# The Differential Geometry of the Crosscap

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## Abstract

The aim of this thesis is to apply the methods of singularity theory to gain results on the differential geometry of 'geometric crosscaps' (sometimes referred to as Whitney Umbrellas). These are the surfaces parametrised by map germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ that are in the $\mathcal{A}$-orbit of the function with normal form $(x, xy, y^2)$.

The first chapter contains background material from singularity theory and differential geometry.

In Chapter 2 we consider differential topological aspects of the crosscap. We determine the tangent cone to the crosscap and the module of vector fields tangent to the Whitney Umbrella, the variety with equation $v^2 - u^2 w = 0$.

In the third chapter we determine a parametrisation of the family of geometric crosscaps. We are then in a position to study their differential geometry by composing the parametrisation with families of mappings on the ambient space $\mathbf{R}^3$. Chapters 4 and 5 are concerned with the flat geometry of the crosscap, that is those geometrical characteristics that can be measured by contact with lines and planes. In Chapter 4 we measure contact with planes by considering height functions on the crosscap. We obtain results on the dual of the crosscap and its parabolic set. In Chapter 5 we consider projections of the crosscap and give a partial result on the asymptotic curves of the crosscap.

In Chapters 6 and 7 we investigate the singularities of the family of distance squared functions. This measures contact with spheres, and yields information on the focal set of the crosscap.

Finally in Chapter 8 we present a different approach to studying the geometry of the crosscap. We classify functions on $\mathbf{R}^3$ up to $\mathcal{R}(X)$-equivalence, where $\mathcal{R}(X)$ is the group of diffeomorphisms on $\mathbf{R}^3$ which preserve the Whitney Umbrella. Similarly we classify map germs $\mathbf{R}^3, 0 \to \mathbf{R}^2, 0$ up to $\mathcal{R}(X)$-$\mathcal{L}$-equivalence. Analysing the geometry of these germs corroborates results obtained in previous chapters and gives information on the geometry of the double point set of the crosscap and how it interacts with the surface.

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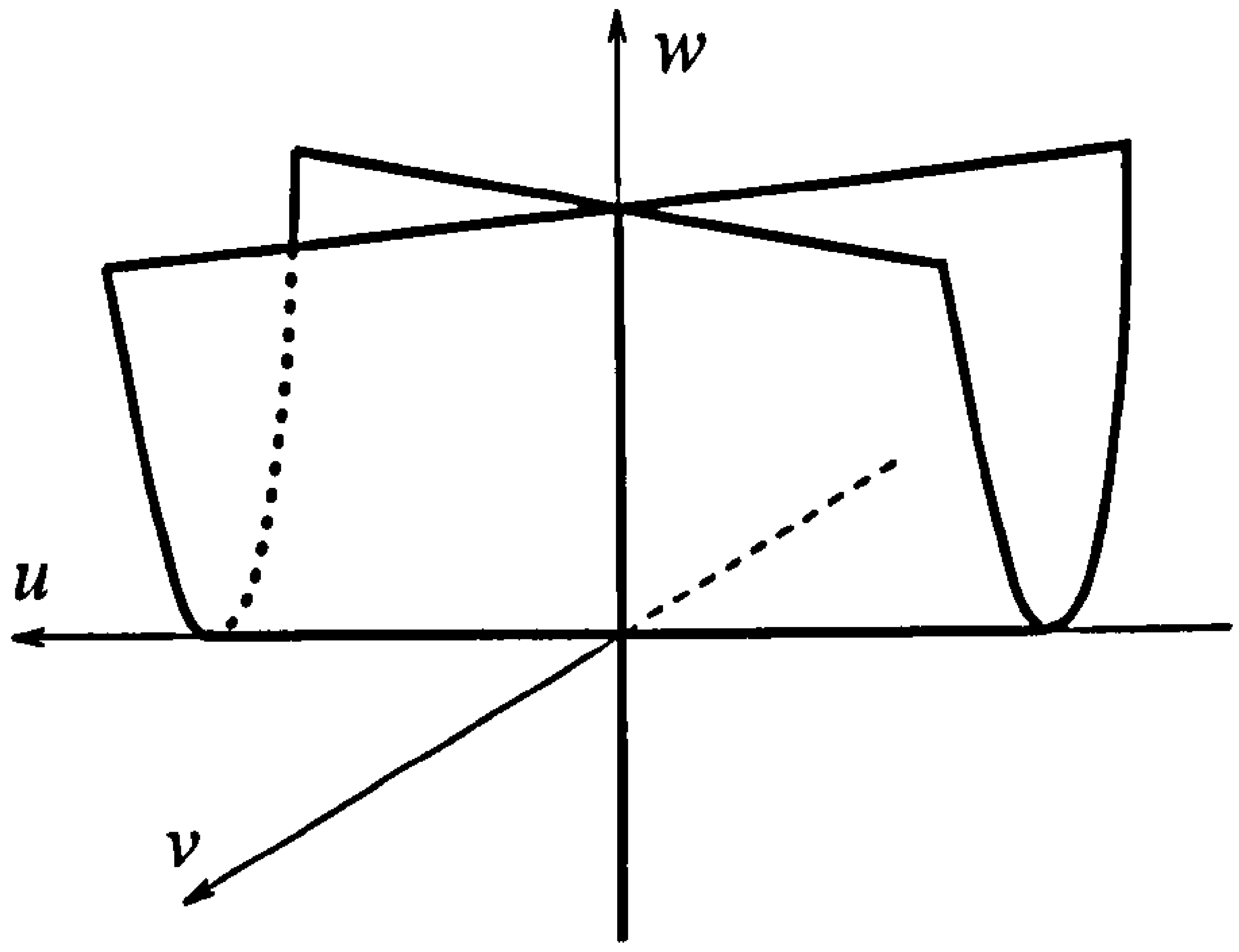
*To Mum, Dad and Christine.*

# Contents

# Prologue: the Crosscap




The *Whitney Umbrella* is the variety defined by the equation $v^2 - u^2w = 0$. The variety intersects the plane $w = 0$ in a repeated line, and the planes $w - k = 0$ in pairs of lines $v^2 = ku^2$ where $k$ is a positive constant. If $c$ is a constant the Whitney Umbrella intersects the planes $u - c = 0$ in parabolas $cw = v^2$. The only part of the Whitney Umbrella that lies in the region $w < 0$ is the $w$-axis. The variety is depicted above.

Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be the map germ defined by $f(x, y) = (x, xy, y^2)$. The image of $f$ is the Whitney Umbrella without its 'handle' (the half line $u = v = 0$, $w < 0$). It can be shown that $f$ is 2-$\mathcal{A}$-determined (see [Mo]). The Jacobian matrix of $f$ is

$$J_f = \begin{pmatrix} 1 & y & 0 \\ 0 & x & 2y \end{pmatrix}.$$

At the origin the rank of this matrix drops from two to one, and so $f$ is a singularity. The tangent space to the image of $f$ at this point is thus the $u$-axis. We define a *crosscap* to be the image of any germ $g$ that is $\mathcal{A}$-equivalent to $f$. We say that $g$ parametrises a crosscap. The *crosscap point* is the image under $g$ of

$(0,0)$. We call the crosscap parametrised by $f$ the standard crosscap. It is clear that the image of the derivative at the origin of a germ parametrising a crosscap is always a line. We abuse terminology by calling this the *tangent line* to the crosscap.

Surfaces in $\mathbf{R}^3$ can arise in various ways. They can be given implicitly, i.e. they can be defined by a single equation $g(u,v,w) = 0$ for some smooth function $g : \mathbf{R}^3, 0 \to \mathbf{R}, 0$. The Whitney Umbrella is an example of such a surface. Sard's Theorem implies that the set $c \in \mathbf{R}$ for which the set $g(u,v,w) = c$ fails to be a manifold has Lebesgue measure zero. Thus we expect 'almost all' surfaces defined implicitly to be manifolds. In particular the Whitney Umbrella surface is not 'generic' in this sense.

Surfaces can also be defined explicitly, or parametrised by a smooth mapping $f : U \to \mathbf{R}^3$, where $U$ is an open subset of $\mathbf{R}^2$. The crosscap arises in this way. In this case, it is not true in general that such parametrisations will yield manifolds. The surface can intersect itself, but this problem is relatively unimportant. A more substantial problem is that such mappings may have crosscaps. Whitney ([Wh]) proved that the crosscap is stable (that is we cannot remove the crosscap singularities by perturbing the map), and that a stable mapping $\psi : \mathbf{R}^2 \to \mathbf{R}^3$ only admits local singularities of this type.

Consequently, when studying the differential geometry of surfaces in 3-space there are good reasons for studying surfaces with crosscaps. It is this problem that we concern ourselves with in this thesis.

# Chapter 1

# Background Results from Singularity Theory

In this chapter, we review the results of singularity theory and differential geometry that will be needed in this thesis. For further reference we recommend the survey article of Wall [Wal], [GG], [Mar] for unfolding theory and [BduPW] for determinacy results. Our notation is drawn from these references.

We denote by $\mathcal{E}_n$ the $\mathbf{R}$-algebra of smooth function germs $\mathbf{R}^n, 0 \to \mathbf{R}$ and $\mathcal{O}_n$ the $\mathbf{C}$-algebra of analytic function germs $\mathbf{C}^n, 0 \to \mathbf{C}$. Both $\mathcal{E}_n$ and $\mathcal{O}_n$ are local rings with maximal ideal $\mathcal{M}_n$, which consists of the germs $f \in \mathcal{E}_n$ (respectively $\mathcal{O}_n$) such that $f(0) = 0$. The set of map germs $\mathbf{R}^n, 0 \to \mathbf{R}^p$ (respectively, $\mathbf{C}^n, 0 \to \mathbf{C}^p$) is a free $\mathcal{E}_n$-module (respectively, $\mathcal{O}_n$-module) and will be denoted $\mathcal{E}(n,p)$ (respectively, $\mathcal{O}(n,p)$). The manifolds $\mathbf{R}^n$ and $\mathbf{R}^p$ will be referred to as the source and target of a map-germ $\mathbf{R}^n, 0 \to \mathbf{R}^p, 0$. Similarly in the complex case $\mathbf{C}^n$ and $\mathbf{C}^p$ will be referred to as the source and target of a map-germ $\mathbf{C}^n, 0 \to \mathbf{C}^p, 0$.

Now suppose that $f \in \mathcal{E}(n,p)$(or $\mathcal{O}(n,p)$) is infinitely differentiable. If we truncate the power series expansion of $f$ at the origin by ignoring terms of degree greater than $k$, we obtain the $k$-jet of $f$, denoted by $j^k f$. The set of all $k$-jets forms a vector space $J^k(n,p)$. Now if $g \in \mathcal{E}_n$, it is not hard to show that

$$j^k g = 0 \Leftrightarrow g \in \mathcal{M}_n^{k+1}.$$

Thus we can identify $J^k(n,p)$ with the product of $p$ copies of $\mathcal{E}_n/\mathcal{M}_n^{k+1}$ (or $\mathcal{O}(n,p)/\mathcal{M}_n^{k+1} \cdot \mathcal{O}(n,p)$).

## 1.1 The Mather Groups

In [MatIII], Mather defined the equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}$ and $\mathcal{K}$ on $\mathcal{M}_n.\mathcal{E}(n,p)$. These equivalence relations have since become standard. We define $\mathcal{R}$ to be the group of germs of diffeomorphisms $\mathbf{R}^n, 0 \to \mathbf{R}^n, 0$, $\mathcal{L}$ to be the group of germs of diffeomorphisms $\mathbf{R}^p, 0 \to \mathbf{R}^p, 0$ and $\mathcal{A}$ to be the direct product $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. These groups act on $f \in \mathcal{M}_n.\mathcal{E}(n,p)$ in the following way:

$$h.f = f \circ h^{-1}, h \in \mathcal{R},$$

$$k.f = k \circ f, k \in \mathcal{L}, \quad \text{and}$$

$$(h,k).f = k \circ f \circ h^{-1}, (h,k) \in \mathcal{A}.$$

We often call $\mathcal{R}$ the group of smooth coordinate changes in the source, or the right group, and $\mathcal{L}$ the group of smooth coordinate changes in the target, or the left group.

We define $\mathcal{C}$ to be the germs of diffeomorphisms $\mathbf{R}^n \times \mathbf{R}^p, 0 \to \mathbf{R}^n \times \mathbf{R}^p, 0$ which project to the identity on $\mathbf{R}^n$ and preserve the subspace $\mathbf{R}^n \times \{0\}$. Thus $H \in \mathcal{C}$ can be written in the form

$$H(x,y) = (x, \tilde{H}(x,y)),$$

where $\tilde{H} : \mathbf{R}^n \times \mathbf{R}^p, 0 \to \mathbf{R}^p, 0$ and $\tilde{H}(x,0) = 0$ for $x \in \mathbf{R}^n$ near zero. We define an action of $\mathcal{C}$ on $\mathcal{M}_n.\mathcal{E}(n,p)$ by

$$(x, H \cdot f(x)) = H(x, f(x)), \quad H \in \mathcal{C}, \quad f \in \mathcal{M}_n.\mathcal{E}(n,p).$$

The group $\mathcal{C}$ can be thought of as the group of germs of diffeomorphisms $\mathbf{R}^p, 0 \to \mathbf{R}^p, 0$ parametrised by $x \in \mathbf{R}^n$. Define $h_x(y) = \tilde{H}(x,y)$, for $x$ near zero; $h_x$ is the germ of a diffeomorphism. The previous formula can be written as

$$H \cdot f(x) = h_x(f(x)).$$

We define $\mathcal{K}$ to be the group of germs of diffeomorphisms $\mathbf{R}^n \times \mathbf{R}^p, 0 \to \mathbf{R}^n \times \mathbf{R}^p, 0$ which can be written in the form

$$H(x,y) = (h(x), \tilde{H}(x,y))$$

where $h$ is a map-germ $\mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ (necessarily a diffeomorphism), $\tilde{H}$ a map-germ $\mathbf{R}^n \times \mathbf{R}^p, 0 \to \mathbf{R}^p, 0$ such that $\tilde{H}(x,0) = 0$ for $x \in \mathbf{R}^n$ near zero. We can define an action of $\mathcal{K}$ on $\mathcal{M}_n.\mathcal{E}(n,p)$ by

$$(x, H \cdot f(x)) = H(h^{-1}(x), f(h^{-1}(x))), \quad H \in \mathcal{K}, \quad f \in \mathcal{M}_n.\mathcal{E}(n,p).$$

The group $\mathcal{K}$ is often called the contact group.

The group $\mathcal{C}$ is a normal subgroup of $\mathcal{K}$, and $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ can be identified with subgroups of $\mathcal{K}$ by identifying $h \in \mathcal{R}$, $h' \in \mathcal{L}$ with the map-germs

$$(x, y) \mapsto (h(x), y), \qquad (x, y) \mapsto (x, h'(y)),$$

of $\mathcal{K}$. With these identifications, we can see that $\mathcal{K}$ is the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$, as $\mathcal{C}$ is a normal subgroup of $\mathcal{K}$ and each element of $\mathcal{K}$ can be written uniquely in the form $h \circ c$ where $h \in \mathcal{R}$ and $c \in \mathcal{C}$.

The equivalence relations $\mathcal{A}$, $\mathcal{C}$, $\mathcal{K}$ ('contact equivalence'), $\mathcal{L}$ ('left equivalence') and $\mathcal{R}$ ('right equivalence') are defined by the actions of the corresponding groups. In other words if $\mathcal{G}$ is a subgroup of $\mathcal{K}$, then two germs are $\mathcal{G}$-equivalent if they lie in the same $\mathcal{G}$ orbit.

## 1.2 Tangent Spaces

One of the aims of singularity theory is to classify map germs up to $\mathcal{G}$-equivalence, where $\mathcal{G}$ is some subgroup of $\mathcal{K}$. This amounts to finding representatives $f$ of the different $\mathcal{G}$ orbits. The study of these orbits has been based on an analogy between $\mathcal{G}$ acting on the map germs, and a Lie group acting on a manifold. The analogue of the Lie group is $\mathcal{G}$, which acts on $\mathcal{E}(n,p)$ (or $\mathcal{O}(n,p)$) and $J^k(n,p)$; now we need to define our tangent spaces.

We define the 'tangent space', $\theta_f$, to $\mathcal{E}(n,p)$ at $f$ to be the $\mathcal{E}_n$-module of germs of smooth vector fields along $f$. So $\xi \in \theta_f$ if $\xi : \mathbf{R}^n, 0 \to T(\mathbf{R}^p)$ and $\pi_p \circ \xi = f$, where $\pi_p : T(\mathbf{R}^p) \to \mathbf{R}^p$ is the natural projection from the tangent bundle $T(\mathbf{R}^p)$ of $\mathbf{R}^p$, to $\mathbf{R}^p$. We define $\theta_n = \theta_{1_{R^n}}$, $\theta_p = \theta_{1_{R^p}}$ where $1_{R^n}$ and $1_{R^p}$ denote the (germs at 0 of the) identity maps on $\mathbf{R}^n$, $\mathbf{R}^p$, respectively.

We define the $\mathcal{E}_n$-homomorphism

$$\begin{aligned} tf : \theta_n &\longrightarrow \theta_f \\ \phi &\mapsto df \circ \phi \end{aligned}$$

and the $\mathcal{E}_p$-homomorphism (via $f^* : \mathcal{E}_p \longrightarrow \mathcal{E}_n$, $\alpha \mapsto \alpha \circ f$ for $\alpha \in \mathcal{E}_p$)

$$\begin{aligned} wf : \theta_p &\longrightarrow \theta_f \\ \psi &\mapsto \psi \circ f. \end{aligned}$$

The tangent spaces to the orbits of the standard Mather groups are then given by

$$L\mathcal{R}\cdot f = tf(\mathcal{M}_n.\theta_n), \qquad L\mathcal{L}\cdot f = wf(\mathcal{M}_p.\theta_p), \qquad L\mathcal{C}\cdot f = f^*(\mathcal{M}_p).\theta_f,$$

$$L\mathcal{A}\cdot f = L\mathcal{R}\cdot f + L\mathcal{L}\cdot f, \qquad L\mathcal{K}\cdot f = L\mathcal{R}\cdot f + L\mathcal{C}\cdot f.$$

(Here we use the notation in [BduPW]; others use the notation $T(\mathcal{G}\cdot f)$).

We need a way to calculate the tangent spaces in practice, so we make the following observations. The tangent space $\theta_f$ is a free $\mathcal{E}_n$-module of rank $p$, for if $(y_1, \ldots, y_p)$ is a system of local coordinates on $\mathbf{R}^p, 0$ then the vector fields

$$(\partial/\partial y_1)\circ f, \ldots, (\partial/\partial y_p)\circ f$$

along $f$ form a free basis for $\theta_f$. We can therefore identify $\theta_f$ with $\mathcal{E}(n,p)$ and the above tangent spaces can be written as

$$\begin{aligned} L\mathcal{R}\cdot f &= \mathcal{M}_n.\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\} \\ L\mathcal{L}\cdot f &= f^*(\mathcal{M}_p).\{e_1, \ldots, e_p\} \\ L\mathcal{C}\cdot f &= f^*(\mathcal{M}_p).\mathcal{E}_n.\{e_1, \ldots, e_p\} \end{aligned}$$

where $e_1, \ldots, e_p$ are the standard basis vectors of $\mathbf{R}^p$ (considered as elements of $\mathcal{E}(n,p)$).

## 1.3 Transversality

Transversality is a central idea in singularity theory. In this section we give the definition of transversality and state a version of the fundamental Thom Transversality Lemma. For a detailed discussion we refer the reader to [GG], from which the following definition is taken.

**Definition 1.3.1** Let $X$ and $Y$ be smooth manifolds, and let $f : X \to Y$ be a smooth mapping. Let $W$ be a submanifold of $Y$ and $x$ a point in $X$. Then $f$ *intersects* $W$ *transversely at* $x$ (or *is transverse to* $W$ *at* $x$) if either

$$f(x) \notin W,$$

or

$$f(x) \in W \quad \text{and} \quad T_{f(x)}Y = T_{f(x)}W + df_x(T_xX).$$

If $A$ is a subset of $X$, then $f$ *intersects* $W$ *transversely on* $A$ if $f$ intersects $W$ transversely at $x$ for all $x \in A$. We say that $f$ *intersects* $W$ *transversely* if $f$ intersects $W$ transversely on $X$. If $B \subset W$ we say that $f$ *intersects* $W$ *transversely on* $B$ if $f$ intersects $W$ transversely at $x$ for all $x$ for which $f(x) \in B$.

**Lemma 1.3.2 (Thom Transversality Lemma)** Let $X \subset \mathbf{R}^r$ and $Y \subset \mathbf{R}^m$ be smooth manifolds and $U$ be an open set in $\mathbf{R}^t$. Suppose that

$$G : X \times U \longrightarrow \mathbf{R}^m$$

is a smooth map transverse to $Y$. Then for almost all $a \in U$ in the sense of Lebesgue measure (i.e. for all $a$ outside a set of measure zero) the maps

$$G_a : X \longrightarrow \mathbf{R}^m$$

given by $G_a(X) = G(x, a)$ are transverse to $Y$.

**Proof** See [GG], page 54. □

## 1.4 Finite Determinacy of Map Germs

When classifying map germs, one of the key ideas of singularity theory has been that they are frequently determined (up to $\mathcal{G}$-equivalence, where $\mathcal{G} \subset \mathcal{K}$) by their $k$-jets. We shall say that they are $k - \mathcal{G}$-determined. More formally, we have

**Definition 1.4.1** Let $\mathcal{G}$ be a subgroup of the contact group $\mathcal{K}$. We shall say that $f \in \mathcal{E}(n,p)$ (or $\mathcal{O}(n,p)$) is $k - \mathcal{G}$*-determined* if $f$ is $\mathcal{G}$-equivalent to any other map germ $g \in \mathcal{E}(n,p)$ (or $\mathcal{O}(n,p)$) such that $j^k f = j^k g$.

This notion is extremely useful, as in many situations once we know that a map germ is $k$-determined for some $k$, we can replace the ambient space of map germs $\mathcal{M}_n.\mathcal{E}(n,p)$ with the finite dimensional vector space of $k$-jets $J^k(n,p)$.

The theorems in [BduPW] give excellent estimates for the determinacy degree (the least $k$ for which a map germ is $k$-determined) in terms of the tangent space to the orbit of $f$.

The following Lemma is useful in reducing determinacy criteria to finite problems in linear algebra. We invoke it in Chapter 8.

**Lemma 1.4.2 (Nakayama's Lemma)** Let $R$ be a commutative ring, $M$ an ideal such that for $x \in M$, $1+x$ is a unit. Let $C$ be an $R$-module, $A$ and $B$ $R$-submodules of $C$ with $A$ finitely generated. If $A \subset B + M.A$ then $A \subset B$.

**Proof** See [AM], page 21. □

In singularity theory applications, $R$ is taken to be the ring $\mathcal{E}_n$ (or $\mathcal{O}_n$ in the complex case), and $M$ the maximal ideal $\mathcal{M}_n$.

## 1.5 Versal Unfoldings

An $s$-parameter unfolding of a map-germ $f_0 \in \mathcal{M}_n.\mathcal{E}(n,p)$ is a map-germ

$$\begin{aligned} F : \mathbf{R}^n \times \mathbf{R}^s, 0 &\longrightarrow \mathbf{R}^p \times \mathbf{R}^s, 0 \\ (x,u) &\mapsto (f(x,u),u) \end{aligned}$$

such that $f_0(x) = f(x,0)$. We often use the notation $f_u(x) = f(x,u)$. The map $f_u$ can be thought of as a deformation of $f_0$, parametrised smoothly by $u \in \mathbf{R}^s$.

In what follows we will consider the case $\mathcal{G} = \mathcal{A}$ . The definitions and results for other subgroups of $\mathcal{K}$ are analogous.

**Definition 1.5.1** Two unfoldings

$$F, G : \mathbf{R}^n \times \mathbf{R}^s, 0 \longrightarrow \mathbf{R}^p \times \mathbf{R}^s, 0$$

of $f_0 \in \mathcal{E}(n,p)$ are said to be *isomorphic* if there exist germs of diffeomorphisms

$$\begin{aligned} \phi : \mathbf{R}^n \times \mathbf{R}^s, 0 &\longrightarrow \mathbf{R}^n \times \mathbf{R}^s, 0 \\ \psi : \mathbf{R}^p \times \mathbf{R}^s, 0 &\longrightarrow \mathbf{R}^p \times \mathbf{R}^s, 0, \end{aligned}$$

which are $s$-parameter unfoldings of the identity maps on $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, and $G = \psi \circ F \circ \phi^{-1}$. So for small $u$, $\phi_0 = 1_{\mathbf{R}^n}$, $\psi_0 = 1_{R^p}$ and $\phi_u$, $\psi_u$ are germs of diffeomorphisms of $\mathbf{R}^n$, $\mathbf{R}^p$ respectively. Thus $g_u = \psi_u \circ f_u \circ \phi_u^{-1}$ for small $u$ and $g_u$ is $\mathcal{A}$-equivalent to $f_u$ via diffeomorphisms in the source and target which are parametrised smoothly by $u \in \mathbf{R}^s$.

Note that for $u \neq 0$ the germs $f_u$, $g_u$, $\phi_u$, $\psi_u$ cannot be considered as germs at 0 with target 0. The situation where the origin is not fixed is called $\mathcal{A}_e$ - equivalence. If we need to keep the origin fixed then the map-germs $\phi$, $\psi$ must satisfy in addition $\phi(0,u) = 0$ and $\psi(0,u) = 0$ for all small $u$. The terms $\mathcal{A}_e$-unfolding and $\mathcal{A}$ -unfolding are used to clarify which situation we are in. We then need to define the $\mathcal{A}_e$-tangent space.

**Definition 1.5.2** The $\mathcal{A}_e$*-tangent space* of $f_0 \in \mathcal{E}(n,p)$ is defined by

$$L\mathcal{A}_e \cdot f_0 = \mathcal{E}_n.\langle \partial f_0/\partial x_1, \ldots, \partial f_0/\partial x_n \rangle + \mathcal{E}_p.\{e_1, \ldots, e_p\}.$$

We now define an equivalence relation on unfoldings of a map germ $f_0$.

**Definition 1.5.3** Given $h : \mathbf{R}^t, 0 \longrightarrow \mathbf{R}^s, 0$ we define the *pull-back* of $F$ by $h$, denoted $h^*F$, to be the $t$-parameter unfolding

$$(h^*F)(x,v) = (f(x,h(v)),v).$$

Let $F$ and $G$ be two $s$-parameter unfoldings of a map germ $f_0$. Then $F$ and $G$ are said to be *equivalent* if there exists a diffeomorphism $h : \mathbf{R}^s, 0 \longrightarrow \mathbf{R}^s, 0$ such that $G$ is isomorphic to $h^*F$.

If $H$ is now some $t$-parameter unfolding of $f_0$ (so $t$ does not necessarily equal $s$), we say $H$ is *induced* from $F$ if there exists a smooth map-germ $h : \mathbf{R}^t, 0 \longrightarrow \mathbf{R}^s, 0$ such that $H$ is isomorphic to $h^*F$.

**Definition 1.5.4** We define the $\mathcal{A}$*-codimension* of $f_0 \in \mathcal{E}(n,p)$ by

$$\mathcal{A}\text{-codim}(f_0) = \dim_{\mathbf{R}} (\mathcal{M}_n.\mathcal{E}(n,p)/L\mathcal{A} \cdot f_0),$$

and the $\mathcal{A}_e$ *-codimension* by

$$\mathcal{A}_e\text{-codim}(f_0) = \dim_{\mathbf{R}} (\mathcal{E}(n,p)/L\mathcal{A}_e \cdot f_0).$$

If $f \in \mathcal{E}(n,p)$ is a mapping of finite codimension, then $f$ has a versal unfolding; that is a family of deformations which contains all deformations of $f$.

**Definition 1.5.5** Let $F$ be an unfolding of $f_0 \in \mathcal{E}(n,p)$. Then $F$ is *versal* if every unfolding of $f_0$ can be induced from $F$.

We say that $F$ is *trivial* if it is isomorphic to the constant unfolding (in $s$ parameters), $(x,u) \mapsto (f_0(x),u)$.

The map germ $f_0$ is *stable* if all unfoldings of $f_0$ are trivial.

Now we state a fundamental theorem on unfoldings. Given an unfolding $F(x,u) = (f(x,u),u)$, the *initial speeds*, $\dot{F}_i \in \mathcal{E}(n,p)$, of $F$ are defined by

$$\dot{F}_i(x) = \partial f/\partial u_i(x,0), \quad \text{for} \quad i = 1,\ldots,s.$$

**Theorem 1.5.6** An $s$-parameter unfolding $F$ of $f_0$ is versal if and only if

$$L\mathcal{A}_e \cdot f_0 + \mathbf{R}.\{\dot{F}_1,\ldots,\dot{F}_s\} = \mathcal{E}(n,p).$$

**Proof** For a proof of this and further discussion on unfoldings, see [Mar] and [Wal]. □

This theorem has the following useful corollaries.

**Corollary 1.5.7** Suppose that $\mathcal{A}_e$-codim$(f_0)$ is finite and $g_1,\ldots,g_c \in \mathcal{E}(n,p)$ form an $\mathbf{R}$-spanning set for the complementary space to $L\mathcal{A}_e \cdot f_0$ in $\mathcal{E}(n,p)$. Defining the unfolding

$$F(x,u) = \left(f(x) + \sum_{i=1}^{c} u_i g_i(x), u\right),$$

we find that $\dot{F}_i = g_i$. Then $F(x,u)$ is a versal unfolding of $f_0$.

**Corollary 1.5.8** The map germ $f_0$ has a versal unfolding if and only if its $\mathcal{A}_e$-codimension is finite.

**Corollary 1.5.9** The map germ $f_0$ is stable if and only if $\mathcal{A}_e$-codim$(f_0) = 0$.

Finally we state the following theorems, the proofs of which can be found in [Mar]. If we define $c = \mathcal{A}_e$-codim$(f_0)$, then the least number of parameters for a versal unfolding of $f_0$ is $c$. We call a $c$-parameter unfolding of $f_0$ a *miniversal* unfolding.

**Theorem 1.5.10** All miniversal unfoldings of $f_0$ are equivalent.

## 1.6 Discriminants and Bifurcation Sets

We can associate to an unfolding its discriminant set and its bifurcation set. If the unfolding arises naturally in a geometric situation, then its discriminant and bifurcation set usually have some geometric significance.

**Definition 1.6.1** Let

$$\begin{array}{rcl} F : \mathbf{R}^n \times \mathbf{R}^s, 0 & \longrightarrow & \mathbf{R}^p \times \mathbf{R}^s, 0 \\ (x,u) & \mapsto & (f(x,u),u) = (f_u(x),u) \end{array}$$

be an $s$-parameter unfolding of the map germ $f_0$. Then the *critical set* is the set

$$\Sigma F = \{u \in \mathbf{R}^s : f_u \quad \text{is singular at some point} \quad x \in \mathbf{R}^n\}.$$

The *discriminant* of $F$, denoted $D(F)$, is the image of $\Sigma F$ under $F$.

The *bifurcation set* of $F$ is the set of points

$$\begin{array}{rl} B(F) \;=\{ & u \in \mathbf{R}^s : f_u \quad \text{is unstable at some point } x \in \mathbf{R}^n \text{ or} \\ & \text{determines an unstable multigerm at some points } x_1 \cdots x_r \in \mathbf{R}^n\}. \end{array}$$

For geometrical applications, the following proposition is useful.

**Proposition 1.6.2** Any two $\mathcal{A}$-versal unfoldings with the same number of parameters have diffeomorphic discriminants and bifurcation sets.

**Proof** See [Mar]. □

## 1.7 Complete Transversals

Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$. Complete transversals provide a useful tool for obtaining the orbits when $\mathcal{G}$ acts on $\mathcal{E}(n,p)$. The method is due to Bruce and duPlessis and is a generalisation of the work of Dimca and Gibson in the $\mathcal{K}$ case (see [DG]). Our main reference for this section is [BduP], but since this is as yet unpublished, we review the relevant results. The proofs may be found in [Wi].

First we state a result due to Mather concerned with Lie group actions and the calculation of orbits.

**Lemma 1.7.1 (Mather Lemma)** Let $G$ be a Lie group which acts smoothly on a finite dimensional manifold $V$. Let $X$ be a connected submanifold of $V$. Then $X$ is contained in a single orbit of $G$ if and only if

(*i*) for each $x \in X$, $T_xX \subset T_x(G \cdot x) = LG \cdot x$;

(*ii*) dim $T_x(G \cdot x)$ remains constant for all $x \in X$.

**Proof** See [MatIV]. □

The following complete transversal theorem is a corollary to the Mather Lemma.

**Theorem 1.7.2** Let $G$ be a Lie group acting smoothly on an affine space $A$, and let $W$ be a subspace of the underlying vector space $V_A$ , with

$$LG \cdot (x + w) = LG \cdot x \tag{1.1}$$

for all $x \in A$ and $w \in W$. Then we have

(*i*) for all $x \in A$

$$x + \{LG \cdot x \cap W\} \subset G \cdot x \cap \{x + W\};$$

(*ii*) if $x_0 \in A$ and $T$ is a vector subspace of $W$ satisfying

$$W \subset T + LG \cdot x_0$$

then for any $w \in W$ there exist $g \in G$ and $t \in T$ such that

$$g \cdot (x_0 + w) = x_0 + t.$$

The vector subspace $T$ is called a *complete transversal.*

**Proof** See [BduP]. □

A classification result now follows. The basic idea is to find a complete list of $k+1$-jets with a given $k$-jet. We can then use an inductive process to classify map germs, using a relevant determinacy theorem to tell us when to stop.

Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $\mathcal{G}_k$ be the normal subgroup of $\mathcal{G}$ consisting of those germs whose $k$-jet is equal to the identity. The standard $k$-jet group is defined to be the quotient group $\mathcal{G}/\mathcal{G}_k$ and is denoted $J^k\mathcal{G}$. This is a Lie group which acts on the affine space $J^k(n,p)$; see [MatIII], section 7. Let $H^k$ denote the image of $\mathcal{M}_n^k.\mathcal{E}(n,p)$ in $J^k(n,p)$. So $H^k$ is the vector subspace of $J^k(n,p)$ consisting of the homogeneous jets of degree $k$.

**Corollary 1.7.3** Let $\mathcal{G}$ be one of the standard Mather groups $\mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}$ or $\mathcal{K}$. Then given $f \in \mathcal{M}_n.\mathcal{E}(n,p)$, and $T \in H^{k+1}$ a vector subspace of $H^{k+1}$ such that

$$H^{k+1} \subset L(J^{k+1}\mathcal{G}_1) \cdot j^{k+1}f + T,$$

we have for every $(k+1)$-jet $j^{k+1}g$ with $j^k g = j^k f$ that $j^{k+1}g$ is in the same $J^{k+1}\mathcal{G}_1$ orbit as $j^{k+1}f + t$ for some $t \in T$.

**Proof** See [BduP]. □

## 1.8 Simple Germs and Moduli

In any classification of map germs, the simple singularities are important. Often when we classify map germs, we only consider the simple singularities. This definition is due to Arnol'd, [AGV], page 184.

**Definition 1.8.1** Let $X$ be a manifold and $G$ be a Lie group which acts on $X$. The *modality* of a point $x \in X$ under the action of $G$ on $X$ is the least number $m$ such that a sufficiently small neighbourhood of $x$ may be covered by a finite number of $m$-parameter families of orbits. The point $x$ is said to be *simple*, if its modality is 0, that is a sufficiently small neighbourhood intersects only a finite number of orbits. If $\mathcal{G}$ is one of the Mather groups, then the *$\mathcal{G}$-modality* of a finitely determined map germ is defined to be the modality of a sufficient jet in the jet space under the action of the jet group.

## 1.9 Some Differential Geometry

A surface $X$ is a two-dimensional manifold. In what follows we shall consider surfaces $X$ embedded in $\mathbf{R}^3$ (via a smooth embedding $g : X \to \mathbf{R}^3$), but abuse notation by not distinguishing between $X$ and $g(X)$. In a neighbourhood $V$ of a point $x$ of a surface $X$ in $\mathbf{R}^3$, there exists a parametrisation

$$\phi : U \to X \cap V,$$

where $\phi$ is a diffeomorphism and $U$ is an open set of $\mathbf{R}^2$. As usual we define the tangent space to $X$ at $x$ to be the image of $d\phi_u$, where $u \in \mathbf{R}^2$ is such that $\phi(u) = x$.

One approach to studying the geometry of surfaces is to look at the properties of curves that lie on the surface. We recall the following definition.

**Definition 1.9.1** Suppose that the map $\alpha : \mathbf{R} \to \mathbf{R}^n$ parametrises the curve $C$, which is unit speed, i.e. $\|\alpha'(t)\| = 1$ for all $t \in \mathbf{R}$. The vector $T(t) = \alpha'(t)$ is called the *unit tangent vector* to $C$.

The *curvature* $\kappa(t)$ of $C$ at the point $\alpha(t)$ is given by

$$\kappa(t) = \|T'(t)\|.$$

Now in the case of plane curves, we can give the curvature a sign.

**Definition 1.9.2** Let $\alpha : \mathbf{R} \to \mathbf{R}^2$ parametrise a unit speed plane curve $C$. Then at a point $\alpha(t) \in C$ we can define a unit normal vector $N(t)$ by rotating the unit tangent vector $T(t)$ anticlockwise through $\frac{\pi}{2}$. Now since $T(t)$ is a unit vector, $T'(t)$ is perpendicular to $T(t)$ for all $t$. Thus there is a real number $\kappa(t)$ such that

$$T'(t) = \kappa(t)N(t).$$

We define the curvature of $C$ at the point $\alpha(t)$ to be $\kappa(t)$.

For more details on the properties of plane and space curves, we refer the reader to any of the standard texts on Differential Geometry, such as [O'N] or [L].

Now suppose that $x$ is a smooth point of a surface $X$. If we intersect the family of planes containing the normal direction to $X$ at $x$ with $X$, we obtain a family of plane curves. We call these curves the *sectional curves* to $X$ at $x$, and their curvatures the *sectional curvatures* of $X$ at $x$.

**Definition 1.9.3** Let $\phi : U \to X \cap V$ be a parametrisation of a surface $X$, where $U$ and $V$ are open sets of $\mathbf{R}^2$ and $\mathbf{R}^3$ respectively. We can construct a *unit normal function* $N$ on $U$

$$N = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}.$$

Then the *coefficients of the first fundamental form* are

$$E = \phi_x . \phi_x, F = \phi_x . \phi_y \quad \text{and} \quad G = \phi_y . \phi_y,$$

and the *coefficients of the second fundamental form* are

$$l = N . \phi_{xx}, m = N . \phi_{xy} \quad \text{and} \quad n = N . \phi_{yy}.$$

**Proposition 1.9.4** Suppose $X$ and $\phi$ are as before, and that $p$ is a point on $X$. Let $N_p$ be the normal to $X$ at $p$ and $\mathbf{a} = a_1\phi_x + a_2\phi_y$ be a vector in $T_pX$. Then the sectional curvature of the curve determined by the plane spanned by $N_p$ and $\mathbf{a}$ is

$$\kappa_n = \frac{la_1^2 + 2ma_1a_2 + na_2^2}{Ea_1^2 + 2Fa_1a_2 + Ga_2^2}$$

**Proof** See [O'N]. □

The extreme values of the sectional curvatures at a point $p \in X$ are called the *principal curvatures.* The corresponding directions are called the *principal directions.*

**Proposition 1.9.5** The principal curvatures are given by the roots of

$$det\left(\begin{pmatrix} l & m \\ m & n \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = 0.$$

**Proof** See [O'N]. □

**Definition 1.9.6** Let $\kappa_1, \kappa_2$ be the principal curvatures at a point $p \in X$. We define the *Gaussian curvature* of $X$ at $p$ to be

$$\kappa = \kappa_1\kappa_2.$$

Note that the signs of the principal curvatures depend on the choice of unit normal to the surface; the sign of their product does not. A *hyperbolic point* is one at which the two principal curvatures have opposite signs, and hence $\kappa$ is negative. At an *elliptic point*, $\kappa$ is positive, and the principal curvatures have the same sign. At a *parabolic point*, at least one of the principal curvatures is zero, so that $\kappa$ is zero. A special type of elliptic point is an *umbilic*, where all the sectional curvatures are equal, and at a *flat umbilic*, the sectional curvatures are all zero.

The Gaussian curvature is a smooth function on $X$. Thus in general we expect parabolic points to lie on curves which separate elliptic and hyperbolic regions.

Gauss's theorema egregium shows that the Gaussian curvature is an intrinsic quantity associated to a surface, that is it depends only on the notion of distance within the surface. For a more precise formulation and proof of this, see [O'N].

**Definition 1.9.7** If a surface is parametrised by the map

$$\phi(x,y) = (x, y, f(x,y)),$$

with $f \in \mathcal{M}_2^2$, then it is said to be in *Monge form.*

Notice that at the origin,

$$f(0,0) = f_x(0,0) = f_y(0,0) = 0.$$

Thus the tangent plane to the surface at the origin is the $(x, y)$-plane. The function $f$ measures the contact between the surface and its tangent plane. We discuss this in more detail later on.

**Definition 1.9.8** The *centres of curvature* above a point $p \in X$ are given by $p + (1/\kappa_i)N$, where $N$ is the unit normal at $p$. The inverses of the principal curvatures, $1/\kappa_1$ and $1/\kappa_2$ are called the *principal radii of curvature.* Note that although $N$ is defined only up to sign, the curvatures change sign with the normal. If $p$ is an umbilic, then there is only one centre of curvature above $p$. We define the *focal set* of $X$ to be the locus of the centres of curvature. The focal set has two sheets; one corresponding to each centre of curvature.

# Chapter 2

# Differential Topological Aspects of the Crosscap

In this chapter we give some preliminary results on the crosscap and the Whitney Umbrella. We determine an important invariant of the crosscap: its tangent cone. The tangent cone turns out to have great significance when considering the geometry of the crosscap. We also determine the module of vector fields that are tangent to the Whitney Umbrella. This is used in Chapter 8 when we classify function and map germs on the crosscap.

Before considering these features of the crosscap, we need the following result. Let $X$ be the standard crosscap, given by the image of the germ $f(x,y) = (x, xy, y^2)$. The image does not determine the parametrisation: it is clear that we can make an arbitrary change of coordinates in the source and retain the same image. The following result, the proof of which is due to J.W. Bruce, shows that two map germs which parametrise the same crosscap are $\mathcal{R}$-equivalent.

**Proposition 2.0.1** Let $g : \mathbf{C}^2, 0 \longrightarrow \mathbf{C}^3, 0$ be an immersion away from the origin, with image set the standard crosscap parametrised by $f : \mathbf{C}^2, 0 \longrightarrow \mathbf{C}^3, 0$, $(x,y) \longmapsto (x, xy, y^2)$. Then $g$ is $\mathcal{R}$-equivalent to $f$. Moreover, the right-equivalence preserves the double point set.

**Proof** It is enough to check that $f$ and $g$ are normalisations of $X$ and hence $\mathcal{R}$-equivalent to any other normalisation (see [Gu], page 34). Alternatively, we can use the fact that $f$ is an immersion to see that there is a well defined map from the complement of the double point set of $f$ to that of $g$ which is a local diffeomorphism at each point. Using the fact that away from the pinch points we have transversely intersecting sheets along the double point curve, we can extend this map to the complement of the origins. By Hartog's theorem (see [F], page 53) this extends to a smooth map between sources, which is invertible (we can use the same construction to get the inverse map). □

## 2.1 The Tangent Cone at the Crosscap Point

In this section, **K** is always **R** or **C**. We define the difference between smooth and singular points of an affine variety $V$. Since the Whitney Umbrella is a hypersurface, we restrict our attention to this case. Suppose that $f$ is an irreducible polynomial in $n$ variables $x_1, \cdots, x_n$, and let $V = V(< f >) \subset \mathbf{K}^n$. Let $p$ be a point on $V$. Then $p$ is a smooth point of $V$ if and only if $\partial f / \partial x_i(p) \neq 0$ for some $i$. Otherwise $p$ is a singular point of $V$. If $p$ is a smooth point of $V$, the tangent space $T_p V$ is defined to be the kernel of $df_p$. In general, in order to distinguish a singular point of a variety from smooth points, we look at the dimension of the tangent space at the point (for more details, see [Har] or [S]). However the tangent space at a singular point does not tell us much about the local geometry at that point. We define the tangent cone, a construction which gives us more information about the local structure of a variety at a singular point.

Let $X$ be an irreducible affine variety in n-dimensional affine space $\mathbf{K}^n$, defined by the ideal $I$, and let $x \in X$. The *tangent cone* to $X$ at $x$ consists of a collection of lines through $x$, which we define in a way which turns out to be equivalent to the limiting position of secants in differential geometry. As such, the tangent cone can be considered to be a refinement of the notion of tangent space.

Following Shafarevich ([S]), we make $\mathbf{K}^n$ into a vector space by choosing $x = (0, \cdots, 0)$ as the origin of coordinates. Now let

$$\tilde{X} = \{(a,t) : a \in \mathbf{K}^n, t \in \mathbf{K}^1, a.t \in X\} \subset \mathbf{K}^{n+1} = \mathbf{K}^n \times \mathbf{K}^1$$

and let $\phi : \tilde{X} \to \mathbf{K}^1$ and $\psi : \tilde{X} \to \mathbf{K}^n$ be the natural projection maps. It is clear that $\tilde{X}$ is a closed subset of $\mathbf{K}^{n+1}$, and if $X \neq \mathbf{K}^n$, it is reducible, and consists of two components, $\tilde{X}_1$ and $\tilde{X}_2$. The set $\tilde{X}_1$ is the closure of $\phi^{-1}(\mathbf{K}^1 \backslash (0))$ in $\tilde{X}$ and $\tilde{X}_2 = \{(a, 0); a \in \mathbf{K}^n\}$. We denote by $\phi_1$ and $\psi_1$ the restrictions of $\phi$ and $\psi$ to $\tilde{X}_1$. If a line $L$ is a secant of $X$ that passes through $x = (0, \cdots, 0)$, then $L = \{at : t \in \mathbf{K}\}$, where a is a fixed point in $\mathbf{K}^n$ and $at' \in X$ for some $t' \in \mathbf{K}$. Thus $\psi_1(\tilde{X}_1)$ is the closure of the set of points on all secants of $X$ that pass through $x$. Finally we define the set $T_x = \psi_1({\phi_1}^{-1}(0))$ to be the tangent cone to $X$ at $x$.

Now if $(a,0) \in \phi_1^{-1}(0)$, then there exists a sequence $((a_i,t_i)) \subset \tilde{X}$ such that $\lim_{i\to\infty} a_i = a$, $\lim_{i\to\infty} t_i = 0$ . For each $i$, $\psi_1(a_i,t_i) = a_i$ is a point on a secant of $X$ that passes through $x$. Thus $\psi_1((a,0)) = a$ is a point on a line that is a limit of secants of $X$ that pass through $x$. Conversely suppose that $L$ is a line that is a limit of secants of $X$ that pass through $x = (0,\cdots,0)$. Then

$$L = \{at : t \in \mathbf{K}\},$$

where $a$ is a fixed point in $\mathbf{K}^n$ such that there exists a sequence $(a_i) \subset X$ for which $\lim_{i\to\infty} a_i = 0$ and

$$\lim_{i\to\infty} \frac{a_i}{\|a_i\|} = a.$$

Thus for each $i$,

$$\left(\frac{a_i}{\|a_i\|}, \|a_i\|\right) \in \tilde{X}_1.$$

Since $\tilde{X}_1$ is a closed set, $(a,0) \in \tilde{X}_1$, and for any $t \neq 0 \in \mathbf{K}$,

$$\lim_{i\to\infty} \left(\frac{a_i t}{\|a_i\|}, \frac{\|a_i\|}{t}\right) = (at,0) \in \tilde{X}_1.$$

So $L$ is indeed contained in $\psi_1({\phi_1}^{-1}(0))$, the tangent cone to $X$ at $x$.

In fact this construction works for any set in $\mathbf{K}^n$, but when the set is an affine variety, we can derive the equations of the tangent cone. The equations defining $\tilde{X}$ are of the form $f(at) = 0$, where $f \in I$. For each $f \in I$ write $f$ as a sum of its homogeneous parts, i.e. $f = f_k + f_{k+1} + \cdots + f_m$, where $f_j$ is a homogeneous polynomial of degree $j$ for $k \leq j \leq m$ and $f_k \neq 0$. Then we can write

$$f(at) = t^k f_k(a) + \cdots + t^m f_m(a).$$

Since $x = (0,\cdots,0) \in X$, $f(0) = 0$. Thus the equation defining $\tilde{X}_2$ is $t = 0$. It is clear that the equations of $T_x$ are the lowest degree terms of the polynomials in $I$. Since $T_x$ is determined by homogeneous equations, it is a cone with vertex at $x$.

The tangent cone to the Whitney umbrella is thus the repeated plane given by the equation $v^2 = 0$. We want to know what the tangent cone to any crosscap point is. In the case of the standard crosscap, we have the following.

**Lemma 2.1.1** Let $X$ be the standard crosscap parametrised by

$$f(x,y) = (x, xy, y^2).$$

Then the tangent cone to $X$ at the origin $T_0X$ is the $(u,w)$-plane.

**Proof** The set $X$ is a subset of the Whitney Umbrella, so $T_0X$ must be contained in the plane defined by $v^2 = 0$. So we just need to show that any line in the $(u,w)$-plane can be achieved as the limiting position of secants. Let $\gamma : \mathbf{R}, 0 \to \mathbf{R}^2, 0$ be the map germ defined by $\gamma(s) = (\alpha s^2, s)$. Then the secant line passing through the point $f(\gamma(s))$ and the crosscap point is the set

$$L_s = \{t(\alpha s^2, \alpha s^3, s^2) : s \in \mathbf{R}\}.$$

To find the limiting position of $L_s$ as $s \to 0$, we need to find

$$\lim_{s\to 0} \frac{(\alpha s^2, \alpha s^3, s^2)}{||(\alpha s^2, \alpha s^3, s^2)||}$$

$$= \lim_{s\to 0} \frac{(\alpha s^2, \alpha s^3, s^2)}{s^2\sqrt{\alpha^2 + 1 + \alpha^2 s^2}}$$

$$= \frac{(\alpha, 0, 1)}{\sqrt{1+\alpha^2}}.$$

So by varying $\alpha$ we have any unit direction in the $(u,w)$-plane except for $(1,0,0)$. The line in this direction is the $u$-axis. However it is clear that this is also contained in $T_0X$, since the standard crosscap contains the $u$-axis. □

**Proposition 2.1.2** Let $X$ be a surface in $\mathbf{R}^3$ with a singularity at the origin. Suppose that

$$\phi : \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$$

is the germ of a diffeomorphism and that $Y$ is the image of $X$ under $\phi$.

Let $T_0X$ be the tangent cone to $X$ at the origin. Then we have

$$T_0Y = d\phi_0(T_0X).$$

**Proof** Suppose that $(x_i) \in X$ is a sequence of points such that $\lim_{i\to\infty} x_i = 0$ and $\lim_{i\to\infty} \frac{x_i}{||x_i||}$ exists. Then we consider

$$d\phi_0\left(\lim_{i\to\infty} \frac{x_i}{||x_i||}\right).$$

Since $d\phi_0$ is a continuous linear map this is equal to

$$\lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|x_i\|}.$$

Since $\phi$ is invertible,

$$\lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|x_i\|} \neq 0,$$

and so

$$\lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|x_i\|} \cdot \lim_{i\to\infty} \frac{\|x_i\|}{\|d\phi_0(x_i)\|}$$

is a unit vector that is in the same direction as

$$d\phi_0 \left( \lim_{i\to\infty} \frac{x_i}{\|x_i\|} \right).$$

Now we have

$$\lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|x_i\|} \cdot \lim_{i\to\infty} \frac{\|x_i\|}{\|d\phi_0(x_i)\|} = \lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|d\phi_0(x_i)\|}.$$

Finally we claim that

$$\lim_{i\to\infty} \frac{d\phi_0(x_i)}{\|d\phi_0(x_i)\|} = \lim_{i\to\infty} \frac{\phi_0(x_i)}{\|\phi_0(x_i)\|}.$$

For by the definition of the derivative,

$$\lim_{i\to\infty} \frac{\phi(x_i) - d\phi_0(x_i)}{\|(x_i)\|} = 0,$$

and therefore

$$\lim_{i\to\infty} \frac{\phi(x_i) - d\phi_0(x_i)}{\|(x_i)\|} \cdot \lim_{i\to\infty} \frac{\|x_i\|}{\|d\phi_0(x_i)\|} = \lim_{i\to\infty} \frac{\phi(x_i) - d\phi_0(x_i)}{\|d\phi_0(x_i)\|} = 0. \tag{2.1}$$

Thus we have

$$\lim_{i\to\infty} \frac{\phi(x_i)}{\|d\phi_0(x_i)\|} = \lim_{i\to\infty} \frac{d\phi(x_i)}{\|d\phi_0(x_i)\|},$$

and

$$\lim_{i\to\infty} \frac{\|\phi(x_i)\| - \|d\phi_0(x_i)\|}{\|d\phi_0(x_i)\|} = 0.$$

Since

$$\frac{\phi(x_i)}{\|\phi(x_i)\|}$$

is a unit vector, we have

$$\lim_{i\to\infty} \frac{\phi(x_i)}{\|\phi_0(x_i)\|} - \frac{\phi(x_i)}{\|d\phi_0(x_i)\|} = 0. \tag{2.2}$$

Adding Equations (2.1) and (2.2) proves the claim. So $d\phi_0(T_0X) \subset T_0(Y)$, and since $\phi$ is a diffeomorphism, $d\phi_0(T_0X) = T_0Y$. □

Applying Proposition 2.1.2 we have

**Corollary 2.1.3** The tangent cone to a crosscap point is always a plane.

and

**Corollary 2.1.4** The tangent line to any crosscap $Y$ is contained in its tangent cone $T_0Y$.

**Proof** If $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ defined by $f(x, y) = (x, xy, y^2)$ parametrises the crosscap $X$, then the tangent cone $T_0X$ is the plane with equation $v = 0$. The tangent line to $X$ is the $u$-axis, which is contained in $T_0X$. The result then follows from the chain rule and Proposition 2.1.2. □

Now we make the following observation. Let $Y$ be a crosscap, parametrised by the map $g : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$, with double point curve $C$. The unit normal function

$$N = \frac{g_x \times g_y}{\|g_x \times g_y\|}$$

assigns two normal vectors to each point of $C$. These determine the tangent planes to the two patches of surface going through the point. Both of these planes have the same limiting position as we approach the crosscap point along $C$. This is in fact the same position as the tangent cone $T_0Y$. More precisely we have

**Lemma 2.1.5** Let $Y$, $g$, $C$, and $N$ be as above. Let $C' \in \mathbf{R}^2$ be the preimage of $C$. Then $C'$ is smooth, since it is the image of the $y$-axis under some diffeomorphism $\phi : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$. We choose a parametrisation of $C'$, $\gamma : \mathbf{R}, 0 \to \mathbf{R}^2, 0$. Then $N(\gamma(t))$ is the normal vector to the crosscap surface at the point $\gamma(t)$. Let $T(t)$ be the plane defined by the unit vector $N(\gamma(t))$. Then we have

$$\lim_{t \to 0} T(t) = T_0Y.$$

**Proof** First we show that the result is true in the case where $X$ is the standard crosscap parametrised by the map $f(x, y) = (x, xy, y^2)$. In this case the preimage of the double point curve is the $y$-axis. We choose $\gamma(t) = (0, t)$ so that

$$f(\gamma(t)) = (0, 0, t^2).$$

Now at any point $f(x,y)$, the normal to the surface is given by

$$N(x,y) = \frac{(2y^2, -2y, x)}{\sqrt{(4y^4 + 4y^2 + x^2)}},$$

so

$$N(\gamma(t)) = \frac{(2t^2, -2t, 0)}{\sqrt{(4t^4 + 4t^2)}} = \frac{(t, -1, 0)}{\sqrt{(t^2 + 1)}}.$$

We will denote the tangent plane to $X$ at $(0,0,t^2)$ defined by $N(\gamma(t))$ by $T_X(t)$. Now it is clear that

$$\lim_{t \to 0} N(\gamma(t)) = (0, -1, 0).$$

This limiting normal defines the plane $v = 0$. So in this case we have

$$\lim_{t \to 0} T_X(t) = T_0 X.$$

Let $\phi : \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ be the germ of a diffeomorphism, and let $Y$ be the crosscap parametrised by $g = \phi \circ f$. Then the double point curve $C$ of $Y$ is parametrised by the map germ $\psi : \mathbf{R}, 0 \to \mathbf{R}^3, 0$ defined by $\psi(t) = \phi(0,0,t^2)$. At any point $\psi(t)$ of $C$, with $t \neq 0$, the tangent plane is given by

$$T(t) = d\phi_{(0,0,t^2)}(T_X(t)).$$

Thus the limiting position of the tangent plane as we approach the crosscap point along $C$ is given by

$$\lim_{t \to 0} d\phi_{(0,0,t^2)}(T_X(t)) = d\phi_0(T_0 X) = T_0 Y.$$

□

**Remark 2.1.6** *Lemma 2.1.5 gives some geometrical insight into the multiplicity of the defining equation of the tangent cone to the Whitney Umbrella and leads us to think intuitively of the tangent cone to the crosscap as a repeated plane.*

## 2.2 The Vector Fields Tangent to the Crosscap

In what follows, $\mathcal{O}_n = \mathcal{O}(x_1, \cdots, x_n)$ will denote the set of smooth function germs $\mathbf{K}^n, 0 \to \mathbf{K}$, where $\mathbf{K}$ is either $\mathbf{R}$ or $\mathbf{C}$. A vector field on a manifold $X$ in $\mathbf{K}^n$ is a smooth map $\xi : X \to \mathbf{K}^n$ such that for each $x \in X$

$$\xi(x) \in T_x X.$$

A germ of a vector field at $x_0 \in X$ is a germ $\eta : X, x_0 \to \mathbf{K}^n$ with the same property. If $V$ is a germ of an analytic variety in $\mathbf{K}^n$, a germ of a vector field $\xi$ on $\mathbf{K}^n$ is tangent to $V$ if for each regular point $v \in V$, $\xi(v) \in T_v V$. The set of such vector fields forms an $\mathcal{O}_n$-module.

Let $h$ be an irreducible polynomial in $\mathbf{K}[x_1, \cdots, x_n]$ defining a hypersurface $X \subset \mathbf{K}^n$. If $\xi$ is an analytic vector field on $\mathbf{K}^n$ that is tangent to $X$, then at any smooth point $x \in X$,

$$\xi h(x) = dh_x(\xi(x)) = 0.$$

Thus $\xi h$ vanishes at all smooth points of $X$. Since the set of smooth points of $X$ is dense in $X$, $\xi h$ vanishes on all of $X$. So if $\xi$ is tangent to $X$ then

$$\xi h \in I(V < h >).$$

Conversely if $\xi h \in I(V < h >)$ then $\xi h$ vanishes on $X$. Hence $\xi h$ vanishes at the regular points of $X$ and so $\xi$ is tangent to $X$. In the complex case the Nullstellensatz says that

$$I(V < h >) = \sqrt{< h >} = < h >,$$

and so $\xi h = \lambda h$ for some function $\lambda$. In the case of the Whitney Umbrella, where $h = v^2 - u^2 w$ then this is also true over the reals.

