Multi–dimensional Motions of the Plane and Space

Thesis submitted in accordance
with the requirements of the
University of Liverpool
for the degree of
Doctor in Philosophy

by

Wendy Hawes

December 1994

Abstract

This thesis is an investigation into the types of singularities which can appear on trajectories of rigid motions of the plane and space. By a *rigid motion* of p–space we mean a mapping from some manifold M to the Lie group $SE(p)$. We consider general n–dimensional motions of the plane and 3–dimensional motions of space.

We obtain several classifications. The first is a complete list of local models for planar motions with 3 degrees of freedom. That is, a classification of map–germs $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$ of 'codimension' ≤ 5 under $\mathcal{A}$–equivalence. This classification is then extended to general n–dimensional motions, using a splitting lemma. We also consider the classification of map–germs $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ of 'codimension' ≤ 6 under $\mathcal{A}$–equivalence using the computer package TRANSVERSAL.

The geometry of the singularities $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$ and $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ is studied. Several geometrical invariants are calculated and the bifurcation curves are found.

For my parents.

A beginning, a muddle, and an end.
Philip Larkin (1922–85)

Acknowledgements

I owe a debt of thanks to my supervisor, Chris Gibson for proposing the problem; his encouragement and guidance over the past three years have been invaluable. I am also grateful to Dorle Gibson for her hospitality and friendship.

I would like to thank the members of the Department of Pure Mathematics, University of Liverpool. In particular I would like to mention Bill Bruce, Neil Kirk, Richard Morris, Farid Tari and the rest of the Singularities Group for many helpful conversations and suggestions. Special thanks to Farid, Janet West and Ian Nutt for proof-reading my thesis.

I would like to thank my family for their support over the years and for bringing me back down to reality whenever I needed it. I cannot forget all my friends in Liverpool who have kept my spirits up during my six years here, especially Joe Monaghan and all my fellow singers and musicians from the Music department for providing many welcome distractions. I am indebted to Rachel Norman for her continuing support (and squash matches!). Finally, my thanks to Ian for being there whenever I needed him.

I acknowledge the Science and Engineering Research Council (SERC) for the financial support given during the writing of this thesis.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 An Example

In mechanical design one mechanism which is commonly used is the *four–bar linkage*. An example of this can be seen in Fig. 1.1, [Hau].

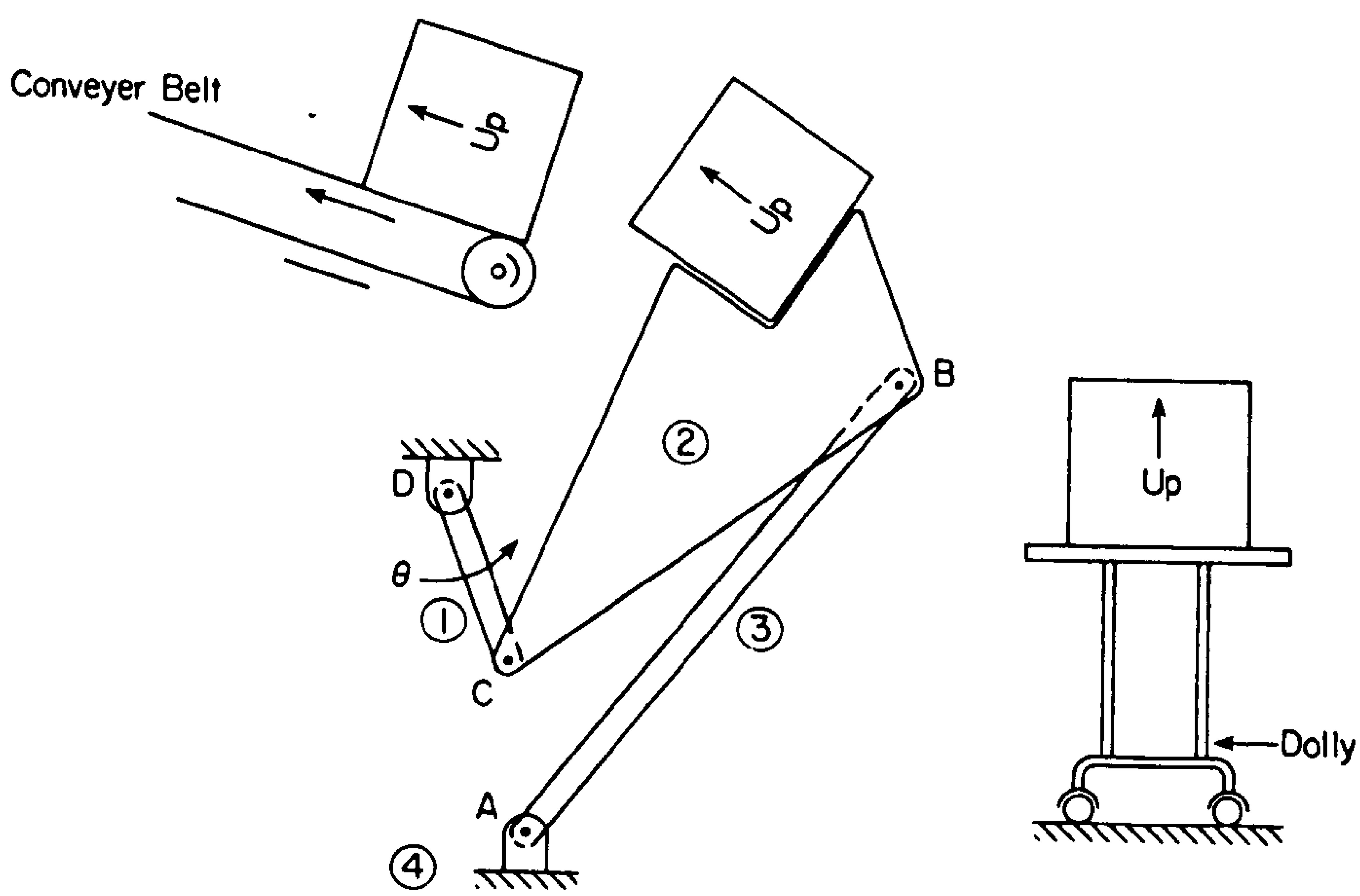


Figure 1.1: A material–handling mechanism

The purpose of this material–handling mechanism is to permit a counterclockwise rotation θ of the crank to lower the material–handling arm to a position that enables loading of cargo from a dolly. Subsequent clockwise rotation of the crank raises the cargo so that it can be transmitted to a conveyer belt. It is geometrically clear that the dimensions of the components have to be carefully chosen so that the mechanism is able to travel the required trajectory.

In order to realize the necessary dimensions of the components needed to follow a given trajectory we need to study the geometry of the *planar 4–bar*, see Fig 1.2.

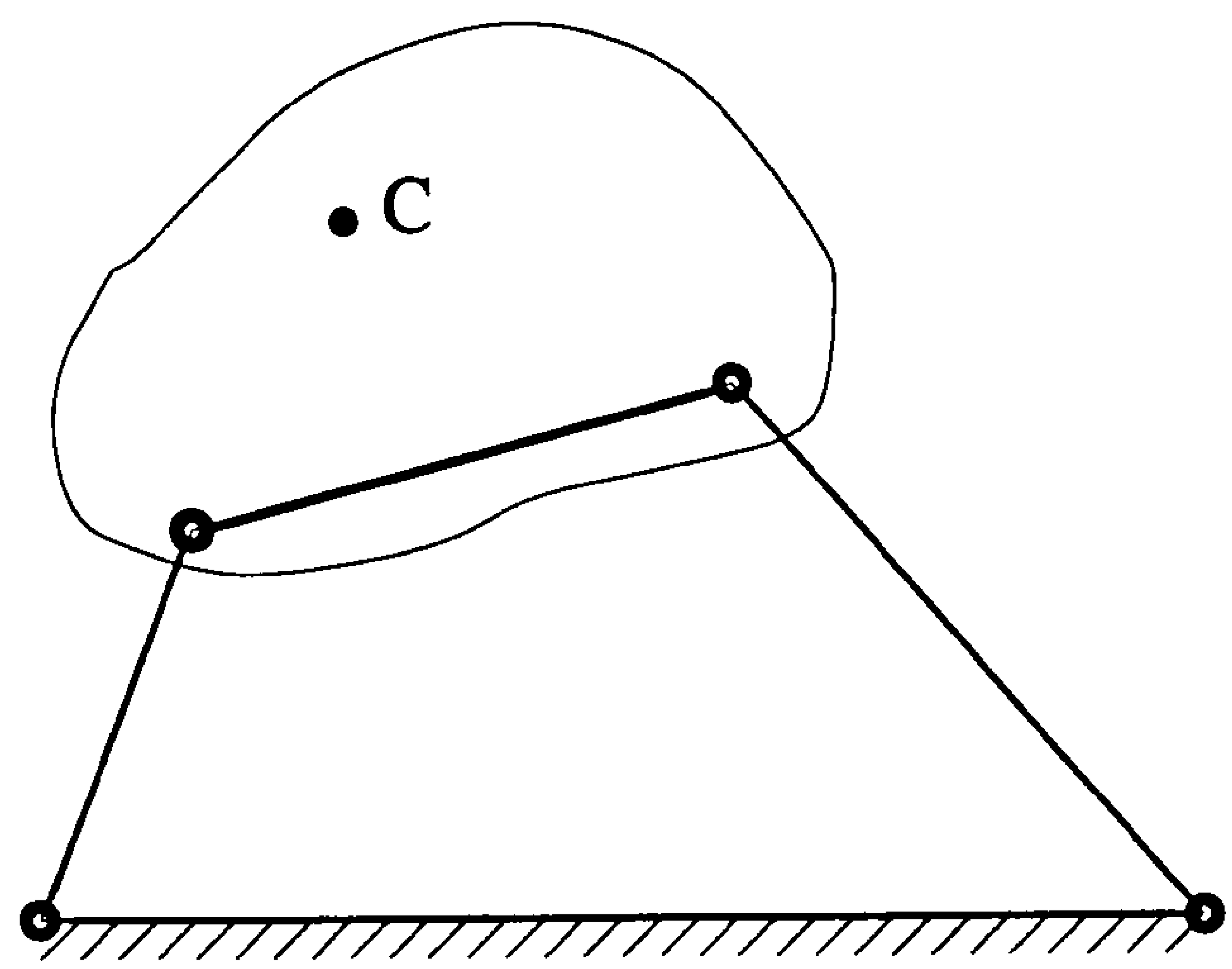


Figure 1.2: The planar 4–bar

One bar is fixed, allowing the others to move in the plane. The mechanism has one degree of freedom, i.e. the point C, known as the *coupler point* will trace out a curve, called the *coupler curve* in the plane, as the motion progresses. This coupler curve is generally of degree 6 and can be quite complex. An example of such a curve is shown in Fig 1.3. The geometry of such coupler curves is studied in [GN].

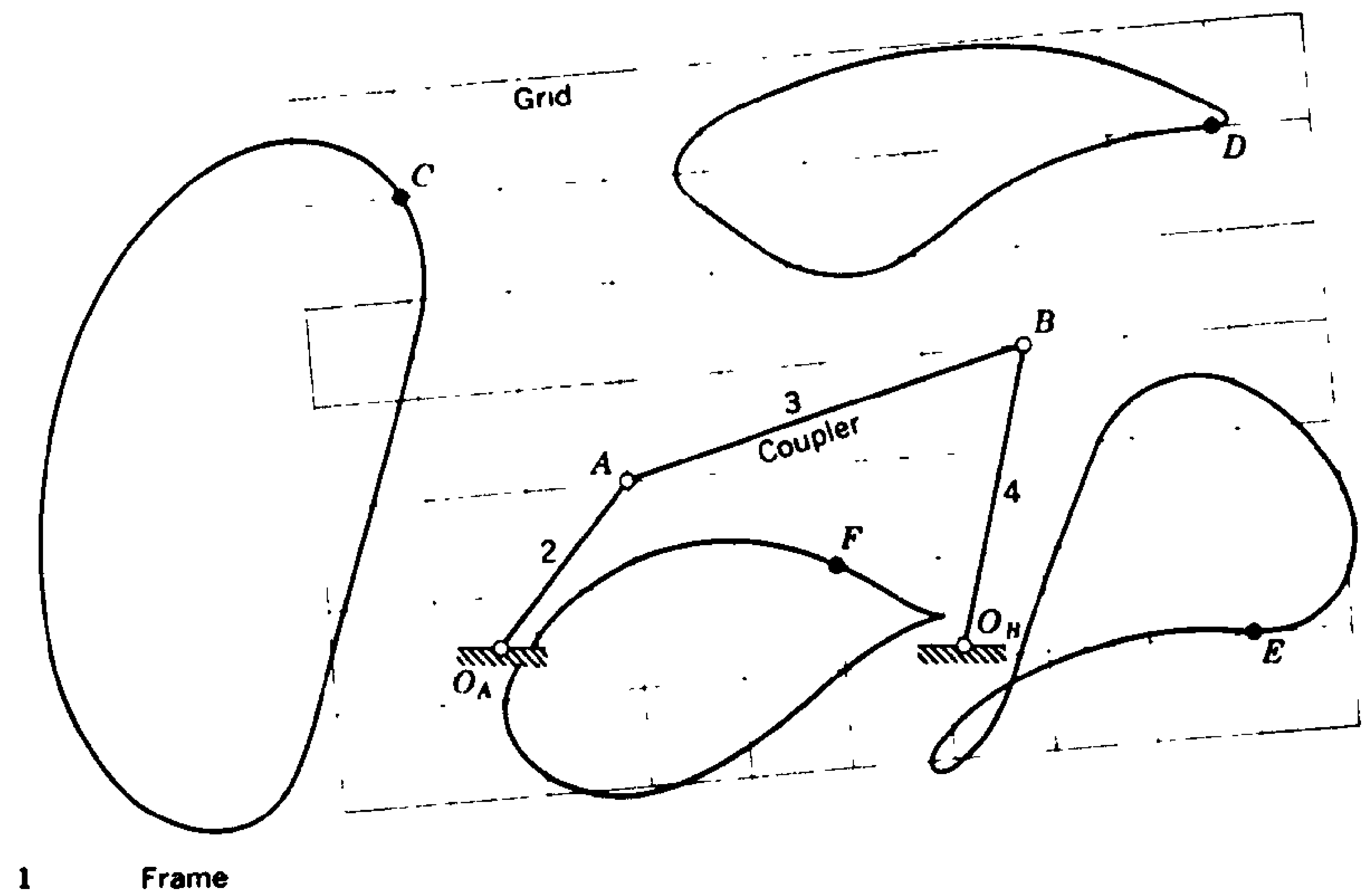


Figure 1.3: Four–bar coupler curves: The transparent grid is part of the coupler plane, link 3. The curves are traced on the plane of link 1.

The 4–bar linkage was first used by James Watt to obtain approximate rectilinear motion from rotations. Although more sophisticated mechanisms can be found nowadays, 4–bar linkages are still used for axle suspension in various high performance cars, [HD] and windscreen wipers, [Hau] and a number of other applications.

1.2 Generalities

We require a formal definition of a rigid motion of the plane. A movement of a rigid body in $\mathbb{R}^2$ can be achieved by an orthogonal transformation (i.e. a rigid rotation about the origin) followed by a translation, see Fig. 1.4. Such a rotation must be a proper one, preserving orientation. Thus the orthogonal transformation is represented by an 2×2 matrix belonging to $SO(2)$. A translation is given by a 2–vector. So a rigid body motion of the plane is given by a pair $(\mathbf{M}, \mathbf{v})$ where $\mathbf{M} \in SO(2)$ and $\mathbf{v} \in \mathbb{R}^2$. The set of all possible motions is denoted $SE(2)$. Formally, a motion of the plane can be thought of as a smooth curve in the Lie group $SE(2)$ of proper rigid motions of the plane. The one-parameter motion can be represented by a smooth mapping $\mu : \mathbb{R} \longrightarrow SE(2)$.

Now one-parameter motions of the plane are relatively well understood geometric objects, forming the core of classical kinematics, but little is known about multi-parameter motions of the plane and spatial motions. So in general, we consider $SE(p)$, the Lie group of proper rigid motions of $\mathbb{R}^p$ and the map

$$\begin{array}{rccc} \mu : & \mathbb{R}^n & \longrightarrow & SE(p) \\ & t & \longmapsto & \mu(t) \end{array}$$

where n is the number of degrees of freedom of the motion. Given $a \in \mathbb{R}^p$ we can apply $\mu(t)$ to it.

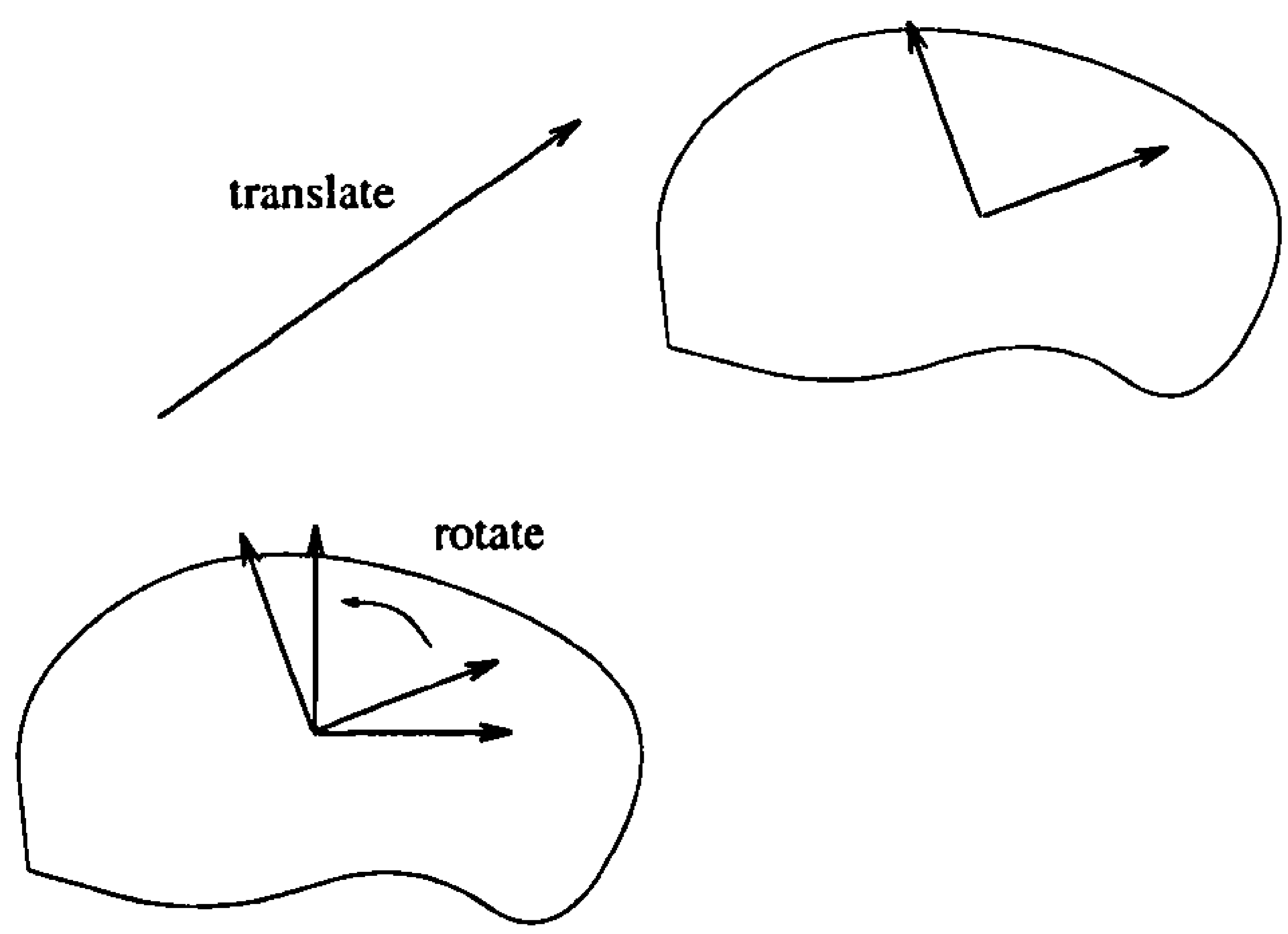
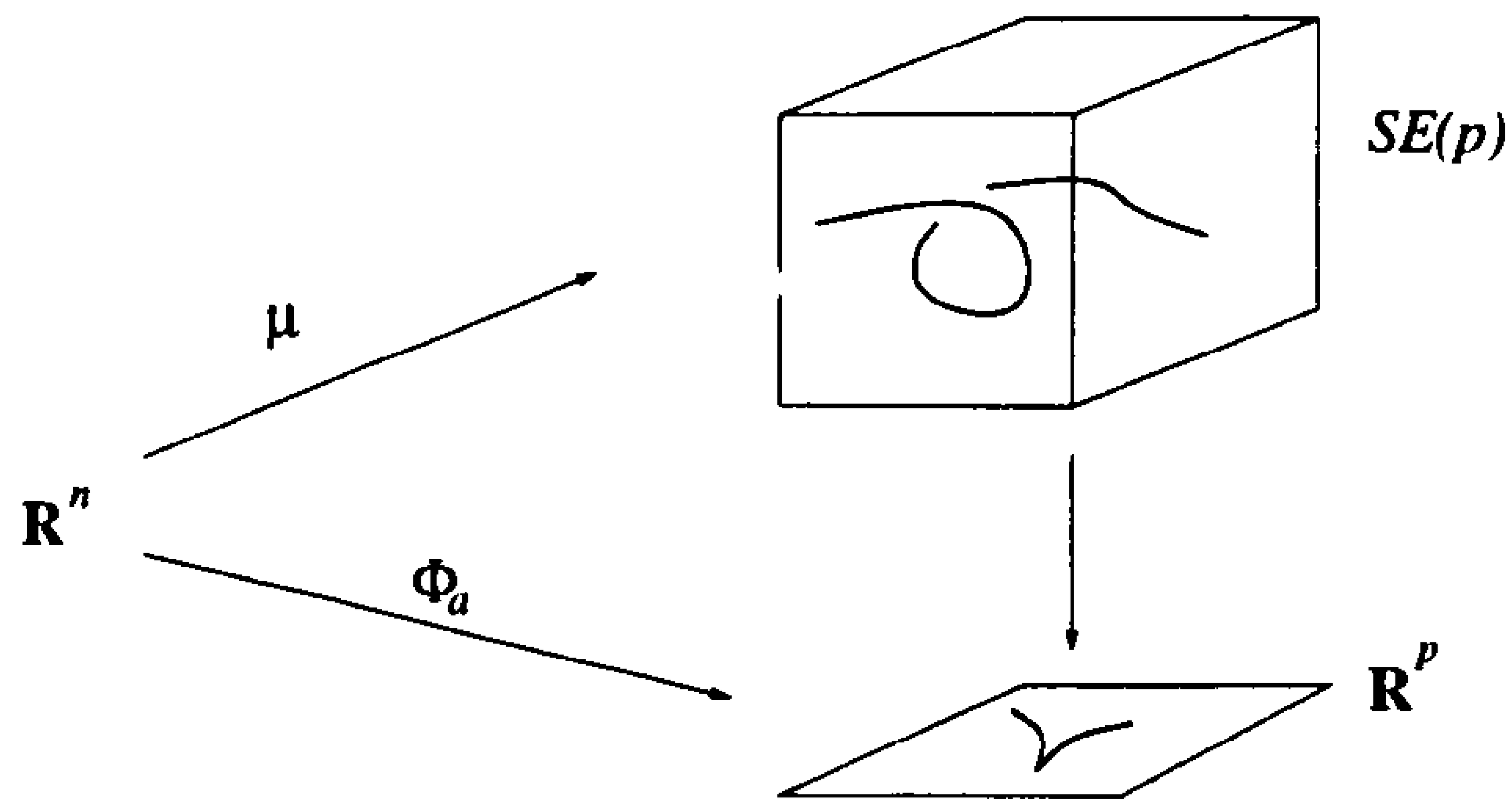


Figure 1.4: Motion of a rigid body in $\mathbb{R}^2$

One should visualize this as follows:



So

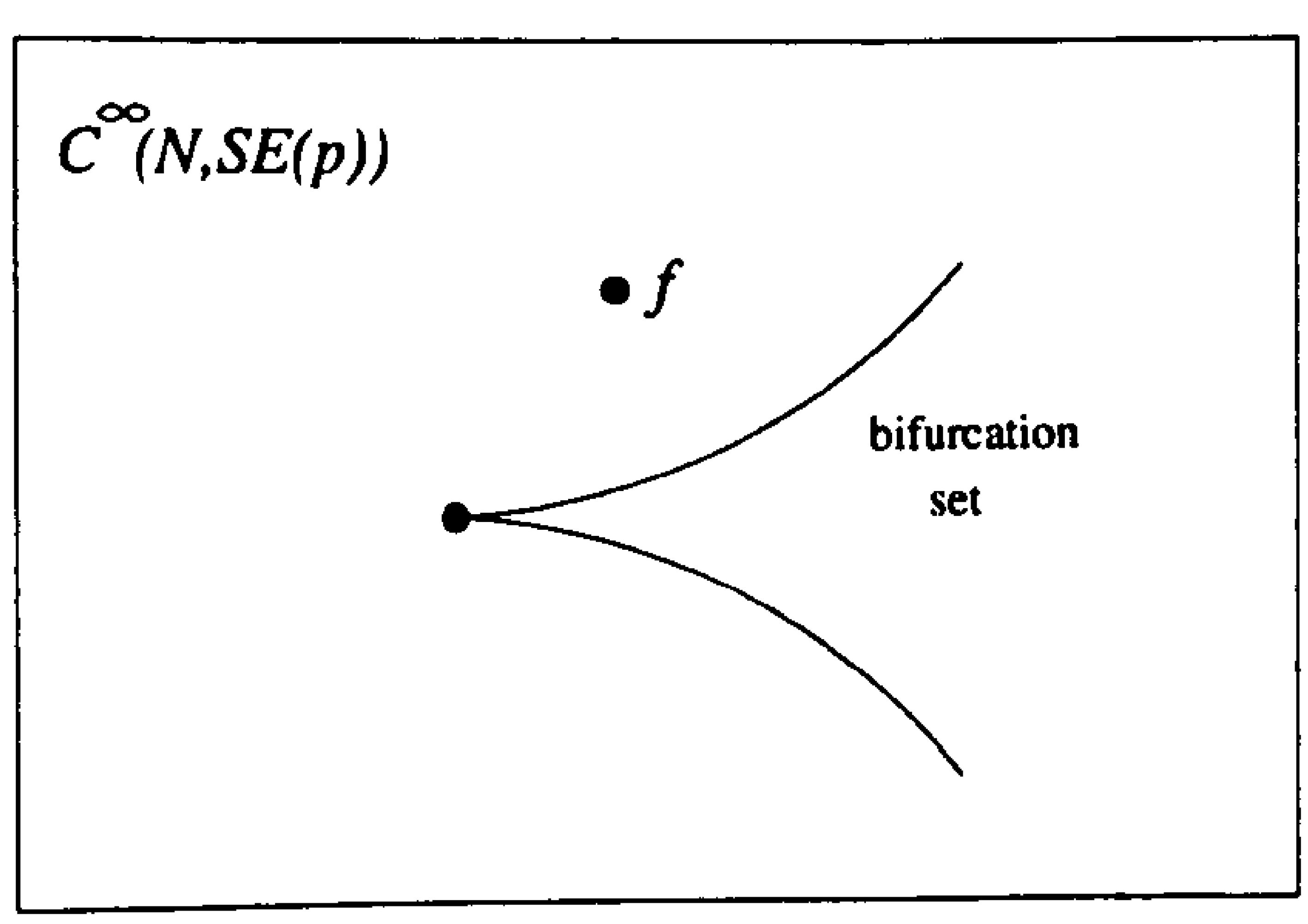
$$\begin{array}{rccc} \phi_a : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^p \\ & t & \longmapsto & \mu(t)(a). \end{array}$$

This map is the *trajectory.* So we have a p–parameter family of trajectories since a is a p–vector. There are two geometric objects to study:

A How does the image of the mapping sit in $SE(p)$?

B What singularities do these trajectories exhibit?

Now **A** has been studied by many authors (Grassman/Clifford/...), the first order information giving rise to the *screw theory* of engineering kinematics, but **B** is less understood as in principle the singularities can be of arbitrary complexity. We wish to describe the local structure and bifurcations for a "generic" motion. The idea of studying a "generic" motion extends from the work of René Thom. This idea can be pictured as follows:



If we take $f \in C^\infty(N, SE(p))$ where N is a smooth manifold of dimension n, away from some bifurcation set (i.e. away from special cases) we should be able to prove general results. In our case, this should lead to a finite list of singularity types. This work was first studied by Donelan [Do1, Do2]. He showed that the only possible local models of trajectories arising from general one–parameter motions of the plane are simple points, ordinary cusps and ramphoid cusps. Hobbs [Hob] extended these studies via the following results.

Definition 1.2.1 Let X and Y be smooth manifolds.

1. Denote by $C^\infty(X, Y)$ the set of smooth mappings from X to Y.
2. Fix a non–negative integer K. Let U be a subset of $J^k(X, Y)$. Then we denote by $M(U)$ the set
$$\left\{ f \in C^\infty(X,Y) \ : \ j^k f(x) \subset U \right\}.$$
3. The family of sets $\{M(U)\}$ where U is an open subset of $J^k(X, Y)$ form a basis for the Whitney C^k topology on $C^\infty(X, Y)$. Denote by W_k the set of open subsets of $C^\infty(X, Y)$ in the Whitney C^k topology.
4. The Whitney C^∞ topology on $C^\infty(X, Y)$ is the topology whose basis is
$$W = \bigcup_{k=1}^{\infty} W_k.$$

Now let $E(p)$ denote the Lie group of proper rigid motions of $\mathbb{R}^p$. By an *n–dimensional motion* of $\mathbb{R}^p$ we mean a smooth mapping $\mu : N \longrightarrow E(p)$, where N is a smooth manifold of dimension n. That yields a smooth mapping $\phi_\mu : N \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ defined by $(t, a) \longmapsto \mu(t)(a)$, and for a fixed choice of a this yields the smooth mapping $\phi_{\mu,a} : N \longrightarrow \mathbb{R}^p$ defined by $t \longmapsto \mu(t)(a)$, *i.e.* the trajectory of a under μ. So we can think of ϕ_μ as a p–parameter family of trajectories. Given positive integers r and k this induces a multijet extension

$${}_rj^k\phi_{\mu,a} : N^{(r)} \longrightarrow {}_rJ^k(N, \mathbb{R}^p)$$

and since $\phi_{\mu,a}$ depends smoothly on a this yields the mapping

$${}_rj^k\phi_\mu : N^{(r)} \times \mathbb{R}^p \longrightarrow {}_rJ^k(N, \mathbb{R}^p).$$

Theorem 1.2.2 *Let X be a smooth submanifold of ${}_rJ^k(N, \mathbb{R}^p)$, where N is a smooth manifold of dimension n. The set of n–dimensional motions $\mu : N \longrightarrow E(p)$ such that the multi-jet extension ${}_rj^k\phi_\mu$ is transverse to X is open and dense in $C^\infty(N, E(p))$, endowed with the Whitney topology.*

Proof. [Hob] □

The theorem tells us that if we find a smooth submanifold of ${}_rJ^k(N, \mathbb{R}^p)$ then generically the mapping ${}_rj^k\phi_\mu$, for a given μ, will be transverse to it. So if we stratify ${}_rJ^k(N, \mathbb{R}^p)$ into $\mathcal{A}$–orbits (which are necessarily smooth submanifolds) we get an induced stratification via $({}_rj^k\phi_\mu)^{-1}$. So to classify singularity types which occur on trajectories upto $\mathcal{A}$–equivalence, it is enough to stratify ${}_rJ^k(N, \mathbb{R}^p)$ into $\mathcal{A}$–orbits.

We need to look a little closer at the codimensions of the singularities which can appear on trajectories. Let N, P, Q be smooth manifolds having dimensions n, p, q respectively. By a q parameter *family of mappings* we mean a smooth mapping $F : Q \times N \longrightarrow P$, where we think of F as the family of mappings $f_z : N \longrightarrow P$ parametrized by the elements $z \in Q$. For each parameter z, and positive integers r and k, we have the multijet extension ${}_rj^kf_z : N^{(r)} \longrightarrow {}_rJ^k(N, P)$ giving a mapping

$${}_rj^kF : Q \times N^{(r)} \longrightarrow {}_rJ^k(N, P).$$

We have the following lemma.

Lemma 1.2.3 *Let X be an $\mathcal{A}$–invariant submanifold of ${}_rJ^k(n,p)$ giving rise to another $\mathcal{A}$–invariant submanifold Y in ${}_rJ^k(n,p)$. Assume that X has modality m, i.e. the codimension of any $\mathcal{A}$–orbit in X takes a constant value m. If ${}_rj^kF$ is transverse to Y, then for any $z \in Q$ with ${}_rj^kf_z \in Y$ the $\mathcal{A}_e$–codimension of f_z is $\leq q + m$.*

Proof. [Hob] □

So when considering motions of the plane we have $q = 2$ and therefore are looking for singularities with $\mathcal{A}_e$–codimension less than or equal to 2. For 1–parameter motions of the plane Hobbs obtained the following list.

Theorem 1.2.4 *On the trajectory of a generic motion of the plane with one degree of freedom we only see germs and multi-germs $\mathcal{A}$ equivalent to the following:*

Normal form	$\mathcal{A}_e$*-codim*
$(t, 0)$	*0*
(t^2, t^3)	*1*
(t^2, t^5)	*2*
$(t, 0; 0, s)$	*0*
$(t, 0; s, s^2)$	*1*
$(t, 0; s, s^3)$	*2*
$(t, 0; s^3, s^2)$	*2*
$(t, 0; 0, s; u, u)$	*1*
$(t, 0; 0, s; u, u^2)$	*2*
$(t, 0; 0, s; u, u; v, \lambda v)$	*3*

Table 1.1: Map–germs :$(\mathbf{R}, 0) \longrightarrow (\mathbf{R}^2, 0)$

1.3 Multi–parameter Motions of the Plane

If we move from classical kinematics, i.e. 1–parameter motions of the plane to multi–parameter motions the story is less clear. Even though multi–parameter motions are closer to the concern of the working kinematician, little is known about them. Whereas for 1–parameter motions we have trajectories represented by curves in $SE(2)$, for 2–parameter motions the trajectories are, in general, 2–dimensional subsets of the plane and the singular sets include the "boundaries" of these sets.

For 2–parameter motions of the plane we have the two simplest engineering examples of the *double 4–bar*, shown in Fig. 1.5 and the *planar 5–bar*, shown in Fig. 1.6.

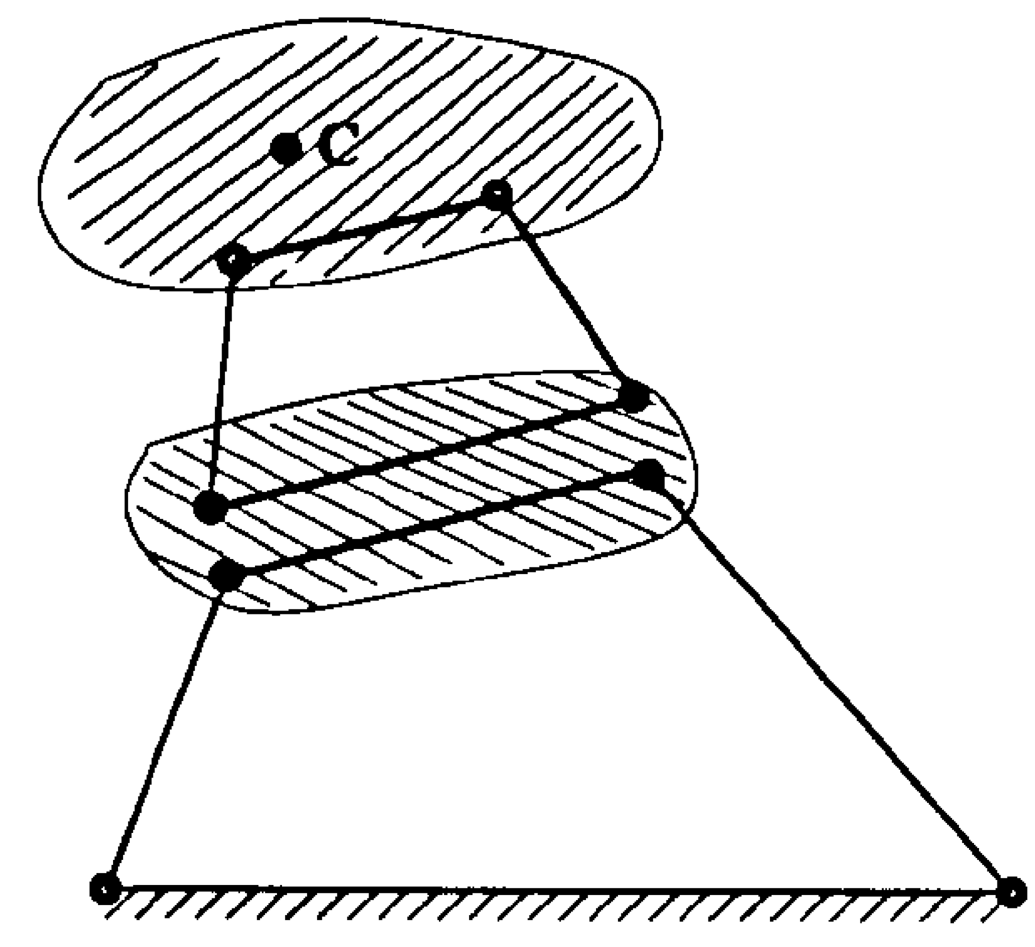


Figure 1.5: Double 4–bar linkage

A special case of the double 4–bar is used in engineering; the following appears in an unpublished article by Donelan.

A Remote Centre Compliance Device

Many constraints apply to practical robots. These include avoidance of obstacles and minimising forces on components and joints. Where the robot task imposes such constraints the robot is required to be *compliant.* This may be achieved by use of sensing devices and control of joints by servo–motors (active compliance), or alternatively the careful choice of robot geometry can frequently assist (passive compliance).

We describe a planar device designed to insert a shaft in a hole. This requires the shaft to be well aligned with the hole at the point of entry in order to avoid jamming, which would put undue stress on the components or damage them.

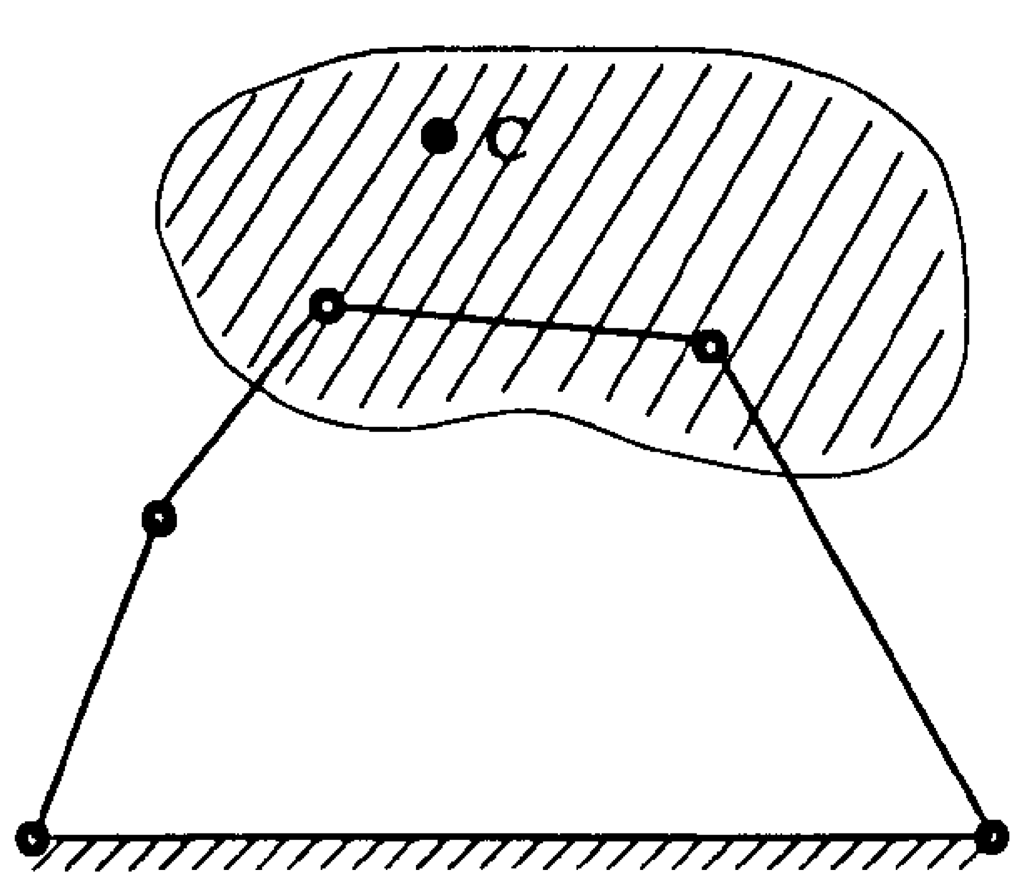


Figure 1.6: 5–bar linkage

The device is designed to give some tolerance on the exact point and the angle of entry, see Fig. 1.7.

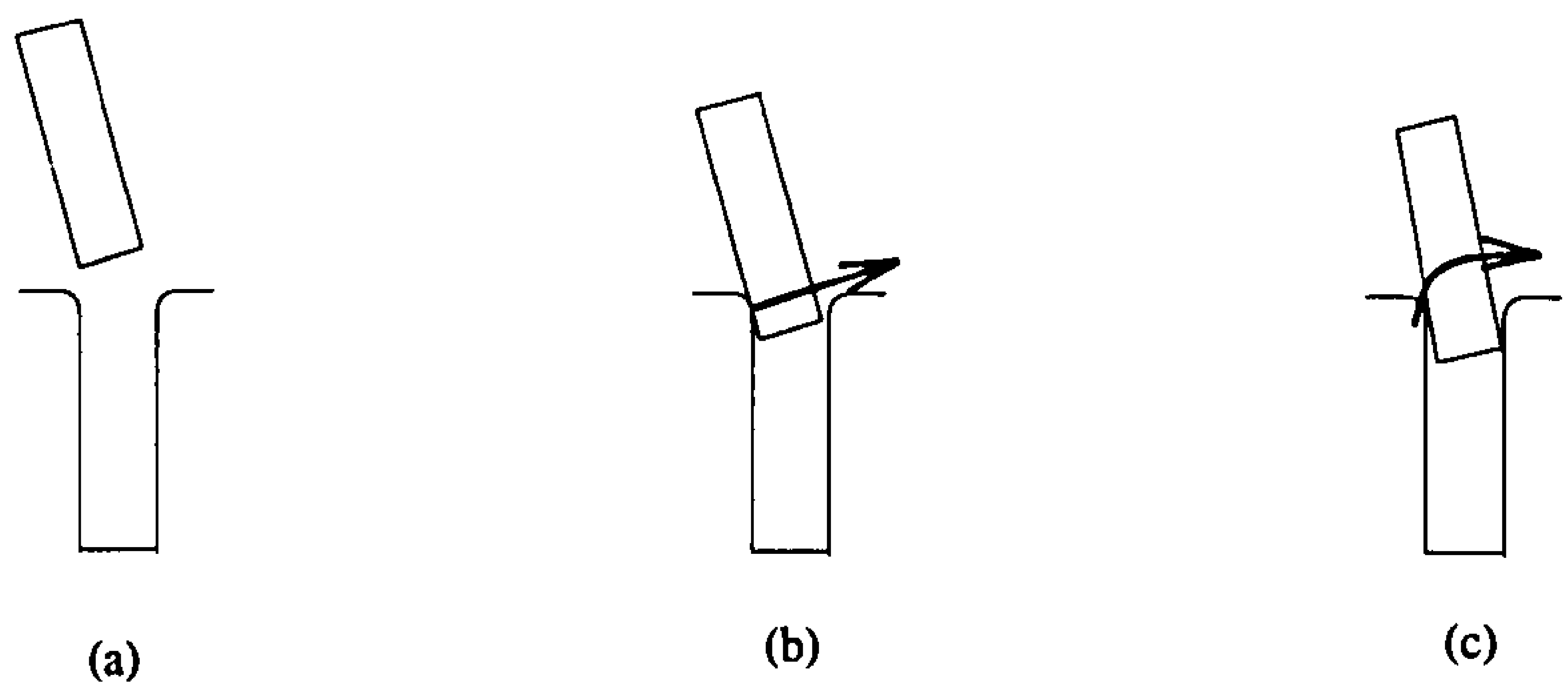


Figure 1.7: Insertion of peg in a hole

While a full analysis of the constraints that apply would require consideration of the forces that impinge on the peg as it enters the hole, it is the underlying geometry which determines its effectiveness. Observe that when one point of the peg contacts the hole there is a force on the peg perpendicular to it, Fig. 1.7(b). At this stage the robot should have the freedom to move laterally in the direction of the force.

A simple device with lateral motion is a 4–bar in the shape of a parallelogram, Fig. 1.8(a). However this would not in itself achieve alignment of the peg as a whole, simply appropriate placement of the tip. As the peg enters the hole it will soon achieve two–point contact and now there is a moment or turning force on the peg about its end–point, Fig. 1.7(c). The geometry of the robot will be *compliant*, that is to say, responsive to the task constraints, if the tip of the peg is designed to be at (or near) a centre of rotation of the component holding the arm.

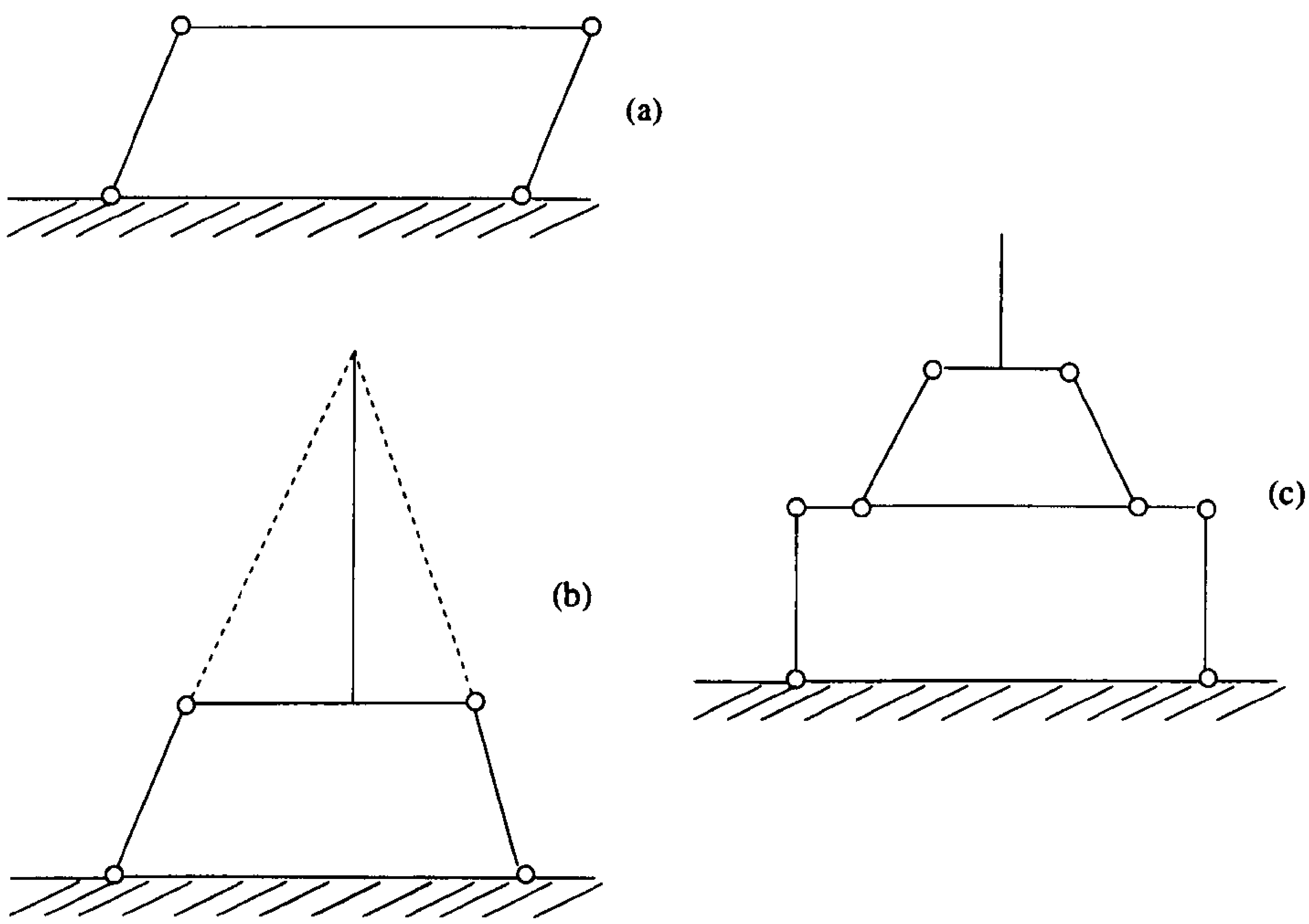


Figure 1.8: Components for the remote centre compliance device

This too can be achieved with the 4–bar, though this one is not a parallelogram, Fig. 1.8(b). A combination of two 4–bars provides a device which is compliant for both the one and two point contact, Fig. 1.8(c). To be effective this device must be mounted on an arm which provides motion towards the hole. This general type of device is known as a remote centre compliance device.

