \mu_i(0) = 2(g(0) + H(0,0) \pm \sqrt{\Delta})$ and so $g(0) - H(0,0) = \pm\sqrt{\Delta}$.

Squaring gives $(g(0) - H(0,0))^2 = (g(0) - H(0,0))^2 + h(0)^2$ and so $h(0) = 0$. Therefore if $h(0) \neq 0$, ξ_i does not vanish at the origin. In this case, some neighbourhood of the origin is foliated by nonsingular integral curves of ξ_i. We have shown that, provided $h(0) \neq 0$, the point m is not a source or sink for lines of curvature. Furthermore, by direct calculation,

$$\xi_i(0,0) = \begin{pmatrix} 0 \\ 4g(0) - \mu_i(0,0) \end{pmatrix}$$

and since this equality holds for both i=1 and i=2, the integral curves of both ξ_1 and ξ_2 are tangent to the v axis at the origin. From (3.43) it follows that the lines of curvature of both systems passing through m have cusps there.

Now suppose that h(0)=0. Let h(u)=ut(u). The two roots of (3.65) are 1/4g(0) and 1/4H(0,0). To avoid confusing μ_1 with μ_2, we choose to distinguish the roots by defining

$a_1 = 1/4g(0)$ and $a_2 = 1/4H(0,0)$.

We are assuming that $a_1 \neq a_2$, so as in the case above when $h(0)\neq 0$, we find that ξ_2 does not vanish at the origin, and so the lines of curvature of the second system form a foliation on M, just as in the case when h(0) is non-zero.

We now consider whether m is a ridge point for the surface M. A ridge point on a nonsingular surface is one at which the equivalent conditions (3.8.i) to (3.8.vii) all hold with j=2. We shall investigate whether condition (3.8.iv) is satisfied. By composing with r, we can consider the two principal curvature functions as being defined on the plane $\mathbb{R}^2$ with co-ordinates (u,v), and similarly one can consider the lines of curvature of M as curves in $\mathbb{R}^2$. Then m is a ridge point for the i^{th} sheet if and only if the derivative of λ_i in the i^{th} principal direction vanishes at the origin.

Let $\mu_i = 1/\lambda_i$. We shall examine the 1-jet at the origin of the functions μ_i (this is where it really pays to look at μ_i instead of λ_i). Substituting for E, L, P, Q, and R in (3.64) gives

$$0 = 4\mu^2 - [16g(0)+16H(0,0)+16(g_1(0)-2h(0)f(0))+14H_v(0,0)v]\mu + 16[g(0)H(0,0)-h(0)^2+(4g_1(0)H(0,0)-4f(0)h(0)H(0,0)-2h(0)h_1(0)]u + 56gH_vv + O(2) \quad (3.68)$$

where $O(2)$ denotes terms involving u^2, uv, v^2, etc.

Solving the quadratic (3.68) gives

$$\mu_i = 2(g(0)+H(0,0)\pm\sqrt{\Delta}) + uZ(0,0) + 15vH_v(0,0)[1\pm(H(0,0)-g(0))/\sqrt{\Delta}]/4 + O(2) \quad (3.69)$$

where $Z = 4g_1-4fh+2H_u\pm(4gg_1-4g_1H-2gH_u+2HH_u+4hh_1-4gh)/\sqrt{\Delta}$

If $h(0) \neq 0$, then m is a ridge point if and only if the derivative of μ_i in the direction of ξ_i vanishes at origin. But since ξ_i is a multiple of $\begin{pmatrix}0\\1\end{pmatrix}$, this derivative is

$\alpha_i = 15H_v(0,0)[1\pm(H(0,0)-g(0))/\sqrt{\Delta}]/4$

This is definitely non-zero.

The i^{th} sheet of the focal set is the image of the map

$$\begin{pmatrix}u\\v\end{pmatrix} \longrightarrow e\begin{pmatrix}u\\v\end{pmatrix} = r\begin{pmatrix}u\\v\end{pmatrix} + \lambda_i\begin{pmatrix}u\\v\end{pmatrix}N\begin{pmatrix}u\\v\end{pmatrix} =$$

$$= \begin{pmatrix}u\\0\\0\end{pmatrix} + \begin{pmatrix}-4a_ig(0)u\\-2a_ih(0)u\\a_i[1 - a_iZ(0,0)u - \alpha_ia_iv]\end{pmatrix} + O(2) \quad (3.70)$$

By inspection this is an immersion near the origin. Hence both sheets of the focal set are nonsingular, and so the RLCs, being geodesics on a nonsingular surface, must themselves be nonsingular. The presence of inflexions can be deduced by working backwards. If inflexions were not present, then by (3.49) M could not possibly have a rhamphoid cuspidal edge. QED.

Those points with $h(0) = 0$ appear to have interesting properties, and merit further study. From equation (3.70), it is clear that one sheet of the focal set (corresponding to μ_2) with $a_2 = 1/4H(0,0)$ is nonsingular and the other (corresponding to μ_1 with $a_1 = 1/4g(0)$) is singular. It appears that these are points where ribs cross the cuspidal edge.

<u>**Theorem 3.71**</u>

Under the same hypotheses as in (3.63), suppose that $h(0) = 0$. Let $h(u) = ut(u)$. Then if $g(0) \neq 0$ and $H_u(0,0) \neq 0$, there is a ridge γ passing through m. The curve γ is the image under r of a nonsingular curve in the (u,v) plane.

