HOMOLOGICAL THEORY OF BOCS REPRESENTATIONS

Thesis submitted in accordance with the requirements of the

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Doctor in Philosophy by

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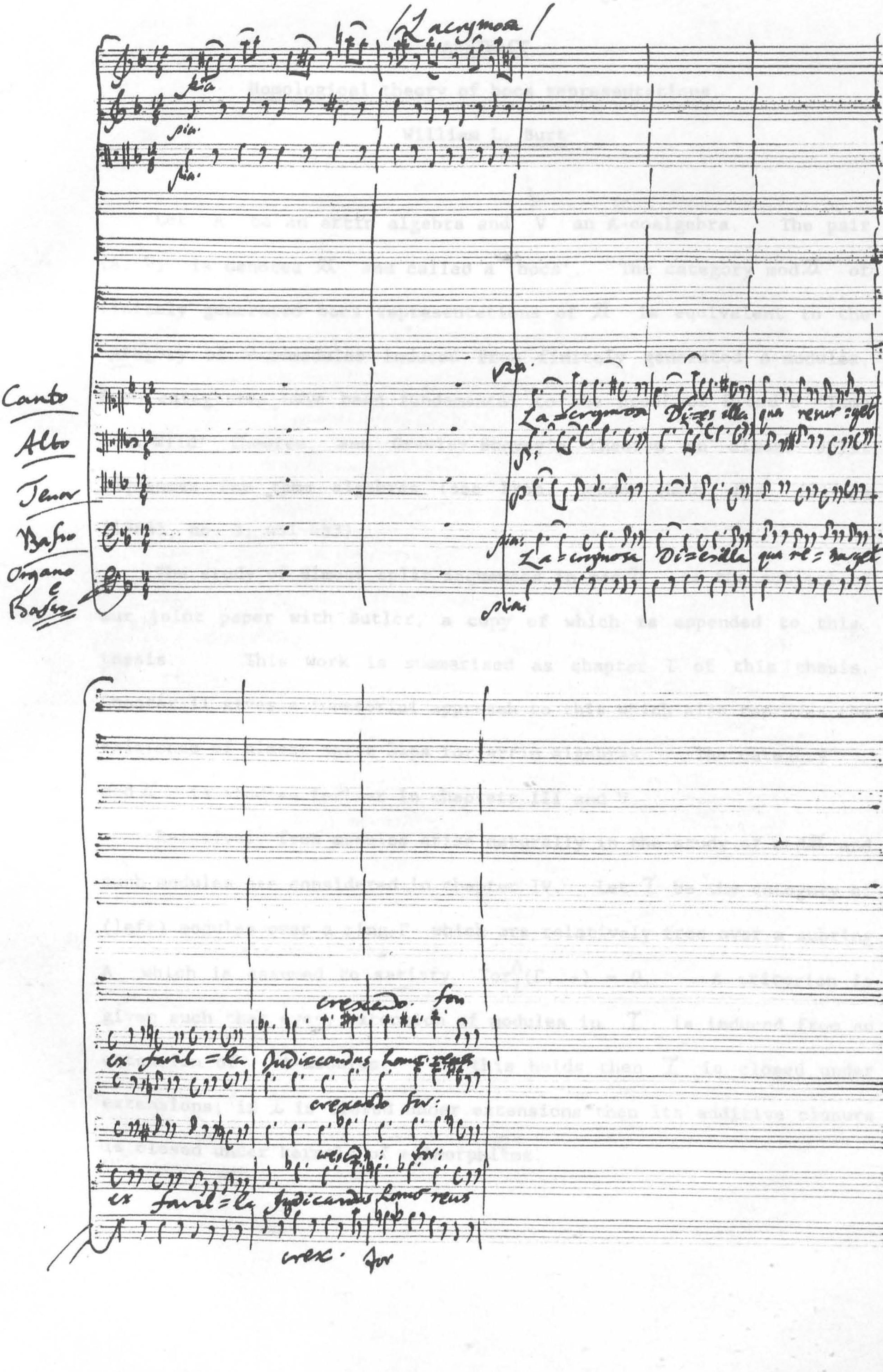
Dedicated to the memory of

WOLFGANG AMADEUS MOZART (1756-1791),

and marking the bicentenary of his death,

5th December 1791.

The illustration on the following page is a handwritten 'facsimile' of the opening of the 'Lacrymosa' from Mozart's Requiem (KV. 626), on which he was working at the time of his death.


Lacrymosa
Canto
Alto
Tenor
Basso
Organo e Basso
Lacrymosa
dies illa
qua resurget
crescendo
ex favil=la
Judicandus homo reus

ABSTRACT

Homological theory of bocs representations.

William L. Burt

Let A be an artin algebra and V an A-coalgebra. The pair (A, V) is denoted $\mathfrak{A}$ and called a 'bocs'. The category mod$\mathfrak{A}$ of finitely generated bocs representations of $\mathfrak{A}$ is equivalent to the category of V-comodules induced from finitely generated A-modules. Such categories have been fundamental to the proof of Drozd's 'Tame and Wild' theorem, and Crawley-Boevey's theorem on almost split sequences for tame algebras (see Proc. London Math. Soc. (3) 56 (1988), no. 3, 451-483).

The study of almost split sequences in mod$\mathfrak{A}$ is the subject of our joint paper with Butler, a copy of which is appended to this thesis. This work is summarised as chapter I of this thesis. Chapter II gives a functorial approach to this which also reproves the existence of almost split maps for artin algebras. The category mod$\mathfrak{A}$ is studied further in chapters III and V.

Relatively free modules arise naturally in the study of mod$\mathfrak{A}$ and such modules are considered in chapter IV. Let $\mathfrak{X}$ be the category of (left) modules over a ring Γ which are relatively free over a subring Λ which is assumed to satisfy $\mathrm{Tor}_1^{\Lambda}(\Gamma, -) = 0$. A criterion is given such that every extension of modules in $\mathfrak{X}$ is induced from an extension of Λ-modules. If this holds then $\mathfrak{X}$ is closed under extensions; if $\mathfrak{X}$ is closed under extensions then its additive closure is closed under kernels of epimorphisms.

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Preface

The work presented here originates from work of M.C.R. Butler on almost split sequences for bocses. This formed the basis for our joint paper [BB] with Butler. This paper has been accepted for publication and a copy of its final form is appended to this thesis with a covering note as to the extent of the present writer's contribution. This paper is summarised in chapter I and §1 of chapter III.

The content of chapters II, III (§§2, 3), IV, V, is new. Appropriate credit is given whenever other work is referred to or used. We except from this the constant use of standard techniques from homological algebra and elsewhere. In particular some results similar (but not identical) to those in chapter IV have been obtained independently by Prof. Mark Kleiner (Syracuse University, New York) - an addendum (§3) to chapter IV discusses this and other matters in detail.

Introduction

As our title implies we are concerned here with the development of homological algebra in the category of representations of a bocs. This category can be realised as a category of induced (or dually coinduced) modules - and some of the theory can, and will, be developed in terms of such categories without any reference to bocses and their representations. We now discuss the motivation for this work.

The concept of a bocs and its representations was introduced by Roiter (see e.g. [Ro]) to formalize the theory of 'reductions' of 'matrix problems', which had been developed by Roiter and others, principally at Kiev. Two of the most striking applications of this methodology are to the representation theory of finite-dimensional algebras (over algebraically closed fields); they are Drozd's 'Tame and Wild' theorem [D], and Crawley-Boevey's theorem [C-B1: Theorem D] concerning the Auslander-Reiten theory of tame algebras. Despite the huge volume of new work in representation theory of finite-dimensional algebras over the past two decades, no proofs of these important results are known, excepting their original proofs using bocses.

The Crawley-Boevey theorem is particularly tantalising, as it is a statement about Auslander-Reiten theory of a class of algebras for which no proof within that theory is known. This has led various researchers to investigate Auslander-Reiten theory for bocses.

The earliest reference for such work that we know is the work of de la Peña and Simson [PS] which covers (implicitly) certain bocses, including Drozd's construction [C-B1: §6] of a bocs corresponding to a given finite-dimensional algebra.

A more general account is that of Bautista and Kleiner [BK] - and a theory along similar lines was developed independently by M.C.R. Butler, which formed the nucleus of the paper [BB] written jointly with the present writer. This paper which was prepared with the benefit of a preprint version of [BK] and discussions with one of its authors, Raymundo Bautista, provides a smoother treatment with additional results not in [BK], and is phrased explicitly in terms of bocs representations throughout. We do not propose to repeat proofs which are already present in the paper [BB]. For the reader's convenience, a copy of this paper is appended to this thesis.

We now discuss the content of this thesis in more detail. The term 'bocs' is derived as an abbreviation for 'bimodule over category with coalgebra structure'. In the papers [BB, BK], and also in this thesis, this is replaced by the usual notion of a coalgebra over a ring, but will continue to be called a bocs. The categories of 'representations' which arise are equivalent to those which occur for the coalgebras over categories - the formal necessity of using coalgebras over categories occurs only in connection with certain types of 'reductions', but these are not our concern here.

After a short section devoted to notation, conventions, and the definition of a bocs and its representations, chapter I consists of a review of the general 'left and right algebra theory', developed in [BB]. Chapter II gives a new approach, in the manner of [A2: §7], to the existence of almost split maps (the basic ingredients of any Auslander-Reiten theory), by studying finitely generated and finitely presented functors on the category of representations of a bocs. It is worth noting that this reproves the existence of almost split maps and hence almost split sequences for artin algebras, first established by Auslander and Reiten in [AR2].

Chapter III is devoted to the further development of the left and right algebra theory in [BB] which we summarised in Chapter I; most of the results are new and not to be found in [BB].

In chapter IV we initiate an 'abstract' study of categories of relatively projective modules (and dually relatively injective) modules. The main concern is to develop a criterion for the category of relatively projective modules to be closed under extensions - a similar question has also been raised by Auslander and Reiten in a recent paper [AR4] and we discuss (briefly) the relationship between our work, the paper [AR4], and some results of Kleiner quoted in [AR4]. The relation of 'relative' almost split sequences of relatively projective modules to 'absolute' one is discussed in §2, and a result of Kleiner concerning group algebras [K] is shown to have a very short proof.

Finally chapter V returns the to the study of bocses, investigating various notions of monomorphism/epimorphism for morphisms of bocs representations, particularly in the case that the bocs satisfies suitable 'triangularity' conditions (often present in applications - e.g. [D, C-B1]).

Acknowledgements

My first acknowledgements must be to my supervisor Dr. Michael Butler for many many hours of discussions on representation theory (among other things) and also to Dr. Sheila Brenner for overseeing my introduction to the representation theory of algebras. Also a sincere thankyou to the Science and Engineering Research Council for their financial support during my time as a research student at Liverpool University.

I should also mention my fellow representation theory students Flavio Coelho and Liu Shiping for their friendship and 'solidarity'.

It has been my privilege to meet several notable exponents of representation theory of algebras and bocses during the period of this research. It is a pleasure to mention particularly Bill Crawley-Boevey, Raymundo Bautista, Idun Reiten, and Maurice Auslander for useful discussions, advice and encouragement.

On a more personal note I would like to express my gratitude to Dr. Christie and the staff of Galbraith ward, Knowle hospital; also Sheila Brenmner, Michael Butler, and Suzanne Tickner for their support over the last year.

A special thank you to the dedicatee of this thesis, Wolfgang Amadeus Mozart, for his sublime music.

Finally an acknowledgement to my typist Wendy Orr who produced the elegant typescript throughout this thesis.

Conventions, bocses and their representations

Throughout this thesis A denotes a fixed ring, (associative, with identity) subject to further qualification for specific results. The symbol $\otimes$ always means $\otimes_A$. For rings B, C, the notations ${}_BX$, Y_C, ${}_BZ_C$ indicate that X, Y, Z are respectively, a left B-module, a right C-module, and a left B, right C-bimodule (more briefly, a B-C-bimodule).

Since a right module over a ring C can be identified as a left C^{op}-module we adopt the following notations:

Given ${}_BX$, ${}_BX'$, $\mathrm{Hom}_B(X, X')$ denotes the group of left B-module morphisms $X \to X'$, likewise the group of right C-module morphisms $Y_C \to Y'_C$ is denoted $\mathrm{Hom}_{C^{op}}(Y, Y')$. Groups of B-C-bimodule morphisms are denoted $\mathrm{Hom}_{B\times C^{op}}(-, -)$.

The category of all left modules over a ring B is denoted Mod B, and the full subcategory of fintely generated modules mod B. The corresponding notations for categories of right B-modules will be Mod B^{op}, mod B^{op}.

Given ${}_AZ$, W_A we often abuse notation by writing $A \otimes Z = Z$, $W \otimes A = W$. As an application of this 'abuse' we now give the definition of a 'bocs'.

Definition By a bocs $\mathfrak{U}$ we mean a quadruple $\mathfrak{U} = (A, V, \mu, \epsilon)$, often denoted just $\mathfrak{U} = (A, V)$, in which V is an A-A-bimodule; the comultiplication $\mu : V \to V \otimes V$ is an A-A-bimodule morphism and is coassociative, that is, $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$; the counit $\epsilon : V \to A$ is an A-A-bimodule morphism satisfying $(1 \otimes \epsilon)\mu = 1_V = (\epsilon \otimes 1)\mu$; we shall always assume that ϵ is surjective.

If A is an algebra over a central subring k, we also assume that k acts centrally on V.

<u>Definition</u> Let $\mathfrak{A} = (A, V)$ be a bocs. The category $\mathrm{Mod}\,\mathfrak{A}$ of <u>representations</u> of $\mathfrak{A}$ is the category whose objects are the left A-modules $X, Y, \ldots,$ and whose morphism groups $\mathfrak{A}(X, Y)$ are given by

$$\mathfrak{A}(X, Y) = \mathrm{Hom}_A(V \otimes X, Y).$$

Given f in $\mathfrak{A}(X, Y)$, g in $\mathfrak{A}(Y, Z)$, their composite $g \circ f$ in $\mathfrak{A}(X, Z)$ is given by

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y \xrightarrow{g} Z .$$

The identity morphism of an object X is $\epsilon \otimes 1 : V \otimes X \to X$.

$\mathrm{Mod}\,\mathfrak{A}$ is an additive category (direct sums being given by the usual direct sum in $\mathrm{Mod}\,A$), but not necessarily fully additive - sufficient conditions are given in [BK §5, BB §6] for this to hold, and these conditions are certainly satisfied by important classes of bocses used in applications; for example, all 'additive Roiter bocses' [CB1: 3.5].

We are usually interested only in the full subcategory $\mathrm{mod}\,\mathfrak{A}$ of $\mathrm{Mod}\,\mathfrak{A}$ defined by the class of finitely generated left A-modules.

<u>Example</u> A trivial, but important, example of a bocs is the <u>principal bocs</u> $\mathfrak{A} = (A, A)$ which is given by setting the counit to be $1 : A \to A$, and the comultiplication to be the natural map $A \to A \otimes A$. Since for this bocs $\mathrm{mod}\,\mathfrak{A}$ ($\mathrm{Mod}\,\mathfrak{A}$) is just (isomorphic to) $\mathrm{mod}\,A$ ($\mathrm{Mod}\,A$), representations of bocses can be viewed as a sort of generalization of modules over rings.

<u>Some category theory</u>

Let $\underline{C}$ be a category, $X \xrightarrow{f} Y$ a morphism in $\underline{C}$, and Z another object of $\underline{C}$. Then there are mappings

$$\underline{C}(Z, f): \underline{C}(Z, X) \to \underline{C}(Z, Y),$$

$$\underline{C}(f, Z): \underline{C}(Y, Z) \to \underline{C}(X, Z),$$

for which we shall often employ the shorthand notations f_* for $\underline{C}(Z, f)$ and f^* for $\underline{C}(f, Z)$. Likewise the natural transformations

$$\underline{C}(-, f): \underline{C}(-, X) \to \underline{C}(-, Y),$$

$$\underline{C}(f, -): \underline{C}(Y, -) \to \underline{C}(X, -),$$

may be denoted f_*, f^* respectively.

Let us fix here the convention that all mappings will be written, and composed, on the left.

Recall that <u>idempotents split</u> in a category $\underline{C}$ if every idempotent endomorphism $e = e^2$ of every $\underline{C}$ - object X admits a factorisation $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $fg = 1_Y$. In particular if $\underline{C}$ is a full subcategory of a module category then idempotents split in add($\underline{C}$), the full subcategory defined by the modules isomorphic to direct summands of finite direct sums of objects in $\underline{C}$.

If $\underline{C}$ is an abelian category there is the usual sequence $\mathrm{Ext}^i(-, -)$, $i = 1, 2, \ldots$ of Ext - functors, defined in terms of sets of equivalence classes of (i-fold) extensions in $\underline{C}$. $\mathrm{Ext}^0(-, -)$ is (identified with) the hom - functor $\underline{C}(-, -)$.

Throughout this thesis we shall use the terminologies 'map' and 'morphism' interchangeably.

Our final remark concerns our convention for endomorphism rings. If $\underline{C}$ is an additive category and X an object of $\underline{C}$ then $\underline{C}(X, X)$ has a natural ring structure where for f, g in $\underline{C}(X, X)$ their product $f.g$ in $\underline{C}(X, X)$ is $f \circ g$, i.e. $X \xrightarrow{g} X \xrightarrow{f} X$. Thus if $\underline{C}$ is Mod Λ (Λ any ring) then X has a natural structure as a left $\mathrm{End}_\Lambda(X)$ - module.

Chapter I: Almost split sequences for bocses - a summary

This chapter summarises, without proofs, the main theory of the paper [BB]. The only change is a slight difference in notation - we use $\mathfrak{A}$ instead of $\mathcal{A}$ to denote our fixed bocs, and the category of finitely generated (left) modules over a given ring (B say) is denoted mod B rather than B-mod. The four sections of this chapter correspond to the first four chapters of [BB].

§1 The left and right algebra of a bocs

Let $\mathfrak{A} = (A, V, \mu, \epsilon)$ be a bocs. We define its left algebra L to be the A-A-bimodule $\mathrm{Hom}_{A^{op}}(V, A)$ with multiplication given by the following rule:

for e, f in L, $e.f$ is the composite

$$V \xrightarrow{\mu} V \otimes V \xrightarrow{f \otimes 1} A \otimes V = V \xrightarrow{e} A.$$

Proposition 1.1 [BB: 1.1] The bimodule ${}_AL_A$, equipped with the above multiplication $L \otimes L \to L$ and the map $a \longmapsto a\epsilon$ of A into L, is an A-algebra with identity ϵ (the counit of $\mathfrak{A}$), and the bimodule structure on L obtained by restriction along the map $a \longmapsto a\epsilon$ conincides with its natural A-A-bimodule structure.

Similarly, the right algebra R of $\mathfrak{A}$ is the A-A-bimodule $\mathrm{Hom}_A(V, A)$, with multiplication given as follows:

for s, t in R, $s.t$ is the composite

$$V \xrightarrow{\mu} V \otimes V \xrightarrow{1 \otimes s} V \otimes A = V \xrightarrow{t} A.$$

Proposition 1.1′ [BB: 1.1′] The bimodule ${}_AR_A$, equipped with the above multiplication $R \otimes R \to R$ and the map $a \longmapsto a\epsilon$ of A into R, is an A-algebra with identity ϵ, and the bimodule structure on R obtained by restriction along $a \longmapsto a\epsilon$ coincides with the natural A-A-bimodule structure on R.

The terminology 'left' and 'right' algebra is justified by the existence of natural left L-module, and right R-module structures on V. They are obtained as follows: first note that there are evaluation maps

$$E_L: L \otimes V \to A \ ; \ e \otimes v \longmapsto e(v),$$

$$E_R: V \otimes R \to A \ ; \ v \otimes s \longmapsto s(v).$$

Hence there is a left L-action on V, given by

$$L \otimes V \xrightarrow{1 \otimes \mu} L \otimes V \otimes V \xrightarrow{E_L \otimes 1} A \otimes V = V,$$

and a right R-action on V, given by

$$V \otimes R \xrightarrow{\mu \otimes 1} V \otimes V \otimes R \xrightarrow{1 \otimes E_R} V \otimes A = V.$$

<u>Proposition 1.2</u> [BB: 1.2] The actions of L and R on V just defined induce an L-R-bimodule structure ${}_L V_R$ on V, compatible with the original A-A-bimodule structure on V after restriction along the maps $a \longmapsto a\epsilon$ of A to L and A to R.

<u>Corollary 1.3</u> [BB: 1.3] $\mu : V \to V \otimes V$ is an L-R-bimodule morphism.

§2 Functorially finite embeddings of mod $\mathfrak{A}$

Throughout §2, $\mathfrak{A} = (A, V)$ is a bocs such that $_AV$ and V_A are finitely generated projective modules.

In view of applications this creates an immediate problem since for the bocses used in, e.g., [C-Bl] A is an algebra over a field k but not necessarily a finite-dimensional one. If A is not finite-dimensional it is usually the case that $_AV$, V_A are not finitely generated modules so our theory will not apply. Thus we restrict our attention to the case that A is a finite-dimensional algebra - or an artin algebra, since the theory still goes through in that case. Our formal assumption is that

A is an artin algebra over a central artinian subring k.

This is a rather more special assumption than that made in the corresponding section of [BB], but it avoids the necessity of including the rather artificial extra conditions in [BB: 2.6].

Our assumptions imply that R_A and $_AL$ are finitely generated projective modules, and the duality maps $_AV_R \to {}_A\mathrm{Hom}_{A^{op}}(R, A)_R$ and $_LV_A \to {}_L\mathrm{Hom}_A(L, A)_A$ are bimodule isomorphisms. Also, given any left A-module X, there are natural maps

$$\beta : R \otimes X \to \mathrm{Hom}_A(V, X) \ ; \ s \otimes x \longmapsto (v \longmapsto s(v)x),$$

$$\alpha : V \otimes X \to \mathrm{Hom}_A(L, X) \ ; \ v \otimes x \longmapsto (e \longmapsto e(v)x),$$

which are isomorphisms of R-modules and L-modules respectively.

Notation Let $\underline{I}$ denote the full subcategory of mod R with objects the induced modules - that is, modules $_RM$ isomorphic to $R \otimes X$ for some finitely generated $_AX$. Let $\underline{C}$ denote the full subcategory of mod L with objects the coinduced modules - that is, modules $_LN$ isomorphic to $\mathrm{Hom}_A(L, X)$ for some finitely generated $_AX$.

Under the assumptions made at the start of this section these subcategories are equivalent to mod $\mathfrak{A}$. The precise results are as follows:

<u>Theorem 2.1</u> [BB: 2.4] The functors $V \otimes_R -$: mod R $\to$ mod L and $\mathrm{Hom}_L(V, -)$: mod L $\to$ mod R restrict to mutually inverse equivalences between $\underline{I}$ and $\underline{C}$.

<u>Theorem 2.2</u> [BB: 2.5] There are equivalences of categories F_C: mod $\mathfrak{A} \to \underline{C}$ and F_I: mod $\mathfrak{A} \to \underline{I}$.

<u>Proof</u> Full details of this will be found in [BB], but for later use we recall the definitions of F_C, F_I. F_C is defined explicitly and then we set F_I to be $\mathrm{Hom}_L(V, -) \circ F_C$.

Given an object X of mod $\mathfrak{A}$ we let $F_C(X) = V \otimes X$, which is isomorphic under the map $\alpha : V \otimes X \to \mathrm{Hom}_A(L, X)$ to an object in $\underline{C}$, and hence is itself in $\underline{C}$. Given f in $\mathfrak{A}(X, Y)$ set $F_C(f)$ to be the composite

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y.$$

This is an L-module morphism by 1.3.

The equivalences in 2.2 above exhibit mod $\mathfrak{A}$ as a full subcategory of each of mod R, mod L. R and L are clearly both artin algebras (over k) and so mod R, mod L, have almost split sequences. In §4 we apply a result of Auslander and Smalø [AS2] to deduce that $\underline{I}$ and $\underline{C}$ sometimes also have almost split sequences - we show here that one of the criteria required is satisfied, namely that $\underline{I}$ and $\underline{C}$ are 'functorially finite' in mod R and mod L respectively. This notion was introduced in [AS1] and for the convenience of the reader we recall the definition.

<u>Definition 2.3</u> Let Λ be an artin algebra and let <u>Ab</u> denote the category of all abelian groups. Let $\underline{D}$ be a full subcategory of mod Λ. A covariant functor F from $\underline{D}$ to <u>Ab</u> is called

finitely generated if there is an object X in $\underline{D}$ and a surjection

$$\mathrm{Hom}_{\Lambda}(X, -)|_{\underline{D}} \to F$$

of functors - thus F is a quotient of a covariant representable functor on $\underline{D}$. The definition of a finitely generated contravariant functor $\underline{D} \to \underline{Ab}$ is dual to this.

$\underline{D}$ is called <u>covariantly finite</u> if, for each M in mod Λ the functor $\mathrm{Hom}_{\Lambda}(M, -)|_{\underline{D}} : \underline{D} \to \underline{Ab}$ is finitely generated; $\underline{D}$ is called <u>contravariantly finite</u> if, for each M in mod Λ the functor $\mathrm{Hom}_{\Lambda}(-, M)|_{\underline{D}} : \underline{D} \to \underline{Ab}$ is finitely generated.

$\underline{D}$ is called <u>functorially finite</u> if $\underline{D}$ is both covariantly and contravariantly finite.

<u>Theorem 2.4 [BB: 2.6]</u> $\underline{C}$ is functorially finite in mod L and $\underline{I}$ is functorially finite in mod R.

As a final remark it can be shown that, under the assumptions made in this section ${}_LV_R$ is a 'balanced' bimodule:

<u>Proposition 2.5</u> [BB: 2.7] The natural ring homomorphisms

$$L \to \mathrm{End}_{R^{op}}(V) \ ; \ s \longmapsto (v \longmapsto sv),$$

$$R^{op} \to \mathrm{End}_L(V) \ ; \ e \longmapsto (v \longmapsto ve),$$

are isomorphisms.

§3 Extension properties

We keep the same assumptions as in §2, thus $\mathfrak{A} = (A, V)$ is a bocs over an artin algebra, the counit is surjective, and ${}_AV$, V_A are finitely generated projective modules. Thus R_A, ${}_AL$ are finitely generated projective and so there are exact functors

$R \otimes - : \text{mod } A \to \underline{I}$,

$\text{Hom}_A(L, -) : \text{mod } A \to \underline{C}$.

$R\otimes -$ thus carries exact sequences of left A-modules to exact sequences of (induced) left R-modules and so induces, for all X, Z in mod A and $n = 0, 1, 2, \ldots$ maps

$$\Gamma^{(n)} : \text{Ext}^n_A(Z, X) \to \text{Ext}^n_R(R \otimes Z, R \otimes X).$$

Likewise $\text{Hom}_A(L, -)$ induces maps

$$\Delta^{(n)} : \text{Ext}^n_A(Z, X) \to \text{Ext}^n_L(\text{Hom}_A(L, Z), \text{Hom}_A(L, X)).$$

In order to state the main results of this section we make the following definitions:

Definition 3.1 The kernel $\bar{V} = {}_A\bar{V}_A$ of a bocs $\mathfrak{A} = (A, V)$ is the kernel of the counit ϵ of $\mathfrak{A}$.

Definition 3.2 Let ${}_AU_A$ be a bimodule. It is called projectivising if for all ${}_AX$, Z_A the A-modules $U \otimes X$ and $Z \otimes U$ are projective.