Hobbs, [Hob] classified both the monogerm and multigerm singularity types for 2–parameter motions of the plane and these are given by the following result.

Theorem 1.3.1 *For a generic motion of the plane with two degrees of freedom, any germ or multi-germ of a trajectory is $\mathcal{A}$-equivalent to one of the following normal forms.*

Type	*Normal Form*	*$\mathcal{A}$-codim*
1	(x, y)	*0*
2	(x, y^2)	*1*
3	$(x, xy + y^3)$	*2*
4_2	$(x, y^3 \pm x^2 y)$	*3*
5	$(x, xy + y^4)$	*3*
4_3	$(x, y^3 + x^3 y)$	*4*
6	$(x, xy + y^5 \pm y^7)$	*4*
11_5	$(x, xy^2 + y^4 + y^5)$	*4*
$\mathrm{I}_{2,2}^{1,1}$	$(x^2 + y^3, x^3 + y^2)$	*4*
$\mathrm{I}_{2,2}^{1}$	$(x^2 - y^2 + x^3, xy)$	*4*
	$(x, y^2; X^2, Y)$	*2*
	$(x, y^2; X, XY + Y^2)$	*3*
	$(x, y^2; X, XY + Y^3)$	*4*
	$(x, y^2; XY + X^3, Y)$	*3*
	$(x, y^2; X, Y^2 + X^3)$	*4*
	$(x, y^2; XY^2 \pm X^3, Y)$	*4*
	$(x, y^2; XY + X^4, Y)$	*4*
	$(x, xy + y^3; XY + X^3, Y)$	*4*
	$(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2)$	*3*
	$(x, y^2; X^2, Y; \tilde{x}, \tilde{x}\tilde{y} + \tilde{y}^2)$	*4*
	$(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{x}\tilde{y} + \tilde{y}^3)$	*4*
	$(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2; \check{X}, \lambda\check{X} + \check{Y}^2)$	5

Table 1.2: Map–germs :$(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$

where $\lambda \neq 0, 1$. □

1.4 Spatial Motions

For 3–parameter motions of the space we have *Manheim motion*, see Fig. 1.9.

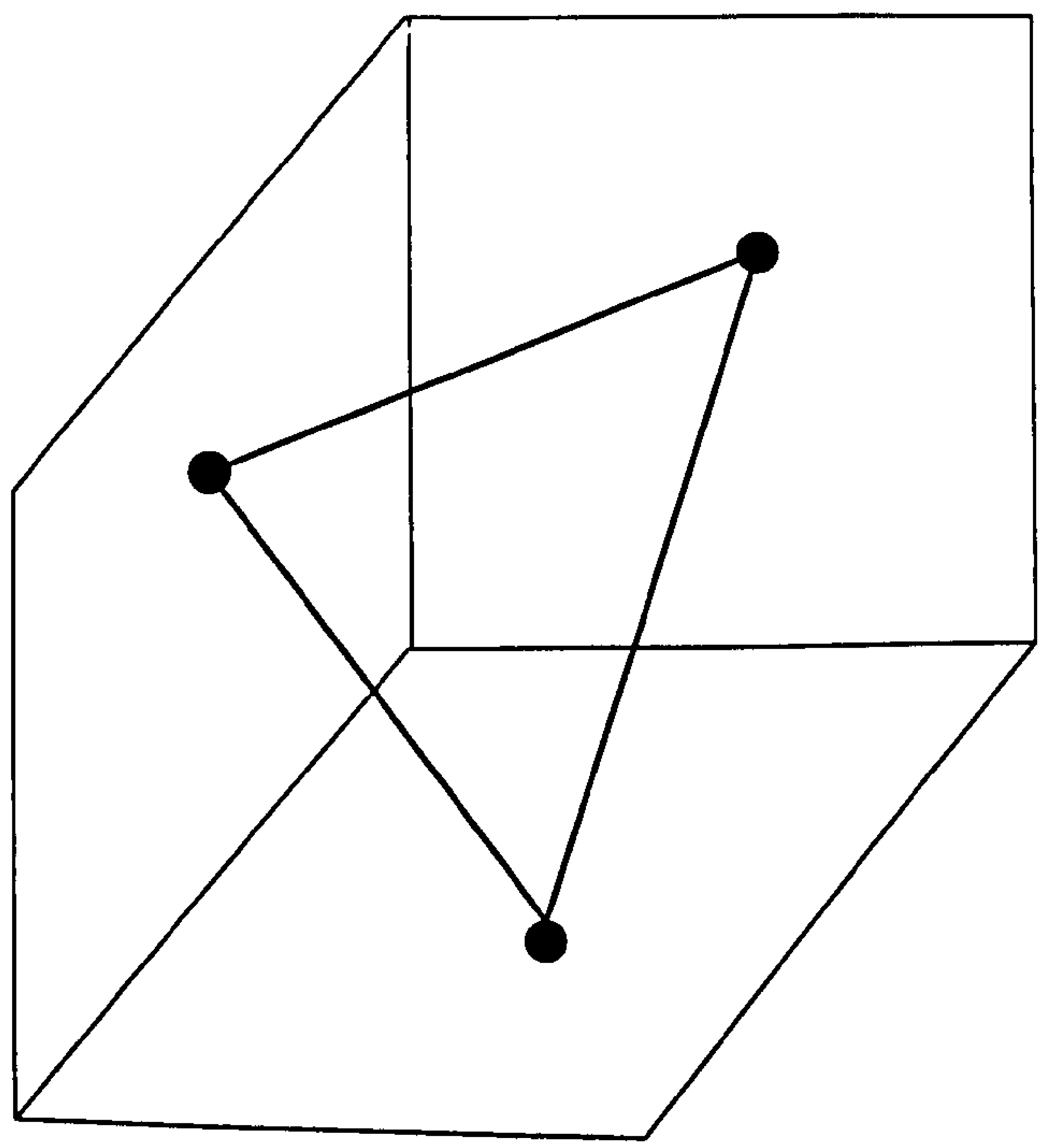


Figure 1.9: Manheim motion

If we take 3 points on a rigid body, and insist that they move on respective planes in 3–space then we have a 3–dimensional motion.

Spatial mechanisms are of great interest, although their motions are much harder to visualize than the motions of planar mechanisms. When we think of a robotic arm we generally think of the following figure, Fig. 1.10, [Hau].

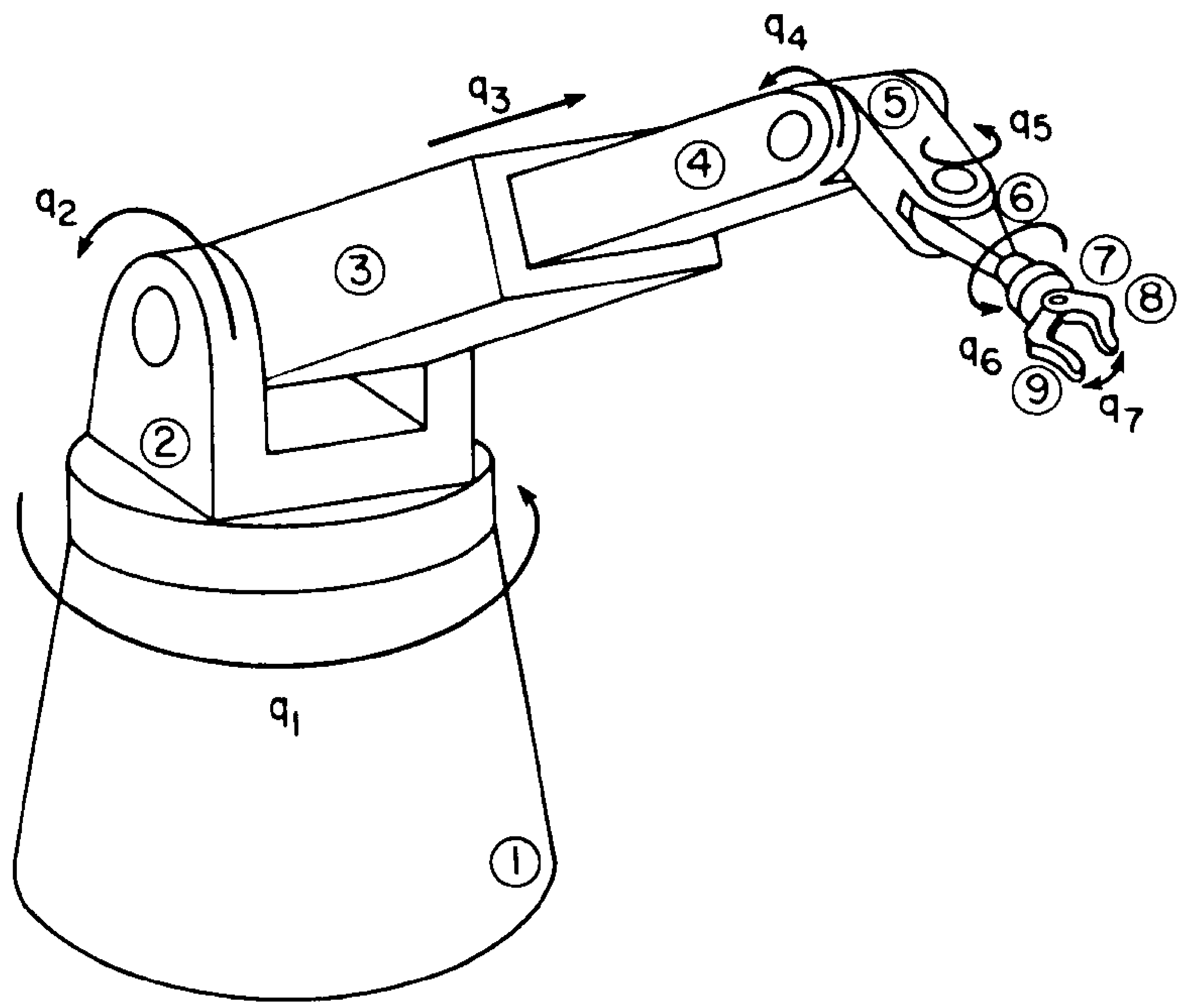


Figure 1.10: Robotic Arm

For complete versatility such an arm must have six degrees of freedom.

It is easier to obtain local models for general 1–parameter motions of space than those of the plane, since there is less opportunity for self–intersection and therefore fewer singularities. Hobbs obtained the following list, [Hob].

Theorem 1.4.1 *On the trajectory of a generic motion of space with one degree of freedom we see only multi–germs $\mathcal{A}$–equivalent to the following.*

Normal Form	$\mathcal{A}_e$*–codim*
$(t,0,0)$	*0*
$(t^2,t^3,0)$	*2*
$(t,0,0;0,s,0)$	*1*
$(t,0,0,;s,s^2,0)$	*3*
$(t,0,0;0,s,0;0,0,u)$	*2*

Table 1.3: Map–germs $:(\mathbf{R},0) \longrightarrow (\mathbf{R}^3,0)$

□

We also have the following local models for 2–parameter motions of space.

Theorem 1.4.2 *For a generic motion of space with two degrees of freedom, any multi-germ of a trajectory is $\mathcal{A}$-equivalent to one of those given in Table 1.4.*

Proof. We refer to [Hob] □

1.5 Bifurcations

For a family of smooth mappings a key role is provided by the *bifurcation set*, i.e. the set of parameters for which the corresponding mapping exhibits non-stable singularities. This bifurcation set represents the boundary between different types of kinematic behaviour, and is therefore of great interest in engineering robotics. By unfolding theory we have that any versal unfolding of a normal form is isomorphic to an unfolding given by varying the parameters of the motion. This means that the pictures given in the unfolding are diffeomorphic to the actual kinematic behaviour exhibited in the different regions of the bifurcation set.

Now the bifurcation curves for 1-parameter motions of $\mathbf{R}^2$ are classical and are the cusp transition, which in engineering is known as the moving centrode, the tacnode transition, which is known in engineering as the transition curve and the triplepoint transition curve. The bifurcations are unknown for motions of $\mathbf{R}^3$ and the bifurcations for ≥ 2 parameter motions are new to kinematics. To study these bifurcations we need to employ computer algebra packages, namely MAPLE, MACAULAY and SINGULAR, all of which were used on a UNIX mainframe. Also the classification of singularity types :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ was done using a MAPLE program called TRANSVERSAL written by N. P. Kirk. All the computer graphics were rendered using programs written by R. J. Morris and GEOMVIEW on a SILICON GRAPHICS INDIGO workstation. I am grateful to Neil Kirk, Richard Morris and Farid Tari for the technical support given in using these computer packages. I would also like to thank Dave Hodgkinson, Gerhard Pfister, Daniel Lazard, Barry Trager and James Davenport for their attention to problems in elimination theory which were uncovered by this work.

Name	*Normal Form*	*$\mathcal{A}_e$-codim*
S_0	(x, y^2, xy)	*0*
$S_1^\pm$	$(x, y^2, y^3 \pm x^2y)$	*1*
S_2	$(x, y^2, y^3 + x^3y)$	*2*
$S_3^\pm$	$(x, y^2, y^3 \pm x^3y)$	*3*
$B_2^\pm$	$(x, y^2, x^2y \pm y^5)$	*2*
$B_3^\pm$	$(x, y^2, x^2y \pm y^7)$	*3*
$C_3^\pm$	$(x, y^2, xy^3 \pm x^3y)$	*3*
H_2	$(x, y^3, xy + y^5)$	*2*
H_3	$(x, y^3, xy + y^8)$	*3*
P_3	$(x, xy + y^3, xy^2 + cy^4)$	*4*
	$(x, y, 0; 0, X, Y)$	*0*
$[A_1^\pm]$	$(x, y, 0; X, Y, X^2 \pm Y^2)$	*1*
$[A_2]$	$(x, y, 0; X, Y, X^2 + Y^3)$	*2*
$[A_3^\pm]$	$(x, y, 0; X, Y, X^2 \pm Y^4)$	*3*
	$(x, y, 0; Y^2, XY + Y^3, X)$	*1*
	$(x, y, 0; Y^2, XY + Y^5, X)$	*2*
	$(x, y, 0; Y^2, XY + Y^7, X)$	*3*
$[S_1^\pm]$	$(x, y, 0; Y^3 \pm X^2Y, Y^2, X)$	*2*
$[S_2]$	$(x, y, 0; Y^3 + X^3Y, Y^2, X)$	*3*
	$(x, y, 0; X, XY, Y^2 + X^3)$	*3*
	$(x, y, 0; X, Y^2, XY + Y^4)$	*3*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; 0, X, Y)$	*0*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, Y + X^2)$	*1*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, Y + X^3)$	*2*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, X^2 \pm Y^2)$	*2*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, Y + X^4)$	*3*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, XY + X^3)$	*3*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, X^2 + Y^3)$	*3*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; 0, X, Y; \breve{X}, \breve{Y}, \breve{X} + \breve{Y})$	*1*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; 0, X, Y; \breve{X}, \breve{Y}, \breve{X} + \lambda\breve{Y}; \mathbf{x}, \mathbf{y}, \mathbf{x} + \mu\mathbf{y})$	*2*
	$(x, y, 0; \tilde{x}, 0, \tilde{y}; 0, X, Y; \breve{X}, \breve{Y}, \breve{X} + \lambda\breve{Y}; \mathbf{x} + \mu\mathbf{y}, \mathbf{x}, \mathbf{y}; \tilde{\mathbf{x}}, \rho\tilde{\mathbf{x}} + \gamma\tilde{\mathbf{y}}, \tilde{\mathbf{y}})$	*2*

Table 1.4: Map–germs :$(\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$

1.6 Review of contents

In Chapter 2 we give a brief overview of the definitions and results of singularity theory which are used in the thesis. We introduce complete transversals, the main technical tool employed in the classifications of Chapters 3 & 6.

In Chapter 3 we consider the classification of map-germs :$(\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^2, 0)$ of $\mathcal{A}_e$-codimension ≤ 2, under $\mathcal{A}$-equivalence. This involves the proof of a splitting lemma from [RR] and includes a list of local models for 3-parameter motions of the plane. We give the calculations involved in finding the discriminants for all the $\mathcal{A}$-simple map-germs :$(\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^2, 0)$ listed in [RR]. We also list the unfoldings of the local models.

In Chapter 4 we study the geometry and bifurcations of planar motions. This includes analysis of the work of Hobbs, [Hob] on the sharksfin and deltoid singularities. The second section of the chapter involves calculating the bifurcation curves and obtaining computer graphics for the singularities listed in Chapter 3.

The work of Chapter 3 is extended to general n-parameter motions of the plane in Chapter 5.

In Chapter 6 we consider 3-parameter motions of space. This involves the work of Kirk, [K] on 'nilpotent' classification methods, introduced by Bruce & du Plessis and which appears in [BduPW], using the MAPLE program TRANSVERSAL, written by Kirk. The full classification is given in detail.

Finally we discuss the geometry of the singularities of Chapter 6 in Chapter 7. We calculate several geometric invariants for those singularities with a smooth critical set and give sketches of the bifurcations which occur.

(Part of the thesis appears in print or in preprints. The discussion of the bifurcations for 2-parameter motions of the plane, of Chapter 4, appears in [GHH], the results from Chapters 3 & 4 are included in [GH1] and those of Chapters 6 & 7 in [GH2].)

Chapter 2

Definitions

2.1 Basic Notation

We will consider smooth map–germs $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ (or germs of analytic maps $F : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^p, 0)$ as needed). Our notation is drawn from [Wal, BduPW, Mart, G].

Let $\mathcal{E}_n$ denote the

$$\mathbf{R}\text{– algebra of smooth function–germs } (\mathbf{R}^n, 0) \longrightarrow \mathbf{R}$$

and θ_n denote the

$$\mathbf{C}\text{– algebra of analytic function–germs } (\mathbf{C}^n, 0) \longrightarrow \mathbf{C}.$$

For each of these we denote the maximal ideal by $\mathcal{M}_n$.

Now the set of map–germs $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ is an $\mathcal{E}_n$–module and is denoted $\mathcal{E}(n, p)$ (similarly for the complex case).

2.2 Standard Mather Groups

There are 5 standard Mather groups: $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$.

$\mathcal{R}$ denotes the group of germs of diffeomorphisms $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$;
$\mathcal{L}$ denotes the group of germs of diffeomorphisms $(\mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$;

and $\mathcal{A}$ is defined as the direct product

$$\mathcal{A} = \mathcal{R} \times \mathcal{L}.$$

The actions of the above groups on $\mathcal{M}_n \cdot \mathcal{E}(n,p)$ are as follows:

$$\begin{array}{rcll} h \cdot f & = & f \circ h^{-1} & h \in \mathcal{R} \\ h' \cdot f & = & h' \circ f & h' \in \mathcal{L} \\ (h,h') \cdot f & = & h' \circ f \circ h^{-1} & (h,h') \in \mathcal{A}. \end{array}$$

($\mathcal{R}$ is often defined as the group of smooth changes of coordinates in the source, and $\mathcal{L}$ as the group of smooth changes of coordinates in the target.)

$\mathcal{C}$ denotes the group of germs of diffeomorphisms

$$(\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p, 0)$$

which project to the identity on $\mathbf{R}^n$ and leave the subspace $\mathbf{R}^n \times \{0\}$ fixed.

So if $H \in \mathcal{C}$ then

$$H(x,y) = (x, \tilde{H}(x,y))$$

where $\tilde{H} : (\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$ with $\tilde{H}(x,0) = 0 \quad \forall x \in \mathbf{R}^n$ close to the origin. The action of $\mathcal{C}$ on $\mathcal{M}_n \cdot \mathcal{E}(n,p)$ is defined by

$$(x, H \cdot f(x)) = H(x, f(x))$$

for $H \in \mathcal{C},\ f \in \mathcal{M}_n \cdot \mathcal{E}(n,p)$.

Finally, $\mathcal{K}$ denotes the group of germs of diffeomorphisms

$$(\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p, 0)$$

which can be written in the form

$$H(x,y) = (h(x), \Theta(x,y))$$

with $h : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ and $\Theta : (\mathbf{R}^n \times \mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$ with $\Theta(x,0) = 0 \; \forall x \in \mathbf{R}^n$ close to the origin.

The action of $\mathcal{K}$ on $\mathcal{M}_n \cdot \mathcal{E}(n,p)$ is defined by

$$(x, H \cdot f(x)) = H(h^{-1}(x), f(h^{-1}(x)))$$

for $H \in \mathcal{K}, \; f \in \mathcal{M}_n \cdot \mathcal{E}(n,p)$.

$\mathcal{K}$ is known as the *contact group.*

We will use the standard Mather groups to define the standard equivalences on $\mathcal{M}_n \cdot \mathcal{E}(n,p)$.

2.3 Tangent Spaces and Equivalence

We will be classifying singularities of smooth map–germs $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ up to $\mathcal{A}$–equivalence. By this we mean, f is $\mathcal{A}$–equivalent to g if there exists diffeomorphisms $h : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ and $k : (\mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^p, 0)$ for which the following diagram commutes.

$$\begin{array}{ccc} (\mathbf{R}^n, 0) & \xrightarrow{f} & (\mathbf{R}^p, 0) \\ \downarrow h & & \downarrow k \\ (\mathbf{R}^n, 0) & \xrightarrow{g} & (\mathbf{R}^p, 0) \end{array}$$

We then say that $f \sim g$.

We formally define $\mathcal{A}$ as

$$\mathcal{A} = \{\textit{Diffeomorphisms} : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)\} \times \{\textit{Diffeomorphisms} : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)\}$$

We can also define a subgroup $\mathcal{A}_1$ of $\mathcal{A}$ by

$$\mathcal{A}_1 = \{\textit{Diffeomorphisms} : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0) \textit{ whose 1 - jet is the identity}\}$$
$$\times\{\textit{Diffeomorphisms} : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0) \textit{ whose 1 - jet is the identity}\}$$

We write $j^k f$ for the k–jet of f. The vector space of such k-jets is denoted by $J^k(n,p)$. This can be identified with the $\mathbb{R}$–vector space of polynomial mappings $(\mathbf{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, whose components have degree $\leq k$. This vector space is acted upon smoothly by the Lie group $\mathcal{A}^k$ which is the set of k-jets of elements of $\mathcal{A}$. Two k-jets are said to be $\mathcal{A}$-equivalent if they lie in the same $\mathcal{A}^k$-orbit. The group $\mathcal{A}_1^k$ is similarly defined.

Using the usual notation,

$$\begin{aligned} \mathcal{E}_n &= \text{ring of function germs} : (\mathbf{R}^n, 0) \longrightarrow \mathbb{R} \\ \& \quad \mathcal{E}_p &= \text{ring of function germs} : (\mathbb{R}^p, 0) \longrightarrow \mathbb{R} \end{aligned}$$

with $\mathcal{M}_n$, $\mathcal{M}_p$ denoting their corresponding maximal ideals. Then let $\mathcal{O}_f$ denote the $\mathcal{E}_n$–module of germs of C^∞–vector fields over f and define $\mathcal{O}_n = \mathcal{O}_{(1_{\mathbf{R}^n},0)}$ and $\mathcal{O}_p = \mathcal{O}_{(1_{\mathbf{R}^p},0)}$. Now we define the following homomorphisms:

$$\begin{array}{rcccl} tf & : & \mathcal{O}_n & \longrightarrow & \mathcal{O}_f \\ & & \phi & \longmapsto & df.\phi \end{array}$$

$$\begin{array}{rcccl} wf & : & \mathcal{O}_p & \longrightarrow & \mathcal{O}_f \\ & & \psi & \longmapsto & \psi \circ f \end{array}$$

(where df if the differential of f).

Then the tangent space of the $\mathcal{A}$-orbit of f is given by

$$T\mathcal{A} \cdot f = tf(\mathcal{M}_n \cdot \mathcal{O}_n) + wf(\mathcal{M}_p \cdot \mathcal{O}_p)$$

and to the $\mathcal{A}_1$-orbit

$$T\mathcal{A}_1 \cdot f = tf(\mathcal{M}_n^2 \cdot \mathcal{O}_n) + wf(\mathcal{M}_p^2 \cdot \mathcal{O}_p).$$

We also need the 'extended' tangent space

$$T\mathcal{A}_e \cdot f = tf(\mathcal{O}_n) + wf(\mathcal{O}_p).$$

So the tangent spaces, given in terms of the Jacobian ideal of f, $\left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle$ are:

$$T\mathcal{A} \cdot f = \mathcal{M}_n \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle + f^* \mathcal{M}_p\{e_1, ..., e_p\}.$$

Similarly,

$$T\mathcal{A}_1 \cdot f = \mathcal{M}_n^2 \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle + f^* \mathcal{M}_p^2\{e_1, ..., e_p\},$$

and

$$T\mathcal{A}_e \cdot f = \mathcal{E}_n \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle + f^* \mathcal{E}_p\{e_1, ..., e_p\}.$$

2.4 Multigerms

We will be mostly concerned with classifying mono–germ singularities, although in analysing the unfoldings of these singularities we will need to consider *multi–germ* singularities. By a *multigerm* of multiplicity m we mean m monogerms with distinct sources and a *common* target. For $m = 1, 2, 3, ..$ we refer to monogerms, bigerms, trigerms, and so on. The concept of a multigerm allows us to discuss the local behaviour which arises from several distinct points on a parameter surface.

To calculate the tangent spaces for multigerms we introduce the following notation for a multigerm. Given a m–germ $f : (\mathbf{R}^n, S) \longrightarrow (\mathbf{R}^p, 0)$ where $S = \{s_1, ..., s_m\}$, we take local coordinates at each s_i. When we act on f with $\mathcal{A}$ we can change coordinates, via diffeomorphism, independently in the source around each s_i, but in the target the same vector field applies to each set of coordinates. So if we have a multigerm with coordinates $(x_1, ..., x_n; X_1, ..., X_n; ...; \tilde{x_1}, ..., \tilde{x_n})$ we calculate the tangent space as follows:

$$
\begin{aligned}
T\mathcal{A}\cdot f &= \mathcal{M}_n\left\langle \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right\rangle + \mathcal{M}_n\left\langle \frac{\partial f}{\partial X_1},\dots,\frac{\partial f}{\partial X_n}\right\rangle + \dots \\
&+ \mathcal{M}_n\left\langle \frac{\partial f}{\partial \bar{x}_1},\dots,\frac{\partial f}{\partial \bar{x}_n}\right\rangle + f^*\mathcal{M}_p\{e_1,\dots,e_p\}
\end{aligned}
.
$$

Similarly for $T\mathcal{A}_1 \cdot f$ and $T\mathcal{A}_e \cdot f$.

As in the case of monogerms we write ${}_mj^k f$ for the k–jet of the m–germ f.

2.5 Determinacy

One of the main ideas within Singularity Theory is to replace the space of map–germs $\mathcal{M}_n \cdot \mathcal{E}(n,p)$ with the space of k-jets $J^k(n,p) = \mathcal{M}_n \cdot \mathcal{E}(n,p)/\mathcal{M}_n^{k+1} \cdot \mathcal{E}(n,p)$, for some k (discussed above). This then allows us to work with a finite dimensional vector space.

We say that f is k–$\mathcal{A}$–determined if any map–germ g with $j^k f = j^k g$ is $\mathcal{A}$–equivalent to f. Once we know a map–germ is k–determined for some k, it is sufficient to work with the k–jet space to classify the $\mathcal{A}$–orbits. The *degree of determinacy* of f is the smallest value of k for f.

For monogerms we have the following results from [BduPW].

Theorem 2.5.1 *Let C be a finitely generated $\mathcal{E}_n$–module, $B \subset C$ a finitely generated $\mathcal{E}_n$–submodule, $A \subset f^*(\mathcal{M}_p) \cdot C$ a finitely generated $\mathcal{E}_p$–submodule (via f^*) and M a proper, finitely generated ideal in $\mathcal{E}_n$ such that for $x \in M$, $1 + x$ is a unit. If*

$$M \cdot C \subset A + B + M \cdot (f^*(\mathcal{M}_p) + M) \cdot C$$

then

$$M \cdot C \subset A + B.$$

Corollary 2.5.2 *A C^r map–germ f is k–$\mathcal{A}_1$–determined if and only if*

$$\mathcal{M}_n^{k+1} \cdot \mathcal{E}_p \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_n^{k+1} \cdot (f^*\mathcal{M}_p \cdot \mathcal{E}_n + \mathcal{M}_n^{k+1}) \cdot \mathcal{E}_n.$$

2.6 Codimension and Unfoldings

In order to distinguish between singularities of map–germs we need invariants. One such invariant is the *codimension* of a map–germ.

Definition 2.6.1 *1. The $\mathcal{A}$-codimension of a map-germ f is given by*

$$\mathcal{A} - codim(f) = dim_{\mathbf{R}} \frac{\mathcal{M}_n \mathcal{O}_f}{T\mathcal{A} \cdot f}$$

2. The $\mathcal{A}_e$–codimension of a map-germ f is given by

$$\mathcal{A}_e - codim(f) = dim_{\mathbf{R}} \frac{\mathcal{E}_n \mathcal{O}_f}{T\mathcal{A}_e \cdot f}$$

In most cases it is more useful to have the $\mathcal{A}_e$–codimension but simpler to calculate the $\mathcal{A}$–codimension. We have the following result which relates the two:

Theorem 2.6.2 *Given a non–stable map–germ $f : (\mathbb{R}^n, S) \longrightarrow (\mathbb{R}^p, 0)$ where $S = (s_1, ..., s_m)$ we have*

$$\mathcal{A}_e - codim(f) = \mathcal{A} - codim(f) + m(p - n) - p.$$

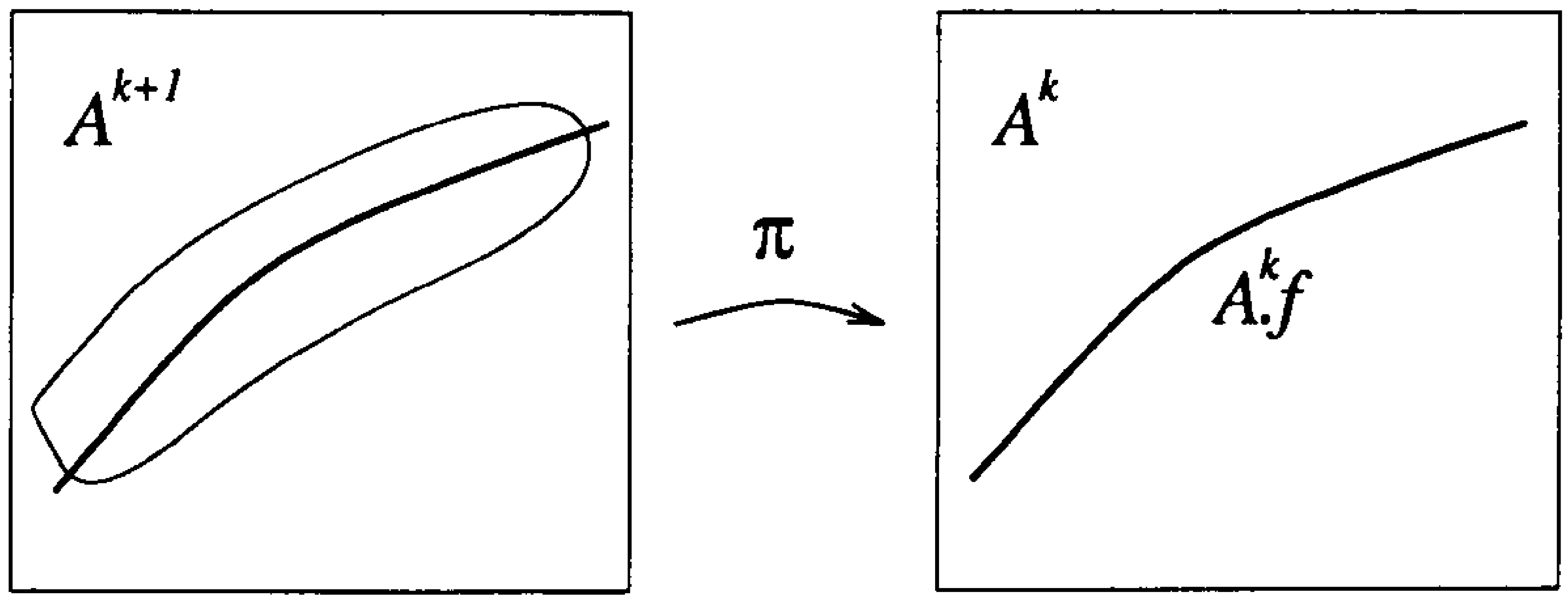
Proof. The case m=1 is noted in Wall [Wal].

A more formal proof can be found in Wilson, [Wi].

□

Lemma 2.6.3 *Suppose f is $\mathcal{A}$–finite and that the codimension of the $\mathcal{A}^k$–orbit of $f = n$, for some k. Then the $\mathcal{A}$–codimension of $f \geq n$.*

Proof. We have the following mental picture



where π is a projection and therefore a submersion. Now the dimension of the $\mathcal{A}^{k+1}$–orbit is $\leq$ the dimension of the $\mathcal{A}^k$–orbit and therefore the codimension is $\geq$ the codimension of $\mathcal{A}^k$–orbit. Now f is $\mathcal{A}$–finite so f is r–$\mathcal{A}$–determined for some r and

$$\mathcal{A}\text{–codimension of } f = \text{codimension } \mathcal{A}^r\text{–orbit} \geq \text{codimension } \mathcal{A}^k\text{–orbit of } f.$$

□

Now, an r–parameter *unfolding* of a map–germ $f_0 \in \mathcal{M}_n \cdot \mathcal{E}(n,p)$ is a map–germ

$$\begin{array}{rrcl} F: & (\mathbf{R}^n \times \mathbf{R}^r, 0) & \longrightarrow & (\mathbf{R}^p \times \mathbf{R}^r, 0) \\ & (x,u) & \longmapsto & (f(x,u),u) \end{array}$$

such that $f_0(x) = F(x,0)$. $f_u = F(x,u)$ can be thought of as a deformation of f_0, parametrized by $u \in \mathbf{R}^r$.

Two unfoldings $F, G : (\mathbf{R}^n \times \mathbf{R}^r, 0) \longrightarrow (\mathbf{R}^p \times \mathbf{R}^r, 0)$ of f_0 are *isomorphic* if there exist germs of diffeomorphisms

$$\phi : (\mathbf{R}^n \times \mathbf{R}^r, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^r, 0),$$

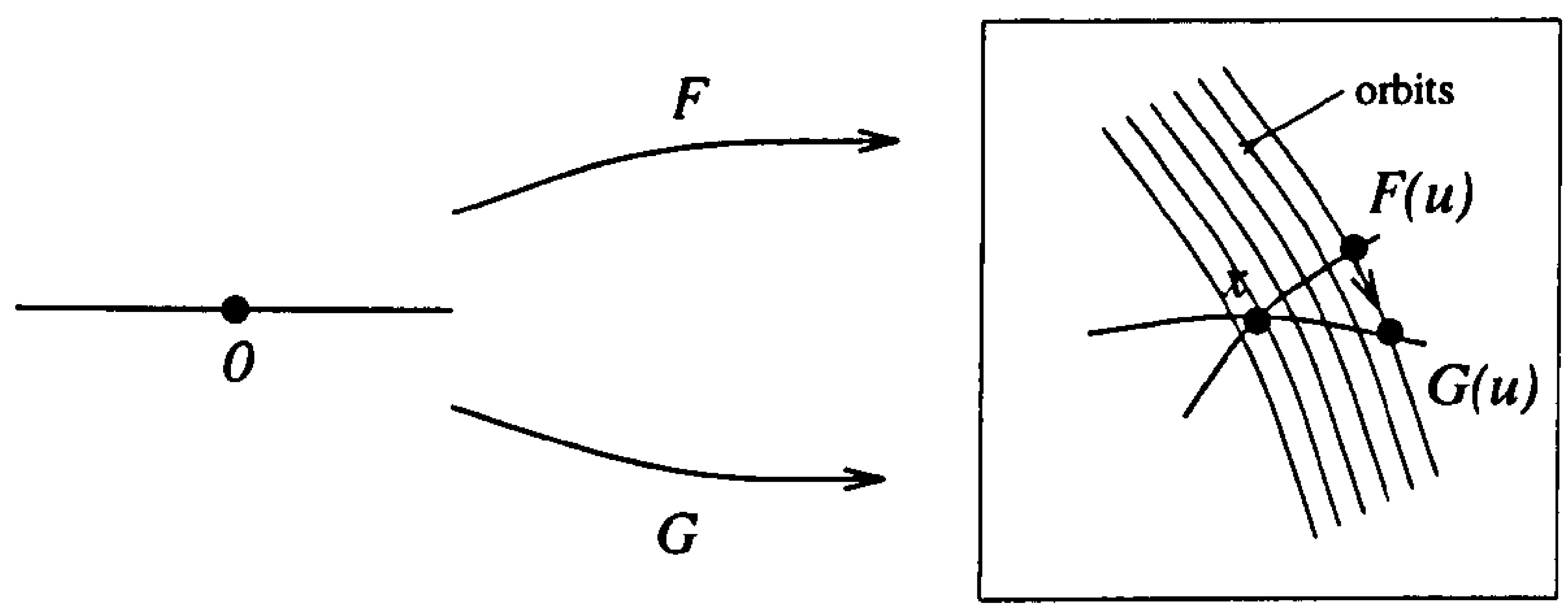
$$\psi : (\mathbf{R}^p \times \mathbf{R}^r, 0) \longrightarrow (\mathbf{R}^p \times \mathbf{R}^r, 0),$$

which are r–parameter unfoldings of the identity maps on $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, and $G = \psi \circ F \circ \phi^{-1}$.

Given $h : (\mathbf{R}^t, 0) \longrightarrow (\mathbf{R}^t, 0)$ we define the *pull-back* of F by h, $h^* F$ to be the t–parameter unfolding

$$(h^* F)(x, u) = (f(x, h(u)), u).$$

F, G are said to be *equivalent* if there exists a diffeomorphism $h : (\mathbf{R}^r, 0) \longrightarrow (\mathbf{R}^r, 0)$ such that G is isomorphic to $h^* F$. Pictorially, we get from F to G by sliding smoothly down the orbits.



If G is now some t–parameter unfolding of f_0 (t not necessarily equal to r), we say G is *induced* from F if there exists a smooth map–germ $h : (\mathbf{R}^t, 0) \longrightarrow (\mathbf{R}^r, 0)$ such that G is isomorphic to $h^* F$.

Definition 2.6.4 1. F is *versal* if every unfolding of f_o is induced from F.

2. F is *trivial* if it is isomorphic to the constant unfolding $(x, u) \longmapsto (f_0(x), u)$.

3. F is *stable* if all unfoldings of f_0 are trivial.

2.7 Complete Transversals

The main method used in the classifications is the method of *complete transversals*. This was first developed by Gibson & Dimca, [DG] for contact equivalence. Bruce & du Plessis, [BduP] have subsequently shown that the same method works for $\mathcal{A}_1$–equivalence.

We start with a result due to Mather.

Lemma 2.7.1 (Mather Lemma)

Let G be a Lie group acting smoothly on a finite dimensional manifold V. Let X be a connected submanifold of V. Then X is contained in a single orbit of G if and only if:

1. *for each $x \in X$, $T_xX \subseteq T_x(G \cdot x) = LG \cdot x$;*
2. *dim $T_x(G \cdot x)$ is constant $\forall x \in X$.*

Proof. We refer to [MaIV] □

The basic complete tranversal theorem is a corollary of the Mather Lemma.

Theorem 2.7.2 *Let G be a Lie group acting smoothly on an affine space A, and let W be a subspace of A with*

$$LG \cdot (x + w) = LG \cdot x,$$

$\forall x \in A$ and $\forall w \in W$. Then

1. *$\forall x \in A$ we have*

$$x + \{LG \cdot x \cap W\} \subset G \cdot x \cap \{x + W\};$$

2. *If $x_0 \in A$ and T is a vector subspace of W satisfying*

$$W \subset T + LG \cdot x_0$$

then for any $w \in W$ there exists $g \in G$ and $t \in T$ such that

$$g \cdot (x_0 + w) = x_0 + t.$$

Proof. We refer to [BduP] □

T is referred to as a *complete transversal.*

Corollary 2.7.3 *Let $\mathcal{G}$ be one of the standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$, and consider $\mathcal{G}_1$. Let $f \in \mathcal{M}_n \cdot \mathcal{E}(n,p)$ and let $T \subset H^{k+1}$ a vector subspace of H^{k+1} such that*

$$H^{k+1} \subset L(J^{k+1}\mathcal{G}_1) \cdot j^{k+1}f + T.$$

Then for every $(k+1)$-jet $j^{k+1}g$ ($g \in \mathcal{M}_n \cdot \mathcal{E}(n,p)$) with $j^k g = j^k f$ that $j^{k+1}g$ belongs to the same $\mathcal{G}_1^{k+1}$ orbit as $j^{k+1} + t$ for some $t \in T$.

Proof. We refer to [BduP] □

2.8 Miscellaneous Results

The standard filtration by degree, which we use in the classification is given in [MaIII]. We filter $\mathcal{M}_n.\mathcal{E}(n,p)$ by a chain of submodules $M_k = \mathcal{M}_n^{k+1}.\mathcal{E}(n,p)$ and filter $\mathcal{K}$ by the normal subgroups $\mathcal{K}_k$ consisting of all $H \in \mathcal{K}$ whose (standard) k-jet at $0 \in \mathbf{R}^n \times \mathbb{R}^p$ is equal to the k-jet at 0 of the identity, i.e. $\mathcal{K}_k = (1_{n+p} + \mathcal{M}_{n+p}^{k+1}.\mathcal{E}(n+p,n+p)) \cap \mathcal{K}$. If we use a subgroup $\mathcal{G}$ of $\mathcal{K}$ this generalises by setting $\mathcal{G}_k = \mathcal{K}_k \cap \mathcal{G}$. So $J^k\mathcal{G}$ is a Lie group and this acts smoothly on $J^k(n,p)$.

A classification using the above standard filtration is achieved using the following inductive step. If at the k-jet-level we have that $j^k f \sim j^k g$ then at the $(k+1)$-jet-level $j^{k+1}f \sim j^{k+1}g + h$ for some $h \in H^{k+1}$. Working with a representative for each orbit at the k-level we can obtain a list of the possible representatives at the $(k+1)$-level. This procedure stops when we reach a determined germ. So if we have a jet-filtration $F = (\{M_k\}, \{\mathcal{G}_k\})$ then we have that a map-germ $f \in \mathcal{M}_n.\mathcal{E}(n,p)$ is k-determined if $j_F^k f = j_F^k g \Rightarrow f \sim g$ for all $g \in \mathcal{M}_n.\mathcal{E}(n,p)$. So the classification of a certain class of finitely determined germs up to a given codimension is equivalent to the classification of the associated orbits in the jet-spaces. This reduces the problem to a finite dimensional problem.

Finally, we need to introduce an important numerical invariant.

Definition 2.8.1 Let $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ be a finitely determined function-germ. The $\mathbf{R}$-algebra $M(f) = \mathcal{E}_n / J_f$ (where J_f is the ideal generated by the partial derivatives of f) is called the *Milnor algebra* of f. Its dimension

$$\mu(f) = dim_{\mathbf{R}} M(f)$$

is known as the *Milnor number* of f.

Chapter 3

3-Parameter Motions of the Plane

3.1 Classification

We wish to study motions of the plane with three degrees of freedom and to find local models for the singularities which we would expect to find on trajectories of such motions. 3–parameter motions of the plane were studied as part of Rieger and Ruas' paper on the classification of $\mathcal{A}$–simple germs from $\mathbb{R}^n$ to $\mathbb{R}^2$, [RR]. In our classification we take this one step further and consider germs with positive modality which can occur on trajectories of such motions. Thm. 1.2.1. reduces the problem to one of classifying singularities from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$ with 'codimension' ≤ 5.

Theorem 3.1.1 *For a generic motion of the plane with three degrees of freedom any mono–germ of a trajectory is $\mathcal{A}$–equivalent to one of the normal forms given in Table 3.1:*

Normal Form	$\mathcal{A}_e$–codim
(x, y)	0
$(x, y^2 \pm z^2)$	0
$(x, xy + y^3 \pm z^2)$	0
$(x, y^3 \pm x^2y \pm z^2)$	1
$(x, xy + y^4 \pm z^2)$	1
$(x, y^3 + x^3y \pm z^2)$	2
$(x, xy^2 + y^4 + y^5 \pm z^2)$	2
$(x, xy + y^5 \pm y^7 \pm z^2)$	2
$(x, xy + z^3 \pm y^2z + ay^3 + y^5)$	2

Table 3.1: Local models for 3–parameter planar motions

The rest of the chapter is devoted to the proof of Thm. 3.1.1.

3.1.1 Jet-space computations

We classify the k–jets : $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$ using $\mathcal{A}_1$– complete transversals (or some nilpotent subgroup of $\mathcal{A}$) of $\mathcal{A}^k$–codimension ≤ 5.

Consider $f : (\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$.

- f has rank 1 so we can assume that its 1-jet is given by $(x, 0)$. (We do not need to consider the 1-jet given by $(0, 0)$ as this has $\mathcal{A}^1$–codimension= 6 already).
- So we have $j^1f = (x, 0)$. We find the complete 2–transversal using the following result from Rieger and Ruas, [RR].

Theorem 3.1.2 *Every $\mathcal{A}$-finite map-germ $f : k^n, 0 \longrightarrow k^2, 0$, $n > 1$, of corank 1 is $\mathcal{A}$-equivalent to a germ of the form*

$$h(x, y, z) = \left(x, g(x, y_1, ..., y_m) + \sum_{i=1}^{n-m-1} \epsilon_i z_i^2\right)$$

where $g(0, y_1, ..., y_m) \in \mathcal{M}_n^3$ and $\epsilon_i = \pm 1$; and $cod(\mathcal{A}, h) = cod(\mathcal{A}, (x, g)) + n - m - 1$.

Proof. The proof proceeds via complete transversals. We need

$$j^k f \sim \left(x, g_k(x, y_1, ..., y_m) + \sum_{i=1}^{n-m-1} \epsilon_i z_i^2 \right).$$

We proceed by induction on k.

- **Base step** $k = 2$.

  f has the form $(x, f_2(x, y))$

  where $f_2(x, y) = ax^2 + 2x\left(\sum b_i y_i\right) + qf(y_1, ..., y_{n-1}) + H.O.T.$

  By a linear change of coordinates the quadratic form can be found to be

  $$qf(y_1, ..y_{n-1}) \sim \{sum \ of \ squares\} \sim \sum_{j=1}^{q} \epsilon_j y_j^2$$

  where $\epsilon_j = \pm 1$, depending on the rank of the quadratic form.

  So $j^2 f \sim (x, ax^2 + 2x\left(\sum b_i' y_i\right) + \sum_{j=1}^{q} \epsilon_j y_j^2)$.

  We can now annihilate the x^2-term using a change of coordinates in the target.

  So $j^2 f \sim (x, 2x\left(\sum_{i=1}^{m} b_i' y_i\right) + \sum_{j=1}^{q} \epsilon_j y_j^2)$.

  For these q terms we can apply a change of coordinates of the form $y_i \mapsto y_i \mp b_i' x$ depending on whether $\epsilon_j = \pm 1$. This gives us the following for each term:

  $$\begin{aligned} 2xb_i' y_i \pm y_1^2 &= 2xb_i'(y_i \mp b_i' x) \pm (y_i \mp b_i' x)^2 \\ &= 2xb_i' y_i \mp 2x^2 {b_i'}^2 \pm y_i^2 \pm {b_i'}^2 x^2 - 2b_i' x y_i \\ &= \pm y_i^2 \mp {b_i'}^2 x^2. \end{aligned}$$

  Again annihilating the x^2–terms using a coordinate change in the target we obtain

  $$j^2 f \sim \left(x, x\left(\sum_{i=1}^{n-q-1} b_i' y_i \right) + \sum_{j=1}^{q} \epsilon_j y_j^2 \right)$$

  giving us the required form with $n - q - 1 = m$.