<u>**Note**</u>

If $g(0) = 0$, then the ridge is still present, but is at infinity. This case has been excluded so as to simplify notation. If $H_u(0,0)$, then the ridge is the image of a curve in the (u,v) plane which is singular at the origin.

Proof

By [26], the ridge points are those for which there is a number μ simultaneously satisfying the three equations

$0 = N.r_2\eta - \mu r_1\eta.r_1$ and

$0 = N.r_3\eta^3 - 3\mu r_1\eta.r_2{}^2$

for some unit vector $\eta \in \mathbb{R}^2$. We want to substitute for the derivatives of r and eliminate two of the variables, namely μ and η, leaving a single equation in u and v. We do this in two stages, as follows. First, eliminate η, leaving two equations, one of which is (3.57), and the other of which is the resultant

$$0 = \begin{vmatrix} N.r_{uuu}-3\mu r_u.r_{vv} & \cdots\cdots & & N.r_{vvv}-3\mu r_v.r_{vv} \\ L-\mu E & M-\mu F & 0 & 0 \\ 0 & L-\mu E & M-\mu F & 0 \\ 0 & 0 & L-\mu E & M-\mu F \end{vmatrix} \qquad (3.72)$$

The first equation is quadratic in μ and the second is cubic in μ. Taking the resultant of these two new equations with respect to μ gives a single equation in u and v for the ridges, given by a 5 by 5 determinant.

Explicitly, (3.57) gives

$0 = 4\mu^2 - (16H(0,0) + 16g(0))\mu + 64g(0)H(0,0) + O(1).$

Consider now the determinant (3.72). The extreme right hand column is divisible by v, since

$N.r_{vvv} = 48vH + 72v^2H_v + 24v^3H_{vv} + 2v^4H_{vvv}$ and

$r_v.r_{vv} = 4v + vO(2)$

Removing this factor, we find after considerable calculation that (3.72) becomes

$$0 = \mu^4 - (12g(0)+4H(0,0))\mu^3 + (48g(0)^2+48g(0)H(0,0))\mu^2 -$$
$$- (64g(0)^3+192g(0)^2H(0,0))\mu + 256g(0)^3H(0,0) + O(1)$$

When $(u,v)=(0,0)$, both (3.57) and (3.72) have $\mu = 4g(0)$ as a root. Therefore we can say without calculating it explicitly that the μ-resultant of (3.57) and (3.72) vanishes at the origin, and so m is a ridge point. We still have to verify that the ridge is a given by a nonsingular curve through the origin in the (u,v) plane. We wish to rule out such possibilities as the resultant of (3.57) and (3.72) being a curve with a double point at the origin (which would correspond to there being two ribs crossing at m), or that the resultant could be identically zero, in which case every point of M is a ridge point.

These possibilities were originally eliminated by explicitly calculating the resultant. However, there is an alternative method which considerably reduces the necessary amount of calculation. The required result can be deduced from (3.70). For suppose that (u,v) is a ridge point of M which is close to the origin. Then the corresponding sheet of the focal set must be singular there. This sheet of focal set is parametrised by the map e given in (3.70), namely

$$e: \begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ u^2f(0) + v^2 \\ u^2g(0) \end{pmatrix} + \mu_1 \begin{pmatrix} -4ug(0) - 2u^2g_1(0) \\ -2u^2t(0) - 4v^2H(0,0) \\ 2 \end{pmatrix} + O(3) \qquad (3.73)$$

where $\mu_1 = (1 - uZ(0,0)/4g(0))/4g(0) + O(2)$ and $Z(0,0) = 4H_u(0,0)$.

Differentiating (3.73) gives

$$e_u = \begin{pmatrix} 2u(H_u(0,0)/g(0)-g_1(0)/2g(0)) + O(2) \\ 2u(f(0)+t(0)/2g(0)) + O(2) \\ -H_u(0,0)/2g(0)^2 + O(1) \end{pmatrix} \quad \text{and}$$

$$e_v = \begin{pmatrix} O(2) \\ 2v(1-H(0,0)/g(0)) + O(2) \\ O(1) \end{pmatrix}$$

It follows, by considering when the vector product $e_u \times e_v$ vanishes, that any ridge point must be the image under r of a point lying on the nonsingular curve $0 = -2v(1-H(0,0)/g(0))H_u(0,0)/2g(0)^2 + O(2)$. It is clear that the image of this curve in M will have a cusp at m. QED.

It has been shown in a number of examples (using (3.45.iii)) that the singular sheet of the focal set has a cuspidal cross cap at e(0,0). We believe that this is always (or almost always) the case for rhamphoid cuspidal edges with h(0)=0, and that it will be possible to prove this using the same kind of algebraic manipulation used above. If this result is true, the evolute of M is none other than the foliated surface considered in (3.35) and (3.36), and we have in fact answered the question which was raised there about the behaviour of the general involutes of this surface.

We have now obtained the *big wave front which we propose as* a candidate to be called H_4. Let us summarise its properties. The family of parallel surfaces is obtained by taking the involutes of a foliated surface F with a cuspidal cross cap at a point P. There are two nonsingular curves γ_c and γ_i on the surface F, such that the geodesics have ordinary cusps along γ_c, inflexions along γ_i, and the geodesic passing through P has a rhamphoid cusp there. An involute M whose starting line σ does not pass through P has a swallowtail where σ crosses the curve γ_c, an H_3 singularity where σ crosses the curve γ_i, and an ordinary cuspidal edge along σ. There is a rhamphoid cuspidal edge along $\pi(\gamma_i)$, and on that edge there is a ridge point $\pi(P)$. Up to diffeomorphism, the point $\pi(P)$ is indistinguishable from the other points on the rhamphoid cuspidal edge.