**Lemma 2.2.1** Let $h$ be an irreducible polynomial in $\mathbf{R}[x_1, \cdots, x_n]$. Denote the sets

$$\{(x_1, \cdots, x_n) \in \mathbf{R}^n : h(x_1, \cdots, x_n) = 0\}$$

and

$$\{(x_1, \cdots, x_n) \in \mathbf{C}^n : h(x_1, \cdots, x_n) = 0\}$$

by $V(h)_{\mathbf{R}}$ and $V(h)_{\mathbf{C}}$ respectively. If $h = v^2 - u^2 w$ then the ideal

$$I(V(h)_{\mathbf{R}}) \subset \mathbf{R}[x_1, \cdots, x_n]$$

is generated by $h$.

**Proof** If $\phi : \mathbf{R}^2 \to \mathbf{R}^3$ is the map defined by $f(x, y) = (x, xy, y^2)$, then $\phi(\mathbf{R}^2) \subset V(h)_{\mathbf{R}}$. Thus if $g$ is a polynomial with real coefficients that vanishes on $V(h)_{\mathbf{R}}$ then the function $g \circ \phi : \mathbf{R}^2 \to \mathbf{R}$ is identically zero. Hence $g \circ \phi : \mathbf{C}^2 \to \mathbf{C}$ is identically zero. So $g$ vanishes on $\phi(\mathbf{C}^2)$. Consequently $g$ vanishes on the Zariski closure of $\phi(\mathbf{C}^2)$ which is $V(h)_\mathbf{C}$. So $g \in I(V(h)_\mathbf{C})$. Hence by the Nullstellensatz $g = \alpha h$, $\alpha \in \mathbf{C}[X_1, \cdots, X_n]$ and since $g$ and $h$ are real,

$$g = \frac{(\alpha + \overline{\alpha})}{2} h \in \mathbf{R}[x_1, \cdots, x_n] < h > .$$

□

It will be useful later on to know which germs of vector fields on $\mathbf{K}^3, 0$ are tangent to the Whitney Umbrella. To determine these we will need the following result.

**Lemma 2.2.2** Consider the linear map

$$L : \mathcal{O}_n^n \to \mathcal{O}_n$$

defined by

$$L(\alpha) = L(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{i=0}^{n} \alpha_i x_i,$$

where $\alpha_1, \cdots, \alpha_n$ are the components of $\alpha$. Then $kerL$ is spanned as an $\mathcal{O}_n$-module by the set of maps

$$\{\gamma_{ij} : \gamma_{ij}(x_1, \cdots, x_n) = x_i e_j - x_j e_i, 1 \leq i < j \leq n\},$$

where $e_k$ is the vector in $\mathbf{K}^n$ with a 1 in the $k$th position and zeros elsewhere.

**Proof** It is clear that the $\gamma_{ij}$ lie in $kerL$. We show that they generate $kerL$ by induction on $n$.

First suppose that $n = 2$, and that

$$\alpha = (\alpha_1, \alpha_2) \in kerL.$$

Then there exists $\lambda \in \mathcal{O}_n$ such that

$$\alpha_2(x_1, x_2) = -x_1 \lambda(x_1, x_2) + \alpha_2'(x_2).$$

Let

$$\alpha_1'(x_1, x_2) = -x_2 \lambda(x_1, x_2) + \alpha_1(x_1, x_2).$$

Then we have

$$L((\alpha_1', \alpha_2')) = L((\alpha_1, \alpha_2) + \lambda \gamma_{12}) = L((\alpha_1, \alpha_2)) = 0.$$

Thus we have

$$x_1\alpha_1'(x_1,x_2)+x_2\alpha_2'(x_2)=0$$

and consequently

$$\alpha_2'(x_2)=0 \quad \text{and} \quad \alpha_1'(x_1,x_2)=0,$$

so $\gamma_{12}$ does indeed span $kerL$.

Now in general suppose that $(\alpha_1,\cdots,\alpha_n)\in kerL$. Then by subtracting suitable multiples of the $\gamma_{ij}$ from the functions $\alpha_k$ we can obtain functions $\alpha_k'$ such that for $2\le k\le n$, $\alpha_k'$ is independent of $x_1$. Then we have

$$x_1\alpha_1'(x_1,x_2,\cdots,x_n)+x_2\alpha_2'(x_2,\cdots,x_n)+\cdots+x_n\alpha_n'(x_2,\cdots,x_n)=0$$

and so $\alpha_1'(x_1,x_2,\cdots,x_n)=0$ and the inductive step is complete. □

**Proposition 2.2.3** The $\mathcal{O}_3$-module of germs at the origin of vector fields on $\mathbf{C}^3$ that are tangent to the Whitney Umbrella is generated by

$$\xi_1=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v},$$

$$\xi_2=v\frac{\partial}{\partial v}+2w\frac{\partial}{\partial w},$$

$$\xi_3=u^2\frac{\partial}{\partial v}+2v\frac{\partial}{\partial w} \quad \text{and}$$

$$\xi_4=v\frac{\partial}{\partial u}+uw\frac{\partial}{\partial v}.$$

**Proof** Let $h(u,v,w)=v^2-u^2w$ be the defining equation of the Whitney Umbrella. Then

$$\xi_1h=2h,\xi_2h=2h,\xi_3h=0 \quad \text{and} \quad \xi_4h=0,$$

so $\xi_1,\xi_2,\xi_3$ and $\xi_4$ are certainly tangent to the Whitney Umbrella.

Now if $\eta$ is a germ at the origin of a vector field on $\mathbf{C}^3$ that is tangent to the Whitney Umbrella, then $\eta h=\lambda h$ for some polynomial $\lambda$. Then

$$(\eta-\frac{1}{2}\lambda\xi_1)h=\eta h-\frac{1}{2}\lambda\xi_1 h=\eta h-\lambda h=0,$$

so we need only check that $\xi_1,\xi_2,\xi_3$ and $\xi_4$ generate all germs of vector fields $\eta$ such that

$$\eta h=0.$$

Writing

$$\eta = \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v} + \eta_3 \frac{\partial}{\partial w},$$

we have $\eta h = 0$ if and only if

$$(-2uw)\eta_1 + (2v)\eta_2 + (-u^2)\eta_3 = 0.$$

Clearly $\eta_2 = u\eta_2'$ for some $\eta_2'$, so $(\eta_1, \eta_2', \eta_3)$ lies in the kernel of the linear map

$$M : \mathcal{O}_3^3 \to \mathcal{O}_3$$

defined by

$$M(f_1, f_2, f_3) = (-f_3)u + (2f_2)v + (-2f_1)w.$$

Now Lemma 2.2.2 implies that $ker M$ is spanned by the maps

$$a(u, v, w) = (v, w, 0),$$

$$b(u, v, w) = (0, u, 2v) \quad \text{and}$$

$$c(u, v, w) = (u, 0, -2w).$$

Substituting each of $a$, $b$ and $c$ for $(\eta_1, \eta_2', \eta_3)$ we see that a germ at the origin of a vector field on $\mathbf{C}^3$ that is tangent to the Whitney Umbrella can indeed be written in terms of $\xi_1$, $\xi_2$, $\xi_3$ and $\xi_4$. □

# Chapter 3

# The Parametrisation of the Crosscap

We wish to look at the local differential geometry of the crosscap. By the crosscap, we mean the image of any map germ which is in the $\mathcal{A}$-orbit of $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ given by $f(x, y) = (x, xy, y^2)$. We will arrive at a family of map germs, all of which are $\mathcal{A}$-equivalent to $f$, but whose images may have differing local differential geometry. We then consider the geometrical properties of the family, and show that in a certain sense, this family cannot be reduced any further. If we consider the flat geometry of the crosscap, that is those properties which can be measured in terms of contact with planes and lines, we have more coordinate changes at our disposal. In the second half of this chapter, we derive a parametrisation which makes use of these coordinate changes.

## 3.1 The Parametrisation

The coordinate changes in the target which preserve the local differential geometry of the image of a map germ are those isometries which fix the origin (the orthogonal transformations). We will consider the differential geometry of an object in terms of its contact with various 'model' submanifolds of $\mathbf{R}^3$, such as spheres, planes and lines. As we shall be looking in particular at degenerate contact with these submanifolds, we also allow dilations which are centred on the origin. These coordinate changes correspond to an action of $O(3) \times \mathbf{R}^+$ on $\mathcal{E}(2, 3)$ which we define in the following way. If we have the function $g \in \mathcal{E}(2, 3)$, the matrix $L \in O(3)$, and $\lambda \in \mathbf{R}^+$, then $(L, \lambda).g = \lambda L(g)$. This is as far as we can go when considering the 'flat' geometry, that is the geometry measured by contact with planes and lines. Diffeomorphic changes of coordinates in the source will not affect the local differential geometry of the image of a map germ.

**Proposition 3.1.1** Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be the map germ defined by $f(x, y) = (x, xy, y^2)$. Let $g$ be a map germ that is $\mathcal{A}$-equivalent to $f$. Then using the coordinate changes described above, we can reduce $g$ to the form

$$g'(x, y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x, y)),$$

where $a$ and $b$ are constants, $p \in \mathcal{M}_1^3$ and $q \in \mathcal{M}_2^3$.

**Proof** Since $g$ is $\mathcal{A}$-equivalent to $f$, it must have rank one. So we can choose a rotation $\rho_1 \in O(3)$ such that

$$\rho_1 \circ g(x, y) = (g_1(x, y), g_2(x, y), g_3(x, y)),$$

where $g_2, g_3 \in \mathcal{M}_2^2$, and $g_1(x, y) = l_1 x + l_2 y + \cdots$, with $l_1, l_2 \in \mathbf{R}$, and not both zero.
Now we choose a diffeomorphism $\phi_1 : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ such that $g_1 \circ \phi_1(x, y) = x$. Then

$$\rho_1 \circ g \circ \phi_1(x, y) = (x, g_2'(x, y), g_3'(x, y)).$$

Next we choose a rotation $\rho_2 \in O(3)$ through an angle $\theta$ about the $u$-axis. Then

$$\rho_2 \circ \rho_1 \circ g \circ \phi_1(x, y) = (x, cos\theta g_2'(x, y) + sin\theta g_3'(x, y), -sin\theta g_2'(x, y) + cos\theta g_3'(x, y))$$

We write the 2-jets of $g_2'$ and $g_3'$ as

$$a_1 x^2 + a_2 xy + a_3 y^2 \quad \text{and}$$

$$b_1 x^2 + b_2 xy + b_3 y^2 \quad \text{respectively,}$$

with $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$. Since $\rho_1 \circ g \circ \phi_1$ is $\mathcal{A}$-equivalent to $f$, $a_2 b_3 - b_2 a_3 \neq 0$. Thus we can choose $\theta$ so that

$$\rho_2 \circ \rho_1 \circ g \circ \phi_1(x, y) = (x, c_1 xy + c_2 x^2 + p_1(x, y), d_1 x^2 + d_2 xy + d_3 y^2 + q_1(x, y)),$$

with $c_1, c_2, d_1, d_2, d_3 \in \mathbf{R}$ and $p_1, q_1 \in \mathcal{M}_2^3$.

To complete the proof, we must show that a map germ $g : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ defined by

$$g(x, y) = (x, c_1 xy + c_2 x^2 + p_1(x, y), d_1 x^2 + d_2 xy + d_3 y^2 + q_1(x, y))$$

can be reduced to the form of $g'$. Making the coordinate change $y' = c_1 y + c_2 x$ we have

$$g(x, y') = (x, xy' + p_2(x, y'), d_1' y'^2 + d_2' xy' + d_3' x^2 + q_2(x, y')),$$

with $d_1' \in \mathbf{R} - \{0\}$, $d_2', d_3' \in \mathbf{R}$ and $p_2, q_2 \in \mathcal{M}_2^3$. Then we need the following lemma.

**Lemma 3.1.2** Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ be defined by $f(x, y) = (x, xy + r(x, y))$, with $r \in \mathcal{M}_2^3$. Then by a change of coordinates in the source, we can reduce $f$ to the normal form

$$f(x, y) = (x, xy + r'(y)), r' \in \mathcal{M}_1^3.$$

**Proof** The proof is an adaptation of the second half of the proof of Lemma 2 from [B2].

Let $x' = x + \partial r/\partial y$ and write $\partial r/\partial y$ as a sum $R_0(y) + x' R_1(x', y)$. Then

$$x + R_0(y) = x'(1 - R_1)$$

which belongs to the Jacobian ideal $J(f)$ of $f$ (this is the ideal of partial derivatives of $f$ - see [Wal]). Now let $x'' = x + R_0$ and write

$$r(x, y) = r_1(y) + x'' r_2(y) + (x'')^2 r_3(x'', y),$$

where $r_i \in \mathcal{M}_2^{4-i}$ for $1 \le i \le 3$.The last component of $f$ is now

$$xy + r_1(y) + x'' r_2(y) + (x'')^2 r_3(x'', y)$$
$$= x(y + r_2(y)) + r_1(y) + R_0(y) r_2(y) + (x'')^2 r_3(x'', y).$$

Next we use the following result, which we quote from [duPW], Corollary 2.8.

**Lemma 3.1.3** Let $f : (N, x_0) \to (P, y_0)$ be a non-submersive smooth map germ. Then $f$ is $\mathcal{M}_N J(f)^2 - \mathcal{R}_{J(f)}$-determined.

Lemma 3.1.3 says that if we add elements of the ideal $\mathcal{M}_N J(f)^2$ to a non-submersive smooth map germ $f$, the result is $\mathcal{R}_{J(f)}$ equivalent to $f$. The group $\mathcal{R}_{J(f)}$ is the subgroup of the Mather group $\mathcal{R}$ whose elements preserve the ideal $J(f)$. Now $(x'')^2 r_3(x, y) \in \mathcal{M}_2 J(f)^2$ so by Lemma 3.1.3 $f$ is $\mathcal{R}$-equivalent to the map germ

$$(x, x(y + r_2(y)) + r_1(y) + R_0(y) r_2(y)).$$

Replacing $y + r_2(y)$ by $y'$ gives a change of coordinates yielding a germ of the required type. □

Returning to the proof of Proposition 3.1.1, we apply the change of coordinates constructed in the proof of Lemma 3.1.2 to the source. This reduces $g$ to the form

$$(x, xy + p(y), d_1''y^2 + d_2''xy + d_3''x^2 + q(x, y)),$$

with $d_1'' \in \mathbf{R} - \{0\}$, $d_2'', d_3'' \in \mathbf{R}$ and $p, q \in \mathcal{M}_2^3$. Finally we 'scale' in the source (replace $x$ by $\lambda_1 x$ and $y$ by $\lambda_2 y$, where $\lambda_1, \lambda_2 \in \mathbf{R} - \{0\}$), and apply a dilation in the target to complete the reduction of $g$ to the required form. □

Parametrising a smooth surface in a neighbourhood of the origin in Monge form has the effect of fixing the position of the tangent plane at the origin. In the case of the crosscap, there is no tangent plane at the origin. However the rotations $\rho_1$ and $\rho_2$ and the diffeomorphism $\phi_1$ do have some geometrical significance.

**Lemma 3.1.4** The effect of $\rho_1$ is to rotate in the target so that the tangent line to the crosscap is the $u$-axis.

**Proof** The tangent line is the image of the linear map $dg_0 : \mathbf{R}^2 \to \mathbf{R}^3$ . It is clear that the image of $d(\rho_1 \circ g)_0$ is the $u$-axis. □

**Lemma 3.1.5** The diffeomorphism $\phi_1$ makes the preimage of the double point curve tangent to the $y$-axis at the origin.

**Proof** Write $g'$ instead of $\rho_1 \circ g \circ \phi_1$. Let $C \subset \mathbf{R}^2$ be the preimage of the double point curve of the crosscap parametrised by $g'$. Since $g'$ is $\mathcal{A}$-equivalent to $f$, $C$ is a smooth curve. A parametrisation for the preimage of the double point curve of the standard crosscap is $\gamma(t) = (0, t)$. Then $f(0, t) = f(0, -t)$. So we can choose a local parametrisation $\gamma : \mathbf{R}, 0 \to \mathbf{R}^2, 0$ of $C$ such that $g'(\gamma(t)) = g'(\gamma(-t))$. Then we write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. By inspecting the first component of $g'$, we see that $\gamma_1(t) = \gamma_1(-t)$ for all $t$. Hence $\gamma_1$ can be written as a function of $t^2$, and the result follows. □

**Lemma 3.1.6** The rotation $\rho_2$ fixes the tangent cone to be the plane $v = 0$.

**Proof** Recall that the tangent cone to any crosscap at the crosscap point is a plane. By considering the secants joining the origin to $g(t, 0)$ and $g(0, t)$, we see that the tangent cone at the origin of the image of

$$g(x, y) = (x, a_1x^2 + a_2xy + a_3y^2 + O(3), b_1x^2 + b_2xy + b_3y^2 + O(3))$$

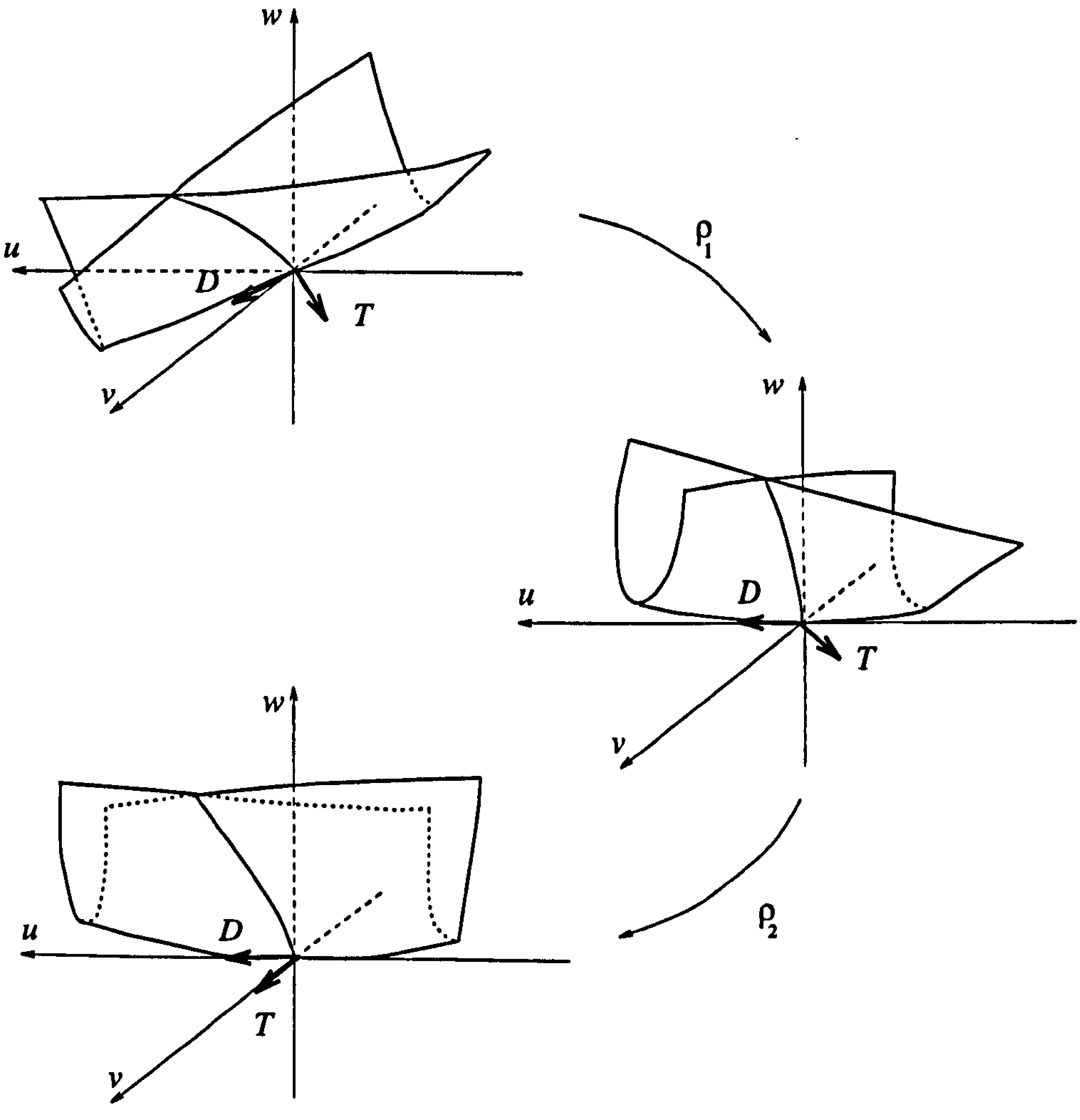



Figure 3.1: The effect of the rotations $\rho_1$ and $\rho_2$ on the crosscap.

contains $(1,0,0)$ and $(0,a_3,b_3)$. It is now clear that by rotating so as to kill off the $a_3$ term we have fixed the tangent cone to be the plane $v=0$. □

Figure 3.1 shows the effect of $\rho_1$ and $\rho_2$ on the crosscap. The direction of the tangent line ($D$) and the normal direction to the tangent cone ($T$) are indicated.

## 3.2 The Isotropy Subgroup

The question of whether we can reduce the normal form any further now arises. To answer this question, we consider the isotropy subgroup of $O(3) \times \mathbf{R}^+ \times \mathcal{R}$ of the family of crosscap maps determined in Proposition 3.1.1. We have defined an action of $O(3) \times \mathbf{R}^+ \times \mathcal{R}$ on $\mathcal{E}(2,3)$, and those map germs with normal form $g'$ determined in Proposition 3.1.1 define an affine subspace $W$ of $\mathcal{E}(2,3)$. The isotropy subgroup of $O(3) \times \mathbf{R}^+ \times \mathcal{R}$ with respect to $W$ consists of those elements $(M, \lambda, \phi) \in O(3) \times \mathbf{R}^+ \times \mathcal{R}$ which leave $W$ invariant.

**Proposition 3.2.1** The isotropy subgroup of $O(3) \times \mathbf{R}^+ \times \mathcal{R}$ with respect to $W$ is isomorphic to $\mathbf{Z}_2^4$.

**Proof** We show that the isotropy subgroup is generated by

$$\{(I, 1, id), (R_1, 1, -id), (R_2, 1, r_1), (R_3, -1, r_2)\},$$

where $I$ is the identity matrix and $R_1, R_2$ and $R_3$ are the matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

respectively, $id : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ is the identity map and $r_1, r_2 : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ are the maps defined by

$$r_1(x, y) = (x, -y) \quad \text{and} \quad r_2(x, y) = (-x, y)$$

respectively. The generators commute and $(R_1, 1, -id), (R_2, 1, r_1)$ and $(R_3, -1, r_2)$ are each of order two, and so the result follows.

We have seen (Lemmas 3.1.4 and 3.1.6) that any element $A$ of $O(3)$ which leaves $W$ invariant under the group action must be an isometry which leaves the configuration of the $u$-axis (the tangent line) and the $(u, w)$-plane (the tangent cone) invariant. Thus $A$ must be in the subgroup generated by the identity, a reflection in the $(v, w)$-plane, a reflection in the $(u, w)$-plane or a reflection in the $(u, v)$-plane. These isometries correspond to the matrices $I, R_1, R_2$ and $R_3$ respectively. So we just need to determine those pairs $(\lambda, \phi) \in \mathbf{R}^+ \times \mathcal{R}$ such that $(M, \lambda, \phi)$ leaves $W$ invariant under the group action for $M = I, R_1, R_2$ and $R_3$.

First consider the case where $M = I$. Let $\phi(x,y) = (\phi_1(x,y), \phi_2(x,y))$. Then for all $a, b \in \mathbf{R}$, $p \in \mathcal{M}^3$ and $q \in \mathcal{M}_2^3$,

$$\lambda(\phi_1(x,y), \phi_1(x,y)\phi_2(x,y) + p(\phi_2(x,y)),$$
$$(\phi_2(x,y))^2 + a(\phi_1(x,y))^2 + b\phi_1(x,y)\phi_2(x,y) + q(\phi_1(x,y), \phi_2(x,y))$$
$$= (x, xy + p'(y), y^2 + a'x^2 + b'xy + q'(x,y)),$$

for some $a', b' \in \mathbf{R}$, $p' \in \mathcal{M}^3$ and $q' \in \mathcal{M}_2^3$.

Comparing the first component implies that

$$\phi_1(x,y) = \lambda^{-1}x.$$

Substituting this in the second component then gives

$$x\phi_2(x,y) + p(\phi_2(x,y)) = xy + p(y).$$

This implies that

$$\phi_2(x,y) = y + \psi(x,y), \psi \in \mathcal{M}_2^2.$$

Suppose that $\psi(x,y) = a_0y^2 + a_1xy + a_2x^2 + O(3)$. Comparing the coefficients of $x^3$ on either side of the equation

$$xy + x\psi(x,y) + p(y + \psi(x,y)) = xy + p(y)$$

shows that $a_2 = 0$. From this it follows that $a_1 = 0$ and $a_0 = 0$. Then by induction on the degree of $x$ in each of the homogeneous parts of $\psi$, using the facts that $p \in \mathcal{M}_2^3$ and $\phi_2$ contains no terms linear in $x$, it can be shown that $\psi(x,y) \equiv 0$. Looking at the third component, we see that $\lambda$ must be 1. Similar considerations for the cases where $M = R_1, R_2$ and $R_3$ complete the result. □

The small size of the isotropy subgroup in this case indicates that we cannot significantly reduce the normal form any further when considering Euclidean properties of the crosscap.

## 3.3 Geometrical Properties of the Family of Crosscaps

We now determine some more geometrical information about the crosscaps parametrised by the family of normal forms of Proposition 3.1.1. In what follows we are considering the family of crosscaps parametrised by

$$f(x,y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x,y)), p \in \mathcal{M}_1^3, q \in \mathcal{M}_2^3.$$

**Lemma 3.3.1** Let $C \in \mathbf{R}^2$ be the preimage of the double point curve, and write $j^3 p = p_3 y^3$. Then $C$ can be written $x = -p_3 y^2 + \psi(y)$ where $\psi \in \mathcal{M}_1^3$ . Hence the limiting tangent to the double point curve in $\mathbf{R}^3$ at 0 is spanned by $(-p_3, 0, 1)$.

**Proof** From Lemma 3.1.5 we know that $C$ is a smooth curve that is tangent to the $y$-axis. Thus $C$ can be written $x = \alpha(y), \alpha \in \mathcal{M}_1^2$.

Now suppose $g'(x,y) = g'(X,Y)$. Then $X = x$ and we have

$$xy + p(y) = xY + p(Y)$$

and

$$ax^2 + bxy + y^2 + q(x,y) = ax^2 + bxY + Y^2 + q(x,Y).$$

So

$$x(Y - y) = -(p(Y) - p(y))$$
$$\Rightarrow x = -p_3(Y^2 + Yy + y^2) + O(3),$$

by the implicit function theorem. We also have

$$Y^2 - y^2 = -bx(Y - y) - q(x,Y) + q(x,y).$$

This implies that

$$Y + y = -bx - \frac{(q(x,Y) - q(x,y))}{(Y - y)}.$$

The right hand side of this equation is of order 2 in $Y$ and $y$ after substitution. So $Y = -y$ to first order and the result now follows. □

**Lemma 3.3.2** If $a > 0$, then there are two transverse curves in the source which are the preimages of curves of parabolic points on the crosscap. The gradients to these curves at the origin are $\pm\sqrt{a}$. In the case where $a < 0$, there are no parabolic points on the crosscap.

**Proof** A point is parabolic when the Gaussian curvature is zero at that point. We calculate the conditions for this to occur. Now away from a singular point, the Gaussian curvature at the point $p = f(x,y)$, $K(x,y)$ is given by

$$K(x,y) = \frac{ln - m^2}{EG - F^2},$$

where $E, F$ and $G$ are the coefficients of the first fundamental form, and $l, m$ and $n$ are the coefficients of the second fundamental form. So

$$K(x,y) = 0 \Leftrightarrow ln - m^2 = 0$$

$$\Leftrightarrow L(x,y) = (f_x \times f_y.f_{xx})(f_x \times f_y.f_{yy}) - (f_x \times f_y.f_{xy})^2 = 0.$$

In our case we can calculate the Gaussian curvature away from the origin. Now

$$f_x = (1, y, 2ax + by + q_x) \quad \text{and} \quad f_y = (0, x + p_y, 2y + bx + q_y).$$

The second derivatives of $f$ are

$$f_{xx} = (0, 0, 2a + q_{xx}), f_{xy} = (0, 1, b + q_{xy}) \quad \text{and} \quad f_{yy} = (0, p_{yy}, 2 + q_{yy}).$$

So

$$f_x \times f_y = (2y^2 - 2ax^2 + yq_y - xq_x - p_y q_x, -(2y + bx + q_y), x + p_y).$$

Then we have

$$j^2(L(x,y)) = 4ax^2 - 4y^2.$$

So if $a \neq 0$, $L(x,y)$ has a Morse singularity at the origin and the result follows.
□

Lemma 3.3.2 leads us to make the following definition:

**Definition 3.3.3** Let $g'(x,y)$ be the normal form obtained in Proposition 3.1.1. Then we define the *parabolic crosscap* to be any crosscap where $a > 0$ in $g'$, and the *hyperbolic crosscap* to be any crosscap where $a < 0$ in $g'$. Examples of the parabolic crosscap (with parabolic set indicated) and the hyperbolic crosscap that were drawn with the Liverpool Surfaces package are shown in Figures 3.2 and 3.3 respectively.

**Lemma 3.3.4** Suppose that $C$ is a non-singular curve in the source that passes through the origin, parametrised by $\gamma(t) = (\alpha t + \cdots, \beta t + \cdots)$, with $\alpha$ and $\beta$ not both zero. Then the limiting tangent plane to the crosscap as we approach the crosscap point along the curve parametrised by $f \circ \gamma$ is given by the equation

$$-(\alpha b + 2\beta)v + \alpha w = 0.$$

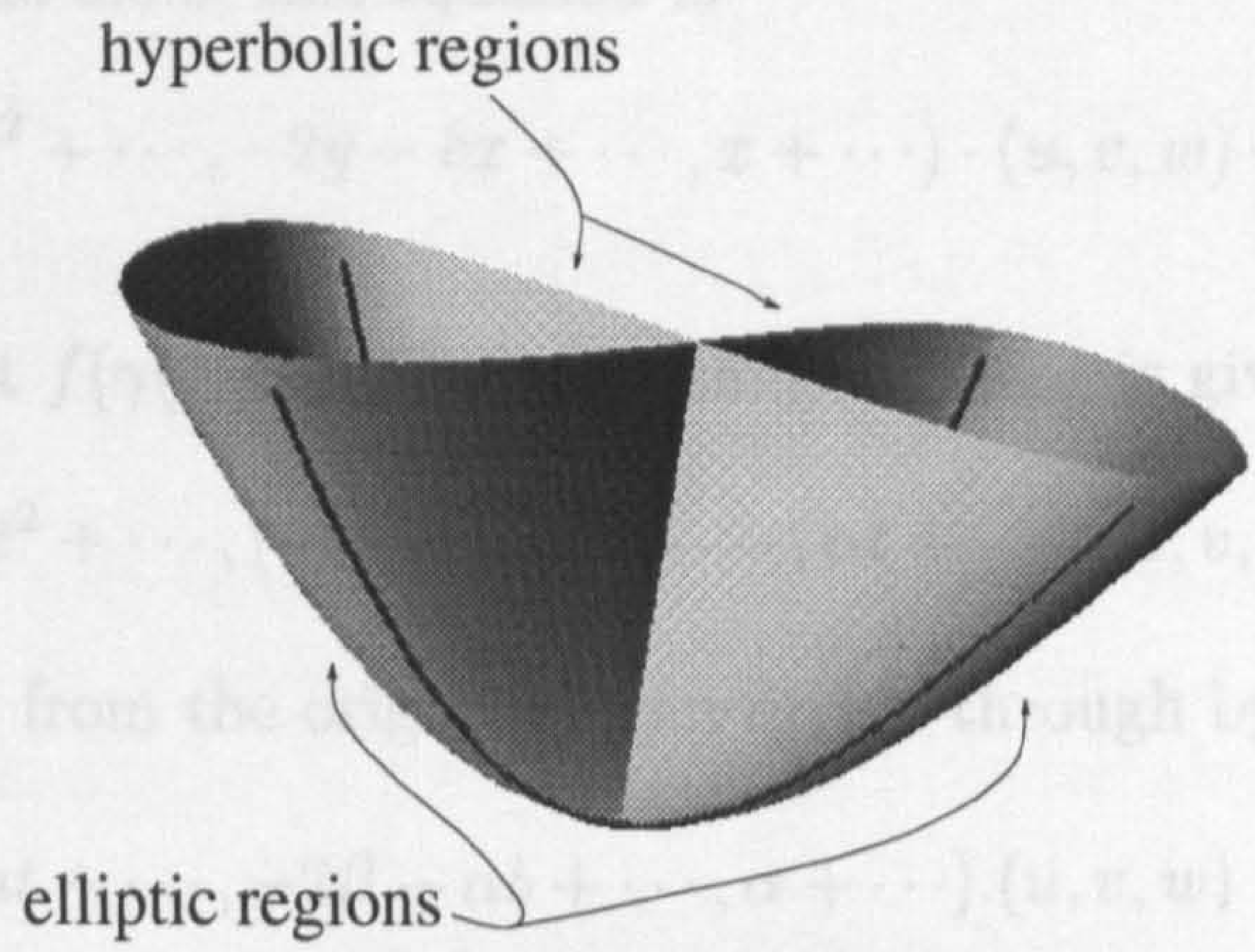



Figure 3.2: The parabolic crosscap parametrised by $(x, y) \mapsto (x, xy, y^2 + x^2)$.

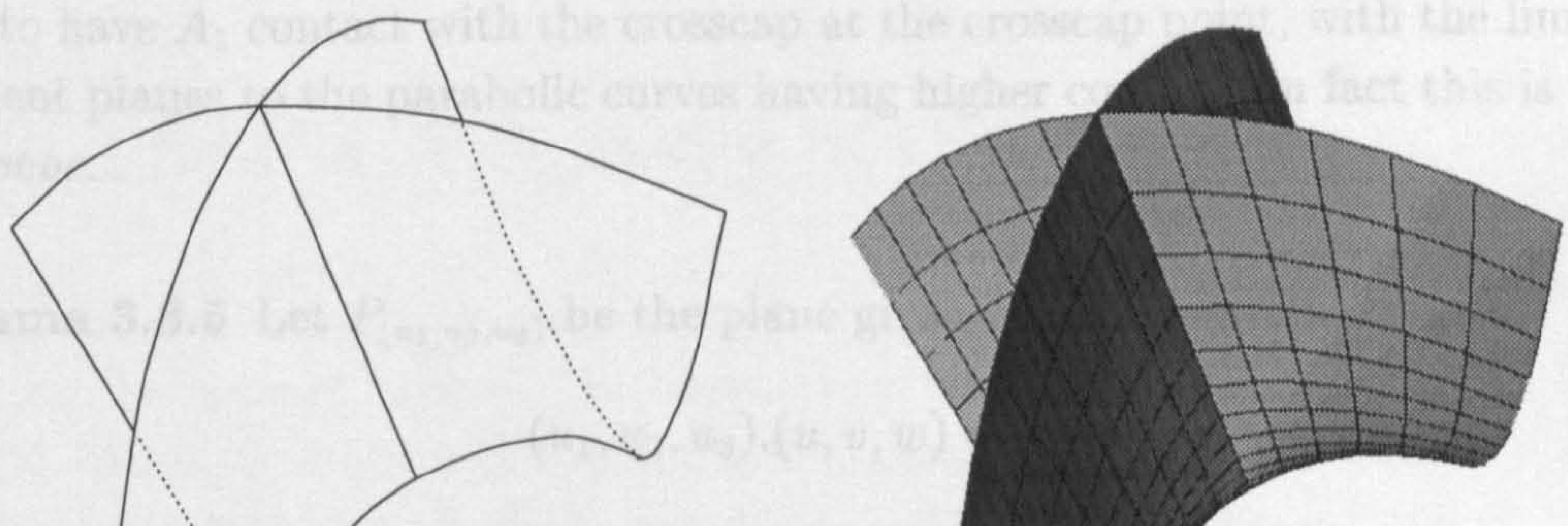


Figure 3.3: The hyperbolic crosscap parametrised by $(x, y) \mapsto (x, xy, y^2 - x^2)$.

**Proof** At any point $p = f(x,y)$ on the crosscap away from the crosscap point, the tangent plane is given by $(f_x \times f_y)\cdot(u,v,w) + K_{(x,y)} = 0$, where $K_{(x,y)}$ is a constant determined by the fact that the plane goes through the point $p$. From the proof of Lemma 3.3.2, this equation is

$$(2y^2 - 2ax^2 + \cdots, -2y - bx + \cdots, x + \cdots)\cdot(u,v,w) + K_{(x,y)} = 0.$$

So at any point $f(\gamma(t))$ on $f(C)$, the tangent plane is given by

$$(2\beta^2t^2 - 2\alpha^2at^2 + \cdots, (-2\beta - \alpha b)t + \cdots, \alpha t + \cdots).(u,v,w) + K_{(x,y)} = 0.$$

Since we are away from the origin, we can divide through by $t$, getting

$$(2\beta^2t - 2\alpha^2at + \cdots, -2\beta - \alpha b + \cdots, \alpha + \cdots).(u,v,w) + K_{(x,y)}/t = 0.$$

Letting $t$ tend to zero gives us the desired result. □

Now we calculate the contact of planes with the crosscap at the crosscap point. This is measured by composing the equation of the plane with the parametrisation of the surface, and seeing what types of singularities arise. If the function obtained in this way is a submersion, the plane is said to be transverse to the surface. When the plane is tangent to the surface, the function has a singularity and we label the contact with the type of this singularity. For more details, see Chapter 4. In the case of the crosscap, we might expect all those planes containing the tangent line to have $A_1$ contact with the crosscap at the crosscap point, with the limiting tangent planes to the parabolic curves having higher contact. In fact this is what happens.

**Lemma 3.3.5** Let $P_{(u_1,u_2,u_3)}$ be the plane given by the equation

$$(u_1,u_2,u_3).(u,v,w) = 0,$$

where $(u_1,u_2,u_3) \in S^2$ is a unit vector. Then all the planes $P_{(u_1,u_2,u_3)}$ are transverse to the crosscap at the crosscap point unless $u_1 = 0$.

The planes $P_{(0,u_2,u_3)}$ have $A_1$ contact with the crosscap parametrised by $f$, except in the case of the parabolic crosscap ($a > 0$) when the limiting tangent planes to the parabolic curves have $A_2$ contact with the crosscap. The equations for these planes are $w + (-b + 2\sqrt{a})v = 0$ and $w + (-b - 2\sqrt{a})v = 0$.

**Proof** Composing the equation of the plane $P_{(u_1,u_2,u_3)}$ with the parametrisation of the crosscap, we obtain the height function

$$H(x,y) = u_1 x + u_2(xy + p(y)) + u_3(y^2 + ax^2 + bxy + q(x,y)).$$

This is a submersion unless $u_1 = 0$, proving the first part of the lemma.

If $u_1 = 0$, we consider the 2-jet of $H(x,y)$. This is

$$u_3 y^2 + (u_2 + bu_3)xy + au_3 x^2.$$

The singularity of $H$ is of type $A_n$, for if not, the 2-jet would be identically zero giving $u_2 = u_3 = 0$: a contradiction. For $H$ to have an $A_{\geq 2}$ singularity, the discriminant of this quadratic form must vanish. So we need

$$(u_2 + bu_3)^2 - 4au_3^2 = 0$$

$$\Rightarrow u_2^2 + 2bu_2u_3 + (b^2 - 4a)u_3^2 = 0.$$

This has solutions if and only if $4b^2 - 4(b^2 - 4a) = 16a > 0$. So if $a > 0$, then $u_2 = (-b \pm 2\sqrt{a})u_3$. By Lemma 3.3.4 the planes defined by these values of $u_2$ and $u_3$ are the limiting tangent planes to the parabolic curve. □

**Remark 3.3.6** *It follows from Lemma 3.3.5 that a plane is transverse to the crosscap at the crosscap point if and only if it does not contain the tangent line to the crosscap.*

## 3.4 The Flat Geometry of the Crosscap

When we restrict our attention to the flat geometry of the crosscap, the normal form can be reduced further. Flat geometrical properties (those properties which can be measured in terms of contact with planes and lines) of the image of a map germ are preserved by all affine changes of coordinates in the target which fix the origin. These coordinate changes correspond to the subgroup $GL(3,\mathbb{R})$ of $\mathcal{L}$.

**Proposition 3.4.1** Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ be the map germ defined by $f(x,y) = (x, xy, y^2)$. Let $g$ be a map germ that is $\mathcal{A}$-equivalent to $f$. Then using affine changes in the target and diffeomorphisms in the source, we can reduce $g$ to the form

$$g'(x,y) = (x, xy + p(y), y^2 + ax^2 + q(x,y)),$$

where $p \in \mathcal{M}_1^4$ and $q \in \mathcal{M}_2^3$. When $a \neq 0$, we can fix the coefficient of $x^2$ in the third component to be 1 if $a > 0$ and $-1$ if $a < 0$. If we write $p(y) = p_4y^4 + \cdots$, and $p_4 \neq 0$, we can fix the coefficient of $y^4$ in the second component to be 1 if $p_4 \geq 0$ and $-1$ if $p_4 \leq 0$. Note that the conditions $p_4 \neq 0$ and $a \neq 0$ are both generic in the sense that they hold for an open dense set of map germs with normal form $g'$.

**Proof** By Proposition 3.1.1 , we can reduce such a map germ $g$ to the form

$$g'(x,y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x,y)),$$

where $a$ and $b$ are constants, $p \in \mathcal{M}_1^3$ and $q \in \mathcal{M}_2^3$. Assuming $a \neq 0$ (this is the generic case), it is not hard to see that $g'(x,y)$ can now be reduced to the form

$$(x, xy + p(y), y^2 \pm x^2 + q(x,y)),$$

where $p \in \mathcal{M}_1^3$ and $q \in \mathcal{M}_2^3$. The coefficient of $x^2$ in the third component is 1 if $a > 0$ and $-1$ if $a < 0$. Suppose that $p(y) = p_3y^3 + p_4y^4 + \cdots$. Replace $x$ by $x - p_3y^2$ to obtain

$$(x - p_3y^2, xy + p_4y^4 + \cdots, y^2 \pm x^2 + q'(x,y)),$$

for some $q' \in \mathcal{M}_2^3$. Next replace $u$ by $u + p_3w$ in the target to obtain $x \pm p_3x^2 + p_3q'(x,y)$ in the first component. We set this equal to $x'$, so that $x = x' \mp p_3x'^2 + r(x',y)$, with $r \in \mathcal{M}_2^3$. Reverting to $x$ our parametrisation is

$$(x, xy \mp p_3x^2y + yr(x,y) + p(y), y^2 \pm x^2 + q'(x,y)),$$

with $p \in \mathcal{M}_1^4$.

Now we can apply Lemma 3.1.2 in the same way as in the proof of Proposition 3.1.1 to obtain the result. The crucial point is that if we replace $y$ by $y \mp p_3xy$, then we can reduce the second component to $xy + ys(x,y)$, with $s \in \mathcal{M}_2^3$, and the reduction of Lemma 3.1.2 will not reintroduce any $y^3$ terms.

We have reduced our normal form to

$$(x, xy + p(y), y^2 + ax^2 + q(x,y)).$$

Let $p(y) = p_4y^4 + \cdots$, and $\lambda$ be such that $\lambda^2 = |p_4|$. Then by applying the coordinate changes

$$(x,y) \mapsto (\lambda x, \lambda y)$$

in the source and

$$(u,v,w) \mapsto (u/\lambda, v/\lambda^2, w/\lambda^2)$$

in the target, we can reduce the parametrisation to the required form. □

We now determine some of the geometrical properties of the crosscap that are fixed by the affine coordinate changes.

**Lemma 3.4.2** In the family of crosscaps parametrised by the normal form determined in Proposition 3.4.1 the limiting tangent to the double point curve is the $w$-axis. The preimage of the double point curve in the $(x,y)$-plane can be written $x = (\pm 2q_3 - p_5)y^4 + \psi$ where

$$\psi \in \mathcal{M}_1^5 \text{ and } p(y) = \pm y^4 + p_5 y^5 + \cdots, q(x,y) = q_3 y^3 + \cdots.$$

**Proof** The first part follows from Lemma 3.3.1 and Proposition 3.4.1.

For the second part we note that the $g'(x,y) = g'(X,Y)$ implies that $x = X$ and

$$x = \frac{-(p(Y) - p(y))}{(Y-y)} \quad \text{and} \quad Y + y = \frac{-(q(x,Y) - q(x,y))}{(Y-y)}.$$

Then

$$x = \mp(y^3 + y^2Y + yY^2 + Y^3) - p_5(y^4 + y^3Y + y^2Y^2 + yY^3 + Y^4) + \cdots$$

from the first equation; the second gives $Y = -y - q_3 y^2 + \cdots$. It follows that

$$x = (\pm 2q_3 - p_5)y^4 + \cdots.$$

□

**Remark 3.4.3** It follows from the proof of Lemma 3.1.5 that the preimage of the double point curve can always be written $x = f(y)$ with $f \in \mathcal{M}_1^2$. Moreover the Taylor series expansion of $f$ must start with an even power of $y$. For we have shown (Lemma 3.1.5) that a parametrisation for this curve is $\gamma(t) = (\gamma_1(t^2), \gamma_2(t))$. Writing $s = \gamma_2^{-1}(t)$, the parametrisation becomes $\gamma(s) = (\gamma_2(\gamma_1^{-1}(t)), t)$. We can also show this by considering the power series in the proof of Lemma 3.4.2.

**Lemma 3.4.4** The crosscap parametrised by the normal form determined in Proposition 3.4.1 has two parabolic curves if and only if $a > 0$. In this case we can fix $a$ to be 1.

Then the curves which are the preimages of the parabolic curves have gradients 1 and $-1$ at the origin and there are two planes with $A_2$ contact at the crosscap point given by the equations $w - 2v = 0$ and $w + 2v = 0$.

**Proof** The first part follows from Lemma 3.3.2 and Proposition 3.4.1. Lemma 3.3.2 then tells us that the gradients at the origin of the preimages of the parabolic curves are 1 and $-1$. Eliminating the $xy$ term in the third component using affine coordinate changes fixes the coefficient $b = 0$. Substituting the values $a = 1$, $b = 0$ into the equations for the planes with $A_2$ contact given in Lemma 3.3.5 completes the proof. □

Definition 3.3.3 can now be reduced to the following.

**Definition 3.4.5** When we are dealing with the flat geometry of the crosscap, we define the *parabolic crosscap* to be any crosscap parametrised by

$$g'(x,y) = (x, xy + p(y), y^2 + x^2 + q(x,y)),$$

and the *hyperbolic crosscap* to be any crosscap parametrised by

$$g'(x,y) = (x, xy + p(y), y^2 - x^2 + q(x,y)),$$

where $p \in \mathcal{M}_1^4$, $q \in \mathcal{M}_2^3$ and $p(y) = \pm y^4 + \cdots$ in both cases.

Now let $V_1$ and $V_2$ be the affine subspaces of $\mathcal{E}(2,3)$ of map germs with normal forms

$$f(x,y) = (x, xy + p(y), y^2 + x^2 + q(x,y))$$

and

$$f(x,y) = (x, xy + p(y), y^2 - x^2 + q(x,y)),$$

respectively, with $p \in \mathcal{M}_1^4, q \in \mathcal{M}_2^3$ and $p(y) = \pm y^4 + \cdots$ in both cases. So $V_1$ is the space of parabolic crosscaps, and $V_2$ is the space of hyperbolic crosscaps.

**Proposition 3.4.6** The isotropy subgroups of $GL(3,\mathbf{R}) \times \mathcal{R}$ with respect to $V_1$ and $V_2$ are both isomorphic to $\mathbf{Z}_2^2$.

**Proof** We show that the isotropy subgroups of $GL(3,\mathbf{R}) \times \mathcal{R}$ with respect to $V_1$ and $V_2$ both consist of the elements $(I, id)$ and $(-I, -id)$ where $I$ is the identity matrix, and $id : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ is the identity map. The generators commute and $(-I, -id)$ is of order two, and so the result follows.

We first consider $V_1$. Then by Lemmas 3.4.4 and 3.1.5, any right coordinate change $\phi$ which leaves $V_1$ invariant must fix the tangent lines at the origin to three curves in the source: the preimages of the double point curve and the two parabolic curves. Thus the derivative $d\phi_0$ must be a multiple of the identity map. So we write

$$\phi(x,y) = (\lambda x + \psi_1, \lambda y + \psi_2),$$

for some $\lambda \in \mathbf{R}/\{0\}$ and functions $\psi_1, \psi_2 \in \mathcal{M}_2^2$.

Lemmas 3.1.4, 3.1.6, 3.4.2 and 3.4.4 imply that any element $A$ of $GL(3,\mathbf{R})$ which leaves $V_1$ invariant must leave the configuration of the $u$-axis, the plane $v = 0$, the $w$-axis, and the planes given by the equations $w \pm 2v = 0$ fixed. These restrictions mean that the matrix must be of the form

$$A = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Now suppose that $(A, \phi)$ is in the isotropy subgroup of $GL(3,\mathbf{R}) \times \mathcal{R}$ with respect to $V_1$, then $A$ and $\phi$ can be written as above, and we have

$$\begin{aligned} x &= a_1(\lambda x + \psi_1) + a_2(\lambda^2 xy + \lambda y\psi_1 + \lambda x\psi_2 + \psi_1\psi_2 \\ &\quad + p(\lambda y + \psi_2)) \qquad (3.1) \\ xy + p'(y) &= b(\lambda^2 xy + \lambda y\psi_1 + \lambda x\psi_2 + \psi_1\psi_2 \\ &\quad + p(\lambda y + \psi_2)) \quad \text{and} \qquad (3.2) \\ y^2 + x^2 + q'(x,y) &= c((\lambda x + \psi_1)^2 + (\lambda y + \psi_2)^2 \\ &\quad + q((\lambda x + \psi_1), (\lambda y + \psi_2))), \qquad (3.3) \end{aligned}$$

for some $p'(y) = \pm y^4 + \cdots$ and $q' \in \mathcal{M}_2^3$. Now Equation (3.1) implies that $a_1 = 1/\lambda$, and gives us the identity

$$\frac{1}{\lambda}\psi_1 + a_2(\lambda^2 xy + \lambda y\psi_1 + \lambda x\psi_2 + \psi_1\psi_2 + p(\lambda y + \psi_2)) = 0, \qquad (3.4)$$

for all functions $p \in \mathcal{M}_1^4$ such that $p(y) = \pm y^4 + \cdots$. If we take $p(y) = y^4$, then Equation (3.4) implies that

$$\frac{1}{\lambda}\psi_1 + a_2(\lambda^2 xy + \lambda y\psi_1 + \lambda x\psi_2 + \psi_1\psi_2 + (\lambda y + \psi_2)^4) = 0.$$

Choosing $p(y) = y^4 + p_5 y^5 + \cdots$, Equation (3.4) implies that

$$a_2(p_5(y + \psi_2)^5 + \cdots) = 0,$$

so that $a_2 = 0$ and $\psi_1 = 0$. From equations (3.2) and (3.3), we see that $b = c = 1/\lambda^2$. Now we consider the equation

$$\frac{1}{\lambda^2}(\lambda x(\lambda y + \psi_2) + p(\lambda y + \psi_2)) = xy + p'(y),$$

for some $p' \in \mathcal{M}_1^4$ such that $p'(y) = \pm y^4 + \cdots$. We can apply a similar inductive argument to that at the end of Proposition 3.2.1 to show that $\psi_2$ must be identically zero. So far we have shown that if $(A, \phi)$ is in the isotropy subgroup of $GL(3, \mathbf{R}) \times \mathcal{R}$ with respect to $V_1$, then $A$ and $\phi$ can be written

$$A = \begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix}$$

and

$$\phi(x, y) = (\lambda x, \lambda y).$$

Equation (3.2) now implies that

$$xy \pm y^4 + \cdots = \frac{1}{\lambda^2}(\lambda^2 xy \pm \lambda^4 y^4 + \cdots),$$

so that $\lambda$ must be $\pm 1$.

When calculating the isotropy subgroup for the space $V_2$, we only know that the right coordinate change $\phi$ fixes the tangent to the preimage of the double point curve, that is the $y$-axis. This means that

$$\phi(x, y) = (\alpha x + O(2), \beta_1 x + \beta_2 y + O(2)).$$

The matrix $A$ must fix the $u$-axis, the $w$-axis and the plane $v = 0$. This means it must be of the form

$$A = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}.$$

So for the hyperbolic crosscaps, the coordinate changes that we can choose are less restricted by geometrical considerations. However the result still follows by comparing coefficients in a similar way to that above. □

# Chapter 4

# Height Functions on the Crosscap and the Dual of the Crosscap

A common approach when studying the flat geometry of smooth surfaces is to consider the composite of a parametrisation of the surface (usually in Monge form) with the 2 parameter families of projections to lines and planes. Now that we have obtained a normal form for the parametrisation of the crosscap, we are in a position to imitate this approach. In this chapter we consider singularities of height functions on the crosscap. This enables us to describe the parabolic set and the dual of the crosscap. We then use techniques from singularity theory to determine further information on the dual of the crosscap.

## 4.1 Height Functions and Duals of Smooth Surfaces

Information about the geometry of a surface can be obtained by considering its dual surface, which can be defined in the following way. Recall that the set of hyperplanes in real projective space $\mathbf{R}P^n$ forms a projective space, called the dual projective space and denoted $\mathbf{R}P^{n*}$. Now if $X$ is a surface in $\mathbf{R}^3$, we can embed $X$ in $\mathbf{R}P^3$ via a natural inclusion of $\mathbf{R}^3$ in $\mathbf{R}P^3$. At smooth points of $X$ each tangent plane corresponds to a point in $\mathbf{R}P^{3*}$. The *dual surface* to $X$, denoted by $X^*$ is the closure of the locus of these points. A fuller discussion of dual varieties can be found in [Har], page 196. In particular this reference contains a proof that the dual of the dual of a variety is the variety itself. So it is certainly the case that the dual of the dual of a smooth parametrised surface $M$ is $M$ itself.

Let $S^2$ be the unit sphere and $X$ be a smooth surface in $\mathbf{R}^3$. We consider the family of height functions

$$\begin{aligned} H : X \times S^2 \times \mathbf{R} &\to \mathbf{R} \\ (x, u, c) &\mapsto x \cdot u - c. \end{aligned}$$

Taking coordinates $(x_1, x_2, x_3)$ on $\mathbf{R}^3$, we can associate to each point

$$(u_1, u_2, u_3, c) \in S^2 \times \mathbf{R}$$

the plane with the equation $(x_1, x_2, x_3) \cdot (u_1, u_2, u_3) = c$ (note that $(u, c)$ and $(-u, -c)$ give the same plane). Thus if $H(x, u, c) = 0$ then the function

$$H_{u,c} : X, x \to \mathbf{R}, 0$$

measures the contact between the surface $X$ and the plane

$$(x_1, x_2, x_3) \cdot (u_1, u_2, u_3) = c.$$

Now we consider the function $h_u : X \to \mathbf{R}$ defined by

$$h_u(x) = H_{u,0}(x) = x \cdot u,$$

and the map

$$\begin{aligned} X \times S^2 &\to \mathbf{R} \times S^2 \\ (x, u) &\mapsto (h_u(x), u). \end{aligned}$$

The discriminant or set of critical values of this map can be thought of as an affine dual to $X$. For $c = h_u(x)$ is the height of the point $x$ above the origin in the direction $u$, and $h_u$ is singular at $x$ if and only if $u$ is normal to the surface $X$ at $x$. This is exactly the information required to reconstruct the tangent plane to $X$ at $x$; this plane is given by the equation $x \cdot u = c$. This approach to studying the geometry of the dual surface was first used in [B1]. Note that the map

$$\begin{aligned} \pi : S^2 \times \mathbf{R} &\to \mathbf{R}P^{3*} \\ (u_1, u_2, u_3, c) &\mapsto [u_1 : u_2 : u_3 : -c] \end{aligned}$$

is at each point a local diffeomorphism, and that $\pi$ doubly covers $X^*$.

In the case where $X$ is a smooth surface, we have the following:

**Proposition 4.1.1** Let $X$ be a generic smooth surface. Let $(x,u) \in X \times S^2$ with $u$ normal to $X$ at $x$. Then the height function $h_u(x)$ can only have one of the following types of singularity (up to $\mathcal{K}$-equivalence):

$$A_1^{\pm} \quad f(x,y) = x^2 \pm y^2,$$

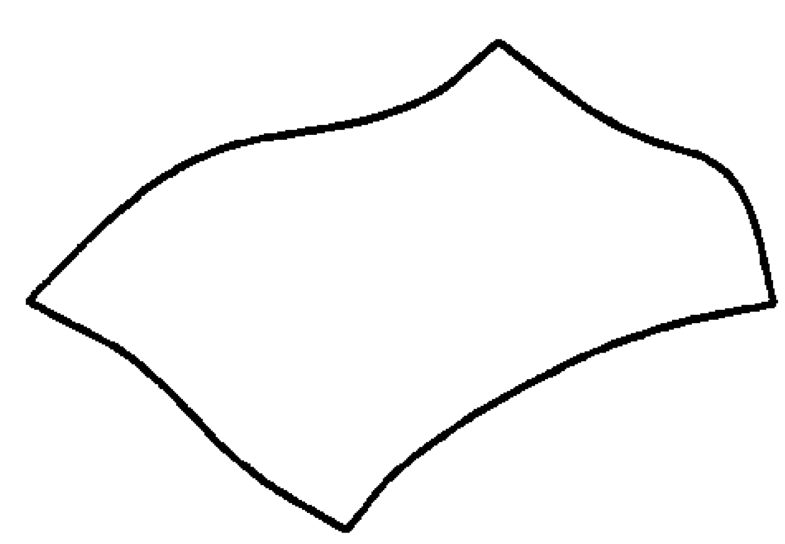
$$A_2 \quad f(x,y) = x^2 + y^3,$$

$$A_3^{\pm} \quad f(x,y) = x^2 \pm y^4.$$

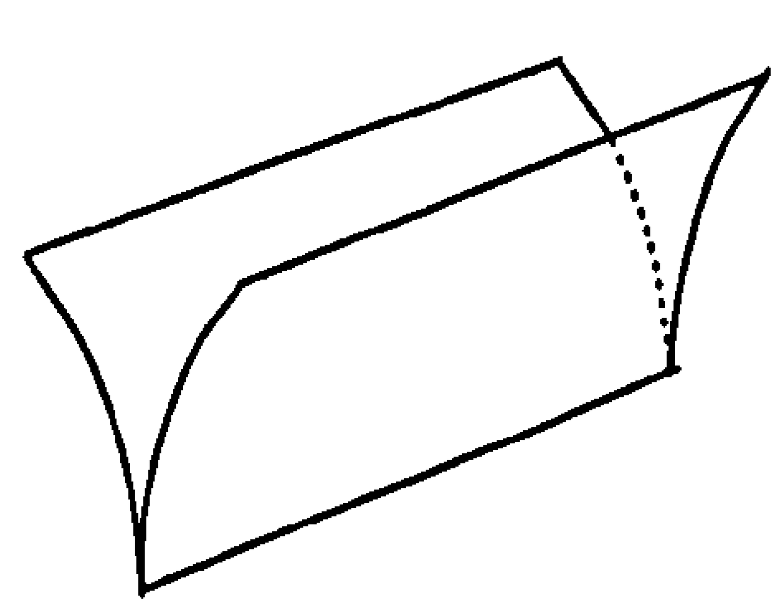
Moreover $h_u(x)$ will be versally unfolded by the family of height functions.