- **Inductive step**

We assume that

$$j^k f \sim \left(x, g_k(x, y_1, ..., y_m) + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2\right)$$

for some k. We now need the complete transversal in J^{k+1}. We have

$$T\mathcal{A}^k \cdot f = \mathcal{M}_n^2 J_f + f^* \mathcal{M}_2^2\{e_1, e_2\}$$

where $J_f = \left\langle \left(1, \frac{\partial g_k}{\partial x}\right), \left(0, \frac{\partial g_k}{\partial y_i}\right), \left(0, z_j\right) \right\rangle$.

Now $\left(1, \frac{\partial g_k}{\partial x}\right)$ gives us $(\phi, 0)$ with $degree(\phi) = k+1$. Also $\left(0, z_j\right)$ yields all $(0, \phi)$ with a factor of z_j. Finally $(0, x^{k+1})$ comes directly from the fact that $x^{k+1} \in f^*(\mathcal{M}_2^2)$. This means that the complete transversal is spanned by $(0, \phi)$ where ϕ are monomials in $y_1, ..., y_m$ only, so

$$j^{k+1} f = j^k f + (0, \Phi)$$

where Φ is a linear combination of monomials in $y_1, ..., y_m$, which is of the required form.

Now the map–germ f is $\mathcal{A}$–finite so at some point we have a k–determined germ and therefore we have our f $\mathcal{A}$–equivalent to the required form.

We provide a proof for the codimension result for our case of $m = 1, n = 3$; thus we have variables x, y, z. We know that any germ of corank 1 is $\mathcal{A}$–equivalent to a germ of the form

$$h(x, y, z) = (x, g(x, y) \pm z^2)$$

where $g(0, y)$ has no constant, linear or quadratic terms.

It is enough to work in some jet–space $J^s(3, 2)$ in which h is $\mathcal{A}$–sufficient. We then look at the tangent space to the $\mathcal{A}^{(s)}$–orbits of h and (x, g). The tangent space to h is given by

$$T\mathcal{A}.h = \mathcal{M}_3\langle(1, g_x), (0, g_y), (0, \pm 2z)\rangle + h * \mathcal{M}_2\{e_1, e_2\}.$$

With computation we find that the tangent space defined above contains all monomials $(0, x^i y^j z^k)$ with factor z, **except** $(0, z)$ and all monomials $(x^i y^j z^k, 0)$ with factor z. We can therefore work modulo such vectors when computing codimensions. Thus for the tf part of the tangent space we need only consider the subspace arising from the expressions not involving z, i.e.

$$(x^i y^j, x^i y^j g_x),$$

$$(0, x^i y^j g_y).$$

In other words the tf part to the $\mathcal{A}^{(s)}$–orbit for the germ $(x, g(x, y))$. Finally, we observe that for any polynomial $p(X, Y)$ we have

$$\begin{aligned} p(x, g(x, y) \pm z^2) &= p(x, g(x, y)) + (\textit{terms in } z^2) \\ &= p(x, g(x, y)) \textit{ modulo terms in } z. \end{aligned}$$

Also, for the wf part we need only consider the subspace arising from expressions $(\phi(X, Y), \psi(X, Y))$ where ϕ, ψ are polynomials, and $X = x, Y = g$, i.e. the wf part of the tangent space to the $\mathcal{A}^{(s)}$–orbit for the germ $(x, g(x, y))$.The formula is immediate, the $+1$ corresponding to the monomial $(0, z)$. □

Thus we have the following possibilities for the 2–jet:

1. $j^2f = (x, xy \pm z^2)$
2. $j^2f = (x, z^2)$
3. $j^2f = (x, y^2 \pm z^2)$ which is 2–$\mathcal{A}_1$–determined. (*)
4. $j^2f = (x, xy)$
5. $j^2f = (x, 0)$

where (*) means that the germ is finitely determined.

Now if we look at the codimensions of these jets in $J^2(3, 2)$ we obtain the following:

	$\mathcal{A}$-codimension	
$(x, xy \pm z^2)$	3	
(x, z^2)	4	
$(x, y^2 \pm z^2)$	3	
(x, xy)	5	
$(x, 0)$	7	($\sharp$)

where ($\sharp$) indicates that the codimension is too high and so we can forget germs whose k–jet is equivalent the form $\sharp$. For the full calculations of the codimensions see Section 3.3.

(3) Firstly we look at $j^2f = (x, xy \pm z^2)$.

We want the complete–3–transversal so we look at

$$T\mathcal{A}_1.f = \mathcal{M}_3^2\langle(1, y), (0, x), (0, \pm 2z)\rangle + f * \mathcal{M}_2^2\{e_1, e_2\}.$$

We find that the only vector missing is $(0, y^3)$, so our complete transversal is $j^3(x, y, z) = (x, xy + ay^3 \pm z^2)$.

1. If $a \neq 0$ we can scale using the following argument to obtain $j^3f = (x, xy + y^3 \pm z^2)$ which we can prove to be 3–$\mathcal{A}_1$–determined and has codimension =3.
   See Appendix B. (*)

If we apply the following changes of co-ordinates, $(x, y, z) \longmapsto (\lambda x, \mu y, \nu z)$ to f we obtain the following:

$$(x, xy \pm z^2 + ay^3) \longmapsto (\lambda x, \lambda\mu xy \pm \nu^2 z^2 + a\mu^3 y^3).$$

Now we want $\lambda\mu = \nu^2 = a\mu^3$, i.e if $a < 0$ we need $\mu < 0$ and $\lambda < 0$; but if $a > 0$ we need $\mu > 0$ and $\lambda > 0$. Since we can assign any values to λ, μ, ν this means that we can scale $(x, xy \pm z^2 + ay^3)$ to $(x, xy \pm z^2 + y^3)$ by applying a change of co-ordinates to the target.

2. If $a = 0$ then we have $j^3 f = (x, xy \pm z^2)$ and this has codimension 4.

Next we need to look at $j^2 f = (x, z^2)$.

The complete-3-transversal is given by $j^3 f = (x, ay^3 + bxy^2 + cx^2 y + z^2)$.

1. If $a, b, c \neq 0$ we can apply a change of coordinates to obtain $(x, uy^3 + vx^2 y + z^2)$. So by scaling we have $j^3 f = (x, y^3 \pm x^2 y \pm z^2)$ which we can prove to be 3-$\mathcal{A}_1$-determined and have codimension 4.

   See Appendix B. (*)

2. If we have $v = 0$ we get $j^3 f = (x, y^3 \pm z^2)$ and this has codimension 5.

3. If $a = 0$ we can apply a change of co-ordinates to obtain $j^3 f = (x, xy^2 \pm z^2)$ and this has codimension 5.

Lastly we need to consider $j^2 f = (x, xy)$.

The complete–3–transversal is given by $j^3 f = (x, xy + ay^3 + by^2 z + cyz^2 + dz^3)$.

- Suppose $d \neq 0$, then by the change of coordinates $z \mapsto z - c/over3dy$, we can annihilate the yz^2 term to give us $j^3 f = (x, xy + a'y^3 + b'y^2 z + dz^3)$. Now assuming a', b' and $d \neq 0$ we can scale this to give $j^3 f = (x, xy + z^3 \pm y^2 z + ay^3)$ where a is a modulus and the stratum has codimension 5.

If $a' = 0$ then we have $j^3f = (x, xy + b'y^2z + dz^3)$ which we can scale to give $(x, xy + z^3 \pm y^2z)$ which has codimension 6. If $b' = 0$ then we have $j^3f = (x, xy + a'y^3 + dz^3)$ which we can scale to give $(x, xy + z^3 + y^3)$ which again has codimension 6.

- If $d = 0$ then we have $j^3f = (x, xy + ay^3 + by^2z + cyz^2)$ which by the change of coordinates $z \mapsto z - \frac{b}{2c}y$ reduces to $j^3f = (x, xy + a'y^3cyz^2)$ assuming that $c \neq 0$. If $a' \neq 0$ then we can scale this to give us $(x, xy + y^3 \pm yz^2)$ which has codinsion 6. If $a' = 0$ then we have $j^3f = (x, xy + cyz^2)$ which scales to give $(x, xy + yz^2)$ which has codimension 7.

  Finally, if $c = 0$ then we have $j^3f = (x, xy + ay^3 + by^2z)$ which reduces to $j^3f = (x, xy + y^2z)$ which has codimension 7.

  So we only need consider the orbit:

$$j^3f = (x, xy + z^3 \pm y^2z + ay^3).$$

(4) So now we consider $j^3f = (x, xy \pm z^2)$.

The complete-4-transversal is given by $j^4f = (x, xy + ay^4 \pm z^2)$.

1. If $a \neq 0$ we can scale to obtain $j^4f = (x, xy + y^4 \pm z^2)$ which we can prove to be 4-$\mathcal{A}$-determined and have codimension 4.

   See Appendix B. (*)
2. If $a = 0$ then we have $j^4f = (x, xy \pm z^2)$ and this has codimension 5.

Next we need to consider $j^3f = (x, y^3 \pm z^2)$.

The complete-4-transversal is given by $j^4f = (x, y^3 + ax^3y \pm z^2)$.

1. If $a \neq 0$ we can scale to obtain $j^4f = (x, y^3 + x^3y \pm z^2)$ which we can prove to be 4-$\mathcal{A}_1$-determined and have codimension 5.

See Appendix B. (∗)

2. If $a = 0$ then we have $j^4 f = (x, y^3 \pm z^2)$ and this has codimension 6. (♯)

Next we need to consider $j^3 f = (x, xy^2 \pm z^2)$.

The complete-4-transversal is given by $j^4 f = (x, xy^2 + ay^4 \pm z^2)$.

1. If $a \neq 0$ then we can scale to obtain $j^4 f = (x, xy^2 + y^4 \pm z^2)$.
2. If $a = 0$ then we have $j^4 f = (x, xy^2 \pm z^2)$ and this has codimension 6. (♯)

Finally in the 3-jet we need to consider $j^3 f = (x, xy + z^3 + ay^3 \pm y^2 z)$.

If we use the nilpotent group containing the element $(\psi, 0)$ where $f(x, y, z) = (\phi, \psi)$ then the complete-4-transversal is empty so we have $j^4 f = (x, xy + z^3 + ay^3 \pm y^2 z)$.

(5) Firstly we consider $j^4 f = (x, xy \pm z^2)$. The complete-5-transversal is given by $j^5 f = (x, xy + ay^5 \pm z^2)$.

1. If $a \neq 0$ we can scale to obtain $j^5 f = (x, xy + y^5 \pm z^2)$.
2. If $a = 0$ then we have $j^5 f = (x, xy \pm z^2)$ and this has codimension 6. (♯)

Next we need to consider $j^4 f = (x, xy^2 + y^4 \pm z^2)$.

The complete-5-transversal is given by $j^5 f = (x, xy^2 + y^4 + ay^5 \pm z^2)$.

1. If $a \neq 0$ then we can scale to obtain $j^5 f = (x, xy^2 + y^4 + y^5 \pm z^2)$ which can be proved to be 5-$\mathcal{A}_1$-determined and have codimension 5.

See Appendix B. (*)

2. If $a = 0$ then we have $j^5 f = (x, xy^2 + y^4 \pm z^2)$ and this has codimension 6. ($\sharp$)

Lastly we look at $j^4 f = (x, xy + z^3 + ay^3 \pm y^2 z)$.

This has complete-5-transversal $j^5 f = (x, xy + z^3 + ay^3 \pm y^2 z + by^5)$.

1. If $b \neq 0$ then we can scale this to $j^5 f = (x, xy + z^3 + ay^3 \pm y^2 z + y^5)$ which we can prove is 5-$\mathcal{A}$-determined with the stratum having codimension 5.

   See Appendix B. (*)

2. If $b = 0$ we have $j^5 f = (x, xy + z^3 + ay^3 \pm y^2 z)$ and this has codimension 6. ($\sharp$)

(6) So now we consider $j^5 f = (x, xy + y^5 \pm z^2)$.

The complete-6-transversal is empty if we use one of the nilpotent subgroups of $\mathcal{A}_1$ so we have $j^6 f = (x, xy + y^5 \pm z^2)$.

(7) So now we consider $j^6 f = (x, xy + y^5 \pm z^2)$.

The complete-7-transversal is $j^7 f = (x, xy + y^5 + ay^7 \pm z^2)$.

1. If $a \neq 0$ we can scale to obtain $j^7 f = (x, xy + y^5 \pm y^7 \pm z^2)$ which we can prove to be 7-$\mathcal{A}$-determined and have codimension 5.

   See Appendix B. (*)

2. If $a = 0$ then we have $j^7 f = (x, xy + y^5 \pm z^2)$ and this has codimension 6. ($\sharp$)

3.2 Discriminants

Definition 3.2.1 *1. Given a map-germ* $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$*, the critical set of* f*,* $\sum f$*, is the set of points in* $\mathbb{R}^n$ *where* df *is not surjective (i.e. when the Jacobian of* f *has less than maximal rank).*

2. With f *as above, the discriminant of* f *is the image of the critical set,* $f(\sum f)$.

We are considering $\mathcal{A}$–finite map–germs :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$, so our critical set $\sum f$ is a curve in 3–space, and the $\mathcal{A}$–class of $(f|\sum f, 0) \longrightarrow (\mathbf{R}^2, 0)$ is an $\mathcal{A}$–invariant of f. Now Bruce and Gaffney, in [BG], proved that the $\mathcal{A}$–equivalence classes of the germs :$(\mathbf{R}, 0) \longrightarrow (\mathbf{R}^2, 0)$ correspond to $\mathcal{K}$–equivalence classes of germs :$(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$, i.e. their defining equations. So the $\mathcal{K}$–class of the defining equations is an invariant, and we wish to find the $\mathcal{K}$–classes of the germs in question.

Lemma 3.2.2 *Critical sets and discriminants are preserved by equivalences, i.e. if we have two germs* $f_0, f_1 : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ *we have*

$$\begin{aligned} r\left(\sum f_0\right) &= \sum f_1 \\ l\left(f_0\left(\sum f_0\right)\right) &= f_1\left(\sum f_1\right). \end{aligned}$$

Rieger and Ruas provided us with a list of simple map–germs and we have the following proposition.

Proposition 3.2.3 *The* $\mathcal{K}$*–classes of the* $\mathcal{A}$*–simple map-germs :*$(\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^2, 0)$ *are given in the following table.*

Normal Form	$\mathcal{K}$–class of defining equation
(x, y)	
$(x, y^2 \pm z^2)$	
$(x, xy + y^3 \pm z^2)$	A_2
$(x, y^3 \pm x^k y \pm z^2)$	A_{3k-1}
$(x, xy + y^4 \pm z^2)$	E_6
$(x, xy + y^5 \pm z^2)$	W_{12}
$(x, xy + y^5 \pm y^7 \pm z^2)$	W_{12}
$(x, xy^2 + y^5 + y^6 \pm z^2)$	W_{17}
$(x, xy^2 + y^5 \pm y^9 \pm z^2)$	W_{17}
$(x, xy^2 + y^5 \pm z^2)$	W_{17}
$(x, x^2y + y^4 \pm y^5 \pm z^2)$	K_{14}
$(x, x^2y + y^4 \pm z^2)$	K_{14}
$(x, xy^2 + y^4 + y^{2k+1} \pm z^2)$	A_{2k}

Table 3.2: $\mathcal{A}$–simple planar map–germs

Proof. The proof is given in the remainder of this section. □

So we now need to work out the discriminants of the map-germs .

3.2.1 Discriminants of the $\mathcal{A}$-simple map-germs

- $f(x, y, z) = (x, y^2 \pm z^2)$

  The critical set for f is given by

$$\begin{aligned} \sum f &= \left\{(x, y, z) \;:\; rank \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2y & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x, y, z) : y = z = 0\} \\ &= \{(t, 0, 0) : t \in \mathbb{R}\}. \end{aligned}$$

  So the discriminant of f is the x–axis

$$\text{i.e.} \quad f|\sum f = \{(t, 0) : t \in \mathbb{R}\}.$$

- $f(x, y, z) = (x, xy + y^3 \pm z^2)$

The critical set for f is given by

$$\begin{aligned}\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y & x+3y^2 & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x,y,z) : x+3y^2 = 0 \ \& \ z=0\} \\ &= \{(-3t^2, t, 0) : t \in \mathbb{R}\}.\end{aligned}$$

and the discriminant is given by

$$f|\sum f = \{(-3t^2, -2t^3) : t \in \mathbb{R}\}.$$

This is $\mathcal{K}$–equivalent to $x^3 + y^2 = 0$ & is therefore of type A_2.

- $f(x,y,z) = (x, y^3 \pm x^k y \pm z^2)$
  The critical set for f is given by

$$\begin{aligned}\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ kx^{k-1}y & \pm x^k + 3y^2 & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x,y,z) : \pm x^k + 3y^2 = 0 \ \& \ z=0\}.\end{aligned}$$

Now the parametrization depends upon the parity of k.

If k is odd, then $3y^2 \pm x^k = 0$ defines a single branch curve (since $3y^2 + x^k \sim 3y^2 - x^k$ via the change of coordinates $x \longmapsto -x$). So we can parametrize by

$$(x,y) = (-t^2, \frac{t^k}{\sqrt{3}}).$$

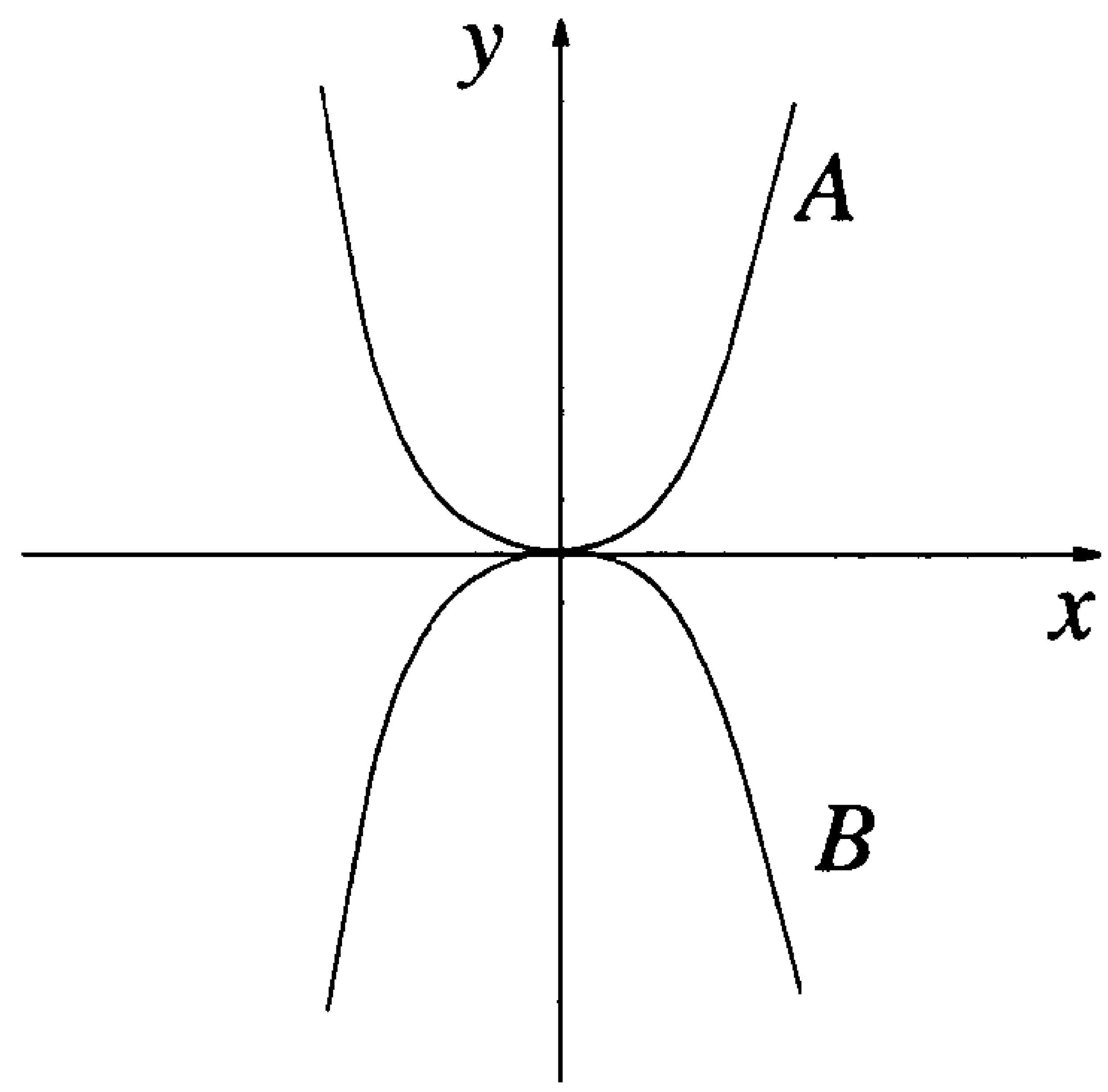
This gives $\sum f = \{(-t^2, \frac{t^k}{\sqrt{3}}, 0) : t \in \mathbb{R}\}$, and the discriminant as

$$f|\sum f = \{(-t^2, \frac{-2t^{3k}}{3\sqrt{3}}) : t \in \mathbb{R}\}.$$

This is $\mathcal{K}$–equivalent to $y^2 + x^{3k} = 0$ & is therefore of the type A_{3k-1}.

If k is even, $k = 2m$ say, then $3y^2 \pm x^{2m} = 0$ defines a double branched curve, i.e. a multigerm. If we take the + case, we have $3y^2 + x^{2m} = 0$ which has no real solutions except for the origin.

If we take the $-$ case, we have $3y^2 - x^{2m} = 0$ which has two branches, e.g. for $k = 4$ we have the following picture,



In this case this defines a tacnode. We must parametrize each branch separately. For the positive branch (branch A in our example) we parametrize by

$$(x, y) = (t^2, \frac{t^k}{\sqrt{3}}).$$

We have the critical set given by

$$\sum f = \{(t^2, \frac{t^k}{\sqrt{3}}, 0) : t \in \mathbf{R}\},$$

and the discriminant given by

$$f | \sum f = \{(t^2, -\frac{2t^k}{3\sqrt{3}}) : t \in \mathbf{R}\}.$$

For the negative branch (branch B in our diagram) we parametrize by

$$(x, y) = (t^2, -\frac{t^k}{\sqrt{3}}),$$

giving the critical set as

$$\sum f = \{(t^2, -\frac{t^k}{\sqrt{3}}, 0) : t \in \mathbf{R}\},$$

and the discriminant as

$$f|\sum f = \{(t^2, \frac{2t^k}{3\sqrt{3}}) : t \in \mathbb{R}\}.$$

These are $\mathcal{K}$–equivalent to $x^{3k} - y^2 = 0$ & is therefore of type A^-_{3k-1}.

- $f(x, y, z) = (x, xy + y^4 \pm z^2)$

The critical set for f is given by

$$\begin{aligned}\sum f &= \left\{(x,y,z) \;:\; rank \begin{pmatrix} 1 & 0 & 0 \\ y & x+4y^3 & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x,y,z) : x + 4y^3 = 0 \;\&\; z = 0\} \\ &= \{(-4t^3, t, 0) : t \in \mathbb{R}\}.\end{aligned}$$

So

$$f|\sum f = \{(-4t^3, -3t^4) : t \in \mathbb{R}\}.$$

This is $\mathcal{K}$–equivalent to $x^4 + y^3 = 0$ and is therefore of type E_6.

- $f(x, y, z) = (x, xy + y^5 \pm z^2)$

The critical set for f is given by

$$\begin{aligned}\sum f &= \left\{(x,y,z) \;:\; rank \begin{pmatrix} 1 & 0 & 0 \\ y & x+5y^4 & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x,y,z) : x + 5y^4 = 0 \;\&\; z = 0\} \\ &= \{(-5t^4, t, 0) : t \in \mathbb{R}\}.\end{aligned}$$

This gives the discriminant parametrized as

$$f|\sum f = \{(-5t^4, -4t^5) : t \in \mathbb{R}\}.$$

This is $\mathcal{K}$–equivalent to $y^4 + x^5 = 0$ & is therefore of type W_{12}.

- $f(x,y,z) = (x, xy + y^5 \pm y^7 \pm z^2)$

  The critical set for f is given by

$$\begin{aligned} \sum f &= \left\{ (x,y,z) \; : \; rank \begin{pmatrix} 1 & 0 & 0 \\ y & x + 5y^4 \pm 7y^6 & \pm 2z \end{pmatrix} < 2 \right\} \\ &= \{(x,y,z) : x + 5y^4 \pm 7y^6 = 0 \;\&\; z = 0\} \\ &= \{(-5t^4 \mp 7t^6, t, 0) : t \in \mathbb{R}\}. \end{aligned}$$

So

$$f|\sum f = \{(-5t^4 \mp 7t^6, -4t^5 \mp 6t^7) : t \in \mathbb{R}\}.$$

In this case we have two polynomials in t. We need to find out if they have any common factors. We do this by looking at the resultant of

$$\begin{array}{rlcl} & X = -5t^4 \mp 7t^6, & \& & Y = -4t^5 \mp 6t^7. \\ i.e. & X + 5t^4 \pm 7t^6 = 0, & \& & Y + 4t^5 \pm 6t^7 = 0. \end{array}$$

The resultant is equal to the determinant of the following matrix:

$$\begin{pmatrix}
X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 5 & 0 & \pm 7 \\
Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6 & 0 & 0 & 0 & 0 & 0 \\
0 & Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6 & 0 & 0 & 0 & 0 \\
0 & 0 & Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6 & 0 & 0 & 0 \\
0 & 0 & 0 & Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6 & 0 \\
0 & 0 & 0 & 0 & 0 & Y & 0 & 0 & 0 & 0 & 4 & 0 & \pm 6
\end{pmatrix}$$

Using the MAPLE computer algebra package the determinant was found to be

$46656x^7 + 13824x^6 + 1024x^5 - 21168y^2x^4 - 1600y^2x^3 + 605052y^4x^2 + 171500y^4x + 823543y^6 + 12500y^4$.

Observe that this germ has the form $Ax^5 + By^4 + HOT$ relative to the weights $weight(x) = 4$, $weight(y) = 5$. We now use Arnold's results found in Appendix A.

Firstly, we create the Newton diagram for the resultant by plotting the points for which the monomial appears. So in this case we have the points $(0, 4)$ corresponding to y^4, $(1, 4)$ corresponding to xy^4, etc. We have the following Newton diagram, Figure 3.1.

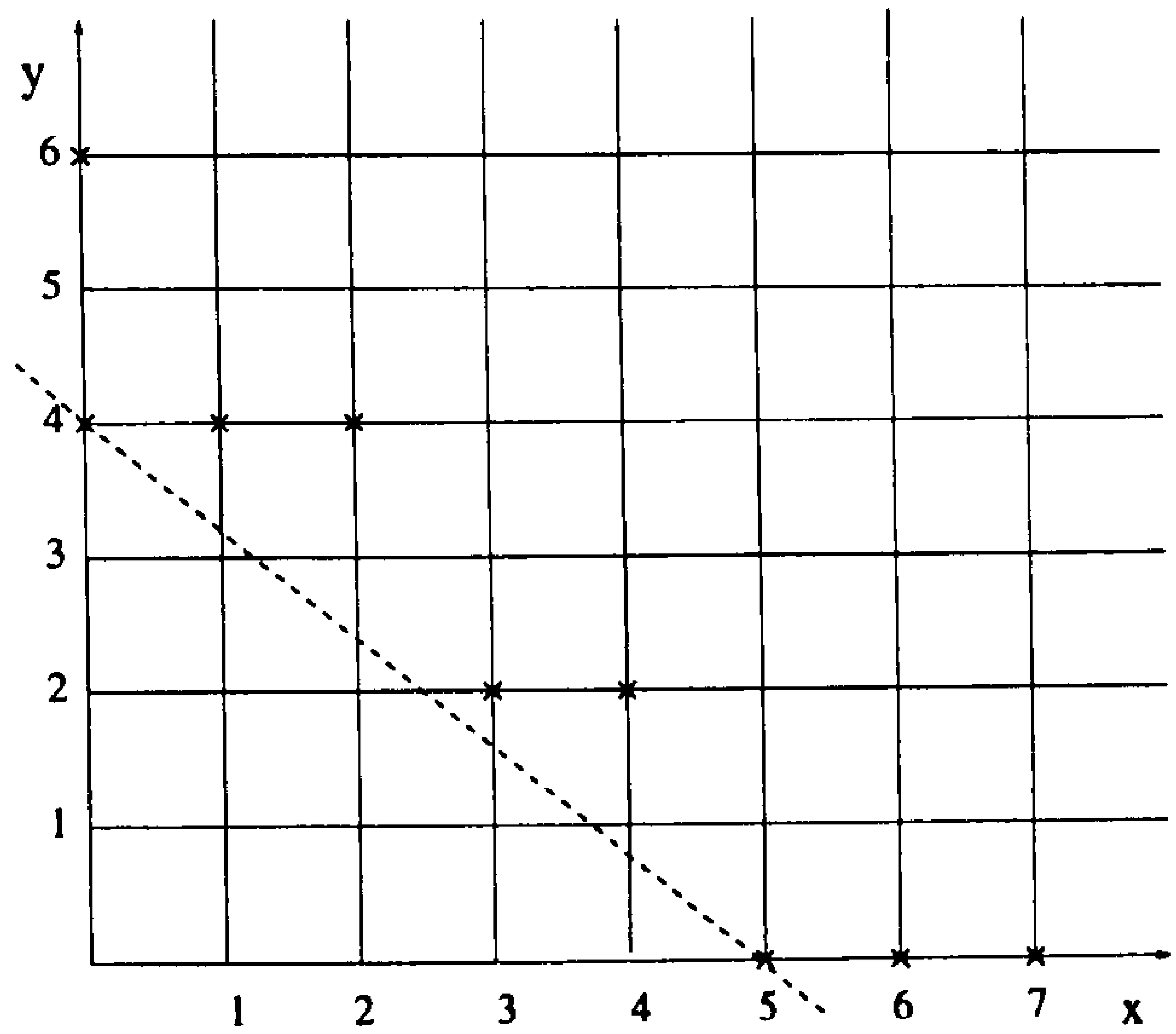


Figure 3.1: Newton diagram

By Cor. A.0.3. we need to look at the local ring for the homogeneous part of f, f_0. In this case, $f_0 = Ax^5 + By^4$, so we need to look at

$$\langle f_x, f_y \rangle = \langle x^4, y^3 \rangle.$$

From Lemma A.0.6. we need to look for which monomials on the diagram are not contained in this ring. The only one missing in this case is x^3y^2. This is the only super–diagonal element, and by Theorem A.0.12. we have that $f \sim x^5 + y^4 + \lambda x^3y^2$. Scaling λ we get $f \sim x^5 + y^4 \pm x^3y^2$, which is of type W_{12}.

- $f(x, y, z) = (x, xy^2 + y^5 + y^6 \pm z^2)$

  The critical set for f is given by

$$
\begin{aligned}
\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y^2 & 2xy+5y^4+6y^5 & \pm 2z \end{pmatrix} < 2\right\} \\
&= \{(x,y,z) : 2xy+5y^4+6y^5=0 \ \& \ z=0\} \\
&= \{(t,0,0) \oplus (-\frac{5}{2}t^3 - 3t^4, t, 0) : t \in \mathbb{R}\}.
\end{aligned}
$$

So

$$f|\sum f = \{(t,0) \oplus (-\frac{5}{2}t^3 - 3t^4, -\frac{3}{2}t^5 - 2t^6) : t \in \mathbb{R}\}.$$

In this case we have two polynomials in t as well as the axis. We need to find out if they have common factors. We do this by looking at the resultant of

$$
\begin{array}{rlcl}
& X = -\frac{5}{2}t^3 - 3t^4, & \& & Y = -\frac{3}{2}t^5 - 2t^6. \\
i.e. & 2X + 5t^3 + 6t^4 = 0, & \& & 2Y + 3t^5 + 4t^7 = 0.
\end{array}
$$

The resultant is equal to the determinant of the following matrix multiplied by y:

$$
\begin{pmatrix}
2X & 0 & 0 & 5 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 2X & 0 & 0 & 5 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2X & 0 & 0 & 5 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 2X & 0 & 0 & 5 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 6 \\
2Y & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 \\
0 & 2Y & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 \\
0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 4 & 0 \\
0 & 0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 4
\end{pmatrix}
$$

Using the MAPLE computer algebra package the determinant was found to be

$16384x^6 - 1728x^5 - 3072yx^4 + 221184y^2x^3 - 24000y^2x^2 - 518400y^3x + 746496y^4 + 50000y^3$.

So that gives $f|\sum f \sim 16384x^6y - 1728x^5y - 3072y^2x^4 + 221184y^3x^3 - 24000y^3x^2 - 518400y^4x + 746496y^5 + 50000y^4$.

Observe that this germ has the form $Ax^5y + By^4 + HOT$ relative to the weights $weight(x) = 3$, $weight(y) = 5$. We now use Arnold's results found in Appendix A.

We have the following Newton diagram, Figure 3.2.

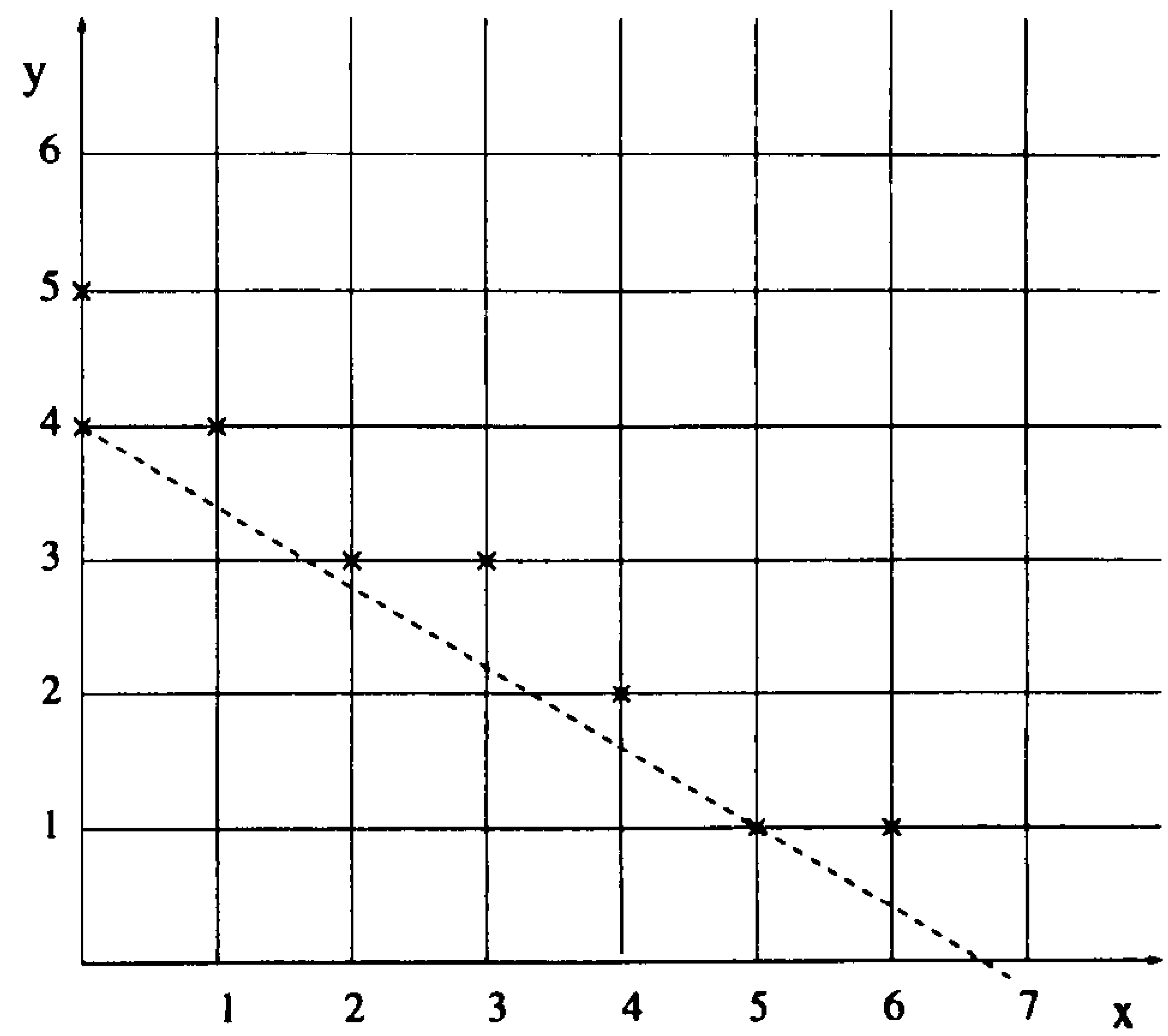


Figure 3.2: Newton diagram

We need to look at the ideal generated by $\langle f_x, f_y \rangle$, namely

$$\langle f_x, f_y \rangle = \langle x^4y, 4y^3 + x^5 \rangle$$

and we find that the only super–diagonal elements of the regular basis is x^2y^3 *and* x^3y^3 So $f \sim y^4 + x^5 + \lambda x^2y^3 + \mu x^3y^3$ and is of type W_{17}.

- $f(x, y, z) = (x, xy^2 + y^5 \pm y^9 \pm z^2)$

  The critical set for f is given by

$$\begin{aligned} \sum f &= \left\{ (x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y^2 & 2xy + 5y^4 \pm 9y^8 & \pm 2z \end{pmatrix} < 2 \right\} \\ &= \{(x, y, z) : 2xy + 5y^4 \pm 9y^8 = 0 \ \& \ z = 0\} \\ &= \{(t, 0, 0) \oplus (-\frac{5}{2}t^3 \mp \frac{9}{2}t^7, t, 0) : t \in \mathbf{R}\}. \end{aligned}$$

So

$$f|\sum f = \{(t,0) \oplus (-\frac{5}{2}t^3 \mp \frac{9}{2}t^7, -\frac{3}{2}t^5 \mp \frac{7}{2}t^9)\}.$$

In this case we have two polynomials in t plus the axis. We need to find out if they have common factors. We do this by looking at the resultant of

$$\begin{array}{rlcl} & X = -\frac{5}{2}t^3 \mp \frac{9}{2}t^7, & \& & Y = -\frac{3}{2}t^5 \mp \frac{7}{2}t^9. \\ i.e. & 2X + 5t^3 \pm 9t^7 = 0, & \& & 2Y + 3t^5 \pm 7t^9 = 0. \end{array}$$

The resultant is equal to the determinant of the following matrix multiplied by y:

$$\begin{pmatrix}
2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2X & 0 & 0 & 5 & 0 & 0 & 0 & \pm9 \\
2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2Y & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \pm7
\end{pmatrix}$$

Using the MAPLE computer algebra package the determinant was found to be

$\pm421654016x^9 + 236027904yx^6 + 3538944x^5 \mp 13538544768y^3x^4 \mp 19660800y^2x^3 - 59997563136y^5x^2 - 6718464000y^4x \mp 49589822592y^7 - 102400000y^3$.

So that gives $f|\sum f = \pm421654016x^9y + 236027904y^2x^6 + 3538944x^5y \mp 13538544768y^4x^4 \mp 19660800y^3x^3 - 59997563136y^6x^2 - 6718464000y^5x \mp 49589822592y^8 - 102400000y^4$.

Observe that this germ has the form $Ax^5y + By^4 + HOT$ relative to the weights $weight(x) = 3,\ weight(y) = 5$. We now use Arnold's results found in Appendix A.

We have the following Newton diagram, Figure 3.3.

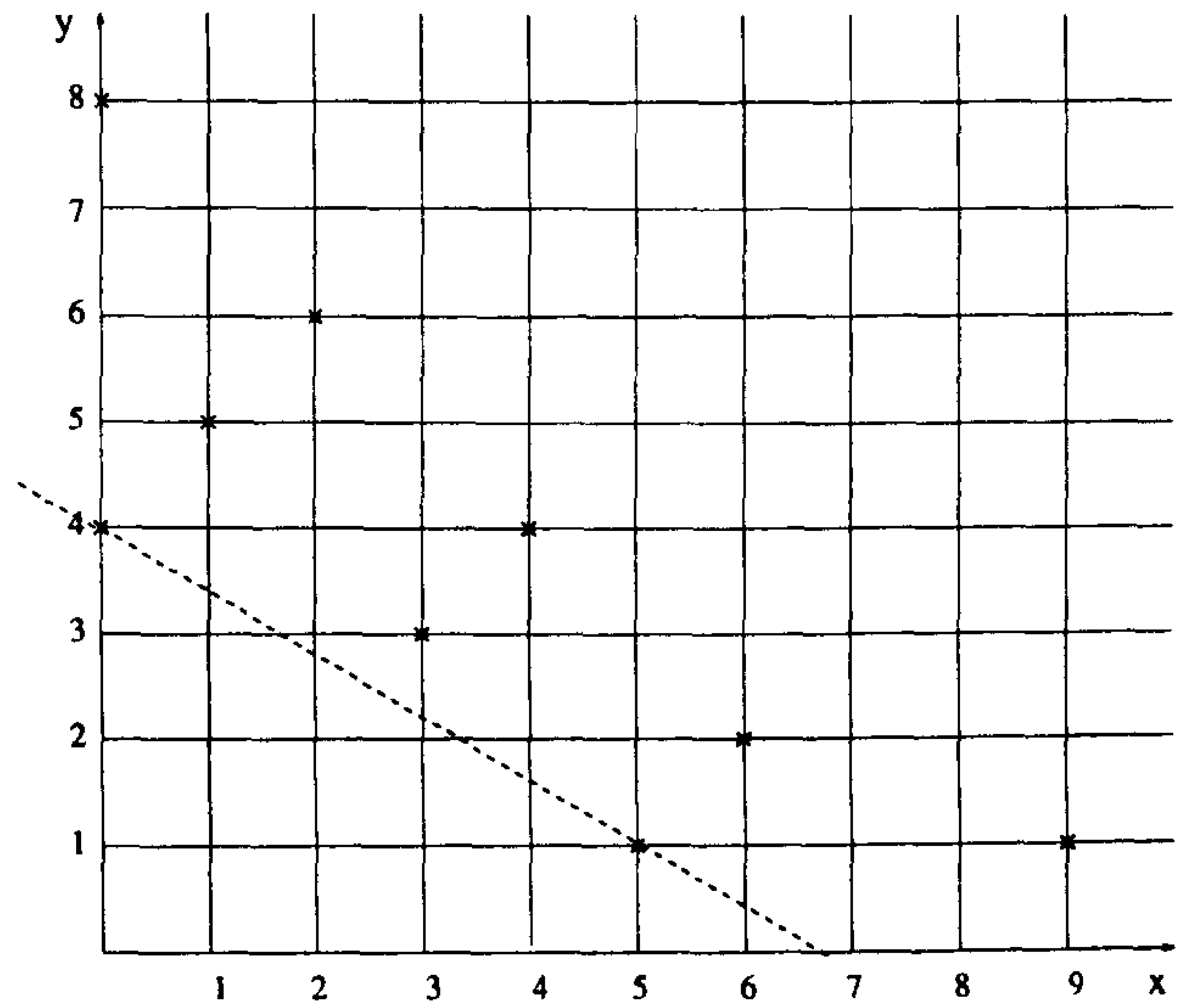


Figure 3.3: Newton diagram

We need to look at the ideal generated by $\langle f_x, f_y \rangle$, namely

$$\langle f_x, f_y \rangle = \langle x^4y, 4y^3 + x^5 \rangle$$

and we find that the only super-diagonal elements of the regular basis is x^2y^3 *and* x^3y^3 So $f \sim y^4 + x^5 + \lambda x^2y^3 + \mu x^3y^3$ and is of type W_{17}.

- $f(x,y,z) = (x, xy^2 + y^5 \pm z^2)$

The critical set for f is given by

$$\begin{aligned}\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y^2 & 2xy+5y^4 & \pm 2z \end{pmatrix} < 2\right\} \\ &= \{(x,y,z) : 2xy + 5y^4 = 0 \ \& \ z = 0\} \\ &= \{(t,0,0) \oplus (-\frac{5}{2}t^3, t, 0) : t \in \mathbb{R}\}.\end{aligned}$$

So

$$f|\sum f = \{(t,0) \oplus (-\frac{5}{2}t^3, -\frac{3}{2}t^5) : t \in \mathbf{R}\}.$$

This is $\mathcal{K}$–equivalent to $x^5y + y^4 = 0$ & therefore has type W_{17}.

So now we have three germs with $\mathcal{K}$-type W_{17}. How do we distinguish between them?

Definition 3.2.4 *Given $\delta > 0$ the semiquasihomogeneous function f is δ right (respectively contact) determined with respect to the weights $\alpha = (\alpha_1, ..., \alpha_n)$ if for any polynomial ϕ of weight $> \delta$ the function $f + \phi$ is right (respectively contact) equivalent to f.*

We are looking at contact equivalence so we let f be of the form $f = f_0 + f_1 + f_2$, where f_0 is the principal part , f_1 is quasihomogeneous of weight $1 + \epsilon$ and f_2 has weight $> 1 + \epsilon$.

We now have the following result from Bruce, [B3]

Theorem 3.2.5 *If $\delta \geq 1 + \epsilon$ is chosen so that weight $\phi > \delta$ implies that $\phi \in \langle \frac{\delta f_0}{\delta x_i}, f_1 \rangle$, then f is δ contact determined with respect to α.*

So for our case we chose $\alpha = (\frac{3}{20}, \frac{5}{20})$.

So we have

$$\begin{aligned} f_0 &= x^5y + y^4 \\ f_1 &= \lambda x^2 y^3 \\ f_2 &= \mu x^3 y^3 \end{aligned}$$

and $I = \langle 5x^4, x^5 + 4y^3, \lambda x^2 y^3 \rangle$.

So we have f_0 with weight 1, f_1 with weight $\frac{21}{20}$ and f_2 with weight $\frac{24}{20}$ which is $> \frac{21}{20}$.

So if we take $\phi = \sum \phi_{p,q} x^p y^q$ with weight $\phi > \frac{21}{20}$ this gives $\frac{3p+5q}{20} > \frac{21}{20} \longmapsto 3p + 5q > 21$.

$$\begin{array}{lll}
p=0 & q \geq 5 & y^5 \in \mathrm{I} \\
p=1 & q \geq 4 & xy^4 \in \mathrm{I} \\
p=2 & q \geq 4 & x^2y^4 \in \mathrm{I} \\
p=3 & q \geq 3 & x^3y^3 \in \mathrm{I} \\
p=4 & q \geq 2 & x^4y^2 \in \mathrm{I} \\
p=5 & q \geq 2 & x^5y^2 \in \mathrm{I} \\
p=6 & q \geq 1 & x^6y \in \mathrm{I} \\
p=7 & q \geq 1 & x^7y \in \mathrm{I} \\
p=8 & q \geq 0 & x^8 \in \mathrm{I}
\end{array}$$

So by the previous result $f \sim x^5y + y^4 + \lambda x^2y^3$ if $\lambda \neq 0$.

By applying a change of co-ordinates we can scale to obtain $\sim x^5y + y^4 + x^2y^3 = 0$ which corresponds to $f(x,y,z) = (x, xy^2 + y^5 + y^6 \pm z^2)$.

If $\lambda = 0$, $\mu \neq 0$ we can scale to obtain $x^5y + y^4 \pm x^3y^3 = 0$ which corresponds to $f(x,y,z) = (x, xy^2 + y^5 \pm y^9 \pm z^2)$.

If $\lambda = \mu = 0$ then we have $x^5y + y^4 = 0$ which corresponds to $f(x,y,z) = (x, xy^2 + y^5 \pm z^2)$.

- $f(x,y,z) = (x, x^2y + y^4 \pm y^5 \pm z^2)$

  The critical set for f is given by

$$\begin{aligned}
\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ 2xy & x^2 + 4y^3 \pm 5y^4 & \pm 2z \end{pmatrix} < 2\right\} \\
&= \{(x,y,z) : x^2 + 4y^3 \pm 5y^4 = 0 \ \& \ z = 0\}. \\
&= \{(\sqrt{(-4t^3 \mp 5t^4)}, t, 0) : t \in \mathbf{R}\}
\end{aligned}$$

So

$$f|\sum f = \{(\sqrt{-4t^3 \mp 5t^4}, -3t^4 \mp 4t^5) : t \in \mathbf{R}\}.$$

In this case we have two functions in t. We need to find out if they have common factors. We do this by looking at the resultant of

$$\begin{array}{rlcl} & X = \sqrt{-4t^3 \mp 5t^4}, & \& & Y = -3t^4 \mp 4t^5. \\ i.e. & X^2 + 4t^3 \pm 5t^4 = 0, & \& & Y + 3t^4 \pm 4t^5 = 0. \end{array}$$

The resultant is equal to the determinant of the following matrix : OB

$$\begin{pmatrix} X^2 & 0 & 0 & 4 & \pm 5 & 0 & 0 & 0 & 0 \\ 0 & X^2 & 0 & 0 & 4 & \pm 5 & 0 & 0 & 0 \\ 0 & 0 & X^2 & 0 & 0 & 4 & \pm 5 & 0 & 0 \\ 0 & 0 & 0 & X^2 & 0 & 0 & 4 & \pm 5 & 0 \\ 0 & 0 & 0 & 0 & X^2 & 0 & 0 & 4 & \pm 5 \\ Y & 0 & 0 & 0 & 3 & \pm 4 & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & 0 & 3 & \pm 4 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 & 0 & 3 & \pm 4 & 0 \\ 0 & 0 & 0 & Y & 0 & 0 & 0 & 3 & \pm 4 \end{pmatrix}$$

Using the MAPLE computer algebra package the determinant was found to be

$256x^{10} - 27x^8 + 36yx^6 - 50y^2x^4 + 2500x^2y^3 + 3125y^4 - 256y^3$.