Thus the involute M has all the distinguishing features one would possibly expect to find if it were one of the sections $\Delta(\epsilon)$ (where $\epsilon \neq 0$) of $\Delta(H_4)$ described in chapter one. Similarly, the involute whose starting line passes through P has all the gross differential geometric features one would expect to find on the section $\Delta(0)$ of $\Delta(H_4)$. Although these surfaces certainly appear to be diffeomorphic to the suggested standard models, we have not proved this here. The conclusion that should be drawn from the lengthy calculations appearing in § 3.4 is that, although this is an area that has been extensively studied for many years, there are still problems worthy of further attention involving the evolutes and involutes of surfaces in $\mathbb{R}^3$.

The objection could be raised, that the group H_3 arises from taking the involutes of a curve with an inflexion. Such a curve is nonsingular, and furthermore is a natural one to consider, since inflexions are present stably on generic nonsingular plane curves. The group H_4 seems to arise from taking the involutes of a surface with a cuspidal cross cap, which is a singular surface. One might say that it is therefore much less natural to look at the involutes of such a surface, indeed how would it ever occur to anyone to take the involutes of such a surface. What is more, if one is to be allowed to unwind geodesics from one type of singular surface, then one must also consider involutes of all other singular surfaces. Then the possibilities are limitless.

But this objection loses most of its validity, when one realises that the cuspidal cross cap is a stable phenomenon in the class of map germs with a cuspidal edge [20]. Because of the possibility of ribs, one should consider the focal set not as an object in the class of nonsingular manifolds, but as a member of the class of manifolds-with-a-cuspidal-edge. It is shown in [20] that, on such manifolds, two types of singular point occur stably: the ordinary cuspidal edge point (3.38) and the cuspidal cross cap (3.41)

As a postscript to the study of involutes and evolutes of cuspidal surfaces, we now have a brief look at the evolute of a cuspidal cross cap.

<u>Proposition 3.74</u>

Let M be a surface given by (3.43) with a cuspidal cross cap at $m=r(0)$. Then the focal set of M blows up at m.

Proof

Proceed as in the proof of (3.55) to derive equation (3.59). Substituting $S = uk + vT$, equation (3.59) becomes

$$\lambda^2[6(uk+vT)L + vP] - \lambda[6(uk+vT)E+vQ] + vR = 0 \qquad (3.75)$$

Approaching (0,0) along the line $u=\alpha v$, the limiting positions of the two points of F lying on $N_m M$ are given by the roots of

$$0 = \lambda^2[6(\alpha k + T)L + P] - \lambda[6(\alpha k + T)E + Q] + R =$$
$$= \alpha k(6\lambda^2 L - 6\lambda E) + 6\lambda^2 LT + 6\lambda^2 P - 6\lambda ET - \lambda Q + R \qquad (3.76)$$

For any $\lambda \in \mathbb{RP}^1$, a value of α can be chosen such that λ is a root of (3.76). (In some cases, we will find α is infinite. This corresponds to approaching (0,0) along the line v=0.) Therefore, by approaching (0,0) in a suitable direction, we can arrive at any point on the normal line $N_m M$, and so the focal set blows up at m. QED.

Here we conclude our brief look at the involutes and evolutes of surfaces in $\mathbb{R}^3$. We have in fact only described only a very few of the configurations which can occur. We have not even exhausted the possibilities of the involutes of a generic nonsingular surface F, because, for example, we would expect to find, in a typical geodesic foliation, isolated points of undulation of the geodesics.

Two methods have been used here to study the relationship between evolute and involute. the first method is to start with a parametrisation $r:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ of a surface M and to compute a parametrisation $e:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ of the evolute using the equations

$$e = r + (1/k)n$$

$$0 = \det(n.r_2 - kr_1.r_1)$$

The second method is to start with the foliated surface F, and to build up an involute of F from its lines of curvature by unwinding individual geodesics.

A third and more powerful method which has only been mentioned in passing is to apply a classification theorem about singularities of functions to the family of distance squared functions on M. This is the method that was used by Shcherbak [29] to describe the big involute of a curve with an inflexion, using Lyashko's classification of functions on cuspidal curves.

A function that appears to be of particular interest is the distance squared function of the surface with parametrisation

$$\begin{pmatrix} u \\ u^2 f(u) + v^2 \\ u^2 g(u) + u^2 v^2 t(u) + v^4 H(u,v) \end{pmatrix}$$

This function will arise in the context of classifying functions on manifolds with a singular boundary (as defined in [18]) where the boundary has a rhamphoid cuspidal edge. This function is certainly adjacent to the A_3 and H_3 strata. The resulting big wave front appears to be very similar to $\Delta(H_4)$, and so we propose that this function should be called 'of type H_4.' This function can clearly be thought of as a one-parameter family of functions on the manifold consisting of $\mathbb{R}^2$ with a singular boundary consisting of the curve $x^2=y^5$. In [18], Lyashko classified all such functions which are simple or unimodal. The H_4 function described above will presumably appear when the classification of [18] is extended to the bimodal case.

3.5 Parallels in Higher Dimensions and Codimensions.

The purpose of this section is to demonstrate the rarity of parallels in general. If $M^k \subset R^n$, there is at most one k-dimensional manifold parallel to M passing through each point of N_mM.

Proposition (2.10) shows that if k=1 or k=n-1, the words 'at most' can be replaced by the word 'precisely'.