**Proof** For the first part see [BGM], or [B4] for a discussion of an argument which uses Monge Taylor expansions. The second part is due to Looijenga, and the proof can be found in [Wa2]. □

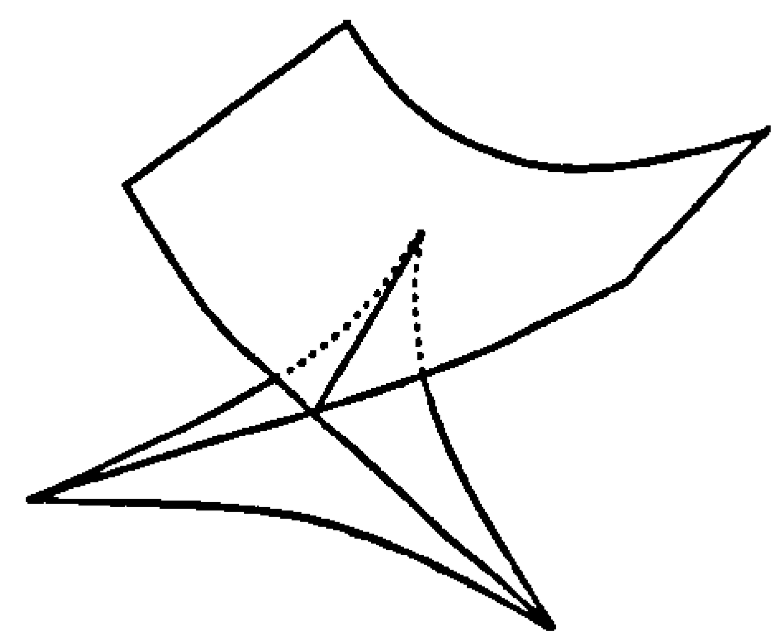
The height function has an $A_1^+$ singularity at an elliptic point, an $A_1^-$ singularity at a hyperbolic point, and a more degenerate singularity at a parabolic point. We expect curves of points where this singularity is $A_2$, and isolated points on these curves where the height function has an $A_3^{\pm}$ singularity, which are called cusps of Gauss. The advantage of these singularities being versally unfolded by the family of height functions is that we already know what their discriminants are, and thus have local models for the duals of the surface. So the dual is locally smooth in the $A_1$ case,



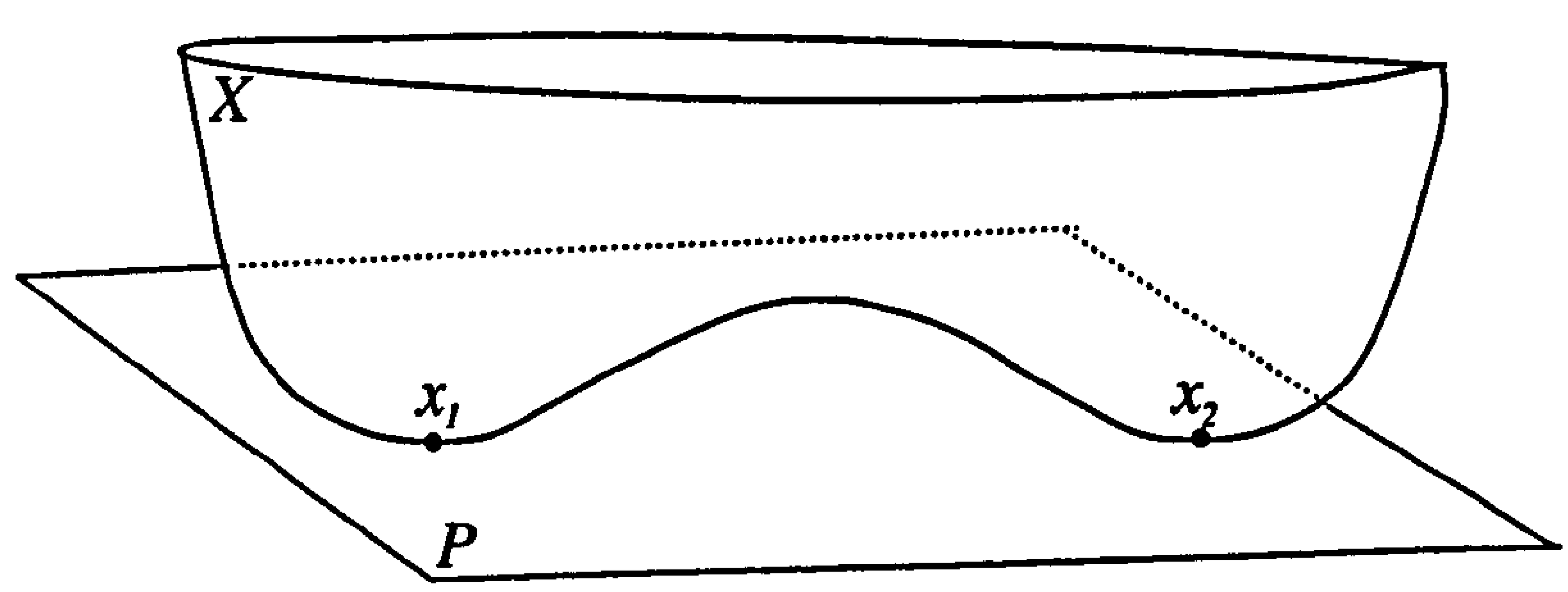
it has a cuspidal edge corresponding to the curves of parabolic points where the height function has an $A_2$ singularity,



and swallowtail points corresponding to the cusps of Gauss.



Height functions can also be used to study the set of planes which are tangent to a surface at more than one point. These correspond to multi-local singularities of the height function (singularities at all of which the height function has the same value). In general, if the plane $P$ is bitangent to a smooth surface $X$ at points $x_1$ and $x_2$, we expect that there will be other bitangent planes whose points of contact will be close to $x_1$ and $x_2$. So we have curves of pairs of points, along which certain planes will be bitangent. Intuitively, think of rolling the plane along the surface.




More precisely, we have

**Proposition 4.1.2** Let $X$ be a generic smooth surface. Then the possible multi-local singularities of the height function are

**(i) Type $A_1^2$.**

The height function $h_u$ has two $A_1$ singularities, at $x = x_1$, $x = x_2$, for which $h_u(x_1) = h_u(x_2)$ . Geometrically, there is a single plane perpendicular to $u$ and tangent to $X$ at $x_1$ and $x_2$. These singularities occur in curves. The family of height functions always versally unfolds such a singularity.

**(ii) Type $A_1A_2$.**

The height function $h_u$ has an $A_1$ singularity at $x_1$ and an $A_2$ singularity at $x_2$, with $h_u(x_1) = h_u(x_2)$. There is a single plane perpendicular to $u$ and tangent to $X$

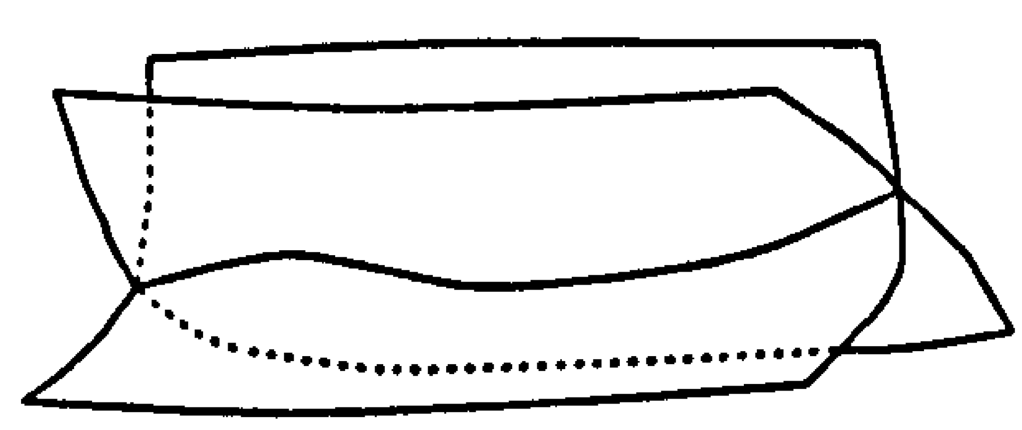
at the non-parabolic point $x_1$ and at the parabolic point $x_2$. These singularities occur at isolated points on the curves of $A_1^2$ singularities. This singularity is generically versally unfolded by the family of height functions as long as the single asymptotic direction at $x_2$ does not pass through $x_1$.

**(iii) Type $A_1^3$.**

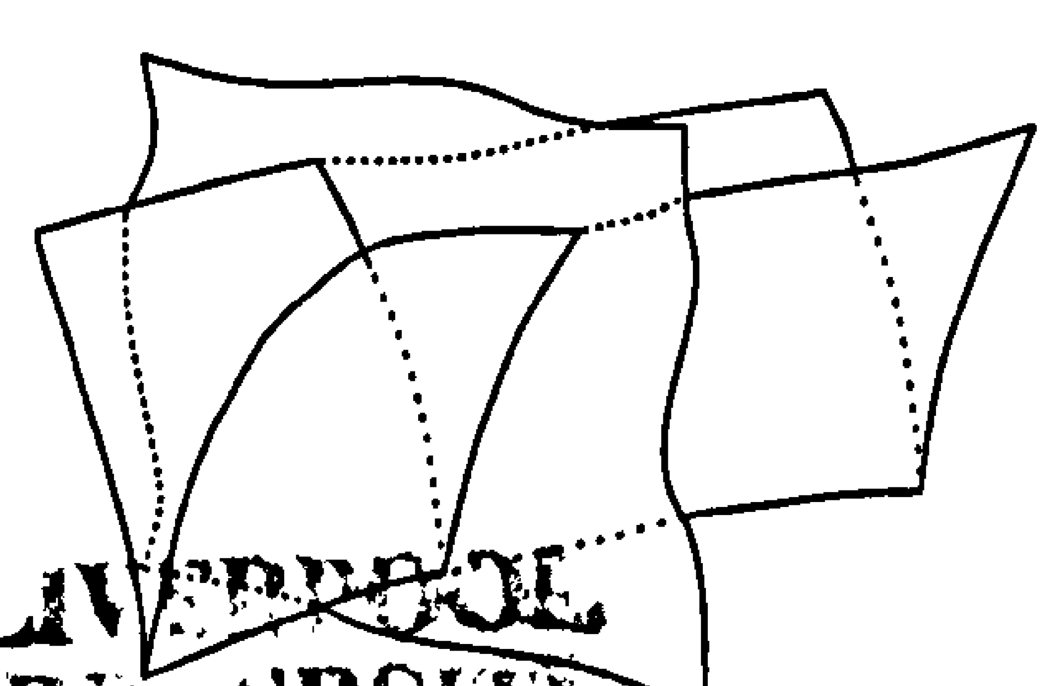
The height function $h_u$ has three $A_1$ singularities, at $x = x_1, x = x_2$ and $x = x_3$, for which $h_u(x_1) = h_u(x_2) = h_u(x_3)$. There is a single plane perpendicular to $u$ which is tangent to $X$ at three non-parabolic points. These singularities occur at isolated points. This singularity is versally unfolded by the family of height functions as long as $x_1$, $x_2$ and $x_3$ are not collinear.

**Proof** See [B1] for a proof that the types of multilocal singularities that can occur are those listed above. The proof of the geometrical discussion is in [BGT]. □

Notice that the conditions for the singularities to be versally unfolded are generic, so we have local models for the duals of generic surfaces when the height function has a multi-local singularity. If the height function has a versally unfolded $A_1^2$ singularity, the dual has a transverse crossing of two smooth sheets.

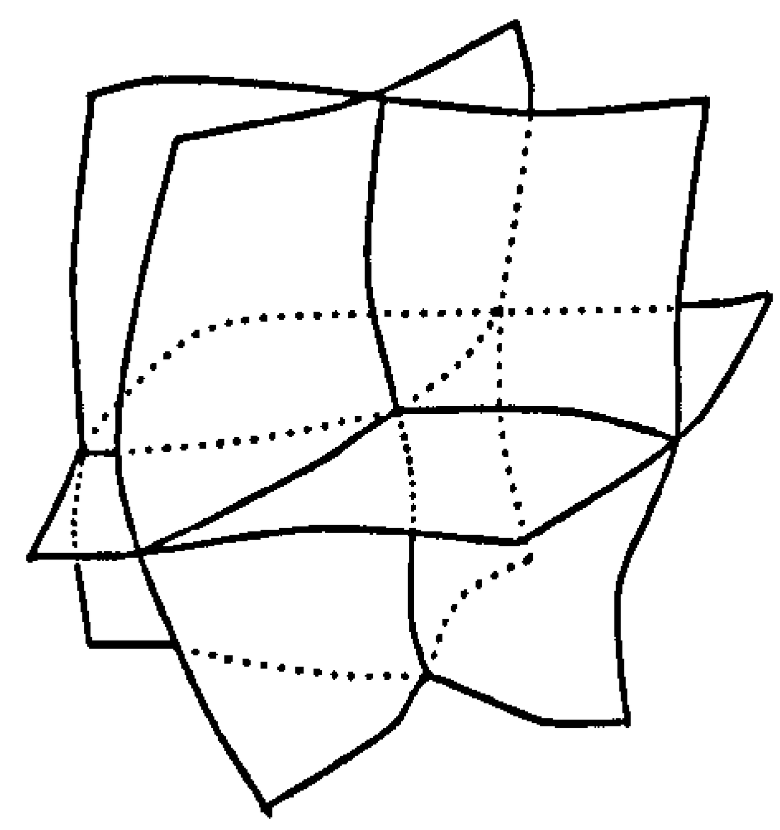


In the case of a versally unfolded $A_1A_2$ singularity, the dual has a crossing of a smooth sheet and a cuspidal edge.





In the case of an $A_1^3$ singularity, the dual has a triple point.



Finally we note the following result

**Proposition 4.1.3** Let $X$ be a smooth surface in $\mathbf{R}P^3$. Let $x$ be a point on $X$, and $x^*$ be the corresponding point on the dual surface to $X$, which we denote $X^*$. Then $x$ is elliptic (respectively hyperbolic) if and only if $x^*$ is elliptic (respectively hyperbolic).

**Proof** See [B3], Theorem 4. □

Note that in Proposition 4.1.3 by the term elliptic (respectively hyperbolic) we mean that the projective plane tangent to $X$ at $x$ has $A_1^+$ (respectively $A_1^-$ contact) with the surface.

## 4.2 Height Functions on the Crosscap

We have already discussed the contact of planes with the crosscap at the crosscap point (Lemma 3.3.5). Now we look at the singularities of the height functions in more detail. Since the height functions measure contact with planes (an aspect of flat geometry), we use the normal form

$$f(x,y) = (x, xy + p(y), y^2 + ax^2 + q(x,y)), p \in \mathcal{M}_1^4, q \in \mathcal{M}_2^3$$

in what follows. We have seen (Proposition 3.4.1) that $a$ could have been taken to be $\pm 1$, but we shall be interested in the transition as $a$ changes from positive to negative. Note also that we can suppose that $p(y) = \pm y^4 + \cdots$, but that this will not affect the geometry investigated in this chapter.

The family of height functions we shall consider is then

$$H(x,y,u_1,u_2,c) = ax^2 + y^2 + q(x,y) + u_1 x + u_2(xy + p(y)) + c.$$

Of course this only includes the directions in the 'hemisphere' $u_3 > 0$. We shall consider the directions this omits separately.

**Proposition 4.2.1** The function $H_{u,c}$ has a singularity at $(x, y) = (0, 0)$ if and only if $u_1 = 0$. This is of type $A_1$ if and only if $u_2^2 - 4a \neq 0$. In particular the singularities are all of type $A_1$ if and only if $a < 0$. Otherwise we expect $A_2$ singularities precisely when $u_2 = \pm 2\sqrt{a}$ and these will be versally unfolded.

**Proof** The first part of this follows from Lemma 3.3.5. For the second part, note that we have the function

$$H_{(0,\pm 2\sqrt{a},1),0}(x,y) = ax^2 + y^2 + q(x,y) \pm 2\sqrt{a}(xy + p(y))$$
$$= (\sqrt{a}x \pm y)^2 + q(x,y) \pm 2\sqrt{a}p(y).$$

Writing $x' = \sqrt{a}x \pm y$ and $y' = y$, this becomes

$$x'^2 + q(\frac{1}{\sqrt{a}}(x' \mp y'), y') \pm 2\sqrt{a}p(y').$$

For an $A_2$ singularity, we need the coefficient of $y'^3$ to be non-zero. This coefficient is $q_3(\mp\frac{1}{\sqrt{a}}, 1)$, where $q_3$ is the homogeneous cubic part of $q$. The condition that this is zero is not generic.

Finally we show that these $A_2$ singularities will be versally unfolded. Since we have an $A_2$ singularity at $(x, y) = (0, 0)$, its Jacobian ideal $J$ contains $\mathcal{M}_2^2$. It is straightforward to check that the partial derivatives $\partial H/\partial u_1$ and $\partial H/\partial u_2$ span the quotient $\mathcal{M}_2/J$ when evaluated at $u_1 = 0$, $u_2 = \pm 2\sqrt{a}$, so that the standard criterion for versality is satisfied. □

Now the family of height functions $H$ misses out those functions corresponding to the directions $(u_1, (1 - u_1^2)^{1/2}, 0)$. In the latter case, the corresponding height function is a submersion. For the directions near to $(0, 1, 0)$ we need to consider the following family:

$$G(x, y, u_1, u_2, c) = u_1 x + u_2(ax^2 + y^2 + q(x, y)) + (xy + p(y)) + c.$$

Now the function $G$ has a singularity at $(x, y) = (0, 0)$ if and only if $u_1 = 0$. This is of type $A_1$ unless $1 - 4au_2^2 = 0$. In particular we see that to get a more degenerate singularity, $u_2 \neq 0$. This case is covered by Proposition 4.2.1, since if $u_2 \neq 0$, $G(x, y, u_1, u_2, c) = u_2 H(x, y, u_1, 1/u_2, c/u_2)$.

From Lemma 3.3.2, we know that the parabolic set in the source is empty when $a < 0$ and consists of two smooth curves of gradient $\pm 2\sqrt{a}$ when $a > 0$. The transition between these two cases is of some interest. When $a = 0$, the family of height functions is now

$$H(x, y, u_1, u_2, c) = y^2 + q(x, y) + u_1 x + u_2(xy + p(y)) + c.$$

**Proposition 4.2.2** The function $H_{u,c}$ has a singularity at $(x, y) = (0, 0)$ if and only if $u_1 = 0$, and is of type $A_1$ unless $u_2 = 0$. When $u_1 = u_2 = 0$ we expect an $A_2$ singularity, which will be versally unfolded.

**Proof** The first part of this is clear. In the case where $u_1 = u_2 = 0$, then for the singularity to be more degenerate than $A_2$, the coefficient of $x^3$ in $q(x, y)$ would have to be zero. This is not a generic condition. A calculation similar to that in the proof of proposition 4.2.1 shows that this $A_2$ singularity is generically versally unfolded. □

As before, we need to check the height functions in the directions missed out by the family $H$. It is clear that the height function corresponding to the direction $(1, 0, 0)$ is a submersion. For the directions near to $(0, 1, 0)$ we must consider the family

$$G(x, y, u_1, u_2, c) = u_1 x + u_2(y^2 + q(x, y)) + (xy + p(y)) + c.$$

The function $G$ has an $A_1$ singularity at $(x, y) = (0, 0)$ if and only if $u_1 = 0$.

We can use the height functions to determine the parabolic set in the source. From Lemma 3.3.2 we know that this consists of two transverse curves intersecting at the origin when $a > 0$ and a single point (the origin) when $a < 0$. Now we consider the transition between these two cases.

**Proposition 4.2.3** When $a = 0$, the parabolic set in the source is a cusp. Moreover the parametrisation of this set is versally unfolded by $a$.

**Proof** Let

$$H(x, y, u_1, u_2, c, a) = y^2 + ax^2 + q(x, y) + u_1 x + u_2(xy + p(y)) + c.$$

The parabolic set can be obtained by projecting the set given by $H_x = H_y = H_{xx}H_{yy} - H_{xy}^2 = 0$ in $(x, y, u_1, u_2, c, a)$-space into $(x, y)$-space. Now we have

$$H_x = 2ax + q_x + u_2 y + u_1, H_y = 2y + u_2(x + p'(y)) + q_y,$$

$$H_{xx} = 2a + q_{xx}, H_{xy} = u_2 + q_{xy} \quad \text{and} \quad H_{yy} = 2 + q_{yy} + p''(y).$$

Considering $H_x = 0$, we see that $u_1 = -(2ax + q_x + u_2 y)$. Since $u_1$ only appears in this equation, we need only consider $H_y = H_{xx}H_{yy} - H_{xy}^2 = 0$. Writing $u$ instead of $u_2$, we get

$$2y + u(x + p'(y)) + q_y = 0 \tag{4.1}$$

and

$$(2a + q_{xx})(2 + q_{yy} + p''(y)) - (u + q_{xy})^2 = 0. \tag{4.2}$$

Now by the implicit function theorem, Equation (4.1) implies that we can write $y$ as a function of $x$ and $u$. Writing

$$q(x, y) = q_0 y^3 + q_1 x y^2 + q_2 x^2 y + q_3 x^3 + q_4 x^4 + \cdots,$$

we see that the lowest order terms of its Taylor expansion are given by

$$y = -1/2(ux + q_2 x^2) + \cdots. \tag{4.3}$$

Expanding Equation (4.2), we get

$$4a + 12q_3 x + 4q_2 y + x^2(24q_4 + 12q_1 q_3 - 4q_2^2) + 4q_1 ax - 4q_2 ux - u^2$$

$$+ y\phi_1(x, y, u, a) + \phi_2(x, a, u) = 0, \phi_1 \in \mathcal{M}_4, \phi_2 \in \mathcal{M}_3^3.$$

Substituting for $y$ gives

$$4a + 12q_3 x + x^2(24q_4 + 12q_1 q_3 - 6q_2^2) + 4q_1 ax - 6q_2 ux - u^2 + \phi_3(x, a, u) = 0,$$

where $\phi_3 \in \mathcal{M}_3^3$. By the implicit function theorem, we can now write $x$ as a function of $u$ and $a$, and the lowest order terms are given by

$$x = -1/12q_3(4a - u^2 - 24q_2 au + 16(q_1 + 24q_4 + 12q_1 q_3 - 6q_2^2)a^2) + \cdots. \tag{4.4}$$

Substituting Equation (4.4) for $x$ in Equation (4.3), we obtain the following parametrisation of the parabolic set in the source;

$$\psi(u, a) = (-1/12q_3(4a - u^2 - 24q_2 au) + \cdots, 1/24q_3(4au - u^3 - 24q_2 au^2) + \cdots).$$

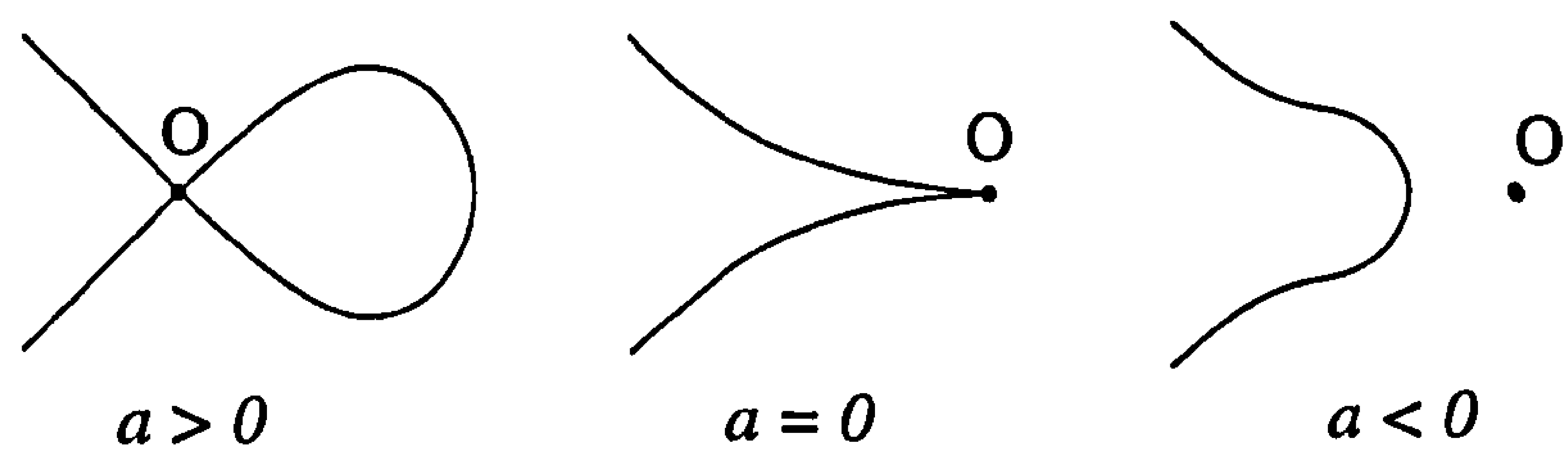
When $a = 0$, this is generically a cusp.

Now for the second part, we note that if $g(t) = (t^2, t^3)$ then $L\mathcal{A}_e.g \supset \mathcal{M}_1^2.\mathcal{E}(1, 2)$. Since $\psi_0 = \psi(u, 0)$ is $\mathcal{A}$-equivalent to $g$, we just need to check that

$$\mathbf{R} \cdot \left\{ \left. \frac{\partial \psi}{\partial a} \right|_{a=0} \right\} + L\mathcal{A}_e.\psi_0 \supset \mathcal{E}(1, 2)/\mathcal{M}_1^2.\mathcal{E}(1, 2).$$

It is clear that this is the case. □

Proposition 4.2.3 gives us a picture of how the parabolic set behaves away from the crosscap point. The versal unfolding of the cusp is shown below, with the origin marked in each case.




So in the case of the parabolic crosscap ($a > 0$), rather than having two distinct parabolic curves in the source, we can think of having one, which self intersects at the origin. As $a$ tends to zero, this crossing becomes a cusp, and then when $a < 0$ (the hyperbolic crosscap) the parabolic set moves away from the crosscap point altogether. Of course, as we are working locally, we cannot really say what happens to the parabolic set away from the crosscap point.

## 4.3 The Dual of the Crosscap

What can we say about the dual of a surface with a crosscap point? Since we are dealing with height functions, we use the normal form that corresponds to flat geometry in what follows. Recall that this is given by

$$f(x, y) = (x, xy + p(y), y^2 + ax^2 + q(x, y)),$$

where $p \in \mathcal{M}_1^4$ and $q \in \mathcal{M}_2^3$. In this case, we have seen (Proposition 4.2.1) that there is a pencil of planes with higher order contact with the crosscap at the crosscap point. So there is a projective line $l$ corresponding to these planes which lies on the dual.

**Proposition 4.3.1** In the case where the crosscap has no parabolic points in a neighbourhood of the crosscap point ($a < 0$ in the normal form $f$) the dual is smooth in a neighbourhood of $l$. When $a > 0$, and there are two parabolic lines which pass through the crosscap point, the dual is smooth in a neighbourhood of $l$ except at the two points which are dual to the two planes with $A_2$ contact at the crosscap point. At these points the line $l$ meets one of the cuspidal edges of the dual surface which correspond to the parabolic lines at the crosscap point.

**Proof** This result follows from Proposition 4.2.1. As we know that the singularities of the family of height functions are versally unfolded, we can use the local models for the dual discussed above. □

**Remark 4.3.2** We can deduce from Proposition 4.3.1 that the dual has no self intersections, and thus there are no bitangent planes in a neighbourhood of the crosscap point.

Note that along the double point curve of the crosscap, the surface consists of a transverse crossing of two smooth sheets. So there will always be a family of bitangent planes to the dual of the crosscap, corresponding to the double point curve. We have shown that the limiting position of the tangent planes to the crosscap along the double point curve as we approach the crosscap point is the tangent cone (Lemma 2.1.5). So there will be curves of pairs of points on the dual, along which we have bitangency. The curve will pass through the point on the projective line $l$ which corresponds to the tangent cone, where we have tangency. It turns out that this point is a flat umbilic. We prove this later on.

## 4.4 The Local Structure of the Dual

We are interested in the structure of the dual in a neighbourhood of the line $l$. We have seen that the dual arises naturally as the discriminant of the family of height functions. Previously, we have considered families of height functions in certain neighbourhoods of the directions $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ which cover the 2-sphere. Lemma 3.3.5 tells us that the height functions in directions close to $(1,0,0)$ are all submersions, and so we only need to consider the other two families when considering the geometry of the dual.

More formally, the crosscap is defined as a parametrised surface in $\mathbf{R}^3$. We can embed $\mathbf{R}^3$ in $\mathbf{R}P^3$ by $(u,v,w) \longrightarrow [u : v : w : 1]$. This embedding takes planes to planes. The point $[\alpha : \beta : \gamma : \delta] \in \mathbf{R}P^{3*}$ represents the projective plane

$$\alpha u + \beta v + \gamma w + \delta t = 0.$$

We can take the affine chart $t = 1$ on $\mathbf{R}P^3$ because of the way we have embedded the crosscap in projective space. This gives us the equation

$$\alpha u + \beta v + \gamma w + \delta = 0.$$

Now we can choose various affine charts on $\mathbf{R}P^{3*}$ to get the equation of a plane in $\mathbf{R}^3$. However we know from Lemma 3.3.5 that if the crosscap is parametrised by the normal form $f$ then $l$ is the set $\{[0:\beta:\gamma:0]:\beta,\gamma\in\mathbf{R}\}$. Thus to look at the structure of the dual in a neighbourhood of $l$, we need only use the affine charts $\beta=1$ and $\gamma=1$. These give rise to the equations

$$\alpha u+v+\gamma w+\delta=0 \quad \text{and}$$

$$\alpha u+\beta v+w+\delta=0.$$

Composing these equations with the parametrisation of the crosscap $f$, we see that we can work in $\mathbf{R}^3$ with the two families of height functions

$$G(x,y,u_1,u_2,c)=xy+p(y)+u_1x+u_2(ax^2+y^2+q(x,y))+c \quad \text{and}$$

$$H(x,y,u_1,u_2,c)=ax^2+y^2+q(x,y)+u_1x+u_2(xy+p(y))+c.$$

We can get local models for the dual by considering the discriminants of these two families. In more down to earth terms, they are obtained from the solution sets of the equations $G=G_x=G_y=0$ and $H=H_x=H_y=0$ in $(x,y,u_1,u_2,c)$-space by projecting into $(u_1,u_2,c)$-space by eliminating the $x$ and $y$ variables. Working this way, the projective line corresponds to the $u_2$-axis in both cases. In what follows, we work mainly with the family $H$. We start with a preliminary result.

**Proposition 4.4.1** The tangent space to the dual of the crosscap is constant along the line $l$ and is given by $\delta=0$ (the plane $c=0$ in $(u_1,u_2,c)$- space).

**Proof** We would expect this to be the case by a duality argument. Consider the case where $X$ is a smooth surface in $\mathbf{R}P^3$. Then at a point $x\in X$, the dual point $x^*\in X^*$ corresponds to the tangent plane $T_xX$. So the tangent plane in the dual space $T_{x^*}X^*$ corresponds to the point $x$. Thus a point $p^*\in T_{x^*}X^*$ corresponds to a plane in $\mathbf{R}P^3$ which contains $x$. So $T_{x^*}X^*$ consists of all planes in $\mathbf{R}P^3$ which pass through $x$. Now in the case of the crosscap, each point on the projective line $l$ corresponds to a plane which is tangent to the crosscap at the crosscap point. It follows that we expect that the tangent plane to $l$ to remain constant, and to correspond to the crosscap point, that is it should consist of all planes that pass through the origin. This plane is in fact given by $\delta=0$.

Now we show that this is indeed the case by working with the family of height functions $H$. We evaluate the matrix

$$\begin{pmatrix} H_x & H_y & H_{u_1} & H_{u_2} & H_c \\ H_{xx} & H_{xy} & H_{xu_1} & H_{xu_2} & H_{xc} \\ H_{yx} & H_{yy} & H_{yu_1} & H_{yu_2} & H_{yc} \end{pmatrix}$$

at all points along the $u_2$ axis, given by $(0,0,0,t,0)$. This matrix is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 2a & t & 1 & 0 & 0 \\ t & 2 & 0 & 0 & 0 \end{pmatrix},$$

which has maximal rank unless $4a - t^2 = 0$. It follows that away from the points $(0,0,0,\pm 2\sqrt{a},0)$ when $a > 0$ (the cuspidal edges in the case of the parabolic crosscap), every point on the $u_2$ axis is a regular point of the map $\mathbf{R}^5 \longrightarrow \mathbf{R}^3$ given by $(x,y,u_1,u_2,c) = (H, H_x, H_y)$. Thus the equations $H = H_x = H_y = 0$ determine a smooth surface at each point $(0,0,0,t,0)$. Now $kerM$ is contained in the set $c = 0$ in $(x,y,u_1,u_2,c)$-space, so projecting into $(u_1,u_2,c)$-space it is clear that the tangent plane to the $u_2$-axis is the plane $c = 0$. The same considerations for the family of height functions $G$ complete the result. □

From the results already established, we know that in the case where the crosscap has two parabolic curves passing through the crosscap point, there are two points $(u_1,u_2,c) = (0,\pm\sqrt{a},0)$ on the dual where the line $l$ meets the corresponding cuspidal edges. We wish to understand how these cuspidal edges meet $l$ and the tangent plane ($c = 0$) to the dual along $l$.

First we consider general smooth curves on cuspidal edges. In what follows, we choose coordinates $t$, $(x,y)$, and $(u,v,w)$ for $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$ respectively. We shall say that $f : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$ parametrises a cuspidal edge if it is $\mathcal{A}$-equivalent to the map germ

$$\begin{aligned} h : \mathbf{R}^2, 0 &\longrightarrow \mathbf{R}^3, 0 \\ (x,y) &\longrightarrow (x, y^2, y^3), \end{aligned}$$

and that the standard cuspidal edge is the surface with defining equation $v^3 = w^2$. Associated to any parametrisation of a cuspidal edge, $f$, there are two distinguished directions through the origin in $\mathbf{R}^2$; the kernel of the derivative $df_0$, and the tangent to the preimage of the cuspidal edge (the critical set of $f$).

**Lemma 4.4.2** Let $f : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$ parametrise a cuspidal surface, and $\alpha : \mathbf{R}, 0 \longrightarrow \mathbf{R}^2, 0$ parametrise a smooth curve $C$. Suppose that $C$ is transverse to $kerdf_0$, and has $k$-point contact with the critical set of $f$. Then we can choose smooth coordinate changes in $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$ taking the cuspidal edge to the standard cuspidal edge and such that $f \circ \alpha$ is reduced to the map $t \mapsto (t, t^{2k}, t^{3k})$.

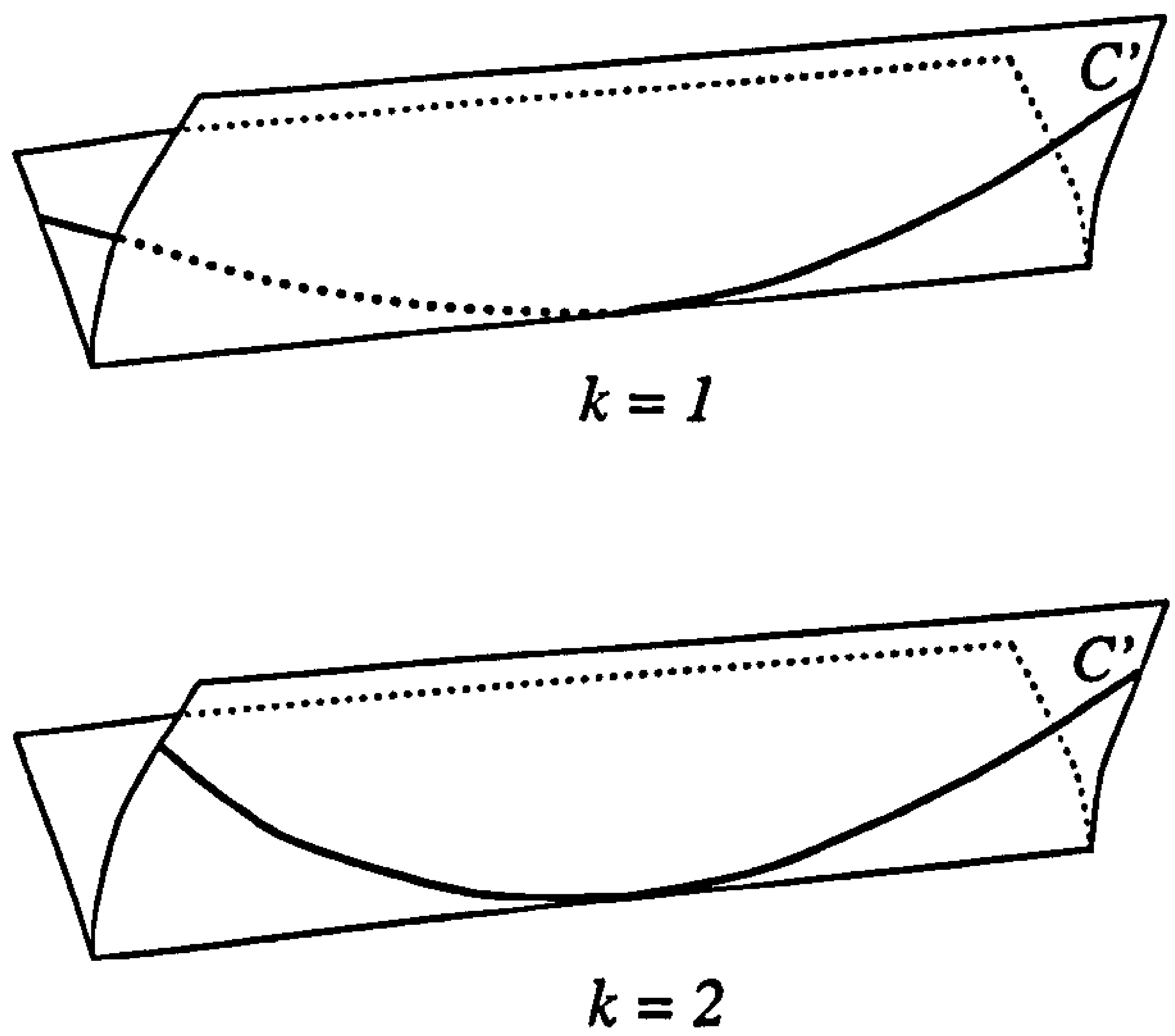



Figure 4.1: The configuration of the curve $C'$ and a cuspidal edge when $k = 1$ and $k = 2$.

**Proof** Since $f$ is $\mathcal{A}$-equivalent to $h$ we may suppose that we are dealing with the standard parametrisation $h(x, y) = (x, y^2, y^3)$ without affecting the hypotheses. Now $kerdh_0$ is the $y$-axis and the preimage of the cuspidal edge is the $x$-axis, so we can choose coordinate changes in $\mathbf{R}$ and $\mathbf{R}^2$ so that $\alpha$ is of the form $(t, t^k\beta(t))$ where $\beta(0) \neq 0$. Then we have

$$h \circ \alpha(t) = (t, t^{2k}\beta^2(t), t^{3k}\beta^3(t)).$$

Finally consider the coordinate change

$$\begin{array}{rcl} \phi : \mathbf{R}^3, 0 & \longrightarrow & \mathbf{R}^3, 0 \\ (u, v, w) & \longrightarrow & (u, v\beta^2(u), w\beta(u)^3). \end{array}$$

This is a diffeomorphism and an automorphism of the standard cuspidal edge, and if we compose $\phi$ with the map $t \mapsto (t, t^{2k}, t^{3k})$, we get $t \mapsto (t, t^{2k}\beta^2(t), t^{3k}\beta^3(t))$. So applying $\phi^{-1}$ to $h \circ \alpha$ gives us the result. □

Lemma 4.4.2 shows that the configuration of the curve $C'$ parametrised by $f \circ \alpha$ and the cuspidal surface is determined (up to diffeomorphism) by the configuration in the source of the curve $C$ and $kerdf_0$, and the degree of contact of $C$ with the preimage of the cuspidal edge. In the cases $k = 1$ and $k = 2$, the configuration is shown in Figure 4.1.

Returning to the dual of the parabolic crosscap, since this surface is locally smooth at all points of $l$ away from where $l$ meets the cuspidal edges, we would expect the configuration of $l$ and the cuspidal edges to be diffeomorphic to that of Figure 4.1, where $k = 1$. We now show that this is indeed the case. We work with the height function $H$, and since we are working with the parabolic crosscap, we can set $a = 1$ to get

$$H(x, y, u_1, u_2, c) = x^2 + y^2 + q(x, y) + u_1 x + u_2(xy + p(y)) + c.$$

**Proposition 4.4.3** Take an affine chart on projective space close to where each cuspidal edge meets the line $l$ . Then locally (close to the points $(0, \pm 2\sqrt{a}, 0)$ in $(u_1, u_2, c)$-space) there exist smooth coordinate changes taking the cuspidal edge on the dual to the standard cuspidal edge, and so that the projective line is taken to the curve parametrised by the map $t \mapsto (t, t^2, t^3)$.

**Proof** To see how $l$ meets the cuspidal edges we apply Lemma 4.4.2 in the case where $k = 1$. To do this we need to be able to locate the the cuspidal surface and the cuspidal edge. Again we work in $(x, y, u_1, u_2, c)$-space. The set of points corresponding to the cuspidal surface, say $S$, is given by $H = H_x = H_y = 0$, with the cuspidal edge points corresponding to the subset given by $\Delta = H_{xx}H_{yy} - H_{xy}^2 = 0$. It follows from the versality of the unfolding (Proposition 4.2.1) that $S$ is a smooth surface (and so diffeomorphic to $\mathbf{R}^2$) and that the projection $\pi : S \longrightarrow \mathbf{R}^3$ to the $(u_1, u_2, c)$-space is a parametrisation of a cuspidal edge. The conditions of Lemma 4.4.2 are diffeomorphism invariant, so we just need to ensure that at the points $(0, 0, 0, \pm 2\sqrt{a}, 0)$ the $u_2$-axis which lies inside $S$ does not coincide with the kernel of $d\pi$ and has 1-point contact with the preimage of the cuspidal edge. The first condition holds trivially. For the second we need to check that $\partial\Delta/\partial u_2$ is non zero at the relevant points. Now we have

$$H_x = 2x + q_x + u_2 y + u_1, H_y = 2y + u_2(x + p'(y)) + q_y,$$

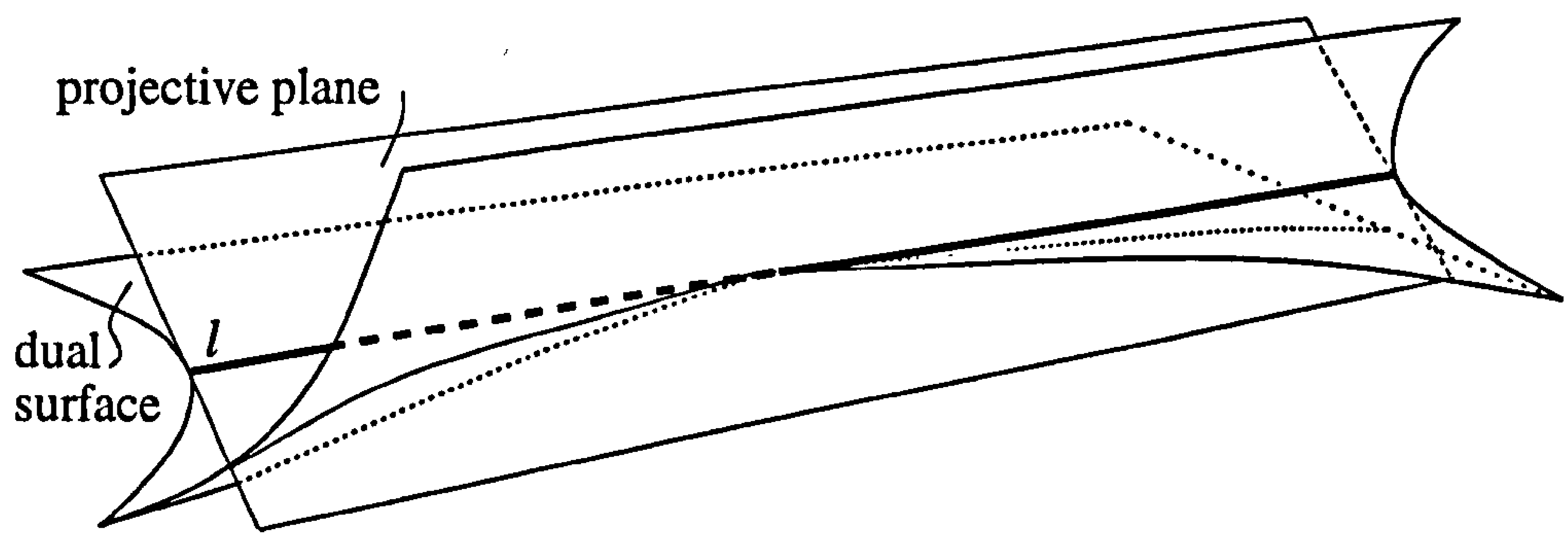
$$H_{xx} = 2 + q_{xx}, H_{xy} = u_2 + q_{xy}, H_{yy} = 2 + q_{yy} + p''(y),$$

so that

$$\Delta = (2 + q_{xx})(2 + q_{yy} + p''(y)) - (u_2 + q_{xy})^2.$$

It follows that $\partial\Delta/\partial u_2 = 2(u_2 + q_{xy}) \neq 0$ at the points $(0, 0, 0, \pm 2\sqrt{a}, 0)$. □

Thus we have the following picture of how $l$ meets the cuspidal edge. The plane (given by $c = 0$) that is tangent to the dual along $l$ is also depicted.




Now we consider the transition between the case of the parabolic crosscap (where there are two cuspidal edges passing through the line $l$) and the hyperbolic crosscap (where there are no cuspidal edges passing through $l$). When $a = 0$, so that the crosscap is parametrised by

$$f(x, y) = (x, xy + p(y), y^2 + x^2 + q(x, y)), p \in \mathcal{M}_1^4, q \in \mathcal{M}_2^3,$$

the height function $H$ becomes

$$H(x, y, u_1, u_2, c) = y^2 + q(x, y) + u_1 x + u_2(xy + p(y)) + c.$$

In this case Proposition 4.2.2 tells us that there is just one cuspidal edge passing through the line $l$ at the point $(0, 0, 0)$ in $(u_1, u_2, c)$-space.

**Proposition 4.4.4** Close to the point $(0, 0, 0)$ in $(u_1, u_2, c)$-space, there exist smooth coordinate changes taking the cuspidal edge on the dual to the standard cuspidal edge, and so that the projective line is taken to the curve parametrised by the map $t \mapsto (t, t^4, t^6)$.

**Proof** This time we check the hypotheses of Lemma 4.4.2 for the case $k = 2$. The set of points corresponding to the cuspidal surface, say $S$, is given by $H = H_x = H_y = 0$, with the cuspidal edge points corresponding to the subset given by $\Delta = H_{xx}H_{yy} - H_{xy}^2 = 0$. It follows from the versality of the unfolding (Proposition 4.2.2) that $S$ is a smooth surface (and so diffeomorphic to $\mathbf{R}^2$) and that the projection $\pi : S \longrightarrow \mathbf{R}^3$ to the $(u_1, u_2, c)$-space is a parametrisation of a cuspidal edge. This time we need to ensure that at the points $(0, 0, 0, 0, 0)$ the $u_2$-axis which lies inside $S$ does not coincide with the kernel of $d\pi$ and has 2-point contact with the preimage of the cuspidal edge. The first condition holds trivially. For the second we need to check that $\partial\Delta/\partial u_2 = 0$, and $\partial^2\Delta/\partial u_2^2 \neq 0$ at $(x, y, u_1, u_2, c) = (0,0,0,0,0)$ . Now we have

$$H_x = q_x + u_2 y + u_1, H_y = 2y + u_2(x + p'(y)) + q_y,$$

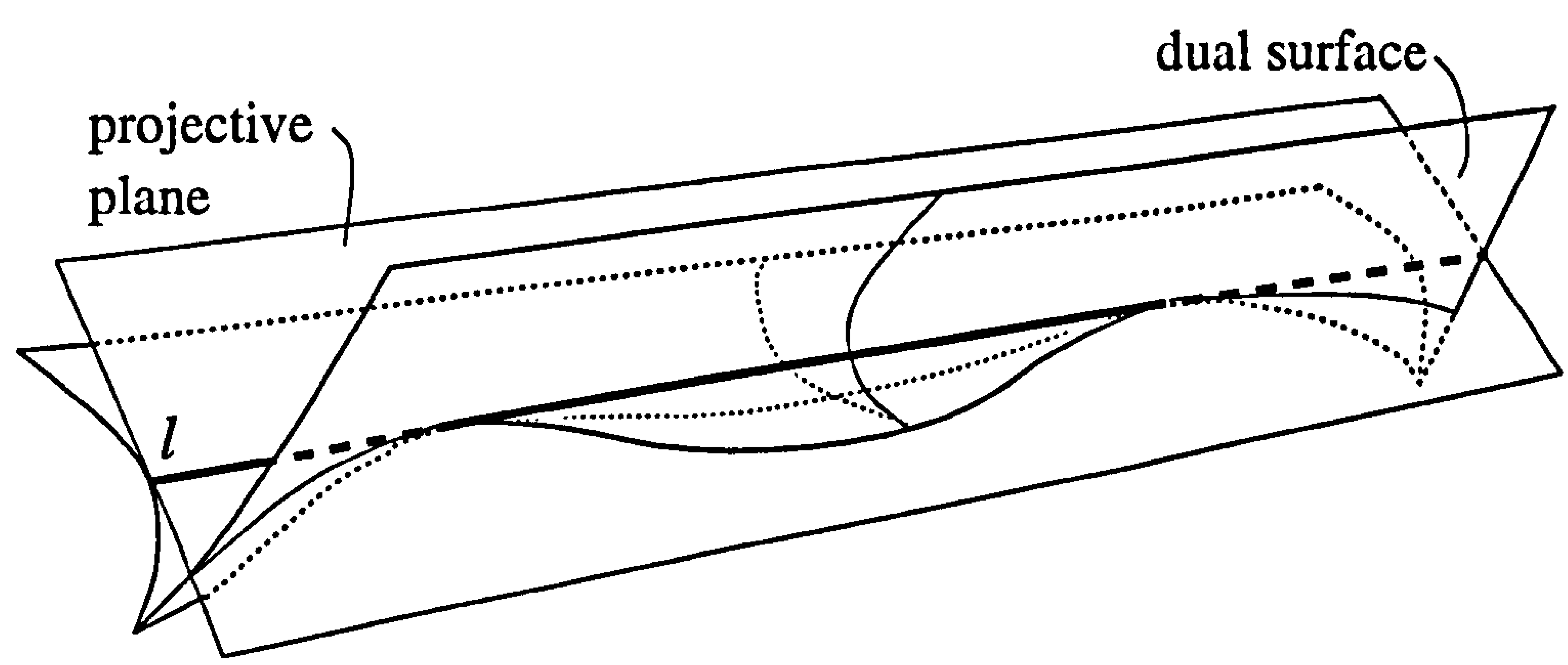
$$H_{xx} = q_{xx}, H_{xy} = u_2 + q_{xy}, H_{yy} = 2 + q_{yy} + p''(y),$$

which implies that

$$\Delta = (q_{xx})(2 + q_{yy} + p''(y)) - (u_2 + q_{xy})^2.$$

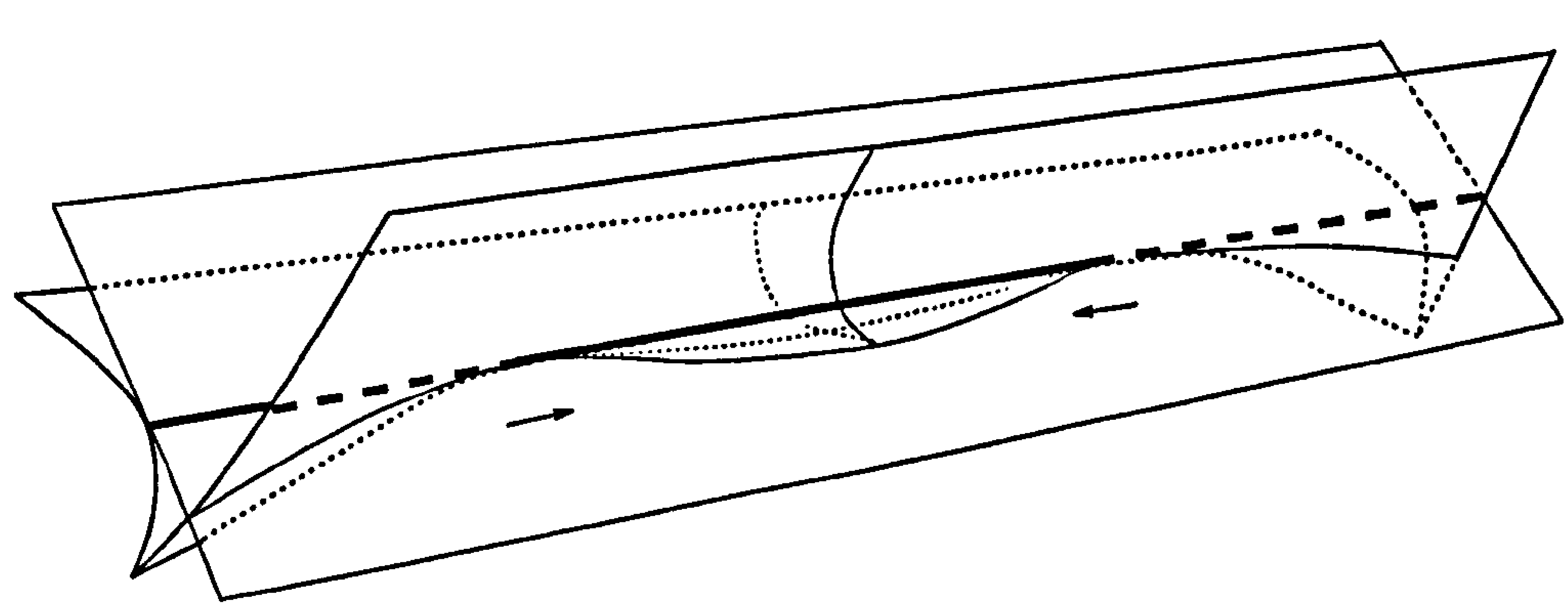
It follows that $\partial\Delta/\partial u_2 = 2(u_2 + q_{xy}) = 0$, and $\partial^2\Delta/\partial u_2^2 = 2$ at $(0,0,0,0,0)$, as required. □

Bearing in mind the result of Proposition 4.2.3, we depict the dual of the parabolic crosscap with one cuspidal edge, which meets $l$ at the two points $(0, \pm\sqrt{a}, 0)$.



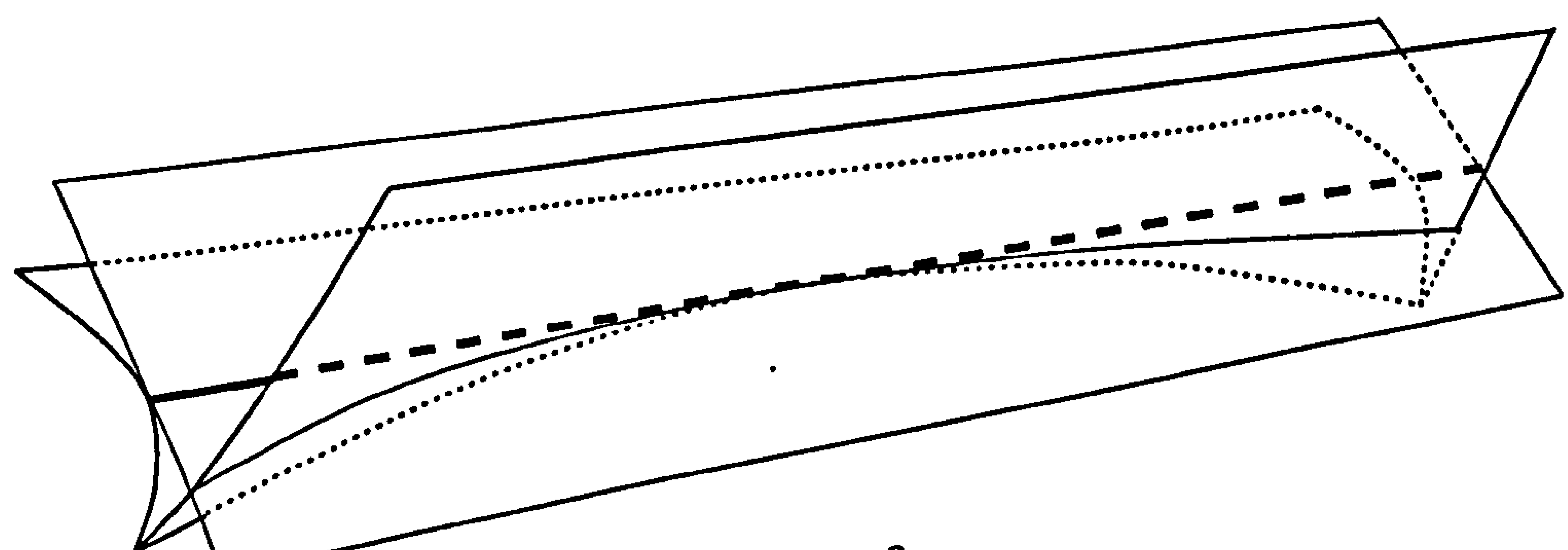

$a > 0$

As $a$ tends to zero, these two points tend towards $(0,0,0)$.



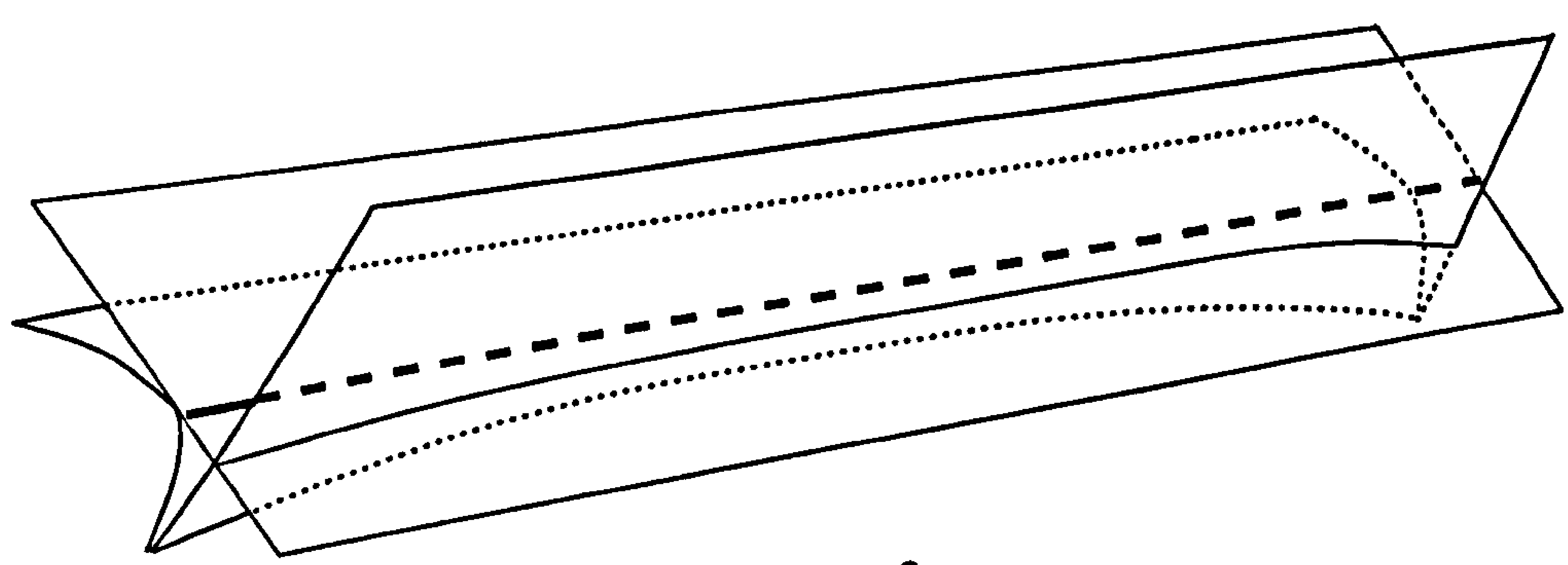
$a \to 0$

By Proposition 4.4.4, when $a = 0$, $l$ remains on one sheet of the cuspidal surface.