So that gives

$$f|\sum f \sim 256x^{10} - 27x^8 + 36yx^6 - 50y^2x^4 + 2500x^2y^3 + 3125y^4 - 256y^3.$$

Observe that this germ has the form $Ax^8 + By^3 + HOT$ relative to the weights $weight(x) = 3$, $weight(y) = 8$. We now use Arnold's results found in Appendix A.

We have the following Newton diagram, Figure 3.4.

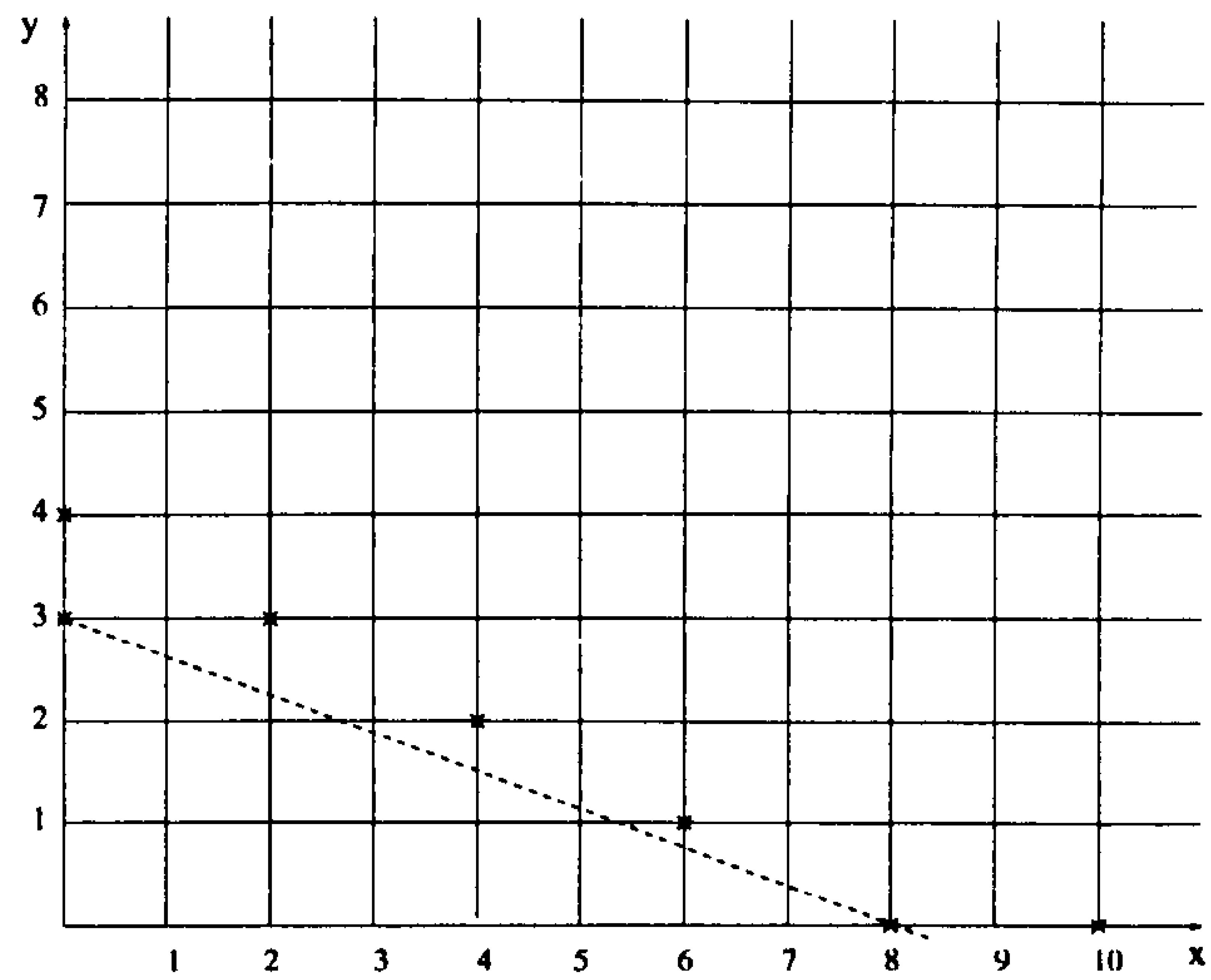


Figure 3.4: Newton diagram

We need to look at the ideal generated by $\langle f_x, f_y \rangle$, namely

$$\langle f_x, f_y \rangle = \langle x^7, y^2 \rangle$$

and we find that the only super-diagonal element of the regular basis is xy^6. So $f \sim x^8 + y^3 + \lambda xy^6$ where $\lambda \neq 0$ and therefore we can scale to give $f \sim x^8 + y^3 + xy^6$ and is of type K_{14}.

- $f(x, y, z) = (x, x^2y + y^4 \pm z^2)$.

$$\begin{aligned} \sum f &= \left\{ (x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ 2xy & x^2 + 4y^3 & \pm 2z \end{pmatrix} < 2 \right\} \\ &= \{(x,y,z) : x^2 + 4y^3 = 0 \ \& \ z = 0\} \\ &= \{(2t^3, -t^2, 0) : t \in \mathbb{R}\} \end{aligned}$$

and the discriminant is given by

$$f|\sum f = \{(-2t^3, -3t^8) : t \in \mathbf{R}\}.$$

This is $\mathcal{K}$–equivalent to $x^8 + y^3 = 0$ and is therefore of type K_{14}.

- $f(x,y,z) = (x, xy^2 + y^4 + y^{2k+1} \pm z^2)$.

$$\begin{aligned}
\sum f &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y^2 & 2xy + 4y^3 + (2k+1)y^{2k} & \pm 2z \end{pmatrix} < 2\right\} \\
&= \{(x,y,z) : 2xy + 4y^3 + (2k+1)y^{2k} = 0 \ \& \ z = 0\} \\
&= \{(t,0,0) \oplus (-2t^2 - \frac{(2k+1)}{2}y^{2k-1}, y, 0) : t \in \mathbf{R}\}.
\end{aligned}$$

and the discriminant is given by

$$f|\sum f = \{(t,0) \oplus (-2t^2 - \frac{(2k+1)}{2}y^{2k-1}, -y^4 - \frac{(2k-1)}{2}y^{2k+1}) : t \in \mathbf{R}\}.$$

This is $\mathcal{K}$–equivalent to $x^{2k+1} + y^2 = 0$ and is therefore of type A_{2k}.

3.3 Codimensions and Unfoldings

Lemma 3.3.1 *The versal unfoldings for our list of map-germs are given in Table 3.3.*

Normal form		Unfolding
(x, y)		(x, y)
$(x, y^2 \pm z^2)$		$(x, y^2 \pm z^2)$
$(x, xy + y^3 \pm z^2)$	A_2	$(x, xy + y^3 \pm z^2)$
$(x, y^3 \pm x^2y \pm z^2)$	A_5	$(x, y^3 \pm x^2y + ay \pm z^2)$
$(x, xy + y^4 \pm z^2)$	E_6	$(x, xy + y^4 + ay^2 \pm z^2)$
$(x, y^3 \pm x^3y \pm z^2)$	A_8	$(x, y^3 \pm x^3y + ay + bxy \pm z^2)$
$(x, xy^2 + y^4 + y^5 \pm z^2)$	A_4	$(x, xy^2 + y^4 + y^5 + ay + by^3 \pm z^2)$
$(x, xy + y^5 \pm y^7 \pm z^2)$	W_{17}	$(x, xy + y^5 \pm y^7 + ay^2 + by^3 \pm z^2)$
$(x, xy + z^3 + ay^3 \pm y^2z + y^5)$		$(x, xy + z^3 \pm y^2z + ay^3 + y^5 + bz + cy^2)$

Table 3.3: Versal unfoldings of map-germs :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^2, 0)$

1. $f(x, y, z) = (x, y)$.

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1,0),(0,1)\rangle + f^*\mathcal{E}_2\{e_1, e_2\}$$

from which we can see that $\mathcal{A}_e$–codimension =0 and therefore f has a trivial unfolding.

2. $f(x, y, z) = (x, y^2 \pm z^2)$.

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1,0),(0,y),(0,z)\rangle + f^*\mathcal{E}_2\{e_1, e_2\}$$

from which again we see that $\mathcal{A}_e$–codimension =0 and therefore f has a trivial unfolding.

3. $f(x, y, z) = (x, xy + y^3 \pm z^2)$.

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1,y),(0,x+3y^2),(0,z)\rangle + f^*\mathcal{E}_2\{e_1,e_2\}.$$

This is 3–$\mathcal{A}$–determined so we work in $J^3(3,2)$.

The following table gives a list of the vectors which we are looking for, a $\surd$ indicates a vector which is contained in the tangent space, whereas a $\times$ indicates that it is missing.

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	$\surd$	$\surd$	x^3	$\surd$	$\surd$
y	$\surd$	$\surd$	y^3	$\surd$	$\surd$
z	$\surd$	$\surd$	z^3	$\surd$	$\surd$
x^2	$\surd$	$\surd$	x^2y	$\surd$	$\surd$
y^2	$\surd$	$\surd$	x^2z	$\surd$	$\surd$
z^2	$\surd$	$\surd$	xy^2	$\surd$	$\surd$
xy	$\surd$	$\surd$	xyz	$\surd$	$\surd$
xz	$\surd$	$\surd$	xz^2	$\surd$	$\surd$
yz	$\surd$	$\surd$	y^2z	$\surd$	$\surd$
			yz^2	$\surd$	$\surd$

We can see that $T\mathcal{A}_e \cdot f$ contains $(x^i,0),(0,x^i)$ for all i, $(0,x\phi)$ from $(0,x+3y^2)$ and $(0,y)$ gives $(\phi,0)$ since $y\phi \in \mathcal{M}_3^4$ and we are working in $J^3(3,2)$. We have $(z^i,0),(0,z^i)$ for all i and $(z\phi,0),(0,z\phi)$.

This means we are looking for $(0,y^2)$ and $(0,y^3)$. We have $(0,x)$ so we have $(0,y^2)$ and we have $(0,xy)$ so this gives $(0,y^3)$.

So f has $\mathcal{A}_e$–codimension $=0$ and therefore has a trivial unfolding.

4. $f(x,y,z) = (x, y^3 \pm x^2y \pm z^2)$.

   This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1,2xy),(0,3y^2 \pm x^2),(0,z)\rangle + f^*\mathcal{E}_2\{e_1,e_2\}.$$

This is 3–$\mathcal{A}_1$–determined so we work in $J^3(3,2)$.

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	$\surd$	$\surd$	x^3	$\surd$	$\surd$
y	$\surd$	$\times$	y^3	$\surd$	$\surd$
z	$\surd$	$\surd$	z^3	$\surd$	$\surd$
x^2	$\surd$	$\surd$	x^2y	$\surd$	$\surd$
y^2	$\surd$	$\surd$	x^2z	$\surd$	$\surd$
z^2	$\surd$	$\surd$	xy^2	$\surd$	$\surd$
xy	$\surd$	$\surd$	xyz	$\surd$	$\surd$
xz	$\surd$	$\surd$	xz^2	$\surd$	$\surd$
yz	$\surd$	$\surd$	y^2z	$\surd$	$\surd$
			yz^2	$\surd$	$\surd$

We can see that $T\mathcal{A}_e \cdot f$ contains $(x^i, 0), (0, x^i)$ for all i, $(0, y^2)$ and $(\phi, 0)$ since $(0, xy\phi) \in \mathcal{M}_3^3$. We have $(0, xy), (z^i, 0), (0, z^i), (z\phi, 0)$ and $(0, z\phi)$. Since we have $(\phi, 0)$ we have $(0, x^2y), (0, xy^2)$. So we are missing $(0, y)$.

So f has $\mathcal{A}_e$–codimension $=1$ and has unfolding

$$f_a(x, y, z) = (x, y^3 \pm x^2y + ay \pm z^2).$$

5. $f(x, y, z) = (x, xy + y^4 \pm z^2.)$

   This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1, y), (0, x + 4y^3), (0, z)\rangle + f^*\mathcal{E}_2\{e_1, e_2\}.$$

   This is 4–$\mathcal{A}$–determined so we work in $J^4(3, 2)$.

   We know that $T\mathcal{A}_e \cdot f$ contains any vector divisible by z so we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	$\surd$	$\surd$	x^4	$\surd$	$\surd$
y	$\star$	$\surd$	x^3y	$\surd$	$\surd$
x^2	$\surd$	$\surd$	x^2y^2	$\surd$	$\surd$
xy	$\surd$	$\surd$	xy^3	$\surd$	$\surd$
y^2	$\surd$	$\star$	y^4	$\surd$	$\surd$
x^3	$\surd$	$\surd$			
x^2y	$\surd$	$\surd$			
xy^2	$\surd$	$\surd$			
y^3	$\surd$	$\surd$			

We have $(x^i,0),(0,x^i)$. Also since we have $(0,\psi)$ for $\phi \in \mathcal{M}_3^7$ we have $(\phi,0)$ for $\phi \in \mathcal{M}_3^6$.

$(0,y^6) \longleftrightarrow (0,xy^3) \longleftrightarrow (xy^2,0) \longleftrightarrow (0,xy^4) \longleftrightarrow (0,y^7)$ which we have.

This also gives $(y^5,0),(xy^3,0)$.

$(0,xy^5) \longleftrightarrow (0,y^8)$ which we have. This also gives $(xy^4,0)$.

$(0,x^2y^4) \longleftrightarrow (0,xy^7)$ which we have. This also gives $(x^2y^3,0)$.

$(0,x^3y^3) \longleftrightarrow (0,x^2y^6)$ which we have. This also gives $(x^3y^2,0)$.

$(0,x^4y^2) \longleftrightarrow (0,x^3y^5)$ which we have. This also gives $(x^4y,0)$.

Have $(0,x^5y)$ since we have $(x^5,0)$.

$(0,y^5) \longleftrightarrow (0,xy^2) \longleftrightarrow (xy,0) \longleftrightarrow (y^4,0)$.

We have $(0,xy^2+4y^5),(xy,xy^2),(xy+y^4,0),(y^4,y^5)$ so $(0,xy^2+4y^5) - 4(y^4,y^5) + 4(xy+y^4,0) - 4(xy,xy^2) = (0,-3xy^2)$ so we have all these vectors.

$(0,xy^4) \longleftrightarrow (0,y^7)$ which we have. This also gives $(xy^3,0)$.

Similarly for $(0,x^2y^3),(0,x^3y^2),(0,x^4y)$.

$(0,y^4) \longleftrightarrow (0,xy)$ which we have. This also gives $(y^3,0)$.

$(0,x^2y^2) \longleftrightarrow (0,xy^5)$ which we have. This also gives $(x^2y,0)$.

$(0,x^3y)$. This comes directly from (x^3,x^3y).

$(0,y^3) \longleftrightarrow (0,x)$ which we have. This also gives $(y^2,0)$.

$(0,x^2y)$. This comes directly from (x^2,x^2y).

$(0,y^2) \longleftrightarrow (y,0)$.

$(0, y) \longleftrightarrow (1, 0)$ which we have.

So f has $\mathcal{A}_e$–codimension $=1$ and has versal unfolding

$$f_a(x, y, z) = (x, xy + y^4 + ay^2 \pm z^2).$$

6. $f(x, y, z) = (x, y^3 \pm x^3 y \pm z^2)$.

This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1, 3x^2 y), (0, 3y^2 \pm x^3), (0, z)\rangle + f^*\mathcal{E}_2\{e_1, e_2\}.$$

This is 4–$\mathcal{A}_1$–determined so we work in $J^4(3, 2)$.

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	√	√	xz^2	√	√
y	√	√	$y^2 z$	√	√
z	√	×	yz^2	√	√
x^2	√	√	x^4	√	√
y^2	√	√	y^4	√	√
z^2	√	√	$x^3 y$	√	√
xy	√	×	$x^3 z$	√	√
xz	√	√	$x^2 y^2$	√	√
yz	√	√	$x^2 yz$	√	√
x^3	√	√	$x^2 z^2$	√	√
y^3	√	√	xy^3	√	√
z^3	√	√	$xy^2 z$	√	√
$x^2 y$	√	√	xyz^2	√	√
$x^2 z$	√	√	xz^3	√	√
xy^2	√	√	$y^3 z$	√	√
xyz	√	√	$y^2 z^2$	√	√
			yz^3	√	√

We can see that $T\mathcal{A}_e \cdot f$ contains $(x^i, 0), (0, x^i)$ for all i, $(0, y^2)$, $(\phi, 0)$ since $(0, x^2 y\phi) \in \mathcal{M}_3^4$. Have $(0, x^2 y), (z^i, 0), (0, z^i), (z\phi, 0), (0, z\phi)$. Since we have $(\phi, 0)$ we have $(0, xy^2), (0, x^3 y), (0, x^2 y^2), (0, xy^3)$. So we are missing $(0, y)$ and $(0, xy)$.

So f has $\mathcal{A}_e$–codimension $=2$ and has unfolding

$$f_{a,b}(x, y, z) = (x, y^3 \pm x^2 y + ay + bxy \pm z^2).$$

7. $f(x,y,z)=(x,xy^2+y^4+y^5\pm z^2)$.

This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e\cdot f=\mathcal{E}_3\langle(1,y^2),(0,2xy+4y^3+5y^4),(0,z)\rangle+f^*\mathcal{E}_2\{e_1,e_2\}.$$

This is 5–$\mathcal{A}_1$–determined so we work in $J^5(3,2)$.

We know that $T\mathcal{A}_ef$ contains any vector divisible by z so we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	√	√	xy^2	√	√	x^5	√	√
y	$*$	×	y^3	√	$*$	x^4y	√	√
x^2	√	√	x^4	√	√	x^3y^2	√	√
xy	√	$*$	x^3y	√	√	x^2y^3	√	√
y^2	√	√	x^2y^2	√	√	xy^4	√	√
x^3	√	√	xy^3	√	√	y^5	√	√
x^2y	√	√	y^4	√	√			

where $*$ indicates dependent vectors. We have $(x^i,0),(0,x^i)$. Also since we have $(0,\phi)$ for $\phi\in\mathcal{M}_3^6$ we have $(\phi,0)$ for $\phi\in\mathcal{M}_3^4$.

$(0,y^5)\longleftrightarrow(0,xy^3)\longleftrightarrow(xy,0)\longleftrightarrow(0,y^4)$ which we have from $(0,2xy^2+4y^4+5y^5)$ and $(0,xy^2+y^4+y^5)$ as we have $(0,xy^2)$.

This also gives $(y^3,0),(y^2,0)$.

$(0,xy^4)\longleftrightarrow(0,x^2y^2)$ which we have. This also gives $(xy^2,0)$. $(0,x^2y^3)\longleftrightarrow(0,xy^5)$ which we have. This also gives $(x^2y,0)$. $(0,x^3y^2)\longleftrightarrow(0,x^2y^4)$ which we have. $(0,x^4y)\longleftrightarrow(0,x^3y^3)$ which we have.

$(0,x^3y)\longleftrightarrow(0,2x^3y+4x^2y^3)$ and we have $(0,x^2y^3)$. $(0,y^3)\longleftrightarrow(0,xy)\longleftrightarrow(0,y)$.

$(0,xy^2)$ we have from $(x,0)$. $(0,x^2y)\longleftrightarrow(0,xy^3)$ which we have. $(0,y^2)$ we have from $(1,y^2)$.

$(0,y)$

So we are missing $(0,y)$ and $(0,y^3)$.

So f has $\mathcal{A}_e$–codimension $=2$ and has unfolding

$$f_{a,b}(x,y,z) = (x, xy^2 + y^4 + y^5 + ay + by^3 \pm z^2).$$

8. $f(x,y,z) = (x, xy + y^5 \pm y^7 \pm z^2)$.

   This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1,y),(0,x+5y^4 \pm 7y^6),(0,z)\rangle + f^*\mathcal{E}_2.$$

   This is 7–$\mathcal{A}$–determined so we work in $J^7(3,2)$.

   We know that $T\mathcal{A}_e f$ contains any vector divisible by z so we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$		$\mathbf{e_1}$	$\mathbf{e_2}$
x	√	√	xy^3	√	√	x^2y^4	√	√
y	*	√	y^4	√	√	xy^5	√	√
x^2	√	√	x^5	√	√	y^6	√	√
xy	√	√	x^4y	√	√	x^7	√	√
y^2	†	*	x^3y^2	√	√	x^6y	√	√
x^3	√	√	x^2y^3	√	√	x^5y^2	√	√
x^2y	√	√	xy^4	√	√	x^4y^3	√	√
xy^2	√	√	y^5	√	√	x^3y^4	√	√
y^3	√	†	x^6	√	√	x^2y^5	√	√
x^4	√	√	x^5y	√	√	xy^6	√	√
x^3y	√	√	x^4y^2	√	√	y^7	√	√
x^2y^2	√	√	x^3y^3	√	√			

where * and † indicate dependent vectors. We have $(x^i,0),(0,x^i)$. Also since we have $(0,\phi)$ for $\phi \in \mathcal{M}_3^8$ we have $(\phi,0)$ for $\phi \in \mathcal{M}_3^7$.

$(0,y^7) \longleftrightarrow (0,y^5) \longleftrightarrow (0,xy)$ which we have. This also gives $(y^4,0),(y^6,0)$.

$(0,xy^6) \longleftrightarrow (0,5y^{10} \pm 7y^{12})$ which we have. This also gives $(xy^5,0)$.

$(0,x^2y^5),(0,x^3y^4)$ etc. follow from arguments similar to the above.

$(0,y^6) \longleftrightarrow (0,y^4) \longleftrightarrow (0,xy^2) \longleftrightarrow (xy,0)$.

So we have $(y^5,y^6) - (xy+y^5,0) + (xy,xy^2) - (0,xy^2+5y^6) = (0,-4y^6)$.

$(0,xy^5) \longleftrightarrow (0,y^9)$ which we have. This also gives $(xy^4,0)$.

$(0, x^2y^4), (0, x^3y^3)$ etc. follow from arguments similar to the above.

$(0, xy^4) \longleftrightarrow (0, x^2)$ which we have. This also gives $(xy^3, 0)$.

$(0, x^2y^3) \longleftrightarrow (0, xy^7)$ which we have . This also gives $(x^2y^2, 0)$.

$(0, x^3y^2)$ etc. follow from arguments similar to the above.

$(0, xy^3) \longleftrightarrow (0, y^7)$ which we have. This also gives $(xy^2, 0)$.

$(0, x^2y^2) \longleftrightarrow (0, xy^6)$ which we have. This also gives $(x^2y, 0)$.

$(0, x^3y)$ comes directly from $(x^3, 0)$.

$(0, y^3) \longleftrightarrow (y^2, 0)$.

$(0, x^2y)$ comes directly from $(x^2, 0)$.

$(0, y^2) \longleftrightarrow (y, 0)$.

$(0, y)$ comes directly from $(1, 0)$.

So we are missing $(0, y^2), (0, y^3)$.

So f has $\mathcal{A}_e$–codimension =2 and has unfolding

$$f_{a,b}(x, y, z) = (x, xy + y^5 \pm y^7 + ay^2 + by^3 \pm z^2).$$

9. $f(x, y, z) = (x, xy + z^3 + ay^3 \pm y^2z + y^5)$.

This has $\mathcal{A}_e$–tangent space

$$T\mathcal{A}_e \cdot f = \mathcal{E}_3\langle(1, y), (0, x + 3ay^2 \pm 2yz + 5y^4), (0, 3z^2 \pm y^2)\rangle + f^*\mathcal{E}_2.$$

This is 5–$\mathcal{A}$–determined so we work in $J^5(3, 2)$.

Using the MAPLE program TRANSVERSAL we are able to work out the unfolding. It is found to be

$$f_{b,c}(x, y, z) = (x, xy + z^3 + ay^3 \pm y^2z + y^5 + bz + cy^2),$$

where a is a modulus.

Chapter 4

Bifurcations of Planar Motions

We recall that the bifurcations of one parameter motions of the plane were discussed in depth in Gibson & Hobbs, [GHo1]. For two parameter motions the corank one mono–germs are well understood in the literature. For multi–germs the bifurcations appear in [GHo2]. The bifurcations for the corank two mono–germs, i.e. the *sharksfin* and the *deltoid*, are not so well understood, but are mentioned in [Hob]. We now present formal arguments establishing the results in [Hob] and present renderings of the bifurcations sets of these corank two mono–germs and for the mono–germs for 3–parameter motions of the plane.

4.1 The sharksfin and the deltoid

We wish to study the bifurcations for the *sharksfin* and the *deltoid* as they give us a graphical link between the mathematics and the kinematics. We are particularly interested in the codimension 1 (multi-)germs as these are the ones which give rise to changes in kinematic behaviour. For planar motions there are six such bifurcations: lips, beaks, swallowtail, tacnodefold, triplepoint and cusp–plus–fold. So we can associate to any motion six bifurcation curves.

The only reference to the unfoldings of the corank 2 planar germs is given in [B2]. Bruce considered any $\mathcal{A}$–finite planar germ in the same $\mathcal{K}$–class as (x^2, y^2), and also noted that neither the cusp–plus–fold stratum, nor the triplepoint stratum, can appear for the sharksfin due to the fact that (x^2, y^2) has multipicity 4 and therefore all nearby types must have multiplicity ≤ 4 which excludes the above mentioned strata.

The sharksfin has versal unfolding $f_{a,b} = (x^2 + y^3 + ay, x^3 + y^2 + bx)$ and its critical set is given by

$$\sum f = \left\{(x,y,z) : 4xy - (3x^2 + b)(3y^2 + a) = 0\right\}.$$

To find the lips/beaks stratum we need to find the points (a, b) for which the critical set $\sum f$ fails to be smooth at some point close to the origin. Now $\sum$ is singular when

$$\begin{array}{lcccl} \sum f & = & 4xy - (3x^2 + b)(3y^2 + a) & = & 0 \\ \sum_x f & = & 4y - 18xy^2 - 6ax & = & 0 \\ \sum_y f & = & 4x - 18x^2y - 6by & = & 0 \end{array}$$

This gives $36xy(3x^2 + b)(3y^2 + a) = 4(3x^2 + b)(3y^2 + a) = 16xy$ so either $3y^2 + a = 0$ or $3x^2 + b = 0$ or $9xy = 1$. We ignore the solutions to $9xy = 1$ since we are working locally at the origin. If $3x^2 + b = 0$, either $x = 0$ or $y = 0$ so either $b = 0$ or $a = 0$. If $3y^2 + a = 0$ then either $x = 0$ or $y = 0$ giving either $a = 0$ or $b = 0$. So our lips/beaks stratum is given by $\{a = 0\} \cup \{b = 0\}$ and since the critical set is given by two transverse curves the transition is a beaks transition and not a lips.

For the tacnode stratum we take two points (x_1, y_1), (x_2, y_2) in the domain of $f_{a,b}(x, y) = (x^2 + y^3 + ay, x^3 + y^2 + bx)$, distinct from each other and from the origin. These two points need to be singular and to give rise to tangent fold curves. The condition for this to happen is given by the rank of the following matrix to be ≤ 1.

$$\begin{pmatrix} 2x_1 & 3y_1^2 + a & 2x_2 & 3y_2^2 + a \\ 3x_1^2 + b & 2y_1 & 3x_2^2 + b & 2y_2 \end{pmatrix}$$

This gives us four equations in x_1, y_1, x_2, y_2.

$$4x_1y_1 - (3x_1^2 + b)(3y_1^2 + a) = 0 \tag{4.1}$$

$$4x_2y_2 - (3x_2^2 + b)(3y_2^2 + a) = 0 \tag{4.2}$$

$$(x_1 - x_2)(3x_1x_2 - b) = 0 \tag{4.3}$$

$$(y_1 - y_2)(3y_1y_2 - a) = 0 \tag{4.4}$$

We also need the two points (x_1, y_1) and (x_2, y_2) to map to the same point in the target, i.e.

$$
\begin{aligned}
x_1^2 + y_1^3 + ay_1 &= x_2^2 + y_2^3 + ay_2 && (4.5)\\
x_1^3 + y_1^2 + bx_1 &= x_2^3 + y_2^2 + bx_2 && (4.6)
\end{aligned}
$$

So we have six equations, one of which is redundant as the rank condition should be given by three conditions, not four. This gives five equations in six variables, x_1, y_1, x_2, y_2, a, b. In principle we should be able to eliminate x_1, y_1, x_2, y_2 to obtain a single polynomial in a and b. It is easy to see that neither $x_1 = x_2$ nor $y_1 = y_2$ leads to a solution of this system of equations, so from Eqns. 4.3 & 4.4 we can say that $b = 3x_1x_2$ and $a = 3y_1y_2$. This yields the following equations from Eqns. 4.1 & 4.2:

$$
\begin{aligned}
x_1y_1\{4 - 9(x_1 + x_2)(y_1 + y_2)\} &= 0\\
x_2y_2\{4 - 9(x_1 + x_2)(y_1 + y_2)\} &= 0
\end{aligned}
$$

We are working locally so we can discard the solution given by the brackets, so $x_1y_1 = 0$ and $x_2y_2 = 0$. We consider the various possibilities:

- $x_1 = 0,\ x_2 = 0$.

  Eqn. 4.6 yields $y_1^2 = y_2^2$ which gives us $y_1 = \pm y_2$. If $y_1 = y_2$ then we contradict the hypothesis that the points are distinct. If $y_1 = -y_2$ then Eqn. 4.5 gives $y_1 = 0$ and $y_2 = 0$, again contradicting the hypothesis.

- $x_1 = 0,\ y_2 = 0$. Eqns. 4.5 & 4.6 yield $y_1^3 = x_2^2$ and $y_1^2 = x_2^3$, the only solutions to which are $x_2 = y_1 = 0$ or $x_2 = y_1 = 1$. If $x_2 = y_1 = 0$ then both points are the origin which contradicts our hypothesis. $x_2 = y_1 = 1$ is excluded as x_1, y_1, x_2, y_2 are assumed to be small.

The remaining two cases $y_1 = 0,\ y_2 = 0$ and $y_1 = 0,\ x_2 = 0$ can be discarded by symmetry. Therefore there are no tacnode folds in the unfolding of the sharksfin.

For the swallowtail stratum, we look for solutions of the equation $f(x,y) = (\alpha,\beta)$ of multiplicity≥ 4. So we want solutions of

$$\begin{aligned} x^2 + y^3 + ay &= \alpha & (4.7) \\ x^3 + y^2 + bx &= \beta & (4.8) \end{aligned}$$

Re–arranging Eqn. 4.8 we have $y^2 = \beta - g(x,b)$, where $g(x,b) = x^3 + bx$. Now re–arranging Eqn.4.7 we have

$(x^2 - \alpha)^2 = (y^3 + ay)^2 = y^2(y^2 + a)^2 = (\beta - g(x,b))(\beta + a - g(x,b))^2 = (\beta - x^3 - bx)(\beta + a - x^3 - bx)^2$.

If we set $\alpha = \beta = 0$ we get $x^4 = x^9$ which has a quadruple root at the origin, so we set

$$P = (x^2 - \alpha)^2 + (x^3 + bx - \beta)(x^3 + bx - a - \beta)^2.$$

The condition that this has a quadruple root is that $P = P' = P'' = P''' = 0$. This yields four equations in five unknowns x, a, b, α, β and in principle we should be able to eliminate x, α and β to give a single polynomial in a and b. This elimination has been found to be impossible to attempt without use of computer algebra. Even attempts with standard computer algebra packages have been unsuccessful, due to restrictions on CPU time, even on an intensive machine. Several experts in the field of computer algebra have been interested in this problem and have obtained the following result. Prof. Pfister has shown that the polynomial has the form $b^5(a^5b + HOT)$. The picture of the unfolding of the sharksfin is given in Fig.4.1.

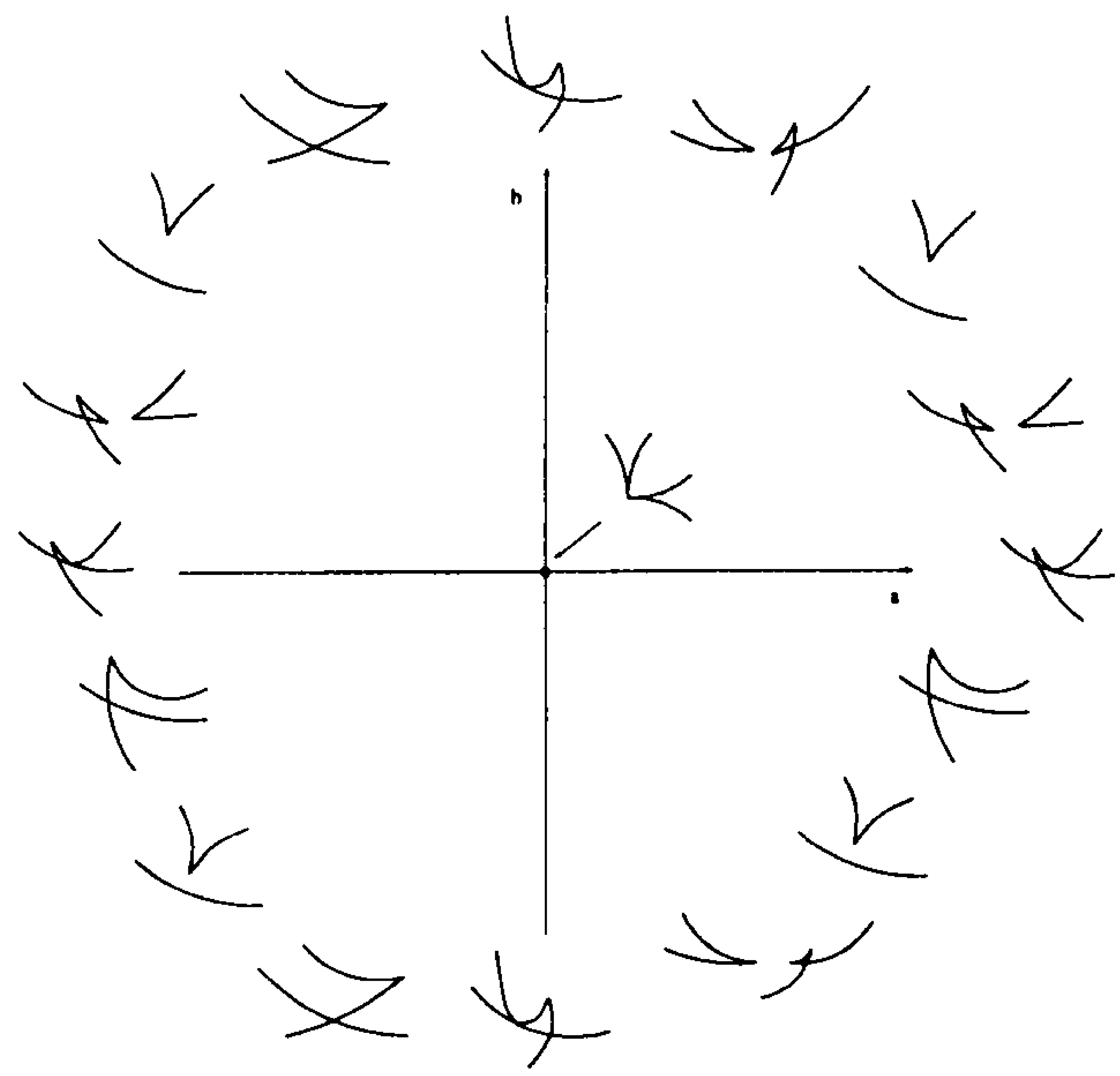


Figure 4.1: The sharksfin unfolding

Now the study of the bifurcations of the deltoid singularity is not covered by [B2] but the same principles apply. Again neither triplepoints nor cusp–plus–fold strata can appear in the unfolding as $(x^2 - y^2, xy)$ has multiplicity 4. The deltoid has versal unfolding $f_{a,b} = (x^2 - y^2 + x^3 + ax, xy + bx)$ and its critical set is given by

$$\sum f = \left\{(x, y, z) : ax + 2by + 2x^2 + 2y^2 + 3x^3 = 0\right\}.$$

To find the lips/beaks stratum we need to find the points (a, b) for which the critical set $\sum f$ fails to be smooth at some point close to the origin. Now $\sum$ is singular when

$$\begin{array}{rcccl} \sum f & = & ax + 2by + 2x^2 + 2y^2 + 3x^3 & = & 0 \\ \sum_x f & = & a + 4x + 9x^2 & = & 0 \\ \sum_y f & = & 2b + 4y & = & 0. \end{array}$$

We can eliminate y to give $9x^2 + 4x + a = 0$ and $12x^3 + 4x^2 + b^2 = 0$. We now find the resultant of these two polynomials with respect to x to give us a polynomial of the form $P = \mu(a^2 + 4b^2) + HOT$ for some $\mu \neq 0$. The only solution to which is the origin so neither lips nor beaks strata can occur.

For the tacnode stratum we take two points (x_1, y_1), (x_2, y_2) in the domain of $f_{a,b}(x, y) = (x^2 - y^2 + x^3 + ax, xy + bx)$, distinct from each other and from the origin. These two points need to be singular and to give rise to tangent fold curves. The condition for this to happen is given by the rank of the following matrix to be ≤ 1.

$$\begin{pmatrix} 2x_1 + 3x_1^2 + a & -2y_1 & 2x_2 + 3x_2^2 + a & -2y_2 \\ y_1 + b & x_1 & y_2 + b & x_2 \end{pmatrix}$$

Since we have one redundant condition (x_1, y_1) and (x_2, y_2) must be linearly dependent, i.e. there exists a scalar $\lambda \neq 0, 1$ such that $x_2 = \lambda x_1$ and $y_2 = \lambda y_1$. So the rank condition gives

$$\begin{aligned} ax_1 + 2by_1 - 3\lambda x_1^3 &= 0 \\ 2(x_1^3 + y_1^2) + 3(1 + \lambda)x_1^3 &= 0. \end{aligned}$$

We also need the two points (x_1, y_1) and (x_2, y_2) to map to the same point in the target, i.e.

$$\begin{aligned} x_1^2 - y_1^2 + x_1^3 + ax_1 &= \lambda^2 x_1^2 - \lambda^2 y_1^2 + \lambda^3 x_1^3 + a\lambda x_1 \\ x_1 y_1 + bx_1 &= \lambda^2 x_1 y_1 + b\lambda x_1. \end{aligned}$$

The only solution to these four equations is $x_1 = y_1 = 0$ which contradicts our hypothesis, so no tacnode folds occur in the unfolding of the deltoid.

As for the sharksfin, looking for the swallowtail stratum involves looking for the solutions of the equation $f(x,y) = (\alpha, \beta)$ of multiplicity ≥ 4. This gives us the equations

$$\begin{aligned} x^2 - y^2 + x^3 + ax &= \alpha & (4.9) \\ xy + bx &= \beta. & (4.10) \end{aligned}$$

Solving Eqn.4.10 for y we have $y = \frac{\beta - bx}{x}$ and substituting into Eqn.4.9 we have a polynomial $P = x^4 - (\beta - bx)^2 + x^5 + ax^3 - \alpha x^2 = 0$ of degree 5 and the condition for the root to have multiplicity 4 is that $P = P' = P'' = P''' = 0$, giving us the following equations:

$$\begin{aligned} x^4 - (\beta - bx)^2 + x^5 + ax^3 - \alpha x^2 &= 0 & (4.11) \\ 4x^3 + 2(\beta - bx)b + 5x^4 + 3ax^2 - 2\alpha x &= 0 & (4.12) \\ 12x^2 - 2b^2 + 20x^3 + 6ax - 2\alpha &= 0 & (4.13) \\ 24x + 60x^2 + 6a &= 0. & (4.14) \end{aligned}$$

So from Eqn. 4.14 we have $a = -4x - 10x^2$, so substituting into Eqns. 4.11, 4.12, 4.13 we have

$$\begin{aligned} x^4 - (\beta - bx)^2 + x^5 + (-4x - 10x^2)x^3 - \alpha x^2 &= 0 & (4.15) \\ 4x^3 + 2(\beta - bx)b + 5x^4 + 3(-4x - 10x^2)x^2 - 2\alpha x &= 0 & (4.16) \\ 12x^2 - 2b^2 + 20x^3 + 6(-4x - 10x^2)x - 2\alpha &= 0. & (4.17) \end{aligned}$$

From Eqn. 4.17 we have $\alpha + b^2 = 6x^2 + 10x^3 + 3(-4x - 10x^2)x = -6x^2 - 20x^3$. Again substituting into Eqns. 4.15 & 4.16 we have

$$\begin{aligned} x^4 - \beta^2 + 2b\beta x - (-6x^2 - 20x^3)x^2 + x^5 + (-4x - 10x^2)x^3 &= 0 & (4.18) \\ 4x^3 + 2\beta b + 5x^4 + 3(-4x - 10x^2)x^2 - 2(-6x^2 - 20x^3)x &= 0. & (4.19) \end{aligned}$$

From Eqn. 4.19 we have $2\beta b = -4x^3 - 5x^4 - 3(-4x - 10x^2)x^2 + 2(-6x^2 - 20x^3)x = -4x^3 - 15x^4$. Finally substituting into Eqn. 4.18 we can solve the following equation for β^2:

$$x^4 - \beta^2 - 4x^4 - 15x^5 + 6x^4 + 15x^5 + x^5 - 4x^4 - 10x^5 = -x^4 - 9x^5 - \beta^2 = 0.$$

This gives us that $\beta^2 = -x^4 - 9x^5$ and substituting these values for $\alpha + b^2, 2\beta b$ and β^2 in the equations we obtain the following parametrization for a, b^2.

$$\begin{aligned} a &= -4x - 10x^2 \\ b^2 &= -\frac{x^2(15x-4)^2}{4(4x+1)} \end{aligned}$$

The resultant of these two polynomials in x gives us a polynomial in a, b^2, namely

$50625a^4 + 180000a^3 + 996000a^2b^2 + 160000a^2 - 1920000ab^2 + 256000ab^4 + 640000b^2 - 96000b^4$.

So the lowest order terms are given by $\lambda(a^2 + 4b^2)$ where $\lambda \neq 0$. From this we can see that no swallowtail stratum appears in the unfolding of the deltoid.

The picture of the unfolding of the deltoid is given in Fig.4.2.

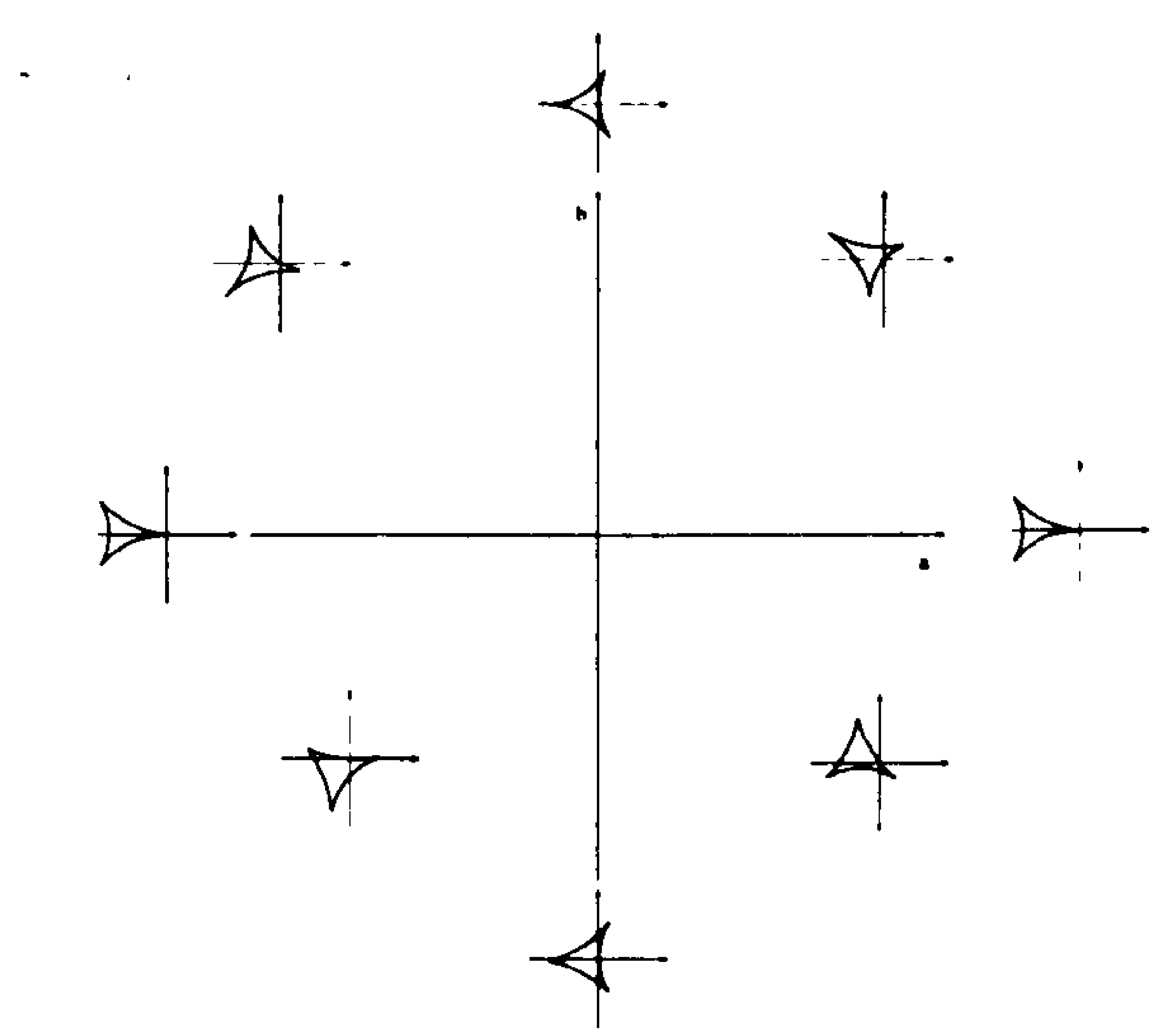


Figure 4.2: The deltoid unfolding

4.2 The bifurcations of 3–dimensional planar motions

Again we wish to study the bifurcations of these motions to find the curves on which the type of kinematic behaviour changes. We have the following list of versal unfoldings:

$\mathcal{K}$–type	$\mathcal{A}_e$ –Cod	Versal unfolding
	0	(x, y)
	0	$(x, y^2 \pm z^2)$
A_2	0	$(x, xy + y^3 \pm z^2)$
A_5	1	$(x, y^3 \pm x^2 y \pm z^2 + ay)$
E_6	1	$(x, xy + y^4 \pm z^2 + ay^2)$
A_8	2	$(x, y^3 + x^3 y \pm z^2 + ay + bxy)$
W_{17}	2	$(x, xy + y^5 \pm y^7 \pm z^2 + ay^2 + by^3)$
A_4	2	$(x, xy^2 + y^4 + y^5 \pm z^2 + ay + by^3)$
	2	$(x, xy + z^3 \pm y^2 z + ay^3 + y^5 + bz + cy^2)$

Table 4.1: Map-germs from $\mathbb{R}^3, 0 \longrightarrow \mathbb{R}^2, 0$

- $f(x, y, z) = (x, y)$.

  This is a submersion ; it has $\mathcal{A}_e$–codimension $= 0$ (i.e. it is stable) and therefore its unfolding is trivial.

- $f(x, y, z) = (x, y^2 \pm z^2)$.

  This has $\mathcal{A}_e$–codimension $= 0$ and therefore its unfolding is trivial.

- $A_2 : f(x, y, z) = (x, xy + y^3 \pm z^2)$.

  This is a cusp, which is stable. It has $\mathcal{A}_e$–codimension $= 0$ and therefore its unfolding is trivial.

- $A_5 : f(x,y,z) = (x, y^3 \pm x^2y \pm z^2)$.