If $2 \leqslant k \leqslant n-2$, the situation is somewhat different. We shall now see that in these cases there are very few parallels. For almost all surfaces M in R^n $(n\geqslant 4)$, any surface M, which is parallel to M, must actually be equal to M. In this section $r:(R^2,0) \longrightarrow (R^n,m)$ is an immersive germ which gives a local parametrisation of $M^2 \subset R^n$.

For each $r(x) \in M$, let $A_{r(x)}$ be the map $N_mM \longrightarrow S^2R^2$ given by $A_{r(x)}(e) = (e-r(x)).r_2(x)-r_1(x).r_1(x)$. The map A_m is affine and the portion $\pi^{-1}(m)$ of the focal set lying in N_mM is the inverse image under A_m of the conic C of degenerate(parabolic) quadratic forms. Taking co-ordinates in R^2,

$$\pi^{-1}(m)=\{e \mid 0=\det A_m(e)= \begin{vmatrix} P_m(e) & Q_m(e) \\ Q_m(e) & R_m(e) \end{vmatrix} = P_m(e)R_m(e)-Q_m(e)^2 \} \tag{3.77}$$

Equation(3.77) shows that after projectivizing, det $A_m(e)$ is quadratic form in e of rank λ where $\lambda \leqslant 3$ and (if M is nonsingular) $\lambda \geqslant 1$ since $A_m(m) = -r_1(0).r_1(0) \neq 0$.

Definition 3.78

If $\lambda = 1$, m is called an **umbilic** and if $\lambda = 2$, m will be called a **semiumbilic**.

Example 3.79

If n=3 every point m is either a semiumbilic (if $\pi^{-1}(m)$ is two distinct points) or an umbilic (if $\pi^{-1}(m)$ is a repeated point).

If n=4, $\pi^{-1}(m)$ is a repeated line, a pair of lines or a conic in the plane N_mM, according to whether $\lambda = 1$, 2 or 3.

For distance generic surface, those points on the focal set at which the distance squared function has a singularity of corank 2 or more (at least D_4) form a subset of F of codimension 2. It follows that not every point of M is semiumbilic. In fact, for almost all surfaces in $\mathbb{R}^n$, the semiumbilics form smooth curves (if n=4) are isolated points (if n=5) and do not occur at all (if $n \geq 6$).

Theorem 3.80

The following conditions (i) to (iv) are equivalent.

(i) There are (n-3) surface germs M_i (i=1,...,(n-3)) parallel to M such that if y_i is the point in which M_i meets N_mM, the points $m, y_1, \ldots y_{n-3}$ do not lie in an (n-4) dimensional affine subspace of N_mM.

(ii) There is an (n-2) parameter family of surface germs parallel to M.

(iii) Every point of M is either semiumbilical or umbilical.

(iv) Let p: $Tub_c(M) \to M$ be the natural projection and let γ_1 and γ_2 be two lines of curvature of $Tub_c(M)$ of the same system. Then either the two curves $p \circ \gamma_1$ and $p \circ \gamma_2$ are disjoint, or they are the same curve.

If the following condition is satisfied then so are (i) to (iv) above.

(v) For any c such that $Tub_c(M)$ is smooth, a local co-ordinate system can be chosen at any point of $Tub_c(M)$ such that the co-ordinate directions are principal directions.

<u>**Remark**</u>

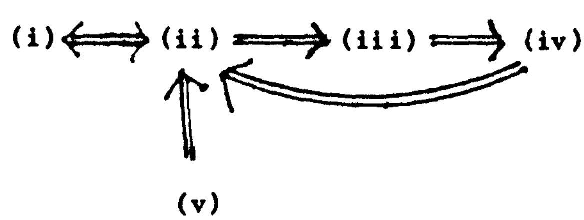
Condition (iv) describes what happens when the lines of curvature on $Tub_c(M)$ are projected to M. By (2·14) there are two nonconstant principal curvature functions on $Tub_c(M)$ and two corresponding systems of lines of curvature on M.

The fibre $p^{-1}(m)$ is a sphere of dimension (n-3) and there are two lines of curvature, one of each system, passing through each point of the fibre $p^{-1}(m)$. If, however, condition (iv) holds, then these infinitely many curves on $Tub_c(M)$ project under p to two curves γ_1 and γ_2.

Since for $c \neq d$, $Tub_c(M)$ and $Tub_d(M)$ are parallel hypersurfaces, and since lines of curvature on parallel hypersurfaces correspond, condition (iv) holds for one value of c if and only if it holds for all values of c.

<u>**Proof**</u>

To simplify notation, it will be assumed that $n = 4$. The proof for $n \geqslant 5$ is essentially the same. We shall prove the following chain of implications.



<u>(i) =>(ii)</u>

By hypothesis M has at least one parallel M_1 passing through $y_1 \in N_m M$. Let y be any point of $N_m M$ not lying on the line my_1. Then positive real numbers c and d can be chosen such that $T=Tub_c(M)$ and $T_1=Tub_d(M_1)$ intersect transversally at y. Then they will intersect in one further point z of the plane $N_m M$, as shown in figure (3.81). The intersection $T \cap T_1$, consists of two surface germs parallel M passing through y and z.

Surface germs passing through points on the line my_1 can now be constructed by repeating the above construction, taking the intersection of tubes whose cores contain the points y and z.