$a = 0$

Finally, when $a < 0$, we have the hyperbolic crosscap, and $l$ does not meet the cuspidal edge at all; there are no parabolic points in a neighbourhood of the crosscap point.



$a < 0$

## 4.5 The Global Structure of the Dual

The next question to arise concerns the global structure of the dual in a neighbourhood of the projective line $l$. Intuitively, we wish to see how many twists the dual has in a neighbourhood of this line. Is the dual topologically equivalent to a cylinder or a Mobius band? Now from Proposition 4.1.3, we know that away from the cuspidal edge and the line $l$, all the points on the dual are hyperbolic or elliptic, since this is the case on the crosscap away from the crosscap point and the possible cuspidal edges. What sort of contact does the tangent plane to the dual have with the dual surface along $l$? We shall see that the answer to this question will help us to determine the global structure of the dual.

Since the tangent plane to the dual is constant along the line $l$, the function obtained by composing a parametrisation of the dual with the equation of the plane will not be one of the standard finitely determined germs. We shall relate the tangency to a different classification, where the relevant germs *are* finitely determined. To do this we need to introduce a new equivalence relation. For the time being we shall revert to $(x, y, z)$ coordinates. We denote by $\mathcal{R}_\delta$ the subgroup of the group of diffeomorphisms $\mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ of the $(x, y)$-plane which preserve the $y$-axis. Using this in place of the usual group $\mathcal{R}$ we obtain a subgroup of the contact group $\mathcal{K}$ denoted by $\mathcal{K}_\delta$. We refer the reader to [A3] for a discussion of the group $\mathcal{R}_\delta$. We note here that $\phi \in \mathcal{R}_\delta$ is a diffeomorphism of the form $\phi(x, y) = (x\phi_1(x, y), \phi_2(x, y))$, where $\phi_1$ and $\phi_2$ are smooth functions. Next we note that for functions, the notion of $\mathcal{K}$-equivalence reduces to the following. Let $f_1 : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ and $f_2 : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ be two smooth functions. Then $f_1$ and $f_2$ are $\mathcal{K}$-equivalent if and only if there exist a smooth function $\lambda : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}$, with $\lambda(0) \neq 0$ and a diffeomorphism $\phi : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ such that $f_1 \circ \phi = \lambda f_2$. For $\mathcal{K}_\delta$-equivalence, we ask further that $\phi$ must be an element of $\mathcal{R}_\delta$.

**Proposition 4.5.1** Suppose that we have a smooth germ $h : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$ and that the surface $z = h(x, y)$ is tangent to the plane $z = 0$ along the $y$-axis.

(a) Under the above hypotheses the function $h$ can be written $h = x^2 f(x, y)$ for some smooth $f$.

(b) Given two such functions $h_i = x^2 f_i(x, y)$, for $i = 1, 2$, let $X_1$ and $X_2$ be the surfaces given by $z = h_1(x, y)$ and $z = h_2(x, y)$ respectively. Then the functions $f_1$ and $f_2$ are $\mathcal{K}_\delta$-equivalent if and only if there is a diffeomorphism $\mathbf{R}^3, 0 \rightarrow \mathbf{R}^3, 0$ taking $X_1$ to $X_2$ and preserving the plane $z = 0$ and the $y$-axis.

**Proof** (a) From the hypothesis, we know that $h(0, y) = 0$ (since the surface contains the $y$-axis), and $h_x(0, y) = h_y(0, y) = 0$ (since the plane $z = 0$ is tangent to the surface along the $y$-axis). We apply Hadamard's Lemma (see [BG2], page 86) to the first of these equations to show that $h$ can be written $xg(x, y)$. The equation $h_x(0, y)$ then shows that $g(0, y) = 0$; we apply Hadamard's Lemma again to get the result.

(b) For the second part suppose that there is a diffeomorphism $\psi = (\psi_1, \psi_2, \psi_3) : \mathbf{R}^3, 0 \rightarrow \mathbf{R}^3, 0$ with the required properties. Then since $\psi$ preserves the plane $z = 0$ we can write $\psi_3(x, y, z) = zA(x, y, z)$. It also preserves the line $x = z = 0$ so we can write $\psi_1(x, y, z) = xB(x, y, z) + zC(x, y, z)$. Note also that $A(0)B(0) \neq 0$.

Now we compose $\psi$ with the parametrisation of $X_1$ to get

$$(xB(x,y,x^2f_1) + x^2f_1C(x,y,x^2f_1), \psi_2(x,y,x^2f_1), x^2f_1(x,y)A(x,y,x^2f_1)).$$

We can rewrite this as

$$(xA'(x,y), B(x,y), x^2f_1(x,y)C'(x,y)),$$

with $A'(0)C'(0) \neq 0$. Now since $\psi$ takes $X_1$ to $X_2$ we have

$$\begin{aligned}(xA'(x,y), B(x,y), x^2f_1(x,y)C'(x,y)) \\ = (x', y', x'^2f_2(x', y'))\end{aligned}$$

for some $(x', y') \in \mathbf{R}^2$, and it is clear that $f_1$ and $f_2$ are $\mathcal{K}_\delta$-equivalent.

Conversely if $f_1$ and $f_2$ are $\mathcal{K}_\delta$-equivalent then we can find a diffeomorphism $\phi(x,y) = (x\phi_1(x,y), \phi_2(x,y))$ and a germ of a non zero function $\lambda : \mathbf{R}^2, 0 \to \mathbf{R}$ with $f_2 \circ (x\phi_1, \phi_2) = (\lambda f_1)(x,y)$. Then the diffeomorphism $(x,y,z) \longmapsto (x\phi_1, \phi_2, z\lambda\phi_1^2)$ preserves the plane $z = 0$, the $y$-axis and takes the surface $X_1$ to the surface $X_2$. □

We now need to determine what type of contact the tangent plane to the dual has with the dual surface along the line $l$. The two simplest types of germs (up to $\mathcal{K}_\delta$-equivalence) are those equivalent to the constant function 1 and $y$. Any function which does not vanish at the origin (respectively is a submersion restricted to the $y$-axis) is equivalent to the first (respectively second) of these. The surfaces given by $z = x^2$ and $z = x^2y$ are depicted in Figure 4.2. Notice that the second of these surfaces is written as the graph of a function with no quadratic terms. Thus both its principal curvatures are zero and it has a flat umbilic at the origin. Moreover the function is of type $D_\infty$. For general smooth surfaces in $\mathbf{R}^3$, this is very degenerate.

**Proposition 4.5.2** We consider the dual of the crosscap by taking the two affine charts on $\mathbf{R}P^3$ which correspond to the families of height functions $G$ and $H$. The affine chart corresponding to $H$ misses out the point [0:1:0:0] on the line $l$. This point corresponds to the tangent cone. Along this part of $l$, a local model for the dual is diffeomorphic to $z = x^2$. The affine chart corresponding to $G$ misses out the point [0:0:1:0]. The origin in $(u_1, u_2, c)$-space corresponds to the tangent cone. Close to the origin on this part of $l$, a local model for the dual is given by $x = x^2y$. Consequently the point on $l$ corresponding to the tangent cone is a flat umbilic.

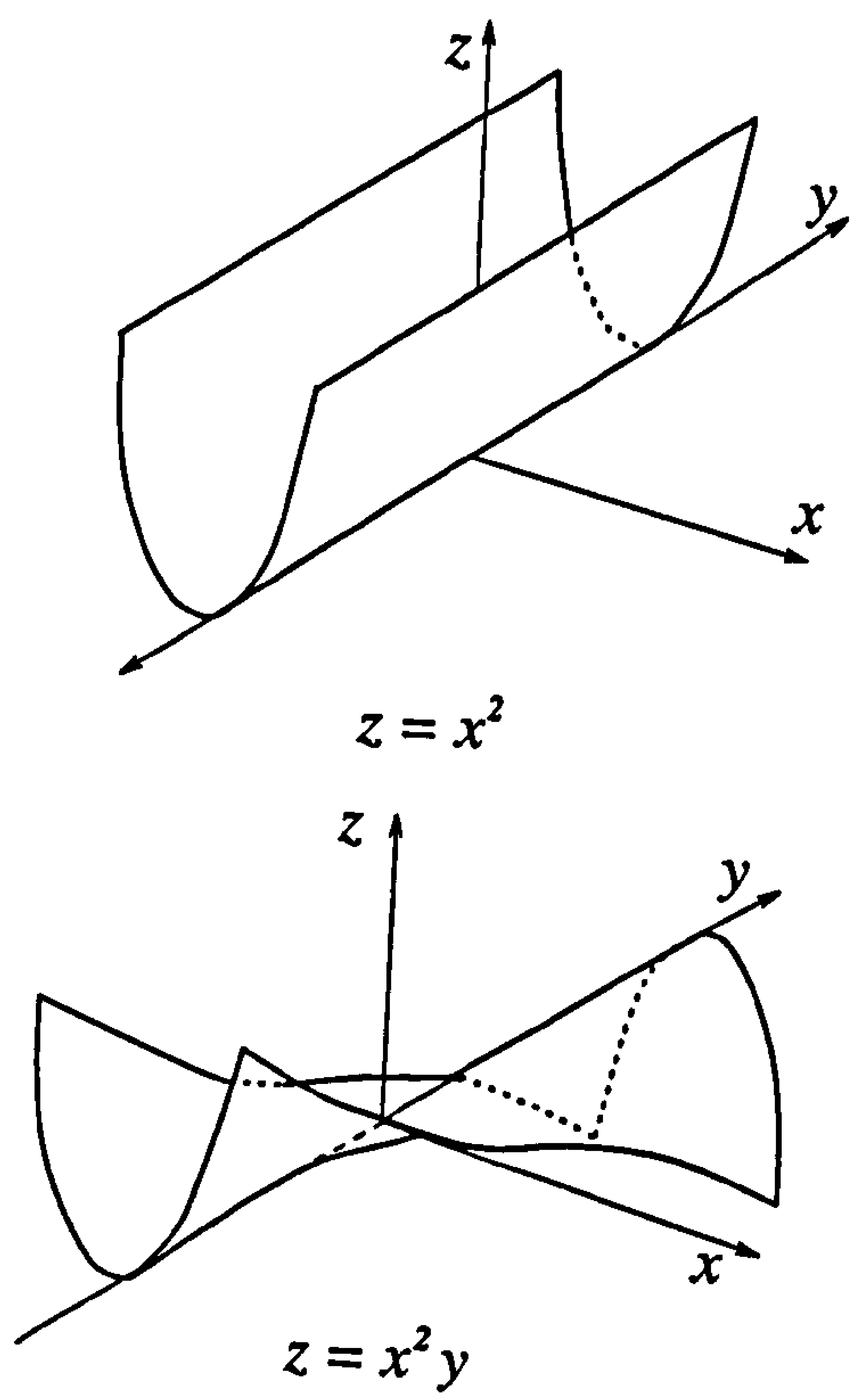



Figure 4.2: The surfaces $z = x^2$ and $z = x^2y$.

**Proof** First we work with the family $H$. The dual is given by the equations $H = H_x = H_y = 0$. This gives the following equations:

$$y^2 + ax^2 + q(x,y) + u_1 x + u_2(xy + p(y)) + c = 0, \tag{4.5}$$

$$2ax + q_x + u_1 + u_2 y = 0 \tag{4.6}$$

and

$$2y + q_y + u_2 x + u_2 p'(y) = 0. \tag{4.7}$$

We wish to look close to each point $(0, t_0, 0)$ on the $u_2$-axis, and so we make the substitution $u_2 = t_0 + t$. By the implicit function theorem, we can write $x$ and $y$ as functions of $u_1$ and $t$ as long as $t_0^2 - 4a \neq 0$, i.e. $t_0$ is away from the cuspidal edges in the case of the parabolic crosscap. Now suppose $x = \psi_1(u_1, t)$ and $y = \psi_2(u_1, t)$. We claim that $\psi_1$ and $\psi_2$ are divisible by $u_1$. To see this, we set $u_1 = 0$ in Equations (4.6) and (4.7) to get

$$2a\psi_1(0,t) + q_x(\psi_1(0,t), \psi_2(0,t)) + (t_0 + t)\psi_2(0,t) = 0 \tag{4.8}$$

and

$$2\psi_2(0,t) + q_y(\psi_1(0,t), \psi_2(0,t)) + (t_0 + t)\psi_1(0,t) + (t_0 + t)p'(\psi_2(0,t)) = 0 \tag{4.9}$$

We assume $\psi_1$ and $\psi_2$ are analytic and we compare the orders of $\psi_1(0,t)$ and $\psi_2(0,t)$, denoted $O(\psi_1(0,t))$ and $O(\psi_2(0,t)$. These cannot be equal. For suppose

$$O(\psi_1(0,t)) = O(\psi_2(0,t) = k.$$

Then we could write

$$\psi_1(0,t) = b_k t^k + \cdots \quad \text{and} \quad \psi_2(0,t) = c_k t^k + \cdots.$$

Substituting into Equations (4.8) and (4.9) gives

$$\begin{aligned} 2ab_k + t_0 c_k &= 0 \\ 2c_k + t_0 b_k &= 0, \end{aligned}$$

and we have a contradiction. If $O(\psi_1(0,t)) < O(\psi_2(0,t))$, then Equation (4.8) implies that $\psi_1(0,t) \equiv 0$ and thus $\psi_2(0,t) \equiv 0$. Alternatively, if $O(\psi_2(0,t)) < O(\psi_1(0,t))$, then Equation (4.9) implies that $\psi_2(0,t) \equiv 0$ and thus $\psi_1(0,t) \equiv 0$. So we deduce that

$$x = \psi_1(u_1, t) = u_1(b_0 + \cdots) \quad \text{and} \quad y = \psi_2(u_1, t) = u_1(c_0 + \cdots),$$

where $b_0$ and $c_0$ are constants.

Substituting back into Equations (4.6) and (4.7), we see that

$$2ab_0 + 1 + t_0 c_0 = 0 \quad \text{and}$$

$$2c_0 + t_0 b_0 = 0,$$

so that

$$b_0 = \frac{2}{(t_0^2 - 4a)} \quad \text{and} \quad c_0 = \frac{-t_0}{(t_0^2 - 4a)}.$$

Finally Equation (4.5) gives us an expression for $c$ in terms of $x, y, u_1$ and $u_2$. We substitute for $x$ and $y$ in this expression and it follows that the dual surface near the point $(0, t_0, 0)$ is given by the equation

$$c = u_1^2(K + \cdots),$$

where $K$ is a non-zero constant. Then we can apply Proposition 4.5.1 to show that there is a diffeomorphism which fixes the plane $c = 0$ and the $u_2$ axis so that the dual surface can be written $c = u_1^2$ (away from the cuspidal edges).

Now we can make the same considerations for the affine chart that includes the tangent cone. We work with the family of height functions $G$. The equations $G = G_x = G_y = 0$ give

$$xy + p(y) + q(x, y) + u_1 x + u_2(y^2 + ax^2) + c = 0, \tag{4.10}$$

$$y + u_1 + u_2(2ax + q_x) = 0 \tag{4.11}$$

and

$$x + p'(y) + u_2(2y + q_y) = 0 \tag{4.12}$$

This time we are just interested in what happens close to the point $(u_1, u_2, c) = (0, 0, 0)$. Equations 4.11 and 4.12 imply that we can write $x$ and $y$ as functions $\psi_1$ and $\psi_2$ respectively of $u_1$ and $u_2$. From Equations (4.11) and (4.12) we have

$$y = -(u_1 + u_2(2ax + q_x)) \quad \text{and}$$

$$x = -(p'(y) + u_2(2y + q_y)).$$

It is clear from this that the functions $\psi_1$ and $\psi_2$ are divisible by $u_1$. Thus we have

$$y = -u_1 + \phi_1(x, y, u_1, u_2, c), \phi_1 \in \mathcal{M}_5^3, \quad \text{and}$$

$$x = 2u_1 u_2 + \phi_2(x, y, u_1, u_2, c), \phi_2 \in \mathcal{M}_5^3.$$

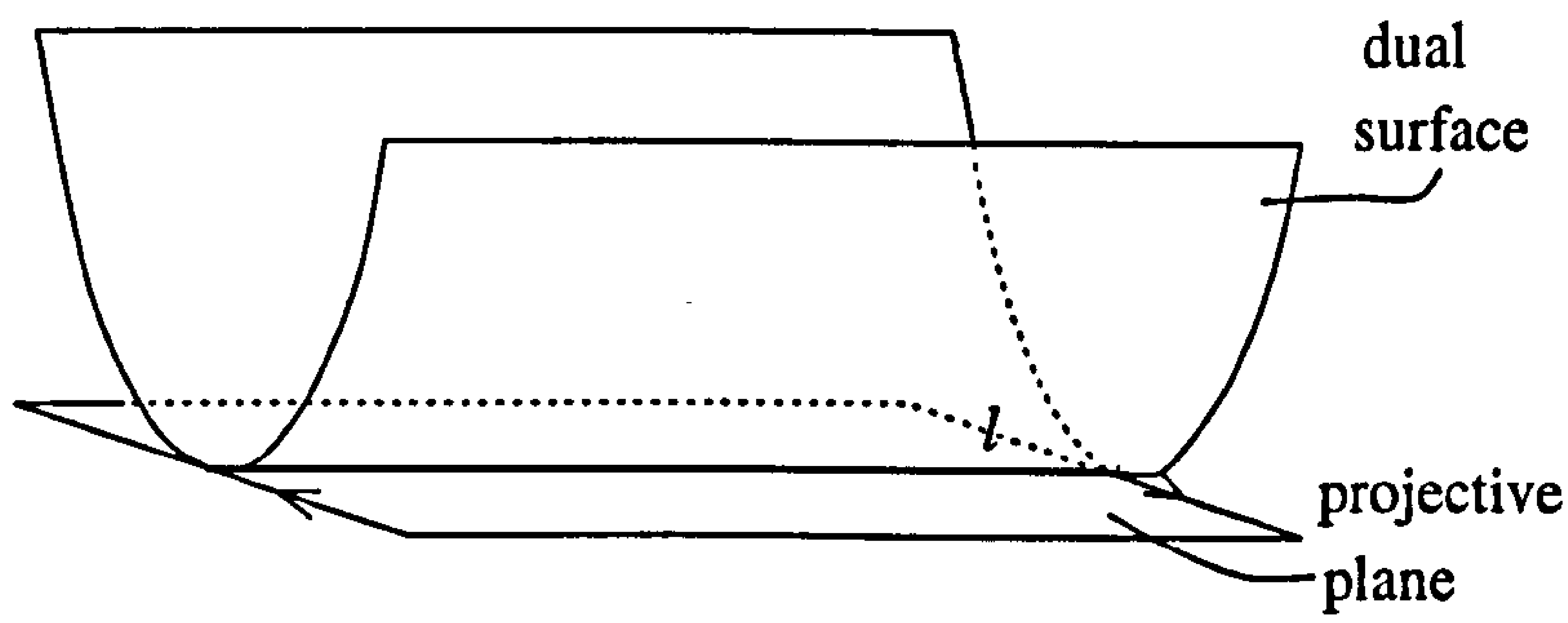



Figure 4.3: The dual surface away from the flat umbilic.

We substitute back into Equation (4.10) to get

$$c = 5u_1^2u_2 + \alpha(u_1, u_2, c), \alpha \in \mathcal{M}_3^4,$$

with $\alpha$ divisible by $u_1^2$. Applying Proposition 4.5.1, we see that there is a diffeomorphism which fixes the plane $c = 0$ and the $u_2$ axis so that the dual surface can be written $c = u_1^2u_2$ at the tangent cone point. □

On generic smooth surfaces, we do not expect flat umbilic points at all. They are not stable; that is they can always be removed by perturbing the surface. However we have seen that we cannot perturb the crosscap to remove the flat umbilic point on the dual, so that within this context, the flat umbilic on the dual of the crosscap is a stable phenomenon. As we mentioned before the statement of Proposition 4.5.2, these flat umbilic points are very degenerate. The considerations of this section allow us to determine some more information about the global structure of the dual. Taking a neighbourhood of the line $l$ and intersecting it with the projective plane $\delta = 0$ gives a Mobius band. Proposition 4.5.2 shows us how the dual lies with respect to this Mobius band. Taking the affine chart that does not include the flat umbilic (corresponding to the height function $H$), the dual surface is as shown in Figure 4.3.

Taking the other affine chart (corresponding to the height function $G$), the dual surface is as shown in Figure 4.4. In this diagram, we have indicated possible curves of points where there are bitangent planes. These correspond to the double point curve. Notice that we have not included the possible cuspidal edges in these diagrams. In the case of the parabolic crosscap, these can be inserted at the relevant points. Considering how the dual lies with respect to the plane $\delta = 0$ that is tangent to the dual along the line $l$ gives

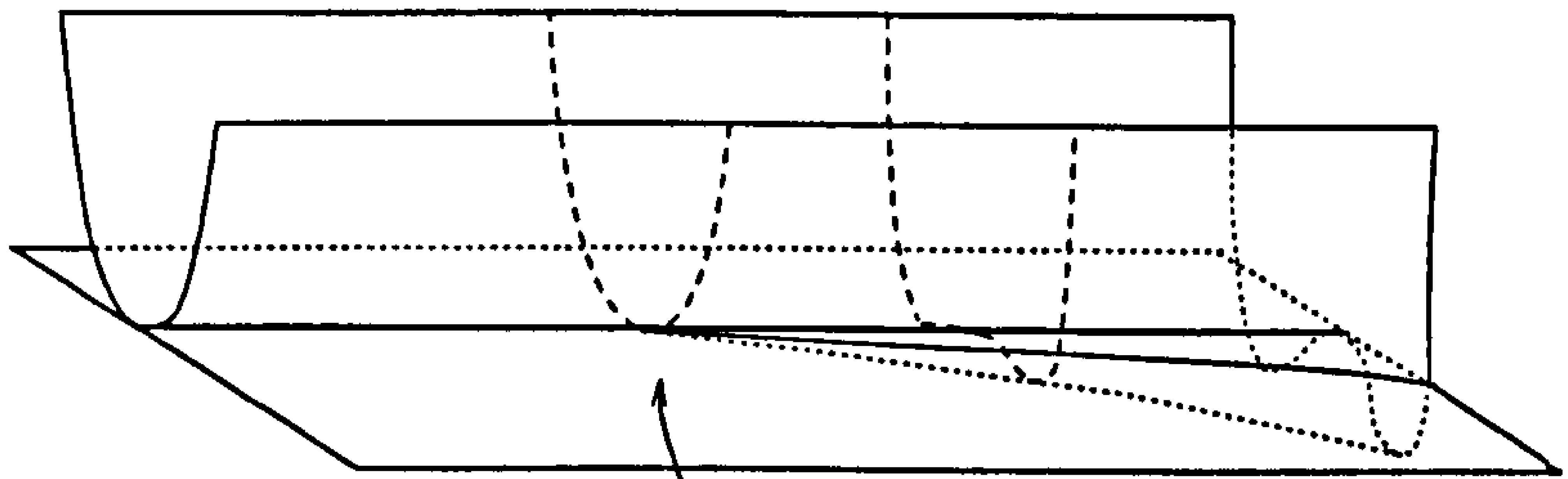



Figure 4.4: The dual in a neighbourhood of the flat umbilic.

**Proposition 4.5.3** In a neighbourhood of the line $l$, the dual to the crosscap is topologically equivalent to a Mobius band.

# Chapter 5

# Projections of the Crosscap and Asymptotic Curves on the Crosscap

In this chapter we consider projections of the crosscap to planes by composing the parametrisation of the crosscap with the family of projections to planes. We also apply some work of Bruce and Tari ([BT]) to obtain a result on the asymptotic curves of the hyperbolic crosscap. We begin with a discussion of projections of smooth surfaces.

## 5.1 Projections of Smooth Surfaces to Planes

Let $\Pi : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear surjection, so that the kernel of $\Pi$ is a line. If $M$ is a fixed surface in $\mathbf{R}^3$ and $p \in M$, then the $\mathcal{A}$-type of $P = \Pi|_M : (M,p) \to \mathbf{R}^2$ depends on the direction of the kernel line of $\Pi$. Now consider orthogonal projections in directions $(1, \alpha, \beta)$, so that we are certainly including all directions close to $(1, 0, 0)$. Instead of projecting to a plane perpendicular to the direction $(1, \alpha, \beta)$ we can, without altering the $\mathcal{A}$-type of the projection, project always to the plane $u = 0$. Thus $\Pi_{(\alpha,\beta)} : \mathbf{R}^3 \to \mathbf{R}^2$ is given by the formula $\Pi_{(\alpha,\beta)}(u, v, w) = (v - \alpha u, w - \beta u)$. Composing this with a parametrisation of our surface $M$ gives a map $P_{(\alpha,\beta)} : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$. Map germs of this type have been classified (see [Re]), and the set of critical values of this map is called the outline or profile of the surface. We recall here the connection between the $\mathcal{A}$ type of the projection and the geometry of $M$, when $M$ is a generic smooth surface. These results can be found in [B4]. They are drawn from [Ke], [Ga] and [McC].

**Theorem 5.1.1** The following normal forms arise in the orthogonal projections of a generic smooth surface $M$ in $\mathbf{R}^3$:

$$\begin{aligned}
\text{Submersion}: &\quad (x,y) \longmapsto (y,x)\\
\text{Fold}: &\quad (x,y) \longmapsto (y,x^2)\\
\text{Cusp}: &\quad (x,y) \longmapsto (y,xy+x^3)\\
\text{Lips/Beaks}: &\quad (x,y) \longmapsto (y,x^3 \pm xy^2)\\
\text{Goose}: &\quad (x,y) \longmapsto (y,x^3 + xy^3)\\
\text{Swallowtail}: &\quad (x,y) \longmapsto (y,xy+x^4)\\
\text{Gulls}: &\quad (x,y) \longmapsto (y,x^2y+x^4+x^5)\\
\text{Butterfly}: &\quad (x,y) \longmapsto (y,xy+x^5 \pm x^7).
\end{aligned}$$

Each of the cases in Theorem 5.1.1 has a geometric interpretation on the surface. The normal forms correspond to projecting a surface $z = f(x,y)$ along the $x$-axis to the $(y,z)$-plane (sending $(x,y)$ to $(y,z)$). The lowest power of $x$ in the second component measures the $\mathcal{K}$-contact between $M$ and the direction of projection. Moreover, for a generic smooth surface $M$ written in Monge form, the singularity will be $\mathcal{A}$-versally unfolded as part of the family of projections in directions $(1,\alpha,\beta)$ for $\alpha$ and $\beta$ close to 0. Following [B4], we give the geometric interpretations (which come from [Ga]) of these results in the form of a table (Figure 5.1). Here $v$ stands for the direction $(1,0,0)$ through the origin.

## 5.2 Projections of the Crosscap

**Theorem 5.2.1** When considering projections of crosscaps near the crosscap point, the following cases occur:
(i) When projecting in a direction that is transverse to the tangent cone, we obtain a fold map.
(ii) When projecting in a direction that is parallel to the tangent cone, we get a cusp except when the direction is either the same as that of the tangent line at the crosscap point, or the limiting tangent to the double point curve.
(iii) When the direction of projection is that of the tangent line, the map is of corank 2.
(iv) When the direction of projection is that of the limiting tangent to the double

| Name | Geometry |
| --- | --- |
| submersion | $v$ not tangent |
| fold | $v$ has 2-point contact with $M$ |
| cusp | $v$ is an asymptotic direction (3-point contact with $M$), 0 is a hyperbolic point |
| lips/beaks | $v$ is asymptotic, 0 parabolic |
| goose | $v$ is asymptotic, 0 hyperbolic, image of parabolic curve under the Gauss map has an inflection |
| swallowtail | $v$ asymptotic, 0 hyperbolic, $v$ has 4-point contact with $M$ |
| gulls | cusp of the Gauss map |
| butterfly | $v$ asymptotic, 0 hyperbolic, $v$ has 5-point contact with $M$ |

Figure 5.1: The geometry of projection maps.

point curve, we get a swallowtail map, which is versally unfolded by the family of orthogonal projections.

**Proof** We can suppose that $M$ is one of the family of crosscaps parametrised by

$$(x, xy + p(y), y^2 + ax^2 + q(x,y)), p \in \mathcal{M}_1^4, q \in \mathcal{M}_2^3,$$

with $a = \pm 1$ and $p(y) = \pm y^4 + \cdots$. Lemmas 3.1.4, 3.1.6 and 3.4.2 imply that the tangent line at the crosscap point of $M$ is the $u$-axis, the tangent cone is the plane $v = 0$ and the limiting tangent to the double point curve at the crosscap point is the $w$-axis. So

$$P_{(\alpha,\beta)}(x,y) = (-\alpha x + xy + p(y), -\beta x + ax^2 + y^2 + q(x,y)).$$

First consider the case where $\alpha \neq 0$. Then the line we are projecting along is transverse to the tangent cone (the plane $v = 0$). Subtracting $\beta/\alpha$ times the first component from the second component, we get

$$(-\alpha x + xy + p(y), \beta/\alpha xy - \beta/\alpha p(y) + ax^2 + y^2 + q(x,y)).$$

Now making the substitution $x' = \alpha^{-1}(x + xy + p(y))$, we see that our map is $\mathcal{A}$-equivalent to the map germ with normal form $(x, y^2)$ which is the fold map.

The next case to consider is $\alpha = 0$, $\beta \neq 0$. Then we are projecting along a line which lies in the tangent cone, but which is not the limiting tangent line to the double point curve (the $w$-axis) or the tangent line (the $u$-axis). Then the map is of the form

$$P_{(0,\beta)}(x,y) = (xy + p(y), -\beta x + ax^2 + y^2 + q(x,y)).$$

Writing $x' = x + \beta^{-1}(ax^2 + y^2 + q(x,y))$, it is clear that we have a map germ which is $\mathcal{A}$-equivalent to $(x, xy+y^3)$, which is $3-\mathcal{A}$-determined. This is the cusp.

Now we consider the case where $\alpha = 0$ and $\beta = 0$. We are projecting along the tangent line. This time we get

$$P_{(0,0)}(x,y) = (xy + p(y), ax^2 + y^2 + q(x,y)).$$

This has corank 2, and we examine this case in more detail in Proposition 5.2.3.

The directions we have missed in our family of projections are of the form $(0,\alpha,\beta)$ or equivalently $(0,1,\alpha)$ and $(0,0,1)$. The first of these corresponds to the linear surjection $(u,v,w) \to (u, w-\alpha v)$. The corresponding projection map is

$$(x,y) \to (x, ax^2 + y^2 + q(x,y) - \alpha(xy + p(y))).$$

It is not hard to see that this is $\mathcal{A}$-equivalent to the fold map.

Finally, projecting along the $w$-axis (the limiting tangent to the double point curve) gives us the map

$$(x,y) \to (x, xy + p(y)) = (x, xy \pm y^4 + \cdots).$$

This is $4-\mathcal{A}$-determined and generically is a swallowtail (with normal form $(x, xy+y^4)$). The family of linear surjections $\Pi_{(\alpha,\beta)} : \mathbf{R}^3 \to \mathbf{R}^2$ close to the direction $(0,0,1)$ is given by $\Pi_{(\alpha,\beta)}(u,v,w) = (u - \alpha w, v - \beta w)$. Applying this to the crosscap gives

$$P_{(\alpha,\beta)}(x,y) = (x - \alpha(ax^2 + y^2 + q(x,y)), xy + p(y) - \beta(ax^2 + y^2 + q(x,y))).$$

It can be checked that for a swallowtail map $f : (\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^2,0)$ we have $L\mathcal{A}_e.f \supset \mathcal{M}_2^3.\mathcal{E}(2,2)$. The same inclusion holds for $L\mathcal{A}_e.P_{(\alpha,\beta)}$, so it is easy to see that the family $P_{(\alpha,\beta)}$ is versally unfolded near $(\alpha,\beta) = (0,0)$. □

Theorem 5.2.1 part (iv) implies that as the view direction passes along a generic curve through that of the limiting tangent to the double point curve, we see a swallowtail transition, which is shown in Figure 5.2.

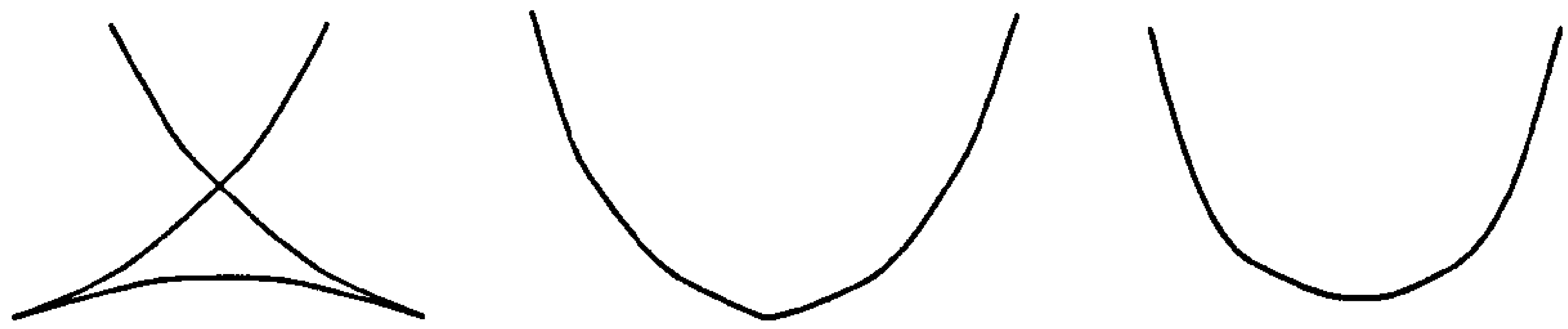


Figure 5.2: The swallowtail transition.

Now we analyse the corank 2 map germ in more detail. We expect this map germ to have low $\mathcal{A}_e$-codimension for a generic crosscap. The two lowest codimension corank 2 map germs $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ are

$$f(x, y) = (x^2 + y^3, y^2 + x^3) \quad \text{and}$$

$$g(x, y) = (xy, x^2 - y^2 + y^3).$$

A calculation shows that their $\mathcal{A}_e$-tangent spaces contain $\mathcal{M}_2^4.\mathcal{E}(2, 2)$.

**Lemma 5.2.2** Let $f, g : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ be defined as above. Then the tangent spaces $L\mathcal{A}_e.f$ and $L\mathcal{A}_e.g$ both contain $\mathcal{M}_2^2.\mathcal{E}(2, 2)$.

**Proof** First we consider $f(x, y) = (x^2 + y^3, y^2 + x^3)$. We have

$$\frac{\partial f}{\partial x} = (2x, 3x^2) \quad \text{and} \quad \frac{\partial f}{\partial y} = (3y^2, 2y).$$

Since $L\mathcal{A}_e.f \supset \mathcal{M}_2^4.\mathcal{E}(2, 2)$, it is clear that $L\mathcal{A}_e.f$ contains $(x\phi, 0)$ and $(0, y\phi)$, where $\phi \in \mathcal{M}_2^2$. The tangent space $L\mathcal{A}_e.f$ also contains $(x^2 + y^3, 0)$, $(0, y^2 + x^3)$, $(2x^2, 3x^3)$ and $(3y^3, 2y^2)$. It can easily be checked that these four vectors are linearly independent. Now we consider terms of degree two. Again we have $(x\phi, 0)$ and $(0, y\phi)$, with $\phi \in \mathcal{M}_2$. Using the vectors $(y^2 + x^3, 0)$, and $(0, x^2 + y^3)$ the result follows for $L\mathcal{A}_e.f$.

In the other case, we have

$$\frac{\partial g}{\partial x} = (y, 2x) \quad \text{and} \quad \frac{\partial g}{\partial y} = (x, -2y + 3y^2).$$

Then $L\mathcal{A}_e.g$ contains $(y^3, 2xy^2)$, $(xy^2, 2x^2y)$, $(x^2y, 2x^3)$, $(x^3, -2x^2y)$, $(x^2y, -2xy^2)$ and $(xy^2, -2y^3)$. It also contains $xye_i$, for $i = 1, 2$, and thus $(0, x^2)$ and $(y^2, 0)$. Thus we have $(0, 2x^2 - 2y^2 + 3y^3)$. Adding this to a multiple of $(0, x^2 - y^2 + y^3)$ gives $(0, y^3)$, and hence $(xy^2, 0)$, $(0, x^2y)$ and $(x^3, 0)$. We also have $(x^2, 3xy^2)$. With

$(y^2,0)$ and $(x^2-y^2+y^3)$ this gives $(y^3,3xy^2)$ and so we have $(y^3,0)$, $(x^2y,0)$, $(0,xy^2)$ and $(0,x^3)$. So $L\mathcal{A}_e.g \supset \mathcal{M}_2^3.\mathcal{E}(2,2)$, and it is now not hard to see that $L\mathcal{A}_e.g \supset \mathcal{M}_2^2.\mathcal{E}(2,2)$. □

Lemma 5.2.2 makes it easier to determine whether the corank 2 map germ obtained from the projection of the crosscap is versally unfolded or not.

**Proposition 5.2.3** In the case where the crosscap has two parabolic curves, the corank 2 map germ is $\mathcal{A}$-equivalent to $f(x,y)=(x^2+y^3,y^2+x^3)$. When there are no parabolic points near the crosscap point, the corank 2 map germ is $\mathcal{A}$-equivalent to $g(x,y)=(xy,x^2-y^2+y^3)$.

In both these cases the singularity is $\mathcal{A}$-versally unfolded by the family of projections in directions close to $(1,0,0)$ .

**Proof** By Lemma 3.4.4, if the crosscap has two parabolic curves passing through it, it can be parametrised by

$$(x,xy+p(y),y^2+x^2+q(x,y)), p\in\mathcal{M}_1^4, q\in\mathcal{M}_2^3.$$

In this case the family of projection maps is given by

$$P_{(\alpha,\beta)}(x,y)=(-\alpha x+xy+p(y),-\beta x+x^2+y^2+q(x,y)).$$

Now the projection map $P_{(0,0)}$ is given by

$$P_{(0,0)}(x,y)=(xy+p(y),x^2+y^2+q(x,y)).$$

Making the substitutions $x=x'+y'$ and $y=x'-y'$, and reverting to $x$ and $y$, we see that $P_{(0,0)}(x,y)$ is $\mathcal{A}$-equivalent to

$$(x^2-y^2+p_1(x,y),2x^2+2y^2+q_1(x,y)), p_1\in\mathcal{M}_1^4, q_1\in\mathcal{M}_2^3.$$

As long as the coefficients of $y^3$ and $x^3$ in the Taylor expansion of $q_1$ are non-zero, this is $\mathcal{A}$-equivalent to $f(x,y)=(x^2+y^3,y^2+x^3)$. It is quite easy to see that $P_{(\alpha,\beta)}$ is versally unfolded near $(\alpha,\beta)=(0,0)$ once we note that by Lemma 5.2.2, the space $L\mathcal{A}_e.P_{(0,0)}\supset\mathcal{M}_2^2.\mathcal{E}(2,2)$.

If the crosscap has no parabolic points near the crosscap point, then it can be parametrised by

$$(x,xy+p(y),y^2-x^2+q(x,y)), p\in\mathcal{M}_1^4, q\in\mathcal{M}_2^3.$$

Then the family of projection maps is

$$P_{(\alpha,\beta)}(x,y) = (-\alpha x + xy + p(y), -\beta x - x^2 + y^2 + q(x,y)).$$

Now

$$P_{(0,0)}(x,y) = (xy + p(y), -x^2 + y^2 + q(x,y)),$$

and it is clear that as long as the coefficient of $y^3$ in the Taylor expansion of $q$ is non-zero, $P_{(0,0)}(x,y)$ is $\mathcal{A}$-equivalent to $g(x,y) = (xy, x^2 - y^2 + y^3)$. Again it is straightforward to see that $P_{(\alpha,\beta)}$ is versally unfolded near $(\alpha,\beta) = (0,0)$, using the inclusion $L\mathcal{A}_e.P_{(0,0)} \supset \mathcal{M}_2^2.\mathcal{E}(2,2)$. □

Results on the bifurcations of the unfoldings of these two corank 2 singularities were obtained in [B5]. They have also been discussed in detail in [Hob], and [Haw]. We reproduce the diagrams found in [Haw] in Figures 5.3 and 5.4. These diagrams show how the critical values of the function $f_{a,b}$ change as we vary the unfolding parameters $a$ and $b$. The bifurcation set of the unfolding $f_{a,b} = (x^2 + y^3 + ay, x^3 + y^2 + bx)$ is discussed in [Haw]. Note that the bifurcation set of the other unfolding consists of a single point; the origin.

## 5.3 Critical Values of the Projection Maps

Now we consider the configuration of the critical values of the projections of the crosscap with the image of the double point curve of the crosscap under the projection.

**Proposition 5.3.1** We can describe the configuration of the outlines of the crosscap with the image of the double point curve as follows:

**(i) The fold case:** the limiting tangent to the double point curve and the tangent to the profile of the crosscap at the origin are transverse.

**(ii) The cusp case:** the limiting tangent to the double point curve and the limiting tangent to the profile of the crosscap at the origin coincide.

**(iii) The swallowtail case:** the limiting tangent to the double point curve is transverse to the tangent cone of the profile of the crosscap.

**(iv) The corank 2 case:**
**(a) The parabolic crosscap:** the critical values consist of two cusps and the

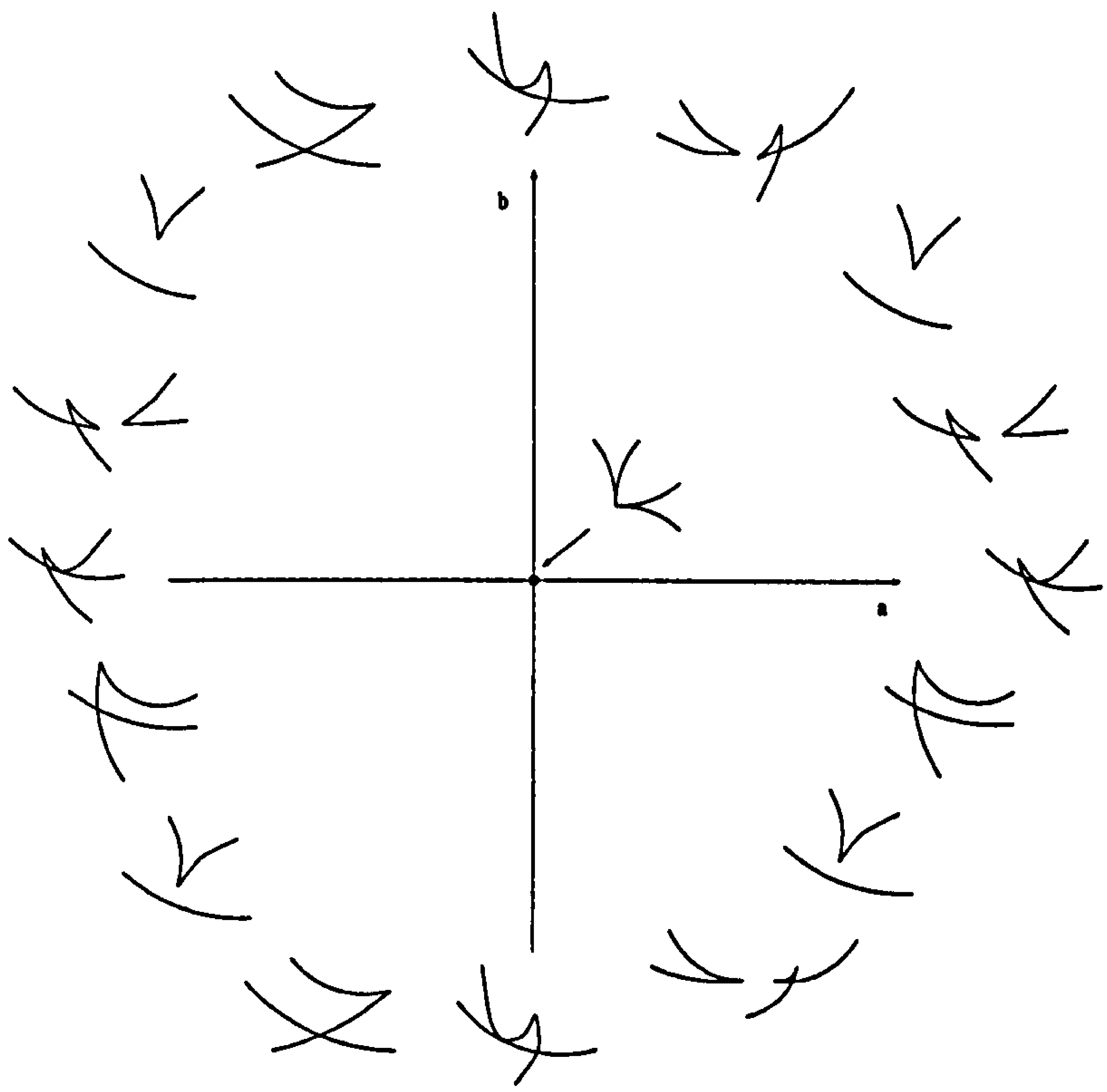



Figure 5.3: Bifurcations of the unfolding $f_{a,b} = (x^2 + y^3 + ay, x^3 + y^2 + bx)$.

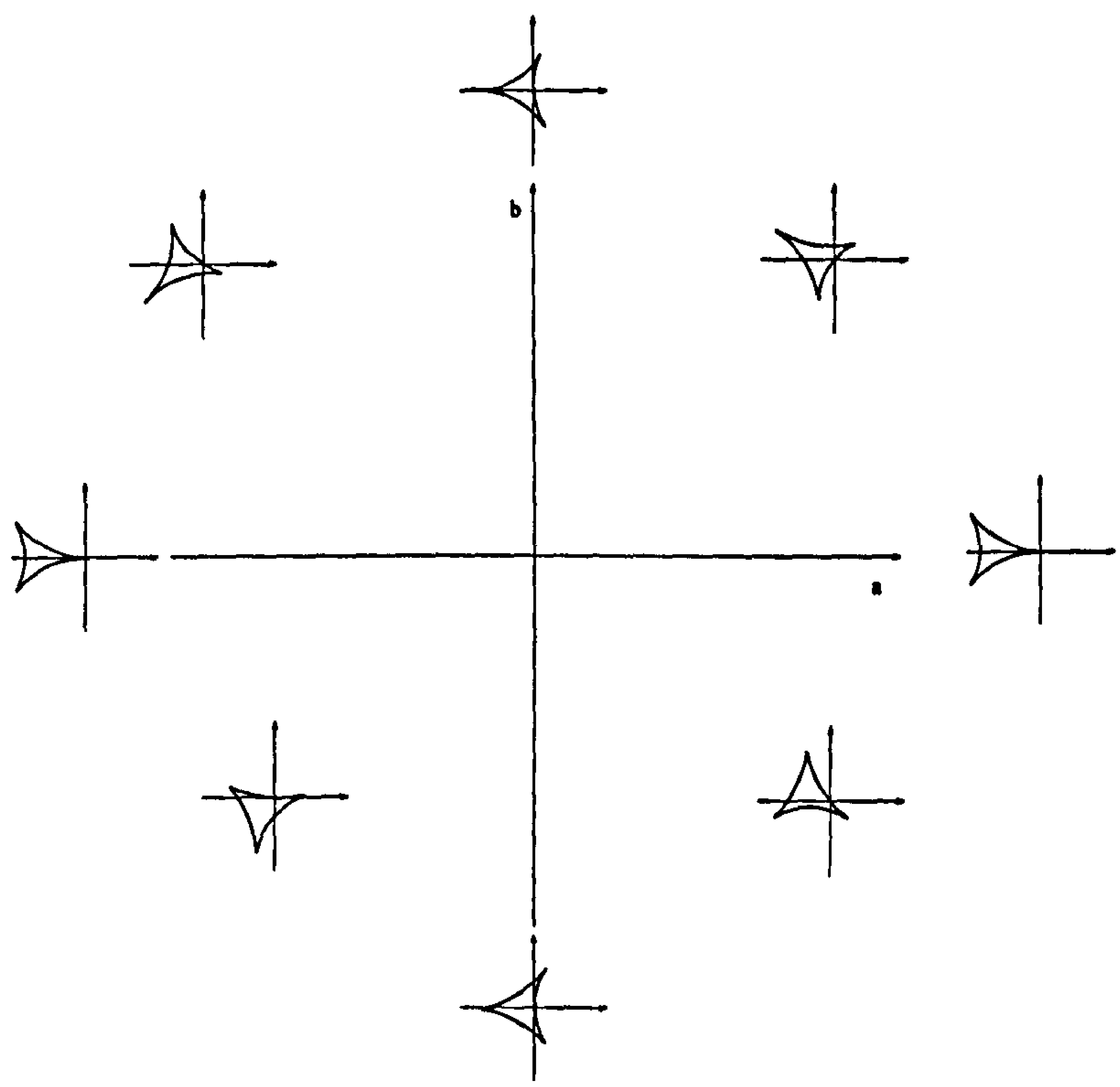



Figure 5.4: Bifurcations of the unfolding $f_{a,b} = (x^2 - y^2 + x^3 + ax, xy + bx)$.

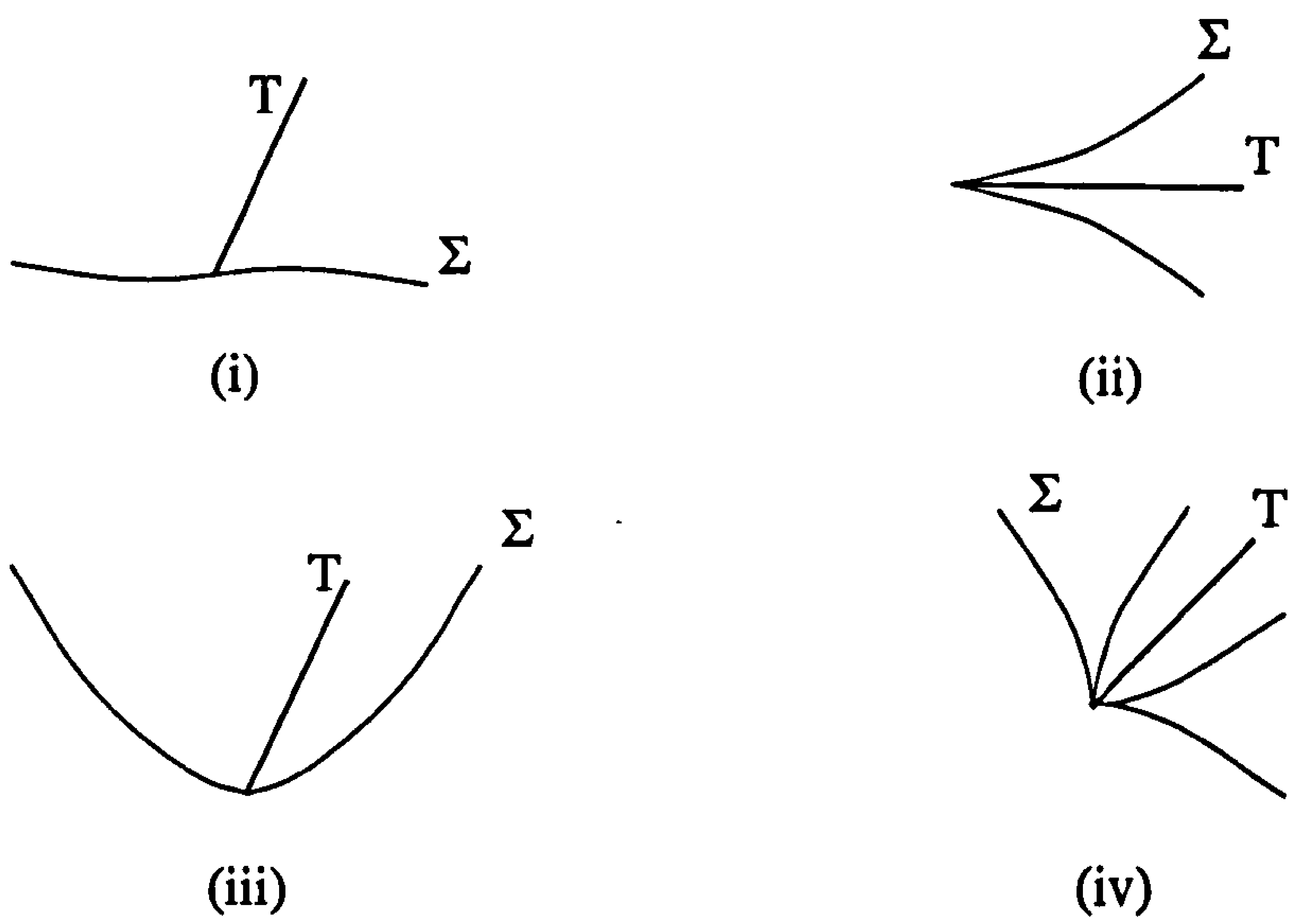



Figure 5.5: Configurations of the outlines of the crosscap ($\Sigma$) with the limiting tangent to the double point curve ($T$), (i) the fold case, (ii) the cusp case, (iii) the swallowtail case and (iv) The corank 2 case for the parabolic crosscap.

limiting tangent to the image of the double point curve under the projection is transverse to the two cuspidal tangents.

**(b) The hyperbolic crosscap**: The critical value of the mapping is the origin in this case.

These configurations are shown in Figure 5.5.

**Proof (i) The fold case:**

We have

$$P_{(\alpha,\beta)}(x,y) = (-\alpha x + xy + p(y), -\beta x + ax^2 + y^2 + q(x,y)),$$

with $\alpha \neq 0$. It follows that

$$\frac{\partial P_{(\alpha,\beta)}}{\partial x} = (-\alpha + y, -\beta + 2ax + q_x) \quad \text{and}$$

$$\frac{\partial P_{(\alpha,\beta)}}{\partial y} = (x + p'(y), 2y + q_y).$$

Now the critical set $\Sigma$ is given by

$$\det\begin{pmatrix} \partial P_{(\alpha,\beta)}/\partial x \\ \partial P_{(\alpha,\beta)}/\partial y \end{pmatrix} = 0,$$

so the initial terms of this equation are

$$-2\alpha y + \beta x + \cdots = 0.$$

Thus we can write

$$x = \beta/2\alpha t + \cdots \quad \text{and} \quad y = t.$$

Thus the critical values are given by $(-\beta/2t + \cdots, -\beta^2/2\alpha t + \cdots)$ and the gradient of the tangent at the origin to the critical set is $\beta/\alpha$. The limiting tangent to the double point curve in $(u, v, w)$-space is the $w$-axis. Since we are not projecting along the $w$-axis, the limiting tangent to the image of the double point curve coincides with the image of the limiting tangent, and is thus the $y$-axis. Thus the limiting tangent to the double point curve and the tangent to the profile of the crosscap at the origin are transverse in this case.

**(ii) The cusp case:**

We have

$$P_{(0,\beta)}(x,y) = (xy + p(y), -\beta x + ax^2 + y^2 + q(x,y)),$$

with $\beta \neq 0$. It follows that

$$\frac{\partial P_{(0,\beta)}}{\partial x} = (y, -\beta + 2ax + q_x) \quad \text{and}$$

$$\frac{\partial P_{(0,\beta)}}{\partial y} = (x + p'(y), 2y + q_y).$$

So $\Sigma$ is given by

$$\beta x + 2y^2 - 2ax^2 + \cdots = 0.$$

So we can write

$$x = -2/\beta t^2 + \cdots \quad \text{and} \quad y = t.$$

Thus the critical values are given by $(-2/\beta t^3 + \cdots, 3t^2 + \cdots)$. This is a cusp at the origin with cuspidal tangent the $y$-axis. The limiting tangent to the image of the double point curve is the image of the $w$-axis under the projection; that is the $y$-axis. Thus the limiting tangent to the double point curve and the limiting tangent to the profile of the crosscap at the origin coincide in this case.

**(iii) The swallowtail case:**

We have

$$P(x,y) = (x, xy + p(y)).$$

Then

$$\frac{\partial P}{\partial x} = (1, y) \quad \text{and} \quad \frac{\partial P}{\partial y} = (0, x + p'(y)).$$

So $\Sigma$ is given by $x + p'(y) = 0$, which can be parametrised by

$$x = -p'(t), y = t.$$

Thus the critical values are given by $(-p'(t), p(t) - tp'(t))$. Since $p \in \mathcal{M}_1^4$, the tangent cone to this set at the origin is the $x$-axis. In this case, we are projecting along the limiting tangent to the double point curve, so to determine the limiting tangent to the image under the projection of the double point curve we need to do some calculations. Recall from Lemma 3.4.2 that generically the preimage of the double point curve is parametrised by $(kt^4 + \cdots, t)$, for some non-zero constant $k$. The image of this under the projection map is then $(kt^4 + \cdots, \pm t^4 + \cdots)$, since $p(y) = \pm y^4 + \cdots$. So generically the limiting tangent to this curve is the line $x = y$, which is transverse to the tangent cone of the profile of the crosscap in this case.

**(iv) The corank 2 case:**

We have

$$P(x, y) = (xy + p(y), ax^2 + y^2 + q(x, y)).$$

Then

$$\frac{\partial P}{\partial x} = (y, 2ax + q_x) \quad \text{and}$$
$$\frac{\partial P}{\partial y} = (x + p'(y), 2y + q_y).$$

The critical set $\Sigma$ is then given by

$$2y^2 - 2ax^2 + \cdots = 0.$$

When $a \neq 0$, this is a Morse singularity.

**(a) The parabolic crosscap:**

In this case, we can fix $a = 1$. Then the two branches of the critical set can be parametrised by $(t + \cdots, t)$ and $(t + \cdots, -t)$. So the images of the two branches under the mapping are parametrised by

$$(t^2 + p(t), 2t^2 + q((t + \cdots, t))$$

and

$$(-t^2 + p(t), 2t^2 + q((t + \cdots, -t)).$$

So the critical values consist of two cusps, with cuspidal tangents given by the equations $2x - y = 0$ and $2x + y = 0$. The limiting tangent to the image of the double point curve under the projection is the $y$-axis. This is transverse to the two cuspidal tangents.

**(b) The hyperbolic crosscap:**

In this case we can fix $a = -1$, so the critical set is the origin. The limiting tangent to the image of the double point curve is, as before, the $y$-axis. □

## 5.4 Binary Differential Equations

In this section, we consider binary differential equations of the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0, \tag{5.1}$$

where $a, b$ and $c$ are smooth functions vanishing at $(0,0)$. We are motivated by the problem of determining the principal and asymptotic curves on a surface near a crosscap point. At each point of a surface the principal directions, and on the hyperbolic part of the surface, the asymptotic directions determine bivalued line fields on the surface, corresponding to a binary differential equation on the parametrising space. For the principal curves, this equation is

$$(Em - lF)dx^2 + (En - lG)dxdy + (Fn - Gm)dy^2 = 0,$$

and for the asymptotic curves, the equation is

$$ldx^2 + 2mdxdy + ndy^2 = 0,$$

where $E, F$ and $G$ and $l, m$ and $n$ are the coefficients of the first and second fundamental forms of the surface.