  This has $\mathcal{A}_e$–codimension=1 and

$$T\mathcal{A}_e.f \;=\; (\mathcal{E}_3, \mathcal{E}_3 - \{y\}).$$

So the versal unfolding is $(x, y^3 \pm x^2y \pm z^2 + ay)$.

We now look at the critical set of this unfolding. We find the critical set in the following way

$$\begin{aligned}\sum f_a &= \left\{(x,y,z)\ :\ rank\begin{pmatrix} 1 & 0 & 0 \\ \pm 2xy & 3y^2 \pm x^2 + a & \pm 2z \end{pmatrix} < 2\right\} \\ &= \left\{(x,y,z)\ :\ 3y^2 \pm x^2 + a = 0\ ;\ z = 0\right\}.\end{aligned}$$

We have two cases for this germ:

1. $f(x,y,z) = (x, y^3 + x^2y \pm z^2)$
   Versal unfolding is $f_a = (x, y^3 + x^2y \pm z^2 + ay)$
   and critical set is defined by $3y^2 + x^2 + a = 0$ and $z = 0$.

   This defines an ellipse in $\mathbb{R}^2$ when $a < 0$. For $a = 0$ we have the origin and for $a > 0$ we have the empty set. See Figure 4.3.
   This has discriminant $(\sqrt{-3y^2 - a}, -2y^3)$.
   Now the algebraic curve is given by

$$\begin{aligned} X &= \sqrt{-a - 3y^2}, & Y &= -2y^3, \\ X^2 &= -a - 3y^2, & Y^2 &= 4y^6. \\ X^2 + a &= -3y^2, & & \\ (X^2 + a)^3 &= -27y^6. & & \end{aligned}$$

$$\Rightarrow \qquad 27Y^2 + 4(X^2 + a)^3 = 0.$$

   The bifurcation set is given by $a = 0$ and the unfoldings are shown in Figure 4.4.

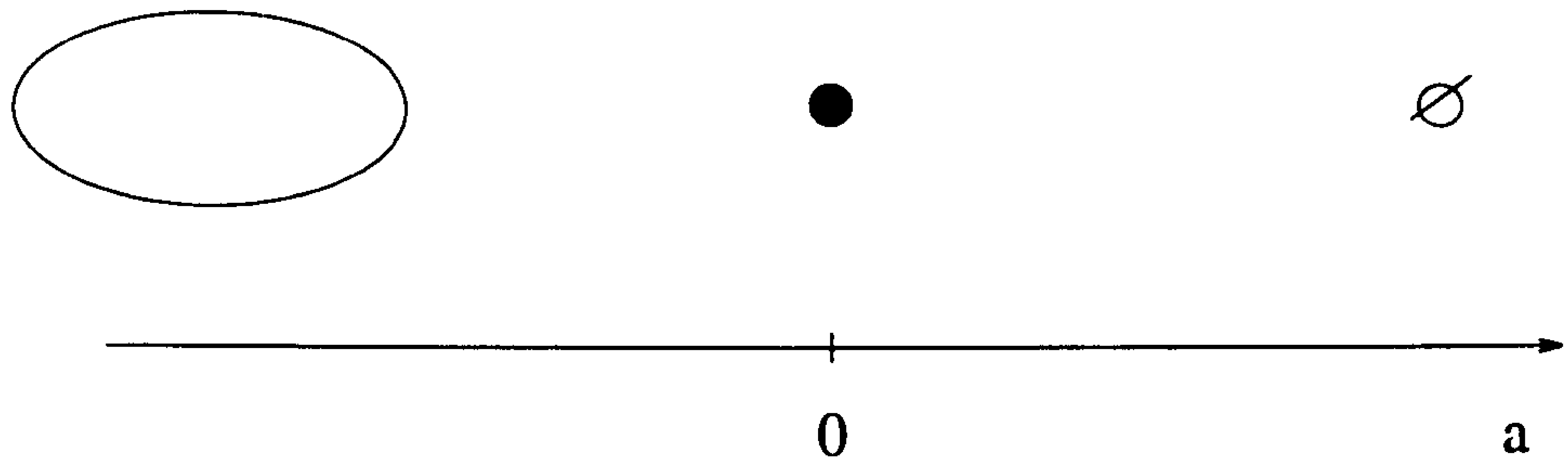


Figure 4.3: Critical Set of $f_a(x,y,z) = (x, y^3 + x^2y \pm z^2 + ay)$

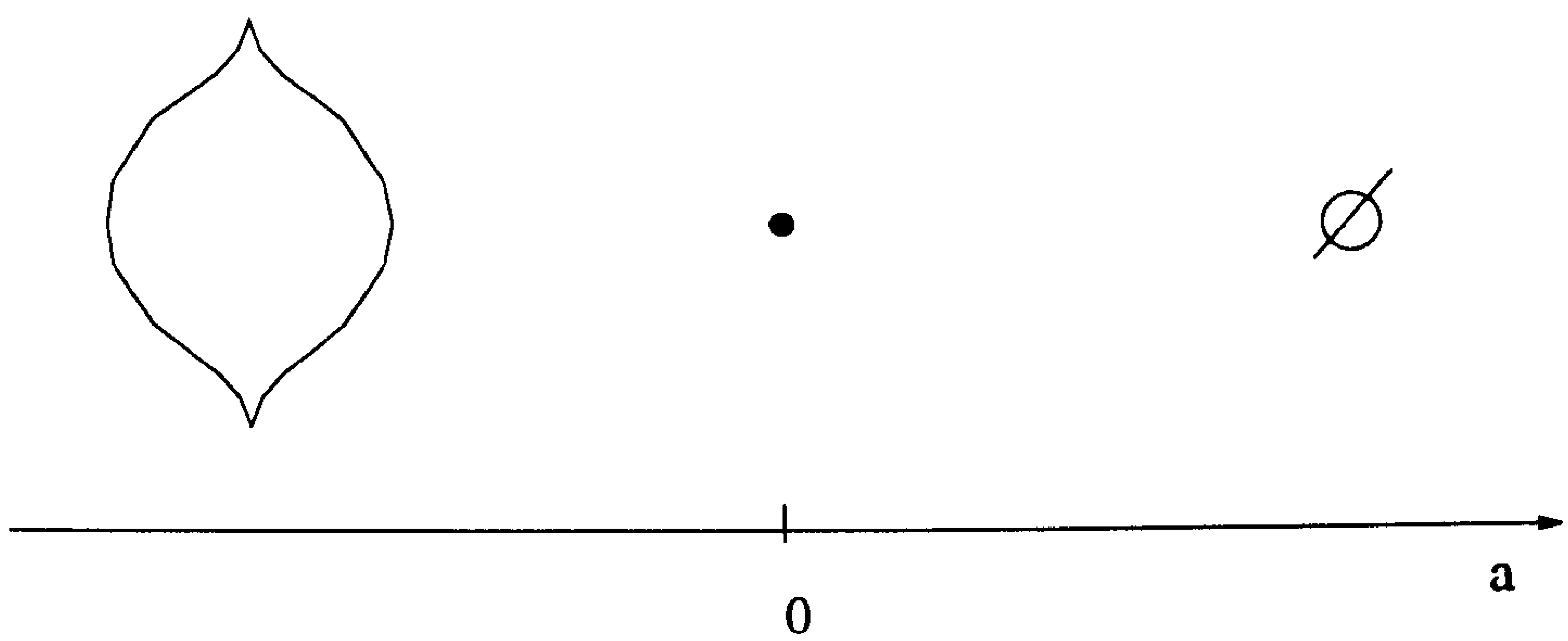


Figure 4.4: Unfolding of $f_a(x,y,z) = (x, y^3 + x^2y \pm z^2 + ay)$

2. $f(x,y,z) = (x, y^3 - x^2y \pm z^2)$
   Versal unfolding is $f_a = (x, y^3 - x^2y \pm z^2 + ay)$
   and critical set is defined by $3y^2 - x^2 + a = 0$ and $z = 0$. This defines a hyperbola in $\mathbb{R}^2$ with 2 branches given by:-
   - For $a < 0$

$$\begin{aligned} 3y^2 &= x^2 - a \\ y &= \pm\frac{1}{\sqrt{3}}(x^2 - a)^{\frac{1}{2}}. \end{aligned}$$

   - For $a = 0$

$$3y^2 = x^2.$$

– For $a > 0$

$$\begin{aligned} x^2 &= 3y^2 + a \\ x &= \pm(3y^2 + a)^{\frac{1}{2}}. \end{aligned}$$

See Figure 4.5
This has discriminant $(\sqrt{3y^2 - a}, 2y^3)$.

Now the algebraic expression is given by

$$\Rightarrow \qquad 27Y^2 - 4(X^2 - a)^3 = 0.$$

The bifurcation set is given by $a = 0$ and the unfoldings are shown in Figure 4.6.

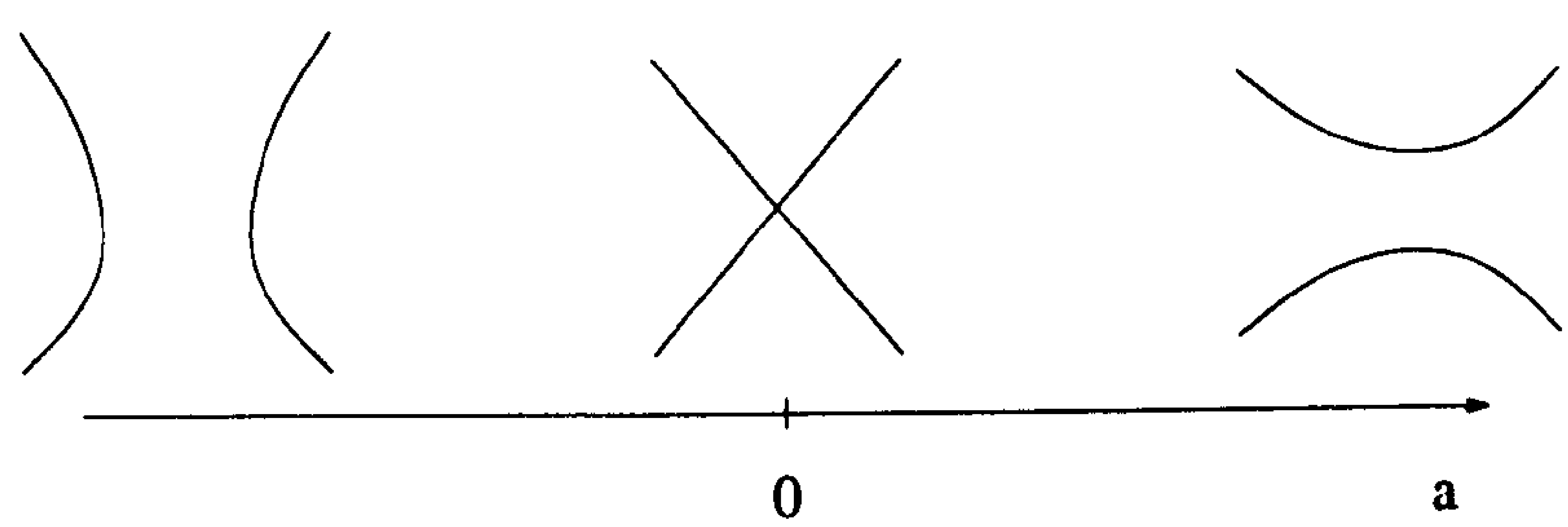


Figure 4.5: Critical Set of $f(x, y, z) = (x, y^3 - x^2y \pm z^2 + ay)$

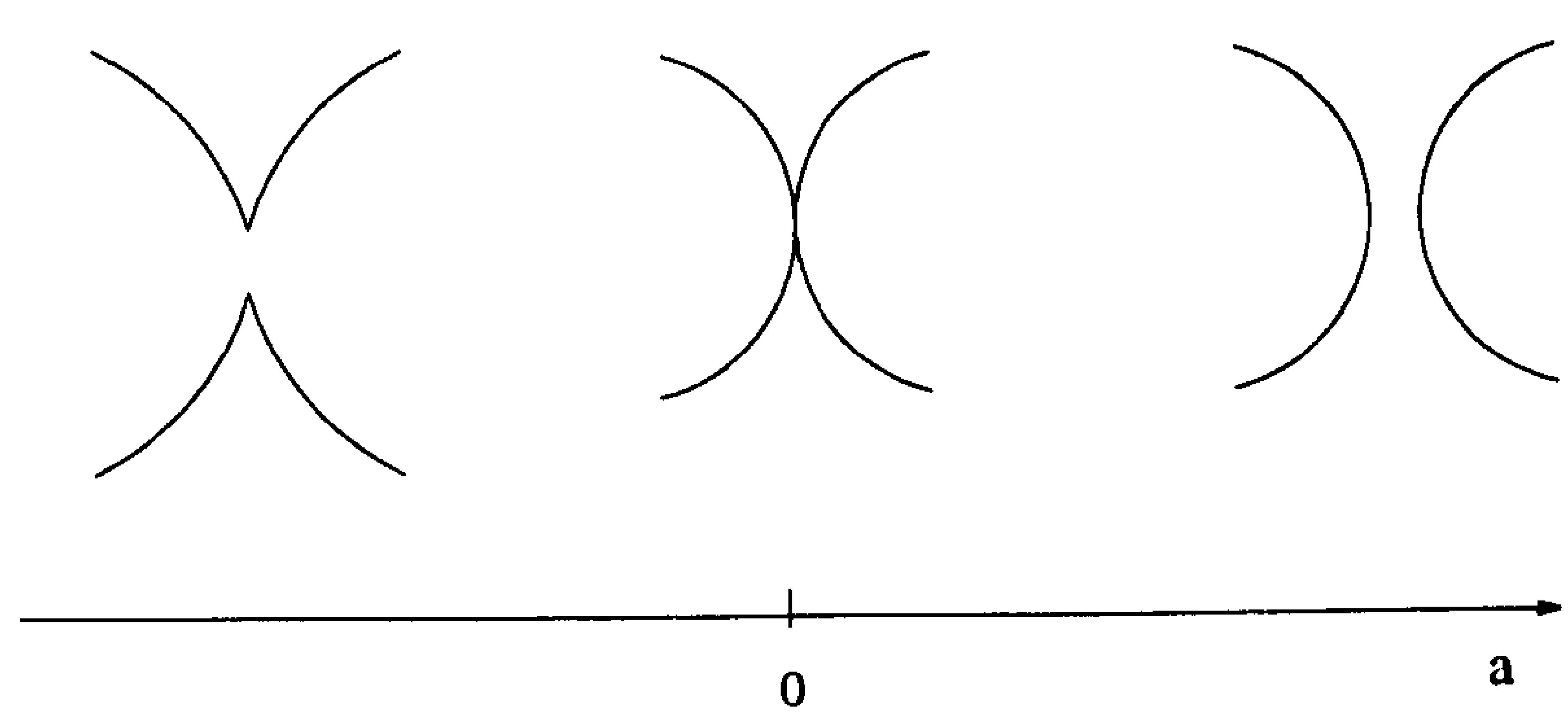


Figure 4.6: Unfolding of $f(x, y, z) = (x, y^3 - x^2y \pm z^2 + ay)$

- $E_6 : f(x,y,z) = (x, xy + y^4 \pm z^2)$.

  This has $\mathcal{A}_e$-codimension=1 and

$$T\mathcal{A}_e.f \quad = \quad (\mathcal{E}_3, \mathcal{E}_3 - \{y^2\}).$$

So the versal unfolding is $(x, xy+y^4 \pm z^2 + ay^2)$ and the critical set is defined by $x + 4y^3 + 2ay = 0$ *and* $z = 0$. See Figure 4.7

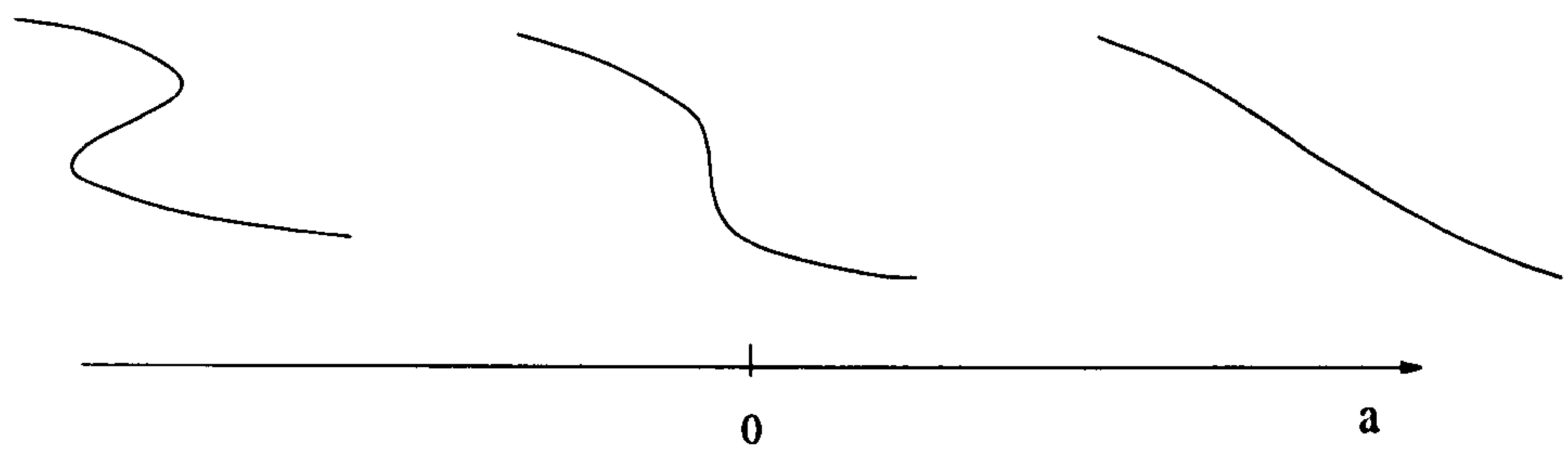


Figure 4.7: Critical Set of $f(x,y,z) = (x, xy + y^4 \pm z^2 + ay^2)$

This has discriminant $(-4y^3 - 2ay, -3y^4 - ay^2)$.

To find the algebraic equation of the curve we put $X = -4y^3 - 2ay$, $Y = -3y^4 - ay^2$ and work out the resultant using a computer algebra package thus eliminating the parameter y. The result is:-

$$16a^4Y + 4X^2a^2 + 128Y^2a^2 + 144aX^2Y + 27X^4 + 256Y^3 = 0.$$

The bifurcation set is given by $a = 0$ and the unfoldings are shown in Figure 4.8.

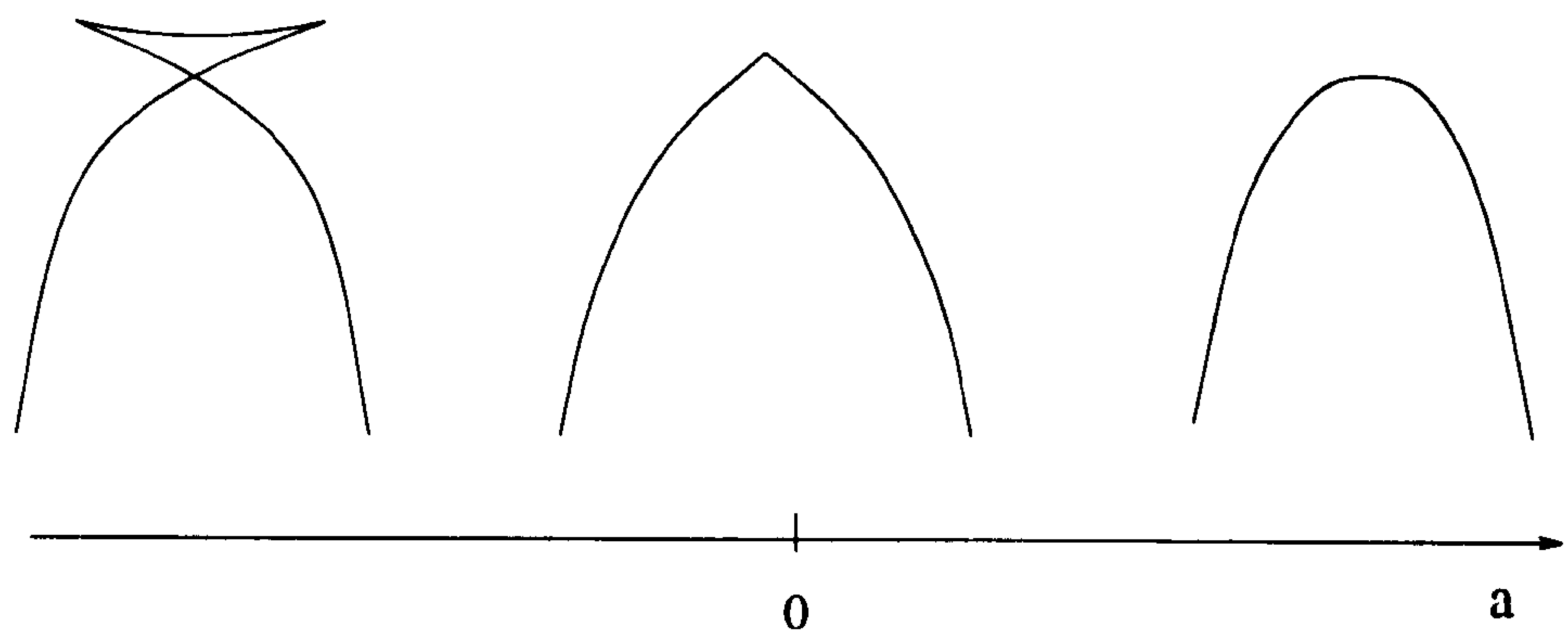


Figure 4.8: Unfolding of $f(x,y,z) = (x, xy + y^4 \pm z^2 + ay^2)$

- $\mathrm{A}_8 : f(x,y,z) = (x, y^3 + x^3y \pm z^2)$.

  This has $\mathcal{A}_e$-codimension=2 and

$$T\mathcal{A}_e.f \quad = \quad (\mathcal{E}_3, \mathcal{E}_3 - \{y, xy\}).$$

So the versal unfolding is $(x, y^3 \pm x^3y \pm z^2 + ay + bxy)$, and the critical set is $\sum f_a = \{(x,y,z) \ : \ 3y^2 + x^3 + a + bx = 0 \ ; \ z = 0\}$. This defines the following curves, see Figure 4.9.

This has discriminant $\left(x, -2\left(\frac{-x^3-bx-a}{3}\right)^{3/2}\right)$.

To obtain the algebraic expression:

$$\begin{aligned} X &= x, & Y &= -2\left(\frac{-x^3-bx-a}{3}\right)^{\frac{3}{2}}, \\ X &= x, & Y^2 &= 4\left(\frac{-x^3-bx-a}{3}\right)^3, \\ X &= x. & 27Y^2 &= 4\left(-x^3-bx-a\right)^3. \end{aligned}$$

$$\Rightarrow \qquad 27Y^2 - 4(-X^3 - bX - a)^3 = 0.$$

To find the lips/beaks bifurcation set we examine when $\sum f_{a,b}$ is singular. This happens when $\frac{\partial f_2}{\partial y} = \frac{\partial^2 f_2}{\partial x \partial y} = 0$. This gives $3y^2+x^3+a+bx = 0$, $6y = 0$ and $3x^2+b=0$ respectively. Solving these three equations for a and b gives $a = 2x^3$ and $b = -3x^2$. So we can parametrize the lips/beaks stratum by $(2x^3, -3x^2)$. See Figure 4.10 for the picture of the unfolding space.

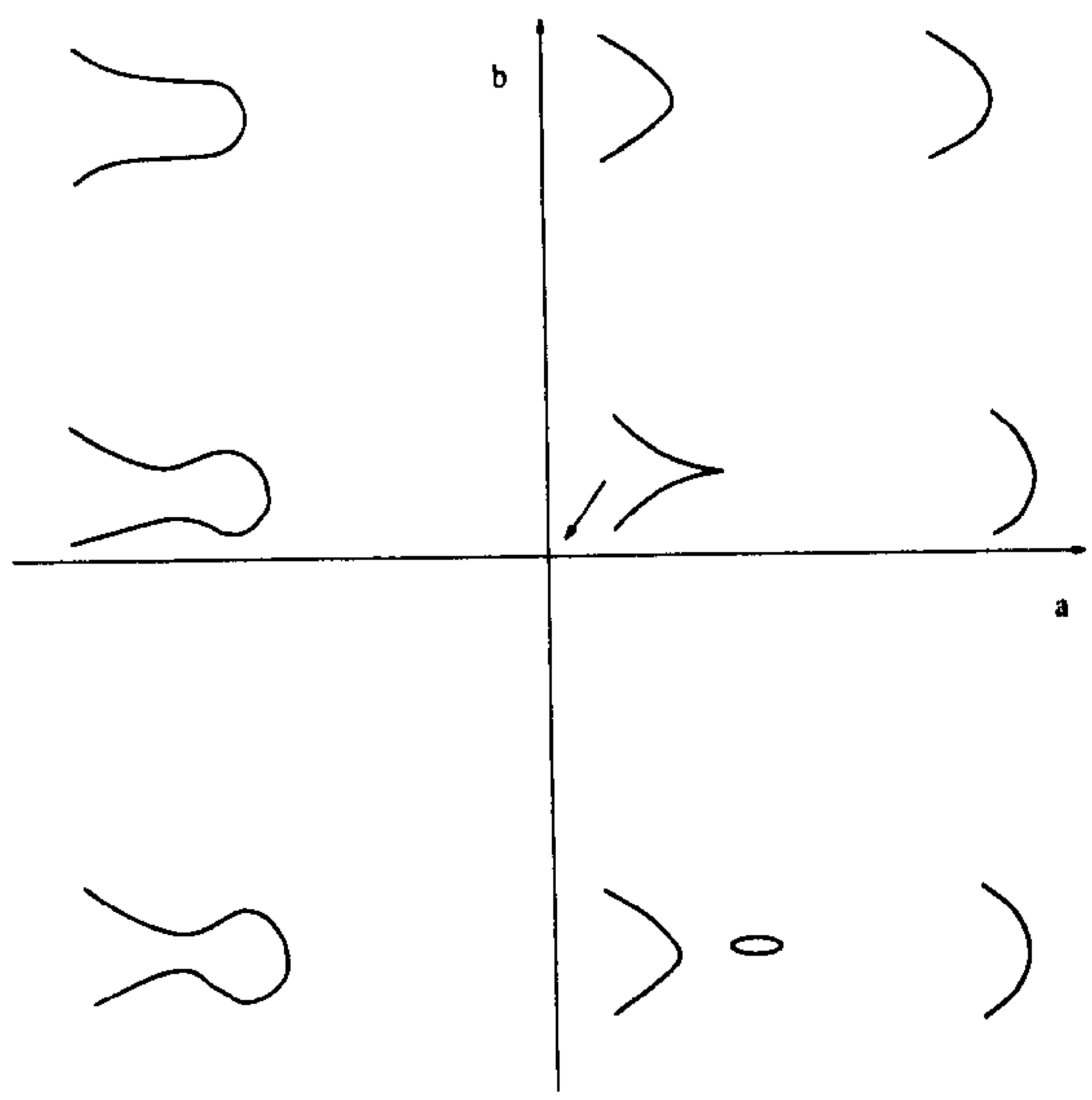


Figure 4.9: Critical set of $f_{a,b} = (x, y^3 + x^3y \pm z^2 + ay + bxy)$

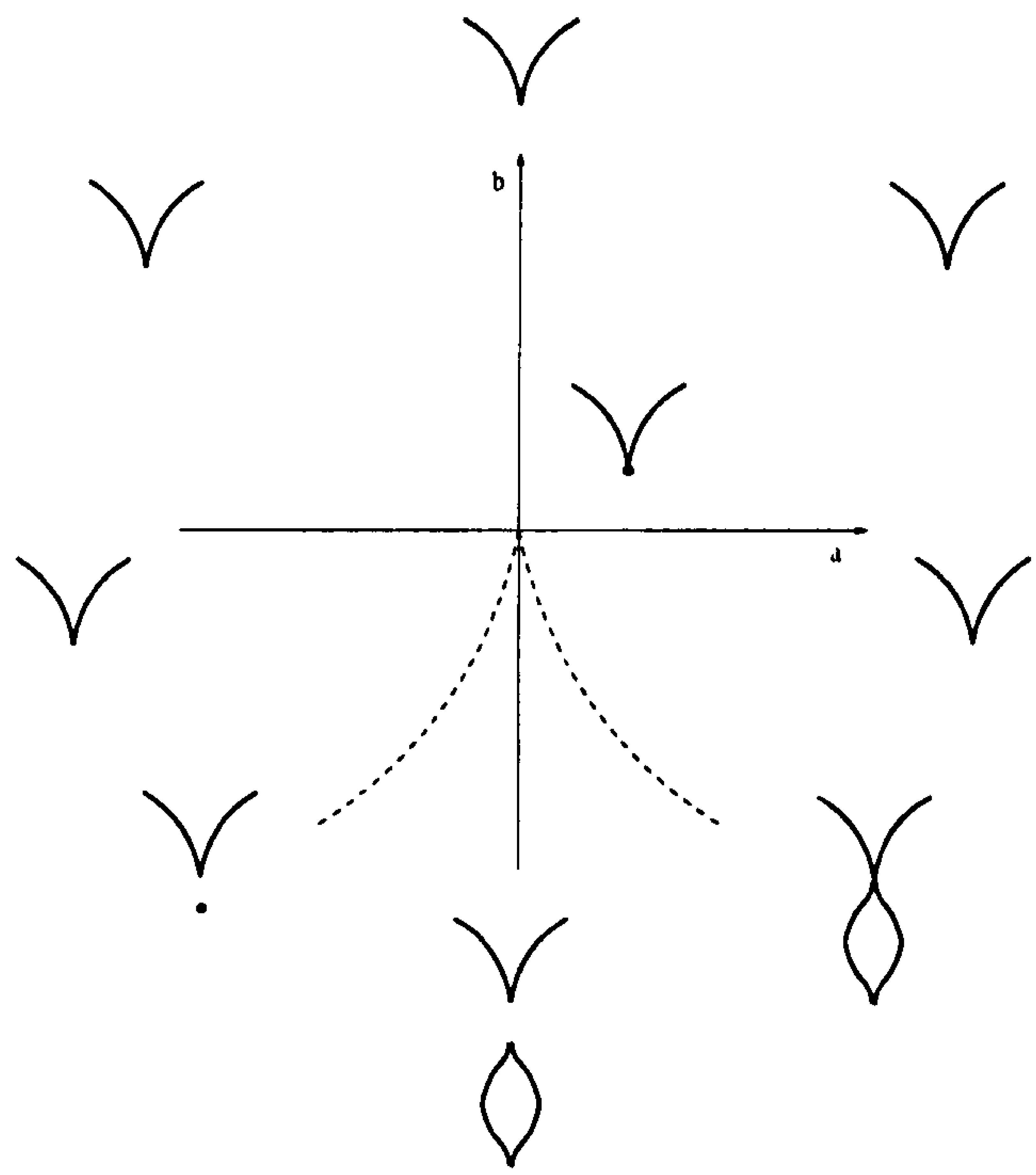


Figure 4.10: Unfolding of $f_{a,b} = (x, y^3 + x^3y \pm z^2 + ay + bxy)$

- $W_{17} : f(x, y, z) = (x, xy + y^5 \pm y^7 \pm z^2)$.

  This has $\mathcal{A}_e$-codimension=2 and

$$T\mathcal{A}_e.f \quad = \quad (\mathcal{E}_3, \mathcal{E}_3 - \{y^2, y^3\}).$$

So the versal unfolding is $(x, xy + y^5 \pm y^7 \pm z^2 + ay^2 + by^3)$ and the critical set is given by $x + 5y^4 \pm 7y^6 + 2ay + 3by^2 = 0$ and $z = 0$.

See Figure 4.11.

This has discriminant given by

$$\left(-2ay - 3by^2 - 5y^4 \mp 7y^6, -ay^2 - 2by^3 - 4y^5 \mp 6y^7\right).$$

The swallowtail stratum is found when

$$\frac{\partial f_2}{\partial y} = \frac{\partial^2 f_2}{\partial y^2} = \frac{\partial^3 f_2}{\partial y^3} = 0.$$

This gives

$$\begin{aligned} \pm 7y^6 + 5y^4 + 3by^2 + 2ay + x &= 0 \\ \pm 42y^5 + 20y^3 + 6by + 2a &= 0 \\ \pm 210y^4 + 60y^2 + 6b &= 0. \end{aligned}$$

So solving for a and b we get $a = 20y^3 \pm 84y^5$ and $b = 10y^2 \mp 35y^4$. So the swallowtail stratum is parametrized by $(20y^3 \pm 84y^5, 10y^2 \mp 35y^4)$, which is locally a cusp.

The cusp–and–fold stratum occurs when f has a cusp at (x_1, y_1, z_1) and a fold at (x_2, y_2, z_2) and $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$ Assuming $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$.) In this case $f(x_1, y_1, z_1) = f(x_2, y_2, z_2) \Leftrightarrow x_1 = x_2$ and $f_2(x_1, y_1, z_1) = f_2(x_2, y_2, z_2)$ (that is $(y_1, z_1) \neq (y_2, z_2)$). For a fold at (x_1, y_2, z_2) we need $\frac{\partial f_2}{\partial y}(x_1, y_2, z_2) = 0$ and $\frac{\partial f_2}{\partial z}(x_1, y_1, z_1) = 0 \Longrightarrow z_1 = 0$. For a cusp at (x_1, y_1, z_1) we need $\frac{\partial f_2}{\partial y}(x_1, y_1, z_1) = \frac{\partial^2 f_2}{\partial y^2}(x_1, y_1, z_1) = 0$ and $\frac{\partial f_2}{\partial z}(x_2, y_2, z_2) = 0 \Longrightarrow z_2 = 0$. So thinking of f_2 as a function of y for a given $x_1 = x_2 = x$, say, there exists a real w such that $f_2(x, y, 0) - w$ has one real repeated root and one triple root. Therefore we have

$$f_2(x, y, 0) - w = (y - y_1)^2 (y - y_2)^3 (\pm y^2 + \alpha y + \beta).$$

Comparing coefficients of this with $f_2(x, y, 0)$ gives us $a = -\frac{10}{27} y_2^3 + \ldots$ and $b = -\frac{5}{3} y_2^2 + \ldots$. So the cusp–and–fold stratum is a cusp, parametrized by

$$\left(-\frac{10}{27} y_2^3 + \ldots, -\frac{5}{3} y_2^2 + \ldots \right).$$

The tacnode stratum is found by looking for two points $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ with $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$ where both are folds and are tangential. This gives us the following equations:

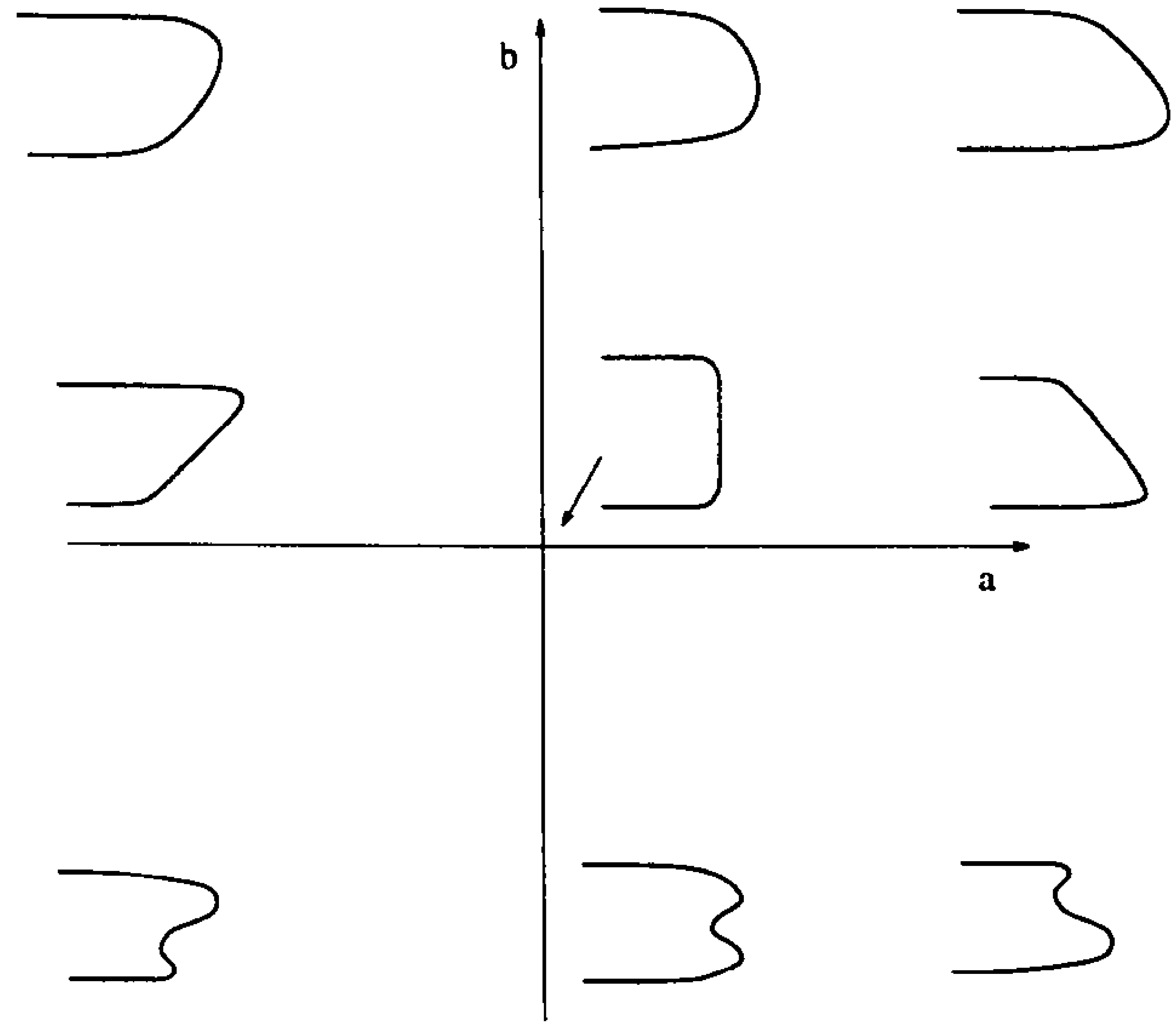


Figure 4.11: Critical Set of $f_{a,b}(x, y, z) = (x, xy + y^5 \pm y^7 \pm z^2 + ay^2 + by^3)$

$$
\begin{aligned}
xy_1 + y_1^5 \pm y_1^7 \pm z_1^2 + ay_1^2 + by_1^3 &= xy_2 + y_2^5 \pm y_2^7 \pm z_2^2 + ay_2^2 + by_2^3, \\
x + 5y_1^4 \pm 7y_1^6 + 2ay_1 + 3by_1^2 &= 0, \\
x + 5y_2^4 \pm 7y_2^6 + 2ay_2 + 3by_2^2 &= 0, \\
z_1 &= 0, \\
z_2 &= 0.
\end{aligned}
$$

Now the two folds are tangent so $\frac{\partial f_2}{\partial x}(x, y_1, z_1) = \frac{\partial f_2}{\partial x}(x, y_2, z_2)$ which gives $y_1 = y_2$ and since both are folds $z_1 = z_2 = 0$ then this contradicts the hypothesis, i.e. we have no tacnode stratum.

See Fig. 4.12.

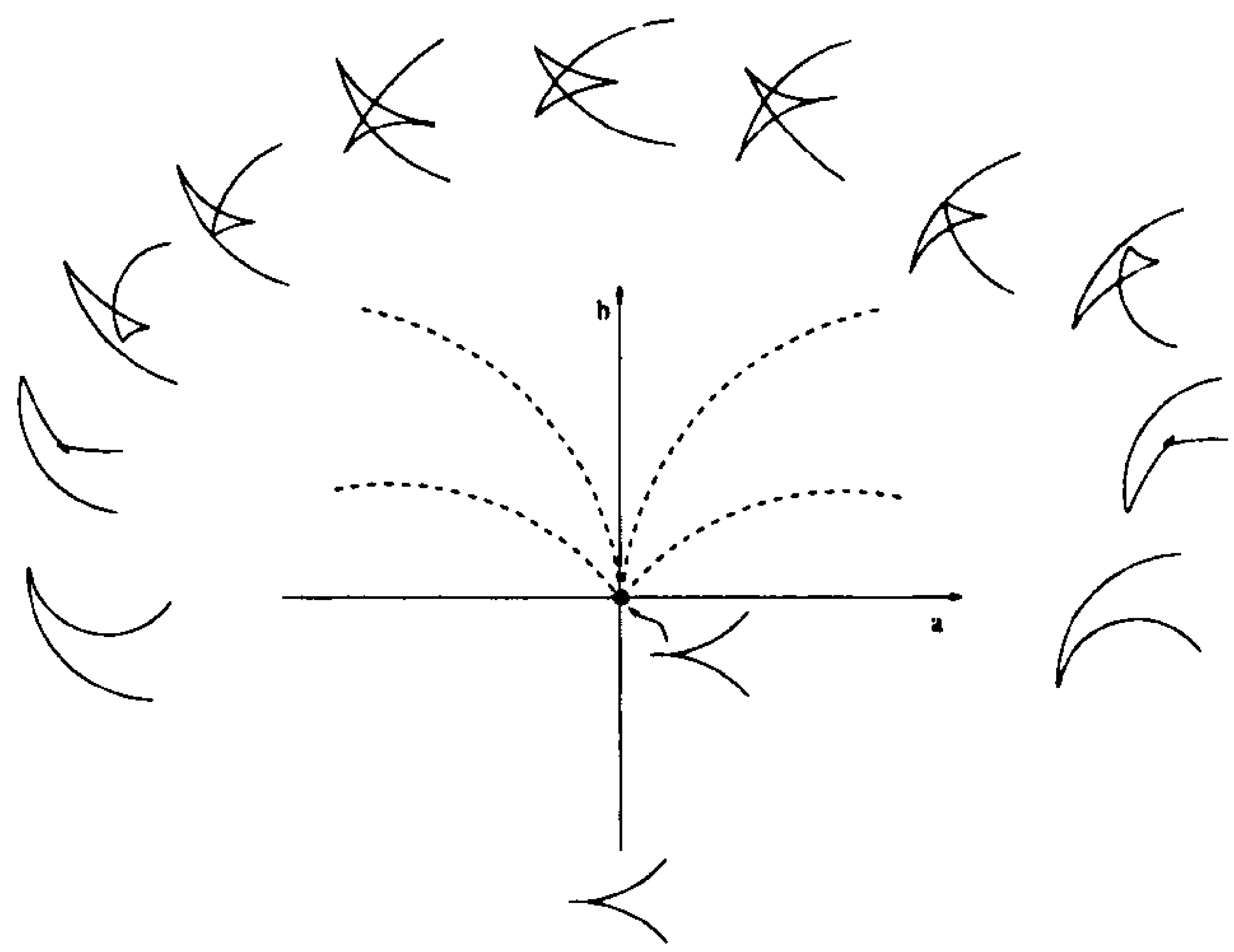


Figure 4.12: Unfolding of $f_{a,b}(x,y,z) = (x, xy + y^5 \pm y^7 \pm z^2 + ay^2 + by^3)$

- $\mathrm{A}_4 : f(x,y,z) = (x, xy^2 + y^4 + y^5 \pm z^2)$.

  This has $\mathcal{A}_e$-codimension=2 and

$$TA_e.f \quad = \quad (\mathcal{E}_3, \mathcal{E}_3 - \{y, y^3\}).$$

So the versal unfolding is $(x, xy^2 + y^4 + y^5 \pm z^2 + ay + by^3)$.

We now look at the critical set of this unfolding. We find the critical set in the following way:

$$\begin{aligned}
\sum f_{a,b} &= \left\{(x,y,z) \ : \ rank \begin{pmatrix} 1 & 0 & 0 \\ y^2 + by & 2xy + 4y^3 + 5y^4 + a + 3by^2 & \pm 2z \end{pmatrix} < 2\right\} \\
&= \left\{(x,y,z) \ : \ 2xy + 4y^3 + 5y^4 + a + 3by^2 = 0 \ ; \ z = 0\right\} \\
&= \left\{\left(\frac{-4y^3 - 5y^4 - a - 3by^2}{2y}, y, 0\right)\right\}.
\end{aligned}$$

See Figure 4.13.

We can now find the discriminant of f_a as follows:

$$f_{a,b}|\sum f_{a,b} = \left(\frac{(-4y^3 - 5y^4 - a - 3by^2)}{2y}, -y^4 - \frac{3}{2}y^5 + \frac{a}{2}y - \frac{b}{2}y^3\right).$$

Now the swallowtail stratum is found when

$$\frac{\partial f_2}{\partial y} + \frac{\partial^2 f_2}{\partial y^2} = \frac{\partial^3 f_2}{\partial y^3} = 0,$$

giving

$$\begin{aligned} 2xy + 4y^3 + 5y^4 + a + 3by^2 &= 0 \\ 2x + 12y^2 + 20y^3 + 6by &= 0 \\ 24y + 60y^2 + 6b &= 0. \end{aligned}$$

Solving these three equations for a and b we find that the swallowtail stratum is parametrized by

$$\left(-4y^3 - 15y^4, -4y - 10y^2\right),$$

which is a cubic through the origin.

The lips/beaks stratum is found when $\sum f_{a,b}$ is singular, i. e. when

$$\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial z} = \frac{\partial^2 f_2}{\partial y^2} = \frac{\partial^2 f_2}{\partial x \partial y} = 0,$$

giving

$$\begin{aligned} 2xy + 4y^3 + 5y^4 + 3by^2 + a &= 0 \\ z &= 0 \\ 2y &= 0 \\ 2x + 12y^2 + 20y^3 + 6by &= 0. \end{aligned}$$

The only solution to these equations is $x = y = z = 0$ and this gives us that $a = 0$, so the lips/beaks stratum is the b-axis. Now the tacnode

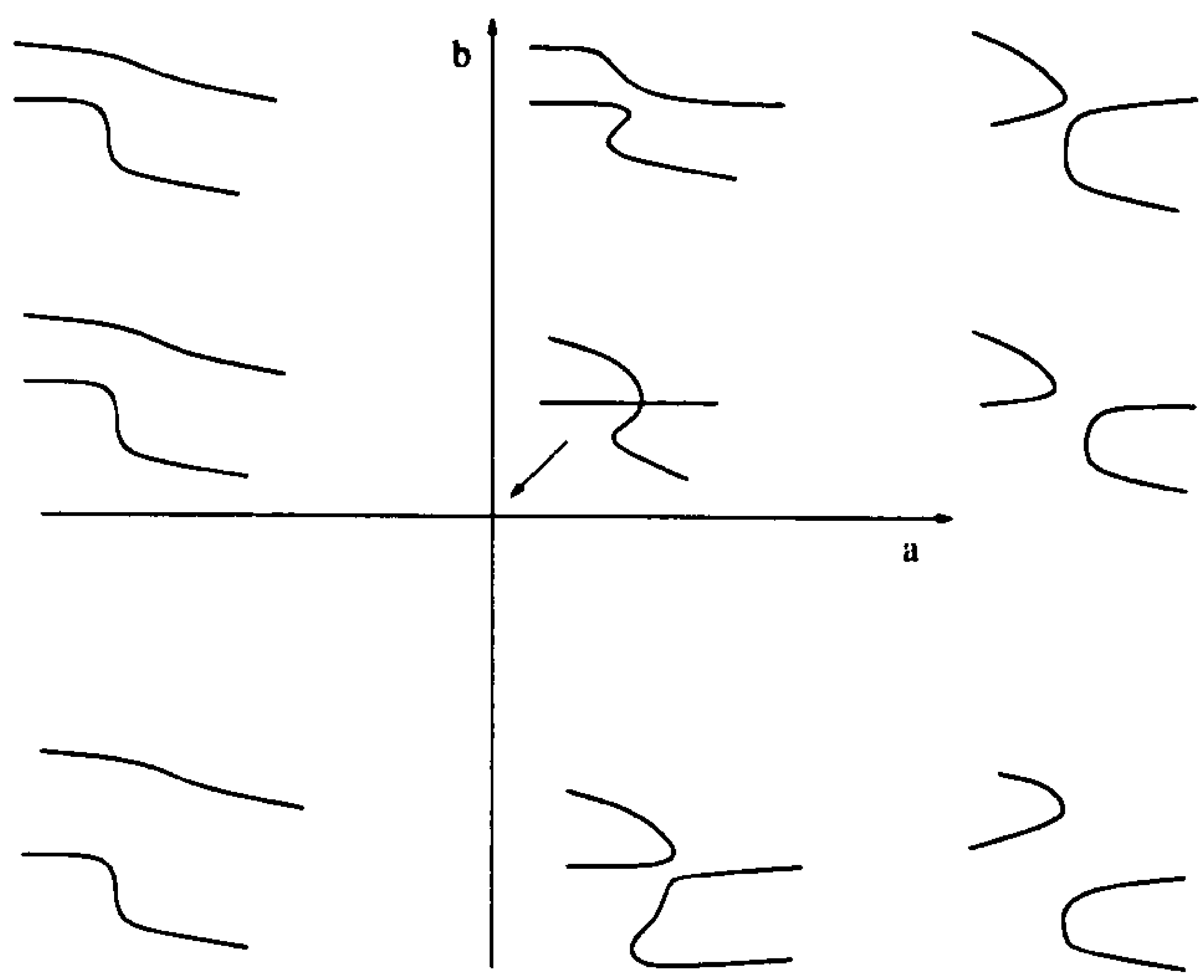


Figure 4.13: Critical Set of $f_{a,b}(x, y, z) = (x, xy^2 + y^4 + y^5 \pm z^2 + ay + bxy)$

stratum occurs when we have $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$ with $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$, where both $f(x_1, y_1, z_1)$ and $f(x_2, y_2, z_2)$ are folds and the folds are tangential. Now the tangent to a fold is given by the image of the tangent of the critical set, $\sum f_{a,b}$, by the Jacobian, Df. Now $\sum f_{a,b}$ is given by $\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial z} = 0$ and the tangent is given by $\left(\frac{\partial^2 f_2}{\partial y^2}, -\frac{\partial^2 f_2}{\partial x \partial y}\right)$ since $\frac{\partial f_2}{\partial z} = 0 \Longrightarrow z = 0$. So the tangent to the fold is given by

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f_2}{\partial y^2} \\ -\frac{\partial^2 f_2}{\partial x \partial y} \end{pmatrix} = \frac{\partial^2 f_2}{\partial y^2} \left(1, \frac{\partial f_2}{\partial x}\right).$$

Now note that the tangent direction at both (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\left(1, \frac{\partial f_2}{\partial x}\right)$, i.e. it is given by $(1, y^2)$. So these two tangents are parallel if and only if $y_1^2 = y_2^2$, i. e. $y_1 = \pm y_2$. Now $y_1 = y_2$ would contradict the hypothesis so y_1 must equal $-y_2$. Now $f_2(x_1, y_1, 0) = f_2(x_1, -y_1, 0)$ and $\frac{\partial f_2}{\partial y}(x_1, y_1, 0) = \frac{\partial f_2}{\partial y}(x_1, -y_1, 0) = 0$ so $f_2(x_1, y, 0) - f_2(x_1, y_1, 0)$ has two repeated roots, namely, y_1 and $-y_1$. Hence

$$f_2(x_1, y, 0) - f_2(x_1, y_1, 0) = (y - y_1)^2 (y + y_1)^2 (y - y_3).$$

Again comparing coefficients of y we find that $a = y_1^4$ and $b = -2y_1^2$. So our tacnode stratum is parametrized by $(y_1^4, -2y_1^2)$. The bifurcation set is shown in Fig. 4.14.