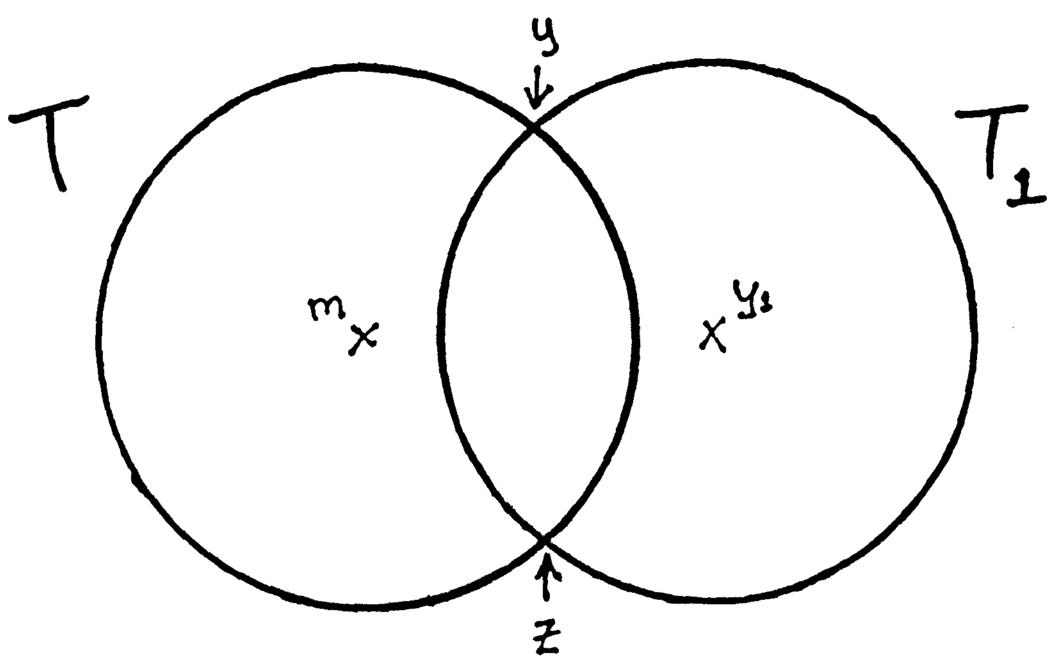


Figure 3.81 Cross-Section of the Tubes T and T_1 by the plane $N_m M$.

<u>(ii)==>(i)</u>

This is trivial.

<u>(ii)==>(iii)</u>

Let C be the conic $\pi^{-1}(m)$.

Let l be a straight line in N_mM passing through m. Then l is a normal to $Tub_c(M)$ and so must intersect the focal set of $Tub_c(M)$ in (n-1) points which must all be real. By (2.15) these (n-1) points are m (counted m-3 times) and the two points $l \cap C$.

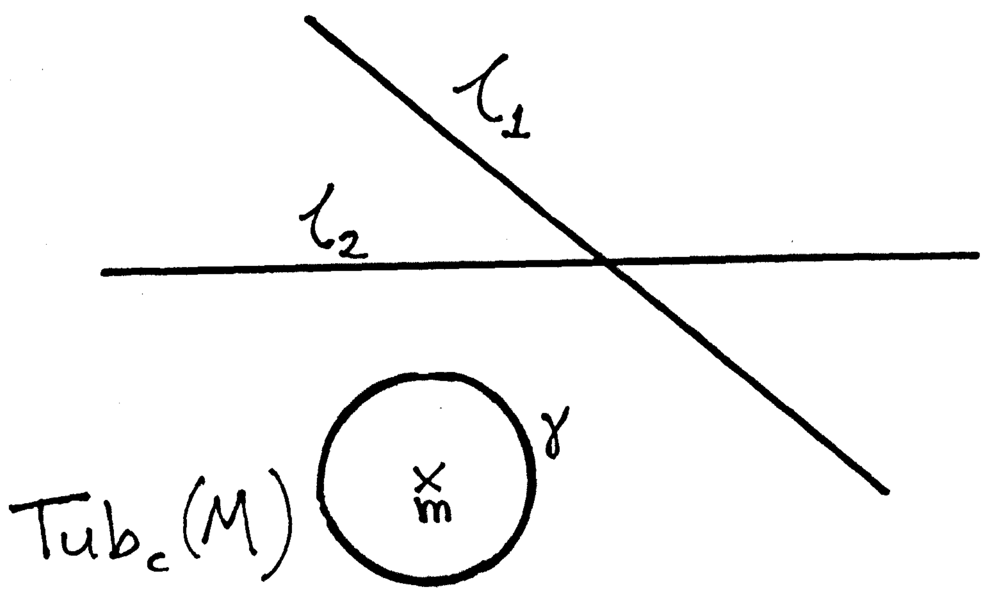
Hence the two points $l \cap C$ are both real.

If M has a two-parameter family of parallels then any line l in N_mM is a normal to $Tub_c(P)$ for some P parallel to M and by the above argument l meets C in two real points. But any conic which meets every line in the plane in two real points is a pair of lines and so m is a semiumbilic.

<u>(iii)==>(iv)</u>

As shown in figure (3.82), the plane N_mM intersects $Tub_c(M)$ in a circle γ and F in a conic which, because of condition (iii), consists of two lines l_1 and l_2.

<u>Figure 3.82</u> Cross-Section of F and $Tub_c(M)$ by the plane N_mM



We wish to show that the projections to M of the principal directions of $Tub_c(M)$ at any point $y \in \gamma$ are two vectors v_1 and v_2 in T_mM which depend only on M and not on y.