Theorem 3.3 [BB 3.8] Let $\mathfrak{A} = (A, V)$ be a bocs (with surjective counit) such that ${}_AV$ and V_A are finitely generated projective and ${}_A\bar{V}_A$ is projectivising. Then for all ${}_AX$, ${}_AZ$ the maps

$$\Gamma^{(n)} : \text{Ext}^n_A(Z, X) \to \text{Ext}^n_R(R \otimes Z, R \otimes X)$$

$$\Delta^{(n)} : \text{Ext}^n_A(Z, X) \to \text{Ext}^n_L(\text{Hom}_A(L, Z), \text{Hom}_A(L, X))$$

exist and are surjective for $n = 1$ and bijective for $n \geq 2$. Moreover $\underline{I}$ and $\underline{C}$ are extension-closed subcategories of mod R and mod L respectively.

§4 Almost split sequences

In [AS2: 1.1] Auslander and Smalø established a criterion sufficient for a subcategory $\underline{D}$ of mod Λ (Λ an artin algebra) to have almost split sequences. Thus for each indecomposable object X of $\underline{D}$ there is a right almost split morphism in $\underline{D}$ terminating at X and a left almost split morphism in $\underline{D}$ starting at X; moreover if there is a non-split exact sequence $0 \to X \to Y \to Z \to 0$ with $Y, Z \in \underline{D}$ (i.e. X is non - Ext - injective in $\underline{D}$) then there is also one which is almost split in $\underline{D}$, and dually there is an almost split sequence in $\underline{D}$ terminating at any indecomposable non - Ext - projective object in $\underline{D}$.

If $\underline{D}$ = add $\underline{D}$ and is functorially finite in mod Λ and closed under extensions then they also showed [AS2: 2.4] that the criterion held, and so we can apply this to the subcategories $\underline{I}$ and $\underline{C}$ arising in connection with mod $\mathfrak{A}$ to obtain the following:

Theorem 4.1 [BB: 4.1] Let $\mathfrak{A} = (A, V)$ be a bocs over an artin algebra with kernel $\overline{V}$ a projectivising bimodule and V being of finite length over the centre of A, then add $\underline{I} \subseteq$ mod R and add $\underline{C} \subseteq$ mod L have almost split sequences. If idempotents split in mod $\mathfrak{A}$ then $\underline{I}$, $\underline{C}$ have almost split sequences, and hence mod $\mathfrak{A}$ (viewed as a full subcategory of mod R or mod L) has almost split sequences.

Chapter II: A functorial approach to Auslander-Reiten theory for bocs representations

In this chapter we develop a new approach to existence of almost split morphisms for mod $\mathfrak{A}$. This is a direct generalization of the proof for finite-dimensional algebras given by Auslander in [A2: §7] and uses a study of functors on mod $\mathfrak{A}$.

The first section recalls some facts about functors and in §§4, 5 Auslander's approach in [A2: §7] is followed to obtain our desired result; §§2, 3 contain some extra technical preliminaries to make the proof carry over to our case. The only missing link is the existence of 'finite presentations' for two particular functors on mod $\mathfrak{A}$, which is not so obvious as in the special case of [A2]. This gap is plugged using the theory of §§1, 2 of Chapter I.

Our assumptions throughout this chapter are that $\mathfrak{A} = (A, V)$ is a bocs such that

(i) A is an artin algebra over its (artinian) centre k,

(ii) ${}_AV$, V_A are finitely generated projectives,

(iii) idempotents split in mod $\mathfrak{A}$.

All functors (categories) in this chapter are k-functors (k-categories).

It will be convenient, as in [A2], to use the notation (-, -) for $\mathrm{Hom}_A(-, -)$.

§1 Functors on k-categories

Let $\mathcal{C}$ be a skeletally small k-category; we consider the category $(\mathcal{C}, \text{mod } k)$ of covariant k-functors from $\mathcal{C}$ to mod k. The category of contravariant k-functors from $\mathcal{C}$ to mod k is identified with $(\mathcal{C}^{op}, \text{mod } k)$ so results about contravariant functors can be inferred from the corresponding ones about covariant functors. We shall feel free to use the 'contravariant' version of the theory outlined here, which follows [A2: §2, §3].

We recall that $(\mathcal{C}, \text{mod } k)$ is an abelian category.

Definition 1.1 A functor F in $(\mathcal{C}, \text{mod } k)$ is called finitely generated if there is an X in $\mathcal{C}$ and an exact sequence

$\mathcal{C}(X, -) \to F \to 0$.

Definition 1.2 A functor F in $(\mathcal{C}, \text{mod } k)$ is called finitely presented if there are X, Y in $\mathcal{C}$ and an exact sequence

$\mathcal{C}(X, -) \to \mathcal{C}(Y, -) \to F \to 0$.

Remarks We shall abbreviate these concepts by f.g. and f.p. respectively. The representable functors $\mathcal{C}(X, -)$ in $(\mathcal{C}, \text{mod } k)$ are projective objects in this category so the notions of f.g., f.p., functors are precise analogues of the corresponding concepts for modules. Hence certain basic results for modules carry over directly for functors. To check that $\mathcal{C}(X, -)$ really is always projective note that if $F \to G \to 0$ is exact in $(\mathcal{C}, \text{mod } k)$ then $\text{Hom}(\mathcal{C}(X, -), -)$ preserves exactness since, by Yoneda's lemma, the resulting sequence of maps is isomorphic to $F(X) \to G(X) \to 0$. The following results are now proved exactly as for modules.

Lemma 1.3 Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence in $(\mathcal{C}, \text{mod } k)$. Then

(i) if F_2 is f.g. so is F_3,

(ii) if F_1, F_3 are f.g. so is F_2,

(iii) if F_1, F_3 are f.p. so is F_2.

Lemma 1.4 Let F be in $(\mathcal{C}, \text{mod } k)$. The following are equivalent

(i) F is f.p.,

(ii) There is an exact sequence $0 \to F_1 \to F_0 \to F \to 0$ with F_0 representable and F_1 f.g.,

(iii) F is f.g. and whenever $0 \to F_1 \to F_0 \to F \to 0$ is exact with F_0 f.g., F_1 is also f.g.

If $\mathcal{C}$ is an abelian category then given an exact sequence $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ in $(\mathcal{C}, \text{mod } k)$ with F_2, F_3 f.p. then also F_1, F_4 are f.p. see e.g. [A2: 3.1(b)]. We now prove this under the weaker hypothesis that $\mathcal{C}$ has 'pseudocokernels', a notion introduced in [AS2] which we now recall:

Definition 1.5 [AS2: §2] We say the category $\mathcal{C}$ has pseudokernels if given a morphism $f : X \to Y$ in $\mathcal{C}$, there exists a morphism $g : Z \to X$ in $\mathcal{C}$ such that the sequence of functors

$$\mathcal{C}(-, Z) \xrightarrow{g_*} \mathcal{C}(-, X) \xrightarrow{f_*} \mathcal{C}(-, Y)$$

is exact; g is called a pseudokernel of f.

Dually we say $\mathcal{C}$ has pseudocokernels if given a morphism $g : Z \to X$ in $\mathcal{C}$ there is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the sequence of functors

$$\mathcal{C}(Y, -) \xrightarrow{f^*} \mathcal{C}(X, -) \xrightarrow{g^*} \mathcal{C}(Z, -)$$

is exact; f is called a pseudocokernel of g.

We will now assume, for the rest of this section that $\mathcal{C}$ has pseudocokernels.

Lemma 1.6 Given f in $\mathcal{C}(Y, X)$ the kernel of

$$f^*: \mathcal{C}(X, -) \to \mathcal{C}(Y, -) \quad \text{is f.g.}$$

Proof Choose g a pseudocokernel of f. Then the sequence

$$\mathcal{C}(Z,\ -) \xrightarrow{g^*} \mathcal{C}(X,\ -) \xrightarrow{f^*} \mathcal{C}(Y,\ -)$$

is exact, from which it is clear that $\ker f^* = \operatorname{im} g^*$ is f.g.

Lemma 1.7 Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be exact in $(\mathcal{C},\ \mathrm{mod}\ k)$ with F_1 f.g. and F_2 f.p. Then F_3 and F_1 are f.p.

Proof Since F_2 is f.p. we can choose an exact sequence $0 \to K \to \mathcal{C}(X,\ -) \to F_2 \to 0$, with K f.g. and form a commutative exact diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & F_1 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] & \mathcal{C}(X,-) \arrow[r] \arrow[d, equal] & F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] & \mathcal{C}(X,-) \arrow[r] & F_3 \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

By the snake lemma we obtain an exact sequence

$$0 \to K \to K' \to F_1 \to 0.$$

K and F_1 are f.g. so K' is f.g. The second row of the above diagram thus shows (by 1.4) that F_3 is f.p.

To show F_1 is f.p. we first consider the case where $F_2 \simeq \mathcal{C}(Y,\ -)$ is a representable functor.

Since F_1 is f.g. choose an epimorphism $\mathcal{C}(Z,\ -) \to F_1$ and consider the commutative exact diagram:

$$\begin{tikzcd}
 & \mathcal{C}(Z,\ -) \arrow[r, equal] \arrow[d] & \mathcal{C}(Z,\ -) \arrow[d] & & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & \mathcal{C}(Y,\ -) \arrow[r] & F_3 \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

By Yoneda's lemma the morphism $\mathcal{C}(Z, -) \to \mathcal{C}(Y, -)$ is f^* for some f in $\mathcal{C}(Y, Z)$. Thus, by 1.6, it has f.g. kernel L say. Applying the snake lemma yields a commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & L & = & L & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{C}(Z, -) & = & \mathcal{C}(Z, -) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & F_1 & \longrightarrow & \mathcal{C}(Y, -) & \to & F_3 \to 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Since L is f.g. the first column of this diagram shows, by 1.4, that F_1 is f.p.

Now consider the general case where

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact with F_2 f.p., F_1 f.g. We know that F_2 is f.p., so by 1.4 there is an exact sequence $0 \to J \to \mathcal{C}(W, -) \to F_2 \to 0$ with J f.g. Consider the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & J & = & J & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & U & \longrightarrow & \mathcal{C}(W, -) & \to & F_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

obtained from an application of the snake lemma.

The first column has J, F_1 f.g. so by 1.3(ii) U is f.g. Now the special case of what we want to prove, which has already been demonstrated, can be applied to the first row of this diagram to deduce that U is f.p. Now we have an exact sequence $0 \to J \to U \to F_1 \to 0$ with U f.p., J f.g. so by part of this lemma already proved above F_1 is f.p. This completes the proof.

It is now easy to deduce the main result of this section:

<u>Proposition 1.8</u> Let $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ be an exact sequence in $(\mathcal{C}, \text{mod } k)$, and assume $\mathcal{C}$ has pseudocokernels. If F_2, F_3 are f.p. then F_1, F_4 are f.p.

<u>Proof</u> Letting $G = \ker (F_3 \to F_4)$

$= \operatorname{coker} (F_1 \to F_2)$

consider the two exact sequences:

$$0 \to F_1 \to F_2 \to G \to 0 \qquad (1.8.1)$$

$$0 \to G \to F_3 \to F_4 \to 0 \qquad (1.8.2)$$

Applying 1.3(i) to (1.8.1) shows G is f.g. Thus, 1.7 applied to (1.8.2) shows G and F_4 are f.p. Since G is f.p. and F_2 is f.p. and thus f.g., 1.4 applied to (1.8.1) shows that F_1 is f.g. Now 1.7 applies to (1.8.1) and we deduce that F_1 is also f.p. as required.

§2 Exact sequence lemmas

Notation 2.1 Given f in (X, Y) there is a corresponding morphism $\hat{f}$ in $\mathfrak{A}(X, Y)$ given by

$$V \otimes X \xrightarrow{\epsilon\otimes 1} A \otimes X = X \xrightarrow{f} Y.$$

Remarks 2.2 The operation $f \longmapsto \hat{f}$ is functorial, i.e. given $g \in (Y, Z)$ $\widehat{gf} = \hat{g} \circ \hat{f}$. $\hat{1}_X = 1_X$, the identity morphism in mod $\mathfrak{A}$ of the object X.

Given h in $\mathfrak{A}(X', X)$ $\hat{f} \circ h$ is in $\mathfrak{A}(X', Y)$ and it is easy to check that

(2.2.1) $\hat{f} \circ h = fh$ (i.e. the composite $V \otimes X' \xrightarrow{h} X \xrightarrow{f} Y$)

Likewise given ℓ in $\mathfrak{A}(Y, Y')$ it is easy to check (2.2.2) $\ell \circ \hat{f} = \ell(1 \otimes f)$.

Lemma 2.3 Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be exact in mod A. Then the sequence of functors

$$0 \to \mathfrak{A}(-, X) \xrightarrow{\hat{f}_*} \mathfrak{A}(-, Y) \xrightarrow{\hat{g}_*} \mathfrak{A}(-, Z) \text{ is exact.}$$

Proof For any X' in mod A there is an exact sequence:

$$0 \to (V \otimes X', X) \xrightarrow{f_*} (V \otimes X', Y) \xrightarrow{g_*} (V \otimes X', Z)$$

which is

$$0 \to \mathfrak{A}(X', X) \to \mathfrak{A}(X', Y) \to \mathfrak{A}(X', Z).$$

The maps in this sequence are $\hat{f}_*$, $\hat{g}_*$ by (2.2.1).

Lemma 2.4 Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be exact in mod A. Then the sequence of functors

$$0 \to \mathfrak{A}(Z, -) \xrightarrow{\hat{g}_*} \mathfrak{A}(Y, -) \xrightarrow{\hat{f}_*} \mathfrak{A}(X, -)$$

is exact.

Proof $V \otimes -$ is a right exact functor and, for any X' in mod A, $(-, X')$ is a left exact functor. Thus there is an exact sequence

$$0 \to (V \otimes Z, X') \xrightarrow{(1\otimes g)^*} (V \otimes Y, X') \xrightarrow{(1\otimes f)^*} (V \otimes X, X').$$

This is

$$0 \to \mathfrak{U}(Z, X') \to \mathfrak{U}(Y, X') \to \mathfrak{U}(X, X'),$$

and by (2.2.2) the maps in this sequence are $\hat{g}^*$ and $\hat{f}^*$.

§3 Existence of pseudokernels and pseudocokernels in mod$\mathfrak{A}$

Definitions 3.1 We define a new category $\widetilde{\text{mod}\mathfrak{A}}$ as follows: its objects are those of mod A, and given two objects X, Y the space of morphisms $X \to Y$ (denoted $\tilde{\mathfrak{A}}(X, Y)$) is given by

$\tilde{\mathfrak{A}}(X, Y) = (X, (V, Y))$.

Given f in $\tilde{\mathfrak{A}}(X, Y)$, g in $\tilde{\mathfrak{A}}(Y, Z)$ we define $g \circ f$ in $\tilde{\mathfrak{A}}(X, Z)$ as the composite

$$X \xrightarrow{f} (V, Y) \xrightarrow{g_*} (V, (V, Z)) \xrightarrow{\sim} (V \otimes V, Z) \xrightarrow{\mu^*} (V, Z).$$

In order to check this is a category observe that there is a natural isomorphism $\tilde{\mathfrak{A}}(X, Y) \simeq \mathfrak{A}(X, Y)$ and that this isomorphism respects composition; to summarise ...

Proposition 3.2 (i) $\widetilde{\text{mod}\mathfrak{A}}$ is a k-category,

(ii) there is an isomorphism $\text{mod}\mathfrak{A} \to \widetilde{\text{mod}\mathfrak{A}}$ given by $X \longmapsto X$ for all ${}_AX$, and on morphisms by the canonical isomorphism

$$\mathfrak{A}(X, Y) = (V \otimes X, Y) \to (X, (V, Y)) = \tilde{\mathfrak{A}}(X, Y).$$

Theorem 3.3 mod$\mathfrak{A}$ has pseudokernels.

Proof By 3.2 it is sufficient to show that $\widetilde{\text{mod}\mathfrak{A}}$ has pseudokernels. Let f be in $\tilde{\mathfrak{A}}(X, Y)$, then define $\tilde{f} : (V, X) \to (V, Y)$ by the composite

$$(V, X) \xrightarrow{f_*} (V, (V, Y)) \xrightarrow{\sim} (V \otimes V, Y) \xrightarrow{\mu^*} (V, Y).$$

Let $K = \ker \tilde{f}$. Then there is $g : K \to (V, X)$, the natural inclusion morphism. Note that g is in $\tilde{\mathfrak{A}}(K, X)$.

We claim that g is a pseudokernel of f, i.e. the sequence $\tilde{\mathfrak{A}}(-, K) \xrightarrow{g_*} \tilde{\mathfrak{A}}(-, X) \xrightarrow{f_*} \tilde{\mathfrak{A}}(-, Y)$ is exact. Since $f_*g_* = (f \circ g)_* = (\tilde{f}g)_* = 0$ ($\tilde{f}g = 0$) im $g_* \subseteq \ker f_*$. Conversely if h in $\tilde{\mathfrak{A}}(X', X)$ is in $\ker f_*$ then $\tilde{f}h = 0$. Thus there is $\ell : X' \to K$ such that $g\ell = h$. Let $\tilde{\ell}$ be the morphism in $\tilde{\mathfrak{A}}(X', K)$

corresponding to $\hat{\ell}$ in $\mathfrak{X}(X', K)$. Then $h = g\ell = g \circ \tilde{\ell}$ is in im g^*.

<u>Theorem 3.4</u> mod $\mathfrak{X}$ has pseudocokernels.

<u>Proof</u> Given f in $\mathfrak{X}(X, Y)$ let $\tilde{f}$ be the composite

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y$$

and $g : V \otimes Y \to C :=$ coker $\tilde{f}$ the natural projection map. Thus g is in $\mathfrak{X}(Y, C)$ and we claim that

$$\mathfrak{X}(C, -) \xrightarrow{g^*} \mathfrak{X}(Y, -) \xrightarrow{f^*} \mathfrak{X}(X, -)$$

is exact, and hence that g is a pseudocokernel of f.

$f^*g^* = (g \circ f)^* = (g\tilde{f})^* = 0^* = 0$ so im $g^* \subseteq$ ker f^*. If h in $\mathfrak{X}(Y, Y')$ is in ker f^* then $f^*(h) = 0$. So $h\tilde{f} = 0$ and we may choose $\ell : C \to Y'$ such that $h = \ell g$. Thus $h = \ell g = \hat{\ell} \circ g$ by (2.2.1), so h is in im g^*.

§4 Bocses with $D\mathfrak{A}(A, -)$ and $D\mathfrak{A}(-, DA)$ f.p.

Remarks 4.1 Given that the functors $D\mathfrak{A}(A, -)$ and $D\mathfrak{A}(-, DA)$ are f.p. it is possible to show, and we do so in this section, that certain injective functors on mod$\mathfrak{A}$ are f.p. Granted this we can follow the strategy of Auslander [A2: §7] to obtain existence of almost split maps in mod $\mathfrak{A}$.

Theorem 4.2 (i) if $D\mathfrak{A}(A, -)$ is f.p. $D\mathfrak{A}(X, -)$ is f.p. for all X in mod $\mathfrak{A}$,

(ii) if $D\mathfrak{A}(-, DA)$ is f.p. $D\mathfrak{A}(-, X)$ is f.p. for all X in mod $\mathfrak{A}$.

Proof (i) Let $A^m \xrightarrow{f} A^n \xrightarrow{g} X \to 0$ be a finite presentation of X. Then by 2.4

$$0 \to \mathfrak{A}(X, -) \xrightarrow{\hat{g}^*} \mathfrak{A}(A^n, -) \xrightarrow{\hat{f}^*} \mathfrak{A}(A^m, -)$$

is exact. Hence there is an exact sequence

$$D\mathfrak{A}(A^m, -) \to D\mathfrak{A}(A^n, -) \to D\mathfrak{A}(X, -) \to 0.$$

$D\mathfrak{A}(A^r, -) \simeq D\mathfrak{A}(A, -) \oplus \ldots \oplus D\mathfrak{A}(A, -)$ (r times)

so it is f.p. Thus by 1.8 $D\mathfrak{A}(X, -)$ is f.p.

(ii) Let $0 \to X \xrightarrow{u} (DA)^r \xrightarrow{v} (DA)^s$ be a finite copresentation of X. Then by 2.3

$$0 \to \mathfrak{A}(-, X) \xrightarrow{\hat{u}_*} \mathfrak{A}(-, (DA)^r) \xrightarrow{\hat{v}_*} \mathfrak{A}(-, (DA)^s)$$

is exact. Hence there is an exact sequence

$$D\mathfrak{A}(-, (DA)^s) \to D\mathfrak{A}(-, (DA)^r) \to D\mathfrak{A}(-, X) \to 0.$$

Thus, by 1.8, $D\mathfrak{A}(-, X)$ is f.p.

Notation 4.3 Let us fix an F in $((\text{mod}\,\mathfrak{A})^{op}, \text{mod}\, k)$ and choose X an object of mod $\mathfrak{A}$.

Define $\alpha : F \to \mathrm{Hom}_k(\mathfrak{A}(X, -), F(X))$ by

$$\alpha_Y : F(Y) \to \mathrm{Hom}_k(\mathfrak{A}(X, Y), F(X))$$

where $\alpha_Y(y)(f) = F(f)(y)$, for all y in F(Y), f in $\mathfrak{A}(X, Y)$.

Then α is a natural transformation of functors.

Let H be a k-submodule of $F(X)$. Then the projection $F(X) \to F(X)/H$ induces a natural transformation $\mathrm{Hom}_k(\mathfrak{A}(X, -), F(X)) \to \mathrm{Hom}_k(\mathfrak{A}(X, -), F(X)/H)$. The composite of this with $\alpha : F \to \mathrm{Hom}_k(\mathfrak{A}(X, -), F(X))$ will be denoted α^H.

<u>Theorem 4.4</u> Suppose $D\,\mathfrak{A}(A, -)$ is f.p. and F is f.g. with $F(X)/H$ a semisimple k-module. Then $\ker \alpha^H$ is f.g.

<u>Proof</u> $\mathrm{Im}\, \alpha^H$ is a f.g. subfunctor of

$$\mathrm{Hom}_k(\mathfrak{A}(X, -), F(X)/H).$$

Let I be the injective envelope of $k/\mathrm{rad}\, k$. Since $F(X)/H$ is semisimple there is an embedding

$$F(X)/H \to I^n \quad \text{(for some } n\text{)}.$$

$\mathrm{Hom}_k(\mathfrak{A}(X, -), I^n) = \oplus\, D\,\mathfrak{A}(X, -)$ (n copies). By 4.2 $D\,\mathfrak{A}(X, -)$ is f.p. By 1.7 it follows that $\mathrm{im}\, \alpha^H$ is f.p. The sequence

$$0 \to \ker \alpha^H \to F \to \mathrm{im}\, \alpha^H \to 0$$

is exact with F f.g. and $\mathrm{im}\, \alpha^H$ f.p. Thus $\ker \alpha^H$ is f.g.

<u>Notation 4.5</u> Dually to 4.3 take G in $(\mathrm{mod}\, \mathfrak{A}, \mathrm{mod}\, k)$ and X in $\mathrm{mod}\, \mathfrak{A}$ and define

$$\beta : G \to \mathrm{Hom}_k(\mathfrak{A}(-, X), G(X))$$

by $\beta_Y(y)(f) = G(f)(y)$ for y in $G(Y)$, f in $\mathfrak{A}(Y, X)$.

If J is a k-submodule of $G(X)$ we define

$$\beta^J : G \to \mathrm{Hom}_k(\mathfrak{A}(-, X), G(X)/J)$$

by following β with the morphism induced by projection $G(X) \to G(X)/J$. Then, with an identical line of proof to 4.4 we have:

<u>Theorem 4.6</u> If G is f.g., $G(X)/J$ semisimple, and $D\,\mathfrak{A}(-, DA)$ is f.p. then $\ker \beta^J$ is f.g.

§5 Almost split maps in mod $\mathfrak{A}$

Theorem 5.1 Let X in mod $\mathfrak{A}$ be an object with local endomorphism ring; suppose $D\mathfrak{A}(A, -)$ is f.p., then there is a right almost split morphism $Z \to X$ in mod $\mathfrak{A}$.

Proof Since $\mathfrak{A}(X, X)$ is local, rad $\mathfrak{A}(X, X)$ - the set of all non-automorphisms of X - is a k-submodule of $\mathfrak{A}(X, X)$. In the notation of 4.3 put $F = \mathfrak{A}(-, X)$, $H = \text{rad}\,\mathfrak{A}(X, X)$. Then 4.4 shows $\ker \alpha^H$ is f.g. Let $\mathfrak{A}(-, Z) \to \ker \alpha^H$ be an epimorphism to $\ker \alpha^H$, and $q : Z \to X$ the image of 1_Z in $\mathfrak{A}(Z, Z)$ under this epimorphism. We claim q is a right almost split map in mod $\mathfrak{A}$.

Let $s : Y \to X$ be a morphism in mod $\mathfrak{A}$. Then s factors through q

iff there is h in $\mathfrak{A}(Y, Z)$ with $q \circ h = s$

iff $\alpha^H(s) = 0$

iff for all t in $\mathfrak{A}(X, Y)$ $\alpha(s)(t)$ is in $H = \text{rad}\,\mathfrak{A}(X, X)$

iff for all t in $\mathfrak{A}(X, Y)$ $s \circ t$ is not an automorphism

iff s is not a split epimorphism.

Since q factors through q, q is not a split epimorphism, and every $s : Y \to X$ which is not a split epimorphism factors through q. Hence q is a right almost split map in mod $\mathfrak{A}$.

Theorem 5.2 Let X be an object of mod $\mathfrak{A}$ with local endomorphism ring; suppose $D\mathfrak{A}(-, A)$ is f.p., then there is a left almost split morphism $X \to Z$ in mod $\mathfrak{A}$.

Proof In the notation of 4.5 put $G = \mathfrak{A}(X, -)$, $J = \text{rad}\,\mathfrak{A}(X, X)$. Then $\ker \beta^J$ is f.g. Let

$$\sigma : \mathfrak{A}(Z, -) \to \ker \beta^J$$

be an epimorphism. Then $\sigma(1_Z)$ is in $\mathfrak{A}(X, Z)$ and, as in 5.1, it can be shown that $\sigma(1_Z)$ is left almost split in mod $\mathfrak{A}$.

§6 Finite presentations of $D\mathfrak{A}(A, -)$, $D\mathfrak{A}(-, DA)$.

In order to apply the criteria 5.1, 5.2, for existence of almost split morphisms in $\operatorname{mod}\mathfrak{A}$, it remains to prove that $D\mathfrak{A}(A, -)$, $D\mathfrak{A}(-, DA)$ are f.p. The easiest approach is to use the embedding theory of chapter I, §2; for $D\mathfrak{A}(A, -)$ consider the embedding $\operatorname{mod}\mathfrak{A} \simeq \underline{I} \subseteq \operatorname{mod} R$. If we regard $D\mathfrak{A}(A, -)$ as a functor on $\underline{I}$ it becomes

$$D \operatorname{Hom}_R(R, -) \mid_{\underline{I}} \simeq \operatorname{Hom}_R(-, DR) \mid_{\underline{I}}.$$

This is the restriction to $\underline{I}$, which is contravariantly finite, of a contravariant representable functor on mod R. This means it is f.p. - let us sketch this argument in general:

Let Λ be an artin algebra and $\underline{D}$ a contravariantly finite subcategory of $\operatorname{mod} \Lambda$. If X is in $\operatorname{mod} \Lambda$ consider $\operatorname{Hom}_\Lambda(-, X)|_{\underline{D}}$. Since $\underline{D}$ is contravariantly finite this functor is f.g. so there exists D in $\underline{D}$ and an exact sequence

$$\operatorname{Hom}_\Lambda(-, D) \mid_{\underline{D}} \to \operatorname{Hom}_\Lambda(-, X) \mid_{\underline{D}} \to 0.$$

By Yoneda's lemma this is realised by a map $D \to X$. Let K be its kernel. Then there is an exact sequence:

$$0 \to \operatorname{Hom}_\Lambda(-. K) \mid_{\underline{D}} \to \operatorname{Hom}_\Lambda(-, D) \mid_{\underline{D}} \to \operatorname{Hom}_\Lambda(-, X) \mid_{\underline{D}} \to 0.$$

The same argument as above applied to $\operatorname{Hom}_\Lambda(-, K) \mid_{\underline{D}}$ shows that this functor is also f.g. Thus by 1.4 the functor $\operatorname{Hom}_\Lambda(-, X)|_{\underline{D}}$ is f.p.