A natural way to study these equations is to lift the bivalued direction fields to a single field on an associated double cover of the surface. This approach is followed by Bruce and Tari ([BT]) in order to study integral curves of binary differential equations. We employ their methods in what follows. Working with binary differential equations of the form given by Equation (5.1), we consider in $\mathbf{R}^2 \times \mathbf{R}P^1$ the set $M$ of points $(x, y, q)$ with $b^2 - ac = 0$ and $q$ a direction determined by Equation (5.1) at $(x, y)$. By taking an affine chart of $\mathbf{R}P^1$, we can consider this double cover surface locally in $\mathbf{R}^3$. So we set $p = dy/dx$ and consider the surface $M$ in $(x, y, p)$-space given by

$$M = \{(x, y, p) : ap^2 + 2bp + c = 0\}.$$

We shall denote by $F$ the function $ap^2 + 2bp + c$. The discriminant of the binary differential equation is the set

$$\Delta = \{(x, y) : b^2 - ac = 0\}.$$

**Proposition 5.4.1** The surface $M$ is smooth in a neighbourhood of $0 \times \mathbf{R}P^1$ if and only if the discriminant function $b^2 - ac$ has a Morse singularity. The natural projection $\pi : M \longrightarrow \mathbf{R}^2$ given by $(x, y, p) \mapsto (x, y)$ is a local diffeomorphism away from $\pi^{-1}(\Delta)$.

**Proof** See [BT]. □

If $M$ is smooth, Bruce and Tari determine a suitable lift $\xi$ of the bivalued fields in $\mathbf{R}^2$ defined by Equation 5.1. They then analyse the type of the zeros of the vector field $\xi$ in order to obtain the different topological normal forms of the binary differential equation. A list of these together with diagrams of their integral curves is given in [BT]. In the next section, we see how far these results can be applied to the crosscap.

## 5.5 Integral Curves on the Crosscap

We first consider the principal curves near the crosscap point. We suppose that we have a crosscap parametrised by the normal form

$$f(x,y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x,y)),$$

where $a$ and $b$ are constants, $p \in \mathcal{M}_1^3$ and $q \in \mathcal{M}_2^3$. As mentioned above, the binary differential equation which gives rise to the principal curves is

$$(Em - lF)dx^2 + (En - lG)dxdy + (Fn - Gm)dy^2 = 0. \qquad (5.2)$$

When working with the crosscap, we run into problems, as there is no well defined normal to the crosscap at the crosscap point. The coefficients $l, m$ and $n$ are defined as $f_{xx} \cdot N$, $f_{xy} \cdot N$ and $f_{yy} \cdot N$, respectively, where $N$ is the unit normal to the surface given by

$$N = \frac{f_x \times f_y}{\|f_x \times f_y\|}.$$

Note that Equation (5.2) is homogeneous in $l, m$ and $n$, so we can replace them by $l_1 = f_{xx} \cdot (f_x \times f_y), m_1 = f_{xy} \cdot (f_x \times f_y)$ and $n_1 = f_{yy} \cdot (f_x \times f_y)$ respectively. At $(x, y) = (0, 0)$, we have $l_1 = m_1 = n_1 = 0$, so that every direction is a solution of the binary differential equation

$$H(x,y,dx,dy) = (Em_1 - l_1F)dx^2 + (En_1 - l_1G)dxdy + (Fn_1 - Gm_1)dy^2 = 0. \quad (5.3)$$

Now we consider the surface M given by

$$M = \{((x,y), [\alpha : \beta]) \in \mathbf{R}^2 \times \mathbf{R}P^1 : H(x,y,\alpha,\beta) = 0\}.$$

**Proposition 5.5.1** The surface $M$ is not smooth.

**Proof** We apply the criteria of Proposition 5.4.1 to the Equation (5.3). From the proof of Proposition 6.3.1, we know that

$$\begin{aligned} E &= 1+y^2+(2ax+by+q_x)^2, \\ F &= xy+yp_y+(2ax+by+q_x)(2y+bx+q_y) \quad \text{and} \\ G &= (x+p_y)^2+(2y+bx+q_y)^2, \end{aligned}$$

and

$$\begin{aligned} l_1 &= (2a+q_{xx})(x+p_y) = 2ax+\cdots, \\ m_1 &= -(2y+bx+q_y)+(x+p_y)(b+q_{xy}) = -2y+\cdots \quad \text{and} \\ n_1 &= -p_{yy}(2y+bx+q_y)+(x+p_y)(2+q_{yy}) = 2x+\cdots. \end{aligned}$$

Now writing

$$\begin{aligned} Em_1-l_1F &= a(x,y) \\ En_1-l_1G &= 2b(x,y) \quad \text{and} \\ Fn_1-Gm_1 &= c(x,y), \end{aligned}$$

we see that the 2-jet of the discriminant function $b^2-ac$ is $4x^2$. Thus the discriminant function is not Morse, so that by Proposition 5.4.1, the surface $M$ is not smooth. □

Unfortunately the techniques available do not cover the case where $M$ is not smooth. However, for the asymptotic curves, we can obtain a result in the case of the hyperbolic crosscap. This time we work with the normal form

$$f(x,y) = (x, xy+p(y), y^2+ax^2+q(x,y)),$$

where $a=\pm 1$, $p \in \mathcal{M}_1^4$ and $q \in \mathcal{M}_2^3$, since the asymptotic curves are invariant under affine changes of coordinates. As there is no well defined unit normal at the crosscap point, we work with the binary differential equation

$$H(x,y,dx,dy) = l_1dx^2+2m_1dxdy+n_1dy^2 = 0, \tag{5.4}$$

where $l_1 = f_{xx}\cdot(f_x\times f_y), m_1 = f_{xy}\cdot(f_x\times f_y)$ and $n_1 = f_{yy}\cdot(f_x\times f_y)$. As before, at $(x,y)=(0,0)$, we have $l_1=m_1=n_1=0$, so that every direction is a solution of this binary differential equation. Notice that the discriminant of this equation is $m_1^2-l_1n_1$, which is positive on the hyperbolic part of the surface. This time we take an affine chart on $\mathbf{R}P^1$ by setting $q=dx/dy$, so that we study the surface

$$M = \{(x,y,q) : F(x,y,q) = l_1q^2+2m_1q+n_1 = 0\}.$$

Now

$$\begin{aligned} l_1 &= 2ax + \cdots, \\ m_1 &= -2y + \cdots \quad \text{and} \\ n_1 &= 2x + \cdots, \end{aligned}$$

so that the discriminant is Morse on the hyperbolic part of the crosscap surface. Proposition 5.4.1 then implies that the double cover surface $M$ is smooth.

Now we give the result from [BT] which determines a suitable lift of the bivalued fields onto $M$.

**Lemma 5.5.2** A suitable lift on $M$ of the bivalued fields defined by equation (5.4) is

$$\xi = qF_q\frac{\partial}{\partial x} + F_q\frac{\partial}{\partial y} - (qF_x + F_y)\frac{\partial}{\partial q}.$$

**Proof** ([BT], Proposition 5.4.2) Suppose that

$$\xi = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial q}$$

is such a lift. Then if $\pi$ is the projection map from $M$ back to $\mathbf{R}^2$, we have

$$d\pi(\xi) = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y},$$

so that $A = qB$. We must ensure that the lifted field is tangent to $M$, so that $\xi F = 0$. This gives

$$qBF_x + BF_y + CF_q = 0,$$

so we set $B = F_q$ and $C = -qF_x - F_y$. □

Next we must determine where the zeros of $\xi$ lie on $M$. To use the methods of [BT], we need only consider those terms of $F$ of degree 1 in $x$ and $y$. This gives us

$$F(x, y, q) = 2axq^2 - 4yq + 2x.$$

**Proposition 5.5.3** In the case of the hyperbolic crosscap ($a = -1$), the lifted vector field $\xi$ has one zero, which lies at the point $[0:1]$ ($q = 0$) on the exceptional fibre. In the case of the parabolic crosscap ($a = 1$), the lifted vector field $\xi$ has three zeros, which lie at the points $[0:1], [1:1]$ and $[-1:1]$ ($q = 0, 1$ and $-1$) on the exceptional fibre.

**Proof** The zeros of $\xi$ are given by the equations

$$F = F_q = (qF_x + F_y) = 0.$$

We can assume that the zeros of the field that lie on the exceptional fibre are isolated (for a discussion of this see [BT]), and so the first two of these equations imply that $x = y = 0$. The third equation gives

$$q(aq^2 + 1) - 2q = 0,$$

$$\text{i.e.} \quad q(aq^2 - 1) = 0.$$

This implies that $q = 0$, and $q = \pm 1/\sqrt{a}$, and the result follows. □

In the case of the parabolic crosscap, this result is unfortunate. Notice that the discriminant of Equation (5.4) corresponds to the parabolic set in the source. Proposition 3.4.4 tells us that this set consists of two transverse curves with gradient $\pm 1$. So when we construct the double cover surface $M$, the zeros of $\xi$ are in the same position on the exceptional fibre as the branches of the discriminant. This means that the methods of [BT] cannot be used. However we do have a result in the case of the hyperbolic crosscap. We need to determine whether the zero of the lifted field is a saddle or a node.

**Proposition 5.5.4** For the hyperbolic crosscap, the zero of $\xi$ is a saddle. We can deduce that the configuration of the asymptotic curves in the source is topologically equivalent to the Lemon, which first arose as one of the configurations of lines of curvature near an umbilic point. The Lemon is depicted in Figure 5.6.

**Proof** Note that in this case, the lower order terms of $F$ are given by

$$F(x, y, q) = -2xq^2 - 4yq + 2x.$$

At the zeros of $\xi$, $q = 0$ and $H_x = -2q^2 + 2 \neq 0$, so we write $M$ locally as the graph of a function

$$x = u(y, q) = \frac{2qy}{1 - q^2} + \cdots.$$

Now we project the lifted field $\xi$ onto the $(y, q)$-plane to obtain

$$\begin{aligned}
\tilde{\xi} &= F_q\left(\frac{2qy}{1-q^2} + \cdots, y, q\right)\frac{\partial}{\partial y} \\
&\quad -\left(qF_u\left(\frac{2qy}{1-q^2} + \cdots, y, q\right) + F_y\left(\frac{2qy}{1-q^2} + \cdots, y, q\right)\right)\frac{\partial}{\partial q} \\
&= A(y, q)\frac{\partial}{\partial y} + B(y, q)\frac{\partial}{\partial q}.
\end{aligned}$$

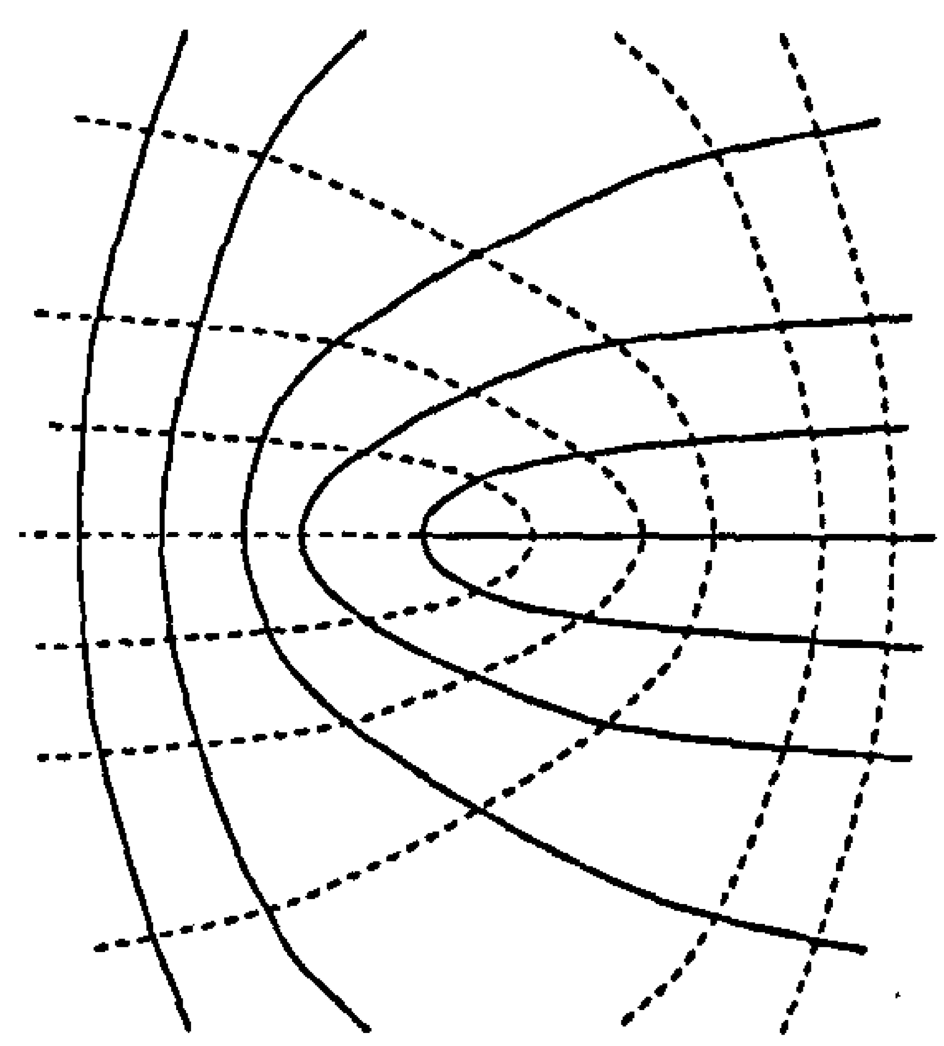


Figure 5.6: The Lemon

The vector field $\xi$ will have the same type of zero at $x = y = q = 0$ as $\tilde{\xi}$, so we just need to evaluate the eigenvalues of $\tilde{\xi}$. Now in this case, $u_y = u_q = 0$ when $x = y = 0$, so

$$\frac{\partial A}{\partial y} = F_{qy} + F_{qu}u_y = -4 \quad \text{and}$$

$$\frac{\partial A}{\partial q} = F_{qq} + F_{qu}u_q = -2x = 0.$$

Thus one of the eigenvalues of $\tilde{\xi}$ is $-4$, and the other is $\partial B/\partial q$. Now

$$\begin{aligned} \frac{\partial B}{\partial q} &= -(F_u(\frac{2qy}{1-q^2} + \cdots, y, q) \\ &\quad + qF_{uq}(\frac{2qy}{1-q^2} + \cdots, y, q) + F_{qy}(\frac{2qy}{1-q^2} + \cdots, y, q) \\ &= -(2(1-q^2) + q(-4q) - 4) = 2 \end{aligned}$$

when $q = 0$. So the eigenvalues have opposite signs, and the zero is indeed a saddle. The deduction about the topological type of the asymptotic curves follows from [BT], Section 4. □

# Chapter 6

# The Focal Set of the Crosscap

Having obtained information on the flat geometry of the crosscap, we move onto an aspect of the spherical geometry: the focal set. For smooth surfaces this has been studied by considering the singularities of the distance squared function. We use this method to obtain a description of the focal set of the crosscap. This is more complicated than the usual focal surface due to the fact that the crosscap does not have a unique normal direction at the crosscap point. The limiting principal radii of curvature at the crosscap point are of interest when considering the focal set, and we calculate these in the last part of this chapter.

First we review some results on focal sets of smooth surfaces.

## 6.1 Focal Sets of Smooth Surfaces

In [P1] and [P2] Porteous studied the focal set by considering the singularities of the distance squared function.

**Definition 6.1.1** Let $X \subset \mathbf{R}^3$ be a surface. We define the distance squared function by

$$\begin{aligned} d : X \times \mathbf{R}^3 &\longrightarrow \mathbf{R} \\ (p, u) &\mapsto \|p - u\|^2 \end{aligned}$$

It is well known (see for example [Mi]) that the focal set can be modelled locally by the bifurcation set of $d$. Choosing a parametrisation $\phi$ of a neighbourhood of $p$ we consider the singularity type (the $\mathcal{R}$- equivalence class) of the function germ

$$\begin{array}{rcl} \mathbf{R}^2, 0 & \longrightarrow & \mathbf{R}, 0 \\ (x,y) & \mapsto & d_u(x,y) = d(\phi(x,y), u) \end{array}$$

for each $u \in \mathbf{R}^3$.

Following Porteous and Looijenga's unpublished thesis (for an account of this, see [Wa2]), we have

**Theorem 6.1.2** Let $X \subset \mathbf{R}^3$ be a generic smooth surface. Then the distance squared function $d_u$ can only have singularities of types $A_1, A_2, A_3, A_4$ and $D_4$, where the $A_k$ singularities are the singularities $\mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$ that are $\mathcal{R}$-equivalent to

$$(x,y) \mapsto \pm x^2 \pm y^{k+1}, k \geq 1,$$

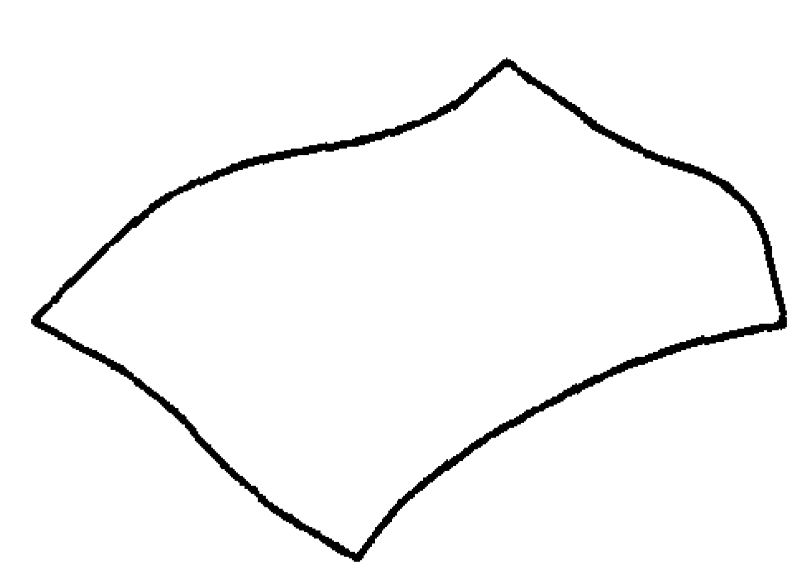
and the $D_4$ singularity is one of the $D_k$ singularities that are $\mathcal{R}$-equivalent to

$$(x,y) \mapsto x^2 y \pm y^{k-1}, k \geq 4.$$

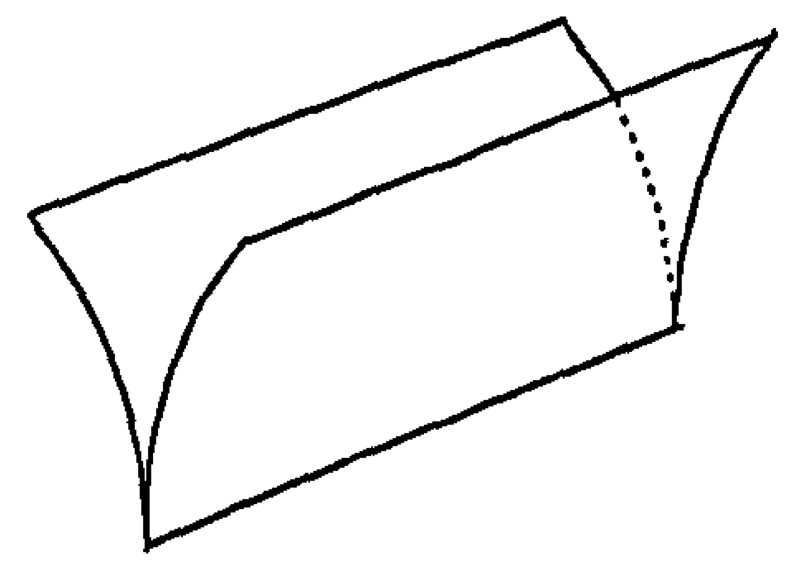
In each case the singularity is versally unfolded by the $u$ parameters.

**Proof** See [BG1]. □

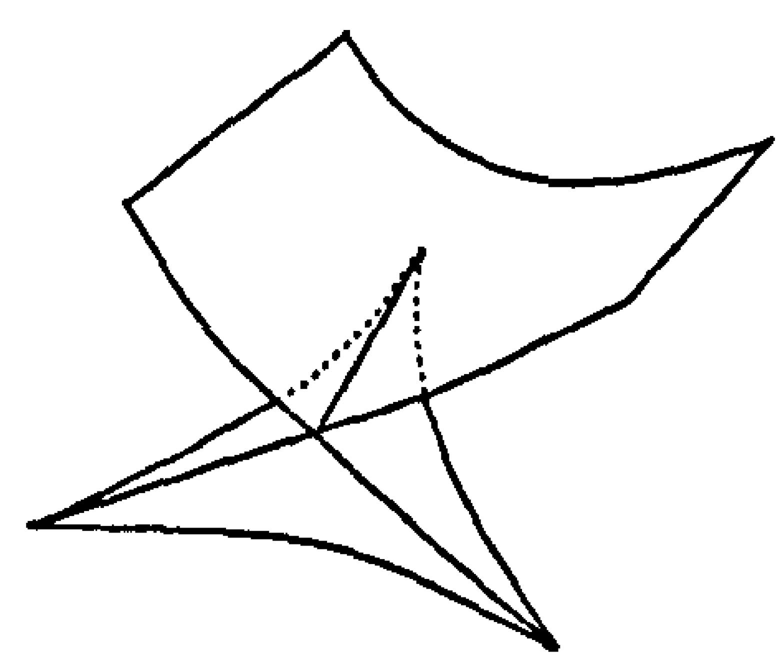
It is not difficult to prove (see [Mi]) that the distance squared function $d_v$ has a singularity of type $A_{\geq 2}$ or is of corank 2 if and only if $v$ is a point of the focal set of $X$. Since the family of distance squared functions is a versal unfolding of each of its singularities, we have local models (up to diffeomorphism) for the focal set, i.e. the bifurcation sets of the standard unfoldings of $A_2, A_3, A_4$ and $D_4$ singularities (see for example [B4]). In the case of an $A_2$ singularity, the focal set is locally smooth,



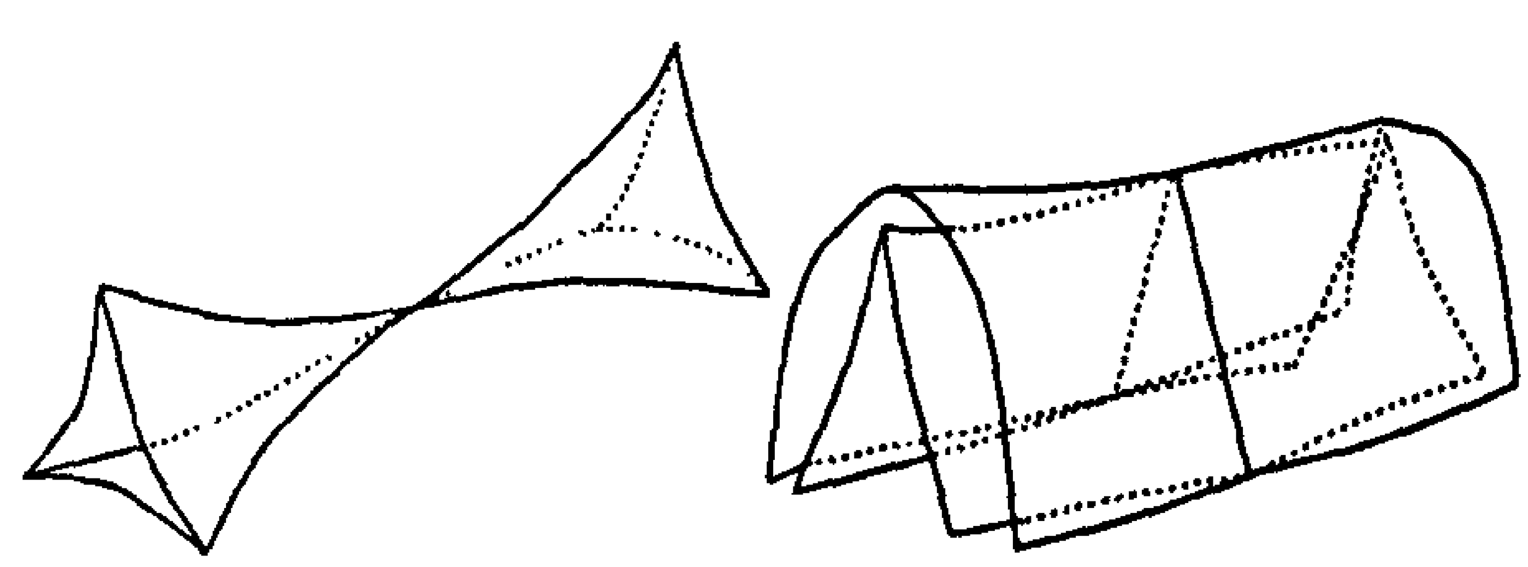
and for an $A_3$ singularity it is locally diffeomorphic to a cuspidal edge.



These points are known as *ridges*. The corresponding points on the focal surface are called *ribs*. When the distance squared function has a singularity of type $A_4$, the focal set has a swallowtail point,



and when the singularity is of type $D_4$, the focal set has one of the two umbilic types.



## 6.2 The Focal Set of the Crosscap

We use the methods described above to study the focal set of the crosscap. In what follows, we work with the normal form

$$f(x,y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x,y)), p \in \mathcal{M}_1^3, q \in \mathcal{M}_2^3,$$

since we are dealing with contact with spheres so can only use the action of the group $O(3) \times \mathbf{R}^+$ (defined in chapter 3) in the target.

**Proposition 6.2.1** The part of the focal set corresponding to the crosscap point is a conic section in the plane orthogonal to the tangent line (the $(u_2, u_3)$- plane) with equation

$$2u_3(1 - 2au_3) + (bu_3 + u_2)^2 = 0.$$

For the hyperbolic crosscap ($a < 0$), this conic section is an ellipse, for the parabolic crosscap ($a > 0$) we have a hyperbola and in the case where $a = 0$, the conic is a parabola.

**Proof** The distance squared function on the crosscap is given by

$$\begin{aligned} F(x,y,u_1,u_2,u_3) &= (x-u_1)^2 \\ &\quad +(xy+p(y)-u_2)^2+(y^2+ax^2+bxy+q(x,y)-u_3)^2. \end{aligned}$$

We look for the singularities of $F(-,u_1,u_2,u_3)$ at $(x,y) = (0,0)$. The partial derivatives of $F$ are given by

$$\begin{aligned} F_x &= 2(x-u_1)+2y(xy+p(y)-u_2) \\ &\quad +2(2ax+by+q_x)(y^2+ax^2+bxy+q(x,y)-u_3). \end{aligned}$$

and

$$\begin{aligned} F_y &= 2(x+p'(y))(xy+p(y)-u_2) \\ &\quad +2(2y+bx+q_y)(y^2+ax^2+bxy+q(x,y)-u_3). \end{aligned}$$

From these expressions, it is clear that

$$F_x(0,0,u_1,u_2,u_3) = F_y(0,0,u_1,u_2,u_3) = 0$$

if and only if $u_1 = 0$, so that the part of the focal surface corresponding to the crosscap point lies in the plane $u_1 = 0$, i.e. the plane normal to the tangent line. We consider the set in the $(u_2,u_3)$-plane for which $F$ has a degenerate singularity at $(x,y) = (0,0)$. This is given by the condition $F_{xx}F_{yy} - F_{xy}^2 = 0$ at $(x,y) = (0,0)$. We have

$$\begin{aligned} F_{xx} &= 2+2y^2+(4a+2q_{xx})(y^2+ax^2+bxy+q(x,y)-u_3) \\ &\quad +2(2ax+by+q_x)^2, \\ F_{xy} &= 2(xy+p(y)-u_2)+2y(x+p'(y)) \\ &\quad +2(b+q_{xy})(y^2+ax^2+bxy+q(x,y)-u_3) \\ &\quad +2(2ax+by+q_x)(2y+bx+q_y), \quad \text{and} \\ F_{yy} &= 2p''(y)(xy+p(y)-u_2)+2(x+p'(y))^2 \\ &\quad +2(2+q_{yy})(y^2+ax^2+bxy+q(x,y)-u_3)+2(2y+bx+q_y)^2, \end{aligned}$$

so when $(x,y) = (0,0)$,

$$F_{xx} = 2-4au_3, F_{xy} = -2bu_3-2u_2 \quad \text{and} \quad F_{yy} = -4u_3.$$

Thus we have

$$\begin{aligned} -4u_3(2-4au_3)-4(bu_3+u_2)^2 &= 0, \\ \text{i.e.}\quad 2u_3(1-2au_3)+(bu_3+u_2)^2 &= 0 \\ \text{or equivalently}\quad (bu_3+u_2)^2-4a(u_3-1/4a)^2+1/4a &= 0, a\neq 0. \end{aligned}$$

This is the equation of a conic section in the $(u_2, u_3)$-plane. In the case of the parabolic crosscap, $a > 0$, so the conic is a hyperbola. For the hyperbolic crosscap, $a < 0$, so the conic is an ellipse. When $a = 0$, the conic is a parabola. □

**Corollary 6.2.2** In the case of the parabolic crosscap, the asymptotes to the hyperbola are the lines in the $(u_2, u_3)$-plane with equations

$$u_2 + u_3(b \pm 2\sqrt{a}) \mp 2\sqrt{a}.$$

**Proof** This is immediate from considering the equation of the hyperbola given in Proposition 6.2.1. □

For the parabolic crosscap, we expect the asymptotes of the hyperbola to correspond to the planes with $A_2$ contact at the crosscap point. These planes can be thought of as spheres of infinite radius, whose centres lie on the asymptotes to the hyperbola. More precisely, if we consider points $p$ on either branch of the hyperbola, and the sphere centred at $p$ which passes through the origin, then as $p$ tends to infinity, the sphere tends to the plane orthogonal to the line joining the origin and $p$. In the limit, this line tends to a line through the origin parallel to one of the asymptotes. Corollary 6.2.2 shows that this is the case. Recall (Lemma 3.3.5) that the planes with $A_2$ contact at the crosscap point are given by

$$u_3 + (-b \pm 2\sqrt{a})u_2 = 0.$$

It is clear that each of these planes is orthogonal to an asymptote of the hyperbola.

We consider the nature of the singularity of the distance squared function when $(x, y) = (0,0)$. The case $(u_2, u_3) = (0,0)$ is dealt with separately.

**Proposition 6.2.3** When $u_1 = 0$, and $(u_2, u_3)$ lies on the conic specified in Proposition 6.2.1, but away from $(0,0)$, the distance squared function has a corank 1 singularity when $(x, y) = 0$. These are of type $A_2$ at all except for 0, 2 or 4 points on the conic (generically), where they are of type $A_3$.

These singularities are versally unfolded by the distance squared functions.

**Proof** Let $F(x, y, u_1, u_2, u_3)$ be the distance squared function defined in the proof of Proposition 6.2.1. The condition for a corank 2 singularity ($F_{xx} = F_{xy} = F_{yy} = 0$) clearly cannot be satisfied when $(x, y) = 0$, so $F$ must have a corank one singularity when $x = y = u_1 = 0$, and $u_2, u_3$ satisfy

$$2u_3(1 - 2au_3) + (bu_3 + u_2)^2 = 0.$$

This means that the quadratic part of $F(x, y, 0, u_2, u_3)$ in the variables $x$ and $y$, that is

$$(1 - 2au_3)x^2 - 2(u_2 + bu_3)xy - 2u_3y^2, \tag{6.1}$$

is the square of a linear form for the appropriate values of $u_2$ and $u_3$. We determine this linear form for the cases of the parabolic crosscap ($a > 0$) and the hyperbolic crosscap ($a < 0$) separately.

### (i) The parabolic crosscap ($a > 0$)

In this case, we write $a = \lambda^2$, so that the part of the focal set corresponding to the crosscap point is the hyperbola lying in the plane $u_1 = 0$ with equation

$$\begin{aligned} (bu_3 + u_2)^2 - 4\lambda^2(u_3 - 1/4\lambda^2)^2 + 1/4\lambda^2 &= 0 \\ \text{that is} \quad 16\lambda^4(u_3 - 1/4\lambda^2)^2 - 4\lambda^2(bu_3 + u_2)^2 &= 1. \end{aligned}$$

We parametrise the hyperbola by setting

$$\begin{aligned} 16\lambda^4(u_3 - 1/4\lambda^2)^2 &= \cosh^2\theta \quad \text{and} \\ 4\lambda^2(bu_3 + u_2)^2 &= \sinh^2\theta, \end{aligned}$$

so we have

$$(u_3 - 1/4\lambda^2) = \frac{\pm\cosh\theta}{4\lambda^2} \quad \text{and} \quad (bu_3 + u_2) = \frac{\sinh\theta}{2\lambda},$$

$$\text{that is} \quad u_3 = \frac{1 \pm \cosh\theta}{4\lambda^2} \quad \text{and} \quad u_2 = \frac{\sinh\theta}{2\lambda} - bu_3.$$

Each choice of sign corresponds to a branch of the hyperbola. Now we substitute for $u_2$ and $u_3$ in Expression (6.1) to get

$$\begin{aligned} &\left(1 - 2\lambda^2\left(\frac{1 \pm \cosh\theta}{4\lambda^2}\right)\right)x^2 - 2\left(\frac{\sinh\theta}{2\lambda} - bu_3 + bu_3\right)xy - 2\left(\frac{1 \pm \cosh\theta}{4\lambda^2}\right)y^2 \\ &= \left(\frac{1 \mp \cosh\theta}{2}\right)x^2 - \left(\frac{\sinh\theta}{\lambda}\right)xy - \left(\frac{1 \pm \cosh\theta}{2\lambda^2}\right)y^2. \end{aligned}$$

Now let

$$t = e^\theta, t \geq 0 \quad \text{and} \quad t = -e^\theta, t < 0,$$

so that we can make the substitution

$$\sinh\theta = \frac{t^2-1}{2t} \quad \text{and} \quad \pm\cosh\theta = \frac{t^2+1}{2t},$$

giving

$$u_3 = \frac{1+\cosh\theta}{4\lambda^2} = \frac{(t+1)^2}{8\lambda^2 t}$$

$$\text{and} \quad u_2 = \frac{\sinh\theta}{2\lambda} - bu_3 = \frac{(t+1)((2\lambda-b)t-(2\lambda+b))}{8\lambda^2 t}.$$

Note that $t = 0$ corresponds to the asymptotes to the hyperbola, and $t = -1$ gives the origin. Then we have

$$\left(\frac{2t-(t^2+1)}{4t}\right)x^2 - \left(\frac{t^2-1}{2\lambda t}\right)xy - \left(\frac{2t+(t^2+1)}{4\lambda^2 t}\right)y^2.$$

This gives us the quadratic form

$$\frac{-1}{t}\left(\frac{(t-1)^2}{4}x^2 + \frac{(t-1)(t+1)}{2\lambda}xy + \frac{(t+1)^2}{4\lambda^2}y^2\right)$$

$$= -\frac{1}{t}\left(\frac{(t-1)}{2}x + \frac{(t+1)}{2\lambda}y\right)^2. \tag{6.2}$$

**(ii) The hyperbolic crosscap ($a < 0$)**

In this case we write $a = -\lambda^2$, so the part of the focal set corresponding to the crosscap point is the ellipse lying in the plane $u_1 = 0$ with equation

$$\begin{aligned} (bu_3+u_2)^2 + 4\lambda^2(u_3+1/4\lambda^2)^2 + 1/4\lambda^2 &= 0 \\ \text{that is} \quad 16\lambda^4(u_3+1/4\lambda^2)^2 + 4\lambda^2(bu_3+u_2)^2 &= 1. \end{aligned}$$

We parametrise the ellipse by setting

$$(u_3+1/4\lambda^2) = \frac{\sin\theta}{4\lambda^2} \quad \text{and} \quad (bu_3+u_2) = \frac{\cos\theta}{2\lambda},$$

so we have

$$u_3 = \frac{\sin\theta-1}{4\lambda^2} \quad \text{and} \quad u_2 = \frac{\cos\theta}{2\lambda} - bu_3.$$

Now we substitute for $a$, $u_2$ and $u_3$ in Expression (6.1) to get

$$\left(1+2\lambda^2\left(\frac{\sin\theta-1}{4\lambda^2}\right)\right)x^2 - 2\left(\frac{\cos\theta}{2\lambda} - bu_3 + bu_3\right)xy - 2\left(\frac{\sin\theta-1}{4\lambda^2}\right)y^2$$

$$= \left(\frac{1-\sin\theta}{2}\right)x^2 - \left(\frac{\cos\theta}{\lambda}\right)xy - \left(\frac{\sin\theta - 1}{2\lambda^2}\right)y^2.$$

Let $t = \tan\theta/2$. Then

$$\sin\theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos\theta = \frac{1-t^2}{1+t^2},$$

and

$$u_3 = -\frac{(1-t)^2}{4\lambda^2(1+t^2)} \quad \text{and} \quad u_2 = \frac{(1-t)((2\lambda-b)+(2\lambda+b)t)}{4\lambda^2(1+t^2)}.$$

Note that $t = 1$ gives the origin. Then we have the quadratic form

$$\left(\frac{(1+t)^2}{2(1+t^2)}\right)x^2 - \left(\frac{(1-t)(1+t)}{\lambda(1+t^2)}\right)xy + \left(\frac{(1-t^2)}{2\lambda^2(1+t^2)}\right)y^2$$

$$= \frac{1}{2(1+t^2)}\left((1+t)x - \frac{(1-t)}{\lambda}y\right)^2. \tag{6.3}$$

The cubic part of $F$ is

$$C(x,y,u_2,u_3) = -2u_2p_3y^3 - 2u_3(q_0x^3 + q_1x^2y + q_2xy^2 + q_3y^3),$$

where $p_3y^3$ and $q_0x^3 + q_1x^2y + q_2xy^2 + q_3y^3$ are the cubic parts of $p$ and $q$ respectively. Now $F$ has an $A_2$ singularity when $(u_2, u_3)$ lies on the conic unless $C$ is divisible by the linear form. To determine whether $F$ has an $A_{\geq 3}$ singularity, we first substitute the parametrisation of the relevant conic for $u_2$ and $u_3$. Suppose that $L(x,y,t)$ is the linear form. We rearrange $L(x,y,t) = 0$ to get an expression for $y$ in terms of $x$ and $t$. We substitute this into $C$, and arrive at a polynomial $P(x,t)$ that is homogeneous and of degree 3 in $x$ . The condition for $L$ to divide $C$ is equivalent to the condition $P(x,t) = 0$ for all $x$, i.e. $P(x,t)/x^3 = 0$.

### (i) The parabolic crosscap

The hyperbola is parametrised by

$$u_2 = \frac{(t+1)((2\lambda-b)t-(2\lambda+b))}{8\lambda^2 t} \quad \text{and} \quad u_3 = \frac{(t+1)^2}{8\lambda^2 t},$$

and the linear form is

$$L(x,y,t) = \frac{(t-1)}{2}x + \frac{(t+1)}{2\lambda}y.$$

Substituting

$$y = -\frac{\lambda(t-1)}{(t+1)}x$$

into $C$, we get

$$\begin{aligned}
P(x,t) &= -2\frac{(t+1)((2\lambda-b)t-(2\lambda+b))}{8\lambda^2 t}p_3\left(-\frac{\lambda(t-1)}{(t+1)}x\right)^3 - 2\frac{(t+1)^2}{8\lambda^2 t}\left(q_0x^3\right. \\
&\left. +q_1x^2\left(-\frac{\lambda(t-1)}{(t+1)}x\right)+q_2x\left(-\frac{\lambda(t-1)}{(t+1)}x\right)^2+q_3\left(-\frac{\lambda(t-1)}{(t+1)}x\right)^3\right) \\
&= \frac{x^3}{4\lambda^2 t}\left(\lambda^3 p_3\frac{((2\lambda-b)t-(2\lambda+b))(t-1)^3}{(t+1)^2}-q_0(t+1)^2\right. \\
&\left. +\lambda q_1(t-1)(t+1)-\lambda^2 q_2(t-1)^2+\lambda^3 q_3\frac{(t-1)^3}{(t+1)}\right).
\end{aligned}$$

Now when $t \neq 0$ or $-1$ (so we are away from the asymptotes and the origin), the condition that $P(x,t)/x^3 = 0$ is equivalent to

$$\begin{aligned}
\lambda^3 p_3((2\lambda-b)t &-(2\lambda+b))(t-1)^3-q_0(t+1)^4+\lambda q_1(t-1)(t+1)^3 \\
&-\lambda^2 q_2(t-1)^2(t+1)^2+\lambda^3 q_3(t-1)^3(t+1)=0.
\end{aligned}$$

Expanding, we get

$$\lambda^3 p_3((2\lambda-b)t^4+(-8\lambda+2b)t^3+12\lambda t^2+(-8\lambda-2b)t+(2\lambda+b))-q_0(t^4+4t^3+6t^2+4t+1)$$

$$+\lambda q_1(t^4+2t^3-2t-1)-\lambda^2 q_2(t^4-2t^2+1)+\lambda^3 q_3(t^4-2t^3+2t-1)=0.$$

It is clear that the coefficients of $t^4, t^3, t^2, t$ and the constant term are linearly independent. Thus we have a general quartic, which has 0, 2 or 4 real roots generically.

### (ii) The hyperbolic crosscap

In this case, we have the parametrisation

$$u_3 = -\frac{(1-t)^2}{4\lambda^2(1+t^2)} \quad \text{and} \quad u_2 = \frac{(1-t)((2\lambda-b)+(2\lambda+b)t)}{4\lambda^2(1+t^2)}.$$

and the linear form

$$L(x,y,t) = (1+t)x - \frac{(1-t)}{\lambda}y.$$

Substituting

$$y = \frac{\lambda(1+t)}{(1-t)}x$$

into $C$, we get

$$\begin{aligned}
P(x,t) &= -2\frac{(1-t)((2\lambda-b)+(2\lambda+b)t)}{4\lambda^2(1+t^2)}p_3\left(\frac{\lambda(1+t)}{(1-t)}x\right)^3 \\
&+2\frac{(1-t)^2}{4\lambda^2(1+t^2)}\left(q_0x^3+q_1x^2\left(\frac{\lambda(1+t)}{(1-t)}x\right)+q_2x\left(\frac{\lambda(1+t)}{(1-t)}x\right)^2\right. \\
&\left.+q_3\left(\frac{\lambda(1+t)}{(1-t)}x\right)^3\right) \\
&= \frac{x^3}{2\lambda^2(1+t^2)}\left(-\lambda^3p_3\frac{((2\lambda-b)+(2\lambda+b)t)(1+t)^3}{(1-t)^2}+q_0(1-t)^2\right. \\
&\left.+\lambda q_1(1+t)(1-t)+\lambda^2q_2(1+t)^2+\lambda^3q_3\frac{(1+t)^3}{(1-t)}\right).
\end{aligned}$$

Now the condition that $P(x,t)/x^3=0$ is equivalent to

$$\begin{aligned}
-\lambda^3p_3((2\lambda-b) &+(2\lambda+b)t)(1+t)^3+q_0(1-t)^4+\lambda q_1(1+t)(1-t)^3 \\
&+\lambda^2q_2(1+t)^2(1-t)^2+\lambda^3q_3(1+t)^3(1-t)=0,
\end{aligned}$$

when $t\neq 1$ (which corresponds to $(u_2,u_3)=(0,0)$). Expanding, we get

$$-\lambda^3p_3((2\lambda-b)+(8\lambda-2b)t^3+12\lambda t^2+(8\lambda+2b)t+(2\lambda+b)^4)+q_0(t^4-4t^3+6t^2-4t+1)$$

$$+\lambda q_1(-t^4-2t^3-2t+1)-\lambda^2q_2(t^4-2t^2+1)+\lambda^3q_3(-t^4-2t^3+2t+1)=0.$$

It is clear that the coefficients of $t^4,t^3,t^2,t$ and the constant term are linearly independent. Thus we have a general quartic, which has 0, 2 or 4 real roots generically.

In each case, each of the roots of the quartic we obtain gives a point on the conic where the contact of a sphere centred at that point with the crosscap at the crosscap point is of type $A_{\geq 3}$. To complete the proof, we note that the transversality results in Appendix A, Section A.1 show that the singularity is of type $A_3$, and that the singularities are indeed versally unfolded by the family of distance squared functions. □

**Proposition 6.2.4** When $(u_1,u_2,u_3)=(0,0,0)$, the distance squared function has an $A_3$ singularity at $(x,y)=(0,0)$. It is not versally unfolded by the $(u_1,u_2,u_3)$-parameters.

To prove this we need to use the notion of weighted homogeneous functions. A fuller treatment of these can be found in [A1].

**Definition 6.2.5** A function $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ is said to be *quasihomogeneous* of degree $d$ with weights $\alpha_1, \ldots, \alpha_n$ if

$$f(\lambda^{\alpha_1}x_1, \cdots, \lambda^{\alpha_n}x_n) = \lambda^d f(x_1, \cdots, x_n) \quad \text{for all} \quad \lambda \in \mathbf{R}.$$

A quasihomogeneous function $f$ is said to be *non-degenerate* if 0 is an isolated critical point.

We say that the monomial $x^k = x_1^{k_1} \ldots x_n^{k_n}$ has *degree d* with respect to $\alpha = (\alpha_1, \cdots, \alpha_n)$ if $< \alpha, k >= \alpha_1 k_1 + \cdots \alpha_n k_n = d$.

A polynomial has *filtration d* if all its monomials are of degree $d$ or higher.

A polynomial $f$ is said to be *semiquasihomogeneous* of degree $d$ with weights $\alpha_1, \ldots, \alpha_n$ if it is of the form $f = f_0 + f'$, where $f_0$ is a non-degenerate quasihomogeneous polynomial of degree $d$ with exponents $\alpha_1, \ldots, \alpha_n$, and $f'$ is a polynomial of filtration greater than $d$.

**Theorem 6.2.6** Suppose that $f = f_0 + f'$ is a semiquasihomogeneous polynomial of degree $d$ with weights $\alpha_1, \ldots, \alpha_n$.

If $f_0$ is finitely $\mathcal{R}$-determined and simple, then $f$ is finitely $\mathcal{R}$-determined, and is $\mathcal{R}$-equivalent to $f_0$.

**Proof** A theorem of Arnol'd ([A1], Theorem 3.1) says that the multiplicity of the critical point 0 of a semiquasihomogeneous function $f$ is that of its quasihomogeneous part. Now we define $f_t = f_0 + tf'$. The function $f_t$ then has the same multiplicity as its quasihomogeneous part. It follows from Lemma 6.2.7 that the $f_t$ are all $\mathcal{R}$-equivalent. □

**Lemma 6.2.7** Let

$$g : \mathbf{R}^n \times [0,1], 0 \times [0,1] \longrightarrow \mathbf{R}, 0$$

be a family of functions, each with an isolated singularity. Suppose that $g_0$ is simple and that $\mu(g_t)$, the Milnor number of $g_t$, is the same for all $t \in [0,1]$. Then the functions $g_t$ are all $\mathcal{R}$-equivalent.

**Proof** Let

$$A = \{t \in [0,1] : g_t \quad \text{is } \mathcal{R}\text{-equivalent to} \quad g_0\}.$$

We show that the set $A$ is open and closed, and hence equal to $[0,1]$.

To prove that $A$ is open, note that if $g_{t_0}$ is simple then there is a neighbourhood $U \subset [0,1]$ of $t_0$ such that all $t \in U \backslash \{t_0\}$, say $t > t_0$ (respectively $t < t_0$) yield $\mathcal{R}$-equivalent germs. Since $\mu(g_t) = \mu(g_{t_0})$, these are all equivalent to $g_0$.

To see that $A$ is closed, let $(t_n) \in A$ be a sequence such that

$$\lim_{n\to\infty} t_n = t_0 \notin A.$$

Then $g_{t_0}$ is a function in the closure of the simple singularity $g_0$ and $\mu(g_{t_0}) = \mu(g_0)$. This would imply that $g_0$ and $g_{t_0}$ have the same local topology, which is not possible by the various topological characterisations of simple singularities. □

**Proof (of Proposition 6.2.4).** The distance squared function on the crosscap is given by

$$\begin{aligned} F(x,y,u_1,u_2,u_3) &= (x-u_1)^2 \\ &\quad +(xy+p(y)-u_2)^2+(y^2+ax^2+bxy+q(x,y)-u_3)^2. \end{aligned}$$

Substituting $(u_1,u_2,u_3) = (0,0,0)$ we get

$$\begin{aligned} f(x,y) = F(x,y,0,0,0) &= x^2+(xy+p(y))^2+(y^2+ax^2+bxy+q(x,y))^2 \\ &= x^2+x^2y^2(1+2a+b^2)+2bxy^3+2abx^3y+y^4+a^2x^4+\cdots. \end{aligned}$$

Assigning weight $1/2$ to $x$ and weight $1/4$ to $y$, we see that $F(x,y,0,0,0)$ is a semiquasihomogeneous polynomial of degree 1, with quasihomogeneous part $f_0 = x^2+y^4$. Now $f_0$ is an $A_3$ singularity, which is $4-\mathcal{R}$-determined, so by Theorem 6.2.6, $F(x,y,0,0,0)$ has an $A_3$ singularity at $(x,y) = (0,0)$. Now the initial speeds of $F$ ($\partial F/\partial u_1, \partial F/\partial u_2, \partial F/\partial u_3$ evaluated at $(u_1,u_2,u_3) = (0,0,0)$) are

$$-2x, -2(xy+p(y)) \quad \text{and} \quad -2(y^2+ax^2+bxy+q(x,y)).$$

It is clear that

$$J(f)+\mathbf{R}.\{\partial F/\partial u_1, \partial F/\partial u_2, \partial F/\partial u_3\} \neq \mathcal{E}_2,$$

so that $f$ is not versally unfolded by the parameters $u_1, u_2$ and $u_3$. □

The result of Proposition 6.2.4 makes more work for us. As the distance squared function is not versally unfolded when $(x,y,u_1,u_2,u_3) = (0,0,0,0,0)$ we cannot use the standard models of the bifurcation set of an $A_3$ singularity to describe the focal set near $(u_1,u_2,u_3) = (0,0,0)$. In fact we devote Chapter 7 to determining a local model for this part of the focal set of the crosscap.

## 6.3 The Limiting Principal Curvatures at the Crosscap Point

To gain more information on the stucture of the focal set at the crosscap point, we revert to more elementary techniques. Recall from Definition 1.9.8 that the focal set is the locus of the centres of curvature and that the focal set has two sheets; one for each centre of curvature. Thus it is of interest to calculate the limiting principal curvatures as we approach the crosscap point along curves lying on the crosscap. The crosscap is parametrised by the normal form

$$f(x,y) = (x, xy + p(y), y^2 + ax^2 + bxy + q(x,y)),$$

where $a$ and $b$ are constants, $p \in \mathcal{M}_1^3$ and $q \in \mathcal{M}_2^3$.

**Proposition 6.3.1** Suppose that $C$ is a non-singular curve in the source that passes through the origin, parametrised by $\gamma(t) = (\alpha t + \cdots, \beta t + \cdots)$, with $\alpha$ and $\beta$ not both zero. Then as we approach the crosscap point along the curve parametrised by $f \circ \gamma$ one principal curvature tends to

$$\frac{2a\alpha^2 - 2\beta^2}{\alpha(\alpha^2 + (2\beta + \alpha b)^2)^{1/2}}$$

and the other tends to infinity.

**Proof** Recall (Proposition 1.9.5) that the principal curvatures are given by the roots of

$$det\left(\begin{pmatrix} l & m \\ m & n \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = 0,$$

where $E, F$ and $G$ are the coefficients of the first fundamental form and $l, m$ and $n$ are the coefficients of the second fundamental form. Expanding this equation, we get

$$\begin{aligned} (l - \lambda E)(n - \lambda G) - (m - \lambda F)^2 &= 0 \quad \text{which gives} \\ (EG - F^2)\lambda^2 + (2mF - nE - lG)\lambda + (ln - m^2) &= 0. \end{aligned}$$

Writing

$$A = EG - F^2, B = 2mF - nE - lG \quad \text{and} \quad C = ln - m^2,$$

the two principal curvatures are

$$\frac{-B \pm (B^2 - 4AC)^{1/2}}{2A} = \frac{B}{2A}\left(-1 \pm \left(1 - \frac{4AC}{B^2}\right)^{1/2}\right). \tag{6.4}$$

To prove the result, we substitute

$$x = \alpha t + \cdots \quad \text{and} \quad y = \beta t + \cdots$$

and determine the limiting values of $A$, $B$ and $C$ as $t$ tends to zero. In our case, the unit normal vector $N$ is given by

$$N = \frac{f_x \times f_y}{\|f_x \times f_y\|}.$$

Now we have

$$f_x = (1, y, 2ax + by + q_x) \quad \text{and} \quad f_y = (0, x + p_y, 2y + bx + q_y).$$

It follows that

$$N(x,y) = \frac{(2y^2 - 2ax^2 + yq_y - xq_x - p_y q_x, -(2y + bx + q_y), x + p_y)}{\|(2y^2 - 2ax^2 + yq_y - xq_x - p_y q_x, -(2y + bx + q_y), x + p_y)\|},$$

$$\begin{aligned} E = \ & f_x \cdot f_x = \ 1 + y^2 + (2ax + by + q_x)^2, \\ F = \ & f_x \cdot f_y = \ xy + yp_y + (2ax + by + q_x)(2y + bx + q_y) \quad \text{and} \\ G = \ & f_y \cdot f_y = \ (x + p_y)^2 + (2y + bx + q_y)^2. \end{aligned}$$

The second derivatives of $f$ are

$$f_{xx} = (0, 0, 2a + q_{xx}), f_{xy} = (0, 1, b + q_{xy}) \quad \text{and} \quad f_{yy} = (0, p_{yy}, 2 + q_{yy}),$$

so we have

$$\begin{aligned} l = \ & N.f_{xx} = \ \frac{(2a + q_{xx})(x + p_y)}{\|f_x \times f_y\|}, \\ m = \ & N.f_{xy} = \ \frac{-(2y + bx + q_y) + (x + p_y)(b + q_{xy})}{\|f_x \times f_y\|} \quad \text{and} \\ n = \ & N.f_{yy} = \ \frac{-p_{yy}(2y + bx + q_y) + (x + p_y)(2 + q_{yy})}{\|f_x \times f_y\|}. \end{aligned}$$

Taking the lowest order terms of $E, F, G, l, m$ and $n$, we have

$$\begin{aligned} E &= 1 + y^2 + (2ax + by)^2 + \cdots, \\ F &= xy + (2ax + by)(2y + bx) + \cdots, \\ G &= x^2 + (2y + bx)^2 + \cdots, \\ l &= \frac{2ax + \cdots}{\|f_x \times f_y\|}, \\ m &= \frac{-2y + \cdots}{\|f_x \times f_y\|}, \\ n &= \frac{2x + \cdots}{\|f_x \times f_y\|} \end{aligned}$$

and

$$\|f_x \times f_y\| = (x^2 + (2y + bx)^2 + \cdots)^{1/2}.$$

Then we have

$$\begin{aligned}
A &= EG - F^2 = \|f_x \times f_y\|^2 = (x^2 + (2y+bx)^2 + \cdots)^{1/2}, \\
B &= 2mF - nE - lG = \frac{-2x + \cdots}{\|f_x \times f_y\|} \quad \text{and} \\
C &= ln - m^2 = \frac{4ax^2 - 4y^2 + \cdots}{\|f_x \times f_y\|^2}.
\end{aligned}$$

Writing $B' = -2x + \cdots$ and $C' = 4ax^2 - 4y^2 + \cdots$, we have

$$B = \frac{B'}{\sqrt{A}} \quad \text{and} \quad C = \frac{C'}{A}.$$

Then from Equation (6.4), the two principal curvatures are

$$\frac{B'}{2A^{3/2}} \left( -1 \pm \left( 1 - \frac{4AC'}{B'^2} \right)^{1/2} \right).$$

Expanding

$$\left( 1 - \frac{4AC'}{B'^2} \right)^{1/2} \quad \text{gives}$$

$$1 - \binom{1/2}{1} \frac{4AC'}{B'^2} + \binom{1/2}{2} \left( \frac{4AC'}{B'^2} \right)^2 + \cdots + (-1)^n \binom{1/2}{n} \left( \frac{4AC'}{B'^2} \right)^n + \cdots,$$

where

$$\binom{1/2}{n} = \frac{1/2 \cdot (-1/2) \cdots (1/2 - n + 1)}{n \cdot (n-1) \cdots 1}.$$

So the two principal curvatures are

$$\frac{B'}{2A^{3/2}} \left( -2 + \binom{1/2}{1} \frac{4AC'}{B'^2} - \binom{1/2}{2} \left( \frac{4AC'}{B'^2} \right)^2 + \cdots \right.$$
$$\left. + (-1)^{n+1} \binom{1/2}{n} \left( \frac{4AC'}{B'^2} \right)^n + \cdots \right)$$

and

$$\frac{B'}{2A^{3/2}} \left( - \binom{1/2}{1} \frac{4AC'}{B'^2} + \binom{1/2}{2} \left( \frac{4AC'}{B'^2} \right)^2 + \cdots \right.$$
$$\left. + (-1)^n \binom{1/2}{n} \left( \frac{4AC'}{B'^2} \right)^n + \cdots \right).$$

Now

$$\frac{B'}{2A^{3/2}}\left(\frac{4AC'}{B'^2}\right)^n = 2^{2n-1}\frac{A^{n-3/2}C'^n}{B'^{2n-1}}.$$

Substituting $x = \alpha t + \cdots$ and $y = \beta t + \cdots$ into the expressions for $A$, $B'$ and $C'$, we see that $A$ is order two in $t$, $B'$ is order one in $t$ and $C'$ is order two in $t$, so that as $t$ tends to zero,

$$\frac{A^{n-3/2}C'^n}{B'^{2n-1}} \quad \text{tends to zero,}$$

unless

$$2n - 1 \geq 2n + 2n - 3 = 4n - 3,$$

i.e. $n \leq 1$.

So to get the limiting principal curvatures, we need to evaluate the limits of

$$-\frac{B'}{A^{3/2}} + \frac{C'}{B'A^{1/2}} \quad \text{and} \quad -\frac{C'}{B'A^{1/2}}$$

as $t$ tends to zero. The first of these is clearly infinite. The second is given by

$$\lim_{t\to 0} - \frac{(4a\alpha^2t^2 - 4\beta^2t^2 + \cdots)}{(-2\alpha t + \cdots)(\alpha^2t^2 + (2\beta t + \alpha bt)^2 + \cdots)^{1/2}}$$

$$= \frac{2a\alpha^2 - 2\beta^2}{\alpha(\alpha^2 + (2\beta + \alpha b)^2)^{1/2}}.$$

□

Now from the definition of the focal set (Definition 1.9.8), we can deduce the following

**Proposition 6.3.2** As we approach the crosscap point along curves on the crosscap, one sheet of the focal set goes through the origin. The other sheet blows up to the conic section described in Proposition 6.2.1.

**Proof** The focal set is the locus of centres of curvature. These are given by $p + 1/\kappa_i N$, where $N$ is the unit normal at $p$. It follows from the proof of Proposition 6.3.1 that one of the radii of curvature (the inverses of the principal curvatures) is always zero, thus one sheet of the focal surface passes through the origin.