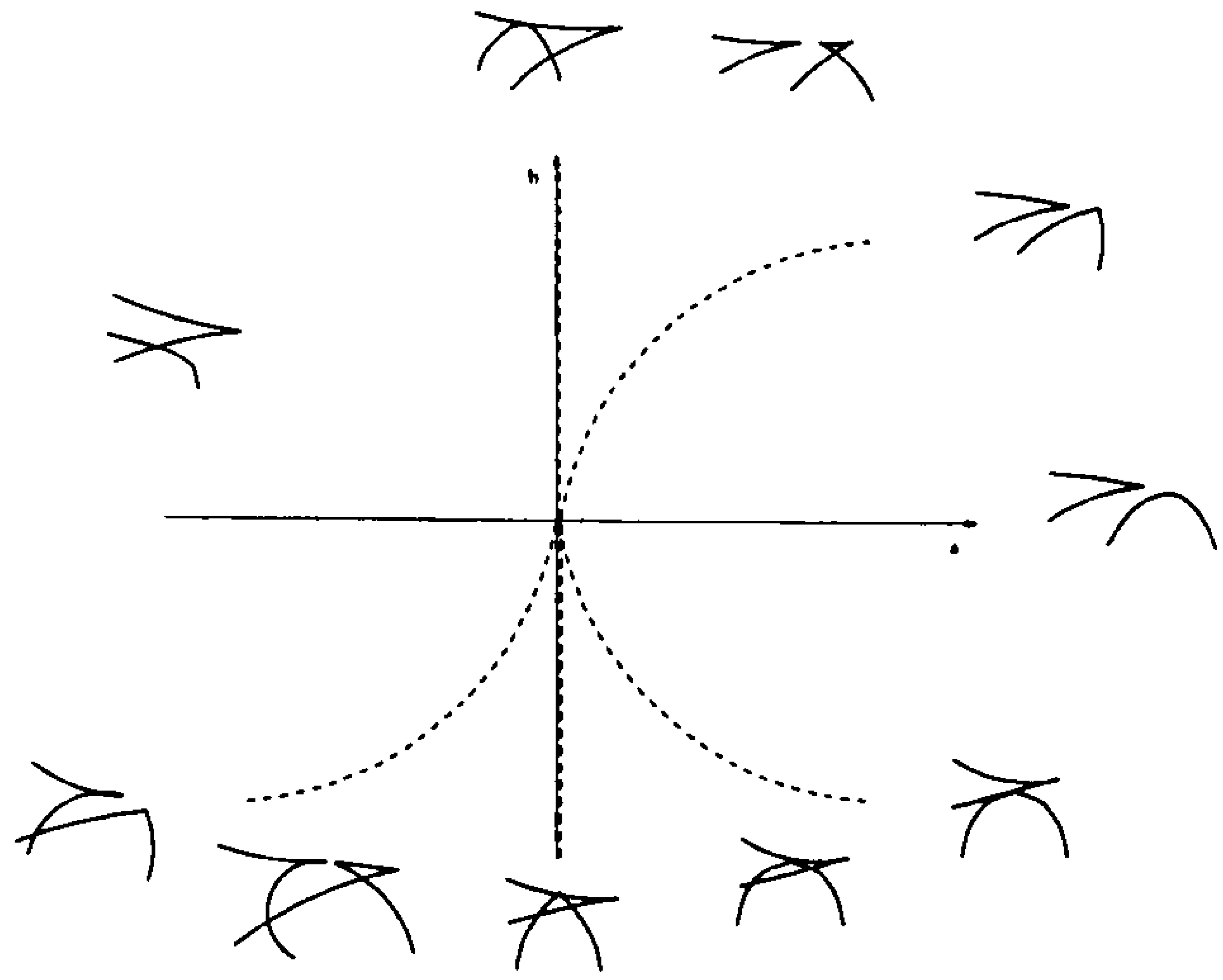


Figure 4.14: Unfolding of $f_{a,b}(x,y,z) = (x, xy^2 + y^4 + y^5 \pm z^2 + ay + bxy)$

- $f(x,y,z) = (x, xy + z^3 \pm y^2 z + ay^3 + y^5)$.

  This has $\mathcal{A}_e$–codimension=2 and

$$T\mathcal{A}_e \cdot f = \left(\mathcal{E}_3, \mathcal{E}_3 - \{y^2, z\}\right).$$

So the versal unfolding is given by $f_{b,c}(x,y,z) = (x, xy + z^3 \pm y^2 z + ay^3 + y^5 + bz + cy^2)$. Now this has critical set, $\sum f$, given by

$$\begin{aligned}\sum f_{b,c} &= \left\{(x,y,z) : rank \begin{pmatrix} 1 & 0 & 0 \\ y & x \pm 2yz + 3ay^2 + 5y^4 + 2cy & 3z^2 \pm y^2 + b \end{pmatrix} < 2\right\} \\ &= \left\{(x,y,z) : x \pm 2yz + 3ay^2 + 5y^4 + 2cy = 0 \;\&\; 3z^2 \pm y^2 + b = 0\right\}\end{aligned}$$

We will consider the + and − cases separately.

1. $f(x,y,z) = (x, xy + z^3 + y^2 z + ay^3 + y^5 + bz + cy^2)$.
   To parametrize the critical set we need $b < 0$, say $b = -w^2$. We substitute $y = w\cos(t)$ and $z = \frac{w}{\sqrt{3}}\sin(t)$. This gives

$$\sum f_{b,c} = \left\{ \begin{array}{r} -\frac{2w^2}{\sqrt{3}}\sin(t)\cos(t) - 3aw^2\cos^2(t) - 5w^4\cos^4(t) - 2cw\cos(t), \\ w\cos(t), \frac{w}{\sqrt{3}}\sin(t) \end{array} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f|\Sigma f = \left\{ \begin{array}{c} -\frac{2w^2}{\sqrt{3}}\sin(t)\cos(t) - 3aw^2\cos^2(t) \\ -5w^4\cos^4(t) - 2cw\cos(t), \\ -\frac{w^3}{\sqrt{3}}\sin(t)\cos^2(t) - 2aw^3\cos^3(t) - 4w^5\cos^5(t) + \\ \frac{w^3}{3\sqrt{3}}\sin^3(t) - cw^2\cos^2(t) - \frac{w^3}{\sqrt{3}}\sin(t) \end{array} \right\}.$$

The bifurcations are given in Fig. 4.15.

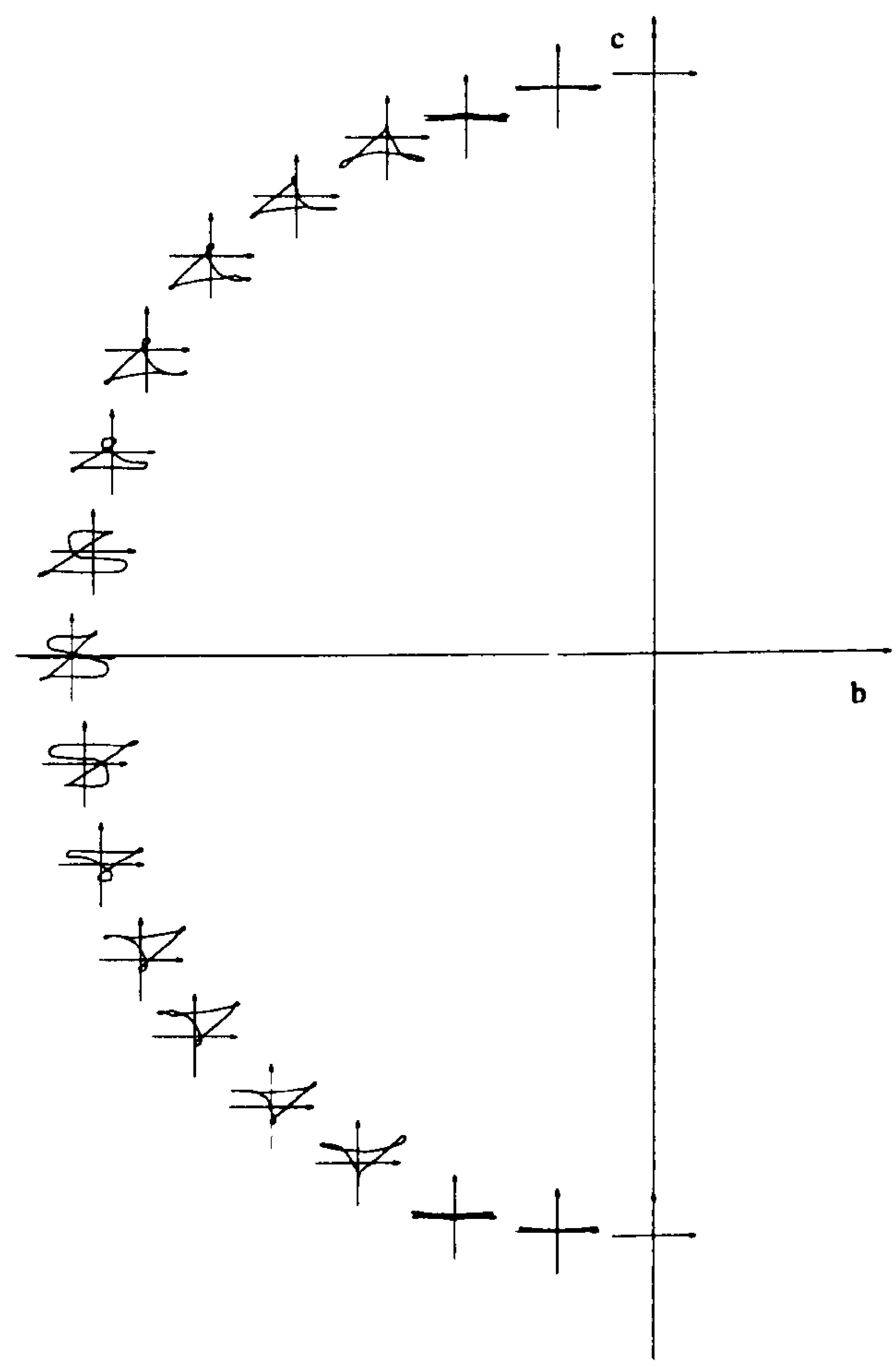


Figure 4.15: Bifurcations for $f(x,y,z) = (x, y, z^3 + y^2z + ay^3 + y^5 + bz + cy^2)$

Now we want to find where the tangent to this map lies in the kernel, and the tangent to the critical set is given by the minors of the following matrix.

$$\begin{pmatrix} 1 & 2z+6ay+20y^3+2c & 2y \\ 0 & 2y & 6z \end{pmatrix}.$$

So our tangent is given by $(12z^2+36ayz+120y^3z+12cz-4y^2, -6z, 2y) = (u, v, w)$ say, and this lies in the kernel if

$$\begin{pmatrix} 1 & 0 & 0 \\ y & x+2yz+3ay^2+5y^4+2cy & 3z^2+y^2+b \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the following two equations:

$$12z^2+36ayz+120y^3z+12cz-4y^2=0, \qquad (4.20)$$

$$6yz^2+18ay^2z+90y^4z-2y^3-6xz+2by=0. \qquad (4.21)$$

From the conditions for a point to lie on the critical set we know that $x=-2yz-3ay^2-5y^4-2cy$, so Equation 4.21 can be written as

$$18yz^2+36ay^2z+120y^4z-2y^3+2by+12cyz=0. \qquad (4.22)$$

We need to solve equations 4.20 and 4.22. We do this using MAPLE, giving the following result.

$$\begin{aligned} b &= -3z^2-y^2, \\ 9ayz &= y^2-3z^2-30y^3z-3cz. \end{aligned}$$

So we have a function $f=3z^2+30y^3z+3cz-y^2+9ayz=0$. Rewriting this as

$$y^2-3z^2-3cz=3yz(10y^2+3a),$$

and squaring both sides we have

$$(y^2 - 3z^2 - 3cz)^2 = 9y^2z^2(10y^2 + 3a)^2.$$

But $y^2 = -3z^2 - b$ so substituting we obtain

$$F = (6z^2 + 3cz + b)^2 + 9z^2(3z^2 + b)(3a - 10b - 30z^2)^2 = 0.$$

So we now have a function of one variable z so we can find its discriminant, again using MAPLE.

$$\Delta F = b^3(50b^2 + 150c^2b - 60ab - 45c^2a + 18a^2)^2 G$$

where G is a polynomial in a, b, c.

Now this gives us the local structure of the bifurcation set at the origin. We need to find any exceptional values for the modulus a. To do this we need to find the points for where the discriminant is singular. We can write G as

$$A_1b^3 + A_2b^2c^2 + A_3bc^4 + A_4c^6 + H.O.T.$$

where A_i are polynomials in a and the Newton diagram of G gives the above equation as quasihomogeneous of the above form. So we consider the lowest order terms and differentiate w.r.t. b and c to get the two equations:

$$\begin{aligned} 3A_1b^2 + 2A_2bc^2 + A_3c^4 &= 0 \\ c(2A_2b^2 + 4A_3bc^2 + 6A_4c^4 &= 0. \end{aligned}$$

Dividing the equations by c^4 and substituting $w = \frac{b}{c^2}$ we obtain:

$$\begin{aligned} 3A_1w^2 + 2A_2w + A_3 &= 0 \\ 2A_2w^2 + 4A_3w + 6A_4 &= 0. \end{aligned}$$

We now solve these equations by taking resultants using MAPLE and factorising to obtain the following conditions on a:
$82944(27a^2+4)^{12}a^2(2287680209a^{10} - 508373271954a^8 + 131800418865a^6 + 114366526020a^4 + 18596183472a^2 + 918330048) = 0.$

This gives the exceptional values of a to be approximately

$$\begin{array}{c} \pm 1.299274747 \\ \pm 0.9693086208 \\ 0 \end{array}$$

Now if $a \neq$ any of these exceptional values then our bifurcation set is diffeomorphic to J_{10}.

The other two factors of our discriminant give us the c–axis.

For multi–germ transitions we expect triplepoints, tacnodes and cusp–and–folds.

For the cusp–and–fold transitions we need $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ with $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$, $\frac{\partial f_2}{y}(x_1, y_1, z_1) = \frac{\partial f_2}{\partial z}(x_1, y_1, z_1) = 0$ and $\frac{\partial f_2}{\partial y}(x_2, y_2, z_2) = \frac{\partial f_2}{\partial z}(x_2, y_2, z_2) = \frac{\partial^2 f_2}{\partial y^2}(x_2, y_2, z_2) = 0$. This gives us the following six equations:

$$\begin{aligned} xy_1 + z_1^3 + y_1^2 z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 + y_2^2 z_2 + ay_2^3 + y_2^5 \\ &\quad + bz_2 + cy_2^2 \\ x + 2y_1 z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\ x + 2y_2 z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\ 2z_2 + 6ay_2 + 20y_2^3 + 2c &= 0 \\ 3z_1^2 + y_1^2 + b &= 0 \\ 3z_2^2 + y_2^2 + b &= 0. \end{aligned}$$

To solve this system of equations we need to employ computer algebra. Whilst attempting to solve this problem we have come across various problems in elimination theory for which the standard computer algebra packages fail. This has led to collaboration with various experts in the field of computer algebra. These people are still working on trying to find a solution to this problem and the others which follow.

For the triplepoint transitions we need $(x_1, y_1, z_1) \neq (x_2, y_2, z_2) \neq (x_3, y_3, z_3)$ with $f(x_1, y_1, z_1) = f(x_2, y_2, z_2) = f(x_3, y_3, z_3)$. This gives us that $x_1 = x_2 = x_3 = x$, say. Now they are all folds so we need $\frac{\partial f_2}{\partial y}(x, y_i, z_i) = \frac{\partial f_2}{\partial z}(x, y_i, z_i) = 0$ for $i = 1, 2, 3$. This gives us the following system of nine equations:

$$
\begin{aligned}
xy_1 + z_1^3 + y_1^2 z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 + y_2^2 z_2 + ay_2^3 + y_2^5 \\
&\quad + bz_2 + cy_2^2 \\
xy_2 + z_2^3 + y_2^2 z_2 + ay_2^3 + y_2^5 + bz_2 + cy_2^2 &= xy_3 + z_3^3 + y_3^2 z_3 + ay_3^3 + y_3^5 \\
&\quad + bz_3 + cy_3^2 \\
xy_1 3z_3^3 + y_3^2 z_3 + ay_3^3 + y_3^5 + bz_3 + cy_3^2 &= xy_1 + z_1^3 + y_1^2 z_1 + ay_1^3 + y_1^5 \\
&\quad + bz_1 + cy_1^2 \\
x + 2y_1 z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\
x + 2y_2 z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\
x + 2y_3 z_3 + 3ay_3^2 + 5y_3^4 + 2cy_3 &= 0 \\
3z_1^2 + y_1^2 + b &= 0 \\
3z_2^2 + y_2^2 + b &= 0 \\
3z_3^2 + y_3^2 + b &= 0.
\end{aligned}
$$

Again we need to employ computer algebra to eliminate the variables $x, y_1, y_2, y_3, z_1, z_2, z_3$ from the above system of equations. This problem is still unsolved.

For the tacnode stratum we need to consider the images of the tangents of the critical set, $\sum f$, by Df. Now our critical set is given by the two equations

$$
\begin{aligned}
x + 2yz + 3ay^2 + 5y^4 + 2cy &= 0 \\
3z^2 + y^2 + b &= 0.
\end{aligned}
$$

We can form the Jacobian of these:

$$
\begin{pmatrix} 1 & 2z + 6ay + 20y^3 + 2c & 2y \\ 0 & 2y & 6z \end{pmatrix}.
$$

We wish to find the kernel vectors of this matrix, i.e. $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & 2z+6ay+20y^3+2c & 2y \\ 0 & 2y & 6z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we have

$$\begin{aligned} u + v(2z+6ay+20y^3+2c) + 2wy &= 0 \\ 2vy + 6wz &= 0. \end{aligned}$$

This gives us the following form for a kernel vector:

$$\begin{pmatrix} \frac{6z^2}{y} + 18az + 60y^2z + \frac{6cz}{y} - 2y \\ -\frac{3z}{y} \\ 1 \end{pmatrix}.$$

So we take two kernel vectors, given by substituting for (x, y_1, z_1) and (x, y_2, z_2) where $(x, y_1, z_1) \neq (x, y_2, z_2)$ as before, and applying Df to them, where Df is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ y & x+2yz+3ay^2+5y^4+2cy & 3z^2+y^2+b \end{pmatrix}.$$

Now we need a common tangent so this means that our vectors must be linearly dependent giving:

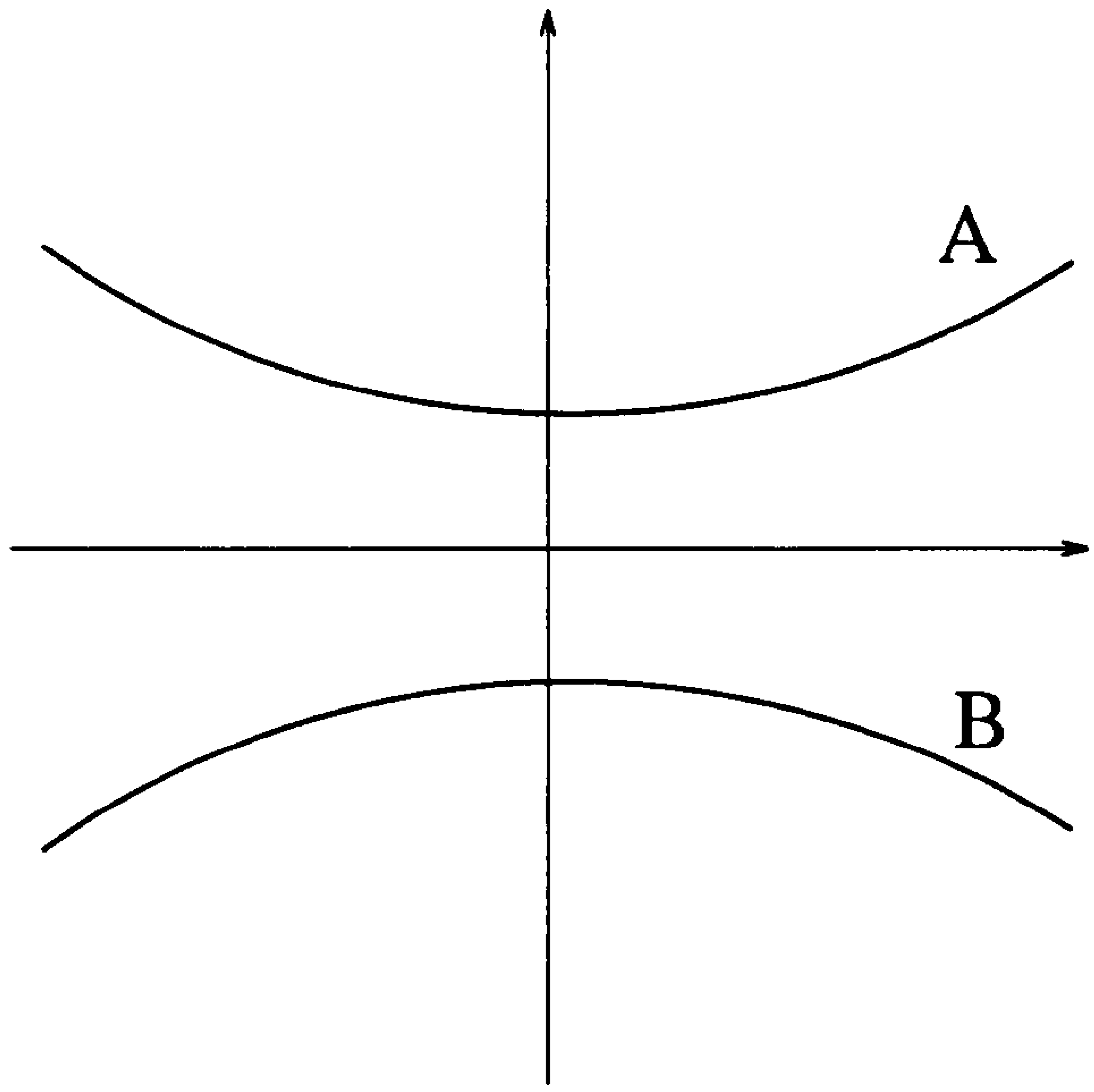
$$\left(\frac{6z_1^2}{y_1} + 18az_1 + 60y_1^2z_1 + \frac{6cz_1}{y_1} - 2y_1\right)\left(3z_2^2 + 9ay_2z_2 + 45y_2^3z_2 - y_2^2 - \frac{3xz_2}{y_2} + b\right) -$$
$$\left(\frac{6z_2^2}{y_2} + 18az_2 + 60y_2^2z_2 + \frac{6cz_2}{y_2} - 2y_2\right)\left(3z_1^2 + 9ay_1z_1 + 45y_1^3z_1 - y_1^2 - \frac{3xz_1}{y_1} + b\right)$$

So we have this equation and five others describing the fact that both branches of the multi–germ are folds:

$$\begin{aligned} xy_1 + z_1^3 + y_1^2z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 + y_2^2z_2 + ay_2^3 + y_2^5 \\ &\quad +bz_2 + cy_2^2 \\ x + 2y_1z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\ x + 2y_2z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\ 3z_1^2 + y_1^2 + b &= 0 \\ 3z_2^2 + y_2^2 + b &= 0. \end{aligned}$$

Again computer algebra is employed to eliminate the variables x, y_1, y_2, z_1, z_2. This problem is still unsolved.

2. $f(x,y,z) = (x, xy + z^3 - y^2 z + ay^3 + y^5 + bz + cy^2)$.
The parametrization of the critical set depends on the sign of b:
   - $b < 0$. Again we say that $b = -w^2$ and since $3z^2 - y^2 - w^2 = 0$ is a hyperbola we look at each branch separately.



(a) For branch A we have

$$
\begin{aligned}
y &= w \sinh(t) \\
z &= \frac{w}{\sqrt{3}} \cosh(t).
\end{aligned}
$$

This gives

$$
\sum f_{b,c} = \left\{ \begin{array}{c} \frac{2w^2}{\sqrt{3}} \sinh(t)\cosh(t) - 3aw^2 \sinh^2(t) \\ -5w^4 \sinh^4(t) - 2cw \sinh(t), w\sinh(t), \frac{w}{\sqrt{3}} \cosh(t) \end{array} \right\}.
$$

Now we map this over to get the following parametrization for the discriminant:

$$
f|\sum f = \left\{ \begin{array}{c} \frac{2w^2}{\sqrt{3}} \sinh(t)\cosh(t) - 3aw^2 \sinh^2(t) \\ -5w^4 \sinh^4(t) - 2cw \sinh(t), \\ \frac{w^3}{\sqrt{3}} \sinh^2(t)\cosh(t) - 2aw^3 \sinh^3(t) - 4w^5 \sinh^5(t) + \\ \frac{w^3}{3\sqrt{3}} \cosh^3(t) - cw^2 \sinh^2(t) - \frac{w^3}{\sqrt{3}} \cosh(t) \end{array} \right\}.
$$

(b) For branch B we have

$$y = w\sinh(t)$$
$$z = -\frac{w}{\sqrt{3}}\cosh(t).$$

This gives

$$\sum f_{b,c} = \left\{ \begin{array}{c} -\frac{2w^2}{\sqrt{3}}\sinh(t)\cosh(t) - 3aw^2\sinh^2(t) \\ -5w^4\sinh^4(t) - 2cw\sinh(t), w\sinh(t), \frac{w}{\sqrt{3}}\cosh(t) \end{array} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f|\sum f = \left\{ \begin{array}{c} -\frac{2w^2}{\sqrt{3}}\sinh(t)\cosh(t) - 3aw^2\sinh^2(t) \\ -5w^4\sinh^4(t) - 2cw\sinh(t), \\ -\frac{w^3}{\sqrt{3}}\sinh^2(t)\cosh(t) - 2aw^3\sinh^3(t) - 4w^5\sinh^5(t) - \\ \frac{w^3}{3\sqrt{3}}\cosh^3(t) - cw^2\sinh^2(t) + \frac{w^3}{\sqrt{3}}\cosh(t) \end{array} \right\}.$$

The bifurcation sets are given in Fig. 4.16 and Fig. 4.17.

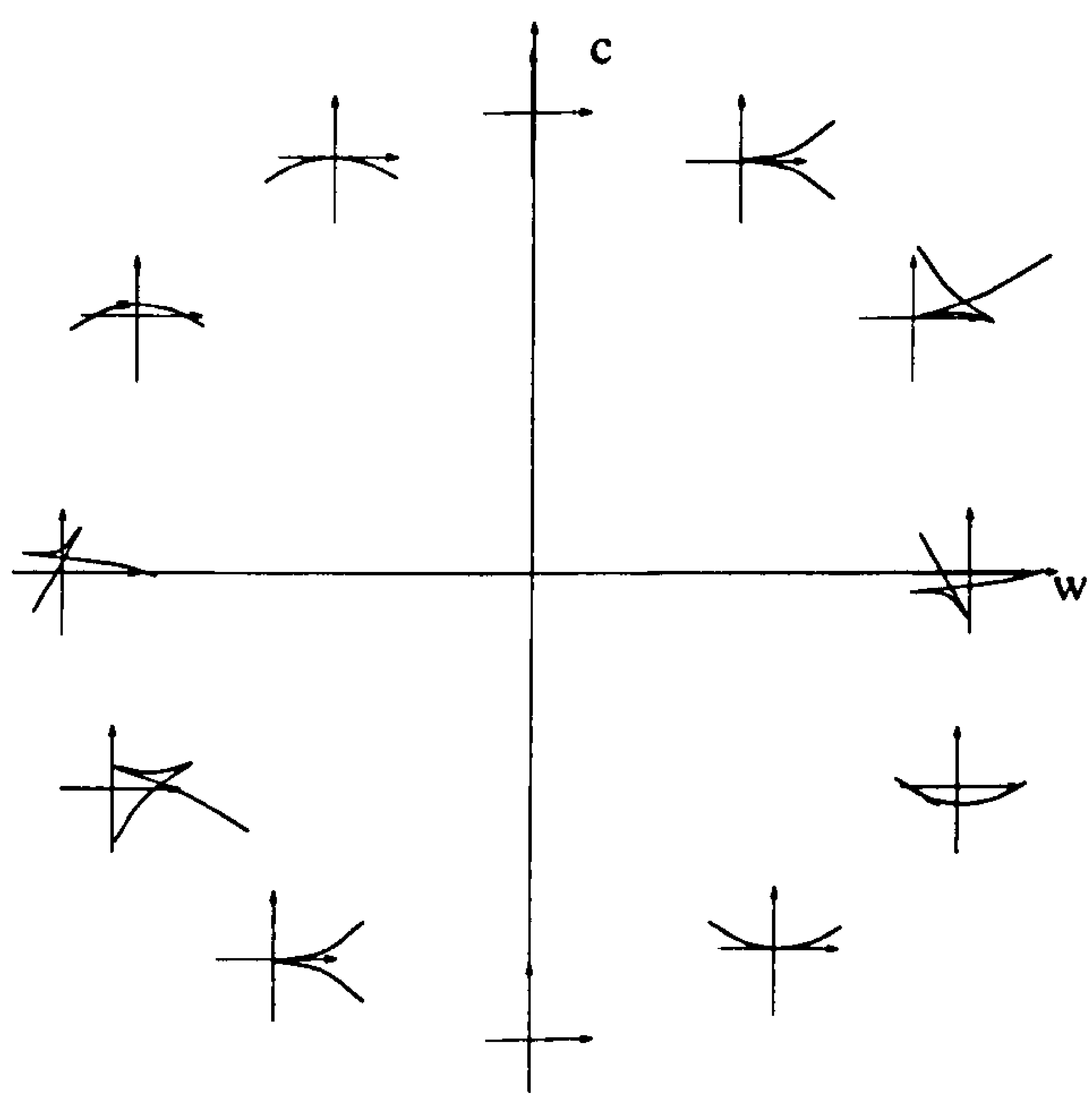


Figure 4.16: Bifurcations for $b < 0$ and branch A

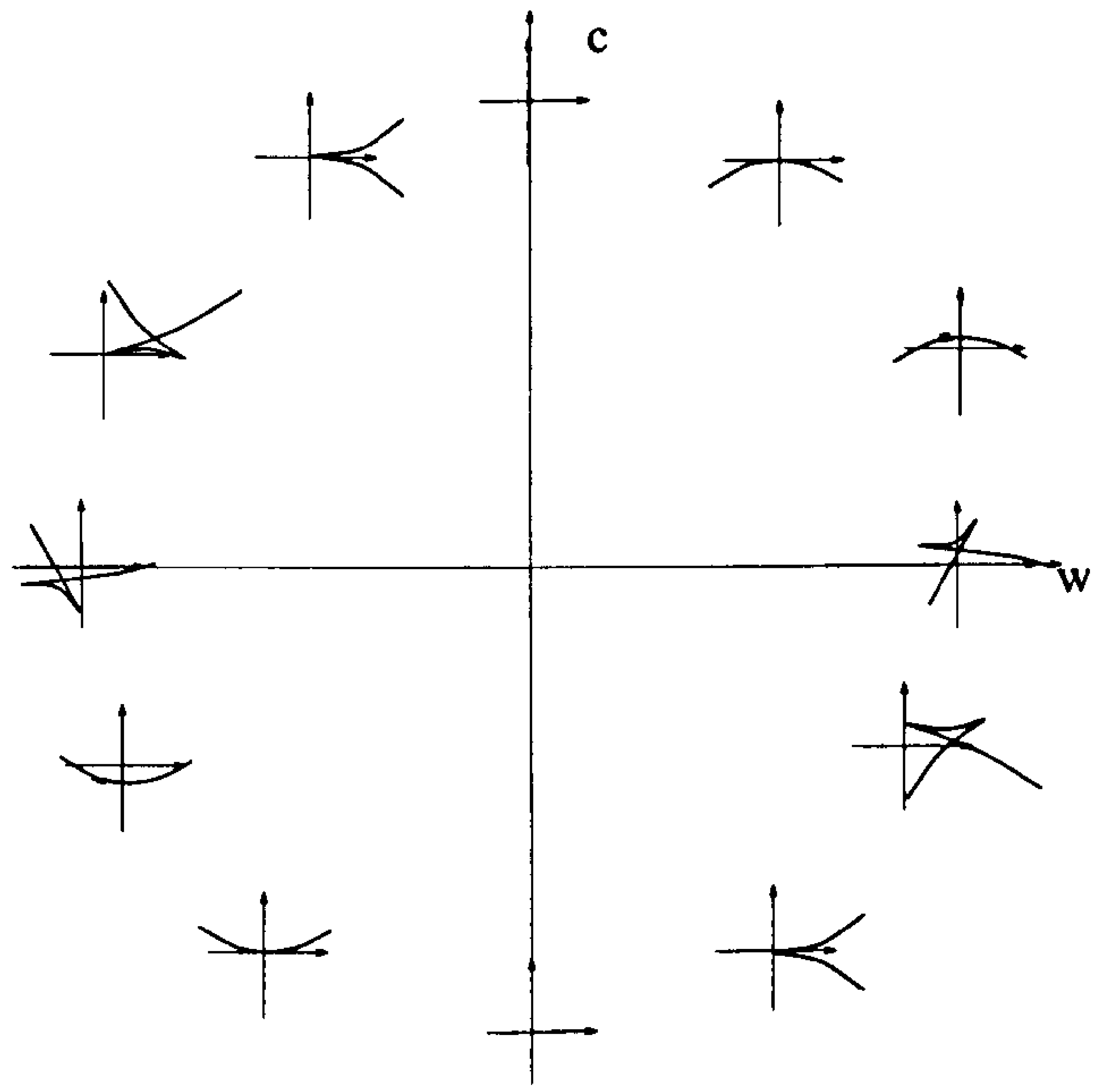
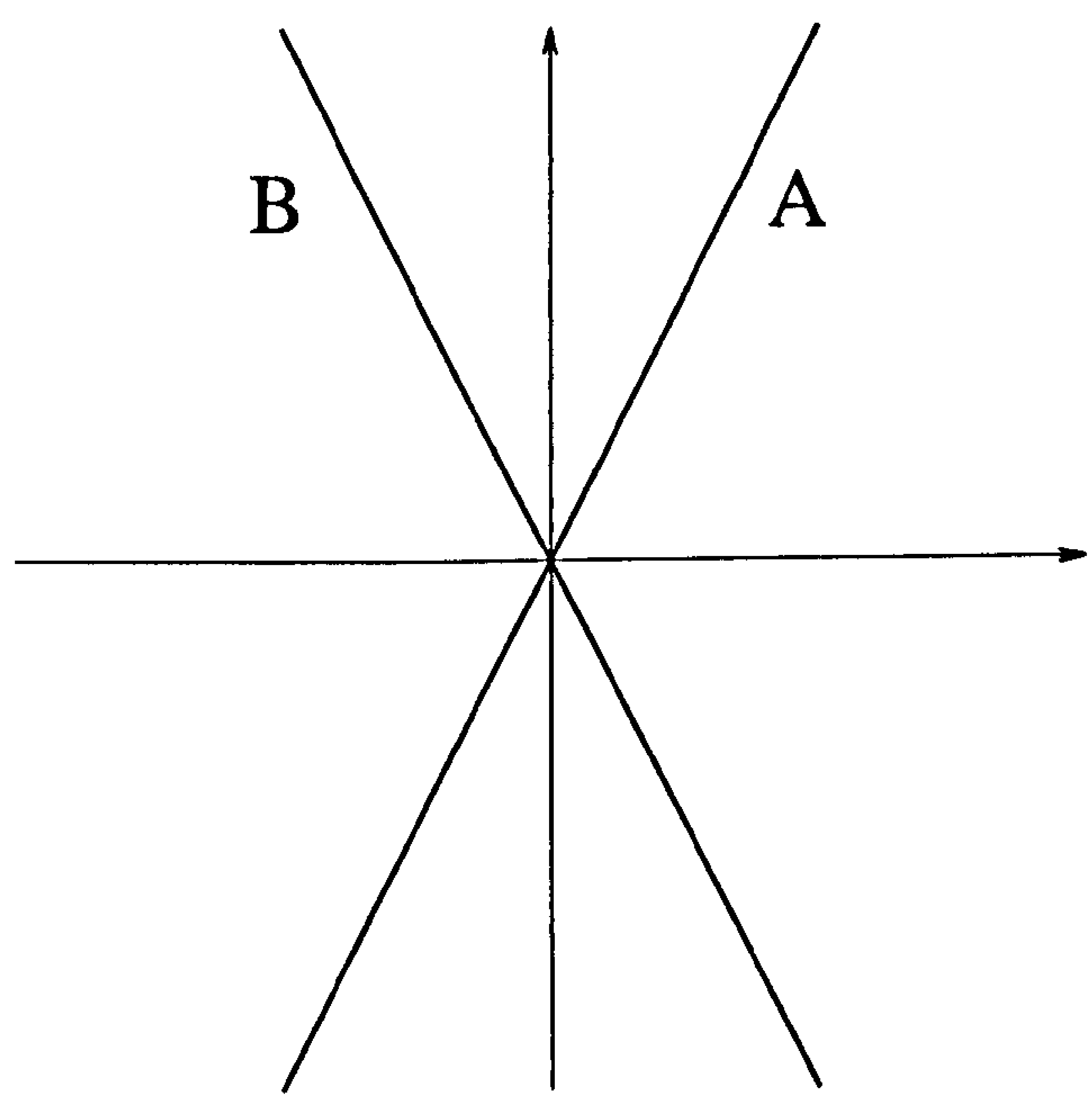


Figure 4.17: Bifurcations for $b < 0$ and branch B

- $b = 0$. Here we have two lines through the origin.



Again we parametrize each line separately

(a) For branch A we have $z = \frac{y}{\sqrt{3}}$ giving us the following parametrization for the critical set:

$$\sum f_c = \left\{ \frac{2y^2}{\sqrt{3}} - 3ay^2 - 5y^4 - 2cy, y, \frac{y}{\sqrt{3}} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f | \sum f = \left\{ \begin{array}{l} 2y^2\sqrt{3} - 3ay^2 - 5y^4 - 2cy, \\ \qquad y^3\left(\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} - 2a\right) - 4y^4 - cy^2. \end{array} \right\}.$$

(b) For branch B we have $z = -\frac{y}{\sqrt{3}}$ giving us the following parametrization for the critical set:

$$\sum f_c = \left\{ -\frac{2y^2}{\sqrt{3}} - 3ay^2 - 5y^4 - 2cy, y, -\frac{y}{\sqrt{3}} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f | \sum f = \left\{ \begin{array}{l} -2y^2\sqrt{3} - 3ay^2 - 5y^4 - 2cy, \\ \qquad -y^3\left(\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + 2a\right) - 4y^4 - cy^2. \end{array} \right\}.$$

For bifurcations see Fig. 4.18 and Fig. 4.19.

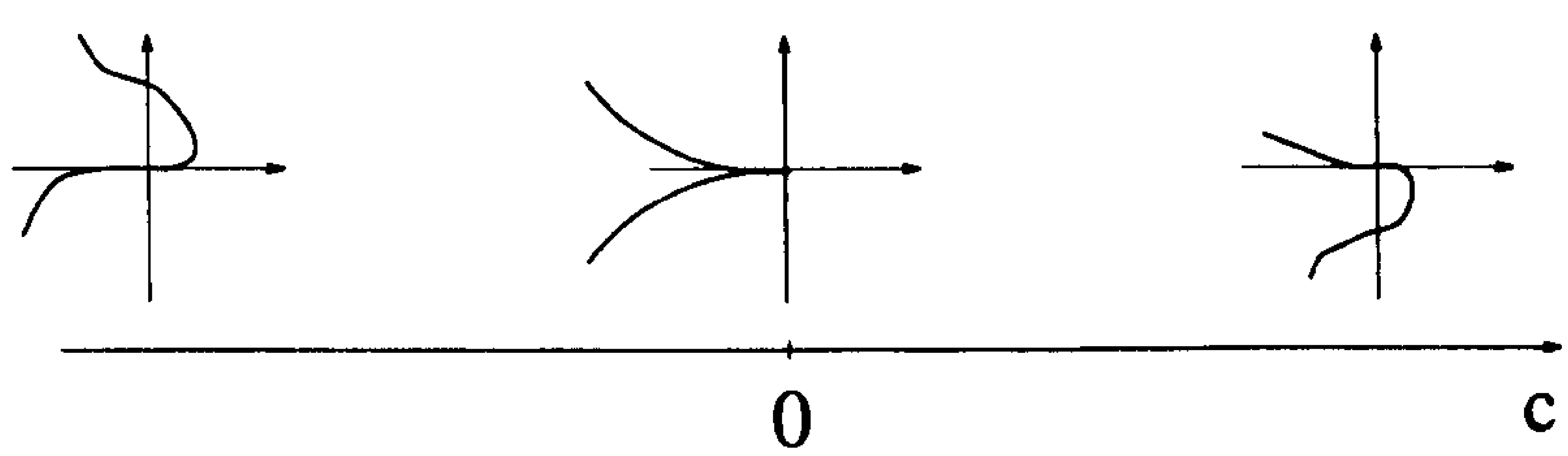


Figure 4.18: Bifurcations for $b = 0$ and branch A

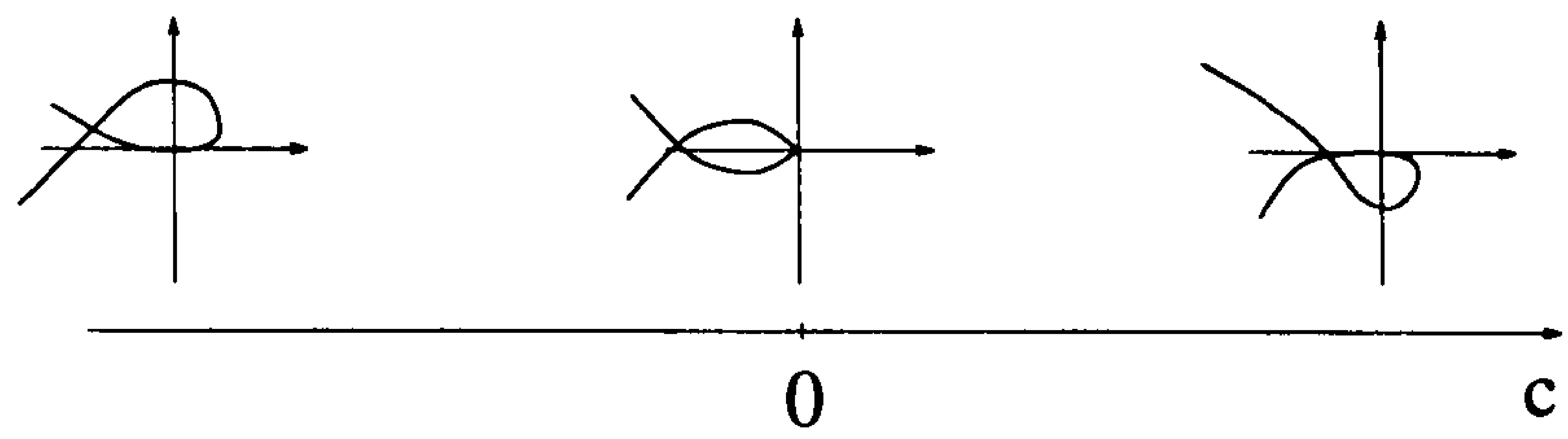
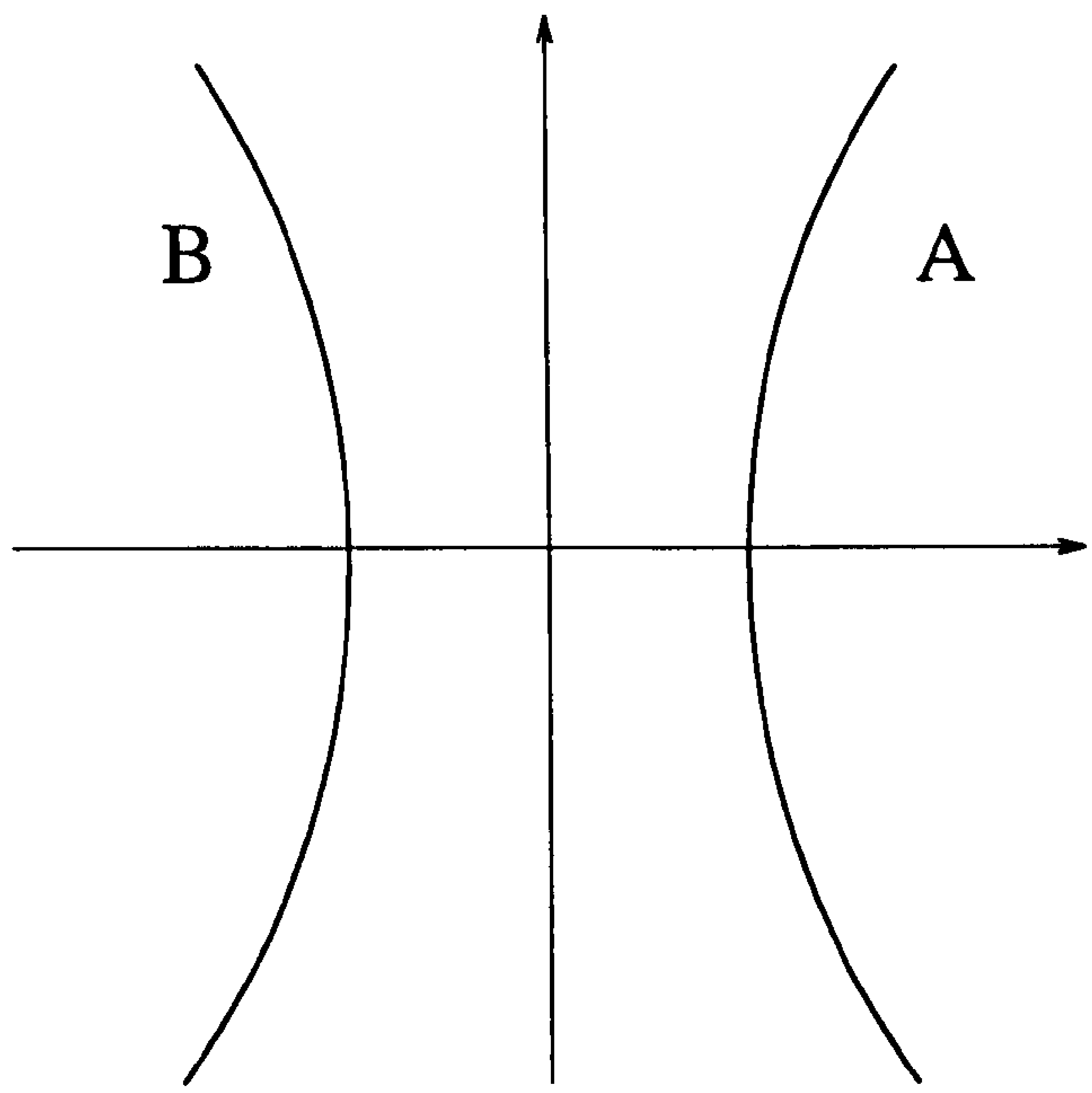


Figure 4.19: Bifurcations for $b = 0$ and branch B

– $b > 0$. Here we say that $b = w^2$ and since $3z^2 - y^2 + w^2 = 0$ is a hyperbola we look at each branch separately.