Likewise, if we consider $D\mathfrak{A}(-, DA)$ as a functor on $\underline{C} \subseteq \operatorname{mod} L$, by way of $\operatorname{mod}\mathfrak{A} \simeq \underline{C}$, $D\mathfrak{A}(-, DA)$ becomes

$$\begin{aligned} &D \operatorname{Hom}_L(-, \operatorname{Hom}_A(L, DA)) \mid_{\underline{C}} \\ &\simeq D \operatorname{Hom}_L(-, DL)|_{\underline{C}} \simeq \operatorname{Hom}_L(L, -)|_{\underline{C}} \end{aligned}$$

which is the restriction to the covariantly finite subcategory $\underline{C}$ of a covariant representable functor on mod L. A dual argument to that sketched above shows that $\operatorname{Hom}_L(L, -)|_{\underline{C}}$ is f.p.

Notice that, in the above, we have not used the facts that $\underline{I}$ is covariantly finite in mod R and $\underline{C}$ is contravariantly finite in mod L. These were, in [BB: §2], the harder properties of $\underline{I}$ and $\underline{C}$ to verify. The facts we have used are standard and easy to verify:

$\underline{I}$ is contravariantly finite in mod R since for any X in mod R, the natural map

$$R \otimes X \to X$$

induces a surjective map of functors

$$\mathrm{Hom}_R(-,\ R \otimes X)|_{\underline{I}} \to \mathrm{Hom}_R(-,\ X)|_{\underline{I}}$$

from a representable contravariant functor on $\underline{I}$.

$\underline{C}$ is covariantly finite in mod L since, for any M in mod L, the natural map

$$M \to \mathrm{Hom}_A(L,\ M)$$

induces a surjective map of functors

$$\mathrm{Hom}_L(\mathrm{Hom}_A(L,\ M),\ -)|_{\underline{C}} \to \mathrm{Hom}_L(M,\ -)|_{\underline{C}}$$

from a representable covariant functor on $\underline{C}$.

Hence we may conclude that:

Theorem 6.1 Under the hypotheses of this chapter mod $\mathfrak{A}$ has almost split morphisms.

An alternative approach

We may dispense with using the embeddings of mod $\mathfrak{A}$, but the proof in this case becomes very technical - however it is not too hard to give the explicit form of the finite presentations for $D\,\mathfrak{A}(-,\ DA)$, $D\,\mathfrak{A}(A,\ -)$ and we sketch this below for $D\,\mathfrak{A}(-,\ DA)$.

Definition 6.2 Define a functor H in (mod $\mathfrak{A}$, mod k) as follows:

$H(X) = V \otimes X$, and if f is in $\mathfrak{A}(X,\ Y)$ then

$H(f) : H(X) \to H(Y)$ is given by the composite

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y.$$

Lemma 6.3 H is indeed a functor, and is isomorphic to the functor $D\,\mathfrak{A}(-, DA)$.

Proof It is convenient to define the 'natural isomorphism' of H with $D\,\mathfrak{A}(-, DA)$ before checking H is a functor. Let $\nu_X: H(X) \to D\,\mathfrak{A}(X, DA)$ be given by

$$\begin{aligned} H(X) = V \otimes X &\simeq A \otimes V \otimes X \\ &\simeq D^2(A \otimes V \otimes X) \\ &\simeq D(V \otimes X, DA) = D\,\mathfrak{A}(X, DA). \end{aligned}$$

It is not hard to check that, for f in $\mathfrak{A}(X, Y)$

$$\nu_Y H(f) = D\,\mathfrak{A}(f, DA)\nu_X.$$

It follows easily that $H(g \circ f) = H(g)H(f)$ for any g in $\mathfrak{A}(Y, Z)$, and it is routine to check that $H(1_X) = 1_{H(X)}$. The maps ν_X, X in mod $\mathfrak{A}$, then define a natural isomorphism as required.

Notation 6.4 There is a natural map $L \to (L, L)$ given by $e \longmapsto (f \longmapsto fe)$, which is L-linear. Consider the L-morphism $L \to V \otimes L$ obtained by composing this with the natural isomorphism $(L, L) \simeq V \otimes L$. Then the image of 1_L is a 'canonical element' in $V \otimes L$, which we shall denote by θ. Let M be the cokernel of the map $L \to V \otimes L$ and $g : V \otimes L \to M$ the natural projection onto M. Define maps

$$\pi_X : \mathfrak{A}(L, X) \to H(X) = V \otimes X$$

by $\pi_X(f) = H(f)(\theta)$. Then this defines a natural morphism $\pi : \mathfrak{A}(L, -) \to H$, and it can be proved that

Proposition 6.5 $\mathfrak{A}(M, -) \xrightarrow{g^*} \mathfrak{A}(L, -) \xrightarrow{\pi} H \to 0$ is exact.

Remarks 6.6 This gives a f.p. for $H \simeq D\,\mathfrak{A}(-, DA)$. Notice that $L \to (L, L)$ is just the map yielding the surjection

$$\mathrm{Hom}_L((L, L), -)|_{\underline{C}} \to \mathrm{Hom}_L(L, -)|_{\underline{C}}$$

in the proof that $\underline{C}$ is contravariantly finite.

$(L, L) \simeq V \otimes L \xrightarrow{g} M$ is the cokernel of $L \to (L, L)$. Thus the sequence

$$0 \to \mathrm{Hom}_L(M, -)|_{\underline{C}} \to \mathrm{Hom}_L((L, L), -)|_{\underline{C}} \to \mathrm{Hom}_L(L, -)|_{\underline{C}} \to 0$$

is exact. $\mathrm{Hom}_L(M, -)|_{\underline{C}}$ is f.g. since $\underline{C}$ is covariantly finite. The sequence given in 6.5 is just this one regarded as a sequence of functors on $\mathrm{mod}\,\mathfrak{A} \simeq \underline{C}$. The proof is therefore the one given above, but 'pulled back' along $\mathrm{mod}\,\mathfrak{A} \xrightarrow{\simeq} \underline{C}$ so as to make no mention of this realisation of $\mathrm{mod}\,\mathfrak{A}$. The only version of this that we have been able to find is technically complex, and we have chosen to omit it from this account.

There is a 'dual' procedure for dealing with $D\mathfrak{A}(A, -)$, which makes use of the category $\widetilde{\mathrm{mod}\,\mathfrak{A}}$ introduced in §3.

§7 Relationship with other existence theorems.

We finish this chapter with a brief discussion of how the existence theorem proved in this chapter relates to similar theorems found in [BB], [BK]. Our theorem is weaker in the sense that it only asserts existence of almost split maps, not almost split sequences.

The main difference between our proof here and those found in [BB], [BK], is that it does not assume existence of almost split sequences in mod Λ, where Λ is an artin algebra. Indeed it is possible to reprove this fact by taking $(A, V) = (\Lambda, \Lambda)$ the principal bocs. The work in this chapter then collapses back to that of [A2 : §7] and we obtain the existence of almost split maps. The existence of almost split sequences can be made to follow from this, although the relationship between the end terms by the operators DTr, TrD of [AR 2] is not exhibited.

The approach in [BB] is to use the Auslander-Smalø criterion [AS2 : 2.4] for existence of almost split sequences in subcategories. The proof of this criterion constructs the whole almost split sequence at once, in terms of a projective resolution of a simple functor on the subcategory in question - this uses results on functors to be found in [AR1, A1].

The existence of right almost split morphisms in the [BK] version is proved as follows. Let X be indecomposable in mod $\mathfrak{A}$. Then $R \otimes X$ is indecomposable in $\underline{I}$ and if $E \to R \otimes X$ is a right almost split morphism in mod R, let $R \otimes E \to E$ be the natural map; the composite $R \otimes E \to E \to R \otimes X$ is then shown to be right almost split in $\underline{I}$. The point of view in [BK] is the study of almost split sequences for relatively projective modules, and the connection with mod$\mathfrak{A}$ is not made very explicit. This means that the situation for left almost split morphisms is not so clear as the realisation of mod $\mathfrak{A}$ as a category of coinduced modules is not available. The authors use instead an elaborate 'duality' theory (using triples and cotriples) to obtain the desired result. Also no attempt is made to prove extension closure for $\underline{I}$ - so the Auslander - Smalø criterion is not available to them. Instead a new criterion for existence of almost split sequences in subcategories is developed in the first section of their paper.

It is perhaps worth commenting that a paper of de la Peña and Simson [PS], predating both [BB] and [BK], also establishes existence of almost split sequences for the category of 'prinjective modules over a triangular matrix ring'. A quick proof of this using the Auslander - Smalø criterion has also been provided by Smalø [S]. Some of these categories are equivalent to mod $\mathfrak{A}$ for certain bocses $\mathfrak{A}$, including the construction, due to Drozd, of a bocs $\mathfrak{A}$ corresponding to a given finite-dimensional algebra [C-B1: §6].

Chapter III Further results in left and right algebra theory

In this chapter we give further results in the left and right algebra theory of bocses developed in [BB: §1-4] and summarised in Chapter I of this thesis. These include the results of [BB: §5] on cotilting theory, and the Ext-projective and Ext-injective modules in $\underline{I}$ and $\underline{C}$. We then give further results not to be found in [BB]. The final section is devoted to a result on how L and R behave under 'induced bocs' constructions.

§1 Cotilting theory and Ext-projectives / Ext-injectives

We assume that $\mathfrak{A} = (A, V)$ is a bocs in which A is an artin algebra and ${}_AV$, V_A are finitely generated projectives. L and R are also artin algebras. We shall denote the usual duality associated with any of these artin algebras A, L, R by D.

Notation 1.1 Set $d = \max(\text{i.d. } {}_AA, \text{ i.d. } A_A)$ where i.d. denotes injective dimension. When $d < \infty$ it is well-known that i.d. ${}_AA = d =$ i.d. A_A, a proof of this can be found sketched in [AR3: 6.9(a)].

For the readers convenience we recall the definition of a 'cotilting module of finite injective dimension'. This is the dual of the concept of a 'tilting module of finite projective dimension' introduced by Miyashita in [Miy].

Definition 1.2 If Λ is an artin algebra a module T in mod Λ is called a cotilting module of finite injective dimension if i.d. $T < \infty$, $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for $i > 0$, and there is an exact sequence

$$0 \to T_d \to \ldots \to T_0 \to D\Lambda \to 0$$

with the T_i in add T.

Theorem 1.3 [BB: 5.1] If $d < \infty$ then ${}_LV$, V_R (i.e. ${}_{R^{op}}V$) are cotilting modules of finite injective dimension.

This injective dimension is at most d for both modules.

Remarks 1.4 In the proof of 1.3 it is not necessary to assume $\overline{V}$ is projectivising, only that ${}_AV$, V_A are projective (and finitely generated since V has finite length over k). A remark after this theorem in [BB] shows that if $d \geq 2$ the injective dimensions of the cotilting modules are exactly d. If $d = 1$, e.g. if A is hereditary - this holds for the 'layered bocses' of [C-B1: 3.6] - ${}_LV$, V_R are either classical cotilting modules or injective modules, since their injective dimensions are at most 1.

A basic property of cotilting modules is that they are balanced - a property already observed for ${}_L V_R$ in I : 2.5. Indeed once we have shown that ${}_L V$ is a cotilting module it follows that V is a cotilting module over $\mathrm{End}_L(V)$ (c.f. [Miy : 1.5]). Since the canonical homomorphism $R^{op} \to \mathrm{End}_L(V)$ is an isomorphism it follows that V is a cotilting R^{op}-module, i.e. V_R is cotilting.

We now describe the Ext-projective and Ext-injective modules in $\underline{I}$ and $\underline{C}$.

<u>Theorem 1.5</u> (c.f. [BB: 5.2]) Suppose idempotents split in mod $\mathfrak{X}$; then the Ext-projectives in $\underline{I}$ are the finitely generated projectives and the Ext-injectives in $\underline{I}$ are the objects of add DV.

<u>Proof</u> In [BB] it was assumed that $\bar{V}$ was projectivising, this being used to show that all the objects of add DV were Ext-injective. This is not necessary. There is a natural isomorphism $\mathrm{Ext}^1_A(Z, DV) \simeq \mathrm{Ext}^1_R(R \otimes Z, DV)$; as V_A is projective ${}_A(DV)$ is injective, hence $\mathrm{Ext}^1_R(-, DV)|_{\underline{I}}$ is zero, i.e. DV is Ext-injective in $\underline{I}$ - note that $DV \simeq R \otimes DA$ is in $\underline{I}$. The rest of the proof now follows [BB].

<u>Theorem 1.5'</u> (c.f. [BB: 5.3]) Suppose idempotents split in mod $\mathfrak{X}$; then the Ext-injectives in $\underline{C}$ are the finitely generated injective L-modules and the Ext-projectives in $\underline{C}$ are the objects of add V.

<u>Proof</u> Dual to that for 1.5.

<u>Remark 1.6</u> If d is finite then L and R are cotilted from each other (by V) and so by the analogue of [Miy: 1.19] for cotilting modules the number of simple left L-modules and the number of simple left R-modules (up to isomorphism) are equal. This number

is finite, equal to s say; it follows that the number of isomorphism classes of indecomposable Ext-projective or Ext-injective modules in either $\underline{I}$ or $\underline{C}$ is s.

§2 Further properties of the categories $\underline{I}$ and $\underline{C}$

We shall assume here that $\mathfrak{A} = (A, V)$ is a bocs over an artin algebra and that ${}_AV$, V_A are finitely generated projectives. We also assume $\overline{V}$ is projectivising and (as usual) $\epsilon : V \to A$ is surjective. Thus, by I: 3.3, every exténsion of modules in $\underline{I}$ ($\underline{C}$) is induced (coinduced) from one in mod A.

<u>Proposition 2.1</u> Suppose idempotents split in mod $\mathfrak{A}$. Let $0 \to X \to Y \to Z \to 0$ be an almost split sequence in mod A such that X and Z are indecomposable in mod $\mathfrak{A}$. Then the induced sequence $0 \to R \otimes X \to R \otimes Y \to R \otimes Z \to 0$ is either split or almost split in $\underline{I}$.

<u>Proof</u> If the induced sequence is not split then, since $R \otimes Z$ is indecomposable and non - Ext-injective in $\underline{I}$, there is (by I: 4.1) an almost split sequence in $\underline{I}$ ending at $R \otimes Z$:

$$0 \to R \otimes X' \to R \otimes Y' \to R \otimes Z \to 0.$$

This sequence may be chosen so that it is induced from a sequence $0 \to X' \to Y' \to Z \to 0$ in mod A, which clearly cannot be split.

Since $R \otimes Y' \to R \otimes Z$ is right almost split in $\underline{I}$ $R \otimes Y \to R \otimes Z$ factors through it and we obtain a commuatative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & R \otimes X \arrow[r] \arrow[d] & R \otimes Y \arrow[r] \arrow[d] & R \otimes Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \otimes X' \arrow[r] & R \otimes Y' \arrow[r] & R \otimes Z \arrow[r] & 0.
\end{tikzcd}$$

Since $Y \to Z$ is right almost split in mod A $Y' \to Z$ factors through it and we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

Applying the exact functor $R \otimes -$ yields a commuative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes X' & \longrightarrow & R \otimes Y' & \longrightarrow & R \otimes Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & R \otimes X & \longrightarrow & R \otimes Y & \longrightarrow & R \otimes Z & \longrightarrow & 0. \end{array}$$

Thus there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes X' & \longrightarrow & R \otimes Y' & \longrightarrow & R \otimes Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & R \otimes X & \longrightarrow & R \otimes Y & \longrightarrow & R \otimes Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & R \otimes X' & \longrightarrow & R \otimes Y' & \longrightarrow & R \otimes Z & \longrightarrow & 0. \end{array}$$

Since $R \otimes Y' \to R \otimes Z$ is right minimal the composite $R \otimes Y' \to R \otimes Y \to R \otimes Y'$ is an isomorphism. Hence the composite $R \otimes X' \to R \otimes X \to R \otimes X'$ is an isomorphism. Since $R \otimes X$ is indecomposable $R \otimes X' \to R \otimes X$ is an isomorphism.

Hence the two induced sequences are isomorphic and the result follows.

<u>Proposition 2.1'</u> Suppose idempotents split in mod $\mathfrak{X}$. Let $0 \to X \to Y \to Z \to 0$ be an almost split sequence in mod A such that X and Z are indecomposable in mod $\mathfrak{X}$. Then the coinduced sequence:

$$0 \to \mathrm{Hom}_A(L, X) \to \mathrm{Hom}_A(L, Y) \to \mathrm{Hom}_A(L, Z) \to 0$$

is either split or almost split in $\underline{C}$.

<u>Lemma 2.2</u> add $\underline{I}$ is closed under kernels of epimorphisms.

<u>Proof</u> This is similar to the argument for a result in Chapter IV (1.8); we omit the proof but give its dual below...

<u>Lemma 2.2'</u> add $\underline{C}$ is closed under cokernels of monomorphisms.

<u>Proof</u> Let $0 \to M \to N \to C \to 0$ be exact with M, N in add $\underline{C}$ and choose K in add $\underline{C}$ and X in mod A such that $\mathrm{Hom}_A(L, X) \simeq M \oplus K$. Then $0 \to M \oplus K \to N \oplus K \to C \to 0$ is exact. Let $0 \to X \to Q \to \Omega^{-1}X \to 0$ be exact in mod A with Q injective. Then, as $\mathrm{Hom}_A(L, X) \simeq M \oplus K$

$$0 \to M \oplus K \to \mathrm{Hom}_A(L, Q) \to \mathrm{Hom}_A(L, \Omega^{-1}X) \to 0$$

is exact, and $\mathrm{Hom}_A(L, Q)$ is injective in mod L.

Consider the pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M \oplus K & \longrightarrow & N \oplus K & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}_A(L, Q) & \longrightarrow & W & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Hom}_A(L, \Omega^{-1}X) & = & \mathrm{Hom}_A(L, \Omega^{-1}X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The module W is in add $\underline{C}$ since add $\underline{C}$ is closed under extensions. The sequence

$$0 \to \mathrm{Hom}_A(L, Q) \to W \to C \to 0$$

splits since $\mathrm{Hom}_A(L, Q)$ is injective. Thus C is in add $\underline{C}$, completing the proof.

Recall from Chapter I: 2.1 that there are equivalences $V \otimes_R -: \underline{I} \to \underline{C}$ and $\mathrm{Hom}_L(V, -): \underline{C} \to \underline{I}$ which are mutually inverse.

<u>Lemma 2.3</u> $V \otimes_R - |_{\underline{I}}$ and $\mathrm{Hom}_L(V, -)|_{\underline{C}}$ preserve exact sequences.

<u>Proof</u> If $0 \to R \otimes X \to R \otimes Y \to R \otimes Z \to 0$ is exact so is $\mathrm{Tor}_1^R(V, R \otimes Z) \to \mathrm{Hom}_A(L, X) \to \mathrm{Hom}_A(L, Y) \to \mathrm{Hom}_A(L, Z) \to 0$. Let P_* be a projective resolution of Z. Then

$$\begin{aligned}
\mathrm{Tor}_1^R(V, R \otimes Z) &= H_1(V \otimes_R R \otimes P_*) \\
&= H_1(V \otimes P_*) \\
&= \mathrm{Tor}_1^A(V, Z) \\
&= 0 \text{ as } V_A
\end{aligned}$$

is projective. Likewise if

$0 \to \mathrm{Hom}_A(L, X) \to \mathrm{Hom}_A(L, Y) \to \mathrm{Hom}_A(L, Z) \to 0$ is exact so is

$$0 \to R \otimes X \to R \otimes Y \to R \otimes Z \to \mathrm{Ext}_L^1(V, \mathrm{Hom}_A(L, X)).$$

Let Q_* be an injective resolution of X. Then

$$\begin{aligned}\mathrm{Ext}^1_L(V, \mathrm{Hom}_A(L, X)) &= H^1(\mathrm{Hom}_L(V, \mathrm{Hom}_A(L, Q_*)))\\ &= H^1(\mathrm{Hom}_A(V, Q_*))\\ &= \mathrm{Ext}^1_A(V, X)\\ &= 0 \quad \text{as } {}_AV \text{ is projective.}\end{aligned}$$

<u>Proposition 2.4</u> Suppose idempotents split in mod $\mathfrak{A}$.

If $0 \to R \otimes X \to R \otimes Y \to C \to 0$ is an exact sequence in mod R and $\mathrm{Tor}^R_1(V, C) = 0$, then there is a Z such that $C \simeq R \otimes Z$.

<u>Proof</u> The condition on $\mathrm{Tor}^R_1(V, C)$ means that applying $V \otimes_R -$ to this sequence preserves exactness. Since $\underline{C}$ is closed under cokernels of monomorphisms (2.2′) it follows that

$V \otimes_R C \simeq \mathrm{Hom}_A(L, Z)$ for some Z. Applying $\mathrm{Hom}_L(V, V \otimes_R -)$ preserves exactness ($\mathrm{Hom}_L(V, -)$ is exact on $\underline{C}$ by 2.3) and thus there is a commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_L(V, V \otimes X) & \to & \mathrm{Hom}_L(V, V \otimes Y) & \to & \mathrm{Hom}_L(V, V \otimes_R C) & \to 0 \\ & \downarrow & & \downarrow & & & \\ 0 \longrightarrow & R \otimes X & \longrightarrow & R \otimes Y & \longrightarrow & C & \longrightarrow 0 \end{array}$$

in which the vertical maps are isomorphisms. Thus the cokernels are isomorphic, that is

$$\begin{aligned}C \simeq \mathrm{Hom}_L(V, V \otimes_R C) &\simeq \mathrm{Hom}_L(V, \mathrm{Hom}_A(L, Z))\\ &\simeq \mathrm{Hom}_A(V, Z)\\ &\simeq R \otimes Z \quad \text{as required.}\end{aligned}$$

<u>Proposition 2.4′</u> Suppose idempotents split in mod $\mathfrak{A}$.

If $0 \to K \to \mathrm{Hom}_A(L, Y) \to \mathrm{Hom}_A(L, Z) \to 0$ is exact in mod L and $\mathrm{Ext}^1_L(V, K) = 0$, then there is X such that $K \simeq \mathrm{Hom}_A(L, X)$.

<u>Proposition 2.5</u> Suppose idempotents split in mod $\mathfrak{A}$.

Let C be of finite projective dimension in mod R. Then C is induced if and only if $\mathrm{Ext}^i_R(C, DV) = 0$ for all $i > 0$.

<u>Proof</u> If $C \simeq R \otimes Z$ then for $i > 0$

$$\mathrm{Ext}^i_R(C,\ DV) \simeq \mathrm{Ext}^i_R(R \otimes Z,\ R \otimes DA)$$

which is a homomorphic image of $\mathrm{Ext}^i_A(Z,\ DA) = 0$ by I: 3.3.

Conversely let

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & P_r \arrow[r] & P_{r-1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dr] & P_0 \arrow[r] & C \arrow[r] & 0 \\
 & & & \Omega^{r-1}C & & \Omega C \arrow[u] & &
\end{tikzcd}$$

be a projective resolution of C (which has finite projective dimension). Then

$$0 = \mathrm{Ext}^i_R(C,\ DV) \simeq \mathrm{Ext}^1_R(\Omega^{i-1}C,\ DV)$$
$$\simeq D\mathrm{Tor}^R_1(V,\ \Omega^{i-1}C).$$

Thus C, ΩC, $\Omega^2 C$, ...all vanish on application of $\mathrm{Tor}^R_1(V,\ -)$. Since P_r, $P_{r-1}, \ldots, P_0$ are all in $\underline{I}$ repeated application of 2.4 shows that
$\Omega^{r-1}C$, $\Omega^{r-2}C, \ldots,$ C are all in $\underline{I}$, in particular C is in $\underline{I}$ as required.

<u>Proposition 2.5'</u> Suppose idempotents split in mod $\mathfrak{A}$. Let K be of finite injective dimension in mod L. Then K is coinduced if and only if $\mathrm{Ext}^i_L(V,\ K) = 0$ for all $i > 0$.

<u>Notation 2.6</u> [AR3 : §3] Let Λ be an artin algebra and $\underline{D}$ a subcategory of mod Λ. Then set ${}^{\perp}\underline{D}$ to be the subcategory consisting of modules M for which $\mathrm{Ext}^i_\Lambda(M,\ -)|_{\underline{D}} = 0$ for all $i > 0$; set $\underline{D}^{\perp}$ to be the subcategory consisting of modules M for which $\mathrm{Ext}^i_\Lambda(-,\ M)|_{\underline{D}} = 0$ for all $i > 0$.

<u>Corollary 2.7</u> Suppose idempotents split in mod $\mathfrak{A}$. If R has finite global dimension then $\underline{I} = {}^{\perp}(\mathrm{add}\ DV)$; if L has finite global dimension then $\underline{C} = (\mathrm{add}\ V)^{\perp}$.

<u>Remarks 2.8</u> If $\mathrm{i.d.}_A A$, $\mathrm{i.d.}A_A$ are finite then L and R are cotilted from each other (by V) so L has finite global dimension if and only if R has finite global dimension.

§3 Induced bocses

In much of this section we do not need to assume that $\mathfrak{A} = (A, V)$ has any special properties. However, to obtain the main result of this section we need to assume that the embeddings of Chapter I §2 are available. Thus we suppose that $\mathfrak{A} = (A, V)$ is a bocs with A an artin algebra over a commutative artin ring k; k acts centrally on V and V has finite length over k. ${}_AV$ and V_A are projective modules.

We let $t : A \to B$ be a homomorphism of artin algebras over k and recall the notion of the 'induced' bocs $\mathfrak{A}^t$. It has right algebra R^t say, which we relate to R. The assumption that B is an artin algebra is not very essential as we point out later.

Given a bimodule ${}_AM_A$ denote $B \otimes M \otimes B$ by ${}^BM^B$ and use a similar notation for maps.

<u>Definition 3.1</u> [C-B1: §3] Given $\mathfrak{A}$, $t : A \to B$ as above define the <u>induced bocs</u> $\mathfrak{A}^t = (B, {}^BV^B)$ by the following data:

the counit is $\quad {}^BV^B \xrightarrow{{}^B\epsilon^B} {}^BA^B \twoheadrightarrow B,$

the comultiplication is

$$ {}^BV^B \xrightarrow{{}^B\mu^B} {}^BV \otimes V^B \xrightarrow{\sim} {}^BV \otimes A \otimes V^B \xrightarrow{{}^B1\otimes t\otimes 1^B} {}^BV \otimes B \otimes V^B $$
$$ \xrightarrow{\sim} {}^BV^B \otimes_B {}^BV^B . $$

<u>Lemma 3.2</u> [C-B1: 3.1] There is an additive functor $F : \operatorname{mod} \mathfrak{A}^t \to \operatorname{mod} \mathfrak{A}$ which is full and faithful.