In the other case, the radius of curvature depends upon the direction in which we approach the crosscap point. It follows from Proposition 6.3.1 that the limiting radius of curvature as we approach the crosscap along the curve parametrised by $f \circ \gamma$, where $\gamma(t) = (\alpha t + \cdots, \beta t + \cdots)$, is

$$\frac{\alpha(\alpha^2 + (2\beta + \alpha b)^2)^{1/2}}{2a\alpha^2 - 2\beta^2}.$$

From Proposition 3.3.4, we know that the limiting unit normal to the crosscap as we approach the crosscap point along $f \circ \gamma$ is

$$\frac{(0, -(2\beta + \alpha b), \alpha)}{(\alpha^2 + (2\beta + \alpha b)^2)^{1/2}}.$$

Thus the limiting centre of curvature is

$$\frac{(0, -\alpha(2\beta + \alpha b), \alpha^2)}{2a\alpha^2 - 2\beta^2}.$$

We write

$$u = \frac{-\alpha(2\beta + \alpha b)}{2a\alpha^2 - 2\beta^2} \quad \text{and} \quad v = \frac{\alpha^2}{2a\alpha^2 - 2\beta^2}.$$

Now let $s = \alpha/\beta$, so that

$$u = \frac{-2s - bs^2}{2as^2 - 2} \quad \text{and} \quad v = \frac{s^2}{2as^2 - 2}.$$

Thus $u$ and $v$ satisfy

$$s^2(2au + b) - 2u + 2s = 0 \quad \text{and}$$

$$s^2(2av - 1) - 2v = 0.$$

We use Maple to eliminate $s$ from these two equations (by taking the resultant of the two equations) to get

$$4v^2b^2 + 8vbu + 4u^2 + 8v - 16av^2 = 0.$$

This gives

$$v^2b^2 + 2vbu + u^2 + 2v - 4av^2 = 0,$$

and we can factorise this to obtain

$$(bv + u)^2 + 2v(1 - 2av) = 0,$$

which is (not surprisingly) precisely the equation of the conic section of Proposition 6.2.1. □

# Chapter 7

# Non-versal Unfoldings of Functions and the Focal Set of the Crosscap

The main problem in trying to construct local models of the focal surface of the crosscap is that the $A_3$ singularity of the distance squared function that occurs at the origin is not versally unfolded. It would be useful to find, up to some finite order, the mapping which takes the bifurcation set that arises in this example to the bifurcation set of the standard unfolding of an $A_3$ singularity. To this end, we implement an algorithm described by Bruce and Giblin in [BG3] to calculate the mappings needed to induce a general unfolding from the universal unfolding of a singularity. The algorithm is described in the case of functions of one variable in [BG2]. We note that a similar algorithm has been described by Cowell and Wright in [CW-I] and [CW-II] and implemented in the computer algebra package REDUCE 3.2. We adapt a certain part of their algorithm to generalise that of [BG2]. We can use the algorithm to determine a local model for the focal set of the crosscap near the origin.

## 7.1 Unfoldings of Functions

We are dealing here with functions, and so the definitions of Section 1.5 can be restricted to the following. For more information, we refer the reader to [Mar].

**Definition 7.1.1** Let $f \in \mathcal{E}_n$ be a germ of a function. A *p-parameter unfolding* of $f$ is a germ $F$ of a smooth function at the origin of $\mathbf{R}^n \times \mathbf{R}^p$ (given coordinates

$(x, t)$) such that

$$F_0(x) = F(x, 0) = f(x).$$

Two $p$-parameter unfoldings, $F$ and $G$ of the same germ $f \in \mathcal{E}_n$ are *isomorphic* if there exists a local diffeomorphism $\phi$ at the origin such that

1. $\phi(x, t) = (\psi(x, t), t)$ and $\psi(x, 0) = x$, i.e. $\phi$ is a $p$-parameter unfolding of the identity map on $\mathbf{R}^n$;
2. $G = F \circ \phi$.

An unfolding is called *trivial* if it is isomorphic to the constant unfolding $(x, t) \mapsto f(x)$.

As before we say that two $p$-parameter unfoldings $F$ and $G$ of a function germ $f$ are *equivalent* if there exists a diffeomorphism $h : \mathbf{R}^p, 0 \longrightarrow \mathbf{R}^p, 0$ such that $G$ is isomorphic to $h^*F$, the pullback of $F$ by $h$. If $H$ is now some $q$-parameter unfolding of $f$, we say that $H$ can be *induced* from $F$ if there exists a smooth map germ $h : \mathbf{R}^q, 0 \longrightarrow \mathbf{R}^p, 0$ such that $H$ is isomorphic to $h^*F$.

We say that an unfolding $F$ of $f \in \mathcal{E}_n$ is a *versal unfolding* of $f$ if any other unfolding $G$ of $f$ can be induced from $f$.

Finally we can state the fundamental theorem on unfoldings of function germs. Given a $p$-parameter unfolding $F(x, t)$ of the function germ $f \in \mathcal{E}_n$, the *initial speeds*, $\dot{F}_i \in \mathcal{E}_n$, of $F$ are defined by

$$\dot{F}_i(x) = \partial F / \partial t_i(x, 0), \quad \text{for} \quad i = 1, \cdots, p.$$

The *Jacobian ideal* $J(f)$ is the ideal in $\mathcal{E}_n$ generated by the partial derivatives $\partial f / \partial x_i$, for $i = 1 \cdots n$.

**Theorem 7.1.2** The unfolding $F$ is versal if and only if

$$J(f) + \mathbf{R}.\{\dot{F}_1, \ldots, \dot{F}_s\} = \mathcal{E}_n.$$

**Proof** For a proof of this see [Mar], Chapter 11. □

Notice that $\mathcal{M}_n \cdot J(f) = L\mathcal{R} \cdot f$ so that Theorem 7.1.2 is analogous to Theorem 1.5.6, for $\mathcal{R}$- and $\mathcal{R}_e$-equivalence.

## 7.2 The Algorithm

In implementing the algorithm, we restrict our attention to the case of an $A_3$ singularity, and assume that the crosscap parametrisation is analytic. We restate and prove the relevant propositions from [BG3], as the proofs outline the construction that we shall implement.

Let $f : \mathbf{R}^q, 0 \longrightarrow \mathbf{R}, 0$ be a smooth function germ of finite codimension with Jacobian ideal $J(f)$. We denote by $\mathcal{E}(x)(= \mathcal{E}_q)$ and $\mathcal{E}(u)(= \mathcal{E}_m)$ the rings of smooth function germs in $q$, respectively $m$ variables, with maximal ideals $\mathcal{M}(x)$ and $\mathcal{M}(u)$ respectively. Also we denote the module of smooth function germs at the origin on $\mathbf{R}^q \times \mathbf{R}^m$ by $\mathcal{E}(x,u)$. Let $\phi_1, \cdots, \phi_k$ be a basis for $\mathcal{E}(x)/J(f)$. We write

$$G(x, s, t) = f(x) + \sum_{i=1}^{k} s_i \phi_i,$$

where $t = (t_1, \cdots, t_n)$ are redundant unfolding parameters. This is the standard versal unfolding of $f$. Let $F(x, u)$, where $u \in \mathbf{R}^m$, be any unfolding of $f$. We use $()_i$ to refer to the $i$th component of a map. We remark now that we will be working with $\mathcal{R}^+$-equivalence (which ignores constant terms), so we are not interested in the terms of $F$ involving only the unfolding parameters $u$.

**Proposition 7.2.1** ([BG3], 5.1) For all $r \geq 0$ we can find mappings $a^r : \mathbf{R}^m \times \mathbf{R}^q, 0 \longrightarrow \mathbf{R}^m, 0$ and $b^r : \mathbf{R}^q, 0 \longrightarrow \mathbf{R}^k, 0$, with $(a^r(x,u))_i \in \mathcal{E}(x)[u_1, \cdots, u_m]$, for $i = 1, \cdots, m$, and $(b^r(u))_j \in \mathbf{R}[u_1, \cdots, u_m]$, for $j = 1, \cdots, k$ such that

$$F(x, u) - G(a^r(x,u), b^r(u)) \in \mathcal{M}^{r+1}(u)\mathcal{E}(x,u)$$

and

$$\frac{\partial a_i^r}{\partial x_j}(0,0) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq q.$$

**Proof** It is enough to find $a^r(x,u), b^r(u)$ with

$$F(a^r(x,u), u) - G(x, b^r(u)) \in \mathcal{M}^{r+1}(u)\mathcal{E}(x,u) \tag{7.1}$$

and

$$\frac{\partial a_i^r}{\partial x_j}(0,0) = \delta_{ij}.$$

For if we then consider the map

$$\begin{aligned} a^r \times id : \mathbf{R}^q \times \mathbf{R}^m, 0 &\longrightarrow \mathbf{R}^q \times \mathbf{R}^m, 0 \\ (x, u) &\longrightarrow (a^r(x,u), u), \end{aligned}$$

it is clear that $a^r \times id$ is a local diffeomorphism. Thus we can define a map $a^{r\prime} : \mathbf{R}^q \times \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^q$ by

$$a^{r\prime} \times id = (a^r \times id)^{-1} : \mathbf{R}^q \times \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^q \times \mathbf{R}^m, 0.$$

Thus if we have (7.1), then

$$F \circ (a^{r\prime} \times id)(a^r(x,u),u) - G \circ (a^{r\prime} \times id)(x, b^r(u))$$

$$= F(x,u) - G(a^{r\prime}(x,b^r(u)), b^r(u)) \in \mathcal{M}^{r+1}(u)\mathcal{E}(x,u),$$

as required.

Now we set $a^0(x,u) = x$ and $b^0(u) = 0$, and proceed by induction on $r$. Suppose that we have found $a^r$ and $b^r$; we may assume that they are of degree at most $r$ in $u$. We seek polynomial maps $\alpha_{r+1}$, $\beta_{r+1}$ that are homogeneous and degree $r+1$ in $u$ so that

$$F((a^r + \alpha_{r+1})(x,u),u) - G(x,(b^r + \beta_{r+1})(u)) \in \mathcal{M}^{r+2}(u)\mathcal{E}(x,u). \tag{7.2}$$

By Taylor's Theorem, we can write

$$F(a^r + \alpha_{r+1}, u) = F(a^r, u) + \sum_{i=1}^{q} (\alpha_{r+1})_i \frac{\partial F}{\partial x_i} + \psi(x,u),$$

where $\psi \in \mathcal{M}(u)^{r+2}\mathcal{E}(x,u)$. Since $(\alpha_{r+1})_i$ is a polynomial that is homogeneous of degree $r+1$ in $u$, we need only consider those terms in $\partial F/\partial x_i$ which do not involve $u$. This gives

$$F(a^r + \alpha_{r+1}, u) = F(a^r, u) + \sum_{i=1}^{q} (\alpha_{r+1})_i \frac{\partial f}{\partial x_i} + \psi(x,u),$$

where $\psi \in \mathcal{M}(u)^{r+2}\mathcal{E}(x,u)$. Similarly, we can write

$$G(x, b^r + \beta_{r+1}) = G(x, b^r) + \sum_{i=1}^{m} (\beta_{r+1})_i \frac{\partial G}{\partial u_i} + \eta(x,u)$$

$$= G(x, b^r) + \sum_{j=1}^{m} (\beta_{r+1})_j \phi_j(x) + \eta(x,u),$$

since $G$ is the standard versal unfolding. It follows that we can rewrite (7.2) as

$$(F(a^r, u) - G(x, b^r)) + \sum_{i=1}^{q} (\alpha_{r+1})_i \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} (\beta_{r+1})_j \phi_j(x) \in \mathcal{M}(u)^{r+2}\mathcal{E}(x,u).$$

By the inductive hypothesis,

$$(F(a^r,u) - G(x,b^r)) \in \mathcal{M}(u)^{r+1}\mathcal{E}(x,u).$$

The terms of degree $r+1$ in $F(a^r,u) - G(x,b^r)$ have the form $\psi_\nu(x)u^\nu$ for some multi-index $\nu$, and $\psi_\nu(x)$ can be written

$$\psi_\nu(x) = -\sum_{i=1}^{q} \gamma_i^\nu(x)\frac{\partial f}{\partial x_i} + \sum_{j=1}^{k} c_j^\nu \phi_j(x),$$

for $\gamma_i^\nu \in \mathcal{E}(x)$ and $c_j^\nu \in \mathbf{R}$, since $\phi_1, \cdots \phi_k$ is a basis for $\mathcal{E}(x)/J(f)$. To complete the inductive construction, we set

$$(\alpha_{r+1}(x,u))_i = \sum_\nu \gamma_i^\nu(x)u^\nu \qquad (\beta_{r+1}(u))_j = \sum_\nu c_j^\nu u^\nu.$$

□

Note that the induction process produces $(\beta_{r+1})_j$ for $j = 1, \cdots, k$. The remaining components of $\beta_{r+1}$ are arbitrary.

Proposition 7.2.1 allows us to compute finite jets of the diffeomorphism $b$. There is one apparent problem: we appear to need arbitrarily high order terms of $x$. We can correct this with the following result.

**Proposition 7.2.2** ([BG3], (5.3)) Let $k_1$ be such that $J(f) \supset \mathcal{M}^{k_1}(x)$ (since $f$ has finite codimension) and let $r \geq 1$. Then the same $b^r(u)$ can be used for any two $F$ with the same $l = (k_1 r + k_1 + r)$-jet.

**Proof** See [BG3]. □

Now we state and prove a stronger version of 7.2.2 for the case where $f : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$ is an $A_k$ singularity. So $f(x,y) = x^2 + y^{k+1}$ and the standard versal unfolding of $f$ is given by

$$\begin{aligned} f : \mathbf{R}^2 \times \mathbf{R}^{k-1} &\longrightarrow \mathbf{R} \\ (x,y,u) &\longrightarrow x^2 + y^{k+1} + \sum_{i=1}^{k-1} u_i y^i \end{aligned}$$

**Lemma 7.2.3** Suppose that we wish to determine $b^m$ for some $m$. Then for each $r$, $1 \leq r \leq m$, $b^m$ is uniquely determined by the terms of

$$F(a^r,u) - G(x,y,b^r)$$

of degree less than or equal to $(m-r)k - 1$ in $y$ and $m$ in $u$.

**Proof** It is clear that we need only consider the terms of degree less than or equal to $m$ in $u$.

Now we consider the algorithm described in Proposition 7.2.1 for the case where $f$ is an $A_k$ singularity. Suppose that we have determined $a^r(x,y,u), b^r(u)$ with

$$F(a^r(x,y,u),u) - G(x,y,b^r(u)) \in \mathcal{M}^{r+1}(u)\mathcal{E}(y,u),$$

where $a^r : \mathbf{R}\times\mathbf{R}^q \longrightarrow \mathbf{R}$ and $b^r : \mathbf{R}^q \longrightarrow \mathbf{R}^{k-1}$. For each term $\psi_\nu(y)u^\nu$ of degree $r+1$ in $u$ in $F(a^r,u) - G(x,y,b^r)$ (where $\nu$ is a multi-index), we write

$$\psi_\nu(y) = (k+1)y^k\gamma^\nu(y) + \sum_{i=1}^{k-1} c_i^\nu y^i,$$

where $\gamma^\nu \in \mathcal{E}(y)$ and $c_i^\nu \in \mathbf{R}$. We then set

$$(\alpha_{r+1}(y,u)) = -\sum_\nu \gamma^\nu(y)u^\nu \qquad (\beta_{r+1}(u))_i = \sum_\nu c_i^\nu u^\nu.$$

So it is clear that we must determine the terms of degree $m$ in $u$ of

$$F(a^{m-1}(x,y,u),u) - G(x,y,b^{m-1}(u))$$

correctly up to degree $k-1$ in $y$, and thus the terms of degree greater than $m-1$ in $u$ of

$$F(a^{m-2}(x,y,u),u) - G(x,y,b^{m-2}(u))$$

correctly up to degree $k+k-1 = 2k-1$ in $y$. Continuing in this way gives us the result. □

The following is now immediate.

**Corollary 7.2.4** Suppose that

$$F(x,y,u) - G(x,y,0) = \sum_{p=1}^{\infty}\sum_{\nu_p} u^{\nu_p}\gamma^{\nu_p}(y), \tag{7.3}$$

where $\nu_p$ is a multi index of degree $p$. Then to determine $b^m$ correctly, we need only know the following part of the power series (7.3)

$$\sum_{p=1}^{m}\sum_{\nu_p} u^{\nu_p}(j^{(k(m-p+1)-1)}(\gamma^{\nu_p}(y))), \tag{7.4}$$

where $j^{(k(m-p+1)-1)}(\gamma^{\nu_p}(y))$ is the $(k(m-p+1)-1)$-jet of the function $\gamma^{\nu_p}$.

The results described so far provide us with an algorithm to determine, up to some finite order, the mappings which induce the standard versal unfolding of a map germ $f$ from any unfolding of $f$. The next question is whether we can extend these methods to deal with unfoldings of map germs that are in the same $\mathcal{R}$-orbit as $f$. This problem is dealt with in generality in [CW-I] and [CW-II], and we adapt their methods for our application, where $g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$ is the $A_3$ singularity given by $g(x, y) = x^2 + y^4$ with standard versal unfolding

$$G(x, v_1, v_2) = x^2 + y^4 + v_1 y + v_2 y^2,$$

and $F(x, y, u_1, u_2, u_3)$ is the family of distance squared functions for the crosscap, given by

$$\begin{aligned} F(x, y, u_1, u_2, u_3) &= (x - u_1)^2 \\ &\quad + (xy + p(y) - u_2)^2 + (y^2 + ax^2 + bxy + q(x, y) - u_3)^2. \end{aligned}$$

First we consider the case where we have an unfolding

$$F(x, y, u) = x^2 + \Phi(y, u) = x^2 + \phi(y) + R(y, u),$$

where $\phi(y)$ is $\mathcal{R}$-equivalent to the normal form $f(y) = y^4$ and

$$R(y, u) \in \mathcal{M}(u) \cdot \mathcal{E}(y, u).$$

There exists a diffeomorphism

$$y \mapsto \psi(y)$$

such that $\phi \circ \psi(y) = y^4$ and

$$\Phi(\psi(y), u) = y^4 + R'(y, u),$$

where $R'(y, u) \in \mathcal{M}(u) \cdot \mathcal{E}(y, u)$. To what order do we need the power series expansions of $\phi$ and $\psi$ to apply the algorithm of Proposition 7.2.1 with accuracy?

**Lemma 7.2.5** Suppose that we wish to determine the map $b^m$ of Proposition 7.2.1 for some $m$. Then we need the $(3m - 1)$-jet of $\psi$. It follows that we must know the $(3m + 2)$-jet of $\phi$.

**Proof** We write

$$R'(y, u) = \sum_{p=1}^{m} \sum_{\nu_p} \gamma^{\nu_p}(y) u^{\nu_p},$$

where $\nu_p$ is a multi index of degree $p$. Substituting $\psi(y)$ into the unfolding $\Phi$, we get

$$\Phi(\psi(y), u) = y^4 + \sum_{p=1}^{m} \sum_{\nu_p} \gamma^{\nu_p}(\psi(y)) u^{\nu_p}.$$

By Corollary 7.2.4, it is clear that we can truncate $\psi(y)$ at degree $3m-1$ without changing $b^m$. If we write $\psi \circ \psi^{-1}(y) = y$, and consider the power series expansion of $\psi \circ \psi^{-1}$, we see that the $k$-jet of $\psi$ is determined by the $k$-jet of $\psi^{-1}$. Now writing $Y = \psi(y)$,

$$y^4 = \phi(Y) = Y^4(\phi_0 + \phi_1 Y + \phi_2 Y^2 + \cdots),$$

so

$$y = Y(\phi_0 + \phi_1 Y + \phi_2 Y^2 + \cdots)^{1/4}.$$

Considering the Taylor series expansion of $(\phi_0 + \phi_1 Y + \phi_2 Y^2 + \cdots)^{1/4}$ we see that to determine the $(3m-1)$-jet of $\psi^{-1}$, we must have the $(3m+2)$-jet of $\phi$. Since the $(3m-1)$-jet of $\psi^{-1}$ determines that of $\psi$, the result follows. □

We have shown (Proposition 6.2.4) that near $(u_1, u_2, u_3) = (0, 0, 0)$, the distance squared function $F$ is a non-versal unfolding of an $A_3$ singularity. We write $u$ for the vector $(u_1, u_2, u_3)$. We note that $F$ is of the form

$$\Phi(y, u) + x^2 + x \sum_{i=2}^{\infty} a_i y^i + x \sum_{p=1}^{\infty} \sum_{j=0}^{\infty} y^j \phi_{pj}(u) + R(x, y, u),$$

where $\phi_{pj}$ is homogeneous and degree $p$ in $u$ and $R \in \mathcal{M}(x)^2 \cdot \mathcal{E}(x, y, u)$.

**Lemma 7.2.6** (c.f. [CW-II], page 350). Suppose that we wish to determine the map $b^m$ of Proposition 7.2.1 for some $m$. Then we truncate $F$ at degree $m$ in $u$, and $\Phi(y, u)$ at the levels specified by Corollary 7.2.4 in $y$. Then there exists a transformation $x' = x - f(y, u)$ reducing $F$ to the form

$$\Phi'(y, u) + x^2 + R(x, y, u), \tag{7.5}$$

where $R \in \mathcal{M}(x)^2 \cdot \mathcal{E}(x, y, u)$. Thus all terms of $F$ that are linear in $x$ can be removed.

**Proof** We first truncate $F$ at degree $m$ in $u$, and $\Phi(y, u)$ at the levels specified by Corollary 7.2.4 in $y$, and then separate the terms linear and non-linear in $x$ so that the unfolding may be written as

$$F(x, y, u) = \Phi(y, u) + x^2 + x \sum_{i=2}^{\infty} a_i y^i + x \sum_{p=1}^{m} \sum_{j=0}^{\infty} y^j \phi_{pj}(u) + R(x, y, u),$$

where $\phi_{pj}$ is homogeneous and degree $p$ in $u$ and $R \in \mathcal{M}(x)^2 \cdot \mathcal{E}(x,y,u)$. For each $i$, we apply the transformation

$$x = x' - \frac{1}{2}a_i y^i,$$

to remove the terms of

$$x \sum_{i=2}^{\infty} a_i y^i$$

from $F$. Then $1 \le p \le m$, we apply the transformation

$$x = x' - \frac{1}{2}y^j \phi_{pj} \tag{7.6}$$

for each $j$. After applying the transformation, we have

$$\begin{aligned} F(x', y, u) &= \Phi(y,u) + (x' - \frac{1}{2}y^j \phi_{pj})^2 + (x' - \frac{1}{2}y^j\phi_{pj}) \sum_{p=1}^{m}\sum_{j=0}^{\infty} y^j \phi_{pj}(u) \\ &\quad + R(x' - \frac{1}{2}y^j \phi_{pj}, y, u). \end{aligned}$$

It is clear that at the $(p,j)$-th level, the transformation removes the term $xy^j\phi_{pj}(u)$. Those terms only involving $y$ and $u$ which are introduced are

$$\frac{1}{4}y^{2j}\phi_{pj}^2 - \frac{1}{2}y^j\phi_{pj}\sum_{p=1}^{m}\sum_{j=0}^{\infty} y^j\phi_{pj}(u),$$

which are of degree higher than $p$ in $u$. So applying the transformation (7.6) successively gives the desired result. □

From the algorithm of Lemma 7.2.6, we see that we can ignore terms of

$$x\sum_{i=2}^{\infty} a_i y^i \quad \text{and} \quad x\sum_{p=1}^{m}\sum_{j=0}^{\infty} y^j \phi_{pj}(u)$$

higher than a certain degree.

**Corollary 7.2.7** We can change the terms of

$$x \sum_{i=3m+1}^{\infty} a_i y^i$$

and the terms of

$$x \sum_{p=1}^{m} \sum_{j=N_p}^{\infty} y^j \phi_{pj}(u),$$

where $N_p = 3(m - p + 1)$ , without affecting $b^m$ .

**Proof** When we apply the transformation

$$x = x' - \frac{1}{2} y^j \phi_{pj},$$

we remove the term $x\phi_{pj}(u)$, and introduce the terms

$$\frac{1}{4} y^{2j} \phi_{pj}^2 - \frac{1}{2} y^j \phi_{pj} \sum_{p=1}^{m} \sum_{j=0}^{\infty} y^j \phi_{pj}(u).$$

The result then follows from Corollary 7.2.4 and Lemma 7.2.5. □

Finally we have

**Lemma 7.2.8** (c.f. [CW-II], page 351) A sequence of transformations can be constructed which removes the terms in $R(x, y, u)$, but does not affect $\Phi(y, u)$ in Equation (7.5).

**Proof** The construction is analogous to the one in Lemma 7.2.6. We write

$$R(x, y, u) = x \sum_{p=0}^{m} \sum_{j=0}^{3(m-p+1)-1} y^j \sum_{i=0}^{\infty} x^i \phi_{pji}(u),$$

where $\phi_{pji}$ is homogeneous and of degree $p$ in $u$. Then for each $p$, each $j$ from 0 to $3(m - p + 1) - 1$, and each $i$ from 0 upwards, the transformation

$$x = x' - \frac{1}{2k}(x^i y^j \phi_{pji}(u))$$

removes the term $x^{i+1} y^j \phi_{pji}(u))$ in $R(x, y, u)$. It can easily be checked that each step removes unwanted terms without reintroducing terms already removed by previous transformations. □

These results can be implemented in a series of Maple routines. These are listed in Appendix B. Note that the final stage of the algorithm (Lemma 7.2.8) reduces the unfolding (7.5) to the form

$$\Phi(y, u) + x^2,$$

without altering $\Phi(y, u)$. In our application, only the form of $\Phi(y, u)$ is of interest. So when $\Phi(y, u)$ has been determined using the algorithm of Lemma 7.2.3, the terms in $R(x, y, u)$ may be discarded. Thus the algorithm can be implemented to reduce a general unfolding of an $A_3$ singularity to one of the form

$$\Phi(y, u) = x^2 + \phi(y) + R(y, u),$$

where $\phi(y)$ is $\mathcal{R}$-equivalent to the normal form $f(y) = y^4$, and $R(y, u) \in \mathcal{M}(u) \cdot \mathcal{E}(y, u)$. We can use the Maple library routine reversion to calculate the relevant terms of $\psi$ up to the degree specified by Lemma 7.2.5. The procedure ourtruncm truncates the unfolding at the degrees specified by Corollary 7.2.4. Then the procedure remlinm reduces the unfolding to the form

$$\Phi(y, u) + kx^2 + R(x, y, u)$$

using the algorithm of Lemma 7.2.6. As we have noted, we can now deal simply with the unfolding $\Phi(y, u) = \phi(y) + x^2 + R(y, u)$, where $\phi(y)$ is an $A_3$ singularity and $R(y, u) \in \mathcal{M}(u) \cdot \mathcal{E}(y, u)$. The procedure revm calculates and applies the diffeomorphisms (up to the degree specified by Lemma 7.2.5) needed to take $\phi(y)$ to the normal form for an $A_3$ singularity, $f(y) = y^4$. Finally the procedure changem applies the algorithm of Lemma 7.2.1.

## 7.3 The Focal Set of the Crosscap

Our next step is to apply the algorithm of Section 7.2 to the problem of finding a local model for the focal set of the crosscap near the origin. We proceed in the following way. Let $F(x, y, u_1, u_2, u_3)$ be the family of distance squared functions for the crosscap, given by

$$\begin{aligned} F(x, y, u_1, u_2, u_3) &= (x - u_1)^2 \\ &\quad + (xy + p(y) - u_2)^2 + (y^2 + ax^2 + bxy + q(x, y) - u_3)^2, \end{aligned}$$

and let

$$G(x, v_1, v_2) = x^2 + y^4 + v_1 y + v_2 y^2,$$

be the standard versal unfolding of an $A_3$ singularity. Let the smooth mappings $a : \mathbf{R}^2 \times \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^2, 0$ and $b : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^2, 0$ be such that

$$F(a(x, y, u_1, u_2, u_3), u_1, u_2, u_3) = G(x, y, b(u_1, u_2, u_3)).$$

Then the focal set of the crosscap is the bifurcation set of $F$, and is thus the pullback of the bifurcation set of $G$ (the cusp with equation $27v_1^2 + 16v_2^3 = 0$) by the mapping $b$. Section 7.2 describes a method of calculating the Taylor Series of $b$ accurate to degree $k$. How large must $k$ be for our application? It turns out that we only need the 2-jet of $b$. In the next two results we calculate this.

**Proposition 7.3.1** Let $F$, $G$, $a$ and $b$ be as above. Then we have

$$j^1(b(u_1, u_2, u_3)) = (0, -2u_3).$$

**Proof** As we have noted, we can discard the terms of $F$ not involving $u$. Then we truncate $F$ at degree 1 in $u_1$, $u_2$ and $u_3$. Consider the terms of $F$ involving only $y$, $u_1$, $u_2$ and $u_3$. Then by Corollary 7.2.4, we need these terms to degree 5 in $y$ only and the terms degree 1 in $u_1$, $u_2$ and $u_3$ to degree 2 in $y$.

Now consider the terms that are linear in $x$. By Corollary 7.2.7, we need terms that are linear in $x$ that are degree 0 in $u_1$, $u_2$ and $u_3$ to degree 2 in $y$, and those that are degree 1 in $u_1, u_2$ and $u_3$ to degree 3 in $y$. Then we can discard all terms of degree greater than 2 in $x$ except for $x^2$ by Lemma 7.2.8.

This leaves us with

$$x^2 + y^4 + 2q_3y^5 - 2u_3y^2 + 2x(by^3 - u_1 - u_2y - bu_3y - u_3q_{31}y^2),$$

where $q_3$ and $q_{31}$ are the coefficients of $y^3$ and $xy^2$ in $q(x, y)$ respectively. The substitution prescribed by Lemma 7.2.6 replaces $x$ with

$$x - (by^3 - u_1 - u_2y - bu_3y - u_3q_{31}y^2).$$

This rids us of the terms linear in $x$, and introduces the terms

$$-(by^3 - u_1 - u_2y - bu_3y - u_3q_{31}y^2)^2.$$

All of these terms are higher degree in $y$ and $u_1$, $u_2$, and $u_3$ than we need, so we can discard them, leaving

$$x^2 + y^4 + 2q_3y^5 - 2u_3y^2.$$

The result now follows easily by considering the algorithm that we have described above. □

Next we calculate the 2-jet of $b$. This calculation requires some brute force. Fortunately, for the distance squared function on the crosscap, we can lower the degrees in $y$ and $u_1, u_2$ and $u_3$ that we need to look at the terms of our general unfolding to. The crucial fact is that there are no terms in our unfolding of the form $u_1y$, $u_2y$ or $u_3y$.

**Proposition 7.3.2** Let $F$, $G$, $a$ and $b$ be as in Proposition 7.3.1. Then

$$\begin{aligned} j^2(b(u_1,u_2,u_3) = (-2(u_1u_2+p_3u_2u_3) \quad , \quad & -2u_3 - \frac{3}{2}b^2u_1^2 - (\frac{3}{2}p_3^2+1)u_2^2 + p_3^2u_3^2 \\ & +(3p_3b+q_3)u_1u_2 + 2p_3u_1u_3 \\ & +(3p_3q_3 - 2p_4)u_2u_3), \end{aligned}$$

where $p(y) = p_3y^3 + p_4y^4 + \cdots$ and $q(x,y) = q_3y^3 + \cdots$ in the parametrisation of the crosscap.

**Proof** Once we have truncated at the relevant degrees, our unfolding is of the form

$$x^2 + j^8(\phi(y)) - 2u_3y^2 + y^3\sum_{i=1}^{3} u_ij^2(\gamma_i(y)) + \sum_{\nu_2} u^{\nu_2}j^2(\gamma^{\nu_2}(y)) + x\sum_{j=2}^{5} a_jy^j$$
$$+x\sum_{p=1}^{2}\sum_{j=0}^{8-3p} y^j\phi_{pj}(u), \quad (7.7)$$

where $\phi$ is an $A_3$ singularity, and the functions $\gamma_i$ and $\gamma^{\nu_2}$ are in $\mathcal{E}(y)$. We show that for this situation, we need only the 6-jet of $\phi$, the 1-jet of the functions $\gamma_i$, and linear terms in $x$. To see this, we first suppose that we have removed all linear terms in $x$, and applied the coordinate change $y \mapsto \psi(y)$, where $\phi \circ \psi(y) = y^4$. It follows from proposition 7.3.2 that the unfolding is then of the form

$$x^2 + y^4 - 2u_3y^2 + y^3\sum_{i=1}^{3} u_ij^2(\beta_i(y)) + \sum_{\nu_2} u^{\nu_2}j^2(\beta^{\nu_2}(y)),$$

where $j^2(\beta)$ is the 2-jet of the function $\beta$, and $\beta_i, \beta^{\nu_2} \in \mathcal{E}(y)$. Applying the algorithm of Proposition 7.2.1, we set

$$b^1(u_1,u_2,u_3) = (0, -2u_3)$$

and

$$a^1(x,y,u_1,u_2,u_3) = (x, y - \frac{1}{4}\sum_{i=1}^{3} u_ij^2(\beta_i(y)).$$

We write $A = -\frac{1}{4}\sum_{i=1}^{3} u_ij^2(\beta_i(y))$, and $y' = y + A$. Then we have

$$\begin{aligned} F(a^1,u_1,u_2,u_3) &= x^2 + y'^4 - 2u_3y'^2 + y'^3\sum_{i=1}^{3} u_ij^2(\beta_i(y')) \\ & \quad + \sum_{\nu_2} u^{\nu_2}j^2(\beta^{\nu_2}(y')) + \cdots. \end{aligned}$$

We must see how this substitution affects $b^2$. The terms of

$$y'^4 - 2u_3y'^2 + y'^3 \sum_{i=1}^{3} u_i j^2(\beta_i(y'))$$

that contribute to $b^2$ are those of degree 2 in $u_1$, $u_2$, $u_3$ and degree up to 2 in $y$. We see that $y'^4$ contributes

$$\frac{6}{16}y^2 \left( \sum_{i=1}^{3} u_i j^0(\beta_i(y)) \right)^2,$$

the term $-2u_3y'^2$ contributes

$$u_3 y \sum_{i=1}^{3} u_i j^1(\beta_i(y)),$$

and $y'^3 \sum_{i=1}^{3} u_i j^2(\beta_i(y'))$ contributes

$$-\frac{3}{4}y^2 \left( \sum_{i=1}^{3} u_i j^0(\beta_i(y)) \right)^2.$$

It follows that we need only know the 1-jets of the functions $\beta_i$. Thus we need only know the 1-jets of the functions $\gamma_i$ in the unfolding (7.7).

Now we determine the degree that we need $\phi(y)$ correct to. Suppose that we have removed all the terms that are linear in $x$, so that our unfolding is of the form

$$x^2 + \phi(y) - 2u_3y^2 + y^3 \sum_{i=1}^{3} u_i j^1(\beta_i(y)) + \sum_{\nu_2} u^{\nu_2} j^2(\beta^{\nu_2}(y)).$$

Now let the function $\psi : \mathbf{R}, 0 \longrightarrow \mathbf{R}, 0$ be such that $\phi \circ \psi(y) = y^4$. Substituting $\psi(y)$ for $y$, we get

$$x^2 + y^4 - 2u_3\psi(y)^2 + \psi(y)^3 \sum_{i=1}^{3} u_i j^1(\beta_i(\psi(y))) + \sum_{\nu_2} u^{\nu_2} j^2(\beta^{\nu_2}(\psi(y))).$$

It is clear from considering this that we only need the 3-jet of $\psi$. From similar considerations to those of Lemma 7.2.5, we see that we must determine the 6-jet of $\phi$.

So far we have shown that we can truncate $F$ so that it is of the form

$$x^2 + j^6(\phi(y)) - 2u_3y^2 + y^3 \sum_{i=1}^{3} u_i j^1(\gamma_i(y)) + \sum_{\nu_2} u^{\nu_2} j^2(\gamma^{\nu_2}(y)) + x \sum_{j=2}^{5} a_j y^j$$
$$+ x \sum_{p=1}^{2} \sum_{j=0}^{8-3p} y^j \phi_{pj}(u).$$

Finally by considering the transformation (Lemma 7.2.6) that removes the terms that are linear in $x$, we see that we need only consider the following:

$$x\sum_{j=2}^{4} a_j y^j + x\sum_{j=0}^{4} y^j\phi_{1j}(u) + x\sum_{j=0}^{2} y^j\phi_{2j}(u).$$

Now we prove the Proposition, by applying the algorithm to our truncated unfolding, which is

$$\begin{gathered}x^2 + y^4 + 2q_3y^5 + (p_3^2 + q_3^2 + 2q_4)y^6 - 2u_3y^2 \\ +y^3(-2(u_2p_3 + u_3q_3) - 2(u_2p_4 + u_3q_4)y) + 2x(by^3 + (p_3 + bq_3 + q_{31})y^4 \\ -u_1 - (u_2 + bu_3)y - u_3(q_{31}y^2 + q_{41}y^3 + q_{51}y^4)),\end{gathered}$$

where $p(y) = p_3y^3 + p_4y^4 + \cdots$ and $q(x,y) = q_3y^3 + q_{31}xy^2 + q_4y^4 + q_{41}xy^3 + q_5y^5 + q_{51}xy^4 + \cdots$ in the parametrisation of the crosscap. Our first step is to remove the terms linear in $x$, by making the substitution

$$x' = x - (by^3 + (p_3 + bq_3 + q_{31})y^4 - u_1 - (u_2 + bu_3)y - u_3(q_{31}y^2 + q_{41}y^3 + q_{51}y^4)).$$

After making this transformation and truncating at the required levels, our unfolding is

$$\begin{aligned}x^2 + y^4 + 2q_3y^5 + (p_3^2 + q_3^2 + 2q_4 - b^2)y^6 - 2u_3y^2 + y^3(-2(u_2p_3 + u_3q_3 - bu_1)) \\ -2y^4(u_2p_4 + u_3q_4 - b(u_2 + bu_3) - (p_3 + bq_3 + q_31)u_1) - 2u_1(u_2 + bu_3)y \\ -(2q_{31}u_1u_3 + (u_2 + bu_3)^2)y^2.\end{aligned}$$

Next we must determine the substitution we have to make for $y$ so that this unfolds a germ with normal form $x^2 + y^4$. Writing

$$y' = \psi(y) = y + b_2y^2 + b_3y^3 + \cdots,$$

then if

$$y^4 = y'^4 + a_1y'^5 + a_2y'^6 + \cdots,$$

considering the power series $y'(1 + a_1y' + a_2y'^2 + \cdots)^{1/4}$, we see that

$$b_2 = -\frac{a_1}{4} \quad \text{and} \quad b_3 = \frac{7a_1^2}{32} - a_24.$$

In our case,

$$a_1 = 2q_3, \quad \text{and} \quad a_2 = (p_3^2 + q_3^2 + 2q_4 - b^2),$$

so we make the substitution

$$y' = \psi(y) = y - \frac{1}{2}q_3y^2 + (-\frac{1}{4}(p_3^2 + 2q_4 - b^2) + \frac{5}{8}q_3^2)y^3).$$

After truncating, we get

$$\begin{aligned} & x^2 + y^4 - 2u_3y^2 + y^3(-2p_3u_2 + 2bu_1) \\ & + y^4(u_1(2p_3 + 2q_{31} - bq_3) + u_2(-2p_4 + 2b + 3q_3p_3) + u_3(b^2 + p_3^2)) \\ & -2yu_1(u_2 + bu_3) - y^2(2u_1u_3q_{31} + (u_2 + bu_3)^2 - q_3u_1(u_2 + bu_3)). \end{aligned}$$

We write the coefficients of $y$, $y^2$, $y^3$ and $y^4$ that are homogeneous and degree 2 in $u$ in the above equation as $A$, $B$, $C$ and $D$ respectively. Then we set

$$a^1(x, y, u_1, u_2, u_3) = (x, y - \frac{1}{4}(C + yD)).$$

The considerations at the beginning of this proof show that

$$b^2(u_1, u_2, u_3) = (A + u_3C, -2u_3 + B - \frac{6}{16}C^2 + u_3D).$$

Substituting

$$\begin{aligned} A &= -2u_1(u_2 + bu_3) \\ B &= -(2u_1u_3q_{31} + (u_2 + bu_3)^2 - q_3u_1(u_2 + bu_3)) \\ C &= -2p_3u_2 + 2bu_1 \quad \text{and} \\ D &= u_1(2p_3 + 2q_{31} - bq_3) + u_2(-2p_4 + 2b + 3q_3p_3) + u_3(b^2 + p_3^2)), \end{aligned}$$

we find that

$$\begin{aligned} j^2(b(u_1, u_2, u_3)) = (-2(u_1u_2 + p_3u_2u_3) \quad , \quad & -2u_3 - \frac{3}{2}b^2u_1^2 - (\frac{3}{2}p_3^2 + 1)u_2^2 + p_3^2u_3^2 \\ & +(3p_3b + q_3)u_1u_2 + 2p_3u_1u_3 \\ & +(3p_3q_3 - 2p_4)u_2u_3), \end{aligned}$$

as required. □

Now that we have the 2-jet of $b$, we make the following definition.

**Definition 7.3.3** A map germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^p, 0$ (where $p \leq n$) is called a *submersion with folds* if it is $\mathcal{A}$-equivalent to a map germ of the form

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{p-1}, x_p^2 \pm \cdots \pm x_n^2).$$

For a discussion of submersions with folds and their stability, see [GG], pages 87-88.

**Corollary 7.3.4 (to Proposition 7.3.1).** Let $F$, $G$, $a$ and $b$ be as in Proposition 7.3.1. Then the mapping $b$ is a submersion with folds.

As we have noted before, the bifurcation set of $G$, denoted $B(G)$, is a cusp with equation $27v_1^2 + 16v_2^3 = 0$. The critical values of $b$ form a smooth curve tangent to the cuspidal tangent of $B(G)$ at the origin.

**Proof** The first part of this is clear from considering the definition of a submersion with folds and the 2-jet of $b$. Notice that in fact we only need the 1-jet of the second component of $b$, and the 2-jet of the first component of $b$.

The cuspidal tangent to $B(G)$ is the $v_2$-axis. It is easy to see that the set of critical values of $b$ is a smooth curve tangent to the $v_2$-axis. □

The following result will be needed in determining the focal set of the crosscap. It is a special case of a result in [duPW].

**Proposition 7.3.5** If $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^p, 0$ is a submersion with folds whose set of critical values is given by $y_p = 0$, then $f$ is $\mathcal{R}$-equivalent to the germ

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{p-1}, \pm x_p^2 \pm \cdots \pm x_n^2).$$

**Proof** Since we have a submersion with folds the critical set is smooth and of dimension $p-1$. By a change of coordinates in the source, we may suppose that the critical set is given by $x_p = \cdots = x_n = 0$, and that the first $p-1$ components of $f$ are $x_1, \cdots, x_{p-1}$ respectively. It follows that the composition of the projection

$$P : (y_1, \cdots, y_p) \rightarrow y_p$$

with $f$ has a Morse singularity. So we can make a change of coordinates in $(x_p, \cdots, x_n)$ space so that

$$P \circ f(0, \cdots, 0, x_p, \cdots, x_n) = \pm x_p^2 \pm \cdots \pm x_n^2.$$

Now we think of the function $P \circ f$ as an unfolding of

$$(P \circ f)_0 = P \circ f(0, \cdots, 0, x_p, \cdots, x_n)$$

with unfolding parameters $x_1, \cdots, x_{p-1}$. A versal unfolding of $(P \circ f)_0$ is given by $\pm x_p^2 \pm \cdots \pm x_n^2 + k$, where $k$ is a constant. So by the theorem on versal unfoldings (Theorem 1.5.6), we can make a coordinate change so that

$$P \circ f(x_1, \cdots, x_{p-1}, x_p, \cdots, x_n) = \pm x_p^2 \pm \cdots \pm x_n^2 + g(x_1, \cdots, x_{p-1}).$$

Note that this coordinate change does not affect the first $p-1$ components of $f$, so that we have reduced $f$ to the form

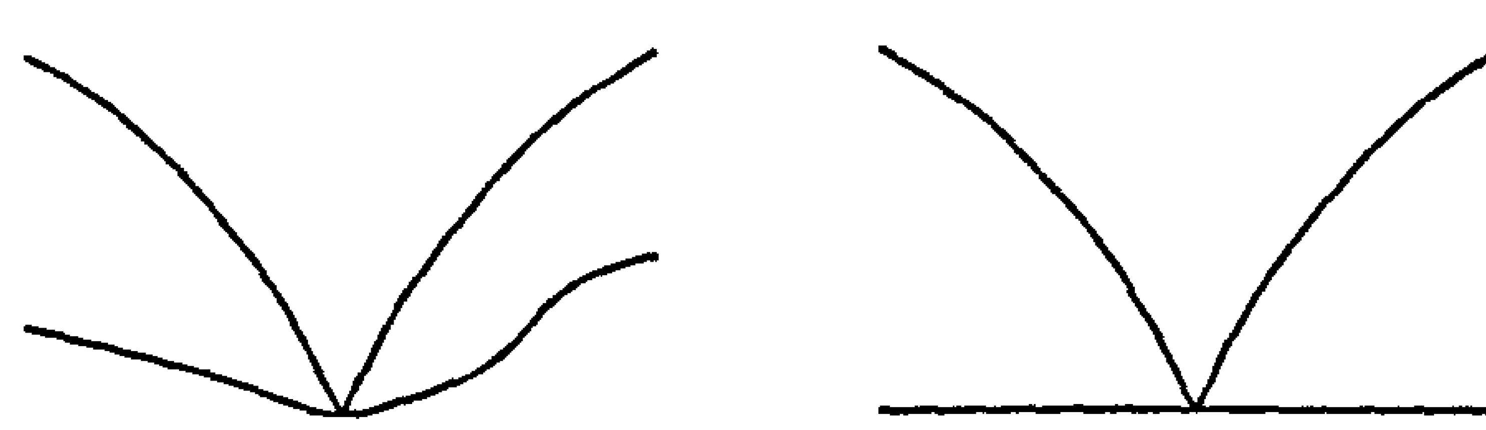
$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{p-1}, \pm x_p^2 \pm \cdots \pm x_n^2 + g(x_1, \cdots, x_{p-1})).$$

Finally, if the set of critical values of $f$ is given by $y_p = 0$, then it follows that the function $g$ is identically zero. □

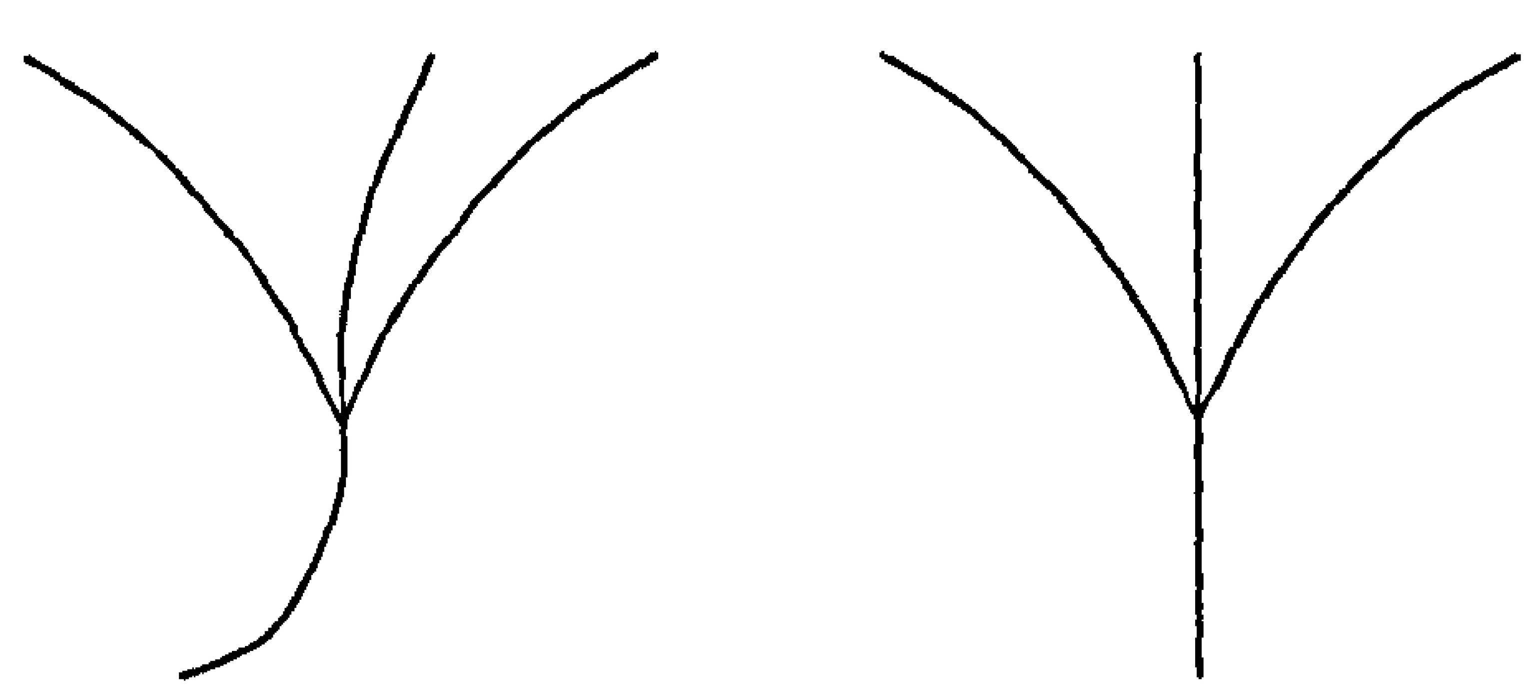
To apply Proposition 7.3.5, we need to be able to apply a diffeomorphism to $\mathbf{R}^2$, which preserves the bifurcation set of $G$, and 'straightens out' the curve of critical values of $b$. In [A2], Arnol'd lists the normal forms of orbits of functions on $\mathbf{R}^2$ which vanish at the origin under the action of the group of plane diffeomorphisms preserving the cusp with cuspidal tangent the $v_2$ axis. The two such normal forms with lowest codimension are

$$\begin{aligned} f(v_1, v_2) &= \pm v_1 \\ f(v_1, v_2) &= \pm v_2, \end{aligned}$$

with codimension 0 and 1 respectively. Intuitively, this means that if we have a smooth curve which passes through the origin and is transverse to the cuspidal tangent then there is a diffeomorphism on $\mathbf{R}^2$ leaving the cusp invariant which 'straightens out' the curve to the line orthogonal to the cuspidal tangent,



and that a smooth curve which passes through the origin and is tangent to the cuspidal tangent there can similarly be 'straightened out' to the cuspidal tangent whilst leaving the cusp invariant.

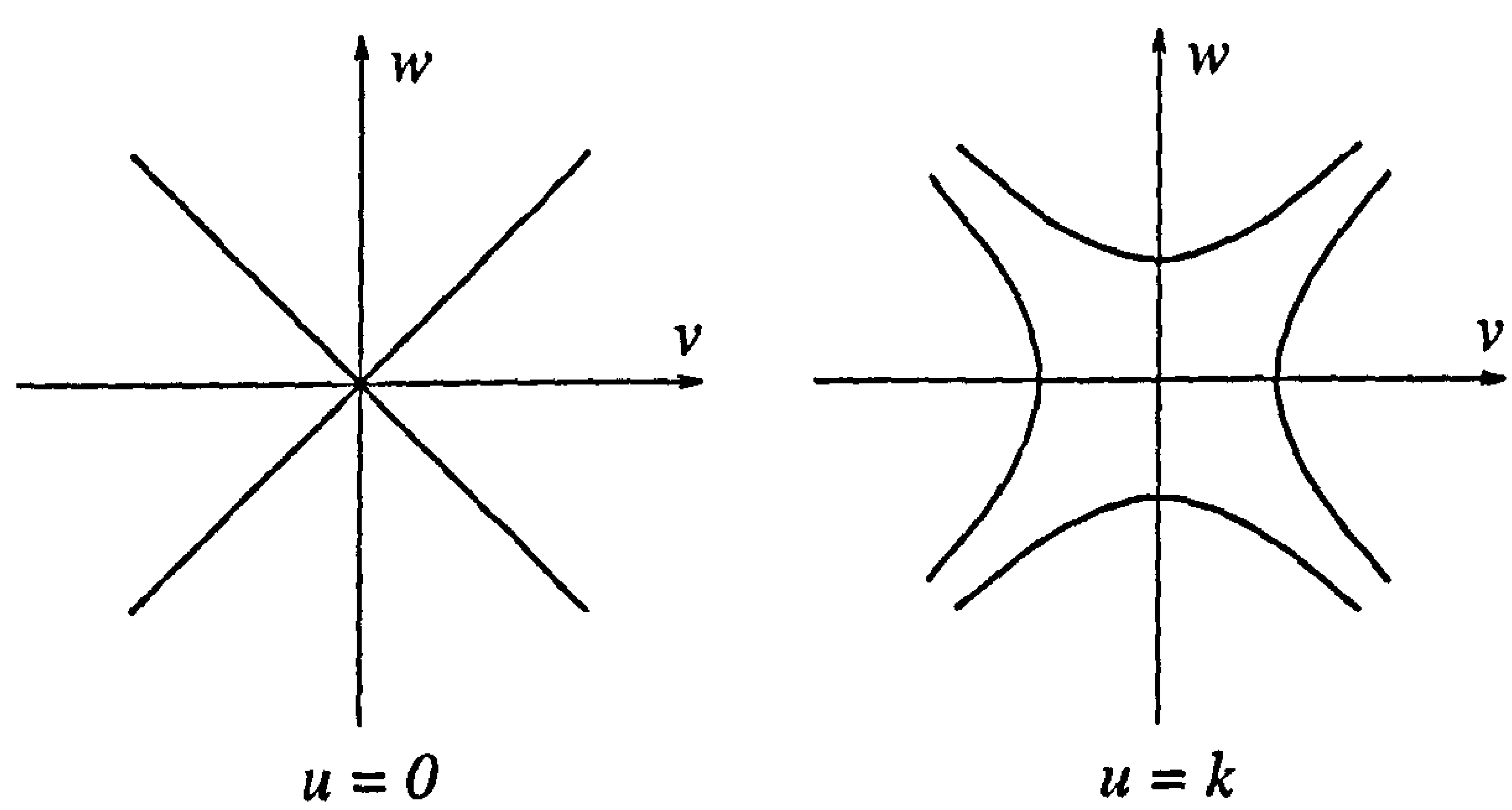


We can write the critical values of $b$ as the set with the equation $h(v_1, v_2) = 0$, where $h : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$, as the set is a smooth curve. Then Arnol'd's results show that $h$ is in the second of these two orbits. Thus we can choose a diffeomorphism $\phi : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ which leaves the cusp invariant with the property that the critical values of $\phi \circ b$ are the $v_2$ axis; the cuspidal tangent to $B(G)$. We are then able to apply 7.3.5 to get

**Theorem 7.3.6** The focal set of the crosscap near to the origin is diffeomorphic to the pullback of the bifurcation set of the standard $A_3$ singularity (the cusp with equation $x^3 - y^2 = 0$) pulled back by the submersion with folds $(u, v, w) \mapsto (u, v^2 - w^2)$. So the focal set near to the origin is locally diffeomorphic to the hypersurface with equation $u^3 - (v^2 - w^2)^2 = 0$.

**Proof** This follows from 7.3.5 and the discussion above. □

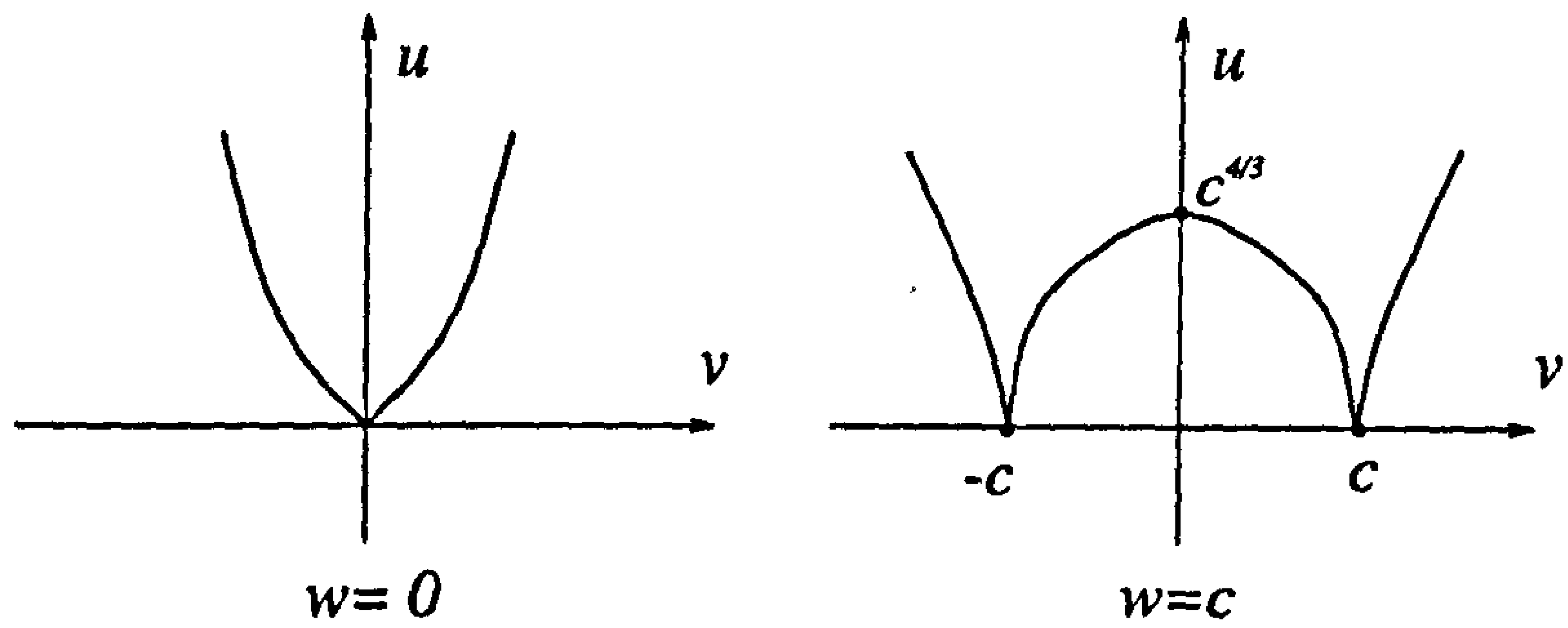
Now we describe the surface with equation $u^3 - (v^2 - w^2)^2 = 0$. This variety is singular when $u = 0$ and $v = \pm w$. The intersection of the surface with the plane defined by $u = k$ (where $k$ is a constant) is empty when $k < 0$, the two lines given by $(v^2 - w^2)^2 = 0$ when $k = 0$ and the two hyperbolae given by $(v^2 - w^2)^2 = k^{3/2}$ when $k > 0$. These sets are shown below.




Intersections of the surface $u^2 - (v^2 - w^2)^2 = 0$
with the planes $u = 0$ and $u = k$ $(k > 0)$.

The intersection of the surface with the plane defined by $w = c$ is the curve defined by $u^3 = (v^2 - c^2)^2$. This has two cusps at $(u, v, w) = (0, \pm c, c)$. When $v = 0$, we have $u = c^{4/3}$. The cuspidal tangents are vertical. As $c$ tends to zero, the cusps approach each other, and when $c = 0$, the section is given by $u^3 = v^4$.

These cross-sections are depicted below.




Intersections of the surface $u^2 - (v^2 - w^2)^2 = 0$
with the planes $w = 0$ and $w = c$.