(a) For branch A we have

$$\begin{aligned} y &= w\cosh(t) \\ z &= \frac{w}{\sqrt{3}}\sinh(t). \end{aligned}$$

This gives

$$\sum f_{b,c} = \left\{ \begin{array}{c} \frac{2w^2}{\sqrt{3}}\sinh(t)\cosh(t) - 3aw^2\cosh^2(t) \\ -5w^4\cosh^4(t) - 2cw\cosh(t), w\cosh(t), \frac{w}{\sqrt{3}}\sinh(t) \end{array} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f|\sum f = \left\{ \begin{array}{c} \frac{2w^2}{\sqrt{3}} \sinh(t)\cosh(t) - 3aw^2\cosh^2(t) \\ -5w^4\cosh^4(t) - 2cw\cosh(t), \\ \frac{w^3}{\sqrt{3}} \sinh(t)\cosh^2(t) - 2aw^3\cosh^3(t) - 4w^5\cosh^5(t) + \\ \frac{w^3}{3\sqrt{3}}\sinh^3(t) - cw^2\cosh^2(t) - \frac{w^3}{\sqrt{3}}\sinh(t) \end{array} \right\}.$$

(b) For branch B we have

$$\begin{aligned} y &= -w\cosh(t) \\ z &= \frac{w}{\sqrt{3}}\sinh(t). \end{aligned}$$

This gives

$$\sum f_{b,c} = \left\{ \begin{array}{c} -\frac{2w^2}{\sqrt{3}}\sinh(t)\cosh(t) - 3aw^2\cosh^2(t) \\ -5w^4\cosh^4(t) + 2cw\cosh(t), -w\cosh(t), \frac{w}{\sqrt{3}}\sinh(t) \end{array} \right\}.$$

Now we map this over to get the following parametrization for the discriminant:

$$f|\sum f = \left\{ \begin{array}{c} -\frac{2w^2}{\sqrt{3}} \sinh(t)\cosh(t) - 3aw^2\cosh^2(t) \\ -5w^4\cosh^4(t) + 2cw\cosh(t), \\ \frac{w^3}{\sqrt{3}} \sinh(t)\cosh^2(t) + 2aw^3\cosh^3(t) + 4w^5\cosh^5(t) + \\ \frac{w^3}{3\sqrt{3}}\sinh^3(t) - cw^2\cosh^2(t) + \frac{w^3}{\sqrt{3}}\sinh(t) \end{array} \right\}.$$

For bifurcations see Fig. 4.20 and Fig. 4.21.

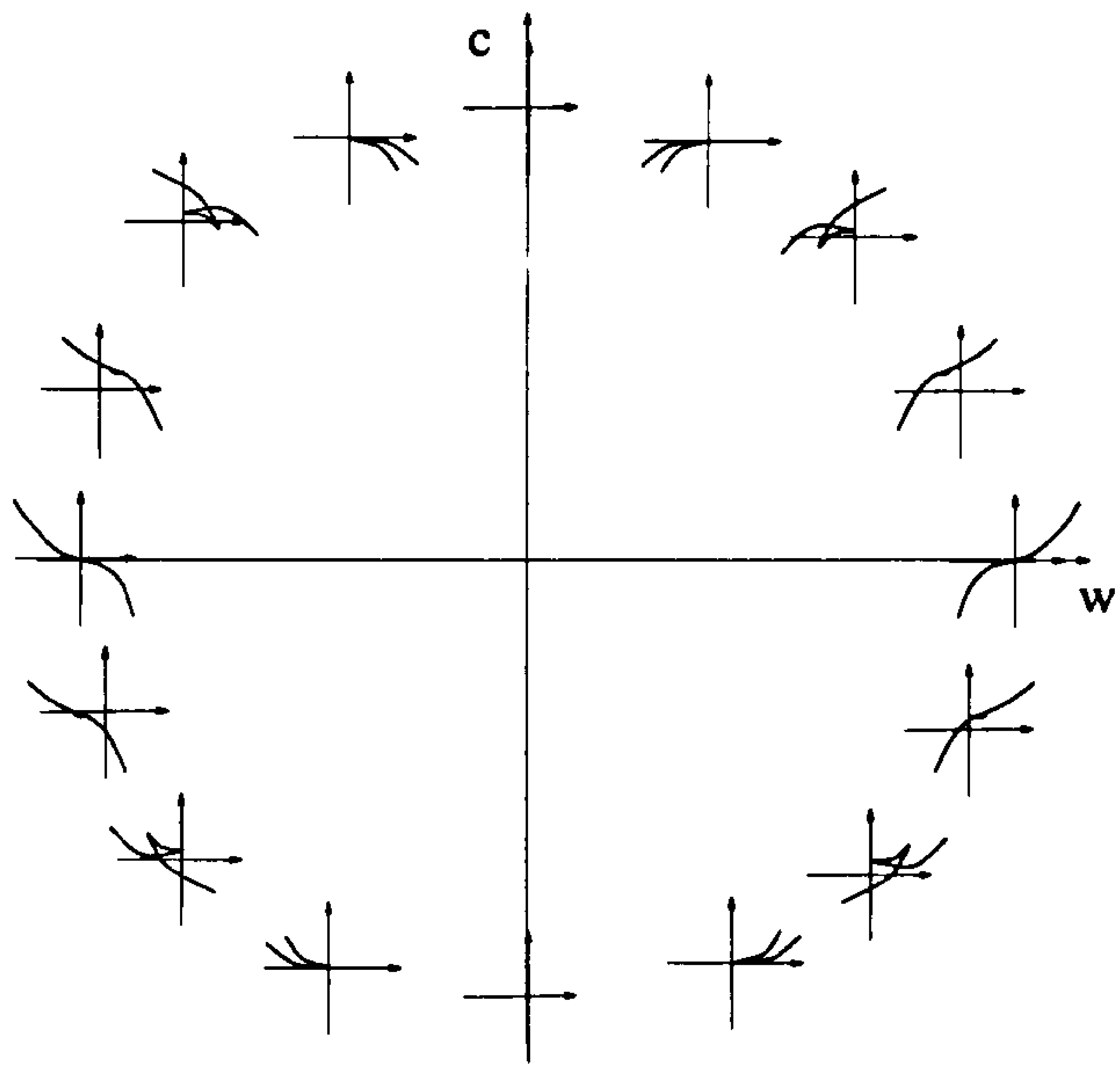


Figure 4.20: Bifurcations for $b > 0$ and branch A

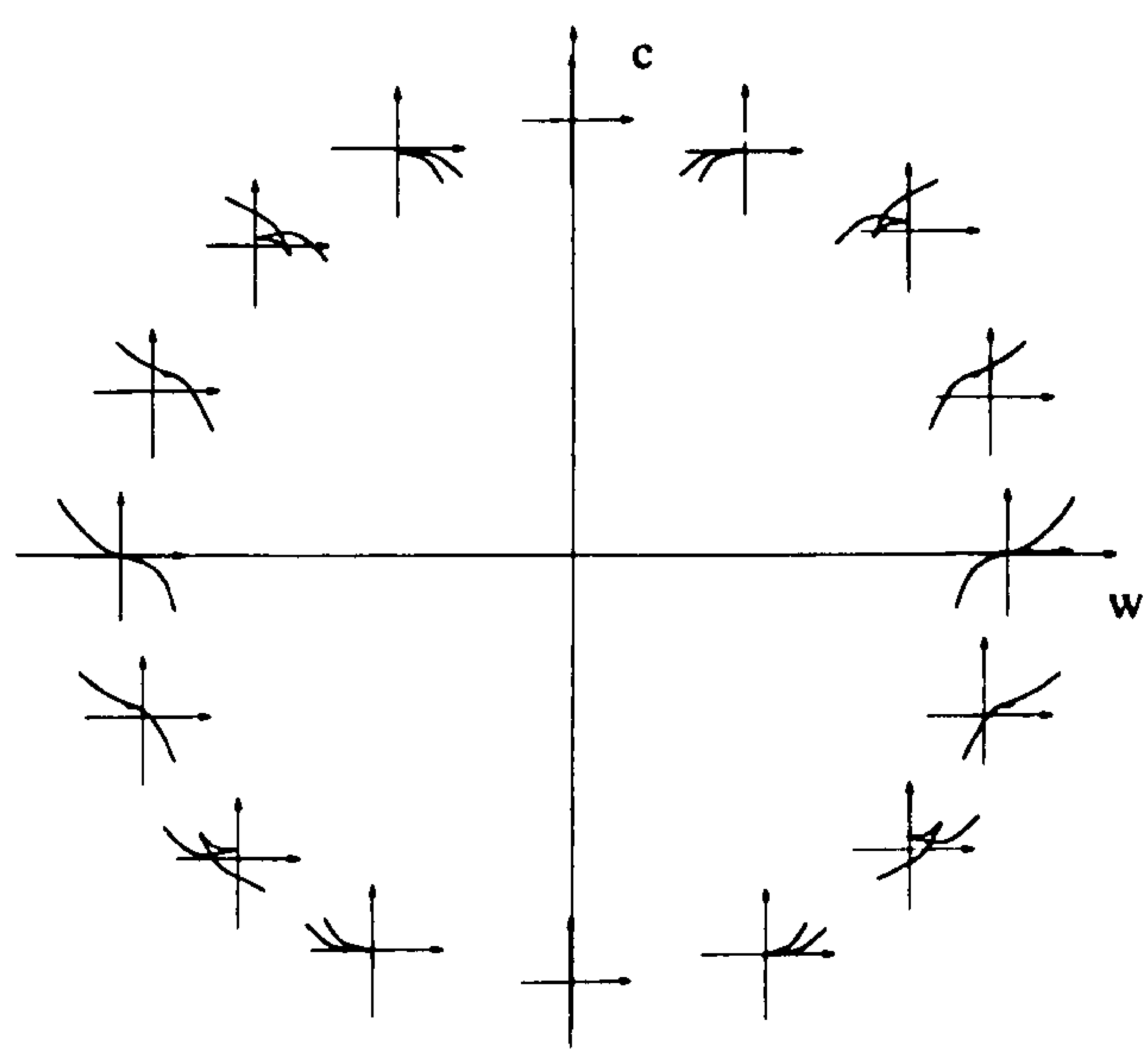


Figure 4.21: Bifurcations for $b > 0$ and branch B

Now the bifurcations for the negative case follow directly from that of the positive:

We want to find where the tangent to this map lies in the kernel, and the tangent to the critical set is given by the minors of the following matrix.

$$\begin{pmatrix} 1 & -2z+6ay+20y^3+2c & -2y \\ 0 & -2y & 6z \end{pmatrix}$$

So our tangent is given by $(-12z^2+36ayz+120y^3z+12cz-4y^2, -6z, -2y) = (u, v, w)$ say, and this lies in the kernel if

$$\begin{pmatrix} 1 & 0 & 0 \\ y & x-2yz+3ay^2+5y^4+2cy & 3z^2-y^2+b \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the following two equations:

$$-12z^2+36ayz+120y^3z+12cz-4y^2=0, \tag{4.23}$$

$$-6yz^2+18ay^2z+90y^4z-2y^3-6xz-2by=0. \tag{4.24}$$

But from the conditions for a point to lie on the critical set we know that $x=2yz-3ay^2-5y^4-2cy$ so equation 4.24 can be written as

$$-18yz^2+36ay^2z+120y^4z-2y^3-2by+12cyz=0. \tag{4.25}$$

We need to solve equations 4.23 and 4.25. We do this using MAPLE giving the following result.

$$\begin{aligned} b &= -3z^2+y^2, \\ 9ayz &= y^2+3z^2-30y^3z-3cz. \end{aligned}$$

So we have a function $f=3z^2-30y^3z-3cz+y^2-9ayz=0$. Rewriting this as

$$y^2+3z^2-3cz=3yz(10y^2+3a),$$

and squaring both sides we have

$$(y^2 + 3z^2 - 3cz)^2 = 9y^2z^2(10y^2 + 3a)^2.$$

But $y^2 = 3z^2 + b$ so substituting we obtain

$$F = -(6z^2 - 3cz + b)^2 + 9z^2(3z^2 + b)(3a + 10b + 30z^2)^2 = 0.$$

So we now have a function of one variable z so we can find its discriminant, again using MAPLE.

$$\Delta F = b^3(50b^2 + 150c^2b + 60ab + 45c^2a + 18a^2)^2G,$$

where G is a polynomial in a, b, c.
Now this gives us the local structure of the bifurcation set at the origin. We need to find any exceptional values for the modulus a. To do this we need to find the points for where the discriminant is singular. We can write G as

$$A_1b^3 + A_2b^2c^2 + A_3bc^4 + A_4c^6 + H.O.T.$$

where A_i are polynomials in a and the newton diagram of G gives the above equation as quasi-homogeneous of the above form. So we consider the lowest order terms and differentiate w.r.t. b and c to get the two equations.

$$\begin{aligned} 3A_1b^2 + 2A_2bc^2 + A_3c^4 &= 0 \\ c(2A_2b^2 + 4A_3bc^2 + 6A_4c^4 &= 0. \end{aligned}$$

Dividing the equations by c^4 and substituting $w = \frac{b}{c^2}$ we obtain:

$$\begin{aligned} 3A_1w^2 + 2A_2w + A_3 &= 0 \\ 2A_2w^2 + 4A_3w + 6A_4 &= 0. \end{aligned}$$

We now solve these equations by taking resultants using MAPLE and factorising to obtain the following conditions on a:
$82944(27a^2-4)^{12}a^2(2287680209a^{10}+5083732719 54a^8+131800418865a^6-114366526020a^4 + 18596183472a^2 - 918330048) = 0.$

This gives the exceptional values of a to be approximately

$$\pm\frac{2}{3\sqrt{3}}$$
$$\pm 0.2660974049$$
$$0$$

Now if $a \neq$ any of these exceptional values then our bifurcation set is diffeomorphic to J_{10}.

The other two factors of our discriminant give us the c–axis.

For multi–germ transitions we expect triplepoints, tacnodes and cusp–and–folds.

For the cusp–and–fold transitions we need $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ with $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$, $\frac{\partial f_2}{y}(x_1, y_1, z_1) = \frac{\partial f_2}{\partial z}(x_1, y_1, z_1) = 0$ and $\frac{\partial f_2}{\partial y}(x_2, y_2, z_2) = \frac{\partial f_2}{\partial z}(x_2, y_2, z_2) = \frac{\partial^2 f_2}{\partial y^2}(x_2, y_2, z_2) = 0$.This gives us the following 6 equations.

$$\begin{aligned}
xy_1 + z_1^3 - y_1^2 z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 - y_2^2 z_2 + ay_2^3 + y_2^5 \\
&\quad +bz_2 + cy_2^2 \\
x - 2y_1 z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\
x - 2y_2 z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\
-2z_2 + 6ay_2 + 20y_2^3 + 2c &= 0 \\
3z_1^2 - y_1^2 + b &= 0 \\
3z_2^2 - y_2^2 + b &= 0.
\end{aligned}$$

To solve this system of equations we need to employ computer algebra. This problem is still unsolved.

For the triplepoint transitions we need $(x_1, y_1, z_1) \neq (x_2, y_2, z_2) \neq (x_3, y_3, z_3)$ with $f(x_1, y_1, z_1) = f(x_2, y_2, z_2) = f(x_3, y_3, z_3)$. This gives us that $x_1 = x_2 = x_3 = x$, say. Now they are all folds so we need $\frac{\partial f_2}{\partial y}(x, y_i, z_i) = \frac{\partial f_2}{\partial z}(x, y_i, z_i) = 0$ for $i = 1, 2, 3$. This gives us the following system of nine equations:

$$
\begin{aligned}
xy_1 + z_1^3 - y_1^2 z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 - y_2^2 z_2 + ay_2^3 + y_2^5 \\
&\quad +bz_2 + cy_2^2 \\
xy_2 + z_2^3 - y_2^2 z_2 + ay_2^3 + y_2^5 + bz_2 + cy_2^2 &= xy_3 + z_3^3 - y_3^2 z_3 + ay_3^3 + y_3^5 \\
&\quad +bz_3 + cy_3^2 \\
xy_1 3z_3^3 - y_3^2 z_3 + ay_3^3 + y_3^5 + bz_3 + cy_3^2 &= xy_1 + z_1^3 - y_1^2 z_1 + ay_1^3 + y_1^5 \\
&\quad +bz_1 + cy_1^2 \\
x - 2y_1 z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\
x - 2y_2 z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\
x - +2y_3 z_3 + 3ay_3^2 + 5y_3^4 + 2cy_3 &= 0 \\
3z_1^2 - y_1^2 + b &= 0 \\
3z_2^2 - y_2^2 + b &= 0 \\
3z_3^2 - +y_3^2 + b &= 0.
\end{aligned}
$$

Again we need to employ computer algebra to eliminate the variables $x, y_1, y_2, y_3, z_1, z_2, z_3$ from the above system of equations. This problem is still unsolved.

For the tacnode stratum we need to consider the images of the tangents of the critical set, $\sum f$, by Df. Now our critical set is given by the two equations:

$$
\begin{aligned}
x - 2yz + 3ay^2 + 5y^4 + 2cy &= 0 \\
3z^2 - y^2 + b &= 0.
\end{aligned}
$$

We can form the Jacobian of these:

$$
\begin{pmatrix} 1 & -2z + 6ay + 20y^3 + 2c & -2y \\ 0 & -2y & 6z \end{pmatrix}.
$$

We wish to find the kernel vectors of this matrix, i.e. $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & -2z+6ay+20y^3+2c & -2y \\ 0 & -2y & 6z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we have

$$\begin{aligned} u+v(-2z+6ay+20y^3+2c)-2wy &= 0, \\ -2vy+6wz &= 0. \end{aligned}$$

This gives us the following form for a kernel vector:

$$\begin{pmatrix} \frac{6z^2}{y} - 18az - 60y^2z - \frac{6cz}{y} + 2y \\ \frac{3z}{y} \\ 1 \end{pmatrix}.$$

So we take two kernel vectors, given by substituting for (x, y_1, z_1) and (x, y_2, z_2) where $(x, y_1, z_1) \neq (x, y_2, z_2)$ as before, and applying Df to them, where Df is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ y & x-2yz+3ay^2+5y^4+2cy & 3z^2-y^2+b \end{pmatrix}.$$

Now we need a common tangent so this means that our vectors must be linearly dependent giving:

$$\left(\frac{6z_1^2}{y_1} - 18az_1 - 60y_1^2z_1 - 6cy_1z_1 + 2y_1\right)\left(3z_2^2 - 9ay_2z_2 - 45y_2^3z_2 + y_2^2 + \frac{3xz_2}{y_2} + b\right) - \left(\frac{6z_2^2}{y_2} - 18az_2 - 60y_2^2z_2 - \frac{6cz_2}{y_2} + 2y_2\right)\left(3z_1^2 - 9ay_1z_1 - 45y_1^3z_1 + y_1^2 + \frac{3xz_1}{y_1} + b\right) = 0.$$

So we have this equation and five others describing the fact that both branches of the multi–germ are folds:

$$\begin{aligned} xy_1 + z_1^3 - y_1^2z_1 + ay_1^3 + y_1^5 + bz_1 + cy_1^2 &= xy_2 + z_2^3 - y_2^2z_2 + ay_2^3 + y_2^5 \\ &\quad + bz_2 + cy_2^2 \\ x - 2y_1z_1 + 3ay_1^2 + 5y_1^4 + 2cy_1 &= 0 \\ x - 2y_2z_2 + 3ay_2^2 + 5y_2^4 + 2cy_2 &= 0 \\ 3z_1^2 - y_1^2 + b &= 0 \\ 3z_2^2 - y_2^2 + b &= 0. \end{aligned}$$

Again computer algebra is employed to eliminate the variables x, y_1, y_2, z_1, z_2. This problem is still unsolved.

Chapter 5

n–Dimensional Planar Motions

We now wish to consider map–germs: $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^2, 0)$, for general n.

Lemma 5.0.1 *Any $\mathcal{A}$–finite map–germ $f : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^2, 0)$ with $n \geq 3$ of $\mathcal{A}$–codimension $\leq n+2$ has corank ≤ 1.*

Proof. Suppose f has corank 2, i.e. rank 0. In $J^1(n,2)$ the jets of rank 0 are those with no linear terms, so they form a subspace of codimension $2n$, containing the $\mathcal{A}^1$–orbit through $j^1 f$. Therefore the $\mathcal{A}$–orbit has codimension $\geq 2n$ in $J^1(n,2)$, so by Lemma 2.6.3 of section 2.6, the $\mathcal{A}$–codimension of f is $\geq 2n$.

Now the $\mathcal{A}$–codimension is $\leq n+2$. We deduce that $2n \leq n+2$, i.e. $n \leq 2$, contradicting the hypothesis.

□

So we only need consider corank 0 and corank 1 map–germs: $(\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^2, 0)$. Now by Theorem 3.1.2 we know all such map–germs have the form

$$h(x,y,z) \sim (x, g(x, y_1, \ldots, y_m) + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$$

where $g(0, y_1, \ldots, y_m) \in \mathcal{M}_n^3$.

Now by considering the restriction to the codimension of our map–germs we have the following theorem.

Theorem 5.0.2 *For map-germs* $h(x, y, z) \sim (x, g(x, y_1, .., y_m) + \sum_{i=1}^{n-m-1} \epsilon_i z_i^2)$ *we need only consider the cases* $m = 0$, 1 *and* 2.

Proof. By the proof of Theorem 3.1.2 we know that m depends on the rank of the quadratic form $qf(y_1, ..., y_{n-1})$, and the 2–jet has the form $j^2h = (x, x(\sum_{i=1}^{m} a_i y_i) + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$ since $g(0, y_1, ..., y_m) \in \mathcal{M}_n^3$. This gives us the following cases for the 2–jet:

1. If we assume that at least one of the a_i's $\neq 0$, say y_1 by reordering, then we have $j^2h = (x, xy_1 + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$ by the change of coordinates $y_1 \mapsto y_1 - \sum_{i=1}^{m} a_i y_i$.
2. If all the a_i's$= 0$ then we have $j^2h = (x, \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$.
3. If we have no quadratic terms we have $j^2h = (x, 0)$.

We will compute the codimension of the orbits in $J^2(n, 2)$ for each of the above cases.

1. $j^2h = (x, xy_1 + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$.

$$T\mathcal{A}^2 \cdot h = \mathcal{M}_n\langle(1, y_1), (0, x), (0, z_1), ..., (0, z_{n-m-1})\rangle + h^*\mathcal{M}_2.$$

So we have

$$\begin{array}{ll} (\phi, 0) & \phi \in \mathcal{M}_n^2 \\ (0, x\psi) & \psi \in \mathcal{M}_n \\ (0, z_j z_k) & \\ (x, 0) & \\ (0, x) & \\ (0, x^2) & \end{array}$$

and we are missing

$$\begin{array}{c}(0, y_i)\\(0, z_j)\\(0, y_i y_l).\end{array}$$

So our codimension is given by

$$cod\ j^2 h = n - 1 + \frac{1}{2}m(m+1)$$

since the number of y_i's and z_j's is $n-1$ and the number of y_i's is m.

2. $j^2 h = (x, \sum_{j=1}^{n-m-1})$.

$$T\mathcal{A}^2 \cdot h = \mathcal{M}_n\langle(1,0), (0, z_1), ..., (0, z_{n-m-1})\rangle + h^* \mathcal{M}_2.$$

So we have

$$\begin{array}{ll}(\phi, 0) & \phi \in \mathcal{M}_n\\(0, \psi z_j) & \psi \in \mathcal{M}_n\\(0, x) & \\(0, x^2) & \end{array}$$

and we are missing

$$\begin{array}{c}(0, y_i)\\(0, z_j)\\(0, x y_i)\\(0, x^2).\end{array}$$

Giving our codimension as

$$cod\ j^2 h = n - 1 + \frac{1}{2}m(m+1) + m = n - 1 + \frac{1}{2}m(m+3).$$

3. $j^2 h = (x, 0)$.

$$T\mathcal{A}^2 \cdot h = \mathcal{M}_n\langle(1,0)\rangle + h^* \mathcal{M}_2.$$

So we have

$$\begin{array}{ll}(\phi, 0) & \phi \in \mathcal{M}_n\\(0, x^i) & i = 1, 2\end{array}$$

and we are missing

$$(0, y_i)$$
$$(0, z_j)$$
$$(0, xy_i)$$
$$(0, xz_j)$$
$$(0, y_i y_k)$$
$$(0, y_i z_j)$$
$$(0, z_j z_l).$$

Giving our codimension as

$$cod\ j^2h = n - 1 + \frac{1}{2}n(n+1) - 1 = \frac{1}{2}n(n+3) - 2.$$

Recall that we need $cod \leq n+2$. We have the following table of codimensions.

Case	m	codim in $J^2(n,2)$
$(x, xy_1 + \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$	0	$n-1$
	1	n
	2	$n+2$
	3	$n+5$
$(x, \sum_{j=1}^{n-m-1} \epsilon_j z_j^2)$	0	$n-1$
	1	$n+1$
	2	$n+4$
$(x, 0)$		$\frac{1}{2}n(n+3) - 2$

So we can see that for Case 1 we only need consider $m = 0,\ 1$ or 2. For Case 2, only $m = 0$ and 1, and finally for Case 3, we need $\frac{1}{2}n(n+3) - 2 \leq n + 2$ which is only true for $n = 1$ or $n = 2$ which contradicts our hypothesis. Giving us that in general we only need consider the cases $m = 0,\ 1$ or 2. □

Chapter 6

3–Parameter Motions of Space

In this chapter we are concerned with the problem of finding a complete list of local models for 3–parameter motions of space. This involves the classification of singularities $(\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$ under $\mathcal{A}$–equivalence. We use the 'nilpotent' classification methods described below, the majority of calculations were done by a computer program called TRANSVERSAL written by N. P. Kirk. Various parts of this classification have been considered by authors [B5, T]. Bruce, [B5] classified germs of $\mathcal{A}_e$–codimension ≤ 1, whereas Marar and Tari, [MT] classified the simple germs of corank 1 mappings. Our classification provides a complete list of singularities upto and including $\mathcal{A}_e$–codimension 2. Some jets for the codimension 3 cases give rise to extremely complicated orbits at a higher jet–level — such jets are excluded from further consideration as studying their geometry and bifurcation would be impossible with the computer software which we have available at present.

6.1 Nilpotent Filtration

We have the standard filtration as described in Chapter 2 but for this classification we will use the non–standard filtration used in Kirk, [K]. Firstly some notation. A filtration of a module M is a strictly decreasing chain of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots$$

and a filtration of a group G is a strictly decreasing chain of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots.$$

Definition 6.1.1 Given a subgroup $\mathcal{G}$ of $\mathcal{K}$ acting on $\mathcal{M}_n.\mathcal{E}(n,p)$ by a jet-filtration we mean a filtration $\{M_k\}$ of the module $\mathcal{M}_n.\mathcal{E}(n,p)$ together with a filtration $\{G_k\}$ of the group $\mathcal{G}$ by *normal* subgroups G_k such that the following hold:

1. Each M_k has finite codimension in $\mathcal{M}_n.\mathcal{E}(n,p)$.
2. Each quotient group $\mathcal{G}/G_k$ is a Lie group and there is a Lie group action

$$\mathcal{G}/G_k \times \mathcal{M}_n.\mathcal{E}(n,p)/M_k \longrightarrow \mathcal{M}_n.\mathcal{E}(n,p)/M_k$$

   induced from the action of $\mathcal{G}$ on $\mathcal{M}_n.\mathcal{E}(n,p)$.

Notation. Given a jet–filtration $F = (\{M_k\}, \{G_k\})$, we define the k-jet space to be the finite–dimensional vector space $\mathcal{M}_n.\mathcal{E}(n,p)/M_k$ and denote this by $J^k_F(n,p)$.

We now have a generalisation of the complete transversal theorem from Kirk, [K].

Theorem 6.1.2 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$, $L \subset L\mathcal{G}$, and $(\{M_i\}, \{G_i\})$ a jet-filtration such that for all $s \geq 0$*

1. *J^sL (a subset of $J^s(L\mathcal{G})$) is a Lie subalgebra of $L(J^s\mathcal{G})$;*
2. *for all $f \in \mathcal{M}_n.\mathcal{E}(n,p)$, $h \in M_s$ and $l \in L$ we have $l.(f+h) - l.f \in M_{s+1}$.*

Then for $f \in \mathcal{M}_n.\mathcal{E}(n.p)$, $k \geq 1$ and T a subspace of $H^{k+1} \subset J^{k+1}(n,p)$ (where H^{k+1} denotes the image of $\mathcal{M}^k_n.\mathcal{E}(n,p)$ in $J^k(n,p)$) such that

$$J^{k+1}L \cdot j^{k+1}f + T \supset H^{k+1},$$

we have any k-jet j^kg with $j^kg \sim_{J^k\mathcal{G}} j^kf$ has $(k+1)$-jet $j^{k+1}g \sim_{J^k\mathcal{G}} j^{k+1}f + t$ for some $t \in T$. Such a space T will be referred to as a complete transversal.

6.1.1 Nilpotent Lie Algebras

Definition 6.1.3 Let V be a finite-dimensional vector space over $\mathbf{R}$. An endomorphism $\alpha \in End(V)$ is called *nilpotent* if $\alpha^n = 0$ for some n. If L is a Lie algebra , with V an L-module, and S a subset of L, then we say that S is *nilpotent on* V if $S^n V = 0$ for some n.

We have the following Proposition from Kirk, [K].

Proposition 6.1.4 *Let $L \subset L\mathcal{K}$ be such that $J^1 L$ is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbf{R}^{n+p}$. Then given an integer $r \geq 1$ there is an integer $k_r \geq 1$ such that*

$$L^{k_r} \cdot (\mathcal{M}_n^r.\mathcal{E}(n,p)) \subset \mathcal{M}_n^{r+1}.\mathcal{E}(n,p).$$

Definition 6.1.5 (Nilpotent Filtration) Let $L \subset L\mathcal{K}$ be as above. We define for integers $r \geq 1$ & $s \geq 0$ the nilpotent filtration

$$M_{r,s}(L) = \sum_{i \geq s} L^i \cdot (\mathcal{M}_n^r.\mathcal{E}(n,p)) + \mathcal{M}_n^{r+1}.\mathcal{E}(n,p).$$

For $r = 0$ we just define $M_{0,0}$ to be $\mathcal{M}_n.\mathcal{E}(n,p)$ for consistency. The associated jet-space $J^{r,s}(n,p)$ is then defined to be $\mathcal{M}_n.\mathcal{E}(n,p)/M_{r,s}(L)$, and we denote the homogeneous terms of degree (r,s) by $H^{r,s}$.

Note that $M_{r,s}$ is just the standard filtration by degree with the added addition of the $\sum L^i \cdot (\mathcal{M}_n^r.\mathcal{E}(n,p))$ terms.

Example 6.1.6 We shall give an example by listing the generators for the space $H^{r,s}$ rather than describing the modules $M_{r,s}(L)$. Each (r,s)-jet-space is just a sharper format of the standard r-jet-space (by degree), the generators for $H^{r,s}$ giving the extra monomials which arise from passing from the $(r,s-1)$-jet-space to the (r,s)-jet-space.

If we take map-germs $(\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^2,0)$ with coordinates (x,y) in the source and (u,v) in the target and the nilpotent Lie sublagebra of $L\mathcal{A}$ given by

$$Sp\{x\partial/\partial y\} \oplus Sp\{v\partial/\partial u\} \oplus LA_1$$

then we obtain the following filtration:

(r, s)	Basis for $H^{r,s}$
(1,0)	$\{(0,0)\}$
(1,1)	$\{(0,y)\}$
(1,2)	$\{(y,0),(0,x)\}$
(1,3) or (2,0)	$\{(x,0)\}$
(2,1)	$\{(0,y^2)\}$
(2,2)	$\{(y^2,0),(0,xy)\}$
(2,3)	$\{(xy,0),(0,x^2)\}$
(2,4) or (3,0)	$\{(x^2,0)\}$
(3,1)	$\{(0,y^3)\}$
(3,2)	$\{(y^3,0),(0,xy^2)\}$
(3,3)	$\{(xy^2,0),(0,x^2y)\}$
(3,4)	$\{(x^2y,0),(0,x^3)\}$
(3,5) or (4,0)	$\{(x^3,0)\}$

As we can see an obvious pattern emerges.

We have the following two lemmas from [BduPW].

Lemma 6.1.7 *Let G be a connected Lie group acting smoothly on an affine space A; let B be a vector subspace of A. Then the action of G on A induces an action on A/B if and only if for all $a \in A$, $b \in B$ and $l \in LG$*

$$l \cdot (a+b) - l \cdot a \in B.$$

Lemma 6.1.8 *Let $L \subset L\mathcal{K}$ be such that J^1L is a Lie subalgebra of $L(J^1\mathcal{K})$ which is nilpotent on $\mathbb{R}^{n+p}$, so that the nilpotent filtration $M_{r,s}(L)$ is defined. Let $f \in \mathcal{M}_n.\mathcal{E}(n,p)$, $h \in M_{r,s}(L)$ & $l \in L$. Then*

$$l \cdot (f+h) - l \cdot f \in \mathcal{M}_{r,s+1}(L).$$

Finally we have the complete transversal result and the determinacy result for these nilpotent Lie algebras, [K].

Theorem 6.1.9 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $L \subset L\mathcal{G}$ such that*

1. *J^rL is a Lie subalgebra of $L(J^r\mathcal{G})$ for all $r \geq 0$;*
2. *the subalgebra J^1L of $L(J^1\mathcal{K})$ is nilpotent on $\mathbb{R}^{n+p}$.*

Then the following complete transversal result holds. For $f \in \mathcal{M}_n.\mathcal{E}(n,p)$ and T a subspace of $H^{r,s+1} \subset J^{r,s+1}(n,p)$ such that

$$J^rL \cdot j^{r,s+1}f + T \supset H^{r,s+1},$$

we have any (r,s)-jet $j^{r,s}g \sim j^{r,s}f$ has $(r,s+1)$-jet $j^{r,s+1}g \sim j^{r,s+1}f + t$ for some $t \in T$. T is known as the $(r,s+1)$ complete transversal.

Example 6.1.10 To illustrate the technique of obtaining nilpotent complete-transversals we will use an example from Kirk, [K].

Consider the 1–jet $(x, 0, 0, 0)$ from $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$; we will show that there exist four possible $J\mathcal{A}^2$–orbits over this 1–jet;

Firstly we consider the jet as a $(2, 2)$–jet; then the $(2, 3)$–transversal gives us the family $(x, ay^2, bxy, 0)$. Scaling this we find that we have four possibilities, namely;

$$\begin{array}{ll} \text{(i)} & (x, y^2, xy, 0) \\ \text{(ii)} & (x, y^2, 0, 0) \\ \text{(iii)} & (x, 0, xy, 0) \\ \text{(iv)} & (x, 0, 0, 0) \end{array}$$

depending on whether a and b are non–zero.

The $(2, s)$–transversals, for $s > 3$, are empty for (i),(ii) and (iii) and we can now consider these as 3–jets. In (iv), the $(2, 4)$–transversal is $\{(0, xy, 0, 0)\}$ giving $(x, axy, 0, 0)$ and scaling gives the following orbits:

$$\begin{array}{ll} \text{(v)} & (x, xy, 0, 0) \\ \text{(vi)} & (x, 0, 0, 0); \end{array}$$

again the higher $(2, s)$–transversals are empty. Clearly (iii) and (v) are in the same orbit and we need only consider (i),(ii),(v) and (vi) as 3–jets.

Corollary 6.1.11 *Let $\mathcal{G}$ be a subgroup of $\mathcal{K}$ and $L \subset L\mathcal{G}$ satisfying the two conditions in the above theorem. Suppose $\mathcal{G}$ satisfies the Mather condition, that is the following conditions are equivalent for $f \in \mathcal{M}_n.\mathcal{E}(n, p)$:*

1. *f is finitely–$\mathcal{G}$–determined;*
2. *$dim_{\mathbf{R}}(\mathcal{M}_n.\mathcal{E}(n, p)/L\mathcal{G}.f) < \infty$;*
3. *there exists $N < \infty$ such that $L\mathcal{G} \cdot f \supset \mathcal{M}_n^N.\mathcal{E}(n, p)$.*

Then a map–germ $f \in \mathcal{M}_n.\mathcal{E}(n, p)$ is k–$\mathcal{G}$–determined if

$$\mathcal{M}_n^{k+1}.\mathcal{E}(n,p) \subset L \cdot f.$$

Example 6.1.12 Again we will take our example to illustrate the nilpotent determinacy result from Kirk, [K].

We consider the 5-jet $(x, y^2, x^2y \pm y^5, 0)$ from $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^4, 0)$. The only non-empty $(7, s)$-transversal is $\{(0,0,0,y^7)\}$ giving two $J\mathcal{A}^7$-orbits:

$$\begin{array}{ll} (x, y^2, x^2 \pm y^5, y^7) & \text{7-determined,} \\ (x, y^2, x^2y \pm y^5, 0) & \text{(A).} \end{array}$$

Continuing to find (r, s)-transversals for (A) gives the following series:

$$\begin{array}{ll} (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}) & \text{(i),} \\ (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3}) & \text{(ii),} \\ (x, y^2, x^2y \pm y^5, y^{2k+1}) & \text{(iii),} \end{array}$$

where

$$\begin{array}{ll} \text{(i)} & (2k+3)\text{-determined, } k \geq 2,\ a^2 \pm 1 \neq 0; \\ \text{(ii)} & (2j+3)\text{-determined, } k \geq 2,\ j \geq k+1,\ a^2 \pm 1 \neq 0; \\ \text{(iii)} & (2k+1)\text{-determined, } k \geq 3. \end{array}$$

To see this we consider $f = (x, y^2, x^2y \pm y^5, 0)$ as a $(2k+1)$-jet for $k \geq 2$. In this example we can always obtain $\{x^a y^b e_i : a \geq 2\}$ (modulo higher order terms) using the $\mathcal{L}_1$-tangent space. So in this case we are only looking for vectors of the form $\{xy^b e_i\}$ and $\{y^b e_i\}$. Now

$$\frac{\partial f}{\partial x} = (1, 0, 2xy, 0), \quad \frac{\partial f}{\partial y} = (0, 2y, x^2 \pm 5y^4, 0),$$

and we can obtain $xy^{2k+1}e_1$ from $\frac{\partial f}{\partial x}$ and $xy^{2k+1}e_2$ from $\frac{\partial f}{\partial y}$. Also,

$$2xy^{2k+1}e_3 = y^{2k}\frac{\partial f}{\partial x} - u_2^k\frac{\partial f}{\partial u_1}$$

where (u_1, u_2, u_3, u_4) are the target coordinates.

So we have $xy^{2k+1}e_i$ for $i = 1, 2, 3$ and $y^{2k+2}e_i \in L\mathcal{L}_1 \cdot f$. This give us the $(2k+2)$–transversal $\{(0, 0, 0, xy^{2k+1})\}$ and the $J\mathcal{A}^{2k+2}$–orbits are :

$$(x, y^2, x^2y \pm y^5, xy^{2k+1}),$$
$$(x, y^2, x^2y \pm y^5, 0).$$

We will consider these separately.

- $(x, y^2, x^2y \pm y^5, xy^{2k+1})$. Now

$$\frac{\partial f}{\partial x} = (1, 0, 2xy, y^{2k+1}), \quad \frac{\partial f}{\partial y} = (0, 2y, x^2 \pm 5y^4, (2k+1)xy^{2k}),$$

which gives us $y^{2k+3}e_1$ and $y^{2k+3}e_2$. Also modulo terms in $\mathcal{M}_2^{2k+4}.\mathcal{E}(2,4)$ we have

$$\pm 4y^{2k+3}e_3 = y^{2k-1}\frac{\partial f}{\partial y} - 2u_2^k\frac{\partial f}{\partial u_2} - u_2^{k-1}u_3\frac{\partial f}{\partial u_3}$$

and we also have $xy^{2k+2}e_i \in L\mathcal{L}_1 \cdot f$. So in this jet–space we cannot obtain $y^{2k+3}e_4$ giving us the one–parameter family of $J\mathcal{A}^{2k+3}$–orbits:

$$(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}) \qquad \text{(2k+3)–determined,} \quad a^2 \pm 1 \neq 0.$$

For the determinacy calculations we must check that

$$\mathcal{M}_2^{2k+4}.\mathcal{E}(2,4) \subset L\mathcal{G} \cdot f + \mathcal{M}_2^{2k+4}.f^*(\mathcal{M}_4).\mathcal{E}(2,4) + \mathcal{M}_2^{4k+8}.\mathcal{E}(2,4).$$

Now $f^*(\mathcal{M}_4).\mathcal{E}_2 = \{x, y^2\}.\mathcal{E}_2$ and $\mathcal{M}_2^{2k+4}.f^*(\mathcal{M}_4).\mathcal{E}_2 \supset \mathcal{M}_2^{2k+6}$ so we can work modulo terms in $\mathcal{M}_2^{2k+6}$; we know we have all vectors of the form $x^ay^be_i$ for $a \geq 2$, so we are only looking for $\{y^{2k+4}e_i\}$, $\{xy^{2k+3}e_i\}$ and $\{y^{2k+5}e_i\}$. The first is trivial and the second we obtain by:

$$xy^{2k+3}e_i = u_2u_4\frac{\partial f}{\partial u_i} - ay^{2k+5}e_i,$$

we need only check $\{y^{2k+5}e_i\}$. Now

$$\begin{aligned} u_1u_4\frac{\partial f}{\partial u_i} &= (x^2y^{2k+1} + ay^{2k+3})e_i, \\ u_2^ku_3\frac{\partial f}{\partial u_i} &= (x^2y^{2k+1} \pm y^{2k+5})e_i, \end{aligned}$$

so

$$(axy^{2k+3} \mp y^{2k+5})e_i \in L\mathcal{L}_1 \cdot f.$$

But we also have that

$$u_2u_4\frac{\partial f}{\partial u_i} = (xy^{2k+3} + ay^{2k+5})e_i$$

which gives us

$$(a^2 \pm 1)y^{2k+5}e_i \in L\mathcal{L}_1 \cdot f.$$

So f is $(2k+3)$–determined for $a^2 \pm 1 \neq 0$.

If $a^2 \pm 1 = 0$ then a $(2k+5)$–transversal is $\{(0,0,0,y^{2k+5})\}$. Generally, if $(x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3})$ is a $(2j+1)$–jet for $j \geq k+1$, then from the above calculations we can see that the $(2j+2)$–transversal is empty. At the $(2j+3)$–level we get the $(2j+3)$–transversal to be $\{(0,0,0,y^{2j+3}\}$ and find the two $J\mathcal{A}^{2j+3}$–orbits to be :

$$\begin{gathered} (x, y^2, x^2 \pm y^5, xy^{2k+1} + ay^{2k+3} \pm y^{2j+3}) \\ (x, y^2, x^2y \pm y^5, xy^{2k+1} + ay^{2k+3}). \end{gathered}$$

The first of these is $(2j+3)$–determined. We can show this by working through a similar determinacy argument to the one above.

- $(x, y^2, x^2y \pm y^5, 0)$. Now

$$\frac{\partial f}{\partial x} = (1, 0, 2xy, 0), \quad \frac{\partial f}{\partial y} = (0, 2y, x^2 \pm 5y^4)$$

and from these we get $y^{2k+3}e_1$ and $y^{2k+3}e_2$; $y^{2k+3}e_3$ as in the above calculation. The only vector which we cannot obtain being $y^{2k+3}e_4$ giving us the $(2k+3)$–transversal and the following two $J\mathcal{A}^{2k+3}$–orbits:

$$\begin{array}{ll} (x, y^2, x^2 \pm y^5, y^{2k+3}) & (2k+3)\text{–determined}, \\ (x, y^2, x^2y \pm y^5, 0). & \end{array}$$

Again the determinacy calculation follows as above.

6.2 Classification Techniques

Consider map-germs :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$, where we shall denote the source co-ordinates by (x, y, z) and the target coordinates by (u, v, w). Throughout the classification we will use the subgroup $\mathcal{G}$ of $\mathcal{A}$ with Lie algebra

$$L\mathcal{A}_1 \oplus Sp\{x\partial/\partial y, x\partial/\partial z, y\partial/\partial z\} \oplus Sp\{v\partial/\partial u, w\partial/\partial u, w\partial/\partial v\},$$

which we shall denote $L\mathcal{G}$. This Lie algebra is nilpotent and we may use the complete transversal techniques and determinacy criterion of Section 6.1.

Theorem 6.2.1 *A map-germ $f : (\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ is k-$\mathcal{G}$-determined if*

$$\mathcal{M}_3^{k+1}.\mathcal{E}(3,3) \subset L\mathcal{G} \cdot f + \mathcal{M}_3^{k+1}.f * (\mathcal{M}_3).\mathcal{E}(3,3) + \mathcal{M}_3^{2k+2}.\mathcal{E}(3,3).$$

Proof. This is just Lemma (2.6) applied to Theorem (2.1) of [BduPW]. □

Although most of the determinacy calculations can be done using the computer the above result was used as a check in some cases.

For the complete transversal techniques we apply Theorem 6.1.9 using the above Lie algebra $L\mathcal{G}$ in the calculations.

6.3 Classification

We will use the following notation:

- f will denote a germ $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$;
- (x, y, z) will denote the source coordinates;
- $J^r(3,3)$ will denote the space of all triples of polynomials in $\mathbf{R}[x, y, z]$ truncated at degree r.

In the notation of [MaIII] we have the following tangent spaces

$$\begin{aligned} L\mathcal{A} \cdot f &= tf(\mathcal{M}_3.\theta_3) + wf(\mathcal{M}_3.\theta_3), \\ L\mathcal{A}_1 \cdot f &= tf(\mathcal{M}_3^2.\theta_3) + wf(\mathcal{M}_3^2.\theta_3), \end{aligned}$$

where $\mathcal{M}_3$ is the maximal ideal of the local ring $\mathcal{E}_3$ and θ_3 denotes the space $\theta_{1_{\mathbf{R}^3}}$ of vector fields along the identity map. In coordinate form our tangent spaces become the following subspaces of $\mathcal{E}(3,3)$.

$$\begin{aligned} L\mathcal{A} \cdot f &= \mathcal{M}_3.\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle + f^*\mathcal{M}_3.\{e_1, e_2, e_3\}, \\ L\mathcal{A}_1 \cdot f &= \mathcal{M}_3^2.\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle + f^*\mathcal{M}_3^2.\{e_1, e_2, e_3\}, \end{aligned}$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis vectors in $\mathbf{R}^3$.

Now the tangent space $L\mathcal{G}$ is an extension of the $L\mathcal{A}_1 \cdot f$ tangent space by vectors from $L\mathcal{A} \cdot f$.

The following theorem summarises the classification.