<u>Proof</u> [C-B1: 3.1] Set $F(X) = {}_AX$ and given $f : {}^BV^B \otimes_B X \to Y$ in $\mathfrak{A}^t(X, Y)$ define $F(f)$ by the composite

$$ V \otimes X \xrightarrow{\sim} A \otimes V \otimes A \otimes X \xrightarrow{t\otimes 1\otimes t\otimes 1} {}^BV^B \otimes X \twoheadrightarrow {}^BV^B \otimes_B X \xrightarrow{f} Y. $$

This is a functor and it is easily checked that the map $f \longmapsto F(f)$ has an inverse which sends g in $\mathfrak{A}({}_AX, {}_AY)$ to ${}^BV^B \otimes_B X \xrightarrow{\sim} B \otimes V \otimes X \xrightarrow{1\otimes g} B \otimes Y \twoheadrightarrow Y$ in $\mathfrak{A}^t(X, Y)$.

Proposition 3.3 R^t is isomorphic to $\mathrm{End}_R(R \otimes B)^{op}$.

Proof Note that for any bocs $\mathfrak{A}$ its right algebra is isomorphic to the endomorphism ring $\mathfrak{A}(A, A)^{op}$. Thus

$$R^t \simeq \mathfrak{A}^t(B, B)^{op} \simeq \mathfrak{A}({}_AB, {}_AB)^{op}$$
$$\simeq \mathrm{End}_R(R \otimes B)^{op},$$

since $\mathrm{mod}\ \mathfrak{A} \simeq \underline{I} \subseteq \mathrm{mod}\ R$.

Remarks 3.4 We indicate that this result can be proved in a slightly more general context. Suppose $t : A \to B$ is a k-algebra homomorphism but we no longer assume that B is finitely generated over k.

It is clear that the full, faithful functor of 3.2 can be defined on $\mathrm{Mod}\ \mathfrak{A}^t$ thus giving a functor $\mathrm{Mod}\ \mathfrak{A}^t \to \mathrm{Mod}\ \mathfrak{A}$ which is full and faithful. Thus $R^t \simeq \mathfrak{A}({}_AB, {}_AB)^{op}$ and using the full embedding of $\mathrm{Mod}\,\mathfrak{A}$ in $\mathrm{Mod}\ R$, which is proved in exactly the same way as the corresponding result in §2 of Chapter I, we obtain

$R^t \simeq \mathrm{End}_R\ (R \otimes B)^{op}$ as before.

Remarks 3.5 The result of 3.3 was first observed in the following form. Suppose ${}_AB$ is projective, then there is a chain of isomorphisms

$$\begin{aligned} R^t &= \mathrm{Hom}_B({}^BV^B, B) \\ &\simeq \mathrm{Hom}_A(V \otimes B, B) \\ &\simeq \mathrm{Hom}_A(B, \mathrm{Hom}_A(V, B)) \\ &\simeq \mathrm{Hom}_A(B, R \otimes B) \\ &\simeq \mathrm{Hom}_A(B, A) \otimes R \otimes B = R' \quad \text{say.} \end{aligned}$$

The algebra structure was then transported to R' and described in terms of that for R.

It was then observed by my supervisor Michael Butler that

$$\begin{aligned} R' &\simeq \mathrm{Hom}_A(B, R \otimes B) \\ &\simeq \mathrm{End}_R(R \otimes B). \end{aligned}$$

This suggested the more direct approach given here, which does not need ${}_AB$ projective. If however ${}_AB$ is projective then $R^t \simeq \mathrm{End}_R(R \otimes B)^{op} \simeq R'$ and the ring structure on R' may be described. The addition is the usual addition (of course) and given $p \otimes r \otimes b$, $q \otimes s \otimes c$ in R' their product is given by

$$q \otimes sp(c)\, r \otimes b,$$

as is readily checked by transporting the ring structure from $\mathrm{End}_R(R \otimes B)^{op}$ across the isomorphism of this with R'.

The procedure used in this section may be carried out to calculate L^t, the left algebra of $\mathfrak{A}^t$ in terms of L; we need, however, an analogue of our theory for 'right representations' of bocses. We sketch the line of argument:

Let $\mathrm{Mod}\ \mathfrak{A}^{op}$ denote the category obtained as for $\mathrm{Mod}\,\mathfrak{A}$ but using right A-modules as objects. Then L^t is the endomorphism ring of B_B in $\mathrm{Mod}(\mathfrak{A}^t)^{op}$.

By the analogue of 3.2 above, this is then the endomorphism ring of B_A in $\mathrm{Mod}\ \mathfrak{A}^{op}$. There is an embedding of this category as the subcategory of $\mathrm{Mod}\ L^{op}$ on the <u>induced</u> L^{op}-modules. This gives the result that $L^t \simeq \mathrm{End}_{L^{op}}(B \otimes L)$.

Chapter IV Homological theory of relatively projective modules

In this section we do not deal explicitly with bocses but consider induced modules abstractly. In the first section we also give (without proof) the dual results for coinduced modules, but in the remainder of the chapter we suppress these for the sake of brevity - the interested reader may formulate these dual results without undue difficulty.

In the first section extensions of induced and relatively projective modules are considered, the context being for arbitrary modules over arbitrary rings. In the second section we consider almost split sequences so we restrict attention to finitely generated modules over artin algebras.

The theory for bocses in chapter I gives a class of examples satisfying the kind of hypotheses needed in this chapter - thus results here carry over to give results in the left and right algebra theory of bocses.

Our final section in this chapter discusses the relation of our work to recent work of Auslander and Reiten [AR4], and some results of Kleiner which they refer to in their paper.

§1 Extensions of relatively projective modules

Definition 1.1 Let Λ be a ring and X, Y in Mod Λ. Let n be an integer greater than zero. An n-fold extension of X by Y is an exact sequence in Mod Λ

$$e : 0 \to X \to E_1 \to \ldots \to E_n \to Y \to 0$$

A morphism of two n-fold extensions is a commutative diagram

$$\begin{tikzcd} e : 0 \arrow[r] & X \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\ e' : 0 \arrow[r] & X' \arrow[r] & E'_1 \arrow[r] & \cdots \arrow[r] & E'_n \arrow[r] & Y' \arrow[r] & 0 . \end{tikzcd}$$

This is thus a morphism $e \to e'$.

Remarks 1.2 1-fold extensions are just short exact sequences.

It is well known that the set of 'equivalence classes' $\mathrm{Ext}^n_\Lambda(Y, X)$ of n-fold extensions of X by Y is a group under the 'Baer sum'. A reference for the general theory of the Ext-functor viewed in this way (i.e. as equivalence classes of extensions) is chapter VII of Mitchell's classic text on category theory [Mit] - in particular we borrow the following terminology from that book:

Definition 1.3 [Mit: chapter VII §3] A morphism with fixed ends is a morphism $e \to e'$ of extensions as in 1.1 such that: $X = X'$, $Y = Y'$ and the mappings $X \to X'$ $(=X)$ and $Y \to Y'(=Y)$ are just 1_X, 1_Y respectively. Thus the morphism looks like

$$\begin{tikzcd} e : 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\ e' : 0 \arrow[r] & X \arrow[r] & E'_1 \arrow[r] & \cdots \arrow[r] & E'_n \arrow[r] & Y \arrow[r] & 0. \end{tikzcd}$$

Recall that the equivalence relation on the class of n-fold extensions of X by Y is defined in terms of morphisms with fixed ends, e and e' are equivalent extensions of X by Y if there is a finite sequence $e = e_0, e_1, \ldots, e_m = e'$ of extensions and for each $i = 1, \ldots, m$ there is either a morphism with fixed ends $e_{i-1} \to e_i$, or a morphism with fixed ends $e_i \to e_{i-1}$.

Notation 1.4 We fix now two ring homomorphisms $A \to B$, $A \to C$ and let $\mathcal{I}$ denote the category of all induced B-modules (i.e. B-modules isomorphic to $B \otimes X$ for some X in Mod A); let $\mathcal{C}$ denote the category of all coinduced C-modules (i.e. C-modules isomorphic to $\mathrm{Hom}_A(C, X)$ for some X in Mod A).

Then add $\mathcal{I}$ is the category of relatively A-projective modules in Mod B, and add $\mathcal{C}$ is the category of relatively A-injective modules in Mod C.

As usual $\otimes$ denotes $\otimes_A$, and we will also write $(-, -)$ instead of $\mathrm{Hom}_A(-, -)$.

Assumptions 1.5 There are two fundamental assumptions we make on the ring morphisms $A \to B$, $A \to C$:

(i) $B \otimes - : \text{Mod } A \to \text{Mod } B$ is exact,

(ii) $(C, -): \text{Mod } A \to \text{Mod } C$ is exact.

Thus if

$$e : 0 \to X \to E_1 \to \dots \to E_n \to Y \to 0$$

is an n-fold extension of left A-modules there are induced and coinduced extensions

$$B \otimes e : 0 \to B \otimes X \to B \otimes E_1 \to \dots \to B \otimes E_n \to B \otimes Y \to 0,$$

$$(C, e): 0 \to (C, X) \to (C, E_1) \to \dots \to (C, E_n) \to (C, Y) \to 0,$$

obtained by applying the exact functors above to the extension e.

Proposition 1.6 Let X be a left A-module. If there is an exact sequence $0 \to X \to B \otimes X \to Q \to 0$ with $X \to B \otimes X$ the canonical morphism $x \longmapsto 1 \otimes x$, and Q an injective left A-module then every n-fold extension of $B \otimes X$ by an induced module is equivalent to an induced n-fold extension.

Proof Let $e : 0 \to B \otimes X \to E_1 \to \dots \to E_n \to B \otimes Z \to 0$ be an n-fold extension of $B \otimes X$ by an induced module. Regard e as an extension in Mod A and consider the pullback e' of e along the canonical morphism $Z \to B \otimes Z$:

$$\begin{tikzcd}
e' : 0 \arrow[r] & B \otimes X \arrow[r] \arrow[d, equal] & E'_1 \arrow[r] \arrow[d] & \dots \arrow[r] & E'_n \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
e \ : 0 \arrow[r] & B \otimes X \arrow[r] & E_1 \arrow[r] & \dots \arrow[r] & E_n \arrow[r] & B \otimes Z \arrow[r] & 0
\end{tikzcd}$$

Since Q is injective the morphism $B \otimes X \to Q$ extends through the monomorphism $B \otimes X \to E'_1$. Let $E'_1 \to Q$ be this extending and E''_1 its kernel. Then there is a commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & & & \\
 & & X \arrow[d] & E''_1 \arrow[d] & & & & \\
e' : & 0 \arrow[r] & B \otimes X \arrow[r] \arrow[d] & E'_1 \arrow[r] \arrow[d] & \dots \arrow[r] & E'_n \arrow[r] & Z \arrow[r] & 0 \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & & & & \\
 & & 0 & 0 & & & &
\end{tikzcd}$$

If $n \geq 2$ set $E''_i = E'_i$ for $2 \leq i \leq n$. We obtain a commutative exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & & & \\
 & 0 \arrow[r] & X \arrow[r] \arrow[d] & E''_1 \arrow[r] \arrow[d] & \dots \arrow[r] & E''_n \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
e' : & 0 \arrow[r] & B \otimes X \arrow[r] \arrow[d] & E'_1 \arrow[r] \arrow[d] & \dots \arrow[r] & E'_n \arrow[r] & Z \arrow[r] & 0 \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & & & & \\
 & & 0 & 0 & & & &
\end{tikzcd}$$

Call the first row of this diagram e". Then this diagram gives a morphism $e" \to e'$. There is a morphism $e' \to e$ (e regarded as an extension of A-modules) since e' is a pullback of e. Applying the exact functor $B \otimes -$ gives morphisms

$$B \otimes e" \to B \otimes e' \to B \otimes e.$$

Let $B \otimes e \to e$ be the morphism induced by the natural family of maps $B \otimes M \to M$, $b \otimes m \longmapsto bm$ (for any left B-module M). Then composing this with the morphism $B \otimes e" \to B \otimes e$ above gives a morphism $B \otimes e" \to e$ which may be checked to have fixed ends. Thus e is equivalent to the induced extension $B \otimes e"$.

<u>Proposition 1.6'</u> Let Z be a left A-module. If there is an exact sequence $0 \to P \to (C, Z) \to Z \to 0$ where $(C, Z) \to Z$ is the natural map $h \longmapsto h(1)$ and P is a projective left A-module, then every extension of a coinduced module by (C, Z) is equivalent to a coinduced extension.

<u>Corollary 1.7</u> Suppose that, for every X in Mod A, there is an exact sequence in Mod A

$$0 \to X \to B \otimes X \to Q_X \to 0$$

in which $X \to B \otimes X$ is $x \longmapsto 1 \otimes x$ and Q_X is injective. Then $\mathcal{I}$ and add $\mathcal{I}$ are closed under extensions.

<u>Proof</u> It is clearly sufficient to show that $\mathcal{I}$ is closed under extensions. Let

$$0 \to B \otimes X \to E \to B \otimes Z \to 0$$

be a 1-fold extension of a pair of modules in $\mathcal{I}$. Then, by 1.6 this is equivalent to an induced extension. By the 5-lemma it follows that the middle terms of these extensions are isomorphic, so E is in $\mathcal{I}$ as required.

<u>Corollary 1.7'</u> Suppose that, for every Z in Mod A, there is an exact sequence in Mod A

$$0 \to P_Z \to (C,\ Z) \to Z \to 0$$

in which $(C,\ Z) \to Z$ is $h \longmapsto h(1)$ and P_Z is projective. Then $\mathcal{C}$ and add $\mathcal{C}$ are closed under extensions.

<u>Proposition 1.8</u> If add $\mathcal{I}$ is closed under extensions it is also closed under kernels of epimorphisms.

<u>Proof</u> Let $0 \to K \to M \to M' \to 0$ be exact, with M, M' in add $\mathcal{I}$; we wish to show K is in add $\mathcal{I}$. Choose ${}_BN$ and ${}_AX$ such that

$$M' \oplus N \simeq B \otimes X.$$

Then there is also an exact sequence

$$0 \to K \to M \oplus N \to B \otimes X \to 0$$

with $M \oplus N$ in add $\mathcal{I}$. Let $0 \to Y \to P \to X \to 0$ in Mod A be exact with P projective. This remains exact on applying the functor $B \otimes -$ (1.5(i)).

Form the commutative exact pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B \otimes Y \arrow[r, equal] \arrow[d] & B \otimes Y \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & B \otimes P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & M \oplus N \arrow[r] \arrow[d] & B \otimes X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $B \otimes P$ is projective the first row of this diagram splits. Hence K is a direct summand of U. But U is in add $\mathcal{I}$ as it is an extension of modules in add $\mathcal{I}$ (look at first column of diagram). Thus, as required, K is in add $\mathcal{I}$.

<u>Proposition 1.8'</u> If add $\mathcal{C}$ is closed under extensions it is also closed under cokernels of monomorphisms.

Remark The reader will observe that the proof of 1.8, 1.8′ is essentially the same as that for 2.2, 2.2′ of chapter III; for variety we gave the proof for the 'coinduced' case there.

§2 Almost split sequences for relatively projective modules

In this section we keep the notation $A \to B$, $A \to C$ but assume that these are homomorphisms of artin algebras over a commutative artinian ring k.

Similarly to 1.5 we assume that

$$B \otimes - : \text{mod } A \to \text{mod } B,$$

$$(C, -) : \text{mod } A \to \text{mod } C,$$

are exact functors.

Let $\mathcal{I}_0$ denote the category $\mathcal{I} \cap \text{mod } B$, and let $\mathcal{C}_0$ denote the category $\mathcal{C} \cap \text{mod } C$. As in §1 we have criteria for $\mathcal{I}_0(\mathcal{C}_0)$ to have the property that every extension in $\mathcal{I}_0(\mathcal{C}_0)$ is (co)induced from one in mod A. These criteria can be deduced directly from 1.6, 1.6'. Rather than work explicitly with sufficient conditions on $A \to B$ $(A \to C)$ for the property to hold we simply insert it as a hypothesis in our theorems. We actually only need it to hold for 1-fold extensions but an easy 'dimension shifting' argument shows that it must then hold for arbitrary length extensions. If $e: 0 \to B \otimes X \to E_1 \to \ldots \to E_n \to B \otimes Y \to 0$ is an extension of induced modules it is not hard to show that $E_2 \to \ldots \to E_n \to B \otimes Y \to 0$ can be taken to be the start of a projective resolution of $B \otimes Y$. If $0 \to \Omega^{n-1} Y \to P_2 \to \ldots \to P_n \to Y \to 0$ is the start of a projective resolution of Y then applying $B \otimes -$ yields such a resolution of $B \otimes Y$. Hence our original extension is equivalent to the splice of $0 \to B \otimes X \to E_1 \to B \otimes \Omega^{n-1} Y \to 0$ with an induced extension. Thus, since all 1-fold extensions were assumed to be equivalent to induced extensions, it follows that e is equivalent to an induced extension.

A similar argument with injective resolutions establishes the corresponding result for $\mathcal{C}_0$.

Minimal morphisms

Let Λ be an artin algebra, we recall some general results about 'right minimal' morphisms in $\operatorname{mod} \Lambda$. These will be used to study almost split sequences in $\mathcal{I}_0$, we suppress the dual results for $\mathcal{E}_0$ which use the dual notion of 'left minimal' morphism.

Definition 2.1 A morphism $f : X \to Y$ in $\operatorname{mod} \Lambda$ is called right minimal if whenever there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \| \\ X & \xrightarrow{f} & Y \end{array}$$

the morphism u is an automorphism.

Lemma 2.2 [AS1 : §1] Let $f : X \to Y$ be a morphism in $\operatorname{mod} \Lambda$; then there is a decomposition $X = M \oplus N$ such that $f|_M$ is right minimal and $f|_N = 0$.

Proof (Sketch of proof from [AS1]) We call a morphism $g : M \to Y$ 'lifting equivalent' to f if 'f factors through g' and 'g factors through f' - that is, there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & Y \\ \downarrow & & \| \\ X & \xrightarrow{f} & Y \\ \downarrow & & \| \\ M & \xrightarrow{g} & Y \end{array} \qquad (*)$$

The collection of such maps form a 'lifting equivalence class' - which is non-empty since f is in it - in which we choose $g : M \to Y$ such that M has minimal length. Then g is right minimal (if u is in $\operatorname{End}_\Lambda(M)$ and satisfies $gu = g$ then g restricted to the image of u is also lifting equivalent to f, thus, by the choice of M, $\operatorname{im}(u) = M$ and u has to be an automorphism).

Using the fact that g is right minimal it follows that in the diagram (*) $M \to X \to M$ is an automorphism. Thus X has a decomposition $M \oplus N$ where N is the kernel of $X \to M$. This is the desired decomposition.

Definition 2.3 A morphism $f : X \to Y$ in mod Λ is called minimal right almost split if it is right almost split and right minimal. Notice that if f is not necessarily minimal and we construct a decomposition $X = M \oplus N$ as in 2.2 then the 'minimal version of f' - that is $g = f|_M$ is still right almost split.

Ringel in [Ri] calls such morphisms 'sink' morphisms, the dual concept of minimal left almost split morphism is called a 'source' morphism. The following lemma is proven in [Ri].

Lemma 2.4 [Ri: 2.3, Lemma 2] If $0 \to X \to Y \to Z \to 0$ is an exact sequence in a full, extension-closed subcategory $\underline{D}$ of mod Λ such that $\underline{D} = \text{add } \underline{D}$ and $Y \to Z$ is minimal right almost split in $\underline{D}$, $X \to Y$ is minimal left almost split in $\underline{D}$.

Proof See [Ri]. A more 'sophisticated' approach is that of Auslander, where this result follows from [Al: II 4.4].

We are now in a position to study the relationship between almost split sequences in the categories mod A, mod B, $\mathcal{I}_0$ (and dually in mod C, $\mathcal{C}_0$ - but we suppress these results). Almost split sequences always exist in mod Λ (Λ any artin algebra) as was established by Auslander and Reiten in [AR2]. We notice that there is a good supply of such sequences in add $\mathcal{I}_0$ - the method by which we obtain a right almost split map is the same as that used by Bautista and Kleiner in [BK: §2].

Given an indecomposable non-projective module N in add $\mathcal{I}_0$ let $M \to N$ be a minimal right almost split map in mod B. Then $B \otimes M \to M \to N$, with $B \otimes M \to M$ the canonical morphism $b \otimes m \longmapsto bm$, is a right almost split in add $\mathcal{I}_0$ - to see this let us check the factorisation property for maps $B \otimes X \to N$ which are not split epimorphisms. This is then sufficient to deduce the property for maps $N' \to N$ from relatively projective modules N' (i.e. summands of $B \otimes X$).

Given the non-split epimorphism $B \otimes X \to N$ we can factorise this through the right almost split morphism $M \to N$. Thus there is $f : B \otimes X \to M$ such that $B \otimes X \xrightarrow{f} M \to N$ is $B \otimes X \to N$.

We can then lift f through $B \otimes M \to M$ by defining $B \otimes X \to B \otimes M$ by $b \otimes x \longmapsto b \otimes f(1 \otimes x)$. Thus $B \otimes M \to N$ is right almost split in $\mathcal{I}_0$. Consider a right minimal version $M' \to N$. Since a projective cover of N will factor through this map, $M' \to N$ is surjective. Thus, if L' is its kernel, there is a short exact sequence

$$0 \to L' \to M' \to N \to 0.$$

If add $\mathcal{I}_0$ is closed under extensions then, using the same argument as in 1.8 we deduce that it is closed under kernels of epimorphisms. Thus L' is in add $\mathcal{I}_0$. Now 2.4 shows that this is an almost split sequence in add $\mathcal{I}_0$ ending at N.

Our first result relates such 'relative' almost split sequences in add $\mathcal{I}_0$ to almost split sequences in mod B.

<u>Theorem 2.5</u> Suppose that every extension in $\mathcal{I}_0$ is equivalent to an induced extension. Let N be a non-projective indecomposable module in add $\mathcal{I}_0$. Then there is an almost split sequence in add $\mathcal{I}_0$

$$e : 0 \to L \to M \to N \to 0$$

and if
$$x : 0 \to DTrN \to E \to N \to 0$$

is the almost split sequence in mod B ending at N then e is a direct summand of the sequence $B \otimes x$.

Proof The existence of e is guaranteed by the argument given above. We make e into an induced extension as follows:

Choose modules L', N' in add $\mathcal{I}_0$ so that for some X, Z in mod A

$$L \oplus L' \simeq B \otimes X, \quad N \oplus N' \simeq B \otimes Z$$

Let $e' : 0 \to L' \to L' \oplus N' \to N' \to 0$ be the split extension of L' by N'. Then

$$e \oplus e' : 0 \to B \otimes X \to M \oplus L' \oplus N' \to B \otimes Z \to 0$$

is an extension in $\mathcal{I}_0$. Thus, by hypothesis there is an extension

$$z : 0 \to X \to Y \to Z \to 0$$

in mod A such that $e \oplus e' \simeq B \otimes z$.

We claim there is a morphism $e \to x$ as follows:

$$\begin{tikzcd}
e : 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
x : 0 \arrow[r] & DTrN \arrow[r] & E \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This is obtained by factorising the non-split map $M \to N$ of e through the right almost split map $E \to N$ of x. Thus there is a morphism

$$B \otimes z \simeq e \oplus e' \twoheadrightarrow e \to x.$$

There is a canonical morphism $B \otimes x \to x$ given by the natural family of maps $B \otimes W \to W$, $b \otimes w \longmapsto bw$, where W is any left B-module.

Thus we have maps of extensions

$$\begin{tikzcd}
& B \otimes x \arrow[d] \\
B \otimes z \arrow[r] & x.
\end{tikzcd}$$

There is a map $B \otimes z \to B \otimes x$ of extensions which makes

$$\begin{tikzcd}
& B \otimes x \arrow[d] \\
B \otimes z \arrow[ur] \arrow[r] & x
\end{tikzcd}$$

commute - to see this notice that if W is any B-module the natural map $B \otimes W \to W$ is 'universal' for maps from induced B-modules $B \otimes U$ to W, for if $\theta : B \otimes U \to W$ is such a map then $\bar{\theta} : B \otimes U \to B \otimes W$ defined by $\theta(b \otimes u) = b \otimes \theta(1 \otimes u)$ makes the diagram

$$\begin{tikzcd} & B \otimes W \arrow[d] \\ B \otimes U \arrow[ur, "\bar{\theta}"] \arrow[r, "\theta"] & W \end{tikzcd}$$

commute. The morphism $B \otimes x \to x$ is given by morphisms $B \otimes W \to W$ for $W = DTrN$, E, N and the morphism $B \otimes z \to x$ is given by morphisms $\theta : B \otimes U \to W$ for $U = X$, $W = DTrN$; $U = Y$, $W = E$; $U = Z$, $W = N$. The liftings $\bar{\theta} : B \otimes U \to B \otimes W$ given above provide morphisms $B \otimes X \to B \otimes DTrN$, $B \otimes Y \to B \otimes E$, and $B \otimes Z \to B \otimes N$, and it is not hard to check that these give rise to a morphism:

$$\begin{tikzcd} B \otimes z : 0 \arrow[r] & B \otimes X \arrow[r] \arrow[d] & B \otimes Y \arrow[r] \arrow[d] & B \otimes Z \arrow[r] \arrow[d] & 0 \\ B \otimes x : 0 \arrow[r] & B \otimes DTrN \arrow[r] & B \otimes E \arrow[r] & B \otimes N \arrow[r] & 0 \end{tikzcd}$$

which makes the diagram

$$\begin{tikzcd} & B \otimes x \arrow[d] \\ B \otimes z \arrow[ur] \arrow[r] & x \end{tikzcd}$$

commute, as claimed.

Hence we have a morphism of extensions $\alpha : e \to B \otimes x$ given by the composite $e \hookrightarrow e \oplus e' \simeq B \otimes z \to B \otimes x$.

Consider the diagram

$$\begin{tikzcd} B \otimes x : 0 \arrow[r] & B \otimes DTrN \arrow[r] & B \otimes E \arrow[r] & B \otimes N \arrow[r] \arrow[d] & 0 \\ e : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

in which $B \otimes N \to N$ is the canonical map.

The composite $B \otimes E \to B \otimes N \to N$ is not a split epimorphism since it is equal to $B \otimes E \to E \to N$ where $E \to N$ is not a split epimorphism. Thus it factors through $M \to N$ (recall e is almost split in add $\mathcal{I}_0$) and we obtain a commutative exact diagram:

$$\begin{tikzcd}
B \otimes x : 0 \arrow[r] & B \otimes DTrN \arrow[r] \arrow[d] & B \otimes E \arrow[r] \arrow[d] & B \otimes N \arrow[r] \arrow[d] & 0 \\
e : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This gives a morphism $\beta : B \otimes x \to e$ and hence we have an endomorphism $\beta\alpha : e \to e$. The reader is invited to check that this endomorphism has the form:

$$\begin{tikzcd}
e : 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
e : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd} ,$$

i.e. the map $N \to N$ at the right hand end is just the identity.

Since $M \to N$ is right minimal the endomorphism $M \to M$ is an automorphism. Hence $L \to L$ is an automorphism. Thus $\beta\alpha$ is an automorphism of e, and so it follows that $\alpha : e \to B \otimes x$ is a split monomorphism. Thus e is a direct summand of $B \otimes x$, which was what we wanted to prove.

Our second result relates almost split sequences in add $\mathcal{I}_0$ to almost split sequences in mod A.

<u>Theorem 2.6</u> Suppose that every extension in $\mathcal{I}_0$ is equivalent to an induced extension. Let

$$e : 0 \to X \to Y \to Z \to 0$$

be an almost split sequence in mod A (so $X = DTrZ$) and assume that there is an indecomposable summand of $B \otimes Z$ such that $B \otimes Y \to B \otimes Z$ followed by projection onto this summand is not a split epimorphism.