Let $y \in Tub_c(M)$ and let

$r:(R^2,0) \longrightarrow (M,m)$ and $\tilde{r}:(R^2 \times S^1,(0,a)) \longrightarrow (Tub_c(M),y)$

be germs of parametrisations, where V and $\tilde{V}$ are the associated distance squared functions for M and $Tub_c(M)$. A straightforward calculation now shows that if $e \in N_mM$,

$$\tilde{V}\rho(0,a)(u) = 0 \iff V\rho\,(0)(u)=0 \qquad (3.83)$$

For a smooth surface M, the image of the map from S^1 to N_mM which associates to θ the projection of $r_2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}^2$ onto the plane N_mM is called the <u>curvature ellipse</u> at m. The fibre $p^{-1}(m)$ of the focal set lying in N_mM is the polar of the curvature ellipse with respect to the conic $\{u \mid r_1(0)(u).r_1(0)(u)=1\}$ in N_mM (see [23] or [33]).

Thus the curvature ellipse at m degenerates to a straight line segment if m is a semiumbilic and a single point if m is an umbilic.

If E is a true ellipse, each tangent direction to M at m contributes one point to E, and the pole-polar relationship gives a one-to-one correspondence between the points on the ellipse and the points on its polar.

Combining this with the one-to-one correspondence between solutions of $V_2(u)=0$ and solutions of $\tilde{V}_2(u)=0$ given by (3.83), it follows that there is a bijective correspondence between projections to M of principal directions of $Tub_c(M)$ at points of

γ and tangent directions to M at m.

If, however m is a semiumbilic, the polars of the points on the lines segment E form part of a pencil of lines in the plane $N_m M$ all passing through same point P. The conic $p^{-1}(m)$ in the plane $N_m M$ will consist of two lines l_1 and l_2 which are the polars of the two end points of the line segment E, and the only solutions of

$V\mathfrak{g}\ (0)(u) = 0$ with $e \in N_m M$, $u \neq 0$

will be

$V\mathfrak{g}(0)\ (u_1) = 0 \iff e \in l_1$ and

$V\mathfrak{g}(0)\ (u_2) = 0 \iff e \in l_2$

for some vectors u_1 and $u_2 \in T_m M$ depending only on m: not on e.

<u>(iv)==>(ii)</u>

We use the Dupin–Darboux theorem which is as follows [31].

Let R_ρ, S_σ, T_τ be three one-parameter families of hypersurfaces in $\mathbb{R}^4$ which are mutually orthogonal. Then for there to exists a fourth one parameter family P_π orthogonal to each of the others, it is necessary (Darboux) and sufficient (Dupin) that for every triple (ρ,σ,τ) the curve $R_\rho \cap S_\sigma \cap T_\tau$ is a line of curvature on R_ρ and on S_σ and on T_τ.

<u>Note</u>

The normal direction to P_π is uniquely determined, since it is perpendicular to the normals to R_ρ and S_σ and T_τ . Therefore the tangent hyperplane to P_π is known at each point. The above theorem given a necessary and sufficient condition for the resulting distribution of hyperplanes to be integrable.

The above theorem is applied to the families R_ρ, S_σ and T_τ where

T_τ ($\tau \in \mathbb{R}^+$) is the tubular hypersurface with core M and radius τ,

$\mathcal{R}$ is the set of curves on M which are the projections of the lines of curvature of T_1 of one system

$\mathcal{S}$ is the set of curves which are projections of lines of curvature of the other system

R_ρ ($\rho \in \mathcal{R}$) is the union of the normal planes to M at points of ρ,

S_σ ($\sigma \in \mathcal{S}$) is the union of the normal planes to M at points of ρ.

The Dupin-Darboux theorem gives a fourth family of hypersurfaces P_π ($\pi \in \mathcal{P}$) with the property that each surface of the form $P_\pi \cap T_\tau$ is a parallel to M.

(V) ⟹ (II)

Suppose (x,y,θ) are local co-ordinates on $Tub_c(M)$ such that the co-ordinate directions are principal directions and x and y are constant on each fibre of $p:Tub_c(M) \longrightarrow M$.

Then the surfaces θ = constant are parallels to M. QED.

For $M^k \subset \mathbb{R}^n$, each fibre $\pi^{-1}(m)$ is an algebraic hypersurface in $N_m M$ of degree k. The proof given above for the case k = 2 shows that for $3 \leqslant k \leqslant (n-2)$, an (n-k) parameter family of parallels to M can exist only if $\pi^{-1}(m)$ is the union of k hyperplanes in $N_m M$ for each point m of M. For a fixed pair (k,n) with $2 \leqslant k \leqslant n-2$ this will not be the case for a general manifold $M \subset \mathbb{R}^n$ (see [33]).

Given a hypersurface F in $\mathbb{R}^4$, one can ask whether it is the focal set of some surface. For this to be the case, F must be foliated by the conics in which would be fibres of the projection p. More generally, if F is the focal set of $M^k \subset \mathbb{R}^n$, the fibres of p are algebraic subvarieties of $\mathbb{R}^n$ of codimension k+1 and degree k. But if $1 \leqslant k \leqslant n-2$, there is some integer N(k,n) depending only on k and n and a dense subset $\mathcal{S}$ of $Imm(\mathbb{R}^{n-1}, \mathbb{R}^n)$, such that, if $r \in \mathcal{S}$, the contact between the image of r and any algebraic subset of $\mathbb{R}^n$ of codimension k+1 and degree k is given by a map-germ of $\mathcal{K}$ codimension at most N. The set $\mathcal{S}$ is obtained from the set of all algebraic varieties of degree k and codimension k+1 in a similar way to that in which the set of distance-generic immersions is obtained from the set of all hyperspheres in $\mathbb{R}^n$ [23]. If $r \in \mathcal{S}$, no algebraic subvariety of codimension k+1 and degree k can lie on the hypersurface Im r. It follows that

Theorem 3.84

Almost all hypersurfaces in $\mathbb{R}^n$ have the property of not being the focal set of any k-manifold in $\mathbb{R}^n$ for any $k \leqslant n-2$.