Figure 7.1 shows the surface viewed from different angles. The pictures on the right hand side were obtained using the Liverpool Surfaces package. We can deduce from this model of the focal surface that the crosscap has two transverse curves of ridge points passing through it.

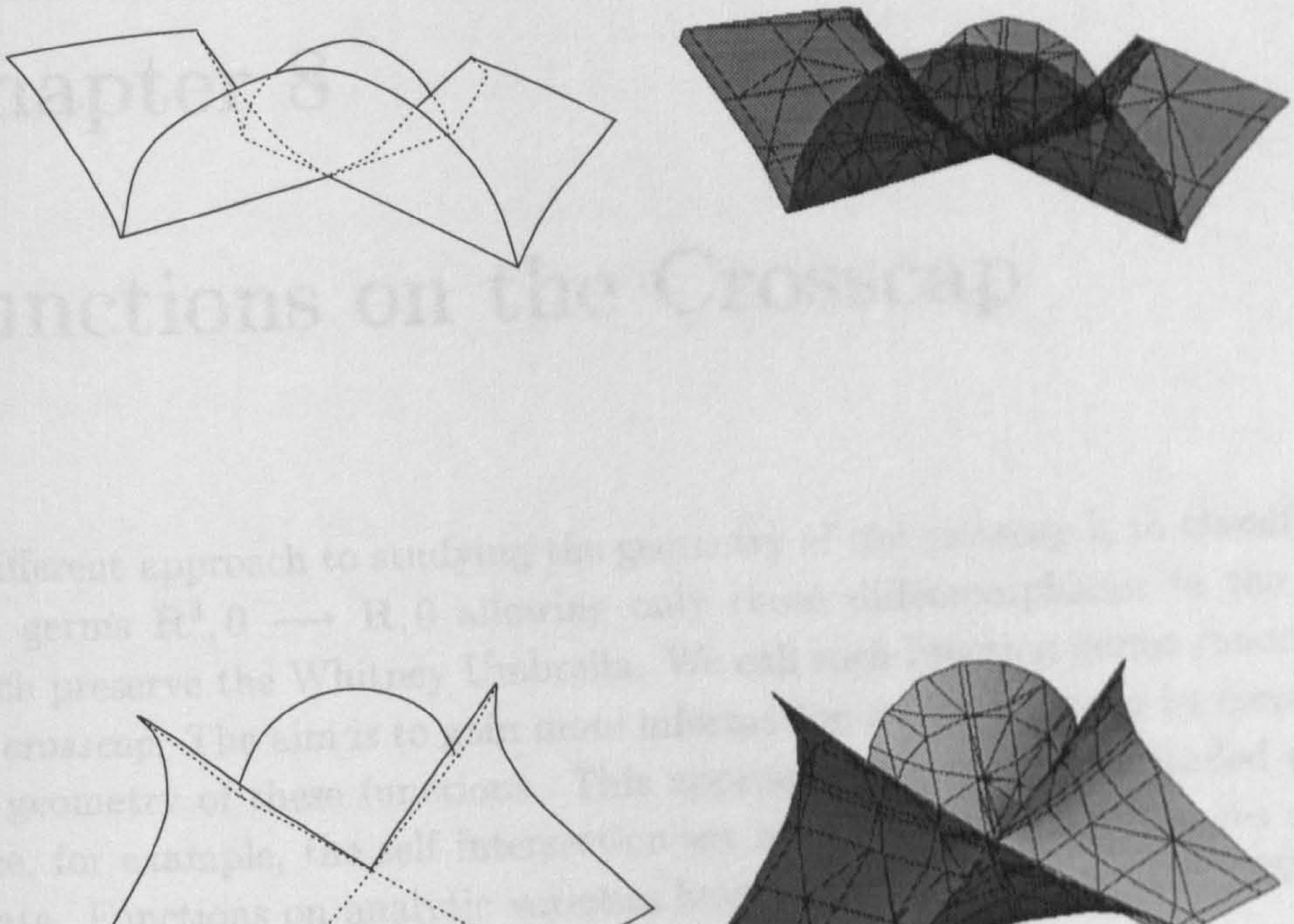


Figure 7.1: The focal surface of the crosscap close to the origin.

# Chapter 8

# Functions on the Crosscap

A different approach to studying the geometry of the crosscap is to classify function germs $\mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ allowing only those diffeomorphisms in the source which preserve the Whitney Umbrella. We call such function germs ***functions on the crosscap***. The aim is to gain more information on the crosscap by considering the geometry of these functions. This approach yields a more detailed picture, since, for example, the self intersection set is preserved by the changes of coordinate. Functions on analytic varieties have been considered in [BR]. We begin with some technical results.

## 8.1 Vector Fields on Varieties

In what follows $\mathbf{K}$ is either $\mathbf{C}$ or $\mathbf{R}$. We denote the algebra of germs of smooth functions on $\mathbf{K}^p$ at 0 by $\mathcal{O}_p$.

Let $X, 0 \subset \mathbf{K}^p, 0$ be the germ of a reduced analytic subvariety of $\mathbf{K}^p$ at 0. We consider analytic function germs $\mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ and say two germs are equivalent if one can be obtained from the other by source coordinate changes which preserve $X$. As usual the theory works well when $\mathbf{K} = \mathbf{C}$.

**Definition 8.1.1** Let $\mathcal{I}$ denote the ideal in $\mathcal{O}_p$ of germs of functions that vanish on $X$. A diffeomorphism $\phi : \mathbf{K}^p, 0 \longrightarrow \mathbf{K}^p, 0$ *preserves* X if $\phi(X)$ and $X$ are equal as germs at 0, i.e. $\phi(X), 0 = X, 0$. Over $\mathbf{C}$ this is equivalent to the assertion that the induced isomorphism $\phi^* : \mathcal{O}_p \longrightarrow \mathcal{O}_p$ satisfies $\phi^*(\mathcal{I}) = \mathcal{I}$. The group of such diffeomorphisms is a subgroup of $\mathcal{R}$, and is denoted $\mathcal{R}(X)$.

Two function germs $f, g \in \mathcal{O}_p$ are said to be $\mathcal{R}(X)$-equivalent if there exists $\phi \in \mathcal{R}(X)$ such that $g \circ \phi = f$.

Let $\delta$ be the germ of an analytic vector field on $\mathbf{K}^p$ at 0. Then $\delta$ is said to be *logarithmic* for $(X, 0)$ if, when considered as a derivation $\delta : \mathcal{O}_p \longrightarrow \mathcal{O}_p$, $f \mapsto \delta \cdot f$, we have $\delta \cdot f \in \mathcal{I}$ for all $f \in \mathcal{I}$, that is, $\delta \cdot f$ vanishes on $X$. The $\mathcal{O}_p$-module of logarithmic vector fields is denoted $\Theta(X)$.

Intuitively, an infinitesimal approach is given by integrating vector fields tangent to $X$ to yield diffeomorphisms which preserve $X$. This is confirmed by the following proposition, which shows that the logarithmic vector fields are precisely those vector fields tangent to $X$.

**Proposition 8.1.2** When $\mathbf{K} = \mathbf{C}$, the germ at 0 of a vector field lies in $\Theta(X)$ if and only if at each smooth point $x$ (sufficiently close to 0) of each irreducible component $X_i$ of $X$ the vector field $\delta$ is tangent to $X_i$ at $x$.

Now if $\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$ suppose that $\delta \in \Theta(X)$ vanishes at 0. Then the flow $\phi_t$ generated by $\delta$ preserves $X, 0$. Thus $\phi_t \in \mathcal{R}(X)$ for all $t$.

**Proof** See [BR], section 1. □

The basic tools for classifying analytic function germs $\mathbf{K}^p, 0 \longrightarrow \mathbf{K}$ up to $\mathcal{R}(X)$-equivalence are generalisations of the standard results concerning of functions under $\mathcal{R}$-equivalence, for which $X = \emptyset$. The group $\mathcal{R}(X)$ is one of Damon's 'geometric subgroups' of $\mathcal{A}$ in [D], so his version of the unfolding and determinacy theorems apply.

**Definition 8.1.3** A function germ $f : \mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ is $k$-$\mathcal{R}(X)$-determined if all function germs with the same $k$-jet as $f$ are $\mathcal{R}(X)$-equivalent to $f$.

**Theorem 8.1.4** A function germ $f : \mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ is finitely $\mathcal{R}(X)$-determined if the ideal

$$J_X(f) = \{\delta \cdot f : \delta \in \Theta(X)\}$$

in $\mathcal{O}_p$ contains some power of the maximal ideal $\mathcal{M}_p$. Moreover if $J_X(f) \supset \mathcal{M}_p^k$ then $f$ is $(k+1) - \mathcal{R}(X)$-determined.

**Proof** See [D]. □

We prove a stronger version of Theorem 8.1.4 in the next section.

The definition and fundamental theorem of $\mathcal{R}(X)$-versality for unfoldings are analogous to those of $\mathcal{R}$-versality; this time the diffeomorphisms must preserve the variety $X$.

**Definition 8.1.5** Let $f \in \mathcal{O}_n$ be a germ of a function. A *p-parameter unfolding* of $f$ is a germ $F$ of a smooth function at the origin of $\mathbf{K}^n \times \mathbf{K}^p$ (given coordinates $(x,t)$) such that

$$F_0(x) = F(x,0) = f(x).$$

We define the *extended pseudo-group of diffeomorphisms preserving* $X$, denoted $\mathcal{R}_e(X)$ to be the pseudo-group obtained by integrating all vector fields $\delta \in \Theta(X)$ as in proposition 8.1.2, but dropping the condition that $\delta(0) = 0$. Note that when $X$ is the Whitney Umbrella, all the vector fields do vanish at the origin, so that in this case $\mathcal{R}_e(X) = \mathcal{R}(X)$.

Two $p$-parameter unfoldings, $F$ and $G$ of the same germ $f \in \mathcal{O}_n$ are *isomorphic* if there exists a local diffeomorphism $\phi : \mathbf{K}^n \times \mathbf{K}^p, 0 \to \mathbf{K}^n \times \mathbf{K}^p, 0$ at the origin such that

1. $\phi(x,t) = (\psi(x,t),t)$ and $\psi(x,0) = x$, i.e. $\phi$ is a $p$-parameter unfolding of the identity map on $\mathbf{K}^n$, this time with the property that the mapping $\psi_t : \mathbf{K}^n \to \mathbf{K}^n$ defined by $\psi_t(x) = \psi(x,t)$ is in $\mathcal{R}_e(X)$ for $t$ near 0 in $\mathbf{K}^p$;
2. $G = F \circ \phi$.

An unfolding is called *trivial* if it is isomorphic to the constant unfolding $(x,t) \mapsto f(x)$.

As before we say that two $p$-parameter unfoldings $F$ and $G$ of a function germ $f$ are *equivalent* if there exists a diffeomorphism $h : \mathbf{K}^p, 0 \longrightarrow \mathbf{K}^p, 0$ such that $G$ is isomorphic to $h^*F$, the pullback of $F$ by $h$. If $H$ is now some $q$-parameter unfolding of $f$, we say that $H$ can be *induced* from $F$ if there exists a smooth map germ $h : \mathbf{K}^q, 0 \longrightarrow \mathbf{K}^p, 0$ such that $h$ is isomorphic to $h^*F$.

We say that an unfolding $F$ of $f \in \mathcal{O}_n$ is a *versal unfolding* of $f$ if any other unfolding $G$ of $f$ can be induced from $f$.

**Theorem 8.1.6** A $p$ parameter unfolding $F(x,u)$ of $f$ is $\mathcal{R}(X)$-versal if and only if the initial speeds

$$\frac{\partial F}{\partial u_1}(x,0),\cdots,\frac{\partial F}{\partial u_p}(x,0)$$

span $\mathcal{O}_n/J_X(f)$.

**Proof** See [D]. □

We shall be considering the geometry of functions on the Whitney Umbrella, so we need to use the notion of $\mathcal{K}(X)$-equivalence. The subgroup $\mathcal{K}(X)$ of $\mathcal{K}$ is defined by replacing $\mathcal{R}$ by $\mathcal{R}(X)$ in the definition of $\mathcal{K}$. Thus $\mathcal{K}(X)$ can be thought of as the semi-direct product of $\mathcal{R}(X)$ and $\mathcal{C}$, and $\mathcal{K}(X)$ acts on $\mathcal{M}_n \cdot \mathcal{O}(n,p)$ as a subgroup of $\mathcal{K}$. In the case where $p=1$, the definition of $\mathcal{K}(X)$-equivalence reduces to the following. Two germs $g_1, g_2 : (\mathbf{K}^n, 0) \longrightarrow (\mathbf{K}, 0)$ are $\mathcal{K}(X)$-equivalent if there exists a map germ $\phi \in \mathcal{R}(X)$ and a smooth function germ $\lambda : \mathbf{K}^n, 0 \longrightarrow \mathbf{K}$, with $\lambda(0) \neq 0$ such that

$$g_1(x) = \lambda(x) g_2(\phi_2(x)).$$

If $f$ is a smooth function germ, then the tangent space to the $\mathcal{K}(X)$-orbit of $f$ is given by

$$L\mathcal{K}(X)\cdot f = L\mathcal{C}\cdot f + L\mathcal{R}(X)\cdot f =< f > + J_X(f).$$

We can define $\mathcal{K}(X)$-versal unfoldings in a similar way to $\mathcal{R}(X)$-versal unfoldings. The group $\mathcal{K}(X)$ is another of Damon's 'geometric subgroups' of $\mathcal{K}$, and so we have the following theorem.

**Theorem 8.1.7** A $p$ parameter unfolding $F(x,u)$ of $f$ is $\mathcal{K}(X)$-versal if and only if the initial speeds

$$\frac{\partial F}{\partial u_1}(x,0),\cdots,\frac{\partial F}{\partial u_r}(x,0)$$

span $\mathcal{O}_n/L\mathcal{K}(X)\cdot f$.

**Proof** See [D]. □

## 8.2 $\mathcal{R}(X)$-triviality and Complete Transversals

To take advantage of the complete transversal methods, we need to imitate the discussion given in Chapter 11 of [BG2].

In what follows we shall be dealing with the germ of an analytic variety $X, 0$ in $\mathbf{R}^n$. We shall suppose that we have a module of vector fields $\Theta(X)$ which are tangent to $X$. By Proposition 8.1.2, when we integrate such fields we obtain a flow which preserves the variety $X$. Note that the module $\Theta(X)$ is not necessarily the collection of all vector fields tangent to $X$. We wish to classify functions $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ up to diffeomorphisms in the source preserving $X$. We shall as usual obtain these diffeomorphisms by integrating vector fields in $\Theta(X)$. For simplicity we suppose that the vector fields all vanish at the origin. This is certainly the case in the application we have in mind, when $X$ is the crosscap.

**Definition 8.2.1** Let $F : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$ be a germ of a smooth 1-parameter family of functions with $F(0,t) = 0$ for small $t$, and let $k$ be an integer $\geq 1$. We say that $F$ is *$\mathcal{R}(X)$-trivial* if we can find a germ of a 1-parameter family of diffeomorphisms $H : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^n, 0$ preserving $X$ with $H(x,0) = x$, $H(0,t) = 0$ (for small $t$) and $F(H(x,t),t) = F(x,0)$.

We say that $F$ is *$k$-$\mathcal{R}(X)$-trivial* if we can find a germ of a 1-parameter family of diffeomorphisms $H : \mathbf{R}^n \times \mathbf{R}, (0,0) \rightarrow \mathbf{R}^n, 0$ preserving $X$ with $H(x,0) = x$, $H(0,t) = 0$ (for small $t$) and $F(H(x,t),t) = F(x,0) + \phi(x,t)$ for some $\phi \in \mathcal{M}_n^{k+1} \subset \mathcal{E}_{n+1}$.

The latter definition says that $F$ is $k$-$\mathcal{R}(X)$-trivial if $F$ is $\mathcal{R}(X)$-trivial up to and including terms of degree $k$. Obviously an $\mathcal{R}(X)$-trivial family is $k$-$\mathcal{R}(X)$-trivial for all $k$. We need criteria for a family to be $\mathcal{R}(X)$-trivial and $k$-$\mathcal{R}(X)$-trivial.

**Proposition 8.2.2** (i) Let

$$F : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$$

be a smooth function with $F(0,t) = 0$ for $t$ small. Let $\xi_1, \ldots, \xi_p$ be vector fields generating $\Theta(X)$. Then the family $F$ is $\mathcal{R}(X)$-trivial if there is a smooth germ

$$\alpha : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^p$$

satisfying

$$\sum_{i=1}^{p} \alpha_i \xi_i F + \frac{\partial F}{\partial t} \equiv 0,$$

or equivalently

$$\frac{\partial F}{\partial t} \in \Theta(X).F \subset \mathcal{E}_{n+1}. \tag{8.1}$$

(ii) The family

$$F : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$$

defined as above is $k$-$\mathcal{R}(X)$-trivial if

$$\frac{\partial F}{\partial t} \in \Theta(X).F + \mathcal{M}_n^{k+1} \subset \mathcal{E}_{n+1}. \tag{8.2}$$

More explicitly, the family

$$F : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$$

is $k$-$\mathcal{R}(X)$-trivial if there is a germ

$$\alpha : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^p$$

satisfying

$$\sum_{i=1}^{p} \alpha_i \xi_i F + \frac{\partial F}{\partial t} \in \mathcal{M}_n^{k+1}.$$

**Proof** (i) Suppose that $\alpha$ exists as in the statement of the Proposition. By the fundamental theorem on the existence of solutions to partial differential equations (see [A4], page 56), if

$$\eta = \sum_{i=1}^{p} \alpha_i \xi_i = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i},$$

then the differential equation

$$\frac{\partial H}{\partial t}(x,t) = \eta(H(x,t),t), \;\; H(x,0) = x \tag{8.3}$$

has a solution defined on some neighbourhood of $(0,0) \in \mathbf{R}^n \times \mathbf{R}$. That is we can find $H : \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}^n, 0$ satisfying (8.3). The vector field $\eta$ is tangent to $X$, so by Proposition 8.1.2, we obtain a diffeomorphism $H_t : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$

preserving $X$ for each small $t$. Now define a new family $G : \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}, 0$ by $G(x,t) = F(H(x,t),t)$. Differentiating with respect to $t$ we obtain

$$\begin{aligned}\frac{\partial G}{\partial t}(x,t) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\frac{\partial H_i}{\partial t}(x,t) + \frac{\partial F}{\partial t}(H(x,t),t) \\ &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\eta_i(H(x,t),t) + \frac{\partial F}{\partial t}(H(x,t),t) \\ &= (\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t})(H(x,t),t) \equiv 0.\end{aligned}$$

Fixing $x$ we see that $G(x,t)$ is constant, i.e. $G(x,t) = G(x,0)$ for all $t$ and $x$. In other words $F(H(x,t),t) = F(H(x,0),0) = F(x,0)$, as required. Note that $\partial H/\partial t(0,t) = \eta(H(0,t),t)$ has the unique solution $H(0,t) \equiv 0$, since the vector fields $\xi_i$ all vanish at the origin, so $H$ does have the required properties.

(ii) Suppose that the germ $\alpha$ exists as in the statement of the Proposition, and $H : \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}^n, 0$ solves the differential equation

$$\frac{\partial H}{\partial t}(x,t) = \eta(H(x,t),t), H(x,0) = x$$

where $\eta = \sum_{i=1}^{p} \alpha_i \xi_i$. As before, since the vector fields $\xi$ are tangent to $X$, we know that $H_t$ is a 1-parameter family of diffeomorphisms preserving $X$. Consider $G(x,t) = F(H(x,t),t)$ and differentiate with respect to $t$. We find that

$$\begin{aligned}\frac{\partial G}{\partial t}(x,t) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\frac{\partial H_i}{\partial t}(x,t) + \frac{\partial F}{\partial t}(H(x,t),t) \\ &= (\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t})(H(x,t),t).\end{aligned}$$

The term

$$\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t}$$

lies in $\mathcal{M}_n^{k+1}$, and hence so does $\partial G/\partial t(x,t)$. In particular we can write $\partial G/\partial t(x,t)$ as a sum $\sum G_I(x,t)x^I$, where $I$ is a multi-index with $|I| = k+1$. So

$$G(x,t) - G(x,0) = \int_0^t \frac{\partial G}{\partial u}(x,u)du = \sum(\int_0^t \frac{\partial G_I}{\partial u}(x,u)du)x^I$$

also lies in $\mathcal{M}_n^{k+1}$, and since $G(x,0) = F(x,0)$ the result follows. □

Now we can use Proposition 8.2.2 to prove a complete transversal result for $\mathcal{R}(X)$-equivalence. Before doing this we make the following definition.

**Definition 8.2.3** If $\Theta(X)$ is the set of vector fields tangent to the variety $X$ then we write $\Theta_1(X)$ for those fields with 1-jet zero. This is a module over the ring $\mathcal{E}_n$.

**Theorem 8.2.4** Let $f : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be a smooth germ (usually a polynomial of degree $k$) and let $\{h_1, \cdots, h_r\}$ be a collection of homogeneous polynomials of degree $k+1$ with the property that

$$\Theta_1(X).f + sp\{h_1, \cdots, h_r\} + \mathcal{M}_n^{k+2} \supset \mathcal{M}_n^{k+1}.$$

Then any germ $g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ with $j^k g(0) = j^k f(0)$ is $\mathcal{R}(X)$-equivalent to a germ of the form

$$f(x) + \sum_{i=1}^{r} u_i h_i(x) + \phi(x)$$

where $\phi(x) \in \mathcal{M}_n^{k+2}$. Moreover this $\mathcal{R}(X)$-equivalence has 1-jet the identity.

**Proof** Given a function germ $g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ with $j^k g(0) = j^k f(0)$ consider the difference $g - f \in \mathcal{M}_n^{k+1}$. We write this as

$$\eta(f) + \sum_{i=1}^{r} u_i h_i + \phi$$

where the vector field $\eta = \sum \eta_i \partial/\partial x_i$ lies in $\Theta_1(X)$, the $u_i$ are real numbers and $\phi$ lies in $\mathcal{M}_n^{k+2}$. Now set

$$F_t(x) = F(x,t) = f(x) + (1-t)(g-f)(x) + t\sum_{i=1}^{r} u_i G_i(x).$$

Note that $F_0 = g$ and $F_1 = f + \sum u_i G_i$ so that it is enough to prove that for each $t_0$ the family $F(x, t+t_0)$ is $k$-$\mathcal{R}(X)$-trivial. To see this first note that

$$\frac{\partial F}{\partial t}(x,t) = (f-g)(x) + \sum u_i G_i, \quad \text{while}$$

$$\eta F = \eta f + \psi(x,t), \quad \text{where} \quad \psi_i \in \mathcal{M}_n^{k+2}.$$

So we have

$$\begin{aligned} \eta F + \frac{\partial F}{\partial t} &= \sum \eta_i \frac{\partial f}{\partial x_i} + (f-g) + \sum u_i G_i + \psi \\ &= -\phi + \psi \in \mathcal{M}_n^{k+2}. \end{aligned}$$

It follows that the family $F(x,t)$ (based at $t_0 = 0$) is $k$-$\mathcal{R}(X)$-trivial, by Proposition 8.2.2.

If we set

$$f' = f + (1 - t_0)(g - f) + t_0 \sum_{i=1}^{r} G_i(x),$$

then it can easily be checked that

$$\Theta_1(X)f' + \mathcal{M}_n^{k+2} = \Theta_1(X)f + \mathcal{M}_n^{k+2},$$

so that each hypothesis of the theorem applies if we replace $f$ by $f'$. Consequently

$$F(x, t + t_0) : \mathbf{R}^n \times \mathbf{R}, 0 \to \mathbf{R}, 0$$

is $k$-$\mathcal{R}(X)$-trivial for each $t_0$, and the result follows by the usual compactness arguments. Since the $\eta$ lies in $\Theta_1(X)$ the diffeomorphisms $H_t$ all have 1-jet the identity. □

**Corollary 8.2.5** If

$$\Theta_1(X) \supset \mathcal{M}_n^{k+1},$$

then $f$ is $k - \mathcal{R}(X)$-determined.

We can weaken the hypothesis of Corollary 8.2.5 by applying Nakayama's Lemma.

**Corollary 8.2.6** (a) If

$$\Theta_1(X) + \mathcal{M}_n^{k+2} \supset \mathcal{M}_n^{k+1},$$

then $f$ is $k - \mathcal{R}(X)$-determined.

(b) In particular, if every vector field in $\Theta(X)$ vanishes at the origin, then if

$$\Theta(X) + \mathcal{M}_n^{k+2} \supset \mathcal{M}_n^{k+1},$$

$f$ is $(k+1) - \mathcal{R}(X)$-determined.

**Proof** (a) We apply Nakayama's Lemma 1.4.2 to Corollary 8.2.5.

(b) Note that if every vector field in $\Theta(X)$ vanishes at the origin then

$$\Theta_1(X) \supset \mathcal{M}_n \cdot \Theta(X).$$

So if we have

$$\Theta(X) + \mathcal{M}_n^{k+2} \supset \mathcal{M}_n^{k+1},$$

then

$$\mathcal{M}_n \cdot \Theta(X) + \mathcal{M}_n^{k+3} \supset \mathcal{M}_n^{k+2},$$

and $f$ is $(k+1) - \mathcal{R}(X)$-determined. □

When $X$ is the Whitney Umbrella, every vector field in $\Theta(X)$ does vanish at the origin, and so we use the determinacy result of Corollary 8.2.6(b) for the classification that follows.

## 8.3 Classifying Functions on the Crosscap

We apply the results of the previous section to the case where $X$ is the Whitney Umbrella; that is the hypersurface with defining equation $v^2 - u^2w = 0$. Recall from Proposition 2.2.3 that $\Theta(X)$ is the $\mathcal{O}_3$ module of germs at the origin of vector fields on $\mathbf{C}^3$ generated by

$$\xi_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

$$\xi_2 = v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w},$$

$$\xi_3 = u^2\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial w} \quad \text{and}$$

$$\xi_4 = v\frac{\partial}{\partial u} + uw\frac{\partial}{\partial v}.$$

To perform the classification, we need to determine at least the 1-jets of the coordinate changes in $\mathcal{R}(X)$. Consider the linear parts of the vector fields generating $\Theta(X)$. These are

$$u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

$$v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w},$$

$$2v\frac{\partial}{\partial w} \quad \text{and} \quad v\frac{\partial}{\partial u}.$$

Integrating these, we get the 1-jets of coordinate changes

$$\begin{aligned}
(u, v, w) &\longmapsto (e^{\lambda}u, e^{\lambda}v, w),\\
(u, v, w) &\longmapsto (u, e^{\mu}v, e^{2\mu}w),\\
(u, v, w) &\longmapsto (u, v, w + \alpha v) \quad \text{and}\\
(u, v, w) &\longmapsto (u + \beta v, v, w).
\end{aligned}$$

**Theorem 8.3.1** Any germ $f : \mathbf{R}^3, X, 0 \to \mathbf{R}, 0$ which is a submersion on the ambient space is $\mathcal{R}(X)$-equivalent to one of the germs in the following three families.

(i) $UW^{\pm}_{k+1}$: $u \pm w^{k+1}$, $k \geq 0$,

(ii) $WU^{\pm}_{k+1}$: $w \pm u^{k+1}$, $k \geq 1$,

(iii) $VW^{\pm}_{k+1}$: $v \pm w^{k+1}$, $k \geq 1$.

Each of these germs is $(k+1) - \mathcal{R}(X)$-determined.

**Proof** We may suppose that we are starting with a 1-jet $au + bv + cw$, with one of $a, b, c$ non-zero. Now if $ac \neq 0$, then using the linear changes of coordinates given above, we may reduce to

$(a)$ $u \pm w$.

If on the other hand $ac = 0$, then using the linear changes given above, we may reduce to one of the following 1-jets

$(b)$ $u$

$(c)$ $w$

$(d)$ $v$.

$(a)$ Suppose that $f(u, v, w) = u \pm w$. Then we have

$$\begin{aligned} \xi_1 f &= u \quad \xi_2 f = \pm 2w \\ \xi_3 f &= 2v \quad \xi_4 f = v \end{aligned}$$

So $J_X(f) \supset \mathcal{M}_3$. Thus $f$ satisfies the criteria of Corollary 8.2.6(b) and is 1-$\mathcal{R}(X)$-determined.

$(b)$ Suppose that $f(u, v, w) = u$. Then we have

$$\begin{aligned} \xi_1 f &= u \quad \xi_2 f = 0 \\ \xi_3 f &= 0 \quad \xi_4 f = v. \end{aligned}$$

Thus if we have a germ with $k$-jet $u$, then a complete transversal is given by $u + \lambda w^{k+1}$. If $\lambda \neq 0$, we can apply the coordinate changes described above to fix $\lambda = \pm 1$. Now let $f(u, v, w) = u \pm w^{k+1}$. Then we have

$$\begin{aligned} \xi_1 f &= u \quad & \xi_2 f &= \pm 2(k+1) w^{k+1} \\ \xi_3 f &= \pm 2(k+1) v w^k \quad & \xi_4 f &= v. \end{aligned}$$

So $J_X(f) \supset \mathcal{M}_3^{k+1}$ and we can see that $f$ is $k+1-\mathcal{R}(X)$-determined.

(c) If $f(u,v,w) = w$, then we have

$$\begin{aligned} \xi_1 f &= 0 & \xi_2 f &= 2w \\ \xi_3 f &= 2v & \xi_4 f &= 0. \end{aligned}$$

Thus if we have a germ with $k$-jet $w$, then a complete transversal is given by $w + \lambda u^{k+1}$. If $\lambda \neq 0$, we can apply the coordinate changes described above to fix $\lambda = \pm 1$. Now let $f(u,v,w) = w \pm u^{k+1}$. Then

$$\begin{aligned} \xi_1 f &= \pm(k+1)u^{k+1} & \xi_2 f &= 2w \\ \xi_3 f &= 2v & \xi_4 f &= \pm(k+1)vu^k, \end{aligned}$$

and again it is clear that $f$ is $k+1-\mathcal{R}(X)$-determined.

(d) Finally we consider the case where $f(u,v,w) = v$. Then we have

$$\begin{aligned} \xi_1 f &= v & \xi_2 f &= v \\ \xi_3 f &= u^2 & \xi_4 f &= uw. \end{aligned}$$

So a complete 2-transversal is $v + \lambda u^2 + \mu uw + \nu w^2$. We think of this as a 1 parameter family $F_\lambda$. We show that this is $\mathcal{R}(X)$-trivial using Proposition 8.2.2. Now we have

$$\begin{aligned} \xi_1 F &= v + 2\lambda u^2 + \mu uw & \xi_2 F &= v + 2\mu uw + 4\nu w^2 \\ \xi_3 F &= u^2 + 2\mu uv + 4\nu vw & \xi_4 F &= uw + 2\lambda uv + \mu vw. \end{aligned}$$

and $\partial F/\partial\lambda = u^2$. It is clear that

$$\partial F/\partial\lambda \in \Theta(X).F + \mathcal{M}_n^3,$$

so that by Proposition 8.2.2, the family $F_\lambda$ is $2-\mathcal{R}(X)$-trivial. Similarly we can show that the family $F_\mu = v + \mu uw + \nu w^2$ is $2-\mathcal{R}(X)$-trivial. We can then scale $\nu$ to get the normal forms $v$ and $v \pm w^2$.

Now if $f(u,v,w) = v$ is a $k$-jet, then a complete $(k+1)$-transversal is given by $g(u,v,w) = v + \lambda w^{k+1}$. Then we have

$$\begin{aligned} \xi_1 g &= v & \xi_2 g &= v \pm 2(k+1)w^{k+1} \\ \xi_3 g &= u^2 \pm 2(k+1)vw^k & \xi_4 g &= uw. \end{aligned}$$

Thus it is clear that $g$ is $k+1-\mathcal{R}(X)$-determined, for all $k \geq 1$. □

Now we discuss our motivation for classifying functions on the Whitney Umbrella. In other sections of this thesis we have considered functions (height functions and distance squared functions) defined geometrically on crosscaps. Our approach has been to compose the parametrisation of the crosscap $g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$ with the geometric function, say $G : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ to obtain a germ $G \circ g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$. Proposition 2.0.1 shows that the result is well defined up to $\mathcal{R}$-equivalence.

The approach in this chapter is to consider functions on the crosscap itself. More precisely, we know that the parametrisation of the crosscap $g$ is $\mathcal{A}$-equivalent to the standard parametrisation $f(x, y) = (x, xy, y^2)$. In other words, there are germs of diffeomorphisms $\phi : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ and $\psi : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^3, 0$ such that $\psi \circ f = g \circ \phi$. Thus the geometric function $G$ defined on the image of $g$ can be thought of as a function on the Whitney Umbrella by replacing $G$ by $G \circ \psi$. The composite $G \circ \psi$ is well defined up to $\mathcal{R}(X)$-equivalence, for given any other diffeomorphisms $\phi_1 : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ and $\psi_1 : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^3, 0$ such that $\psi_1 \circ f = g \circ \phi_1$, we have $h \circ \psi_1 = h \circ \psi \circ (\psi^{-1} \circ \psi_1)$ and the diffeomorphism $\psi^{-1} \circ \psi_1$ preserves the Whitney Umbrella.

Now we wish to distinguish the $\mathcal{R}(X)$-types of the functions on the crosscap. Proposition 2.0.1 shows that if $H : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ is a function on the crosscap, then we can associate with it two germs which are well defined up to $\mathcal{R}$-equivalence. First we can choose any parametrisation of the crosscap $g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$ and compose it with $H$ to obtain a function germ $H \circ g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$. Secondly, we can compose $H$ with the parametrisation of the double point set $\alpha : \mathbf{R}, 0 \longrightarrow \mathbf{R}^3, 0$ to obtain the germ $H \circ \alpha : \mathbf{R}, 0 \longrightarrow \mathbf{R}, 0$. For the standard crosscap parametrised by $(x, y) \mapsto (x, xy, y^2)$, the parametrisation of the double point set is given by $\alpha(y) = (0, 0, y^2)$, so that $\alpha(y) = \alpha(-y)$. Since any crosscap may be obtained from the standard crosscap by a diffeomorphism in the target, we may always suppose that the parametrisation of the double point set of a general crosscap, $\beta : \mathbf{R}, 0 \rightarrow \mathbf{R}^3, 0$ satisfies $\beta(y) = \beta(-y)$ (we used this in the proof of Lemma 3.1.5). So if $H : \mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ is any function germ, then the composite $H \circ \beta : \mathbf{R}, 0 \longrightarrow \mathbf{R}, 0$ is $\mathbf{Z}_2$-equivariant with respect to the involution $y \mapsto -y$ on $\mathbf{R}, 0$.

**Theorem 8.3.2** The function germs $UW^{\pm}_{k+1}$, $WU^{\pm}_{k+1}$ and $VW^{\pm}_{k+1}$ of Theorem 8.3.1 are all $\mathcal{R}(X)$-inequivalent.

**Proof** In each case we compose the germ with the parametrisation of the standard crosscap ($(x, y) \mapsto (x, xy, y^2)$) and the parametrisation of the double point curve ($y \mapsto (0, 0, y^2)$) to obtain

(i) $UW^{\pm}_{k+1}$: the germs $x \pm y^{2k+2}$ and $\pm y^{2k+2}$, i.e. a submersion and a singularity of type $A^{\pm}_{2k+1}$;

(ii) $WU^{\pm}_{k+1}$: the germs $y^2 \pm x^{k+1}$ and $y^2$, i.e. an $A^{\pm}_k$ singularity and an $A_1$ singularity;

(iii) $VW^{\pm}_{k+1}$: the germs $xy \pm y^{2k+2}$ and $\pm y^{2k+2}$, i.e. an $A_1$ singularity and an $A^{\pm}_{2k+1}$ singularity.

The result then follows from the discussion above. □

Now we consider $\mathcal{R}(X)$-versal unfoldings of these germs.

**Theorem 8.3.3** The following are $\mathcal{R}(X)$-versal unfoldings of the germs of Theorem 8.3.1 and are of minimal dimension. They are also $\mathcal{K}(X)$-versal of minimal dimension.

(i) $UW^{\pm}_{k+1} : u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i$,

(ii)$WU^{\pm}_{k+1} : w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i$,

(iii)$VW^{\pm}_{k+1} : v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu$.

**Proof** By Theorem 8.1.6, we need to find a complement to $J_X(f)$ in $\mathcal{O}_3$ for each of the function germs.

(i) Let $f(u, v, w) = u \pm w^{k+1}$. Then we have

$$\begin{aligned} \xi_1 f &= u & \xi_2 f &= \pm 2(k+1) w^{k+1} \\ \xi_3 f &= \pm 2(k+1) v w^k & \xi_4 f &= v. \end{aligned}$$

So a complement to $J_X(f)$ is given by $1, w, w^2, \cdots, w^k$, giving us the unfolding

$$F(u, v, w, a) = u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i.$$

(ii) When $f(u, v, w) = w \pm u^{k+1}$, we have $J_X(f) = < u^{k+1}, v, w >$, so we do indeed have the unfolding

$$F(u, v, w, a) = w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i.$$

(iii) When $f = v \pm w^{k+1}$ we have $J_X(f) = < v, w^{k+1}, u^2, uw >$ with complement spanned by $1, w, w^2, ..., w^k, u$, giving the unfolding

$$F(u,v,w,a) = v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu.$$

Note that in each case $f \in J_X(f)$ so the resulting unfoldings are $\mathcal{K}(X)$-versal, and of minimal dimension. □

## 8.4 The Geometry of the Functions on the Cross-cap

Now we analyse the discriminants of the singularities $UW^{\pm}_{k+1}$, $WU^{\pm}_{k+1}$ and $VW^{\pm}_{k+1}$. If $f : \mathbf{R}^3, X, 0 \to \mathbf{R}, 0$ is a germ and $F : \mathbf{R}^3 \times \mathbf{R}^p, 0 \to \mathbf{R}, 0$ is an unfolding of $f$, we can identify a number of interesting subsets of the unfolding space $\mathbf{R}^p$.

**Definition 8.4.1** (i) The set $\mathcal{D}_1(F)$ is defined by

$$\mathcal{D}_1(F) = \{a : F = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = \frac{\partial F}{\partial w} = 0 \quad \text{at some point} \quad (u,v,w,a)\}.$$

Now set $G = F(x, xy, y^2, a)$.

(ii) The set $\mathcal{D}_2(F)$ is defined by

$$\mathcal{D}_2(F) = \{a : G = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0 \quad \text{at some point} \quad (x,y,a)\}.$$

Set $H = F(0, 0, y^2, a)$

(iii) The set $\mathcal{D}_3(F)$ is defined by

$$\mathcal{D}_3(F) = \{a : H = \frac{\partial H}{\partial y} = 0 \quad \text{at some point} \quad (y,a)\}.$$

(iv) The set $\mathcal{D}_4(F)$ is defined by

$$\mathcal{D}_4(F) = \{a : F(0,0,0,a) = 0\}.$$

So $\mathcal{D}_1(F)$ is the usual discriminant of $F$ as a function on $\mathbf{R}^3$, $\mathcal{D}_2(F)$ is the discriminant of $F$ when composed with the parametrisation of the standard crosscap, $\mathcal{D}_3(F)$ is the discriminant of $F$ restricted to the double point set and $\mathcal{D}_4(F)$ corresponds to the fibres of $F$ that go through the origin. These are clearly well defined subsets of the versal unfolding space. We now identify these discriminants for the families $UW^{\pm}_{k+1}$, $WU^{\pm}_{k+1}$ and $VW^{\pm}_{k+1}$. Since the germs are submersions on the ambient space, $\mathcal{D}_1(F)$ is empty in all cases. Also it is clear that $\mathcal{D}_4(F)$ is given by the equation $a_0 = 0$. Now we analyse the discriminants $\mathcal{D}_2(F)$ and $\mathcal{D}_3(F)$ for each of the unfoldings.

(i) $UW^{\pm}_{k+1} : F(u, v, w, a) = u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i$.
We associate to $F$ the unfoldings

$$\begin{aligned} G(x,y,a) &= x \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i} \quad \text{and} \\ H(y,a) &= \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i}. \end{aligned}$$

The deformation $G$ unfolds the germ $x \pm y^{2k+2}$, which is a submersion at $(0,0)$, so $\mathcal{D}_2(F)$ is empty in this case.

Now $H$ is a versal unfolding of a $B_{k+1}$ singularity in the terminology of [A3], and thus $\mathcal{D}_3(F)$ is the discriminant of such a singularity. For the cases $k = 1$ and $k = 2$, we calculate these discriminants explicitly; the pictures are well known, see for example, [AGV] or [Wi].

(a) $k = 1$.

The discriminant $\mathcal{D}_3(F)$ is defined in this case by the equations

$$\begin{aligned} H(y,a) &= \pm y^4 + a_0 + a_1 y^2 = 0 \quad \text{and} \\ \frac{\partial H}{\partial y} &= \pm 4y^3 + 2a_1 y = 0. \end{aligned}$$

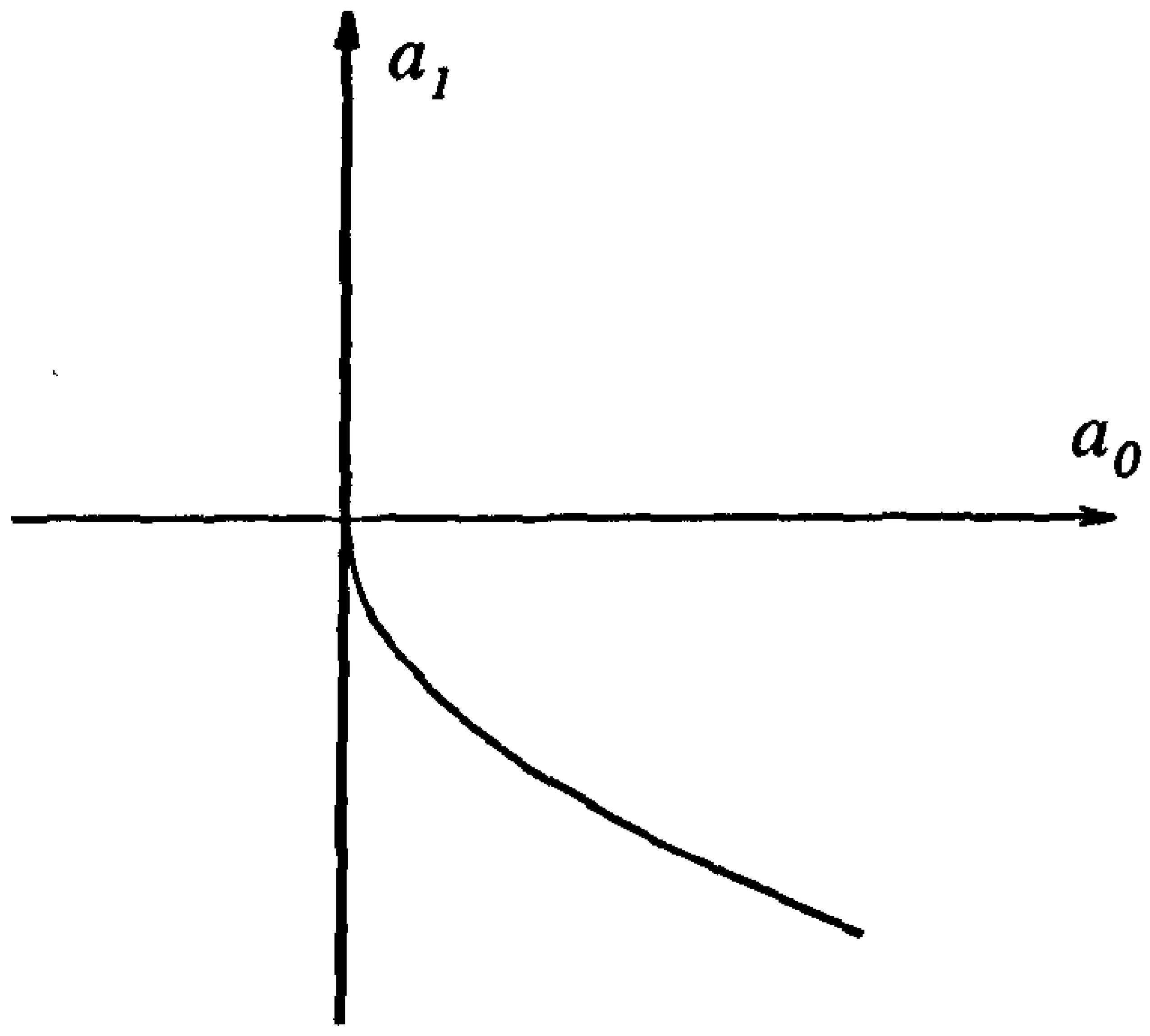
So $\mathcal{D}_3(F)$ consists of the $a_1$-axis and the curve parametrised by

$$(a_0, a_1) = (y^4, -2y^2) \quad \text{for} \quad UW_2^+,$$

and the $a_1$-axis and the curve parametrised by

$$(a_0, a_1) = (-y^4, 2y^2) \quad \text{for} \quad UW_2^-.$$

The set $\mathcal{D}_3(F)$ is shown below for $UW_2^+$.



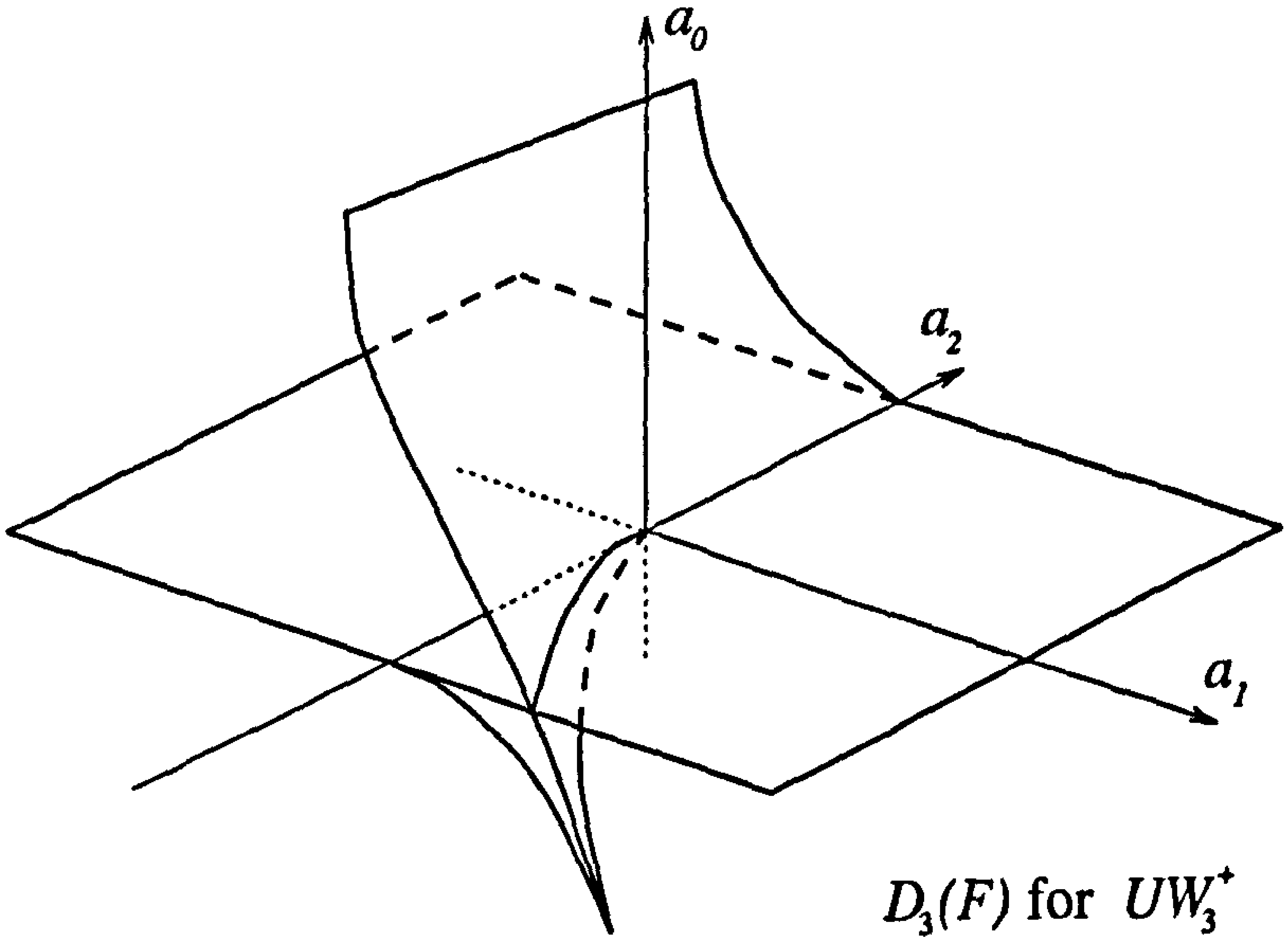

$D_3(F)$ for $UW_2^+$

(b) $k = 2$.

The discriminant $\mathcal{D}_3(F)$ is defined in this case by the equations

$$\begin{aligned} H(y,a) &= \pm y^6 + a_0 + a_1 y^2 + a_2 y^4 = 0 \quad \text{and} \\ \frac{\partial H}{\partial y} &= \pm 6y^5 + 2a_1 y + 4a_2 y^3 = 0. \end{aligned}$$

So $\mathcal{D}_3(F)$ contains the plane $a_0 = 0$. The surface also has a cuspidal edge given by the vanishing of $H, \partial H/\partial y$ and $\partial^2 H/\partial y^2$; this can be parametrised by $y \mapsto (-y^6, 3y^4, -3y^2)$ for $UW_3^+$ and $y \mapsto (y^6, -3y^4, 3y^2)$ for $UW_3^-$. The set $\mathcal{D}_3(F)$ is shown below for $UW_3^+$.




$D_3(F)$ for $UW_3^+$

(ii)$WU^{\pm}_{k+1} : F(u,v,w) = w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i$.
We associate to $F$ the unfoldings

$$\begin{aligned} G(x,y,a) &= y^2 \pm x^{k+1} + \sum_{i=0}^{k} a_i x^i \quad \text{and} \\ H(y,a) &= \pm y^2 + a_0. \end{aligned}$$

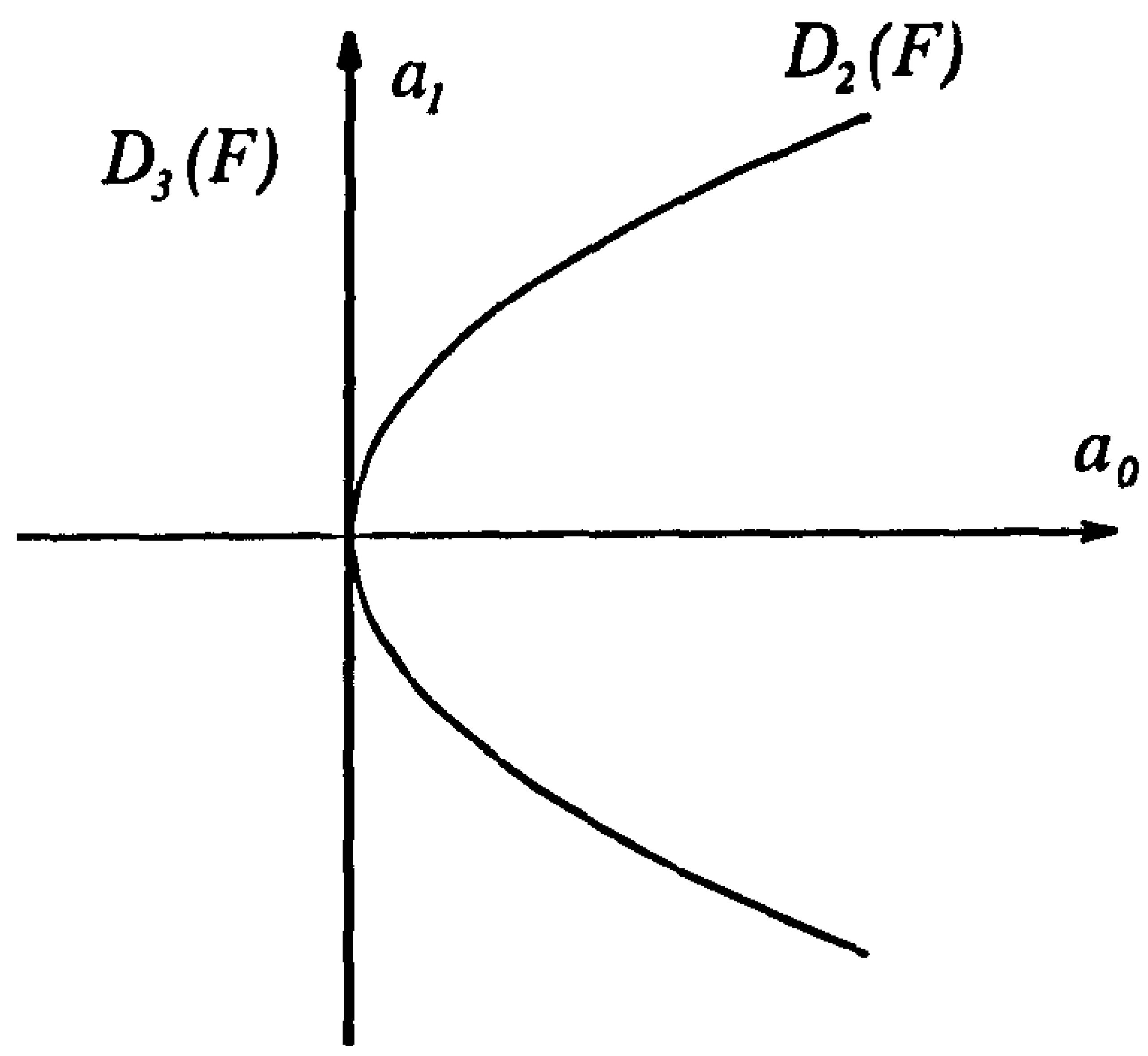
We see immediately that $\mathcal{D}_3(F)$ is the hyperplane $a_0 = 0$. The deformation $G$ is an $\mathcal{R}$-versal unfolding of an $A_k$ singularity. The discriminant $\mathcal{D}_2(F)$ is the standard discriminant of an $A_k$ singularity. We calculate $\mathcal{D}_2(F)$ more explicitly for the cases $k = 1$ and $k = 2$.

(a) $k = 1$.

In this case $\mathcal{D}_2(F)$ is defined by the equations

$$\begin{aligned} G(x,y,a) &= y^2 \pm x^2 + a_0 + a_1 x = 0, \\ \frac{\partial G}{\partial x} &= \pm 2x + a_1 = 0 \quad \text{and} \\ \frac{\partial G}{\partial y} &= 2y = 0. \end{aligned}$$

So $\mathcal{D}_2(F)$ consists of the curve parametrised by $(a_0, a_1) = (x^2, -2x)$ in the case of $WU_2^+$, and the curve parametrised by $(a_0, a_1) = (-x^2, 2x)$ in the case of $WU_2^-$. The sets $\mathcal{D}_2(F)$ and $\mathcal{D}_3(F)$ are shown below for $WU_2^+$.



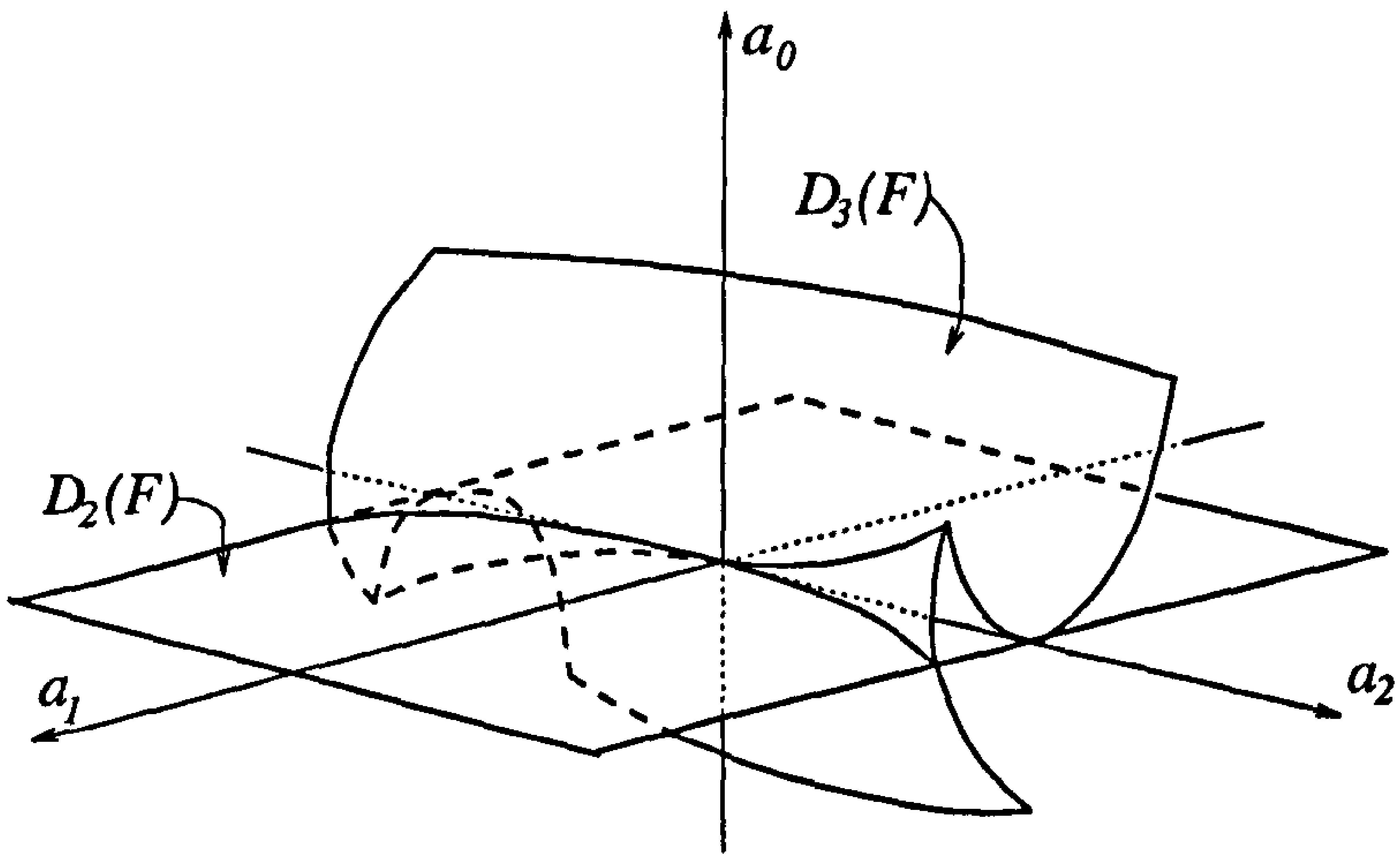

$D_2(F)$ and $D_3(F)$ for $WU_2^+$

(b) $k = 2$.