Then $B \otimes e$ is the direct sum of two extensions, one of which is almost split in add $\mathcal{I}_0$.

Proof Let Z' be an indecomposable summand of $B \otimes Z$ with the property described above. Since Z' is not projective there is an almost split sequence

$$x : 0 \to X' \to Y' \to Z' \to 0$$

in add $\mathcal{I}_0$. Factorising $B \otimes Y \to B \otimes Z \xrightarrow{\text{projection}} Z'$ through the right almost split map $Y' \to Z'$ gives rise to a commutative exact diagram

$$\begin{array}{ccccccccccc} B \otimes e : & 0 & \to & B \otimes X & \to & B \otimes Y & \to & B \otimes Z & \to & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ x : & 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \end{array}$$

and thus a morphism $B \otimes e \to x$.

We now construct a morphism $x \to B \otimes e$.

Choose Z'' such that $Z' \oplus Z'' \simeq B \otimes Z$ and X'' such that $X' \oplus X''$ is isomorphic to an induced module ($B \otimes T$ say).

Define z to be the split extension

$$z : 0 \to X'' \to X'' \oplus Z'' \to Z'' \to 0.$$

Then $x \oplus z$ is isomorphic to an extension of $B \otimes T$ by $B \otimes Z$. By our hypothesis this extension can be chosen to be an induced one, so that $x \otimes z \simeq B \otimes t$ for some extension

$$t : 0 \to T \to U \to Z \to 0$$

in mod A. Since x is not split, $B \otimes t$ and hence t are not split.

The extension

$$e : 0 \to X \to Y \to Z \to 0$$

is almost split in mod A so by factorising the map $U \to Z$ of t through the right almost split map $Y \to Z$ of e we obtain a commutative exact diagram

$$\begin{tikzcd}[row sep=small]
t : 0 \arrow[r] & T \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
e : 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

This is a morphism $t \to e$, so there is a morphism $B \otimes t \to B \otimes e$ and thus a composite

$$x \hookrightarrow x \oplus z \simeq B \otimes t \to B \otimes e.$$

This is a morphism $x \to B \otimes e$ which we show is a split monomorphism. The composite of this with the morphism $B \otimes e \to x$ constructed above is an endomorphism of x of the form:

$$\begin{tikzcd}[row sep=small]
x : 0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, equal] & 0 \\
x : 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0.
\end{tikzcd}$$

As in 2.5 the right minimality of the map $Y' \to Z'$ shows that this endomorphism is an automorphism. Thus $x \to B \otimes e$ is a split monomorphism and $B \otimes e$ has a summand which is almost split in add $\mathcal{I}_0$ as required.

Remarks 2.7 The result 2.6 above is similar in both statement and proof to the result III: 2.1 on bocses. With a little care this earlier result can be deduced from 2.6; the essential point is that the automorphism $x \to B \otimes e \to x$ we constructed has to be the composite of two isomorphisms - this is because the end terms of $B \otimes e$ are assumed indecomposable in III: 2.1.

Of course, by putting $A \to B$ to be $A \to R$ a result on bocses can be deduced from 2.5 as well.

Proposition 2.8 Suppose idempotents split in mod $\mathfrak{A}$ - where $\mathfrak{A}$ satisfies the basic assumptions of chapter III - and Z is indecomposable in mod $\mathfrak{A}$ such that $R \otimes Z$ is not projective in mod R. Then there is an almost split sequence in $\underline{I}$

$$e : 0 \to R \otimes X \to R \otimes Y \to R \otimes Z \to 0,$$

and if $x : 0 \to DTr(R \otimes Z) \to E \to R \otimes Z \to 0$ is the almost split sequence in mod R ending at $R \otimes Z$ then e is a direct summand of $R \otimes x$.

§3 Addendum: recent results of Auslander, Reiten, and Kleiner

While this thesis was in preparation the preprint [AR4] came to our attention. It contains the following result:

Proposition 3.1 [AR4: 4.11] Let $i : \Lambda \to \Gamma$ be a ring homomorphism between artin algebras Λ and Γ, and assume that Γ is a finitely generated projective right Λ-module. If the subcategory $\underline{X}$ of mod Γ of relatively projective modules is closed under extensions, then it is resolving.

The term 'resolving' means that $\underline{X}$ is closed under extensions, contains the projectives in mod Γ, and is closed under kernels of epimorphisms.

This proposition is just a version (for finitely generated modules over artin algebras) of the result 1.8 of this chapter, and the same line of proof will clearly suffice to establish it. The proof given by Auslander and Reiten in [AR4] uses the methodology of their earlier paper [AR3].

In [AR4] the result is attributed to Kleiner and the reader is referred to the remarks made in [AR4] immediately after [AR4: 4.11] for further details.

The category $\underline{X}$ above is contravariantly finite in mod Γ (for any M in mod Γ it is easy to check that the morphism of functors

$$\mathrm{Hom}_\Gamma(-, \Gamma \otimes_\Lambda M)|_{\underline{X}} \to \mathrm{Hom}_\Gamma(-, M)|_{\underline{X}}$$

induced by the natural map $\Gamma \otimes_\Lambda M \to M$ is surjective). The paper [AR3] studies such 'contravariantly finite resolving' subcategories extensively, and so Auslander and Reiten raise in [AR4] the question 'When is $\underline{X}$ closed under extensions?' - if $\underline{X}$ is extension closed then it is an example of a contravariantly finite resolving subcategory. They give the following criterion due to Kleiner:

Theorem 3.2 (Kleiner, [AR4:4.14]) Let Γ, Λ, $\underline{X}$ be as above (so

Γ_Λ is projective). If, for each C in $\underline{X}$, the kernel K_C of the natural map $\Gamma \otimes_\Lambda C \to C$ is injective as a Λ-module, then $\underline{X}$ is closed under extensions.

Details of Kleiner's proof are not given, and there is also no reference. We have failed to find our own proof for this result so we are unable to say much about its relation to our result (1.7) along these lines. However, as pointed out in [AR4], it does generalise the criterion obtained for representations of bocses in [BB: §3]. This is not substantiated in [AR4] so we give a demonstration here.

In [BB: §3] the criterion for $\underline{I}$ to be extension - closed in mod R is that in the exact sequences [BB: 3.3(a)]

$$0 \to X \xrightarrow{\epsilon^*} \mathrm{Hom}_A(V, X) \to \mathrm{Hom}_A(\overline{V}, X) \to 0$$

the cokernel term $\mathrm{Hom}_A(\overline{V}, X)$ is injective; this must hold for any ${}_AX$. We now show that Kleiner's criterion holds if we only know this for X of the form ${}_AR \otimes Y \simeq \mathrm{Hom}_A(V, Y)$. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(V, Y) & \xrightarrow{\quad \epsilon^* \quad} & \mathrm{Hom}_A(V, \mathrm{Hom}_A(V, Y)) \\ & & \uparrow \wr \\ \uparrow \wr & & \mathrm{Hom}_A(V, R \otimes Y) \\ & & \uparrow \wr \\ R \otimes Y & \xrightarrow{\quad f \quad} & R \otimes R \otimes Y \end{array}$$

in which the map f is given by $f(r \otimes y) = \epsilon \otimes r \otimes y$ (recall that ϵ is the identity element in R).

Thus there is an exact sequence

$$0 \to R \otimes Y \xrightarrow{f} R \otimes R \otimes Y \to \mathrm{Hom}_A(\overline{V}, X) \to 0$$

where $X = {}_AR \otimes Y$. This is split, since the canonical map $R \otimes R \otimes Y \to R \otimes Y$, $r \otimes s \otimes y \longmapsto rs \otimes y$ is a left inverse for f. Thus, as an A-module, the kernel of this natural map is $\mathrm{Hom}_A(\overline{V}, {}_AR \otimes Y)$ which is injective. This verifies the criterion in

Kleiner's result.

It may be asked which pairs of rings satisfy the criteria (either Kleiner's or our's) for the category of relatively projectives to be extension - closed. Our result (1.7) is a 'generalisation' of the criterion for bocses in [BB], and was originally obtained by liberating the work of [BB: §3] from its bocs - theoretic context. If in 1.7 we take $A \to B$ to be $A \to R$ (R the right algebra of a bocs $\mathfrak{U} = (A, V)$) then the sequence

$$0 \to X \to B \otimes X \to Q_X \to 0$$

in the statement of 1.7 is isomorphic to the sequence

$$0 \to X \xrightarrow{\epsilon^*} \mathrm{Hom}_A(V, X) \to \mathrm{Hom}_A(\bar{V}, X) \to 0$$

of [BB: 3.3(a)] - where we assume that $\mathfrak{U}$ satisfies the same hypotheses as the bocses used in [BB: §3]. Thus we require that, for all ${}_AX$, $\mathrm{Hom}_A(\bar{V}, X)$ is injective, and as shown in [BB: 3.6], this holds if $\bar{V}$ is a projectivising bimodule.

At the time of writing this provides the only (non-trivial) class of examples we know satisfying 1.7, but we also mention the following class of examples:

Let k be a field of characteristic p (a prime) which divides $|G|$, where G is a finite group. Let H be a subgroup of G; one may ask when the category $\underline{X}$ of relatively H-projective modules in mod kG is closed under extensions. It is possible to verify the criterion 1.7 provided that for each g in $G \backslash H$ the order of the subgroup $H \cap gHg^{-1}$ is not divisible by p. This condition is, however, strong enough to ensure that either $p \nmid |G : H|$ and thus $\underline{X} = \text{mod } kG$, or $p \nmid |H|$ and thus $\underline{X} = \text{add}(kG)$. In fact Auslander and Reiten have proved:

<u>Proposition 3.3</u> [AR4: 2.16] Let $i : \Lambda \to \Gamma$ be a homomorphism of artin algebras, where Γ is a symmetric algebra, Λ is a

self-injective algebra, and Γ_Λ is projective. If $\underline{X}$, the category of relatively Λ-projective modules in mod Γ is closed under extensions then $\underline{X}$ = add(Γ) or mod Γ.

The proof is based on the fact that, for a symmetric artin algebra Λ and any module M in mod Λ. $DTrM = \Omega^2 M$. This fact can also be used to provide a quick demonstration of the following recent result of Kleiner's.

<u>Theorem 3.4</u> (Kleiner [K]) Let k be a field of characteristic a prime p dividing $|G|$, where G is a finite group. Let H be a subgroup of G and $\underline{X}$ the category of relatively H-projective modules in mod kG.

A short exact sequence

$$e : 0 \to L \to M \to N \to 0$$

of modules in X is almost split in $\underline{X}$ if and only if it is an almost split sequence in mod kG.

<u>Proof</u> (W.L. Burt 1991) It is clear that if e is an almost split sequence whose terms lie in $\underline{X}$ then e is almost split in $\underline{X}$, so we prove the converse.

Let $0 \to \Omega^2 N \to E \to N \to 0$ be the almost split sequence in mod kG ending at the non-projective module N (N is non-projective since e does not split).

Since $M \to N$ factors through the right almost split map $E \to N$ there is a commutative exact diagram

$$\begin{array}{ccccccc} e : 0 \to & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & \Omega^2 N & \longrightarrow & E & \longrightarrow & N & \longrightarrow 0. \end{array}$$

If $L \to \Omega^2 N$ is not a split monomorphism then, as $\Omega^2 N$ is in $\underline{X}$ and the sequence e is almost split in $\underline{X}$ the map $L \to \Omega^2 N$ lifts to a map $M \to \Omega^2 N$ such that
$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[dl] \\ \Omega^2 N & \end{tikzcd}$$
commutes.

But this would mean that the pushout along $L \to \Omega^2 N$ of e would split. Since this pushout has already been seen to be the almost split sequence

$$x : 0 \to \Omega^2 N \to E \to N \to 0,$$

this cannot happen.

Thus $L \to \Omega^2 N$ is a split monomorphism and hence, since $\Omega^2 N$ is indecomposable, it is an isomorphism. Thus e is isomorphic to the almost split sequence x ending at N. Hence, as claimed, e is almost split in mod kG.

Chapter V Morphisms in mod $\mathfrak{A}$ and triangular bocses

§1 Morphisms in mod $\mathfrak{A}$

This chapter is concerned with properties of mod $\mathfrak{A}$ which are usually present in applications [D, C-Bl]. We restrict attention therefore, to bocses $\mathfrak{A} = (A, V)$ over k-algebras (with k a field) and consider only the finite dimensional representations of $\mathfrak{A}$.

As was commented in Chapter I §2 the embeddings proven there are not available for all the bocses considered in [C-Bl]. The bocses used in the applications in [C-Bl] satisfy the property that V is finitely generated <u>as an A-A-bimodule</u> (and also ${}_AV$, V_A are projective A-modules). The A-modules ${}_AV$, V_A are finitely generated if A is finite-dimensional over k, but are usually not so if A is not finite-dimensional over k. The embeddings of I§2 are not available in this latter case. Also the category 'R($\mathfrak{A}$)' of representations of $\mathfrak{A}$ considered in [C-Bl] is the category of finite dimensional representations of $\mathfrak{A}$; thus if A is finite dimensional R($\mathfrak{A}$) is just mod $\mathfrak{A}$ and if A is not finite dimensional then R($\mathfrak{A}$) is the subcategory f.d. (mod $\mathfrak{A}$) defined by those X in mod $\mathfrak{A}$ for which dim X (i.e. $\dim_k X$) is finite.

Hence we structure this section as follows - we start by studying f.d. (mod $\mathfrak{A}$) for A an arbitrary k-algebra and then consider the case that A is finite dimensional, ${}_AV$ and V_A are projective, and dim V is finite, noting that in this case f.d. (mod $\mathfrak{A}$) is mod $\mathfrak{A}$.

<u>Definition 1.1</u> A morphism f in $\mathfrak{A}(X, Y)$ is called <u>regular</u> if there exist isomorphisms h in $\mathfrak{A}(X', X)$ and ℓ in $\mathfrak{A}(Y, Y')$ such that $\ell \circ f \circ h$ in $\mathfrak{A}(X', Y')$ is given by

$$V \otimes X' \xrightarrow{\epsilon \otimes 1} A \otimes X' = X' \xrightarrow{g} Y'$$

for some g in $\mathrm{Hom}_A(X', Y')$.

If g is a monomorphism (epimorphism) f will be called a regular mono (epi).

Notation 1.2 If $t : X \to Y$ is a morphism in mod A the morphism $\hat{t}$ in $\mathfrak{A}(X, Y)$ is defined by

$$V \otimes X' \xrightarrow{\epsilon \otimes 1} A \otimes X = X \xrightarrow{t} Y.$$

This is the same notation as was introduced in II 2.1.

Proposition 1.3 Regular maps in f.d. (mod $\mathfrak{A}$) have kernels and cokernels in f.d. (mod $\mathfrak{A}$). The kernel is a regular mono and the cokernel is a regular epi.

Proof It is clearly sufficient to prove this for maps of the form $\hat{f}$ in $\mathfrak{A}(X, Y)$ where f is in $\mathrm{Hom}_A(X, Y)$. Let $K = \ker f$, $C = \mathrm{coker}\, f$ and

$$0 \to K \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{p} C \to 0$$

the canonical exact sequence. Then $\hat{i}$ is a regular mono, $\hat{p}$ a regular epi and we claim that $\hat{i}$ is a kernel of $\hat{f}$ and $\hat{p}$ a cokernel of $\hat{f}$.

To see that $\hat{i}$ is a kernel of $\hat{f}$ note that $\hat{f} \circ \hat{i} = \widehat{fi} = 0$. Now suppose g is in $\mathfrak{A}(W, X)$ and $\hat{f} \circ g = 0$. Then it is easy to check that $\hat{f} \circ g$ is $V \otimes W \xrightarrow{g} X \xrightarrow{f} Y$. Thus $fg = 0$. Hence g factors through $K = \ker f$, i.e. there is a map $h : V \otimes W \to K$ such that $ih = g$. Note that $h \in \mathfrak{A}(W, K)$ and that it is easy to check that $\hat{i} \circ h = g$. Thus g factors through $\hat{i}$.

The only thing that remains to be checked is that the factorisation of g through $\hat{i}$ is unique. Suppose $g = \hat{i} \circ h'$ for some $h' \in \mathfrak{A}(W, K)$. Then as $\hat{i} \circ h' = ih'$ we have that $g = ih = ih'$. Since i is a kernel of f it follows that $h = h'$, thus $\hat{i}$ is a kernel of $\hat{f}$ in f.d. (mod $\mathfrak{A}$).

We now check that $\hat{p}$ is a cokernel of $\hat{f}$. $\hat{p} \circ \hat{f} = \widehat{pf} = 0$, so now suppose q is in $\mathfrak{A}(Y, Z')$ and $q \circ \hat{f} = 0$. Then it is easy to check that $q \circ \hat{f}$ is $V \otimes X \xrightarrow{1 \otimes f} V \otimes Y \xrightarrow{q} Z'$. Notice that

$V \otimes X \xrightarrow{1\otimes f} V \otimes Y \xrightarrow{1 \otimes p} V \otimes Z \longrightarrow 0$ is exact so $1 \otimes p$ is the cokernel of $1 \otimes f$. Since $q(1 \otimes f) = 0$ there is $r : V \otimes Z \to Z'$ such that $r(1 \otimes p) = q$. Notice that $r(1 \otimes p) = r \circ \hat{p}$ so q factors through $\hat{p}$.

It now remains to check that this factorisation is unique. Suppose $r' : V \otimes Z \to Z'$ also satisfies $r' \circ \hat{p} = q$. Then as $r' \circ \hat{p} = r'(1 \otimes p)$ we have $r'(1 \otimes p) = r(1 \otimes p)$, hence $r' = r$ and the proof is complete.

The class of bocses used in [C-B1] all have what are called there 'grouplikes' [C-B1: §3]. Thus we make the following definition.

Definition 1.4 An element ω in V, where $\mathfrak{A} = (A, V)$ is a bocs, is called a grouplike element if

$$\epsilon(\omega) = 1, \ \mu(\omega) = \omega \otimes \omega.$$

Note that this guarantees that ϵ is surjective.

There is a functor ω^*: f.d. $(\mathrm{mod}\,\mathfrak{A}) \to \mathrm{mod}\ k$ associated with a grouplike element ω defined as follows:

$\omega^*(X) = X$ regarded as a k-vector space, and if f is in $\mathfrak{A}(X, Y)$ $\omega^*(f)(x) := f(\omega \otimes x)$. Since k acts centrally on V (by assumption) $\omega^*(f)$ is k-linear and it is easy to check ω^* is a functor.

Definition 1.5 [C-B1: 3.2] ω is called a reflector if whenever the map $\omega^*(f)$ is an isomorphism in mod k f is an isomorphism in f.d. $(\mathrm{mod}\ \mathfrak{A})$. The bocses used in [C-B1] always have this property.

We now identify an analogue of the notion of 'short exact sequence' for f.d. $(\mathrm{mod}\ \mathfrak{A})$.

Definition 1.6 Let f be a morphism in f.d. $(\mathrm{mod}\,\mathfrak{A})$. If $\omega^*(f)$ is a monomorphism (epimorphism) we call f a proper mono (epi). A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in f.d. (mod $\mathfrak{A}$) is called a proper sequence if

$$0 \to \omega^* X \xrightarrow{\omega^* f} \omega^* Y \xrightarrow{\omega^* g} \omega^* Z \to 0$$

is an exact sequence in mod k, and $g \circ f = 0$ in $\mathfrak{A}(X, Z)$.

Proposition 1.7 Suppose there exists a reflector ω for the bocs $\mathfrak{A} = (A, V)$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a proper sequence with either f or g regular. Then both f and g are regular, f is a kernel of g and g is a cokernel of f.

Proof We suppose that f is regular, the proof for the case that g is regular being similar. f has a cokernel $Y \xrightarrow{c} C$ which is regular by 1.3. Since $g \circ f = 0$ g factors through c. Thus there is some g' in $\mathfrak{A}(C, Z)$ such that $g' \circ c = g$. Thus $\omega^*(g) = \omega^*(g')\omega^*(c)$. $\omega^*(g)$ is surjective so $\omega^*(g')$ is surjective. But dim C = dim Y - dim X = dim Z so $\omega^*(g')$ is an isomorphism. Since ω is a reflector g' is an isomorphism. Hence g is also a cokernel of f and is regular. Now apply the dual argument to show that f is a kernel of g.

Remark 1.8 If all proper epis in f.d. (mod $\mathfrak{A}$) are regular epis then ω is a reflector. If all proper monos in f.d. (mod $\mathfrak{A}$) are regular monos then ω is a reflector.

Proof Let f be in $\mathfrak{A}(X, Y)$ and $\omega^* f$ an isomorphism. Then f is a proper epi and so by hypothesis is a regular epi (if we suppose all proper epis are regular epis). Thus there are isomorphisms h in $\mathfrak{A}(X',X)$ and ℓ in $\mathfrak{A}(Y, Y')$ such that $\ell \circ f \circ h = \hat{t}$ for some t in $\mathrm{Hom}_A(X', Y')$. Thus $t = \omega^*(\hat{t}) = \omega^*(\ell)\omega^*(f)\omega^*(h)$ is an isomorphism, since $\omega^*(\ell)$, $\omega^*(f)$, $\omega^*(h)$ are all isomorphisms. Thus $\hat{t}$ is an isomorphism (its inverse is just $\widehat{t^{-1}}$). Thus $f = \ell^{-1} \circ \hat{t} \circ h^{-1}$ is also an isomorphism. The remaining part of the remark is established similarly.

We now restrict attention to bocses $\mathfrak{A} = (A, V)$ in which the k-algebra A is finite dimensional. Our assumptions for the rest of this section are laid out below:

Assumptions 1.9 A is finite-dimensional over k; V is finite-dimensional over k; ${}_AV$, V_A are projective A-modules; there is a grouplike element ω in V.

Thus f.d. (mod $\mathfrak{A}$) = mod $\mathfrak{A}$ and there are embeddings of this category F_I : mod $\mathfrak{A} \simeq \underline{I} \subseteq$ mod R and F_C : mod $\mathfrak{A} \simeq \underline{C} \subseteq$ mod L. It will be useful to have specific descriptions of these embeddings.

Recall that [BB: 2.5] F_C is given by sending objects X in mod $\mathfrak{A}$ to $V \otimes X$ in $\underline{C}$. If f is in $\mathfrak{A}(X, Y)$ $F_C(f)$ is the composite

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y.$$

F_I is defined as mod $\mathfrak{A} \xrightarrow{F_C} \underline{C} \xrightarrow{\mathrm{Hom}_L(V,-)} \underline{I}$. Thus $F_I(X) = \mathrm{Hom}_L(V, V \otimes X)$ which may be identified as $\mathrm{Hom}_A(V, X)$ by the series of natural isomorphisms

$\mathrm{Hom}_L(V, V \otimes X) \simeq \mathrm{Hom}_L(V, \mathrm{Hom}_A(L, X)) \simeq \mathrm{Hom}_A(V, X)$. Regarded as a map from $\mathrm{Hom}_A(V, X)$ to $\mathrm{Hom}_A(V, Y)$ $F_I(f)$ is given by

$$\begin{aligned}\mathrm{Hom}_A(V, X) &\xrightarrow{\tilde{f}_*} \mathrm{Hom}_A(V, \mathrm{Hom}_A(V, Y)) \\ &\simeq \mathrm{Hom}_A(V \otimes V, Y) \\ &\xrightarrow{\mu^*} \mathrm{Hom}_A(V, Y)\end{aligned}$$

where $\tilde{f}(x)(v) = f(v \otimes x)$, as may be checked.

Notice that if $f = \hat{t}$ for some t in $\mathrm{Hom}_A(X, Y)$ then

$F_C(f) = 1 \otimes t : V \otimes X \to V \otimes Y$ and

$F_I(f) = t_*: \mathrm{Hom}_A(V, X) \to \mathrm{Hom}_A(V, Y)$.

With these preliminary remarks in mind we now make the following definitions:

Definition 1.10 Let f be in $\mathfrak{A}(X, Y)$. We say f is $\underline{I}$-injective (surjective) if the map $F_{\underline{I}}(f)$ of left R-modules is injective (surjective). Likewise we say is $\underline{C}$ - injective (surjective) if the map $F_{\underline{C}}(f)$ of left L-modules is injective (surjective).

There are now five notions of monomorphism' (dually 'epimorphism') for mod $\mathfrak{A}$, namely 'regular mono', '$\underline{I}$ - injective', '$\underline{C}$-injective', 'proper mono' and the usual notion of monomorphism in the sense of abstract category theory ('categorical monomorphism'). These concepts may be related as follows:

Proposition 1.11 If f in $\mathfrak{A}(X, Y)$ is a regular mono then it is $\underline{C}$ - injective; if f is $\underline{C}$ - injective then it is $\underline{I}$ - injective; f is $\underline{I}$ - injective if and only if f is a categorical monomorphism.

Proof Suppose f is a regular mono. Then there is a monomorphism g in $\mathrm{Hom}_A(X', Y')$ and a commutative diagram in mod $\mathfrak{A}$

$$\begin{tikzcd} X' \arrow[r, "\hat{g}"] & Y' \arrow[d, "\wr"] \\ X \arrow[u, "\wr"] \arrow[r, "f"] & Y \end{tikzcd}$$

Applying $F_{\underline{C}}$ to this and noting that $F_{\underline{C}}(\hat{g})$ is $1 \otimes g$ and that this is a monomorphism the fact that f is $\underline{C}$-injective follows.

If f is $\underline{C}$-injective then $F_{\underline{C}}(f)$ is injective so $F_{\underline{I}}(f) = \mathrm{Hom}_L(V, F_{\underline{C}}(f))$ is injective, thus f is $\underline{I}$-injective.

If f is $\underline{I}$-injective then it is clearly a categorical monomorphism. Conversely if f is a categorical monomorphism consider the following commutative exact diagram of left R-modules in which $K = \ker F_{\underline{I}}(f)$ and $\mathrm{Hom}_A(V, K) \to K$ is given by $\mathrm{Hom}_A(V, K) \simeq R \otimes K \to K$, where the latter map is the natural one.

$$\begin{array}{ccccccc} & & \mathrm{Hom}_A(V,K) & = & \mathrm{Hom}_A(V,K) & & \\ & & \downarrow & & \downarrow \ell & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathrm{Hom}_A(V,X) & \xrightarrow{F_I(f)} & \mathrm{Hom}_A(V,Y) \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

$F_I(f)\ell = 0$ so $\ell = 0$. Thus $K = 0$ and f is $\underline{I}$-injective.

<u>Proposition 1.11'</u> If f in $\mathfrak{A}(X, Y)$ is a regular epi then it is $\underline{I}$-surjective, if f is $\underline{I}$-surjective it is $\underline{C}$-surjective, and it is $\underline{C}$-surjective if and only if it is a categorical epimorphism.

<u>Proof</u> Dual to 1.11 - note that if f is $\underline{I}$-surjective $\mathrm{Hom}_L(V, F_C(f))$ is surjective so $V \otimes_R \mathrm{Hom}_L(V, F_C(f))$ is surjective.

Now use fact that $V \otimes_R \mathrm{Hom}_L(V, -)|_{\underline{C}}$ is naturally isomorphic to the identity functor on $\underline{C}$ (I:2.1). Thus $F_C(f)$ is surjective, so f is $\underline{C}$-surjective.

<u>Propostion 1.12</u> If f is $\underline{C}$-injective then f is a proper mono.

<u>Proof</u> Let f in $\mathfrak{A}(X, Y)$ be $\underline{C}$-injective, so $F_C(f)$ is injective. Define, for any ${}_AZ$ a k-linear map $i_Z : Z \to V \otimes Z$ by $i_Z(z) = \omega \otimes z$. Then i_Z followed by $V \otimes Z \xrightarrow{\epsilon\otimes 1} A \otimes Z = Z$ is the identity on Z, so i_Z is injective.