APPENDIX ONE

A SET OF BASIC INVARIANTS FOR THE GROUP H_4

(see proposition 1.42)

The polynomials p_2, p_{12}, p_{20}, and p_{30}, listed below form a basic set of invariants for the group H_4. See chapter one for definitions of the polynomials f_2, f_6, f_8, f_{12}, and j_{12} and for a description of the methods used to calculate p_2, p_{12}, p_{20}, and p_{30}.

$$p_2 = f_2$$

$$p_{12} = 72f_{12} + 121f_6^2 - 22f_8f_2^2 + 50688j_{12}$$

$$p_{20} = 17q_{20}/4608 + 53295j_{12}r_8$$

$$p_{30} = q_{30}/1179648 + 435435j_{12}r_{18}/256$$

where

$$q_{20} = 263670638f_2^{10} + 12969f_2^7f_6 - 12078737f_2^6f_8 + 66317790f_2^4f_6^2$$
$$+ 39553776f_2^4f_{12} - 198f_2^3f_6f_8 - 418f_2^2f_8^2 - 40095f_2f_6^3$$
$$- 158840f_2f_6f_{12} + 99275f_8f_6^2 + 7128f_{12}f_8$$

$$r_8 = 1926f_2^4 - 3f_2f_6 - f_8$$

$$
\begin{aligned}
q_{30} = {} & 592837910885018f_2^{15} + 6624163727181f_2^{12}f_6 \\
& - 118016981250168f_2^{11}f_8 + 592505399425940f_2^{9}f_6^{2} \\
& + 397930270177440f_2^{9}f_{12} - 92598501996f_2^{8}f_8f_6 \\
& - 113658288744f_2^{7}f_8^{2} - 17267750570070f_2^{6}f_6^{3} \\
& - 78697932513200f_2^{6}f_6f_{12} + 48348971110440f_2^{5}f_8f_6^{2} \\
& + 3794115124800f_2^{5}f_8f_{12} - 860366815308f_2^{4}f_8^{2}f_6 \\
& + 57428186816f_2^{3}f_8^{3} - 208391933750f_2^{3}f_6^{4} \\
& + 4309388123040f_2^{3}f_6^{2}f_{12} + 66408218240f_2^{3}f_{12}^{2} \\
& - 5699994300f_2^{2}f_8f_6^{3} + 97537440f_2^{2}f_6f_8f_{12} \\
& - 4040836680f_2f_8^{2}f_6^{2} - 467026560f_2f_8^{2}f_{12} \\
& + 372323952f_8^{3}f_6 + 3606888285f_6^{5} + 1683682000f_6^{3}f_{12} \\
& - 804323520f_6f_{12}^{2}
\end{aligned}
$$

$$
\begin{aligned}
r_{18} = {} & 133986522f_2^{9} - 10076755f_2^{6}f_6 - 3496188f_2^{5}f_8 + 754410f_2^{3}f_6^{2} \\
& + 449360f_2^{3}f_{12} + 210f_2^{2}f_6f_8 + 48f_2f_8^{2} - 1435f_6^{3} - 600f_6f_{12}
\end{aligned}
$$

APPENDIX TWO : NOTATION AND ABBREVIATIONS

Mappings are written on the left, so that $f \circ g$ denotes g followed by f.

A numerical superscript denotes the dimension of a manifold.

Numerical subscripts usually denote either differentiation (e.g. e_i is the i^{th} derivative of the map e) or the degree of a polynomial (e.g. $p_2, p_{12}, p_{20}, p_{30}$ are polynomials of degrees 2, 12, 20 and 30 respectively). However, in the notation k_i for the i^{th} principal curvature function, the subscript does not stand for either of these.

Alphabetic subscripts denote partial derivatives,

e.g. $r_u = \dfrac{\partial r}{\partial u}$.

θ	(3.7)	A map $\mathbb{R} \to \mathbb{R}^{n-1}$ such that $t \to r(\theta(t))$ is a line of curvature on M
ϕ	(3.7)	A map $\mathbb{R} \to \mathbb{R}^{n-1}$ such that $t \to r(\phi(t))$ is a line of curvature on M
$\langle ... \rangle$		Usual scalar product on $\mathbb{R}^n$
$\vert ... \vert$		The number of elements in a finite group or set

$\bar{}$	(3.7)	A bar denotes composition with θ
$\sim$	(3.7)	A tilde denotes composition with ϕ
$\sim$	(1.43)	$\tilde{G}$ denotes the pre-image of G under π_n: Spin(n)--> SO(n)
A_m	p165	The map $N_m M \longrightarrow (S^2R^2)^*$ which associates to each point of $N_m M$ the second fundamental form of M at that point
[c,s]		Homogeneous co-ordinates in RP^n ($c \in R^n$, $s \in R$)
$\Delta(G)$	p20	The pseudodiscriminant of the group G
$\Delta(\varepsilon)$	p35	The section of $\Delta(G)$ by the hyperplane $f_2=\varepsilon$
$\Delta_1(\varepsilon)$	p36	A certain subset of $\Delta(\varepsilon)$
$\Delta_2(\varepsilon)$	p36	A certain subset of $\Delta(\varepsilon)$
Disc		The usual discriminant of a polynomial
d(1),....,d(n)	p46	The degrees of a set of basic invariants for the group G
d_{2i}	(1.38)	The polynomial $\sigma_i(x_1^2,\ldots,x_n^2)$
d_2,d_4,d_6,d_8	(1.38)	A set of basic invariants for B_4
e_2,e_4,e_6,e_8	(1.38)	Another set of basic invariants for B_4
E	(2.27)	The space evolute of the curve M