In this case $\mathcal{D}_2(F)$ consists of a cuspidal surface. The discriminant is defined by the equations

$$\begin{aligned} G(x,y,a) &= y^2 \pm x^3 + a_0 + a_1 x + a_2 x^2 = 0, \\ \frac{\partial G}{\partial x} &= \pm 3x^2 + a_1 + 2a_2 x = 0 \quad \text{and} \\ \frac{\partial G}{\partial y} &= 2y = 0. \end{aligned}$$

The cuspidal edge is defined by $G = \partial G/\partial x = \partial^2 G/\partial x^2 = 0$. It is easy to see that this cuspidal edge is parametrised by $(a_0, a_1, a_2) = (-x^3, 3x^2, -3x)$ in the case of $WU_3^+$, and $(a_0, a_1, a_2) = (x^3, -3x^2, 3x)$ in the case of $WU_3^-$. The cuspidal surface meets the plane $a_0 = 0$ along the $a_2$-axis; looking at sections of the surface along this axis, we see that the plane $a_0 = 0$ is tangential here. The cuspidal surface also meets the plane $a_0 = 0$ transversely along the curve parametrised by $(a_1, a_2) = (x^2, -2x)$ in the case of $WU_3^+$, and along the curve parametrised by $(a_1, a_2) = (-x^2, 2x)$ in the case of $WU_3^-$. The sets $\mathcal{D}_2(F)$ and $\mathcal{D}_3(F)$ are shown below for $WU_3^+$




$D_2(F)$ and $D_3(F)$ for $WU_3^+$

(iii)$VW_{k+1}^{\pm} : F(u,v,w,a,b) = v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu$.
We have

$$G(x,y,a,b) = xy \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i} + bx.$$

This is an $\mathcal{R}$-versal unfolding of an $A_1$ singularity, so $\mathcal{D}_2(F)$ is diffeomorphic to a hyperplane in $(a,b)$-space. Indeed by considering the equations $G = \partial G/\partial x =$

$\partial G/\partial y = 0$, we see that $\mathcal{D}_2(F)$ is defined by the equation

$$\pm b^{2k+2} + \sum_{i=0}^{k} a_i b^{2i} = 0.$$

As usual the discriminant $\mathcal{D}_4(F)$ is the hyperplane with equation $a_0 = 0$. The discriminant $\mathcal{D}_2(F)$ meets $\mathcal{D}_4(F)$ in the set defined by

$$b^2(\pm b^{2k} + \sum_{i=1}^{k} a_i b^{2i-2}) = 0.$$

The unfolding $H$ is given by

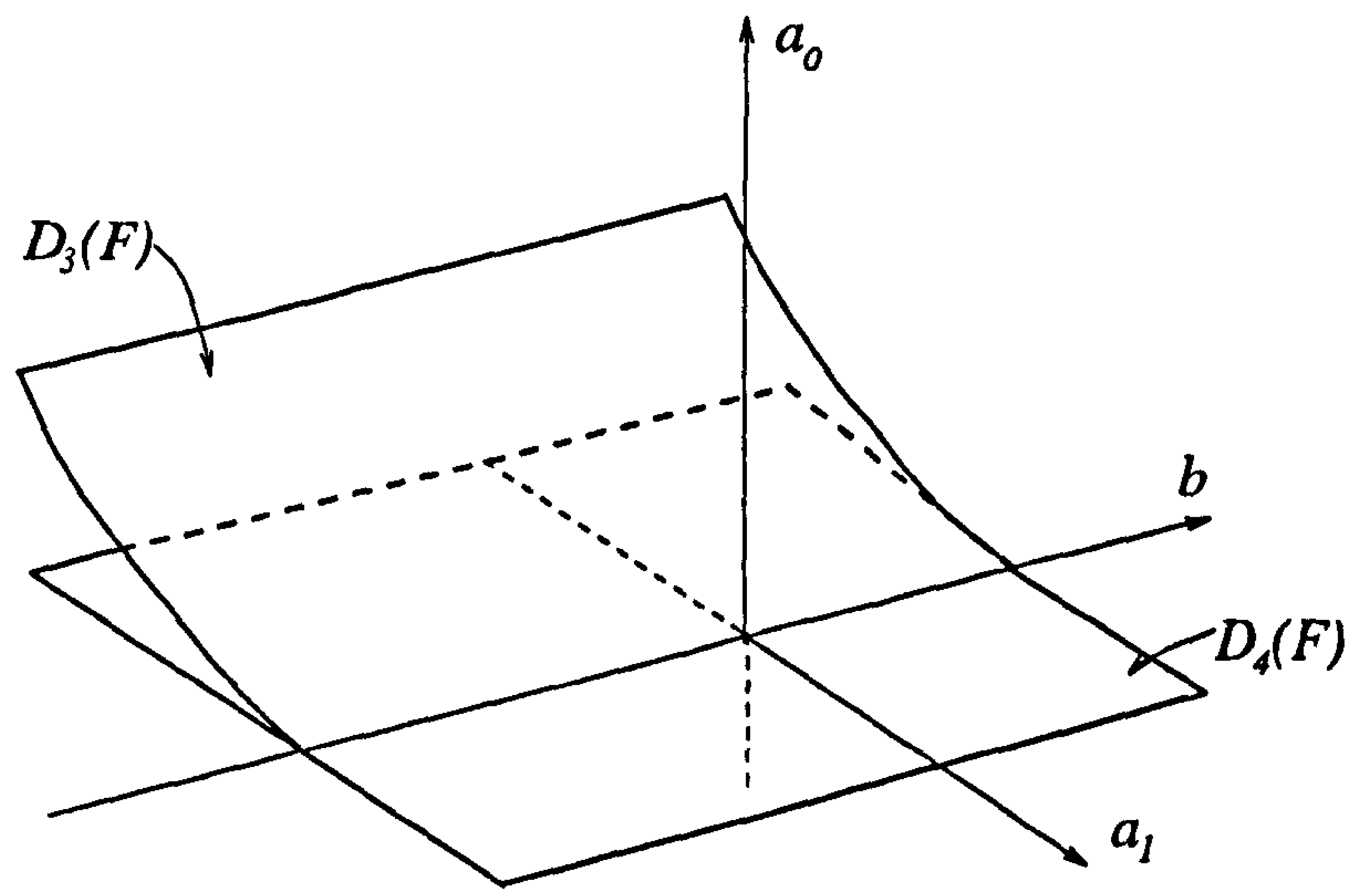
$$H(y, a, b) = \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i}.$$

As in the case of the family $UW_{k+1}$, $H$ is a versal unfolding of a $B_{k+1}$ singularity.

Now we consider the case where $k = 1$ in more detail. Then we have

$$\begin{aligned} F(u, v, w, a, b) &= v \pm w^2 + a_0 + a_1 w + bu, \\ G(x, y, a, b) &= xy \pm y^4 + a_0 + a_1 y^2 + bx \quad \text{and} \\ H(y, a) &= \pm y^4 + a_0 + a_1 y^2. \end{aligned}$$

The discriminant $\mathcal{D}_3(F)$ is parametrised by $(a_0, a_1, b) = (\pm y^4, \mp 2y^2, b)$. For $VW_2^+$ the configuration of $\mathcal{D}_3(F)$ and $\mathcal{D}_4(F)$ (the plane $a_0 = 0$) is given below.




The discriminants $D_2(F)$ and $D_4(F)$ for $VW_2^+$.

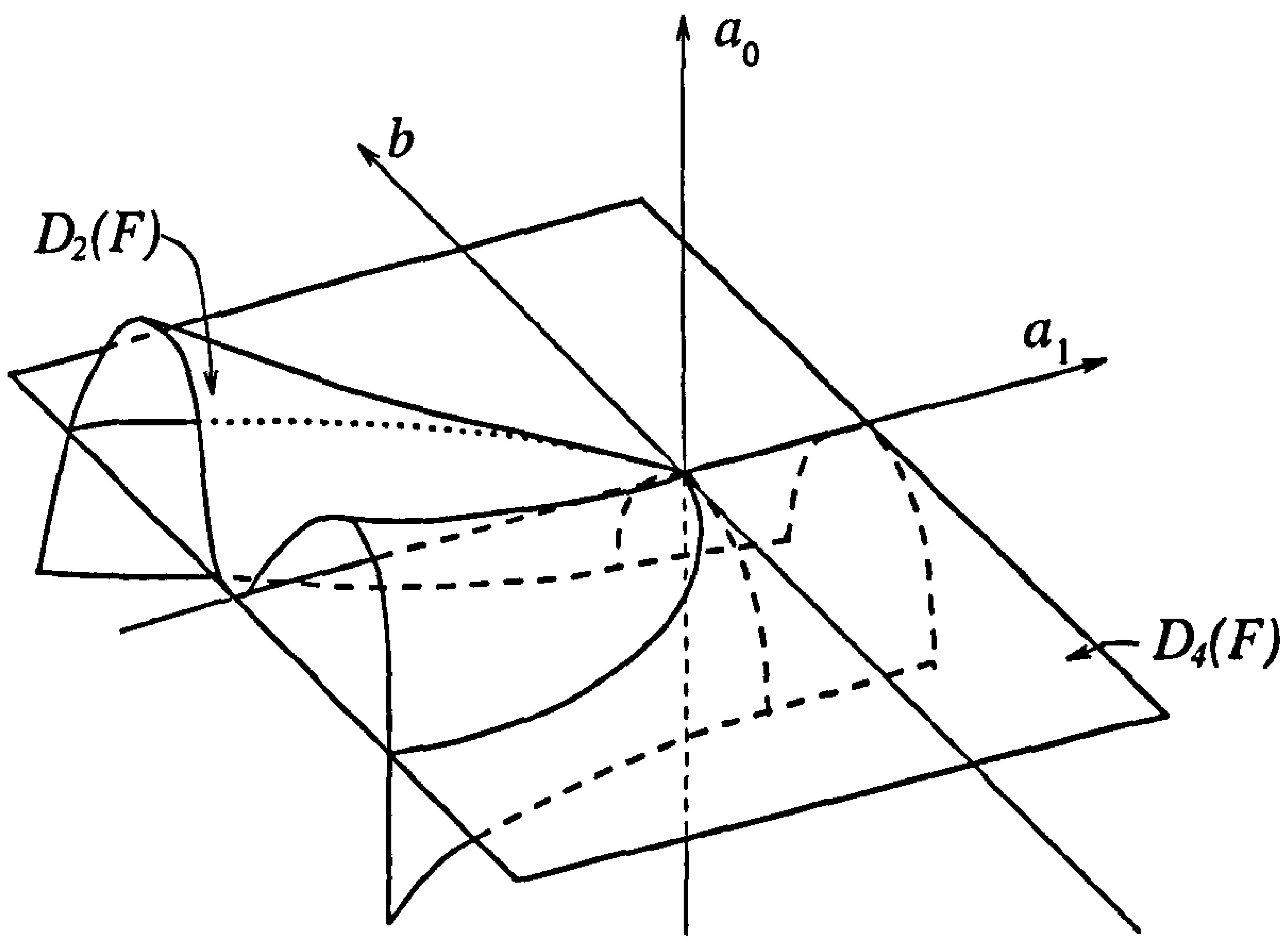
In the case $k = 1$, $\mathcal{D}_4(F)$ meets $\mathcal{D}_2(F)$ in the curve with equation

$$b^2(\pm b^2 + a_1) = 0,$$

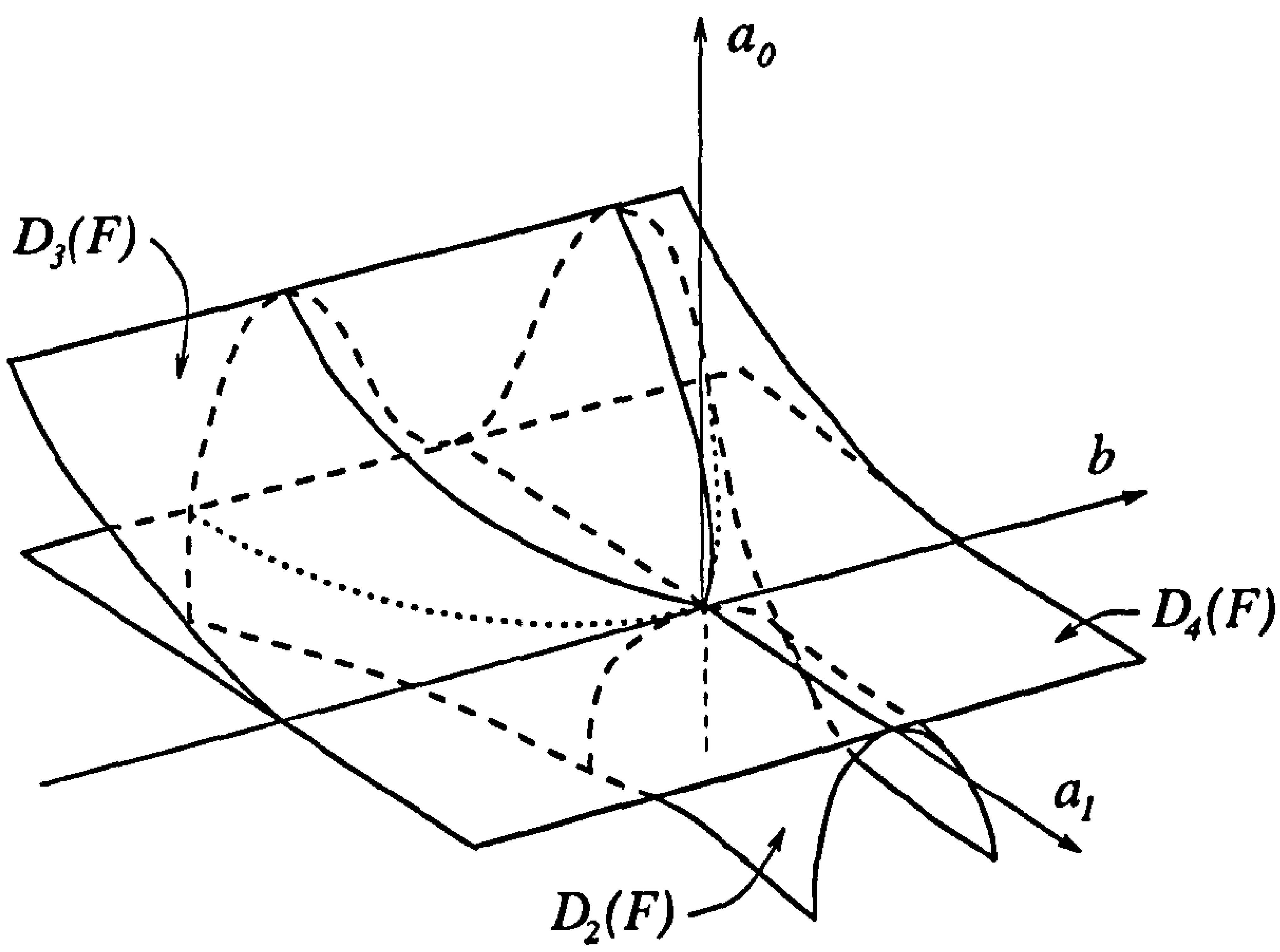
and $\mathcal{D}_3(F)$ meets $\mathcal{D}_2(F)$ in the curve parametrised by

$$(a_0, a_1, b) = (\pm y^4, \mp 2y^2, -y)$$

The pictures look like this:




The discriminants $D_2(F)$ and $D_4(F)$ for $VW_2^+$




The discriminants $D_2(F)$, $D_3(F)$ and $D_4(F)$ for $VW_2^+$.

## 8.5 The Flat Geometry of the Crosscap

We now wish to apply the results on the classification of functions on the crosscap to the flat geometry of the crosscap. This approach requires a transversality result and an understanding of the stratification of the jet-space given by our classification of germs. These results have been supplied by J.W. Bruce, and can be found in Appendix A. The approach we take is as follows.

Let $X, 0$ denote the standard crosscap in $\mathbf{R}^3$, and let $\phi : \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ be the germ of a diffeomorphism taking $X$ to the geometric crosscap $\phi(X)$. Let $f : \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold $Z$ which we wish to apply to the geometric crosscap. Then we can consider the family

$$\begin{aligned} f_\phi : \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}(x). \end{aligned}$$

By studying this family of mappings on the standard crosscap we can describe the interaction between the geometric crosscap and the fibres of the family of functions $f_z : \mathbf{R}^3 \to \mathbf{R}$. We are interested in the family of height functions at the origin. Recall that this family is parametrised by the 2-sphere $S^2$ and is given by

$$\begin{aligned} h : \mathbf{R}^3 \times S^2 &\to \mathbf{R} \\ (x, u) &\mapsto x \cdot u. \end{aligned}$$

Thus the height functions $h_u$ are a family of submersions parametrised by the 2 dimensional manifold $S^2$.

**Proposition 8.5.1** Let $f : \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold $Z$. Suppose that the function $f_z$ is a submersion at the origin for each $z \in Z$ and the dimension of $Z$ is at most 2. We define the family of functions $f_\phi$ by

$$\begin{aligned} f_\phi : \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}(x). \end{aligned}$$

Then generically the function $f_{\phi z}$ at the origin must be $\mathcal{R}(X)$-equivalent to a germ of one of the following types:

(i) $UW_{k+1}$, $k = 0, 1, 2$,

(ii) $WU_{k+1}$, $k = 1, 2$ and

(iii) $VW_2$.

Generically the functions $f_{\phi z}$ are $\mathcal{R}(X)$-versally unfolded by the family $f_\phi$.

**Proof** See Appendix A, Proposition A.2.2. □

Now we can make some interesting deductions about the dual of the crosscap, and also the dual of the double point curve.

**Proposition 8.5.2** At the origin the family of height functions are $\mathcal{R}(X)$- equivalent to the singularities listed in Proposition 8.5.1 and they are $\mathcal{R}(X)$-versally unfolded. The singularities have the following geometric interpretations.

(i) $UW_1$: the plane is transverse to both the crosscap surface and its double point curve.

(ii) $UW_2$: the plane is transverse to the surface and is in the pencil of planes obtained as limiting tangents to the double point curve.

(iii) $UW_3$: the plane is transverse to the surface and is the limiting osculating plane to the double point curve.

(iv) $WU_2$: the plane is tangent to the surface and transverse to the double point curve.

(v) $WU_3$: the plane has parabolic tangency with the surface and is transverse to the double point curve.

(vi) $VW_2$: the plane is the tangent cone, i.e. is tangent to both the surface and the double point curve.

**Proof** The geometrical results follow from composing the listed functions with the parametrisation of the standard crosscap. Note that we have changed coordinates so the fibres of these functions are no longer planes. However the fibres represent planes for our application, and the functions we obtain measure contact with planes in the usual way.

For the $UW_{k+1}$ series, we get $x \pm y^{2k}$ and $\pm y^{2k}$ when composing the function with the parametrisations of the standard crosscap and its double point curve respectively. It is clear that for $k = 0, 1, 2$, the function $UW_{k+1}$ corresponds to a plane that is transverse to the surface. We remark that the self intersection curve is 'doubly parametrised' and is thus highly degenerate. Nevertheless we can consider contact of planes with this curve. If $F$ is an unfolding of $f$, then the discriminant $\mathcal{D}_3(F)$ discussed in Section 8.4 will give us a model for the dual of the double point curve. The plane $a_0 = 0$ corresponds to the set of planes that pass through the crosscap point (the discriminant $\mathcal{D}_4(F)$). So the dual to the double point curve is obtained by removing the plane $a_0 = 0$ that appears as part of the discriminants $\mathcal{D}_3(F)$. Note that in general an osculating plane has $A_2$ contact with a smooth space curve, and so the dual of the space curve will have a cuspidal edge. Considering the discriminant $\mathcal{D}_3(F)$ for $UW_2$ and $UW_3$ completes the proof of results (i)-(iii).

For the $WU_{k+1}$-series we get $y^2 \pm x^k$ and $y^2$. It follows that the corresponding planes are transverse to the double point curve and have $A_1$ (for $k=1$) and $A_2$ (for $k=2$) contact with the surface at the crosscap point. This gives results (iv) and (v).

Finally for $VW_2$ the composites are $xy \pm y^4$ and $y^4$. Here we can see that the corresponding plane is tangent to the double point line (set $x = 0$) and is tangent to the surface. This corresponds to the tangent cone plane by Lemma 2.1.5. □

We have seen that if $F$ is an unfolding of one of the functions listed above then the discriminant $\mathcal{D}_3(F)$ corresponds to the dual of the double point curve, so that this approach gives additional information on the double point curve. Similarly the discriminant $\mathcal{D}_2(F)$ discussed in Section 8.4 will give us a model for the dual of the crosscap. Hence the discussion of these discriminants in Section 8.4 gives an alternative treatment of structure of the dual. Notice for example that when the discriminants $\mathcal{D}_2(F)$ and $\mathcal{D}_3(F)$ intersect at a point, the corresponding plane is tangent to both the double point curve and the surface. Since the results obtained correspond exactly to results in previous chapters, we omit the details.

## 8.6 Map Germs on the Crosscap

In the same way that we gained information on the height functions by classifying functions on the crosscap, we can consider the geometry of projections of the crosscap by carrying out a classification of mappings from 3-space to planes, up to diffeomorphisms in the target and changes of coordinate in the source preserving the crosscap.

In order to do this we need some refinements on the complete transversal methods outlined earlier. This time we need also to take into account the changes of coordinate in the target.

Since we are considering projection maps, we need only consider those map germs which are submersions on the ambient space. We have the following result.

**Theorem 8.6.1** Any map germ $f : \mathbf{R}^3, X, 0 \longrightarrow \mathbf{R}^2, 0$ which is a submersion on the ambient space is $\mathcal{R}(X)$-$\mathcal{L}$ equivalent to one of the following map germs.

(a) $f_1(u,v,w) = (u,w)$,

(b) $f_2(u,v,w) = (u+w,v)$,

(c) $f_3(u,v,w) = (u+w^2+aw^3, v+w^2), \quad a \neq 0$,

(d) $f_4(u,v,w) = (w \pm u^2 + au^3, v+u^3), \quad a^2-4 \neq 0$.

In cases (c) and (d), $a$ is a modulus, and the conditions on $a$ are necessary for the map germ to be determined.

**Proof** The proof requires the use of a computer and Neil Kirk's Maple package Transversal. For a description of this package, see [Ki]. There are a great many calculations involved, most of which are not at all illuminating. What we do here is show how one normalises the 1-jets.

This is essentially a problem in linear algebra: in the target we have the whole of the general linear group available, whereas in the source we have the subgroup generated by integrating the 1-jets of the vector fields which we discussed earlier. Recall that this group is generated by the 1-parameter subgroups given by

$$(u,v,w) \mapsto (e^{\lambda}u, e^{\lambda}v, w)$$

$$(u,v,w) \mapsto (u, e^{\lambda}v, e^{2\lambda}w)$$

$$(u,v,w) \mapsto (u, v, w+\alpha v)$$

$$(u,v,w) \mapsto (u+\beta v, v, w).$$

Now the initial 1-jet is of the form

$$(u,v,w) \mapsto (a_1u+a_2v+a_3w, b_1u+b_2v+b_3w).$$

Since it is of rank 2 one of the $2\times2$ minors of the corresponding matrix is non-zero. We go through the 3 possibilities:

(i) If $a_1b_3 - a_3b_1 \neq 0$ then by a linear change of coordinates in the target we can reduce to the 1-jet $(u + Av, w + Bv)$ for some $A$ and $B$. It is now clear that by allowable changes of coordinate in the target we can ensure that $A$ and $B$ are both zero, that is we have the normal form $(u, w)$.

(ii) If $a_1b_2 - a_2b_1 \neq 0$ then by a linear change of coordinates in the target we can reduce to the 1-jet $(u + Aw, v + Bw)$ for some $A$ and $B$. If $B \neq 0$ then we can reduce to case (i) again. So we have two new normal forms $(u + w, v)$ and $(u, v)$.

(iii) Finally if $a_2b_3 - a_3b_2 \neq 0$ then by a linear change of coordinates in the target we can reduce to the 1-jet $(v + Au, w + Bu)$ for some $A$ and $B$. If $A$ or $B \neq 0$ then we can reduce to case (i) or (ii) again. So we have one new normal forms $(v, w)$.

Note that these linear forms all have geometric interpretations. Recall that the pencil of planes tangent to the crosscap at the base point is given by $\alpha v + \beta w = 0$, the limit of the tangent line to the double point set is the $w$-axis, while the tangent cone is given by $v = 0$. Clearly the condition that the kernel of the projection is contained in one of these subspaces is invariant under the changes of coordinates. (Indeed one can easily check that the linear changes of coordinates allowed in the source are precisely those which preserve this configuration.)

Now for the normal form $(u, w)$ the kernel is clearly transverse to the tangent cone and does not coincide with either of the other geometric lines. For the normal form $(u + w, v)$ the kernel lies in the tangent cone but does not coincide with the other two lines. For the normal form $(u, v)$ the kernel is the limiting tangent line to the double point curve, and for the normal form $(v, w)$ the kernel is the tangent direction to the crosscap point.

The rest of the proof is technical, but does use the following determinacy result, which can be derived from [BduPW]. □

**Theorem 8.6.2** Let $\Theta_1$ denote the submodule of the vector fields tangent to the crosscap spanned by $\{u\xi_1, v\xi_1, w\xi_1, u\xi_2, v\xi_2, w\xi_2, \xi_3, \xi_4\}$.

Given a map germ $f : \mathbf{R}^3, 0 \to \mathbf{R}^2$ defined on the crosscap then it is $k$-$\mathcal{R}(X)$-$\mathcal{L}$-determined if the following inclusion holds

$$\mathcal{M}_3^{k+1}\mathcal{E}(3,2) \subset$$
$$\Theta_1.f + f^*\mathcal{M}_2\{e_1, e_2\} + sp\{f_1e_2\} + \mathcal{M}_3^{k+1}.(f^*\mathcal{M}_2.\mathcal{E}(3,1) + \mathcal{M}_3^{k+1}).\mathcal{E}(3,2).$$

The results obtained from using the Transversal package also lead us to make the following conjecture

**Conjecture 8.6.3** The family of map germs

$$f(u,v,w) = (u + w^{2k+1}, v + w^2)$$

is $(2k+1)$-$\mathcal{R}(X)$-$\mathcal{L}$-determined.

**Remark 8.6.4** Using the Transversal package, we know that this is certainly the case for $k = 1, 2, 3, 4$.

We can now use Transversal to calculate $\mathcal{R}(X)$-$\mathcal{L}$ versal unfoldings of the map germs of Theorem 8.6.1. The existence of such unfoldings follows from the results of Damon in [D].

**Theorem 8.6.5** The following are $\mathcal{R}(X)$-$\mathcal{L}$ versal unfoldings of the map germs of Theorem 8.6.1. They are of minimal dimension.

(*a*) $F_1(u,v,w) = (u,w)$: this map germ is stable.

(*b*) $F_2(u,v,w,\alpha) = (u+w, v+\alpha u)$,

(*c*) $F_3(u,v,w,\alpha,\beta,\gamma) = (u + w^2 + (a+\alpha)w^3 + \beta w, v + w^2 + w), \quad a \neq 0$,

(*d*) $F_4(u,v,w,\alpha,\beta,\gamma) = (w \pm u^2 + (a+\alpha)u^3 + \beta u, v + u^3 + \gamma u), \quad a^2 - 4 \neq 0.$

Now we analyse the geometry of the map germs $f_1$, $f_2$, $f_3$ and $f_4$. First we compose these map germs with the parametrisation of the standard crosscap $g(x,y) = (x, xy, y^2)$ to obtain a map germ $f_i \circ g : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$, for $(i = 1, \cdots, 4)$. We then calculate the critical values of this mapping, and the image of the double point curve (the positive $w$-axis parametrised by $y \longrightarrow (0,0,y^2)$) under the mapping $f_i$. This is in fact 'half' of the image of the $w$-axis under $f_i$.

**Proposition 8.6.6** (*a*) For $f_1(u,v,w) = (u,w)$ the composite is $(x,y) \mapsto (x,y^2)$. This is a fold map. The set of critical values is the $x$ axis. The image of the double point curve is the half line $y \mapsto (0,y^2)$.

(*b*) For $f_2(u,v,w) = (u+w,v)$, the composite is $(x,y) \mapsto (x+y^2, xy)$. This is $\mathcal{A}$-equivalent to the cusp map $(x,y) \mapsto (x, y^3 + xy)$. The set of critical values is the cusp with cuspidal tangent the $x$-axis. The image of the double point curve is the half line $y \mapsto (0, y^2)$ and so coincides with the cuspidal tangent.

(c) For $f_3(u,v,w) = (u+w^2+aw^3, v+w^2), \quad a \neq 0$, the composite is $(x,y) \mapsto (x + y^4 + ay^6, xy + y^4)$. This is $\mathcal{A}$-equivalent to the swallowtail map $(x,y) \mapsto (x, xy + y^4)$. The set of critical values is diffeomorphic to the image of the curve $t \mapsto (t^3, t^4)$. The image of the double point curve is a half branch of a cusp parametrised by $y \mapsto (y^4 + ay^6, y^4)$.

(d) For $f_4(u,v,w) = (w \pm u^2 + au^3, v + u^3), \quad a^2 - 4 \neq 0$, the composite is $(x,y) \mapsto (y^2 \pm x^2 + ax^3, xy + x^3)$. This is a corank 2 map germ. The set of critical values either consists of two cusps with cuspidal tangents given by the equations $x - 2y = 0$ and $x + 2y = 0$ or is empty, depending on the sign of $u^2$ in the first component.The image of the double point curve is the half line given by $y \mapsto (y^2, 0)$.

**Proof** We prove this by direct computation.

(a) $\quad f_1(u,v,w) = (u,w)$.

We have $f_1 \circ g(x,y) = (x, y^2)$. Thus we have

$$\frac{\partial(f_1 \circ g)}{\partial x} = (1,0) \quad \text{and} \quad \frac{\partial(f_1 \circ g)}{\partial y} = (0, 2y).$$

Part (a) follows straightforwardly.

(b) $\quad f_2(u,v,w) = (u+w, v)$.

We have $f_2 \circ g(x,y) = (x + y^2, xy)$. Note that replacing $x + y^2$ by $X$ changes the second component to $Xy - y^3$ and we do obtain the cusp singularity.

Now we find

$$\frac{\partial(f_2 \circ g)}{\partial x} = (1,y) \quad \text{and} \quad \frac{\partial(f_2 \circ g)}{\partial y} = (2y, x).$$

Thus the critical set of this mapping is given by the equation $x - 2y^2 = 0$. So the set of critical values is parametrised by $t \mapsto (3t^2, 2t^3)$. This is a cusp with cuspidal tangent the $x$-axis. The image of the double point curve is given by $y \mapsto (y^2, 0)$, and so coincides with the cuspidal tangent.

(c) $\quad f_3(u,v,w) = (u + w^2 + aw^3, v + w^2), \quad a \neq 0$.

Then we have $f_3 \circ g(x,y) = (x+y^4+ay^6, xy+y^4)$. Here replacing $x+y^4+ay^6$ by $X$ changes the second component to $Xy + y^4 - y^5 + ay^6$, and the germ is $\mathcal{A}$-equivalent to $(x, xy + y^4)$ as asserted.

Calculation yields

$$\frac{\partial(f_3 \circ g)}{\partial x} = (1, y) \quad \text{and} \quad \frac{\partial(f_3 \circ g)}{\partial y} = (4y^3 + 6ay^5, x + 4y^3).$$

Thus the critical set in this case is given by the equation $x+4y^3-4y^4-6ay^6 = 0$. So the set of critical values is given by

$$t \mapsto (-4t^3 + 5t^4 + 7at^6, -3t^4 + 4t^5 + 6at^7).$$

The image of this parametrisation is diffeomorphic to the image of the standard $(t^3, t^4)$ curve. The image of the double point curve is a half cusp. We expect this, since $f_3 \circ g$ is a Swallowtail map, and so corresponds to the case where we are projecting in a direction that is parallel to the limiting tangent to the double point curve. Considering these two curves we see why we must expect a modulus along with this germ.

**Claim** As $a$ changes, the configuration of double point set and set of critical values are not diffeomorphic to each other, at least over the complex numbers.

The point is that we now have (over the complexes) a curve $(t^3, t^4)$ with tangent $y = 0$ and a cusp with tangent $x = y$. The product of the defining equations is a function with isolated singularity but which has modulus; deformations of $a$ will result in distinct composite curves.

$(d)(i) \quad f_4(u,v,w) = (w + u^2 + au^3, v + u^3), \quad a^2 - 4 \neq 0.$

Then $f_4 \circ g(x,y) = (y^2+x^2+ax^3, xy+x^3)$. This is a corank 2 map germ. To compute the critical values, we proceed as follows:

$$\frac{\partial(f_4 \circ g)}{\partial x} = (2x + 3ax^2, y + 3x^2), \quad \text{and} \quad \frac{\partial(f_4 \circ g)}{\partial y} = (2y, x)$$

Thus the critical set of this mapping is given by the equation

$$\phi(x,y) = 2x^2 + 3ax^3 - 2y^2 - 6x^2y = 0.$$

Note that $\phi(x,y)$ is Morse, and so we can parametrise the two branches of the critical set by

$$t \mapsto (t + \cdots, t + \cdots) \quad \text{and} \quad t \mapsto (t + \cdots, -t + \cdots).$$

So we can get the lowest order terms of a parametrisation of the critical values by substituting into $f_4 \circ g(x, y)$. We get

$$t \mapsto (2t^2 + at^3 + \cdots, t^2 + t^3 + \cdots) \quad \text{and} \quad t \mapsto (2t^2 + at^3 + \cdots, -t^2 + t^3 + \cdots).$$

So the set of critical values consists of two cusps with cuspidal tangents given by the equations $x - 2y = 0$ and $x + 2y = 0$.

**Claim** As $a$ changes, these configurations are not diffeomorphic to each other.

Again a pair of cusps, together with a general line yields a composite curve with moduli.

$(d)(ii)$ $f_4(u, v, w) = (w - u^2 + au^3, v + u^3)$, $a^2 - 4 \neq 0$.

Then $f_4 \circ g(x, y) = (y^2 - x^2 + ax^3, xy + x^3)$. Again this is a corank 2 map germ. We have

$$\frac{\partial(f_4 \circ g)}{\partial x} = (-2x + 3ax^2, y + 3x^2), \quad \text{and} \quad \frac{\partial(f_4 \circ g)}{\partial y} = (2y, x)$$

Thus the critical set of this mapping is given by the equation

$$-2x^2 + 3ax^3 - 2y^2 - 6x^2y = 0.$$

We note that in a neighbourhood of the origin, the only solution to this equation is $(x, y) = (0, 0)$. So for the critical values of this map we get an isolated point (the origin). The image of the double point curve is given by $y \mapsto (y^2, 0)$, so it is the positive $x$-axis. □

We see that these results give information which corresponds precisely with the results in Chapter 5. Case $(a)$ corresponds to projections in directions transverse to the tangent cone. Case $(b)$ corresponds to the projections in directions lying in the tangent cone, but not the tangent line or the limiting tangent to the double point curve. We have already mentioned that case (c) corresponds to projecting along the limiting tangent to the double point curve. By considering the critical values of the mapping, we see that case $(d)$ corresponds to projecting along the tangent line.

Finally we make the same considerations for the conjectured family.

**Proposition 8.6.7** For the family of map germs given by

$$f(u, v, w) = (u + w^{2k+1}, v + w^2),$$

the composite is $(x,y) \mapsto (x+y^{4k+2}, xy+y^4)$. This is $\mathcal{A}$-equivalent to the swallowtail map $(x,y) \mapsto (x, xy+y^4)$. The set of critical values is diffeomorphic to the image of the curve $t \mapsto (t^3, t^4)$. The image of the double point curve is a half branch of a degenerate cusp parametrised by $y \mapsto (y^{4k+2}, y^4)$.

**Proof** On composing with $g$ we get $f \circ g(x,y) = (x+y^{4k+2}, xy+y^4)$. So

$$\frac{\partial(f \circ g)}{\partial x} = (1, y) \quad \text{and} \quad \frac{\partial(f \circ g)}{\partial y} = ((4k+2)y^{(4k+1)}, x+4y^3).$$

Thus the critical set is given by the equation $x + 4y^3 - (4k+2)y^{(4k+2)} = 0$. So the critical values are given by

$$t \mapsto (-4t^3 + (4k+3)t^{(4k+2)}, -3t^4 + (4k+2)t^{(4k+3)}),$$

and the image of the double point curve is given by $y \to (y^{(4k+2)}, y^4)$. We note that this family seems to pick out the degree of contact of the double point curve with its limiting tangent line. □

## 8.7 Ideas for Further Work

It is clear that the work on the integral curves of the crosscap at the end of Chapter 5 is far from complete. We noted that the methods of [BT] could not be applied to the problems of determining the principal curves on the crosscap and the asymptotic curves on the parabolic crosscap. Thus we have motivation for trying to extend the work of [BT] to cover these cases. Note also that the model of asymptotic curves for the hyperbolic crosscap is a *topological* model of the asymptotic curves in the parametrising space. We cannot yet say what these curves look like on the crosscap or even where the preimage of the double point curve lies with respect to the configuration. Theorem 5.2.1 says that we have a circle of directions for which the outline of the surface is a cusp or worse. Does this mean that in some sense these directions are the limiting asymptotic directions? It would also be interesting to determine the asymptotic and principal curves on the dual surface to the crosscap.

In Chapters 6 and 7 we have considered mainly the local aspects of the focal set. Some more work could be done on the global structure of the focal set. Also we could try to determine where the conic lies on the part of the focal surface near to the origin. There are several aspects of spherical geometry left to consider. We have mentioned ridges, but we could investigate the condition for 0, 2 or 4 ridges more thoroughly. In [Wi] the sub-parabolic lines of a surface are considered via folding maps. This work could be extended to surfaces with crosscaps. Also we have not considered the wavefronts (or parallels) of the crosscap. The focal surface is the union of the singular points of the family of wavefronts, so considering this family may deepen our understanding of the focal surface. We note here that a wavefront will have a 'hole' corresponding to the circle of limiting normal directions at the crosscap point. Another approach to the geometry of the focal surface is to extend the work in this chapter to cover functions and mappings on the crosscap which are not submersions on the ambient space (the distance squared function is an example of such a function).

Crosscaps are generic phenomena when a surface in $n$-space (where $n > 3$) is projected into 3-space. The geometry of the crosscap could be related to these surfaces in higher dimensional space and their projections.

The techniques of Chapter 7 are very powerful and yet the 'answer' they provide (Corollary 7.3.4) is relatively simple. David Mond has suggested that there may be a geometric reason behind this. He has also pointed out that Lemma 2.2.1 might be extended to cover other polynomials. We are grateful to him for these comments.

# Appendix A

# Transversality Results for Chapters 6 and 8

## A.1 Distance Squared Functions on the Crosscap

This section contains the transversality results necessary to complete the proof of Proposition 6.2.3. The proofs were supplied by J.W. Bruce.

Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be a parametrisation of some 'geometric' crosscap. We choose an open neighbourhood of $0 \in \mathbf{R}^2$ and a representative of $f$, which we also denote by $f$ with $f : U \to \mathbf{R}^3$. We may suppose that the image of $f$ is contained in the closed unit ball $B$ centred at the origin in $\mathbf{R}^3$. Let $P_k$ denote the set of polynomial mappings $\mathbf{R}^3 \to \mathbf{R}^3$ of degree $\leq k$ which fix the origin. Clearly $P_k$ is a finite dimensional vector space. An element of $P_k$ consists of a triple of polynomials of degree $\leq k$ with zero constant term. It is not hard to see that there is an open neighbourhood of the identity in $P_k$ consisting of polynomial mappings which map the unit ball $B$ diffeomorphically onto its image; denote such a set by $V_k$.

The idea is to use the family of diffeomorphisms in $V_k$ to deform the original parametrisation $f$. This gives us a family of parametrisations

$$F : U \times V_k \longrightarrow \mathbf{R}^3$$

given by $F(u, \phi) = \phi(f(u))$. We now wish to show that this is a sufficiently large family to obtain good deformations of $f$ with respect to the family of distance squared functions. Of course there is one problem: Proposition 6.2.4 tells us that the distance squared function from the origin is not versally unfolded by the family of all distance squared functions. For this reason we will have to exclude this particular point in $\mathbf{R}^3$. So we shall consider the family

$$d : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbf{R}$$

given by $d(u,z) = ||f(u) - z||^2$. To show that this has the required versality properties for generic $f$ we consider the new mapping

$$D : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \times V_k \longrightarrow \mathbf{R}$$

given by $D(u,z,\phi) = ||\phi(f(u)) - z||^2$. As usual the versality properties are established using the corresponding jet-map

$$j_1^k D : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \times V_k \to J^k(U,\mathbf{R})$$

given by considering $D(-,z,\phi)$ as a function of $u$ only and taking the $k$-jet. We would like to prove that this map is transverse to the relevant submanifolds of the jet-space. For we can then apply the Thom Transversality Lemma to deduce that for almost all $\phi$ in the sense of Lebesgue measure (i.e. outside a set of measure zero) the same will remain true for the restriction

$$j_1^k D_\phi : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \longrightarrow J^k(U,\mathbf{R})$$

where $D_\phi$ denotes the mapping obtained from $D$ by setting the third component equal to $\phi$. We will have a little further work to do then, because of the non-compactness $\mathbf{R}^3 \setminus \{(0,0,0)\}$, but this is essentially the basic strategy.

Before moving on we mention a couple of related results in the literature.

### Looijenga's Result

In the case when we are considering the case of a smooth surface in $\mathbf{R}^3$ and its contact with spheres, the relevant transversality result is due to Looijenga (see [Wal]). We can use the above notation and simply suppose that $f$ above is an immersion (as our function is away from a punctured neighbourhood of the origin). It is then true that the relevant map is transverse to submanifolds of the jet-space, provided that these submanifolds are invariant under addition of constants. In other words given any $k$-jet in the submanifold we must be able to alter the constant term in the jet and still stay in the manifold. The reason for this is that the distance squared functions give some problem when they are centred at a point on the surface. Basically they will have an $A_1$ singularity, and changing the embedding will not yield any constant tangent vectors required for the transversality result. However this is no real problem as all of the natural submanifolds of the jet-space have this property.

### Montaldi's Result

In [Mt] Montaldi proved a very fine generalisation of Looijenga's result. He replaced the family of distance squared functions by any family of mappings which are versally unfolded with respect to one of Mather's subgroups. The family of distance squared functions is a special case since each of them is actually $\mathcal{A}$-stable. The conclusion is (roughly) that for a generic immersion the corresponding jet-extension map is transverse to $\mathcal{G}$-invariant submanifolds of the jet-space.

In our situation we no longer have an immersion, but we do know that our mapping $f$ is stable. A little thought convinces one that we might hope for a transversality result here, and this is what we now establish.

First a brief word about jet-spaces. If $X$ and $Y$ are manifolds of dimension $n$ and $p$ respectively then the jet-space $J^k(X,Y)$ is a fibre bundle over the product $X \times Y$, with fibre the space $J^k(n,p)$ consisting of polynomial mappings $\mathbf{R}^n \to \mathbf{R}^p$ of degree $\leq k$ with no constant term. The jet extension associates to each smooth map $f : X \to Y$ a map $j^k f : X \to J^k(X,Y)$. Roughly speaking this assigns to a point $x \in X$ the points $(x, f(x)) \in X \times Y$ the base of the fibration, and the Taylor series of $f$ at $x$ truncated to degree $k$ in the fibre. In the case when the target $Y$ is the set of reals $\mathbf{R}$ then one can think of the jet bundle as a fibre bundle over $X$ with fibre consisting of polynomial mappings $\mathbf{R}^n \to \mathbf{R}^p$ of degree $\leq p$ (so we allow a constant term).

**Theorem A.1.1** Let $X \subset J^k(2,1)$ be an $\mathcal{R}^k$-invariant submanifold which is also invariant under addition of constants. Then the map

$$j_1^k D : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \times V_k \longrightarrow J^k(U, \mathbf{R})$$

is transverse to the submanifolds $U \times \mathbf{R} \times X \subset J^k(U, \mathbf{R})$.

**Proof** We choose a point $(x, z, \phi)$ in the domain of $j_1^k D$ and need to prove transversality there. In fact we may as well suppose that $\phi$ is the identity, since we shall only use the fact that $\phi \circ f$ is $\mathcal{A}$-equivalent to a crosscap below. There are two cases to consider depending on whether $f(x) = z$ or not as we shall see.

We start by computing the image of the tangent vectors to $V_k$ in $J^k(2,1)$. Of course the tangent space at the identity to $V_k$ is $P_k$, since $V_k$ is an open subset of this vector space. If $p \in P_k$ we use the path in this space given by $\gamma(s) = id + sp$ and consider

$$\lim_{s \to 0} \frac{j_1^k D(x, z, id + sp) - j_1^k D(x, z, id)}{s}.$$

This becomes

$$\lim_{s \to 0} \frac{j_1^k(||f(x) + sp(f(x)) - z||^2 - ||f(x) - z||^2)}{s}$$

$$= \lim_{s \to 0} \frac{j_1^k(2s(f(x) - z) \cdot p(f(x)) + s^2 p(f(x)) \cdot p(f(x)))}{s}$$

$$= 2j_1^k((f(x) - z) \cdot p(f(x))).$$

We consider what this expression gives in each of the cases $x \neq 0$ and $f(x) \neq z$, $x \neq 0$ and $f(x) = z$ and $x = 0$. We show that each time we obtain all of the elements in the fibre space $J^k(2,1)$, and hence transversality to the manifold.

Now if $x \neq 0$ and $f(x) \neq z$ then $f$ is an immersion and it is easy to see that if we allow $p$ to vary over $P_k$ we obtain all of the elements in the fibre space $J^k(2,1)$, and hence transversality to the manifold.

If $f(x) = z$ then since $z \neq 0$ the distance squared function $||f(x) - z||^2$ has an $A_1$ singularity at $x$, and the derivatives with respect to the $x$ variables together with the tangent space to the orbit (which if it meets $X$ is a subset of $X$ by hypothesis) will give the required vectors.

The remaining case to consider is when $x = 0$, so that $f(x) = 0$ also. Here we first note that since we have a crosscap the pullback $f^*\mathcal{M}_3$ contains the ideal $\mathcal{M}_2^2$. It follows that by a suitable choice of $p$ we can obtain any polynomial triple of degree $\geq 2$ and $\leq k$ from $j^k(p(f))(x)$. Again using the fact that $z \neq 0$ it is easy to see that we shall obtain all of the vectors in the fibre space satisfying the same restriction on the degree. So we are now looking for the linear terms in $J^k(2,1)$.

By a change of coordinates in the source and an orthogonal change of coordinates in the target we may suppose that the 2-jet of $f$ is in the form

$$f(x_1, x_2) = (x_1, a_0 x_1^2 + a_1 x_1 x_2 + a_2 x_2^2, b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2)$$

so that $f^*\mathcal{M}_2$ also contains $x_1$. Now consider the tangent vectors provided by the $x$ variables. We only require a vector with a non-zero $x_2$ coefficient, so we consider the 2-jet of the distance squared function $||f(x) - z||^2$. The conditions that this should fail are now easily determined to be $z_1 = 0$ (for otherwise we have a submersion anyway) and $a_1z_2+b_1z_3 = a_2z_2+b_2z_3 = 0$. In particular, since $z \neq 0$ we can deduce that $a_1b_2 - a_2b_1 = 0$ which contradicts the fact that $f$ has a crosscap at the origin. □

As we have seen the normal plane at the crosscap point is of particular interest. We also need to check things are as we would expect them to be in this plane.

As before we write the crosscap in the form $(x_1, f_2(x_1,x_2), f_3(x_1,x_2))$ with $f_1,\ f_2 \in \mathcal{M}_2^2$. We wish to deform the crosscap in such a way that the tangent line (the $u_1$-axis) remains fixed, and the above form is retained. So we consider the polynomial mappings of the form $(u_1,u_2,u_3) \mapsto (u_1, p_2(u), p_3(u))$ where the $p_j$ are polynomials in the ideal $\langle u_2, u_3, u_1^2\rangle$. We denote the collection of polynomial maps of this form where the degree of $p_j$ is $\leq k$ by $Q_k$, and an open neighbourhood of the identity mapping the unit ball diffeomorphically onto its image by $W_k$. We now consider the map

$$E : \mathbf{R}^2 \setminus \{(0,0)\} \times W_k \longrightarrow J^k(2,1)$$

defined by $E(z,\phi) = j^k(||f(x) - z||^2)(0)$, where $z = (0, z_2, z_3)$. Note that the image of $E$ is actually a subset of the singular jets, i.e. those with no linear terms, and which we denote by $\Sigma(2,1)$. Using the same ideas as above we can prove the following.

**Proposition A.1.2** The map

$$E : \mathbf{R}^2 \setminus \{(0,0\} \times W_k \longrightarrow \Sigma(2,1)$$

is a submersion.

**Proof** Just use the fact, as above, that the pullback $f^*\mathcal{M}_3$ contains the ideal $\mathcal{M}_2^2$, and $z \neq 0$. □

We now use these transversality results to establish some geometric corollaries.

**Corollary A.1.3** For almost all $\phi$ in the sense of Lebesgue measure the family of distance squared functions at the origin can yield only $A_1$, $A_2$ and $A_3$ singularities. Moreover the union of the $A_2$ and $A_3$ points in the normal plane is a smooth curve, with the $A_3$ points isolated on that curve.

**Corollary A.1.4** For almost all $\phi$ in the sense of Lebesgue measure the family of distance squared functions

$$D_\phi : U \times \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$$

has only $A_1$, $A_2$, $A_3$, $A_4$ and $D_4$ singularities, and these are versally unfolded by the family. Moreover we can replace the parametrising set $U$ by a smaller neighbourhood of the origin and ensure that we only have $A_1$, $A_2$ and $A_3$ singularities, which are all versally unfolded.

**Proof** We know (Proposition 6.2.4) that the distance squared function from the origin has an $A_3$ singularity at the origin. There is always the possibility that there may be a collection of $A_4$ and $D_4$ points having the origin as an accumulation point. Denote these points by $u_n$ with $z_n$ the corresponding centres of curvature. If the $z_n$ have some finite accumulation point say $z$ then since the origin only yields $A_1$, $A_2$ and $A_3$ points in the normal plane we obtain a contradiction. Otherwise we find that some subsequence of the $z_n$'s go off to infinity, and then the corresponding $z_n/||z_n||$ have a limit point say $a$ in the unit sphere. So the spheres centred at the $z_n$'s and through the origin have limit the plane through the origin orthogonal to $a$. So we deduce that in this case there is a plane through the origin having contact more degenerate than $A_4$ or $D_4$ with the surface at the origin. However this contradicts our findings on the contact of planes with the crosscap in Lemma 3.3.5. □

## A.2 Functions on the Crosscap

This section contains the results needed to prove Proposition 8.5.1. The proofs were supplied by J.W. Bruce.

Let $X,0$ denote the standard crosscap in $\mathbf{R}^3$, and let $\phi : \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ be the germ of a diffeomorphism taking $X$ to the geometric crosscap $\phi(X)$. Let $f : \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold $Z$ which we wish to apply to the geometric crosscap. Then we can consider the family

$$\begin{aligned} f_\phi : \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x,z) &\mapsto f(\phi(x),z) = f_{\phi z}. \end{aligned}$$

By studying this family of mappings on the standard crosscap we can describe the interaction between the geometric crosscap and the fibres of the family of functions $f_z : \mathbf{R}^3 \to \mathbf{R}$.

As usual we wish to prove a result concerning the generic cross-cap, so we need a suitably large family of cross-caps to consider. So we let $P_k$ denote the set of polynomial mappings $\mathbf{R}^3 \to \mathbf{R}^3$ of degree at most $k$, with no constant terms, so that the elements of $P_k$ preserve the origin. Let $V_k$ denote a neighbourhood of the identity which maps the unit ball at the origin diffeomorphically onto its image. Consider the family

$$\begin{aligned} F : \mathbf{R}^3 \times Z \times V_k &\to \mathbf{R}, \\ (u, z, p) &\mapsto f_z(p \circ \phi(u)). \end{aligned}$$

As before we shall use the coordinates $(x, y)$ in the parametrising space $\mathbf{R}^2$ and the coordinates $(u, v, w)$ in the ambient space $\mathbf{R}^3$.

**Proposition A.2.1** If the germs $f_z$ are all submersions then the jet-extension map

$$j_1^k F : Z \times V_k \to J^k(3, 1)$$

obtained by evaluating the $k$-jet of $F$ at $(0, 0, 0)$ is a submersion.

**Proof** Recall that $J^k(3, 1)$ is the vector space of polynomial functions with zero constant term of degree at most $k$. We shall see that we may as well suppose that the diffeomorphism $\phi$ is the identity. As in the proof of Theorem A.1.1 we consider for $p \in P_k$ the limit

$$\lim_{s \to 0} \frac{j_1^k(f_z(x + sp(x)) - f_z(x))}{s}$$

$$= j^k \left( \frac{\partial f_z}{\partial u_1} p_1 + \frac{\partial f_z}{\partial u_2} p_2 + \frac{\partial f_z}{\partial u_3} p_3 \right) (0).$$

As $f_z$ is a submersion then we obtain the result as one of the $\partial f_z / \partial u_i \neq 0$. □

We now wish to use the above result, and so we need to stratify the jet space. We do this using the above classification. In what follows we shall actually assume that each function in the family $f_z$ is a submersion. In fact we are really interested in the family of height functions, so the $f_z$ are indeed submersions and we have a 2-parameter family.

**Proposition A.2.2** We can stratify $J^3(3, 1)$ into strata consisting of the singular jets, submersions of type $UW_{k+1}$, $k = 0, 1, 2$; $WU_{k+1}$, $k = 1, 2$ and $VW_2$, and the orbits of $u$, $v$ and $w$.

Let $f : \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold $Z$. Suppose that the function $f_z$ is a submersion at the origin for each $z \in Z$ and the dimension of $Z$ is at most 2. We define the family of functions $f_\phi$ by

$$\begin{aligned} f_\phi : \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}. \end{aligned}$$

Then generically, the function $f_{\phi z}$ at the origin must be $\mathcal{R}(X)$-equivalent to a germ of one of the following types:

(i) $UW_{k+1}$, $k = 0, 1, 2$,

(ii) $WU_{k+1}$, $k = 1, 2$ and

(iii) $VW_2$.

Generically, the functions $f_{\phi z}$ are $\mathcal{R}(X)$-versally unfolded by the family $f_\phi$.

**Proof** The group $\mathcal{R}(X)$ has well defined jet groups $J^k\mathcal{R}(X)$. For the orbits we consider those of the $k$-jets we have classified. We are only interested in those of codimension at most 2 since there are no problems with moduli.

The relevant germs are $UW_{k+1}$, $k = 0, 1, 2$; $WU_{k+1}$, $k = 1, 2$ and $VW_2$. The $\mathcal{R}(X)$-codimension is $k$ in each case. Now stratify $J^3(3, 1)$ by the corresponding orbits and their complement, which consists of the singular germs (of codimension 3) and the orbits of $u$, $v$ and $w$. We now ask that the jet-extension map is transverse to these strata (it will be for a generic choice of polynomial mapping, in other words for a generic choice of cross-cap).

The result about the versality follows because of the infinitesimal criterion we gave before. Note that our jet-extension map does not have a copy of the source space $\mathbf{R}^3$ present as is usual in these situations. On the other hand the groups $\mathcal{R}(X)$ and $\mathcal{R}_e(X)$ coincide, so the transversality criterion for versality does not involve tangent vectors from the source space. □

# Appendix B

# Maple Program Implementing the Algorithm of Chapter 7

```
# maple file
# fullcalcm
  readlib(mtaylor);


  read ourtruncm;

  read remlinm;
  # removes linear terms from general unfolding of an A3

  read revm;
  # reduces unfolding F so that F(y,0)=y^4

  read changem;
  # change calculates the Njets of functions that
  #  take any unfolding of
  # an A3 singularity to the universal unfolding

  fullcalcm := proc(unf,M)
     local SQ,SP,SR,ST,SV,ans;

  # unf is any unfolding of an A3
  # M is the degree we wish to calculate inducing mappings to
```

```
  SP := remlinm(unf,M,{u,v,w});

  SQ := coeff(SP,x,0);

  SR := revm(SQ,M);
  ST:= changem(SR,M);
  SV:= op(1,ST);
  ans:=[SP,SQ,SR,ST,SV];
  ans;
  end;

#ourtruncm

# we wish to truncate a taylor series,TS, at degree M in the
# variables in set L.
# a degree i term (in L) will then have as coefft a taylor series
# in the variables in set X.
# We then truncate this series at degree 3*(M-i+1)-1.

ourtruncm := proc(TS,L,X,M)
   local i,g1,g,p,q,h,J;

   i:=0;
   g1:=expand(mtaylor(TS,L,M+1));
   J:=3*(M-i+1)-1;
   g:=expand(mtaylor(g1,X,J+1));
   p:=0;
   q:=0;
   h:=0;

   while i<=M do

    # find degree i part of TS
    p:= expand(mtaylor(g,L,i+1));
    # find coeffts of each term up to degree J=3*(M-i+1)-1
    J:=3*(M-i+1)-1;
    q:= expand(mtaylor(p,X,J+1));
    h:= h + q;
```

```
    g:= g-p;
    i:=i+1;
   od;

   h;
 end;
# remlinm

# removing linear terms in x from general unfolding poly
# with unfolding parameters L={u1,...,un} to degree M
# in unfolding parameters

remlinm := proc(poly,M,L)
   local  F,l,i,Ml,X,g,h;


   F:= ourtruncm(poly,L,y,M);
   F:= expand(F);
   F:= sort(F,x);
   X:=0;
   l:=0;
   while l<=M do
     F:=expand(F);
     F:=sort(F,x);
     g:=coeff(F,x,1);
     h:=expand(mtaylor(g,{u,v,w},l+1));
     h:=sort(h,y);
     i:=0;

     Ml:= 3*(M-l+1)-1;
     while i<=Ml do
        p:=expand(mtaylor(h,y,i+1));
        X:=x-(p/2);
        F:=subs(x=X,F);
F:= ourtruncm(F,L,y,M);
        F:=expand(F);
        F:=sort(F,x);
        g:=coeff(F,x,1);
        h:=expand(mtaylor(g,L,l+1));
```

```
        h:=sort(h,y);
        i:=i+1;
      od;
      l:=l+1;
    od;
    F;
    end;


# revm

# after removing terms in x from our unfolding of an A3
# we now use reversion to make sure that F(y,0)=y^4

revm:= proc(poly,M)
      local G,k,Y,F,H,g,h,f,a,f1,N;
with(powseries):
        G:= expand(mtaylor(poly,{u,v,w},1));
    g:=expand(G);
  h:=expand(g/y^4);
h:=powpoly(h,y);
f:=evalpow(h^(1/4));
a:=powpoly(y,y);
  f1:=multiply(a,f);
Y:=reversion(f1);
        N:=3*(M+1)-1;
        Y:=tpsform(Y,y,N);
Y:=convert(Y,polynom);



        F:=subs(y=Y,poly);
        H:=ourtruncm(F,{u,v,w},y,M);
        H;
    end;
# changem

# changem calculates the Mjets of the diffeomorphisms (c1,c2)
# which are such that F(a(y,u,v,w),u,v,w) = G(y,c1,c2)
# where F is any unflding of an A3, and G is the standard
```

```
# versal unfolding  of an A3 singularity.
# before executing changem, we must make sure that all
# linear terms in x have been removed(using remlin).

changem := proc(F,M)
   local a,c1,c2,H,G,p,J1,m,ans;


    a:=y;
    c1:=0;
    c2:=0;
    p:=0;
    G:=y^4 + c1*y +c2*y^2;
  while p<=M do
     H:= subs(y=a,F)-G;

     m:= (M-p+1)*3 - 1;
     J:= expand(mtaylor(H,{u,v,w},p+1));
     J1:= expand(mtaylor(J,{y},m+1));
     c1:=c1+ coeff(J1,y,1);
     c2:=c2+ coeff(J1,y,2);
     G:=y^4 + c1*y +c2*y^2;
     a:=a-((J1-(coeff(J1,y,1))*y -coeff((J1,y,2))*y^2)/4);
     p:=p+1;
    od;
   ans:=[G,a]
   ans;
end;
```

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