It is not hard to check that

$$i_Y \omega^*(f) = F_C(f) i_X,$$

and so, as i_X, i_Y, and $F_C(f)$ are all injective so is $\omega^*(f)$. Thus f is a proper mono.

<u>Proposition 1.12'</u> If f is $\underline{I}$-surjective then f is a proper epi.

<u>Proof</u> Let f in $\mathfrak{A}(X, Y)$ be $\underline{I}$-surjective, so $F_I(f)$ is surjective. Define, for any ${}_AZ$ a k-linear map p_Z: $\mathrm{Hom}_A(V, Z) \to Z$ by $h \longmapsto h(\omega)$. Then the composite of $Z \simeq \mathrm{Hom}_A(A, Z) \xrightarrow{\epsilon^*} \mathrm{Hom}_A(V, Z)$ with p_Z is the identity on Z, so p_Z is surjective.

It is not hard to check that

$$\omega^*(f)p_X = p_Y F_I(f),$$

and so, as p_X, p_Y, and $F_I(f)$ are all surjective so is $\omega^*(f)$. Thus f is a proper epi.

§2 Triangular bocses

This section develops further the theory of 'triangular' bocses initiated in [BB: §6]. For the notations, conventions, and terminology used here we refer the reader to pp. 24-28 of the appended copy of [BB] noting the following slight differences in notation:

in [BB] $\mathcal{A}$ is used instead of $\mathfrak{A}$,

the category denoted $\mathcal{A}$- k -mod in [BB]

is the category we denote in this chapter by f.d. (mod $\mathfrak{A}$).

In [BB: §6] $(-,-)$ is used for $\mathrm{Hom}_k(-,-)$, this should not be confused with the use of $(-,-)$ for $\mathrm{Hom}_A(-,-)$ elsewhere in this thesis - or indeed with $\mathfrak{A} = (A, V)$, our fixed bocs.

As in [BB: §6] we keep fixed a basefield k and a commutative k-algebra S. Each algebra considered in this section is an S-algebra on which k acts centrally. We do not assume that these k-algebras have finite k-dimension. As usual we only consider bimodules on which k acts centrally. Each bocs $\mathfrak{A} = (A, V)$ will have grouplike element ω and we assume that this is centralised by S, that is, for any s in S $s\omega = \omega s$.

It will be convenient to make the following definitions:

Definition 2.1 $\mathfrak{A}$ is called a triangular bocs if there is a finite sequence $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n = \mathfrak{A}$ of bocses such that $\mathfrak{A}_0$ is a principal bocs, and for $0 \le i \le n-1$ $\mathfrak{A}_{i+1}$ is a triangular kernel or algebra extension of $\mathfrak{A}_i$.

Definition 2.2 If, in 2.1, every triangular algebra extension $\mathfrak{A}_{i+1}$ of $\mathfrak{A}_i$ is a triangular tensor algebra extension, then $\mathfrak{A}$ is called a strictly triangular bocs.

In terms of these definitions the main results of [BB: §6] are:

Proposition 2.3 [BB: 6.3] If $\mathfrak{A}$ is a triangular bocs then the grouplike ω is a reflector.

Theorem 2.4 [BB: 6.4] If $\mathfrak{A}$ is a strictly triangular bocs then idempotents split in f.d. (mod $\mathfrak{A}$).

Notice that, as an easy inductive proof demonstrates, if $\mathfrak{A}$ is a triangular bocs and S is semisimple then the kernel $\bar{V}$ is a projectivising bimodule.

It is possible to deduce further information about the relationship between proper monos (epis) and the other concepts in §1 in the case that $\mathfrak{A}$ is a (strictly) triangular bocs. The first result is:

Proposition 2.5 If $\mathfrak{A}$ is a triangular bocs then proper epis (monos) are categorical epimorphisms (monomorphisms).

Proof The proof for proper monos is dual to that for proper epis and will be omitted. Since the statement of the proposition is clearly true for principal bocses it is sufficient, by induction, to show that if $\mathfrak{A} = (A, V)$ is a triangular kernel or algebra extension of $\mathfrak{A}' = (A', V')$ then if proper epis in f.d. (mod $\mathfrak{A}'$) are categorical epimorphisms then the same holds for f.d. (mod $\mathfrak{A}$).

Suppose $\mathfrak{A}$ is a triangular kernel extension of $\mathfrak{A}'$. Thus $A = A'$ and there is an S-S-subbimodule $\bar{U}$ of $\bar{V}$ such that $\bar{V}' \oplus (A \otimes_S \bar{U} \otimes_S A) \to \bar{V}$ is an isomorphism of A-A-bimodules and $d(\bar{U}) \subseteq \bar{V}' \otimes \bar{V}'$. Let θ in $\mathfrak{A}(Y, Z)$ be a proper epi and let γ in $\mathfrak{A}(Z, X)$ satisfy $\gamma \circ \theta = 0$. Thus $\gamma' \circ \theta' = 0$ and so $\gamma' = 0$. We must show that $\gamma = 0$. It now suffices to prove $\bar{\gamma}(U) = 0$; let u be in $\bar{U}$, we have

$$\gamma(\omega)\, \bar{\theta}(u) + \bar{\gamma}(u)\theta(\omega) + c(\bar{\gamma} \otimes \bar{\theta})\, (d(u)) = 0,$$

since $\gamma \circ \theta = 0$. Since $\gamma(\omega) = 0$ and $d(u)$ is in $\bar{V}' \otimes \bar{V}'$ we deduce $\gamma(u)\theta(\omega) = 0$. But $\theta(\omega)$ is surjective so $\gamma(u) = 0$ as required.

Now suppose $\mathfrak{A}$ is a triangular algebra extension of $\mathfrak{A}'$.

Thus there is an S-S-subbimodule E of A such that $A' \cup E$ generates A and $d(E) \subseteq \overline{V}'$; the natural map $A \otimes' \overline{V}' \otimes' A \to \overline{V}$ is an isomorphism. Given θ, γ as above with $\gamma \circ \theta = 0$, then $\gamma' \circ \theta' = 0$ so $\gamma' = 0$. Thus $\gamma_0 = \gamma_0' = 0$, and for v in $\overline{V}'$ $\gamma'(v) = 0$, so $\overline{\gamma}(1 \otimes' \overline{V}' \otimes' 1) = 0$. Since γ is an A-A-bimodule homomorphism $\overline{\gamma}(A \otimes' \overline{V} \otimes' A) = 0$, i.e. $\overline{\gamma} = 0$. Thus $\gamma = 0$ as required:

The proof that idempotents split in f.d. (mod $\mathfrak{A}$) (for suitable $\mathfrak{A}$) (2.4 above, i.e. [BB: 6.4]) is essentially a proof that all idempotents are regular - this is also the method used for proving this result in [BK : §5]. Likewise 2.3 above is a proof that maps f with $\omega^*(f)$ an isomorphism are regular. This is a similar idea to that used in the proof of the corresponding result in [BK: §5] and in [Ro]. The next result is also along similar lines - the proof (which we include) comes from unpublished work of M.C.R. Butler.

Theorem 2.6 Let $\mathfrak{A}$ be a strictly triangular bocs and suppose S is a semisimple algebra. Then every proper epi (mono) in f.d.(mod $\mathfrak{A}$) is regular.

Proof As in 2.5 we prove the result for proper epis - the case of proper monos being dual.

We shall in fact prove the following:

Let $\mathfrak{A}$ be a strictly tiangular bocs and θ in $\mathfrak{A}(X, Y)$ a proper epi. Then there exists ϕ in $\mathfrak{A}(X, Z)$ an isomorphism such that $\theta \circ \phi^{-1}$ in $\mathfrak{A}(Z, Y)$ has the form $(p, 0)$ for some p in $\mathrm{Hom}_A(Z, Y)$.

Since this clearly holds for principal bocses it is sufficient, by induction, to show that if it holds for a strictly triangular bocs $\mathfrak{A}'$ and $\mathfrak{A}$ is a triangular kernel or tensor algebra extension of $\mathfrak{A}'$ then the property also holds for $\mathfrak{A}$.

Consider first the case that $\mathfrak{A}$ is a triangular kernel extension of $\mathfrak{A}'$. Thus $A' = A$, and there is an S-S-subbimodule U of $\overline{V}$ such that

(a) the natural map $\overline{V}' \oplus (A \otimes_S U \otimes_S A) \to \overline{V}$ induced by the inclusions $\overline{V}' \subsetneq \overline{V}$ and $U \subsetneq \overline{V}$ is an isomorphism;

(b) $d(U) \subsetneq \overline{V}' \otimes \overline{V}'$,

Let θ in $\mathfrak{A}(X, Y)$ be as above. Then $\theta = (\theta_0, \overline{\theta})$ and its restriction $\theta' = (\theta_0', \overline{\theta}')$ in $\mathfrak{A}'(X, Y)$ is also a proper epi. Thus, by hypothesis, there is $\phi' = (\phi_0', \overline{\phi}')$ in $\mathfrak{A}'(X, Z)$ an isomorphism such that $\theta' \circ (\phi')^{-1}$ is $(p, 0)$ for some p in $\mathrm{Hom}_A(Z, Y)$ - note that $A' = A$.

We shall now define $\phi = (\phi_0, \overline{\phi})$ in $\mathfrak{A}(X, Z)$ such that ϕ is an isomorphism (since $\mathfrak{A}$ is (strictly) triangular it is enough (by 2.3) that ϕ_0 be an isomorphism).

Set $\phi_0 = \phi_0'$ and choose s in $\mathrm{Hom}_S(Y, Z)$ such that $ps = 1_Z$ (this can be done since S is semisimple). Now define $\overline{\phi}$ on $\overline{V} = \overline{V}' \oplus A \otimes_S U \otimes_S A$ by $\overline{\phi}'$ on $\overline{V}'$ and on u in U by $\phi(u) = s\theta(u)$. $\phi|_U : U \to (X, Y)$ has a canonical extension to an A-A-bimodule morphism $A \otimes_S U \otimes_S A \to (X, Y)$. This defines $\overline{\phi}$ completely; and $(\phi_0, \overline{\phi})$ gives a morphism in $\mathfrak{A}(X, Z)$ since $a\phi_0 - \phi_0 a = a\phi_0' - \phi_0' a = \overline{\phi}'(d(a)) = \overline{\phi}(d(a))$ as $d(a)$ is in $\overline{V}'$. ϕ is an isomorphism since ϕ_0 is an isomorphism, and it is easily checked that $\theta = (p, 0) \circ \phi$ i.e. $\theta \circ \phi^{-1} = (p, 0)$ as required.

Now suppose $\mathfrak{A}$ is a triangular tensor algebra extension of $\mathfrak{A}'$. Thus there is an S-S-subbimodule E of A such that the morphism

$$\otimes_{A'} A' \otimes_S E \otimes_S A' \to A$$

of A'-algebra induced by $E \subsetneq A$ is an isomorphism. The natural map $A \otimes' \overline{V}' \otimes' A \to \overline{V}$ is an isomorphism and $d(E) \subsetneq \overline{V}'$.

Given θ in (X, Y) a proper epi its restriction θ' in $\mathfrak{A}'(X, Y)$ is a proper epi. Thus, by induction, there is a left A'-module Z and an isomorphism ϕ' in $\mathfrak{A}'(X, Z)$ such that $\theta \circ (\phi')^{-1} = (p, 0)$ for some p in $\mathrm{Hom}_{A'}(Z, Y)$.

First we extend the action of A' on Z to a left A-module action. To do this let e in E act by the endomorphism

$$z \longmapsto \phi_0' \, e((\phi_0')^{-1}(z)) + \phi'(d(e))(\phi_0')^{-1}(z)$$

(note that $d(E) \subseteq \overline{V}'$). This gives an S-S-bimodule map $E \to (Z, Z)$, which then gives a canonical extension of the A'-action on Z to an $A = \otimes_{A'} A' \otimes_S E \otimes_S A'$ - action.

$\phi' = (\phi_0', \overline{\phi}')$ in $\mathfrak{A}'(X, Z)$ is now extended to $\phi = (\phi_0, \overline{\phi})$ in $\mathfrak{A}(X, Z)$ as follows:

Let $\phi_0 = \phi_0'$, and $\overline{\phi}: V \to (X, Z)$ the canonical extension of $\overline{\phi}' : \overline{V} \to (X, Z)$. It must now be verified that ϕ is indeed a morphism, i.e. for all a in A

$$a\phi_0 - \phi_0 a = \overline{\phi}(d(a)) \qquad (2.6.1).$$

If (2.6.1) holds for a, b in A it is easily seen that it holds for ab, $a + b$. Since $A' \cup E$ generates the ring A, and (2.6.1) holds for all a in A' it is sufficient to check that, for e in E,

$$e\phi_0 - \phi_0 e = \overline{\phi}(d(e))$$

i.e. that $e\phi_0 = \overline{\phi}(d(e)) + \phi_0 e$. This follows by the choice made for the action of e on Z.

Hence ϕ is morphism, and it is an isomorphism since ϕ_0 is an isomorphism.

Notice that the map p in $\mathrm{Hom}_{A'}(Z, Y)$ is in fact in $\mathrm{Hom}_A(Z, Y)$ as given z in Z and e in E we have

$$\begin{aligned} p(ez) &= p(\phi_0(e\phi_0{}^{-1}(z))) + p\bar{\phi}(d(e))(\phi_0{}^{-1}(z)) \\ &= \theta_0(e\phi_0{}^{-1}(z)) + \theta(d(e))\phi_0{}^{-1}(z) \\ &= e\theta_0\phi_0{}^{-1}(z) \\ &= ep(z). \end{aligned}$$

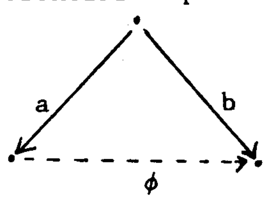
Now we claim $\theta \circ \phi^{-1} = (p, 0)$, i.e. $(p \;\; 0) \circ \phi = \theta$. This follows provided $\bar{\theta}(v) = p\bar{\phi}(v)$ for all v in $\bar{V}$. Since $\bar{V} \simeq A \otimes' \bar{V}' \otimes A$ it is enough to check this on $v = av'b$ (a, b in A, v' in V') which follows since

$$\begin{aligned} \bar{\theta}(av'b) &= a\bar{\theta}(v')b = a\bar{\theta}'(v')b = ap\bar{\phi}'(v')b \\ &= pa\bar{\phi}'(v')b = pa\bar{\phi}(v')b = p\phi(av'b). \end{aligned}$$

The results 2.5, 2.6 may now be combined with results in §1 to deduce further relations between the various concepts of epimorphism (monomorphism) introduced there, for suitably triangular bocses.

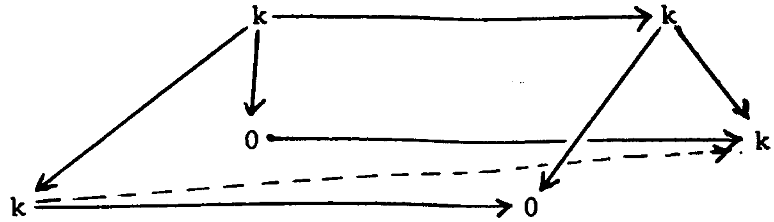
A final example shows that it is not always the case that these concepts are all equivalent:

Example 2.10 Let $\mathfrak{A}$ be the bocs given symbolically by the differential biquiver [C-B2: §3]



$d(b) = a\phi$, $d(a) = 0$, $d(\phi) = 0$.

$\mathfrak{A}$ is a strictly triangular bocs. Consider the morphism in f.d. (mod $\mathfrak{A}$) written symbolically as



in which every morphism $k \to k$ is the identity.

This morphism is neither a proper mono nor a proper epi, but it is both a categorical monomorphism and a categorical epimorphism.

Covering note on

ALMOST SPLIT SEQUENCES FOR BOCSES

by

W.L. Burt and M.C.R. Butler.

The appended copy of this article is in its final form as accepted for publication.

The main results of §§1-4 were already present in a handwritten manuscript of work by M.C.R. Butler dated winter 1988 and hereinafter referred to as 'Butler's manuscript'.

The entire theory in this article was in place by the end of September 1989.

The precise attribution of the work is as follows:

All the results in §§1-4 except 1.3, 2.4 are proved in Butler's manuscript; 1.3, 2.4 are due to the present writer who has also given the proofs of the results in §2 as presented in this paper - excepting the remarks on p.11 and the proof of Proposition 2.7.

§§3, 4 essentially reproduce the theory in Butler's manuscript.

§5 is due to the present writer although the identities $\mathrm{Ext}^i_L(V, V) = 0$, $\mathrm{Ext}^i_{R^{op}}(V, V) = 0$ for $i > 0$ were proved in Butler's manuscript.

§6 is Butler's simplified version of [BK: §5] - which has been further abridged by the present writer - who also distinguished between k and S, a distinction not present in the initial version, but essential for applications.

Excepting §6 the theory of this paper has been developed independently of work along these lines by Bautista and Kleiner [BK] and de la Peña and Simson [PS].

<u>Errata</u>

p.9, 2.4, line 1: 'R-mod' should be 'R-Mod'

p.11, 2.6, line 4: 'R-mod' and 'L-mod' should be interchanged;

p.22, Remarks (iii) line 2: 'projective' should be 'injective';

p.30, line 4 : 'd' should be 'd'';

p.32, line 6 '$\beta\zeta\beta^{-1}$' should be '$\overline{\beta\zeta\beta^{-1}}$';

p.32, line 15 : final $\bar{V}$ should be $\bar{V}'$.

Almost Split Sequences for Bocses

W.L. BURT AND M.C.R. BUTLER

This paper contains a different treatment, with some new results, of the Auslander-Reiten theory for bocses given by Bautista and Kleiner in [BK]. The basic idea in both papers is to realise the representations of a bocs within a larger category, then use criteria like those given by Auslander and Smalø in [AS] for a subcategory to possess almost split sequences. In [BK] bocs representations are realised as induced R-modules for a suitable algebra inclusion $A \to R$, and also as induced comodules over the underlying A-coalgebra V of the bocs (a point of view previously developed in [K]). We will also use the induced R-module realisation, but induced comodules will be replaced by coinduced L-modules for another algebra inclusion $A \to L$ associated with the bocs. We call L the left algebra and R the right algebra of the bocs. One new type of result we obtain concerns the relationship of L and R; Theorem 5.1 asserts that, for the most interesting bocses covered by our theory, V is a left L and right R cotilting bimodule in the sense of

Miyashita [M]. The other main new result is Theorem 3.8, where - for a large and important class of bocses - we show that the subcategories of R-mod and of L-mod which realise bocs representations are closed under extensions.

Throughout the paper the term bocs will mean an A-coalgebra V over an *algebra* A (subject to axioms stated at the end of this introduction) so differing at a formal level from standard accounts such as those of [R, D, C-B] in which A is a category. The main results apply to all 'additive Roiter bocses' [C-B], in which A is a finite-dimensional algebra over a field, hence also to the 'Drozd bocs' of a finite dimensional algebra (see [D, C-B] for details), and to bocses of finite representation type arising from many natural matrix problems. We do not, however, discuss bocses over orders, as is done in [D, BK].

§1 contains the formal definition of the left and right algebras, L and R, of a bocs $\mathcal{A} = (A, V)$, and exhibits a natural bimodule structure ${}_LV_R$ on V compatible with ${}_AV_A$. Through §§2-5, however, our main results require that ${}_AV$ and V_A are finitely generated projective modules; this ensures that ${}_AV$, R_A and V_A, ${}_AL$ are dual pairs of modules with respect to A. It then follows from Theorem 2.5 that the representation category $\mathcal{A}$-mod of $\mathcal{A}$ is equivalent to each of the subcategories $\underline{I}$ of induced modules in R-mod and $\underline{C}$ of coinduced modules in L-mod; and with some additional finiteness assumptions, Theorem 2.6 states that $\underline{I}$ and $\underline{C}$ are functorially finite subcategories of R-mod and L-mod, respectively. (As is well-known, functorial finiteness of, say, $\underline{I}$ in R-mod allows one to push down almost split morphisms from R-mod to $\underline{I}$, and this property is also one of the criteria in [AS]).

In §3 we study conditions under which $\underline{I}$ and $\underline{C}$ are closed under extensions in R-mod and L-mod, respectively. The crucial condition for this, in Theorem 3.8, is that the kernel $\overline{V}$ of the counit $\epsilon : V \to A$ be 'projectivising', in the sense that the functors $\overline{V}\otimes_A -$ and $- \otimes_A \overline{V}$ map all A-modules to projective A-modules. This is the case for the 'free finitely generated' kernels of bocses which occur in [R, D, C-B]. As explained at the beginning of §3, the result of Theorem 3.8 is in fact far stronger than closure of $\underline{I}$ and $\underline{C}$ under extensions; the latter, though, is the specific fact needed to use [AS].

In the brief §4 we extract from §§2,3 an existence theorem for almost split sequences, Theorem 4.1: if A and V are finite-dimensional over a central subfield k, and $\overline{V}$ is projectivising, then add($\underline{I}$) and add($\underline{C}$) admit almost split sequences; if, additionally, idempotents split in $\mathcal{A}$-mod, then $\underline{I}$, $\underline{C}$ and therefore $\mathcal{A}$-mod also admit almost split sequences.

In §5, we again extract from §§2, 3 most of the information needed to formulate and prove the cotilting result Theorem 5.1, though finiteness of coresolutions requires an assumption that ${}_AA$ and A_A be of finite injective dimension.

§6 deals with the rather different and technically complicated problem of characterising bocses for which idempotents split in their categories of finite-dimensional representations, a key criterion in both [BK] and our Theorem 4.1. As Roiter observed in [R], idempotent splitting seems to depend on suitable triangularity and freeness properties of the bocs, and Bautista and Kleiner have described in §5 of [BK] a class of 'left triangular tensor' coalgebras with such properties. In our §6, we adopt left-right symmetric conditions and use a strategy of step-wise construction, to give a re-organised account of this complicated topic.

Thanks are offered to the S.E.R.C. (U.K.) for funding a research studentship for the first-named author, and also a visiting fellowship enabling Raymundo Bautista to visit the University of Liverpool for four weeks in 1989; to Raymundo Bautista, to U.N.A.M. (Mexico), and to the British Council for enabling the second-named author to spend six weeks at U.N.A.M. Instituto de Matematicas in 1988; to the authors of [BK] for sending us a preprint of their paper, which has strongly influenced the work presented here; and finally special thanks to Wendy Orr for typing our manuscript.

Conventions. Bocses and their representations

Throughout the paper A denotes a fixed ring, associative, with identity, subject to further qualification for specific results. The symbol $\otimes$ always means $\otimes_A$. For rings B, C, the notations $_BX$, Y_C, $_BZ_C$ indicate that X, Y, Z are respectively, a left B-module, a right C-module, and a left B, right

C-bimodule (more briefly, a B-C-bimodule).

Given ${}_BX$, ${}_BX'$, $\mathrm{Hom}_B(X, X')$ denotes the group of left B-module morphisms $X \to X'$; likewise the group of right C-module morphisms $Y_C \to Y'_C$ is denoted $\mathrm{Hom}_{C^{op}}(Y, Y')$. Groups of B-C-bimodule morphisms are denoted $\mathrm{Hom}_{B\times C^{op}}(-,-)$.

Given ${}_AZ$, W_A we often abuse notation by writing $A \otimes Z = Z$, $W \otimes A = W$.

By a <u>bocs</u> $\mathcal{Q}$ we mean a quadruple $\mathcal{Q} = (A, V, \mu, \epsilon)$, often denoted just $\mathcal{Q} = (A, V)$, in which V is an A-A-bimodule; the <u>comultiplication</u> $\mu : V \to V \otimes V$ is an A-A-morphism and is coassociative, that is, $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$; the <u>counit</u> $\epsilon : V \to A$ is an A-A-morphism, $(1 \otimes \epsilon)\mu = 1_V = (\epsilon \otimes 1)\mu$; <u>and ϵ is assumed throughout to be surjective</u>.

In case A is an algebra over a central subring k, we also assume tacitly that k acts centrally on V.

The categories of (finitely generated) left [right] modules over any ring B will be denoted B-Mod (B-mod) [Mod-B (mod-B)].

Now let $\mathcal{Q} = (A, V)$ be a bocs. We define $\mathcal{Q}$-mod to be the category with objects the finitely generated left A-modules X, Y, ..., and morphism groups $\mathcal{Q}(X, Y)$ given by

$$\mathcal{Q}(X, Y) = \mathrm{Hom}_A(V \otimes X, Y),$$

(we use a variant of this definition in §6). The composition gf of $f \in \mathcal{Q}(X, Y)$ and $g \in \mathcal{Q}(Y, Z)$ is, as usual, given by the composite of the maps

$$V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y \xrightarrow{g} Z;$$

composition is associative, the identity of X is $\epsilon \otimes 1: V \otimes X \to X$, and in case A is a k-algebra then $\mathcal{C}$-mod is a k-category.

Finally let B be a category; recall that idempotents split in B if every idempotent endomorphism $e = e^2$ of every object X of B admits a factorisation $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $fg = 1_Y$. In particular if B is a full subcategory of a module category then idempotents split in add(B), by which we mean the full subcategory containing all modules isomorphic to direct summands of finite direct sums of objects in B.

§1 The left and right algebras of a bocs

Let $\mathcal{C} = (A, V, \mu, \epsilon)$ be a bocs. We define its left algebra L to be the A-A-bimodule $\mathrm{Hom}_{A^{op}}(V, A)$ with multiplication given by the following rule:

for $e, f \in L$, $e.f$ is the composite

$$V \xrightarrow{\mu} V \otimes V \xrightarrow{f\otimes 1} A \otimes V = V \xrightarrow{e} A.$$

We omit the purely formal verification of the following facts: this multiplication is associative, with the counit ϵ as identity, and $a\epsilon = \epsilon a$ for all $a \in A$; $(a\epsilon)(b\epsilon) = (ab)\epsilon$ for all $a, b \in A$; and $\forall\, a \in A,\ e, f \in L$, $(ea)f = e(af)$, $(a\epsilon)f = af$, and $e(a\epsilon) = ea$. To summarise, we obtain:

1.1 Proposition The bimodule ${}_AL_A$, equipped with the above multiplication $L \otimes L \to L$ and the map $a \longmapsto a\epsilon$ of A into L, is an A-algebra with identity ϵ, and the bimodule structure on L induced along the map $a \longmapsto a\epsilon$ coincides with the natural A-A-bimodule structure on L.

Similarly, the <u>right algebra</u> R of $\mathcal{A}$ is the A-A-bimodule $\mathrm{Hom}_A(V, A)$, with multiplication given as follows:

for $s, t \in R$, $s.t$ is the composite

$$V \xrightarrow{\mu} V \otimes V \xrightarrow{1\otimes s} V \otimes A = V \xrightarrow{t} A.$$

1.1′ <u>**Proposition**</u> The bimodule ${}_AR_A$, equipped with the above multiplication $R \otimes R \to R$ and the map $a \longmapsto a\epsilon$ of A into R, is an A-algebra with identity ϵ, and the bimodule structure on R induced along the map $a \longmapsto a\epsilon$ coincides with the natural A-A-bimodule structure on R.

<u>**Remark**</u> Note that, for $a \in A$, the map $a\epsilon : V \to A$ has different values according to whether it is viewed as an element of R or of L; for $v \in V$, $(a\epsilon)(v) = a\epsilon(v)$ for $a\epsilon \in L$, whereas $(a\epsilon)(v) = \epsilon(v)a$ for $a\epsilon \in R$.