e	(2.27)	a parametrisation of E such that $\pi \circ e = r$
e	(3.5)	a parametrisation of one sheet of F such that $\pi \circ e = r$
F	(2.4)	the focal set of M
$Fix_G(x)$	(1.3)	The stabiliser of the point x under the action of the group G on $\mathbb{C}^n$
f_2	(1.9)	The polynomial $x_1^2 + \ldots x_n^2$
f_2, f_6, f_8, f_{12}	(1.39)	A set of basic invariants for F_4
G		An arbitrary Coxeter group
G^0	(1.43)	The group $G \cap SO(n)$
$Imm(\mathbb{R}^k, \mathbb{R}^n)$		The set of immersive maps from $\mathbb{R}^k$ to $\mathbb{R}^n$.
i_2, i_6, i_{10}	(1.17)	A set of basic invariants for H_3
k_i	p107	The i^{th} principal curvature function on M
$\mathfrak{m}$		The maximal ideal in the ring of C^∞ map-germs from $\mathbb{R}^i$ to $\mathbb{R}^j$ for some pair (i,j)
M		An arbitrary k-dimensional submanifold of $\mathbb{R}^n$, having F as focal set
M^*	(3.46)	The Legendrian submanifold of $PT^*\mathbb{R}^n$ determined by M

$M_1,\ldots,M_N$	p23	The mirrors of the group G
M_i'	p23	The intersection $M_1 \cap M_i$ of two mirrors of G
NM	p65	The normal bundle of M, to be considered as a subset of $M \times \mathbb{R}^n$
$N_m M$		The normal space to M at m: the fibre of $\pi_1 : NM \longrightarrow M$ above the point $m \in M$
$\mathcal{O}(n)$		An arbitrary element of $\mathcal{M}^n$
$p_2, p_{12}, p_{20}, p_{30}$	(1.42)	A set of basic invariants for H_4
$p_i^y(x)$	p46	A G-invariant polynomial of degree i
π_G	p19	A quotient map for the action of G on $\mathbb{C}^n$
π	p66	The projection map $F \longrightarrow M$ where F is the focal set of M
π_1	p65	The projection map $NM \longrightarrow M$
π_2	p65	The projection map $NM \longrightarrow \mathbb{R}^n$ where NM is considered as a subset of $M \times \mathbb{R}^n$
π_n	(1.43)	the projection map $Spin(n) \longrightarrow O(n)$
Pu q	(1.44)	The pure part of the quaternion q
q	(1.6)	A parametrisation of $\Delta(G)$ defined in chapter 1

q_ε	p35	A parametrisation of a hyperplane section of $\Delta(G)$ (the restriction of q to the hyperplane $f_1 = \varepsilon$)
r		A local parametrisation of the manifold M
$(\mathbb{R}^k)^*$		The dual of the vector space $\mathbb{R}^k$
R(G)	p18	The ring of G-invariant polynomials
Re q	(1.44)	The real part of the quaternion q
RLC	(3.7)	Raised line of curvature
ρ	(3.7)	$1/k_i$ where k_i is the i^{th} principal curvature
$\rho(q)$	(1.44)	An element of SO(3) given by $x \dashrightarrow qx\bar{q}$ where q, x are quaternions, x is pure, and $\lvert q \rvert = 1$.
$S^2\mathbb{R}^n$		The symmetric product of two copies of the vector space $\mathbb{R}^n$
TM		The tangent bundle to M
T_mM		The tangent space to M at m
$Tub_c(M)$	(2.13)	The tubular hypersurface with core M and radius c.
τ		$(1 + \sqrt{5})/2$
V	(2.8)	The family of distance-squared functions for the manifold M

$v^{[c,s]}$	(2.5)	A function from a family which includes the distance squared functions on M and the height functions on M
v^y	p65	The distance squared function on M given by the point $y \in \mathbb{R}^n$
Z(G)		The centre of the group G

APPENDIX THREE : LIST OF DEFINITIONS

big involute	see (2.42)
big (wave) front	see page 73
canal (hyper)surface	tubular (hyper)surface
complementary (hyper)surface	see page 122
(locally) congruent	see (2.1)
conjugate directions (on a surface)	see (3.24)
Coxeter group	see page 17
curvature ellipse	see page 170
cuspidal cross-cap	see (3.41)
cuspidal edge of type 3/2	ordinary cuspidal edge
cuspidal edge of type 5/2	rhamphoid cuspidal edge
cuspidal pinch point	cuspidal cross cap
developable cone	see page 75
developable cylinder	see page 76
developable hypersurface	see (2.17)
developable surface	see (3.4)
discriminant	see (1.2)
distance-generic	see (2.8)
(locally) diffeomorphic	see page 63
distance-squared function	see page 65
evolute = focal set	see page 65
evolvent = involute	involute

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Abbreviations Used:

FAP = Funktsional'nyi Analiz i ego Prilozheniya

FAA = Functional Analysis and its Applications

UMN = Uspekhi Matematicheskykh Nauk

RMS = Russian Mathematical Surveys

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