To define actions of L and R on V, we first note that there are evaluation maps

$$E_L : L \otimes V \longrightarrow A \ ; \quad e \otimes v \longmapsto e(v),$$

$$E_R : V \otimes R \longrightarrow A \ ; \quad v \otimes s \longmapsto s(v).$$

Hence there is a left action of L on V, given by

$$L \otimes V \xrightarrow{1\otimes\mu} L \otimes V \otimes V \xrightarrow{E_L\otimes 1} A \otimes V = V,$$

and a right action of R on V, given by

$$V \otimes R \xrightarrow{\mu\otimes 1} V \otimes V \otimes R \xrightarrow{1\otimes E_R} V \otimes A = V.$$

Again we omit the entirely routine verification of the following statements:

1.2 **Proposition** The actions of L and R on V just defined induce an L-R-bimodule structure ${}_LV_R$ on V, compatible with the original bimodule structure ${}_AV_A$ after restriction along the maps $a \longmapsto a\epsilon$ of A into L and A into R.

1.3 **Corollary** $\mu : V \to V \otimes V$ is an L-R-bimodule morphism.

§2 Functorially finite imbeddings of $\mathcal{U}$-mod

Throughout §2, $\mathcal{U} = (A, V)$ is a bocs such that

2.1 <u>${}_AV$ and V_A are finitely generated projective modules.</u>

Therefore R_A and ${}_AL$ are finitely generated projectives, and the duality maps

$${}_AV_R \longrightarrow {}_A\mathrm{Hom}_{A^{op}}(R, A)_R \quad \text{and} \quad {}_LV_A \longrightarrow {}_L\mathrm{Hom}_A(L, A)_A$$

are bimodule isomorphisms. Also, given any left A-module X, there are natural maps

$\beta : R \otimes X \longrightarrow \mathrm{Hom}_A(V, X)\ ;\ s \otimes x \longmapsto (v \longmapsto s(v)x),$

$\alpha : V \otimes X \longrightarrow \mathrm{Hom}_A(L, X)\ ;\ v \otimes x \longmapsto (e \longmapsto e(v)x),$

and repeated use will be made of the following fact:

2.2 **Proposition** For any ${}_AX$, β is an isomorphism of R-modules and α an isomorphism of L - modules.

2.3 **Notation** Let $\underline{I}$ denote the full subcategory of R-mod with objects the induced modules - that is, modules ${}_RM$ isomorphic to $R \otimes X$ for some finitely generated ${}_AX$. Let $\underline{C}$ denote the full subcategory of L-mod with objects the coinduced modules - that is, modules ${}_LN$ isomorphic to $\mathrm{Hom}_A(L, X)$ for some finitely generated ${}_AX$.

Remark 2.1 will need to be supplemented for the proof in Theorem 2.6 that $\underline{I}$ is functorially finite in R-mod, and $\underline{C}$ is so in L-mod. We need to ensure that restriction from R to A maps R-mod into A-mod, and that $\mathrm{Hom}_L(V,-)$ maps L-mod into R-mod.

2.4 **Theorem** The functors $V \otimes_R - :$ R-mod $\to$ L-Mod and $\mathrm{Hom}_L(V,-)$: L-Mod $\to$ R-Mod restrict to mutually inverse equivalences between $\underline{I}$ and $\underline{C}$.

Proof These functors map $\underline{I}$ and $\underline{C}$ into one another, since $V \otimes_R (R \otimes X) \cong V \otimes X \cong \mathrm{Hom}_A(L, X)$ and $\mathrm{Hom}_L(V, \mathrm{Hom}_A(L, X)) \cong \mathrm{Hom}_A(V, X) \cong R \otimes X$. To prove that they induce equivalences, we consider the natural transformations

$$\nu : 1_{R\text{-Mod}} \longrightarrow \mathrm{Hom}_L(V, {}_L V \otimes_R -),$$

given on ${}_R M$ by $\nu_M(m)(v) = v \otimes_R m$ (for $m \in M$, $v \in V$), and

$$\sigma : V \otimes_R \mathrm{Hom}_L(V, -) \longrightarrow 1_{L\text{-Mod}},$$

given on ${}_L N$ by $\sigma_N(v \otimes_R h) = h(v)$ (for $v \in V$, $h \in \mathrm{Hom}_L(V, N)$).

For $M = R \otimes X \in \underline{I}$, ν_M is the composite of the chain of isomorphisms

$$R \otimes X \xrightarrow{\beta} \mathrm{Hom}_A(V, X) \longrightarrow \mathrm{Hom}_L(V, \mathrm{Hom}_A(L, X))$$
$$\xrightarrow{\alpha_*^{-1}} \mathrm{Hom}_L(V, V \otimes X) \longrightarrow \mathrm{Hom}_L(V, V \otimes_R (R \otimes X)),$$

and for $N = \mathrm{Hom}_A(L, X) \in \underline{C}$, σ_N is the composite of a second chain of isomorphisms,

$$V \otimes_R \mathrm{Hom}_L(V, \mathrm{Hom}_A(L, X)) \longrightarrow V \otimes_R \mathrm{Hom}_A(V, X)$$
$$\xrightarrow{1 \otimes_R \beta} V \otimes_R (R \otimes X) \longrightarrow V \otimes X \xrightarrow{\alpha} \mathrm{Hom}_A(L, X).$$

It follows that $(V \otimes_R -)|_{\underline{I}}$ and $\mathrm{Hom}_L(V, -)|_{\underline{C}}$ are mutually inverse equivalences.

2.5 <u>**The Imbedding Theorem**</u> There are equivalences of categories $F_C : \mathcal{A}\text{-mod} \to \underline{C}$ and $F_I : \mathcal{A}\text{-mod} \to \underline{I}$.

<u>**Proof**</u> We construct F_C explicitly, then use the previous theorem and define $F_I = \mathrm{Hom}_L(V, -)|_{\underline{C}} \circ F_C$. On an object $X = {}_AX$ in $\mathcal{A}$-mod, we let $F_C(X) = V \otimes X$, which by Proposition 2.2 is in $\underline{C}$. On a morphism $f \in \mathcal{A}(X, Y) = \mathrm{Hom}_A(V \otimes X, Y)$, we let $F_C(f)$ be the composite of the chain of maps $V \otimes X \xrightarrow{\mu\otimes 1} V \otimes V \otimes X \xrightarrow{1\otimes f} V \otimes Y$; this is an L-morphism by 1.3. It is routine to verify that $F_C : \mathcal{A}\text{-mod} \to \underline{C}$ is a functor and is dense, so we need only show that it is full and faithful. Define

$$G : \mathrm{Hom}_L(V \otimes X,\ V \otimes Y) \to \mathcal{A}(X, Y)\ ;\ f' \longmapsto (\epsilon \otimes 1) \circ f'.$$

Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_L(\mathrm{Hom}_A(L, X), \mathrm{Hom}_A(L, Y)) \arrow[r, "\text{corestrict}"] & \mathrm{Hom}_A(\mathrm{Hom}_A(L, X), Y) \arrow[dd, "\cong"', "\alpha^*"] \\
\mathrm{Hom}_L(\mathrm{Hom}_A(L, X), V\otimes Y) \arrow[u, "\cong", "\alpha_*"'] \arrow[d, "\cong"', "\alpha^*"] & \\
\mathrm{Hom}_L(V \otimes X, V \otimes Y) \arrow[r, "G"'] & \mathrm{Hom}_A(V \otimes X, Y) = \mathcal{A}(X, Y)
\end{tikzcd}$$

in which all maps but G are isomorphisms, so G is also so. However $GF_C(f) = f$, from which it follows that F_C is full and faithful.

In view of 2.5, the next result exhibits $\mathcal{Q}$-mod as a full functorially finite subcategory of each of R-mod and L-mod, provided some finiteness conditions on $\mathcal{Q}$ hold.

2.6 <u>Functorial Finiteness Theorem</u> Let $\mathcal{Q} = (A, V)$ be a bocs (satisfying 2.1) such that (a) $_AR$ is finitely generated, (b) $\mathrm{Hom}_L(V, -)$ maps L-mod to R-mod. Then $\underline{C}$ and $\underline{I}$ are functorially finite in R-mod and L-mod respectively.

Remark (a) and (b) can be replaced by (a) and (b') : A is left noetherian. For $\mathrm{Hom}_L(V, N) \subset \mathrm{Hom}_A(V, N) \cong R \otimes N$, which is in A-mod provided $N \in$ L-mod and (a),(b') hold; hence $\mathrm{Hom}_L(V, N)$ is also in R-mod.

Proof (i) $\underline{C}$ is covariantly finite in L-mod. Neither (a) nor (b) is needed for the well-known proof that for each $N \in$ L-mod, the covariant functor $\mathrm{Hom}_L(N, -)|_{\underline{C}}$ is a finitely generated functor on $\underline{C}$. By 2.2, $\mathrm{Hom}_A(L, N) \cong V \otimes N \in$ L-mod. So the map $N \to \mathrm{Hom}_A(L, N)$; $n \mapsto (e \mapsto en)$, induces a surjective functor morphism

$$\mathrm{Hom}_L(\mathrm{Hom}_A(L, N), -)|_{\underline{C}} \twoheadrightarrow \mathrm{Hom}_L(N, -)|_{\underline{C}}$$

from a <u>representable</u> functor on $\underline{C}$, as finite generation of $\mathrm{Hom}_L(N, -)|_{\underline{C}}$ requires.

(ii) $\underline{I}$ is contravariantly finite in R-mod. We only need (a). Then for any M in R-mod, $R\otimes M$ is in R-mod. Let $g : R \otimes M \to M$ be the map $s \otimes m \longmapsto sm$. Then the functor morphism induced by g,

$$\mathrm{Hom}_R(-, R\otimes M)|_{\underline{I}} \longrightarrow \mathrm{Hom}_R(-, M)|_{I},$$

is well-known to be surjective. Again using (a) to see that $R \otimes M \in \underline{I}$, we see that the contravariant functor $\mathrm{Hom}_R(-, M)|_{\underline{I}}$ is finitely generated.

(iii) $\underline{I}$ is covariantly finite in R-mod. We need neither (a) nor (b). For any R-module M, consider the map

$$\begin{array}{ll} h : M \longrightarrow \mathrm{Hom}_A(V, V \otimes_R M); & m \longmapsto (v \longmapsto v \otimes_R m) \\ \qquad\qquad \Vert\wr & \\ \qquad\qquad R \otimes V \otimes_R M & \text{by } (2.2) \end{array}$$

For $M \in$ R-mod, finite generation of ${}_A V$ implies that $R \otimes V \otimes_R M \in$ R-mod, in fact, that $R \otimes V \otimes_R M \in \underline{I}$. We show now that h actually induces a surjection of covariant functors

$$\mathrm{Hom}_R(\mathrm{Hom}_A(V, V \otimes_R M), -)|_{\underline{I}} \xrightarrow{h^*} \mathrm{Hom}_R(M, -)|_{\underline{I}},$$

which is the required result. By 2.2, modules in $\underline{I}$ are of the form $\mathrm{Hom}_A(V, X)$, where $X \in$ A-mod. Let $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_A(V, X))$, then $f = \tilde{f}_* \circ h = h^*(\tilde{f}_*)$, where $\tilde{f}(v \otimes_R m) = f(m)(v)$ and $\tilde{f}_* = \mathrm{Hom}_A(1_V, \tilde{f})$. Hence h^* is surjective.

(iv) $\underline{C}$ is contravariantly finite in L-mod. We need (a) and (b). For an L-module N, consider the map

$$l : \tilde{N} = V \otimes \mathrm{Hom}_L(V, N) \longrightarrow N; \; v \otimes k \longmapsto k(v),$$

where $\tilde{N} \cong \mathrm{Hom}_A(L, \mathrm{Hom}_L(V, N))$ by 2.2.

The morphism l_* of covariant functors induced by l,

$$\mathrm{Hom}_L(-, \tilde{N})|_{\underline{C}} \longrightarrow \mathrm{Hom}_L(-, N)|_{\underline{C}},$$

is surjective, for taking, by 2.2, a typical object of $\underline{C}$ to have the form $V \otimes X$, $X \in$ A-mod, any L-morphism $g : V \otimes X \to N$ lifts over l_* to $1_V \otimes \tilde{g}$, where

$\tilde{g} : X \to \mathrm{Hom}_L(V, N)$ is given by $\tilde{g}(x)(v) = g(v \otimes x)$. So to prove the contravariant finiteness of $\underline{C}$ in L-mod it is only necessary to check that, for $N \in$ L-mod, we have $\tilde{N} \in \underline{C}$. By (b) $\mathrm{Hom}_L(V, N) \in$ R-mod, so by (a) ${}_A\mathrm{Hom}_L(V, N) \in$ A-mod. Hence $\tilde{N} = V \otimes \mathrm{Hom}_L(V, N)$ is in $\underline{C}$ as required.

This completes the proof of theorem 2.6.

We conclude this section with the observation that the bimodule ${}_LV_R$ is 'balanced'.

2.7 **Proposition** The natural ring homomorphisms

$$L \to \mathrm{End}_{R^{op}}(V) \quad ; \quad s \longmapsto (v \longmapsto sv) \ ,$$

$$R^{op} \to \mathrm{End}_L(V) \quad ; \quad e \longmapsto (v \longmapsto ve) \ ,$$

are isomorphisms.

Proof $L \to \mathrm{End}_{R^{op}}(V)$ is the composite of the isomorphisms

$$L = \mathrm{Hom}_{A^{op}}(V, A) \cong \mathrm{Hom}_{R^{op}}(V, \mathrm{Hom}_{A^{op}}(R, A)) \cong \mathrm{End}_{R^{op}}(V),$$

and likewise

$R^{op} \to \mathrm{End}_L(V)$ is the composite of the isomorphims

$$R^{op} = \mathrm{Hom}_A(V, A) \cong \mathrm{Hom}_L(V, \mathrm{Hom}_A(L, A)) \cong \mathrm{End}_L(V).$$

Remark $R^{op} \to \mathrm{End}_L(V)$ is also realised by the map ν_R in the proof of 2.4; an analogue of 2.4 for mod-R, mod-L would include the corresponding statement about $L \to \mathrm{End}_{R^{op}}(V)$.

§3 Extension Properties

In Theorem 3.8, we give conditions on a bocs $\mathcal{U} = (A, V)$ which ensure that the subcategories $\underline{I}$ of R-mod and $\underline{C}$ of L-mod are closed under extensions. The conditions imply 2.1, so that the functors $R \otimes -$ and

$\mathrm{Hom}_A(L, -)$ are exact on A-mod, and we will in fact obtain the much stronger result that the maps induced by these functors from Ext_A^n to $\mathrm{Ext}_R^n|_{\underline{I}\times\underline{I}}$ and to $\mathrm{Ext}_L^n|_{\underline{C}\times\underline{C}}$ are surjective for $n = 1$, and bijective for $n \geq 2$.

We need some generalities about induction and coinduction, starting with the following well-known lemma of Eckmann and Shapiro.

3.1 __Proposition__ (a) Let R be an A-algebra such that R_A is projective. Given ${}_AZ$, ${}_RM$, the induction isomorphism

$$\Gamma : \mathrm{Hom}_A(Z, M) \xrightarrow{\sim} \mathrm{Hom}_R(R \otimes Z, M)$$

induces isomorphisms

$$\Gamma^{(n)} : \mathrm{Ext}_A^n(Z, M) \xrightarrow{\sim} \mathrm{Ext}_R^n(R \otimes Z, M)$$

for all $n \geq 0$.

(b) Let L be an A-algebra such that ${}_AL$ is projective.

Given ${}_AX$, ${}_LN$, the coinduction isomorphism

$$\Delta : \mathrm{Hom}_A(N, X) \xrightarrow{\sim} \mathrm{Hom}_L(N, \mathrm{Hom}_A(L, X))$$

induces isomorphisms

$$\Delta^{(n)} : \mathrm{Ext}_A^n(N, X) \xrightarrow{\sim} \mathrm{Ext}_L^n(N, \mathrm{Hom}_A(L, X))$$

for all $n \geq 0$.

3.2 __Notation__ For A-modules ${}_AZ$, ${}_AX$, there are natural transformations

$$\gamma : {}_AX \longrightarrow {}_AR \otimes X \quad ; x \longmapsto 1 \otimes x,$$

$$\delta : \mathrm{Hom}_A(L, Z) \longrightarrow Z \quad ; h \longmapsto h(1),$$

inducing n-tural transformations

$$\gamma^{(n)} : \mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_A(Z, R \otimes X),$$

$$\delta^{(n)} : \mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_A(\mathrm{Hom}_A(L, Z), X)$$

for each $n \geq 0$.

We give conditions on the bocs $\mathcal{A} = (A, V)$ of which L, R are the left, right algebra (respectively) to ensure that $\gamma^{(1)}$, $\delta^{(1)}$ are surjective and $\gamma^{(n)}$, $\delta^{(n)}$ bijective for $n \geq 2$.

3.3 __Definition__ The __kernel__ $\overline{V} = {}_A\overline{V}_A$ __of a bocs__ $\mathcal{A} = (A, V)$ is the kernel of the counit ϵ of $\mathcal{A}$. Since ϵ is surjective, the exact sequence of bimodules,

$$0 \longrightarrow \overline{V} \longrightarrow V \xrightarrow{\epsilon} A \longrightarrow 0,$$

splits as a sequence of left A-modules, and also as a sequence of right A-modules. So, for all ${}_AZ$, ${}_AX$, there are exact sequences of left A-modules

3.3 (a) $0 \longrightarrow X \xrightarrow{\epsilon^*} \mathrm{Hom}_A(V, X) \longrightarrow \mathrm{Hom}_A(\overline{V}, X) \longrightarrow 0$

3.3 (b) $0 \longrightarrow \overline{V} \otimes Z \longrightarrow V \otimes Z \xrightarrow{\epsilon\otimes 1} Z \longrightarrow 0.$

In fact, ϵ^* and $\epsilon \otimes 1$ are given by

$$\epsilon^* = \beta\gamma \ , \ \epsilon \otimes 1 = \delta\alpha$$

with β, α as defined in §2, and γ, δ as in 3.2.

3.4 __Proposition__ Let $\overline{V}$ be the kernel of the bocs $\mathcal{A} = (A, V)$, and R and L the right algebra and left algebra, respectively, of $\mathcal{A}$.

(a) Assume ${}_AV$ is finitely generated projective, and X a left A-module such that ${}_A\mathrm{Hom}_A(\overline{V}, X)$ is injective. Then, for all ${}_AZ$, $\gamma^{(1)}$ is surjective and $\gamma^{(n)}$ is bijective for all $n \geq 2$.

(b) Assume V_A is finitely generated projective and Z a left A-module such that ${}_A\overline{V} \otimes Z$ is projective. Then, for all ${}_AX$, $\delta^{(1)}$ is surjective and

$\delta^{(n)}$ is bijective for all $n \geq 2$.

<u>**Proof**</u> (a) Since ${}_A V$ is finitely generated projective, Proposition 2.2 shows that β is an isomorphism, so $\gamma = \beta^{-1}\epsilon^{*}$. Hence it suffices to prove the result for the $\epsilon^{*(n)}$. The last term of 3.3 (a) is injective, so the long exact sequence obtained by applying $\mathrm{Hom}_A(Z, -)$ to 3.3 (a) shows that $\epsilon^{*(1)}$ is surjective and $\epsilon^{*(n)}$ bijective for $n \geq 2$, as required.

(b) In this case, the hypotheses imply that α is an isomorphism, so $\delta = (\epsilon \otimes 1)\ \alpha^{-1}$; the result follows by applying $\mathrm{Hom}_A(-, X)$ to 3.3 (b), in which the first term is, by hypothesis, projective.

We now want to combine the results of 3.1 and 3.4. On taking $M = R \otimes X$ in 3.4 (a) and $N = \mathrm{Hom}_A(L, Z)$ in 3.4(b), we obtain <u>at degree</u> 0, maps

$$\Gamma\gamma^{(0)} : \mathrm{Hom}_A(Z, X) \longrightarrow \mathrm{Hom}_R(R \otimes Z, R \otimes X)$$

$$\Delta\delta^{(0)} : \mathrm{Hom}_A(Z, X) \longrightarrow \mathrm{Hom}_L(\mathrm{Hom}_A(L, Z), \mathrm{Hom}_A(L, X))$$

which are easily verified to be given by

$$(\Gamma\gamma)(h) = 1_R \otimes h \quad , \quad (\Delta\delta)(h) = \mathrm{Hom}_A(1_L, h)$$

for $h \in \mathrm{Hom}_A(Z, X)$. Of course 3.1 gives conditions under which these composites induce maps on Ext^n - groups, and 3.4 gives further conditions under which the induced maps are surjective when $n = 1$, and bijective for $n \geq 2$. We can, however, describe these induced maps more explicitly using Yoneda's interpretation of Ext^n, $n \geq 1$, as a group of equivalence classes of 'n-fold extensions'- by which we mean exact sequences of the form

$$E_* : 0 \to U_n \to U_{n-1} \to \dots \to U_1 \to U_0 \to U \to 0.$$

3.5 **Corollary** With $\mathcal{A}$, L, R, as in Proposition 3.4

(a) assume ${}_AV$ is finitely generated projective and ${}_AX$ is a module for which ${}_A\mathrm{Hom}_A(\overline{V}, X)$ is injective. Then R_A is projective, the 'induction' maps

$$\Gamma\gamma : \mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_R(R \otimes Z, R \otimes X)$$

are defined for all ${}_AZ$ and all $n \geq 1$, are surjective for $n = 1$ and bijective for $n \geq 2$, and are induced by the map $E_* \longmapsto R \otimes E_*$ of n-fold extensions E_*.

(b) assume V_A is finitely generated projective and ${}_AZ$ is a module such that ${}_A\overline{V} \otimes Z$ is projective. Then ${}_AL$ is projective, the 'coinduction' maps

$$\Delta\delta : \mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_L(\mathrm{Hom}_A(L, Z), \mathrm{Hom}_A(L, X))$$

are defined for all ${}_AX$ and all $n \geq 1$, are surjective for $n = 1$ and bijective for $n \geq 2$, and are induced by the map $E_* \longmapsto \mathrm{Hom}_A(L, E_*)$ of n-fold extensions E_*.

Proof All but the last assertion in each part follows immediately from 3.1 and 3.4. For (a), let E_* be an n-fold extension of X by Z representing some element $\varsigma \in \mathrm{Ext}^n_A(Z, X)$; we must show that $R \otimes E_*$ represents $(\Gamma\gamma)(\varsigma)$. Choose a projective resolution (P_*, d_*) of ${}_AZ$, then there is a commutative diagram

$$\begin{tikzcd}
\dots P_{n+1} \arrow[r, "d_n"] \arrow[d] & P_n \arrow[r, "d_{n-1}"] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d] & \dots\dots \arrow[r] & P_0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & Y_{n-1} \arrow[r] & \dots\dots \arrow[r] & Y_0 \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

in which the lower row is E_* and the cocycle f_n represents ζ. Since R_A is projective, $R \otimes -$ is exact and $R \otimes P_*$ is an R-projective resolution of $R \otimes Z$, so on applying $R \otimes -$ to the above diagram, we see that the n-fold extension $R \otimes E_*$ is a Yoneda representative of the element of $\mathrm{Ext}^n_R(R \otimes Z, R \otimes X)$ determined by the cocycle $1_R \otimes f_n$. Since $1_R \otimes f_n = (\Gamma\gamma)(f_n)$, this element is $(\Gamma\gamma)(\zeta)$, as required. The proof of (b) is similar, using an injective resolution of ${}_AX$.

For the application to bocses in Theorem 3.8 below, we will symmetrise the hypotheses given separately in (a) and (b) of the preceding proposition. We begin with a definition:

Definition An A-A-bimodule U will be called a projectivising bimodule if, for all ${}_AZ$, W_A, the A-modules $U \otimes Z$ and $W \otimes U$ are projective.

3.6 **Lemma** Let ${}_AU_A$ be a bimodule. It is projectivising if and only if, for all ${}_AX$, Y_A, $\mathrm{Hom}_A(U,X)$ and $\mathrm{Hom}_{A^{op}}(U, Y)$are injective.

Proof We just sketch the 'only if' part. If ${}_AU_A$ is projectivising ${}_AU$ and U_A are projective, so $- \otimes U$ and $U \otimes -$ are exact. For any short exact sequence ${}_AZ_*$ and module ${}_AX$, there is an isomorphism

$$\mathrm{Hom}_A(Z_*, \mathrm{Hom}_A(U, X)) \cong \mathrm{Hom}_A(U \otimes Z_*, X).$$

Since U_A is projective, $U \otimes Z_*$ is exact, and split exact since U is projectivising. Hence, for all X and Z_*, the left hand side is exact, and so for each fixed ${}_AX$, $\mathrm{Hom}_A(U, X)$ is injective. Similarly, $\mathrm{Hom}_{A^{op}}(U, Y)$ is injective for each Y_A.

3.7 **Examples** For any algebra A over a semisimple artinian subring k, $A \otimes_k A$ is projectivising, as also is any A-A-bimodule summand of any direct sum of copies of $A \otimes_k A$.

3.8 **Extension Closure Theorem**

Let $\mathfrak{A} = (A, V)$ be a bocs such that ${}_AV$ and V_A are finitely generated, and ${}_A\bar{V}_A$ is a projectivising bimodule. Then for all ${}_AX$ and ${}_AZ$, the maps

$$\mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_R(R \otimes Z, R \otimes X)$$

$$\mathrm{Ext}^n_A(Z, X) \longrightarrow \mathrm{Ext}^n_L(\mathrm{Hom}_A(L, Z), \mathrm{Hom}_A(L, X))$$

induced by induction and coinduction exist, and are surjective for $n = 1$ and bijective for $n \geq 2$. Moreover, the subcategories $\underline{I}$ of R-mod and $\underline{C}$ of L-mod are closed under extensions.

Proof We have already noted that ${}_A\bar{V}$, $\bar{V}_A$ are projective, so ${}_AV$ and V_A are finitely generated projective modules. Hence all conditions of 3.5 hold, proving the assertion about the mappings of Ext -groups. We prove that $\underline{I}$ is closed under extensions.
Let $0 \to R \otimes X \to M \to R \otimes Z \to 0$ be a short exact sequence of R-modules with $X, Z \in A$-mod. Then 3.5(a) for $n = 1$ shows that there is a short exact sequence
$0 \to X \to Y \to Z \to 0$ in A-mod and an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R \otimes X & \longrightarrow & R \otimes Y & \longrightarrow & R \otimes Z & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & R \otimes X & \longrightarrow & M & \longrightarrow & R \otimes Z & \longrightarrow & 0
\end{array}$$

Hence $M \cong R \otimes Y$, so $M \in \underline{I}$. The proof for $\underline{C}$ in L-mod is similar.

§4 Almost split sequences

Let k be a field, and $\mathcal{C} = (A, V)$ a finite-dimensional oocs over k, by which we mean that A is a finite-dimensional k – algebra and $\dim_k(V) < \infty$. Then L and R are also finite-dimensional k-algebras. We now briefly give conditions for the existence of almost split sequences in $\mathcal{C}$-mod, $\underline{I}$, and $\underline{C}$, using the Auslander-Smalø criteria [AS:2.4].

4.1 **Theorem** Let $\mathcal{C} = (A, V)$ be a finite-dimensional bocs over a central subfield k, with kernel $\bar{V}$ a projectivising A-A-bimodule. Then add($\underline{I}$) and add($\underline{C}$) admit almost split sequences. If, additionally, idempotents split in $\mathcal{C}$-mod, then $\mathcal{C}$-mod, $\underline{I}$ and $\underline{C}$ admit almost split sequences.

Proof We just prove the assertions pertaining to the subcategory $\underline{I}$ of R-mod, those pertaining to $\underline{C}$ being similar. Since R is a finite-dimensional algebra, we must verify - following [AS:2.4] - that add($\underline{I}$) is a full subcategory of R-mod, closed under isomorphisms, direct summands, and extensions in R-mod, and is functorially finite in R-mod.

Since $\bar{V}$ is projectivising and V finite-dimensional, (2.1) holds. Therefore all conditions of Theorems 2.6 and 3.8 are satisfied, so that $\underline{I}$ - and hence add($\underline{I}$)- is functorially finite in R-mod and is closed under extensions. The other Auslander-Smalø requirements are, of course, true by definition of $\underline{I}$ and add($\underline{I}$). Therefore add($\underline{I}$) admits almost split sequences. If idempotents split in $\mathcal{C}$-mod, then idempotents split in $\underline{I}$, so since $\underline{I}$ is full in R-mod, $\mathcal{C}\text{-mod} \cong \underline{I} = \text{add}(\underline{I})$.

4.2 **Remarks** (i) $\bar{V} \in \text{add}\ (A \otimes_k A)$ is the most frequently occurring type of projectivising kernel of a bocs.

(ii) the R-modules in add($\underline{I}$) are often called relatively projective modules, the L-modules in add($\underline{C}$) are similarly called relatively injective modules ; they are the (finitely generated) projectives and injectives for the relative homological algebras based on short exact sequences in R-mod and L-mod, respectively, which split on restriction to A.

(iii) The entirely different sort of problem of describing bocses $\mathcal{A}$ for which idempotents split in $\mathcal{A}$-mod is the subject of §6.

§5 Cotilting Properties

We assume here that $\mathcal{A} = (A, V)$ is a bocs in which A is an artin algebra and ${}_AV$, V_A are finitely generated projective.

Let $d = \max\ (\text{i.d.}\ {}_AA,\ \text{i.d.}\ A_A)$ where i.d. denotes injective dimension. When $d < \infty$ it is well known that $\text{i.d.}\ {}_AA = d = \text{i.d.}\ A_A$.

Note that L, R are also artin algebras; we shall denote the usual duality associated with any of the artin algebras A, L, R by D. Our main result is as follows:

5.1 **Cotilting Theorem** Under the above hypotheses and assuming that $d < \infty$, ${}_LV$, V_R are generalised cotilting modules of i.d. at most d.

Proof We must show that ${}_LV$ satisfies

(i) $\text{i.d.}\ {}_LV \leq d$

(ii) $\text{Ext}^i_L(V, V) = 0 \quad (i \geq 1)$

(iii) There is an exact sequence

$$0 \longrightarrow V_d \longrightarrow \dots \longrightarrow V_0 \longrightarrow DL \longrightarrow 0$$

with $V_i \in \operatorname{add} V$.

Consider the isomorphism of 3.1(b)

$$\operatorname{Ext}^i_A(N, X) \xrightarrow{\sim} \operatorname{Ext}^i_L(N, \operatorname{Hom}_A(L, X))$$

To prove (i) put $X = A$; since ${}_L\operatorname{Hom}_A(L, A) \cong {}_LV$, this shows that i.d. ${}_LV \leq$ i.d. ${}_AA = d$; (ii) follows by putting $N = {}_LV$, which is projective as a left A-module, and $X = A$.

Since i.d. $A_A = d$ there is a projective resolution

$$0 \longrightarrow P_d \longrightarrow \dots \longrightarrow P_0 \longrightarrow {}_A(DA) \longrightarrow 0$$

of DA. Applying $\operatorname{Hom}_A(L, -)$ to this yields a resolution of $\operatorname{Hom}_A(L, DA) \cong DL$ in $\operatorname{add}(\operatorname{Hom}_A(L, A))$ which is $\operatorname{add}({}_LV)$. This proves (iii).

The proof of the corresponding result for V_R is similar, using the analogue of 3.1(b) for right R and right A-modules, and a projective resolution of DA as a right A-module.

Remarks (i) When $\bar{V}$ is a projectivising bimodule we have (for $i \geq 2$):

$$\operatorname{Ext}^i_A(Z, X) \xrightarrow{\sim} \operatorname{Ext}^i_L(\operatorname{Hom}_A(L, Z), \operatorname{Hom}_A(L, X))$$

so putting $X = A$ and $i = d$ shows that, if $d \geq 2$, ${}_LV$ has i.d. equal to d.

(ii) Since $R \cong \operatorname{End}_L(V)^{op}$ the fact that V_R is a cotilting module also follows at once from ${}_LV$ being a cotilting module.

(iii) In applications A is usually hereditary and $\bar{V}$ projectivising so ${}_LV$, V_R are either projective modules, or classical cotilting modules.

We determine here the Ext-projective and Ext-injective modules in $\underline{I}$ and $\underline{C}$. Full details will be given for $\underline{I}$ and the proof for $\underline{C}$ (which is similar) will be omitted.

Recall that $M_1(M_3)$ in $\underline{I}$ is Ext-injective (Ext-projective) if every short exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ with each $M_i \in \underline{I}$ is split.

5.2 **Theorem** Suppose idempotents split in $\mathcal{A}$-mod and $\bar{V}$ is a projectivising bimodule; then the Ext-projectives in $\underline{I}$ are the finitely-generated projective R-modules and the Ext-injectives in $\underline{I}$ are the objects of add(DV).

Proof Since $\bar{V}$ is projectivising there is a surjection

$$\mathrm{Ext}^1_A(Z, X) \longrightarrow \mathrm{Ext}^1_R(R \otimes Z, R \otimes X)$$

Putting $X = DA$, so $R \otimes X \cong DV$, we see that all the objects of add(DV) are Ext-injective. The finitely generated projective R-modules are obviously Ext-projectives in $\underline{I}$.

If $M = R \otimes Z$ is Ext-projective consider a short exact sequence

$$0 \longrightarrow K \longrightarrow A^r \longrightarrow Z \longrightarrow 0.$$

Applying $R \otimes -$ yields a sequence in $\underline{I}$ with middle term in add(R). As $R \otimes Z$ is Ext-projective the induced sequence splits and so M is in add(R).

Likewise considering a short exact sequence

$$0 \longrightarrow X \longrightarrow (DA)^r \longrightarrow C \longrightarrow 0$$

and applying $R \otimes -$ shows that if $R \otimes X$ is Ext-injective then it is in add(DV).

5.3 **Theorem** Suppose idempotents split in $\mathcal{Q}$-mod and $\bar{V}$ is a projectivising bimodule; then the Ext-projectives in $\underline{C}$ are the objects of add(V) and the Ext-injectives in $\underline{C}$ are the finitely generated injective L-modules.

Remark Since L and R are cotilted from each other the number of isomorphism classes of simple left L-modules and simple left R-modules are equal by the analogue of [M : 1.19] for cotilting modules. If this number is s then s is finite and is also equal to the number of isomorphism classes of Ext-projective or Ext-injective indecomposables in $\underline{I}$ or $\underline{C}$, provided $\mathcal{Q}$ satisfies the hypotheses needed for Theorems 5.2 and 5.3.

§6 Splitting of Idempotents

In 6.4 we give sufficient conditions on bocses over fields for idempotents to split in their categories of finite-dimensional representations. The construction of such bocses starts with a so-called principal bocs and proceeds via finite sequences of elementary extensions of two basic types, which we call triangular kernel and triangular tensor algebra extensions. The bocses so obtained include all 'additive Roiter bocses' [CB: 3.5].

As mentioned before, the construction is essentially a different presentation of the material relevant to bocses over fields in [BK, §5]. As in that account, some rather extensive preliminaries are needed before the main result can be stated and proved.

Throughout §6 we keep fixed a basefield k and a commutative k-algebra S (S is an analogue of the underlying minimal category of the bocses considered by

Crawley-Boevey [C-B: 2.1, 3.6]). Each algebra considered in §6 is an S-algebra on which k acts centrally, but these algebras need not have finite k-dimension. As usual we only consider bimodules on which k acts centrally. More fundamentally we also assume that each bocs $\mathcal{A} = (A, V)$ possesses a <u>grouplike element centralised by S</u>, that is, an element ω of V such that

$$\epsilon(\omega) = 1, \ \mu(\omega) = \omega \otimes \omega, \ s\omega = \omega s \quad \text{for} \quad s \in S.$$

Hence V has direct decompositions

$${}_A V_S = {}_A \overline{V}_S + A\omega, \quad {}_S V_A = {}_S \overline{V}_A + \omega A,$$

and ${}_S V \otimes V_S$ a direct decomposition

$$V \otimes V = \overline{V} \otimes \overline{V} + \overline{V} \otimes \omega A + \omega A \otimes \overline{V} + \omega A \otimes \omega;$$

(in which we have identified $\overline{V} \otimes \overline{V}$ with its canonical image in $V \otimes V$).

We also need the <u>differentials</u> [C-B: 3.3] $d_1 : A \to \overline{V}$, $d_2 : \overline{V} \to \overline{V} \otimes \overline{V}$ defined by:

$$d_1(a) = a\omega - \omega a,$$

$$d_2(v) = \mu(v) - v \otimes \omega - \omega \otimes v,$$

which one may check do indeed have images inside $\overline{V}$ and $\overline{V} \otimes \overline{V}$ [C-B : 3.4(2)]. Often we denote either differential by d; both differentials are S-S-bimodule morphisms.

We will study only representations $X = {}_A X$ of of finite k-dimension; these determine a full subcategory, to be denoted $\mathcal{A}$-k-mod, of $\mathcal{A}$-mod. For any ${}_A X$, ${}_A Y$ write (X, Y) for $\mathrm{Hom}_k(X, Y)$. Then (X, Y) is an A-A-bimodule, and throughout this section we will take $\mathcal{A}(X, Y)$ to be a natural transpose of $\mathrm{Hom}_A(V \otimes X, Y)$, namely,

$$\mathcal{C}(X, Y) = \mathrm{Hom}_{A \times A^{op}}(V, (X, Y)).$$

In this form the composite of $\theta \in \mathcal{C}(X, Y)$ with $\varphi \in \mathcal{C}(Y, Z)$ is the morphism

$$V \xrightarrow{\mu} V \otimes V \xrightarrow{\varphi\otimes\theta} (Y, Z) \otimes (X, Y) \xrightarrow{c} (X, Z),$$

where c is composition of k-linear maps.

We also need a description of $\theta \in \mathcal{C}(X, Y)$ as a pair $\theta = (\theta_0, \bar{\theta})$ where $\theta_0 = \theta(\omega) \in (X, Y)$ and $\bar{\theta} = \theta|_{\bar{V}}$. Since $a\omega - \omega a = d(a)$, θ_0 and $\bar{\theta}$ are related by

$$a\theta_0 - \theta_0 a = \bar{\theta}(d(a)) \quad \text{for every} \quad a \in A.$$

Conversely any pair $(\theta_0, \bar{\theta})$ satisfying this relation determines a bocs morphism, and the correspondence set up in this way is bijective. Note that θ_0 is S-linear.

The formula for the composite of $\theta = (\theta_0, \bar{\theta}) \in \mathcal{C}(X, Y)$ and $\varphi = (\varphi_0, \bar{\varphi}) \in \mathcal{C}(Y, Z)$ is given by

$$(\varphi\theta)_0 = \varphi_0\theta_0,$$

$$\overline{\varphi\theta}(v) = \varphi_0\bar{\theta}(v) + \bar{\varphi}(v)\theta_0 + c(\bar{\varphi} \otimes \bar{\theta})\, d(v) \quad \text{for} \quad v \in \bar{V};$$

the identity morphism in $\mathcal{C}(X, X)$ is $\epsilon_X = (1_X, 0)$.

These formulae show that there is a functor, denoted ω^*,

$$\omega^* : \mathcal{C}\text{-k-mod} \to \text{k-mod}$$

given by $X \to {}_kX$, $\theta \longmapsto \theta_0$. In what follows we first exhibit a class of bocses for which ω^* reflects isomorphisms - i.e. if θ is such that $\omega^*(\theta)$ is an isomorphism then θ is an isomorphism in $\mathcal{C}$-k-mod, which means that the isomorphisms in the category

$\mathcal{A}$-k-mod are exactly the morphisms θ for which θ is invertible.

We need several concepts of one bocs being an extension of another.

6.1(i) **Definition** Let $\mathcal{A} = (A, V)$ be a bocs. We say that $\mathcal{A}$ is an *extension* of the bocs $\mathcal{A}' = (A', V', \mu', \epsilon')$ if

(i) A' is an S-subalgebra of A,

(ii) V' is an A'-A'-subbimodule of V containing ω (the group-like of $\mathcal{A}$), and the following diagrams commute ($\otimes'$ is the tensor product over A'):

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\epsilon'"'] & V \arrow[d, "\epsilon"] \\ A' \arrow[r] & A \end{tikzcd} \quad , \quad \begin{tikzcd} V' \arrow[rr] \arrow[d, "\mu'"'] & & V \arrow[d, "\mu"] \\ V' \otimes' V' \arrow[r] & V \otimes' V \arrow[r] & V \otimes V \end{tikzcd}$$

Remarks It is clear that $\bar{V}' = \bar{V} \cap V'$, and the differentials satisfy $d_1' = d_1|_{A'}$, and $d_2|_{\bar{V}'}$ is the composite

$$\bar{V}' \xrightarrow{d_2'} \bar{V}' \otimes' \bar{V}' \longrightarrow \bar{V} \otimes \bar{V}.$$

Notice that there is a 'restriction' functor $\mathcal{A}$-k-mod $\to \mathcal{A}'$-k'-mod, defined by $X \mapsto {}_{A'}X$ and $\theta \mapsto \theta'$ where $\theta' = \theta|_{V'} = (\theta_0', \bar{\theta}')$ and $\theta_0' = \theta_0$, $\bar{\theta}' = \bar{\theta}|_{\bar{V}'}$.

6.1 (ii) **Definition** The extension $\mathcal{A}$ of $\mathcal{A}'$ is called a *triangular kernel extension* if $A' = A$, and there is an S-S-subbimodule U of $\bar{V}$ such that

(a) the natural map $\bar{V}' \oplus (A \otimes_S U \otimes_S A) \to \bar{V}$ induced by the inclusions $\bar{V}' \subset \bar{V}$ and $U \subset \bar{V}$ is an isomorphism;

(b) $d(U) \subseteq \bar{V}' \otimes \bar{V}'$ (viewed as a subbimodule of $\bar{V} \otimes \bar{V}$ via (a)).

6.1 (iii) **Definition** The extension $\mathcal{Q}$ of $\mathcal{Q}'$ is called a triangular algebra extension (triangular tensor algebra extension) if

(a) there is an S-S-subbimodule E of A such that the morphism of A'-algebras

$$\otimes_{A'} (A' \otimes_S E \otimes_S A') \longrightarrow A$$

induced by the inclusion $E \to A$ is surjective (bijective);

(b) the natural map $A \otimes' \bar{V}' \otimes' A \to \bar{V}$ is an isomorphism;

(c) $d(E) \subseteq \bar{V}'$.

6.2 **Definition** The principal bocs of an algebra B, is the bocs (B, B) with comultiplication $B \cong B \otimes_B B$ and counit $1 : B \to B$. This has a grouplike (namely 1) and ω^* clearly reflects isomorphisms since (B, B)-mod is just B-mod.

6.3 **Proposition** Let $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_n = \mathcal{Q}$ be a finite sequence of bocses such that $\mathcal{Q}_0$ is principal and for $i = 0, 1, \ldots, n-1$, $\mathcal{Q}_{i+1}$ is a triangular kernel or algebra extension of $\mathcal{Q}_i$. Then $\omega^* : \mathcal{Q}\text{-k-mod} \to \text{k-mod}$ reflects isomorphisms.

Proof Since $\omega^* : \mathcal{Q}_0 - \text{k-mod} \to \text{k-mod}$ reflects isomorphisms ($\mathcal{Q}_0$ is a principal bocs) it is sufficient, by induction, to prove the following lemma.

Lemma Let $\mathcal{Q}$ be a triangular kernel or algebra extension of $\mathcal{Q}'$, then if

$$\omega^* : \mathcal{Q}'\text{-k-mod} \to \text{k-mod}$$

reflects isomorphisms so does

$$\omega^* : \mathcal{A}\text{-k-mod} \to \text{k-mod}.$$

Proof Let $\theta = (\theta_0, \bar{\theta}) \in \mathcal{A}(X, Y)$ and suppose θ_0 is invertible. We need to show that θ is an isomorphism. The restriction $\theta' = (\theta_0', \bar{\theta}') \in \mathcal{A}'(X, Y)$ is an isomorphism as $\theta_0' = \theta_0$ is an isomorphism. We will show how to extend an inverse $\varphi' = (\varphi_0', \bar{\varphi}') \in \mathcal{A}'(Y, X)$ for θ' to a left inverse $\varphi = (\varphi_0, \bar{\varphi})$ in $\mathcal{A}(Y, X)$ for θ; a similar construction gives a right inverse for θ, so θ is invertible.

We set $\varphi_0 = \varphi_0'$ so $\varphi_0\theta_0 = 1_X$, $\theta_0\varphi_0 = 1_Y$.

Case (i): Let $\mathcal{A}$ be a triangular kernel extension of $\mathcal{A}'$, so $A' = A$, $\bar{V} = \bar{V}' \oplus (A \otimes_S U \otimes_S A)$ and $d(U) \subseteq \bar{V}' \otimes \bar{V}'$.

Define $\bar{\varphi} = \bar{\varphi}'$ on the summand $\bar{V}'$ of $\bar{V}$. For $u \in U$, since $d(u) \in \bar{V}' \otimes \bar{V}'$ we may set

$$\bar{\varphi}(u) = -\varphi_0\bar{\theta}(u)\varphi_0 - c(\bar{\varphi}' \otimes \bar{\theta}')(d(u))\,\varphi_0 ;$$

this gives an S-S-bimodule map $U \to (Y, X)$, and so has a canonical extension to an A-A-bimodule map of the second summand of $\bar{V}$ to (Y, X). This completes the definition of $\bar{\varphi}$ and of $\varphi = (\varphi_0, \bar{\varphi})$; φ is a bocs morphism because

$$\bar{\varphi}(d(a)) = \bar{\varphi}'(d(a)) = a\varphi_0' - \varphi_0'a = a\varphi_0 - \varphi_0 a$$

for each $a \in A$.

A short calculation shows that $\overline{\varphi\theta} : \bar{V} \to (X, X)$ is zero on $\bar{V}'$ and U, and hence on $\bar{V}$. Therefore $\varphi\theta = \epsilon_X$, as required.

Case (ii) :Let $\mathcal{A}$ be a triangular algebra extension of $\mathcal{A}'$, so A is generated by A' and the S-S-subbimodule E of A, $\bar{V} \cong A \otimes' \bar{V}' \otimes' A$, and $d(E) \subseteq \bar{V}'$. For this case we extend $\bar{\varphi}'$ in the natural way to obtain

$$\bar{\varphi} : \bar{V} \cong A \otimes' \bar{V}' \otimes A \longrightarrow (Y, X).$$

Since φ' is a left inverse of θ' we have, for $v' \in \bar{V}'$, the equation:

(1) $$0 = \varphi_0'\bar{\theta}'(v') + \bar{\varphi}(v')\theta_0' + c(\bar{\varphi}' \otimes' \bar{\theta}')\, d(v')$$

Let $e \in E$ then $d(e) \in V'$, and if we put $v' = d(e)$ in (1) and recall that $\bar{\theta}'(d(e)) = \bar{\theta}(d(e)) = e\theta_0 - \theta_0 e$, $\theta_0' = \theta_0$, $\varphi_0' = \varphi_0 = \theta_0^{-1}$, and $dd(e) = 0$ we see that the relation

(2) $$\bar{\varphi}(d(a)) = a\varphi_0 - \varphi_0 a$$

is satisfied for $a \in E$. Also (2) holds for $a \in A'$, and if it holds for a_1 and a_2 then it holds for $a_1 + a_2$ and $a_1 a_2$. Since $A' \cup E$ generates A, (2) holds for all $a \in A$. Therefore $\varphi = (\varphi_0, \bar{\varphi}) \in \mathcal{Q}(Y, X)$. By definition $(\varphi\theta)_0 = 1_X$ and $\varphi\theta|_{\bar{V}'} = \overline{\varphi'\theta'} = 0$. Since $\bar{V}'$ generates $\bar{V}$ over A we conclude that $\overline{\varphi\theta} = 0$. So finally we conclude that $\varphi\theta = \epsilon_X$, as required.

We now give our main theorem concerning idempotent splitting, and will need Proposition 6.3 in its proof.

6.4 <u>Theorem</u> Let $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_n = \mathcal{Q}$ be a finite sequence of bocses such that $\mathcal{Q}_0$ is principal and, for $i = 0, 1, \ldots, n-1$, $\mathcal{Q}_{i+1}$ is either a triangular kernel or triangular tensor algebra extension of $\mathcal{Q}_i$. Then idempotents split in $\mathcal{Q}$-k-mod.

Proof Following [BK] we prove that if $\sigma \in \mathcal{Q}(X, X)$ is an idempotent there is an isomorphism $\alpha \in \mathcal{Q}(X, Y)$, for some Y, such that $\overline{\alpha\sigma\alpha^{-1}} = 0$. It then follows that $(\alpha\sigma\alpha^{-1})_0$ is an A-module idempotent and if $Y \xrightarrow{f_0} Z \xrightarrow{g_0} Y$ is a splitting of this idempotent then

$$Y \xrightarrow{(f_0,0)} Z \xrightarrow{(g_0,0)} Y$$

is a splitting of $\alpha\sigma\alpha^{-1}$. Hence $\sigma = (\alpha^{-1}(g_0, 0))((f_0,0)\alpha)$ is a splitting of σ. Since this property holds for all principal bocses it is enough, by induction, to prove the following lemma <u>(in which $\mathcal{Q}' = \mathcal{Q}_i$ for some $i \geq 0$)</u>.

<u>Lemma</u> Let $\mathcal{Q}$ be a triangular kernel or tensor algebra extension of $\mathcal{Q}'$. Suppose that for every idempotent σ' in $\mathcal{Q}'$-k-mod there is an isomorphism α' such that $\overline{\alpha'\sigma'(\alpha')^{-1}} = 0$; then for every idempotent σ in $\mathcal{Q}$-k-mod there is an isomorphism α such that $\overline{\alpha\sigma\alpha^{-1}} = 0$.

<u>Proof</u> Given $\sigma = \sigma^2 \in \mathcal{Q}(X, X)$ consider the restriction $\sigma' \in \mathcal{Q}'(X, X)$.

By hypothesis there is an <u>isomorphism</u> $\alpha' \in \mathcal{Q}'(X, Y)$, for some Y, such that $\overline{\alpha'\sigma'(\alpha')^{-1}} = 0$.

<u>Case (i)</u> : $\mathcal{Q}$ is a triangular kernel extension of $\mathcal{Q}'$, so that $A' = A$, $\overline{V} = \overline{V}' \oplus (A \otimes_S U \otimes_S A)$, $d(U) \subseteq \overline{V}' \otimes \overline{V}'$. Hence Y is already an A-module, and we define $\alpha \in (X, Y)$ by $\alpha_0 = \alpha'_0$, $\overline{\alpha}|_{\overline{V}'} = \overline{\alpha}'$ and $\overline{\alpha} = 0$ on $A \otimes_S U \otimes_S A$; it is easy to see that $\alpha = (\alpha_0, \overline{\alpha})$ is in $\mathcal{Q}(X, Y)$ and since $\mathcal{Q}' = \mathcal{Q}_i$ for some i and $\alpha_0 = \alpha'_0$ is an isomorphism α is also an isomorphism (by 6.3).

Let $\varsigma = \alpha\sigma\alpha^{-1}$. Then $\varsigma = \varsigma^2 \in \mathcal{Q}(Y, Y)$ and $\overline{\varsigma}|_{\overline{V}'} = 0$.

We must further modify ς, and will use the fact that, since $d(A) \subseteq \overline{V}'$, if φ, $\theta \in \mathcal{Q}(Y, Y)$ satisfy

$\bar{\varphi}|_{\bar{V}'} = 0 = \bar{\theta}|_{\bar{V}'}$, then $\overline{\varphi\theta}|_{\bar{V}'} = 0$, and hence that

$$\overline{\varphi\theta}(v) = \varphi_0\bar{\theta}(v) + \bar{\varphi}(v)\theta_0 \quad \text{for all} \quad v \in \bar{V}.$$

Define $\beta \in \mathcal{C}(Y, Y)$ by $\beta_0 = 1_X$, $\bar{\beta}(v) = \zeta_0\bar{\zeta}(v) - \bar{\zeta}(v)\zeta_0$ for $v \in \bar{V}$. Then $2\beta - \beta^2 = \epsilon_Y$, so $\beta(v)$ is an automorphism of Y with $\beta^{-1} = 2\,\epsilon_X - \beta$. We claim that $\overline{\beta\zeta\beta^{-1}} = 0$. Using the remark above, one sees that

(3) $\overline{\beta\zeta\beta^{-1}}(v) = \bar{\zeta}(v) - \zeta_0\,\bar{\zeta}(v) - \bar{\zeta}(v)\zeta_0 + 2\zeta_0\bar{\zeta}(v)\zeta_0$.

Since $\zeta^2 = \zeta$ we have $\bar{\zeta}(v) = \zeta_0\bar{\zeta}(v) + \bar{\zeta}(v)\zeta_0$, and as $\zeta_0^2 = \zeta_0$ it follows that $\zeta_0\bar{\zeta}(v)\zeta_0 = 0$. Using these two relations in (3) shows that $\overline{\beta\zeta\beta^{-1}} = 0$.

Let $\gamma = \beta\alpha$, then $\overline{\gamma\sigma\gamma^{-1}} = 0$ as required.

<u>Case (ii)</u> :$\mathcal{C}$ is a triangular tensor algebra extension of $\mathcal{C}'$, so that

$A = \otimes_{A'}(A' \otimes_S E \otimes_S A')$, $\bar{V} = A\otimes' \bar{V}' \otimes' A$, and $d(E) \subseteq \bar{V}$.

We have first to extend the action of A' on Y to an action of A, and for this it suffices to define a suitable S-S-bimodule morphism of E to (Y, Y). We take this morphism to be

$$e \longmapsto \alpha_0' e(\alpha_0')^{-1} + \bar{\alpha}'\,(d(e))\,\alpha_0'^{-1}.$$

Next we define $\gamma \in \mathcal{C}(X, Y)$ by taking $\gamma_0 = \alpha_0'$, and $\bar{\gamma}$ to be the canonical extension of $\bar{\alpha}'$ to $\bar{V} = A \otimes' \bar{V}' \otimes' A$. We need to check the relation

$$\bar{\gamma}(d(a)) = a\gamma_0 - \gamma_0 a \quad \text{for } a \in A.$$

As in the proof of 6.3 it is sufficient to do this for $a \in A'$ (which is easy:
$\bar{\gamma}(d(a)) = \bar{\alpha}'(d(a)) = a\alpha'_0 - \alpha'_0 a = a\gamma_0 - \gamma_0 a$
if $a \in A'$), and for $a \in E$ which is a consequence of the definition of the action of E on Y.

Thus $\gamma \in \mathcal{C}(X, Y)$ is an isomorphism by 6.3 and $\overline{\gamma\sigma\gamma^{-1}} = 0$ on $\overline{V}'$ since $\overline{\gamma\sigma\gamma^{-1}}|_{\overline{V}'} = \overline{\alpha'\sigma'(\alpha')^{-1}} = 0$. Since $\overline{V}'$ generates $\overline{V}$ over A we conclude that $\overline{\gamma\sigma\gamma^{-1}} = 0$, as required.

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