Theorem 6.3.1 *The following tables give, for each k, a list of $\mathcal{A}$-invariant strata for the jet-space $J^k(3,3)$. Determined jets are indicated by the appearance of the determinacy degree in the column marked* det. *For each $J\mathcal{A}^k$-orbit we state the $J\mathcal{A}^k$-codimension. The coefficients a,b,c,.. appearing in some jets are moduli. In such cases the $J\mathcal{A}^k$-codimension is the codimension of the stratum. A $\sharp$ in the first column indicates that the codimension has exceeded the limit which we are studying, and where $\flat$ means we do not carry this computation any further.*

The following tables give a list of the singularities of codimension less than or equal to 3.

	det	*stratum*	*codim*	*label*
	1	(x,y,z)	*0*	-
	-	$(x,y,0)$	*1*	*A*
	-	$(x,0,0)$	*4*	*B*
	-	$(0,0,0)$	*9*	*($\sharp$)*

Table 6.1: 1–Jets

	det	stratum	codim	label
A	2	(x, y, z^2)	1	-
	-	(x, y, yz)	2	A
	-	$(x, y, 0)$	4	B
B	2	$(x, yz, y^2 + z^2 + xy)$	4	-
	2	$(x, yz, y^2 - z^2 + xy)$	4	-
	-	$(x, yz, y^2 + z^2 + xy + xz)$	5	C
	-	$(x, yz, y^2 + z^2)$	6	D
	-	$(x, yz, y^2 - z^2)$	6	D'
	-	$(x, yz, y^2 + xy + xz)$	5	E
	-	$(x, yz, y^2 + xy)$	6	F
	-	$(x, yz, y^2 + xz)$	5	G
	-	(x, y, y^2)	7	(♮)
	-	$(x, y^2 + z^2, xy)$	6	H
	-	$(x, y^2 + z^2, 0)$	8	(♮)
	-	$(x, yz, xy + xz)$	6	I
	-	(x, yz, xy)	7	(♮)
	-	$(x, yz, 0)$	8	(♮)

Table 6.2: 2–Jets

	det	*stratum*	*codim*	*label*
A	3	$(x, y, yz + z^3)$	2	-
	-	$(x, y, yz + xz^2)$	3	A
	-	(x, y, yz)	4	B
B	3	$(x, y, z^3 + (y^2 + x^2)z)$	4	-
	3	$(x, y, z^3 + (y^2 - x^2)z)$	4	-
	-	$(x, y, z^3 + y^2 z)$	5	C
	-	(x, y, z^3)	7	$(\natural)$
	-	$(x, y, z^3 - x^2 z)$	5	C'
	3	$(x, y, z^3 - (y^2 - x^2)z)$	4	-
	3	$(x, y, z^3 - (y^2 + x^2)z)$	4	-
	-	$(x, y, yz^2 + x^2 z)$	5	D
	-	(x, y, yz^2)	6	E
	-	$(x, y, y^2 z + xyz + x^2 z)$	7	$(\natural)$
	-	$(x, y, y^2 z + xyz)$	7	$(\natural)$
	-	$(x, y, y^2 z + xyz - x^2 z)$	7	$(\natural)$
	-	$(x, y, y^2 z + x^2 z)$	7	$(\natural)$
	-	$(x, y, y^2 z)$	8	$(\natural)$
	-	$(x, y, y^2 z - x^2 z)$	7	$(\natural)$
	-	(x, y, xyz)	7	$(\natural)$
	-	$(x, y, 0)$	10	$(\natural)$
C	3	$(x, yz + y^3, y^2 + z^2 + xy + xz)$	5	-
	3	$(x, yz + y^3 - x^2 y, y^2 + z^2 + xy + xz)$	5	-
	-	$(x, yz + y^3 - \frac{3}{16}x^2 y, y^2 + z^2 + xy + xz)$	6	F
	-	$(x, yz + x^2 y, y^2 + z^2 + xy + xz)$	6	G
	-	$(x, yz, y^2 + z^2 + xy + xz)$	7	$(\natural)$
D/D'	3	$(x, yz \pm y^3 + ax^2 y, y^2 \pm z^2 + y^3 + x^2 y)$	6	-
		$a \neq \frac{1}{2}$		
	-	$(x, yz \pm y^3 \pm x^2 y, y^2 \pm z^2 + y^3)$	7	$(\natural)$
	-	$(x, yz \pm y^3, y^2 \pm z^2 + y^3)$	8	$(\natural)$
	3	$(x, yz \pm y^3 + ax^2 y, y^2 \pm z^2 + y^3 - x^2 y)$	6	-
	-	$(x, yz \pm \frac{1}{2}y^3 \pm x^2 y, y^2 \pm z^2 + y^3 + x^2 y)$	7	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3 \pm \frac{1}{2}x^2 y, y^2 \pm z^2 + y^3 + x^2 y)$	8	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3 + x^2 y)$	7	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3 \pm x^2 y, y^2 \pm z^2 + y^3)$	8	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3)$	8	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3 \pm x^2 y, y^2 \pm z^2 + y^3 - x^2 y)$	7	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3 \pm \frac{1}{2}x^2 y, y^2 \pm z^2 + y^3 - x^2 y)$	8	$(\natural)$
	-	$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3 - x^2 y)$	7	$(\natural)$

Table 6.3: 3–Jets

	det	stratum	codim	label
	3	$(x, yz + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$	6	-
	-	$(x, yz \pm x^2y, y^2 \pm z^2 + y^3)$	7	(♯)
	-	$(x, yz, y^2 \pm z^2 + y^3)$	8	(♯)
	3	$(x, yz + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$	6	-
	-	$(x, yz + y^3, y^2 \pm z^2 + x^2y)$	7	(♯)
	-	$(x, yz + y^3 + \frac{1}{2}x^2y, y^2 \pm z^2 + x^2y)$	8	(♯)
	-	$(x, yz + y^3, y^2 \pm z^2 + x^2y)$	7	(♯)
	-	$(x, yz + y^3 + x^2y, y^2 \pm z^2)$	7	(♯)
	-	$(x, yz + y^3, y^2 \pm z^2)$	8	(♯)
	-	$(x, yz + y^3 - x^2y, y^2 \pm z^2)$	7	(♯)
	-	$(x, yz \pm x^2y, y^2 \pm z^2 + x^2y)$	8	(♯)
	-	$(x, yz \pm \frac{1}{2}x^2y, y^2 \pm z^2 + x^2y)$	9	(♯)
	-	$(x, yz, y^2 \pm z^2 + x^2y)$	8	(♯)
	-	$(x, yz \pm x^2y, y^2 \pm z^2)$	8	(♯)
	-	$(x, yz, y^2 \pm z^2)$	9	(♯)
	-	$(x, yz \pm x^2y, y^2 \pm z^2 - x^2y)$	8	(♯)
	-	$(x, yz \pm \frac{1}{2}x^2y, y^2 \pm z^2 - x^2y)$	9	(♯)
	-	$(x, yz, y^2 \pm z^2 - x^2y)$	8	(♯)
E	-	$(x, yz, y^2 + xy + xz + z^3)$	5	H
	-	$(x, yz, y^2 + xy + xz)$	6	I
F	-	$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z)$	6	J
	-	$(x, yz, y^2 + xy + z^3 + x^2z)$	7	(♯)
	-	$(x, yz, y^2 + xy + z^3)$	8	(♯)
	-	$(x, yz, y^2 + xy + z^3 - x^2z)$	7	(♯)
	-	$(x, yz, y^2 + xy + x^2z)$	8	(♯)
	-	$(x, yz, y^2 + xy + x^2z)$	9	(♯)
	-	$(x, yz, y^2 + xy)$	10	(♯)
G	-	$(x, yz, y^2 + xz + z^3)$	5	K
	-	$(x, yz, y^2 + xz)$	6	L
H	-	$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3)$	6	M
	-	$(x, y^2 + z^2, xy + y^2z + ay^3)$	7	(♯)
	-	$(x, y^2 + z^2, xy + ay^3)$	8	(♯)
	-	$(x, y^2 + z^2, xy)$	9	(♯)
I	-	$(x, yz, xy + xz + z^3 + ay^2z + by^3)$	6	N
	-	$(x, yz, xy + xz + y^2z + ay^3)$	7	(♯)
	-	$(x, yz, xy + xz + ay^3)$	8	(♯)
	-	$(x, yz, xy + xz)$	9	(♯)

Table 6.4: 3–Jets continued

	det	*stratum*	*codim*	*label*
A	4	$(x, y, yz + xz^2 + z^4)$	3	-
	-	$(x, y, yz + xz^2)$	4	A
B	4	$(x, y, yz + z^4 + x^2z^2)$	4	-
	-	$(x, y, yz + z^4)$	5	B
	4	$(x, y, yz + z^4 - x^2z^2)$	4	-
	-	$(x, y, yz + xz^3)$	5	C
	-	$(x, y, yz + x^2z^2)$	6	D
	-	(x, y, yz)	7	$(\natural)$
C	4	$(x, y, z^3 + (y^2 + x^3)z)$	5	-
	-	$(x, y, z^3 + y^2z)$	6	E
C'	4	$(x, y, z^3 + (x^2 - y^3)z)$	5	-
	-	$(x, y, z^3 - x^2z)$	6	E'
D	-	$(x, y, yz^2 + x^2z + z^4)$	5	F
	-	$(x, y, yz^2 + x^2z + xz^3)$	6	G
	-	$(x, y, yz^2 + x^2z)$	7	$(\natural)$
E	-	$(x, y, yz^2 \pm z^4 + xz^3 + ax^3z)$	6	H
	-	$(x, y, yz^2 + z^4 + xz^3)$	7	$(\natural)$
	-	$(x, y, yz^2 + z^4 + x^3z)$	7	$(\natural)$
	-	$(x, y, yz^2 + z^4)$	8	$(\natural)$
	-	$(x, y, yz^2 + xz^3 + x^3z)$	7	$(\natural)$
	-	$(x, y, yz^2 + xz^3 - x^3z)$	7	$(\natural)$
	-	$(x, y, yz^2 + xz^3)$	8	$(\natural)$
	-	$(x, y, yz^2 + x^3z)$	8	$(\natural)$
	-	(x, y, yz^2)	9	$(\natural)$
F	4	$(x, yz + y^3 - \frac{3}{16}x^2y + x^3y,$ $y^2 + z^2 + xy + xz)$	6	-
	-	$(x, yz + y^3 - \frac{3}{16}x^2y, y^2 + z^2 + xy + xz)$	7	$(\natural)$
	4	$\frac{(x,yz+y^3-3}{16x^2y-x^3y,}$ $y^2 + z^2 + xy + xz)$	6	-
G	-	$(x, yz + x^2y, y^2 + z^2 + xy + xz)$	6	I
H	4	$(x, yz, y^2 + xy + xz + z^3 + ayz^3)$	5	-
I	-	$(x, yz, y^2 + xy + xz + z^4 + yz^3)$	6	J
	-	$(x, yz, y^2 + xy + xz + z^4)$	7	$(\natural)$
	-	$(x, yz, y^2 + xy + xz + z^4 - yz^3)$	6	J'
	-	$(x, yz, y^2 + xy + xz + yz^3)$	7	$(\natural)$
	-	$(x, yz, y^2 + xy + xz)$	8	$(\natural)$
	-	$(x, yz, y^2 + xy + xz - yz^3)$	7	$(\natural)$

Table 6.5: 4–Jets

	det	stratum	codim	label
J	-	$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z)$	6	K
K	4	$(x, yz, y^2 + xz + z^3 + z^4 + ayz^3)$	5	-
	4	$(x, yz, y^2 + xz + z^3 + yz^3)$	6	-
	-	$(x, yz, y^2 + xz + z^3)$	7	$(\natural)$
L	-	$(x, yz, y^2 + xz + z^4 + yz^3)$	6	L
	-	$(x, yz, y^2 + xz + z^4)$	7	$(\natural)$
	-	$(x, yz, y^2 + xz + yz^3)$	7	$(\natural)$
	-	$(x, yz, y^2 + xz)$	8	$(\natural)$
	-	$(x, yz, y^2 + xz - z^4 + yz^3)$	6	L
	-	$(x, yz, y^2 + xz - z^4)$	7	$(\natural)$
M	-	$(x, y^2 + z^2, xy + z^3 + ay^2z$ $+by^3 + y^3z + cy^4)$	6	M
	-	$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + cy^4)$	7	$(\natural)$
	-	$(x, y^2 + z^2, xy + z^3 + ay^2z$ $+by^3 - y^3z + cy^4)$	6	M
N	-	$(x, yz, xy + xz + z^3 + ay^2z$ $+by^3 + y^3z + cy^4)$	6	N
	-	$(x, yz, xy + xz + z^3 + ay^2z + by^3 + cy^4)$	7	$(\natural)$
	-	$(x, yz, xy + xz + z^3 + ay^2z$ $+by^3 - y^3z + cy^4)$	6	M

Table 6.6: 4–Jets continued

	det	*stratum*	*codim*	*label*
A	5	$(x, y, yz + xz^2 + z^5)$	4	-
	-	$(x, y, yz + xz^2)$	5	A
B	5	$(x, y, yz + z^4 + x^3z^2)$	5	-
	-	$(x, y, yz + z^4)$	6	B
C	-	$(x, y, yz + xz^3 + z^5)$	5	C
	-	$(x, y, yz + xz^3)$	6	D
D	-	$(x, y, yz + x^2z^2 + z^5)$	6	E
	-	$(x, y, yz + x^2z^2 + xz^4)$	7	($\sharp$)
	-	$(x, y, yz + x^2z^2)$	8	($\sharp$)
E	5	$(x, y, z^3 + (y^2 + x^4)z)$	6	-
	-	$(x, y, z^3 + y^2z)$	7	($\sharp$)
	5	$(x, y, z^3 + (y^2 - x^4)z)$	6	-
E'	5	$(x, y, z^3 - (x^2 - y^4)z)$	6	-
	-	$(x, y, z^3 - x^2z)$	7	($\sharp$)
	5	$(x, y, z^3 - (x^2 + y^4)z)$	6	-
F	5	$(x, y, yz^2 + x^2z + z^4 + z^5)$	5	-
	-	$(x, y, yz^2 + x^2z + z^4)$	6	F
	5	$(x, y, yz^2 + x^2z + z^4 - z^5)$	5	-
G	-	$(x, y, yz^2 + x^2z + xz^3 + az^5 + xz^4)$	6	G
	-	$(x, y, yz^2 + x^2z + xz^3 + az^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + x^2z + xz^3 + az^5 - xz^4)$	6	G'
H	-	$(x, y, yz^2 + z^4 + xz^3 - x^3z + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + \frac{1}{10}x^3z + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + x^3z + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + \frac{1}{6}x^3z + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + x^3z + z^5)$	7	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3 + \frac{1}{6}x^3z)$	8	($\sharp$)
	-	$(x, y, yz^2 + z^4 + xz^3)$	8	($\sharp$)

Table 6.7: 5–Jets

	det	stratum	codim	label
I	5	$(x, yz + x^2y + y^5, y^2 + z^2 + xy + xz)$	6	-
	-	$(x, yz + x^2y, y^2 + z^2 + xy + xz)$	7	($\natural$)
	5	$(x, yz + x^2y - y^5, y^2 + z^2 + xy + xz)$	6	-
J	-	$(x, yz, y^2 + xy + xz + z^4 + yz^3$ $+az^5 + byz^4)$	6	H
J'	-	$(x, yz, y^2 + xy + xz + z^4 - yz^3$ $+az^5 + byz^4)$	6	H
K	5	$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5)$ $a \neq \frac{3}{8}$	6	-
L	-	$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4)$	6	I
M	-	$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3$ $+y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	J
N	-	$(x, yz, xy + xz + z^3 + ay^2z + by^3$ $+y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	K

Table 6.8: 5–Jets continued

	det	stratum	codim	label
A	-	$(x,y,yz+xz^2+z^6)$	5	A
	-	$(x,y,yz+xz^2+xz^5)$	6	B
	-	$(x,y,yz+xz^2)$	7	$(\natural)$
B	6	$(x,y,yz+z^4+x^4z^2)$	6	-
	-	$(x,y,yz+z^4)$	7	$(\natural)$
	6	$(x,y,yz+z^4-x^4z^2)$	6	-
C	6	$(x,y,yz+xz^3+z^5+z^6)$	5	-
	-	$(x,y,yz+xz^3+z^5)$	6	C
D	-	$(x,y,yz+xz^3+z^6)$	6	D
	-	$(x,y,yz+xz^3)$	7	$(\natural)$
E	-	$(x,y,yz+x^2z^2+z^5+z^6)$	6	E
	-	$(x,y,yz+x^2z^2+z^5)$	7	$(\natural)$
	-	$(x,y,yz+x^2z^2+z^5-z^6)$	6	E'
F	-	$(x,y,yz^2+x^2z\pm z^4)$	6	F
G	-	$(x,y,yz^2+x^2z+xz^3-2z^5+xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-2z^5+xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-z^5+xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-z^5+xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{2}{5}z^5+xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{2}{5}z^5+xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3+z^5+xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3+z^5+xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{5}{4}z^5+xz^4+z^6)$	8	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{1}{2}z^5+xz^4+z^6)$	8	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{3}{10}z^5+xz^4+z^6)$	8	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3+z^5+xz^4)$	8	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{5}{4}z^5+xz^4)$	9	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{1}{2}z^5+xz^4)$	9	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{3}{10}z^5+xz^4)$	9	$(\natural)$
G'	-	$(x,y,yz^2+x^2z+xz^3-2z^5-xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-2z^5-xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-z^5-xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-z^5-xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{2}{5}z^5-xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3-\frac{2}{5}z^5-xz^4-z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3+z^5-xz^4+z^6)$	7	$(\natural)$
	-	$(x,y,yz^2+x^2z+xz^3+z^5-xz^4-z^6)$	7	$(\natural)$

Table 6.9: 6–Jets

	det	stratum	codim	label
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 - xz^4 + z^6)$	8	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 - xz^4 + z^6)$	8	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 - xz^4 + z^6)$	8	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 + z^5 - xz^4)$	8	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 - xz^4)$	9	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 - xz^4)$	9	(♯)
	-	$(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 - xz^4)$	9	(♯)
H	6	$(x, yz, y^2 + xy + xz + z^4 + yz^3$ $+az^5 + byz^4 + cz^6)$	6	-
H'	6	$(x, yz, y^2 + xy + xz + z^4 - yz^3$ $+az^5 + byz^4 + cz^6)$	6	-
I	6	$(x, yz, y^2 + xz + z^4 + yz^3$ $+az^5 + byz^4 + cz^6)$	6	-
I'	6	$(x, yz, y^2 + xz + z^4 - yz^3$ $+az^5 + byz^4 + cz^6)$	6	-
J	-	$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3$ $+y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	♭
J'	-	$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3$ $-y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	♭
K	-	$(x, yz, xy + xz + z^3 + ay^2z + by^3$ $+y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	♭
K'	-	$(x, yz, xy + xz + z^3 + ay^2z + by^3$ $-y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5)$	6	♭

Table 6.10: 6–Jets continued

	det	*stratum*	*codim*	*label*
A	-	$(x, y, yz + xz^2 + z^6)$	5	A
B	-	$(x, y, yz + xz^2 + xz^5 + z^7)$	6	B
	-	$(x, y, yz + xz^2 + xz^5)$	7	($\sharp$)
C	7	$(x, y, yz + xz^3 + z^5)$	6	-
D	-	$(x, y, yz + xz^3 + z^6 + z^7)$	6	C
	-	$(x, y, yz + xz^3 + z^6)$	7	($\sharp$)
E	7	$(x, y, yz + x^2z^2 + z^5 + z^6 + az^7)$ $a \neq 0$	6	-
E'	7	$(x, y, yz + x^2z^2 + z^5 - z^6 + az^7)$ $a \neq 0$	6	-
F	7	$(x, y, yz^2 + x^2z + z^4 + z^7)$	6	-
	-	$(x, y, yz^2 + x^2z + z^4)$	7	($\sharp$)
	7	$(x, y, yz^2 + x^2z + z^4 - z^7)$	6	-

Table 6.11: 7–Jets

	det	*stratum*	*codim*	*label*
A	-	$(x, y, yz + xz^2 + z^6 + z^8)$	5	A
	-	$(x, y, yz + xz^2 + z^6)$	6	B
	-	$(x, y, yz + xz^2 + z^6 - z^8)$	5	A'
B	-	$(x, y, yz + xz^2 + xz^5 + z^7)$	6	C
C	-	$(x, y, yz + xz^3 + z^6 + z^7 + az^8)$	6	D

Table 6.12: 8–Jets

	det	*stratum*	*codim*	*label*
A	*9*	$(x, y, yz + xz^2 + z^6 + z^8 + az^9)$	*5*	-
A'	*9*	$(x, y, yz + xz^2 + z^6 - z^8 + az^9)$	*5*	-
B	*9*	$(x, y, yz + xz^2 + z^6 + z^9)$	*6*	-
	-	$(x, y, yz + xz^2 + z^6)$	*7*	*(♮)*
C	-	$(x, y, yz + xz^2 + xz^5 + z^7 + az^9)$	*6*	*A*
D	-	$(x, y, yz + xz^3 + z^6 + z^7 + az^8)$	*6*	*B*

Table 6.13: 9–Jets

	det	*stratum*	*codim*	*label*
A	-	$(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9 + bz^{10})$	*6*	*A*
B	-	$(x, y, yz + xz^3 \pm z^6 + z^7 + az^8)$	*6*	*B*

Table 6.14: 10–Jets

	det	*stratum*	*codim*	*label*
A	-	$(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9 + bz^{10} + cz^{11})$	*6*	♭
B	-	$(x, y, yz + xz^3 \pm z^6 + z^7 + az^8 + bz^{11})$	*6*	♭

Table 6.15: 11–Jets

So we have the following list of finite map-germs of $\mathcal{A}_e$-codim ≤ 3:

Singularity	**A**	**D**
(x, y, z)	*0*	*1*
(x, y, z^2)	*0*	*2*
$(x, y, yz + z^3)$	*0*	*3*
$(x, y, yz + xz^2 + z^4)$	*0*	*4*
$(x, yz, y^2 + z^2 + xy)$	*1*	*2*
$(x, yz, y^2 - z^2 + xy)$	*1*	*2*
$(x, y, z^3 + (y^2 + x^2)z)$	*1*	*3*
$(x, y, z^3 + (y^2 - x^2)z)$	*1*	*3*
$(x, y, z^3 - (y^2 - x^2)z)$	*1*	*3*
$(x, y, z^3 - (y^2 + x^2)z)$	*1*	*3*
$(x, y, yz + z^4 + x^2z^2)$	*1*	*4*
$(x, y, yz + z^4 - x^2z^2)$	*1*	*4*
$(x, y, yz + xz^2 + z^5)$	*1*	*5*
$(x, yz + y^3, y^2 + z^2 + xy + xz)$	*2*	*3*
$(x, yz + y^3 - x^2y, y^2 + z^2 + xy + xz)$	*2*	*3*
$(x, y, z^3 + (y^2 + x^3)z)$	*2*	*4*
$(x, yz, y^2 + xy + xz + z^3 + ayz^3)$	*2*	*4*
$(x, yz, y^2 + xz + z^3 + z^4 + ayz^3)$	*2*	*4*
$(x, y, yz + z^4 + x^3z^2)$	*2*	*5*
$(x, y, yz^2 + x^2z + z^4 + z^5)$	*2*	*5*
$(x, y, yz^2 + x^2z + z^4 - z^5)$	*2*	*5*
$(x, y, yz + xz^3 + z^5 + z^6)$	*2*	*6*
$(x, y, yz + xz^2 + z^6 + z^8 + az^9)$	*2*	*9*
$(x, y, yz + xz^2 + z^6 - z^8 + az^9)$	*2*	*9*

Table 6.16: Singularities from $(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$

Singularity	A	D
$(x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$	3	3
$a \neq \frac{1}{2}$		
$(x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$	3	3
$a \neq \frac{1}{2}$		
$(x, yz + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$	3	3
$a \neq \frac{1}{2}$		
$(x, yz + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$	3	3
$a \neq \frac{1}{2}$		
$(x, yz + y^3 - \frac{3}{16}x^2y + x^3y, y^2 + z^2 + xy + xz)$	3	4
$(x, yz + y^3 - \frac{3}{16}x^2y - x^3y, y^2 + z^2 + xy + xz)$	3	4
$(x, yz, y^2 + xz + z^3 + yz^3)$	3	4
$(x, y, z^3 + (y^2 + x^4)z)$	3	5
$(x, y, z^3 + (y^2 - x^4)z)$	3	5
$(x, y, z^3 - (x^2 - y^4)z)$	3	5
$(x, y, z^3 - (x^2 + y^4)z)$	3	5
$(x, yz + x^2y + y^5, y^2 + z^2 + xy + xz)$	3	5
$(x, yz + x^2y - y^5, y^2 + z^2 + xy + xz)$	3	5
$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5)$	3	5
$a \neq \frac{3}{8}$		
$(x, y, yz + z^4 + x^4z^2)$	3	6
$(x, y, yz + z^4 - x^4z^2)$	3	6

Table 6.17: Table 6.16 continued

Singularity	A	D
$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6)$	3	6
$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6)$	3	6
$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6)$	3	6
$(x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6)$	3	6
$(x, y, yz + xz^3 + z^5)$	3	6
$(x, y, yz + x^2z^2 + z^5 + z^6 + az^7)$ $a \neq 0$	3	7
$(x, y, yz + x^2z^2 + z^5 - z^6 + az^7)$ $a \neq 0$	3	7
$(x, y, yz^2 + x^2z + z^4 + z^7)$	3	7
$(x, y, yz^2 + x^2z + z^4 - z^7)$	3	7
$(x, y, yz + xz^2 + z^6 + z^9)$	3	9

Table 6.18: Table 6.16 continued

where A is the $\mathcal{A}_e$-codimension and D is the degree of determinacy.

In the remainder of this chapter we will describe the proof. First we make the following observation on bounds on the codimension. For a map-germ f consider the map

$$\begin{array}{rcl} \pi : (J\mathcal{A}^{k+1} \cdot j^{k+1}f) & \to & (J\mathcal{A}^k \cdot j^k f) \\ j^{k+1}g & \mapsto & j^k g \end{array}$$

from the $J\mathcal{A}^{k+1}$-orbit of f to the $J\mathcal{A}^k$-orbit of f. Any $j^k g \in (J\mathcal{A}^k \cdot j^k f)$ may be written in the form $j^k\phi \cdot j^k f$ for some $\phi \in \mathcal{A}$. But

$$\pi(j^{k+1}\phi \cdot j^{k+1}f) = j^k(\phi \cdot f) = j^k\phi \cdot j^k f$$

so π is surjective and $\pi^{-1}((J\mathcal{A}^k)\cdot j^k f)$ is a submanifold of $J^{k+1}(n,p)$ with codimension equal to the codimension of $(J\mathcal{A}^k) \cdot j^k f$ in $J^k(n,p)$. But $\pi^{-1}((J\mathcal{A}^k) \cdot j^k f) \supset (J^{k+1}\mathcal{A}) \cdot j^{k+1}f$ and it follows that $J\mathcal{A}^k$-codim$(f) \leq J^{k+1}\mathcal{A}$-codim$(f)$. Now if a map-germ f is k–$\mathcal{A}$– determined then,

$$\mathcal{M}_3^{k+1}.\mathcal{E}(3,3) \subset L\mathcal{A} \cdot f,$$

hence, $J^k\mathcal{A}$–codim(f) = $\mathcal{A}$–codim(f). This last results allows us to use the computer to calculate the $\mathcal{A}$–codim(f).

6.3.1 The 1–Jets

Assuming f is of corank 0, 1, 2 or 3 and using linear algebra, we see that upto $\mathcal{A}$–equivalence the $J\mathcal{A}^1$–orbits of f are

(x, y, z)	1–determined,
$(x, y, 0)$	(A),
$(x, 0, 0)$	(B),
$(0, 0, 0)$	($\sharp$),

where $\sharp$ indicates that we have exceeded our codimension limit.

We now consider the corank 1 case (A).

6.3.2 The 2–Jets

There are three $J\mathcal{A}^2$–orbits over $(x, y, 0)$.

(x, y, z^2)	2–determined,
(x, y, yz)	(A),
$(x, y, 0)$	(B).

Remark. This is easy to show using the computer and $\mathcal{A}_1$–complete transversal methods, the resulting transversal giving the $J\mathcal{A}^2$–orbits to be of the form

$$(x, y, az^2 + byz + cxz).$$

If we then look at the nilpotent filtration, at the (2,1)-level we obtain (x, y, az^2) which we can scale to (x, y, z^2) if $a \neq 0$. We find that the $(2, s)$-transversals, for $s > 1$ are empty and working further we find that this is actually 2-determined. If $a = 0$ then at the (2,2)-level we obtain (x, y, byz) which we can scale to (x, y, yz) if $b \neq 0$. Again we find that the higher $(2, s)$-transversals are empty. This leaves the case when $a = b = 0$ which gives $(x, y, 0)$.

The numbering system of the classification will be as follows. Each of the jets (A) and (B) above will be considered in the section below. The jets which arise from the higher jet-levels will be numbered with respect to that section. Each branch will be studied until the germ is found to be determined or the complexity suggests a natural stopping point.

6.3.3 The 3–Jets

A $(3,1)$–transversal for (x,y,yz) is $\{(0,0,z^3)\}$ giving two orbits to consider as $(x,y,yz+z^3) \sim_{\mathcal{A}} (x,y,yz-z^3)$ by the change of coordinates $(x,y,z) \longmapsto (x,y,-z)$:

- $(x,y,yz+z^3)$. All higher $(3,s)$–transversals are empty.
- (x,y,yz). This has $(3,3)$–transversal $\{(0,0,xz^2)\}$ again giving only two orbits as $(x,y,yz+xz^2) \sim_{\mathcal{A}} (x,y,yz-xz^2)$ by the change of coordinates $x \longmapsto -x$.

So there are three $J\mathcal{A}^3$–orbits over (x,y,yz):

$$\begin{array}{ll} (x,y,yz+z^3) & \text{3–determined}, \\ (x,y,yz+xz^2) & \text{(A)}, \\ (x,y,yz) & \text{(B)}. \end{array}$$

A $(3,1)$–transversal for $(x,y,0)$ is $\{(0,0,z^3)\}$; giving two orbits to consider as $(x,y,z^3) \sim_{\mathcal{A}} (x,y,-z^3)$;

- (x,y,z^3). This has $(3,3)$–transversal $\{(0,0,y^2z)\}$ giving three orbits to consider;

$$\begin{array}{ll} (x,y,z^3+y^2z) & \text{(i)}, \\ (x,y,z^3) & \text{(ii)}, \\ (x,y,z^3-y^2z) & \text{(iii)}. \end{array}$$

(i) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving three orbits;

$$\begin{array}{ll} (x,y,z^3+y^2z+x^2z) & \text{3–determined}, \\ (x,y,z^3+y^2z) & \text{(C)}, \\ (x,y,z^3+y^2z-x^2z) & \text{3–determined}. \end{array}$$

(ii) has $(3,4)$–transversal $\{(0,0,xyz)\}$ giving us two orbits to consider;

$$\begin{array}{ll} (x,y,z^3+xyz) & \text{(a)}, \\ (x,y,z^3) & \text{(b)}. \end{array}$$

(a) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving the $\mathcal{A}$–trivial family:

$$(x,y,z^3+xyz+ax^2z).$$

giving us one orbit (x,y,z^3+xyz) which is 3–determined and equivalent to $(x,y,z^3+x^2z-y^2z)$ via the change of coordinates $(x,y,z)\mapsto(x+y,x-y,z)$.
(b) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving three orbits;

$$\begin{array}{ll} (x,y,z^3+x^2z) & \mathcal{A}\text{–equivalent to (C)}, \\ (x,y,z^3) & (\sharp), \\ (x,y,z^3-x^2z) & (\mathrm{C}'). \end{array}$$

(iii) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving three orbits;

$$\begin{array}{ll} (x,y,z^3-y^2z+x^2z) & \text{3–determined}, \\ (x,y,z^3-y^2z) & \mathcal{A}\text{–equivalent to } (\mathrm{C}'), \\ (x,y,z^3-y^2z-x^2z) & \text{3–determined}. \end{array}$$

- $(x,y,0)$. This has $(3,2)$–transversal $\{(0,0,yz^2)\}$ giving two orbits as $(x,y,yz^2)\sim_{\mathcal{A}}(x,y,-yz^2)$;

$$\begin{array}{ll} (x,y,yz^2) & \text{(i)}, \\ (x,y,0) & \text{(ii)}. \end{array}$$

(i) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving two orbits as $(x,y,yz^2+x^2z)\sim_{\mathcal{A}}(x,y,yz^2-x^2z)$ via the change of coordinates $z\longmapsto -z$);

$$\begin{array}{ll} (x,y,yz^2+x^2z) & \text{(D)}, \\ (x,y,yz^2) & \text{(E)}. \end{array}$$

All higher $(3,s)$–transversals are empty.

(ii) has $(3,3)$–transversal $\{(0,0,y^2z),(0,0,xz^2)\}$ giving four orbits to consider as $(x,y,y^2z+xz^2) \sim_{\mathcal{A}} (x,y,y^2z-xz^2) \sim_{\mathcal{A}} (x,y,-y^2z+xz^2) \sim_{\mathcal{A}} (x,y,-y^2z-xz^2)$ etc. by changes of coordinates in the source and target;

(x,y,y^2z+xz^2)	$\mathcal{A}$–equivalent to (D),
(x,y,y^2z)	(a),
(x,y,xz^2)	$\mathcal{A}$–equivalent to (E),
$(x,y,0)$	(b).

(a) has $(3,4)$–transversal $\{(0,0,xyz)\}$ giving two orbits (by a change of coordinates $x \longmapsto -x$ $(x,y,y^2z+xyz) \sim_{\mathcal{A}} (x,y,y^2z-xyz)$;

(x,y,y^2z+xyz)	(α),
(x,y,y^2z)	(β).

(α) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving three orbits;

$(x,y,y^2z+xyz+x^2z)$	$(\natural)$,
(x,y,y^2z+xyz)	$(\natural)$,
$(x,y,y^2z+xyz-x^2z)$	$(\natural)$.

(β) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ which gives three orbits;

(x,y,y^2z+x^2z)	$(\natural)$,
(x,y,y^2z)	$(\natural)$,
(x,y,y^2z-x^2z)	$(\natural)$.

(b) has $(3,4)$–transversal $\{(0,0,xyz)$ which gives two orbits as $(x,y,xyz) \sim_{\mathcal{A}} (x,y,-xyz)$;

(x,y,xyz)	(α),
$(x,y,0)$	(β).

(α) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving two orbits;

$(x,y,xyz+x^2z)$	$\mathcal{A}$–equivalent to (x,y,y^2z+xyz),
(x,y,xyz)	$(\natural)$.

(β) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ which gives us two orbits;

(x,y,x^2z)	$\mathcal{A}$–equivalent to (E),
$(x,y,0)$	$(\natural)$.

So we have seventeen $J\mathcal{A}^3$–orbits over $(x, y, 0)$:

$(x, y, z^3 + (y^2 + x^2)z)$	3–determined,
$(x, y, z^3 + (y^2 - x^2)z)$	3–determined,
$(x, y, z^3 + y^2 z)$	(C),
(x, y, z^3)	($\sharp$),
$(x, y, z^3 - x^2 z)$	(C′),
$(x, y, z^3 - y^2 z + x^2 z)$	3–determined,
$(x, y, z^3 - y^2 z - x^2 z)$	3–determined,
$(x, y, yz^2 + x^2 z)$	(D),
(x, y, yz^2)	(E),
$(x, y, y^2 z + xyz + x^2 z)$	($\sharp$),
$(x, y, y^2 z + xyz)$	($\sharp$),
$(x, y, y^2 z + xyz - x^2 z)$	($\sharp$),
$(x, y, y^2 z + x^2 z)$	($\sharp$),
$(x, y, y^2 z)$	($\sharp$),
$(x, y, y^2 z - x^2 z)$	($\sharp$),
(x, y, xyz)	($\sharp$),
$(x, y, 0)$	($\sharp$).

6.3.4 The 4–Jets

The $(4,1)$–transversal for (A) is given by $\{(0,0,z^4)\}$; higher $(4,s)$– transversals being empty. This gives orbits: $(x, y, yz + xz^2 + z^4)$ and $(x, y, yz + xz^2)$ ($(x, y, yz + xz^2 + z^4) \sim_{\mathcal{A}} (x, y, yz + xz^2 - z^4)$ via the change of coordinates $(x, y, z) \longmapsto (-x, -y, z)$ and $(u, v, w) \longmapsto (-u, -v, -w)$).

$(x, y, yz + xz^2 + z^4)$	4–determined,
$(x, y, yz + xz^2)$	(A).

The $(4,1)$–transversal for (B) is $\{(0,0,z^4)\}$ giving two orbits to consider, again $(x, y, yz + z^4) \sim_{\mathcal{A}} (x, y, yz - z^4)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$ and $(u, v, w) \longmapsto (u, -v, -w)$;

- $(x, y, yz + z^4)$ has $(4,5)$–transversal $\{(0,0,x^2 z^2)\}$ giving three orbits; all higher $(4,s)$–transversals are empty;

$(x, y, yz + z^4 + x^2 z^2)$	4–determined,
$(x, y, yz + z^4)$	(B),
$(x, y, yz + z^4 - x^2 z^2)$	4–determined.

- (x, y, yz) has $(4,3)$–transversal $\{(0, 0, xz^3)\}$ giving again two orbits to consider $((x, y, yz + xz^3) \sim_{\mathcal{A}} (x, y, yz - xz^3)$ via the change of coordinates $x \longmapsto -x$;

  - $(x, y, yz + xz^3)$ (C): all higher $(4, s)$–transversals are empty.
  - (x, y, yz) has $(4,5)$–transversal $\{(0, 0, x^2z^2)\}$ which gives two orbits since $(x, y, yz+x^2z^2) \sim_{\mathcal{A}} (x, y, yz-x^2z^2)$ via the change of coordinates $y \longmapsto -y$ and $(u, v, w) \longmapsto (u, -v, -w)$;

$$\begin{array}{ll} (x, y, yz + x^2z^2) & \text{(D)}, \\ (x, y, yz) & (\sharp). \end{array}$$

This gives us seven $J\mathcal{A}^4$–orbits over (x, y, yz):

$$\begin{array}{ll} (x, y, yz + z^4 + x^2z^2) & \text{4-determined}, \\ (x, y, yz + z^4 - x^2z^2) & \text{4-determined}, \\ (x, y, yz + z^4) & \text{(B)}, \\ (x, y, yz + xz^3) & \text{(C)}, \\ (x, y, yz + x^2z^2) & \text{(D)}, \\ (x, y, yz) & (\sharp). \end{array}$$

The $(4,1)$–transversal for (C) is empty, the only higher non–empty $(4, s)$–transversal being the $(4,7)$–transversal $\{(0, 0, x^3z)\}$; giving two orbits $(x, y, z^3 + (y^2 + x^3)z)$ (as this is equivalent to $(x, y, z^3 + y^2z - x^2z)$ via the change of coordinates $x \longmapsto -x$) and $(x, y, z^3 + y^2z)$ to consider. All higher $(4, s)$–transversals in both cases are empty.

$$\begin{array}{ll} (x, y, z^3 + (y^2 + x^3)z) & \text{4–determined}, \\ (x, y, z^3 + y^2z) & \text{(E)}. \end{array}$$

The 4–transversals for (C ′) are the same as above, so we get the following two $J\mathcal{A}^4$–orbits over $(x, y, z^3 - x^2z + y^3z)$;

$$\begin{array}{ll} (x, y, z^3 - x^2z + y^3z) & \text{4–determined}, \\ (x, y, z^3 - x^2z) & \text{(E}'\text{)}. \end{array}$$

But $(x, y, z^3 - x^2z + y^3z) \sim_{\mathcal{A}} (x, y, z^3 + y^2z + x^3z)$.

The $(4,1)$–transversal for (D) is $\{(0,0,z^4)\}$ giving two orbits to consider (as $(x,y,yz^2+x^2z+z^4) \sim_{\mathcal{A}} (x,y,yz^2+x^2z-z^4)$ via the change of coordinates $(x,y,z) \longmapsto (x,-y,-z)$ and $(u,v,w) \longmapsto (u,-v,-w)$);

- $(x,y,yz^2+x^2z+z^4)$; whose higher $(4,s)$–transversals are all empty.
- (x,y,yz^2+x^2z) which has $(4,3)$–transversal $\{(0,0,xz^3)\}$ which gives us two orbits ; all higher $(4,s)$–transversals being empty in both cases.

So we have three $J\mathcal{A}^4$–orbits over (x,y,yz^2+x^2z):

$$\begin{array}{ll} (x,y,yz^2+x^2z+z^4) & \text{(F)}, \\ (x,y,yz^2+x^2z+xz^3) & \text{(G)}, \\ (x,y,yz^2+x^2z) & (\sharp). \end{array}$$

The $(4,1)$–transversal for (E) is $\{(0,0,z^4)\}$ which gives two orbits to consider (as $(x,y,yz^2+z^4) \sim_{\mathcal{A}} (x,y,yz^2-z^4)$);

- (x,y,yz^2+z^4). This has $(4,3)$–transversal $\{(0,0,xz^3)\}$ which gives us two orbits again as $(x,y,yz^2+z^4+xz^3) \sim_{\mathcal{A}} (x,y,yz^2+z^4-xz^3)$;

$$\begin{array}{ll} (x,y,yz^2+z^4+xz^3) & \text{(i)}, \\ (x,y,yz^2+z^4) & \text{(ii)}. \end{array}$$

(i) has $(4,7)$–transversal $\{(0,0,x^3z)\}$ which gives us the 1–parameter family

$$(x,y,yz^2+z^4+xz^3+ax^3z).$$

(ii) has $(4,7)$–transversal $\{(0,0,x^3z)\}$ which gives us two orbits since $(x,y,yz^2+z^4+x^3z) \sim_{\mathcal{A}} (x,y,yz^2+z^4-x^3z)$ via the change of coordinates $(x,y,z) \longmapsto (-x,y,z)$);

$$\begin{array}{ll} (x,y,yz^2+z^4+x^3z) & (\sharp), \\ (x,y,yz^2+z^4) & (\sharp). \end{array}$$

- (x, y, yz^2). This has $(4,3)$–transversal $\{(0,0,xz^3)\}$ giving two orbits to consider $((x,y,yz^2+xz^3) \sim_{\mathcal{A}} (x,y,yz^2-xz^3))$;

$$\begin{array}{ll} (x,y,yz^2+xz^3) & \text{(i)}, \\ (x,y,yz^2) & \text{(ii)}. \end{array}$$

(i) has $(4,7)$–transversal $\{(0,0,x^3z)\}$ which gives three orbits;

$$\begin{array}{ll} (x,y,yz^2+xz^3+x^3z) & (\sharp), \\ (x,y,yz^2+xz^3) & (\sharp), \\ (x,y,yz^2+xz^3-x^3z) & (\sharp). \end{array}$$

(ii) has $(4,7)$–transversal $\{(0,0,x^3z)\}$ which gives two orbits;

$$\begin{array}{ll} (x,y,yz^2+x^3z) & (\sharp), \\ (x,y,yz^2) & (\sharp). \end{array}$$

So we have nine $J\mathcal{A}^4$–orbits over (x,y,yz^2):

$$\begin{array}{ll} (x,y,yz^2+z^4+xz^3+ax^3z) & \text{(H)}, \\ (x,y,yz^2+z^4+xz^3) & (\sharp), \\ (x,y,yz^2+z^4+x^3z) & (\sharp), \\ (x,y,yz^2+z^4) & (\sharp), \\ (x,y,yz^2+xz^3+x^3z) & (\sharp), \\ (x,y,yz^2+xz^3-x^3z) & (\sharp), \\ (x,y,yz^2+xz^3) & (\sharp), \\ (x,y,yz^2+x^3z) & (\sharp), \\ (x,y,yz^2) & (\sharp). \end{array}$$

6.3.5 The 5–Jets

The $(5,1)$–transversal for (A) is $\{(0,0,z^5)\}$; all higher $(5,s)$–transversals are empty. This gives us two $J\mathcal{A}^5$–orbits over $(x,y,yz+xz^2)$ since $(x,y,yz+xz^2+z^5) \sim_{\mathcal{A}} (x,y,yz+xz^2-z^5)$ via the change of coordinates $(x,y,z) \longmapsto (x,-y,-z)$:

$$\begin{array}{ll} (x,y,yz+xz^2+z^5) & \text{5–determined,} \\ (x,y,yz+xz^2) & \text{(A).} \end{array}$$

The $(5,1)$–transversal for (B) is empty, the first non-empty $(5,s)$–transversal being the $(5,7)$–transversal $\{(0,0,x^3z^2)\}$; all higher $(5,s)$–transversals are empty. This gives us two $J\mathcal{A}^5$–orbits over $(x,y,yz+z^4)$ since $(x,y,yz+z^4+x^3z^2) \sim_{\mathcal{A}} (x,y,yz+z^4-x^3z^2)$ via the change of coordinates $(x,y,z) \longmapsto (-x,y,z)$:

$$\begin{array}{ll} (x,y,yz+z^4+x^3z^2) & \text{5–determined,} \\ (x,y,yz+z^4) & \text{(B).} \end{array}$$

The $(5,1)$–transversal for (C) is $\{(0,0,z^5)\}$ which gives us two orbits to consider (since $(x,y,yz+xz^3+z^5) \sim_{\mathcal{A}} (x,y,yz+xz^3-z^5)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,-z)$ and $(u,v,w) \longmapsto (-u,v,-w)$; all the higher $(5,s)$–transversals being empty in both cases. Thus we have two $J\mathcal{A}^5$–orbits over $(x,y,yz+xz^3)$:

$$\begin{array}{ll} (x,y,yz+xz^3+z^5) & \text{(C),} \\ (x,y,yz+xz^3) & \text{(D).} \end{array}$$

The $(5,1)$–transversal for (D) is $\{(0,0,z^5)\}$ giving us two orbits to consider, as $(x,y,yz+x^2z^2+z^5) \sim_{\mathcal{A}} (x,y,yz+x^2z^2-z^5)$ via the change of coordinates $(x,y,z) \longmapsto (x,-y,-z)$.

- $(x,y,yz+x^2z^2+z^5)$. All higher $(5,s)$–transversals are empty.
- $(x,y,yz+x^2z^2)$. This has $(5,3)$–transversal $\{(0,0,xz^4)\}$ giving two orbits to consider as $(x,y,yz+x^2z^2+xz^4) \sim_{\mathcal{A}} (x,y,yz+x^2z^2-xz^4)$ via the change of coordinates $(x,y,z) \longmapsto (-x,y,z)$;

$$\begin{array}{ll} (x,y,yz+x^2z^2+xz^4) & (\sharp), \\ (x,y,yz+x^2z^2) & (\sharp). \end{array}$$

So we have three $J\mathcal{A}^5$–orbits over $(x, y, yz + x^2z^2)$:

$$\begin{array}{ll} (x, y, yz + x^2z^2 + z^5) & \text{(E)}, \\ (x, y, yz + x^2z^2 + xz^4) & (\sharp), \\ (x, y, yz + x^2z^2) & (\sharp). \end{array}$$

The $(5, 1)$–transversal for (E) is empty; the first non-empty $(5, s)$–transversal being the $(5, 9)$–transversal $\{(0, 0, x^4z)\}$; all higher $(5, s)$–transversals being empty. This gives three $J\mathcal{A}^5$–orbits over $(x, y, z^3 + y^2z)$:

$$\begin{array}{ll} (x, y, z^3 + (y^2 + x^4)z) & \text{5–determined}, \\ (x, y, z^3 + y^2z) & (\sharp), \\ (x, y, z^3 + (y^2 - x^4)z) & \text{5–determined}. \end{array}$$

The $(5, 1)$–transversal for (E ′) is empty; the first non-empty $(5, s)$–transversal being the $(5, 9)$–transversal $\{(0, 0, y^4z)\}$; all higher $(5, s)$–transversals being empty. This gives three $J\mathcal{A}^5$–orbits over $(x, y, z^3 - x^2z)$:

$$\begin{array}{ll} (x, y, z^3 - (x^2 - y^4)z) & \text{5–determined}, \\ (x, y, z^3 - x^2z) & (\sharp), \\ (x, y, z^3 - (x^2 + y^4)z) & \text{5–determined}. \end{array}$$

The $(5, 1)$–transversal for (F) is $\{(0, 0, z^5)\}$ giving three orbits to consider;

- $(x, y, yz^2 + x^2z + z^4 + z^5)$; all higher $(5, s)$–transversals being empty.
- $(x, y, yz^2 + x^2z + z^4)$; all higher $(5, s)$–transversals being empty.
- $(x, y, yz^2 + x^2z + z^4 - z^5)$; all higher $(5, s)$–transversals being empty.

So we have three $J\mathcal{A}^5$–orbits over $(x, y, yz^2 + x^2z + z^4)$:

$$\begin{array}{ll} (x, y, yz^2 + x^2z + z^4 + z^5) & \text{5–determined}, \\ (x, y, yz^2 + x^2z + z^4) & \text{(F)}, \\ (x, y, yz^2 + x^2z + z^4 - z^5) & \text{5–determined}. \end{array}$$

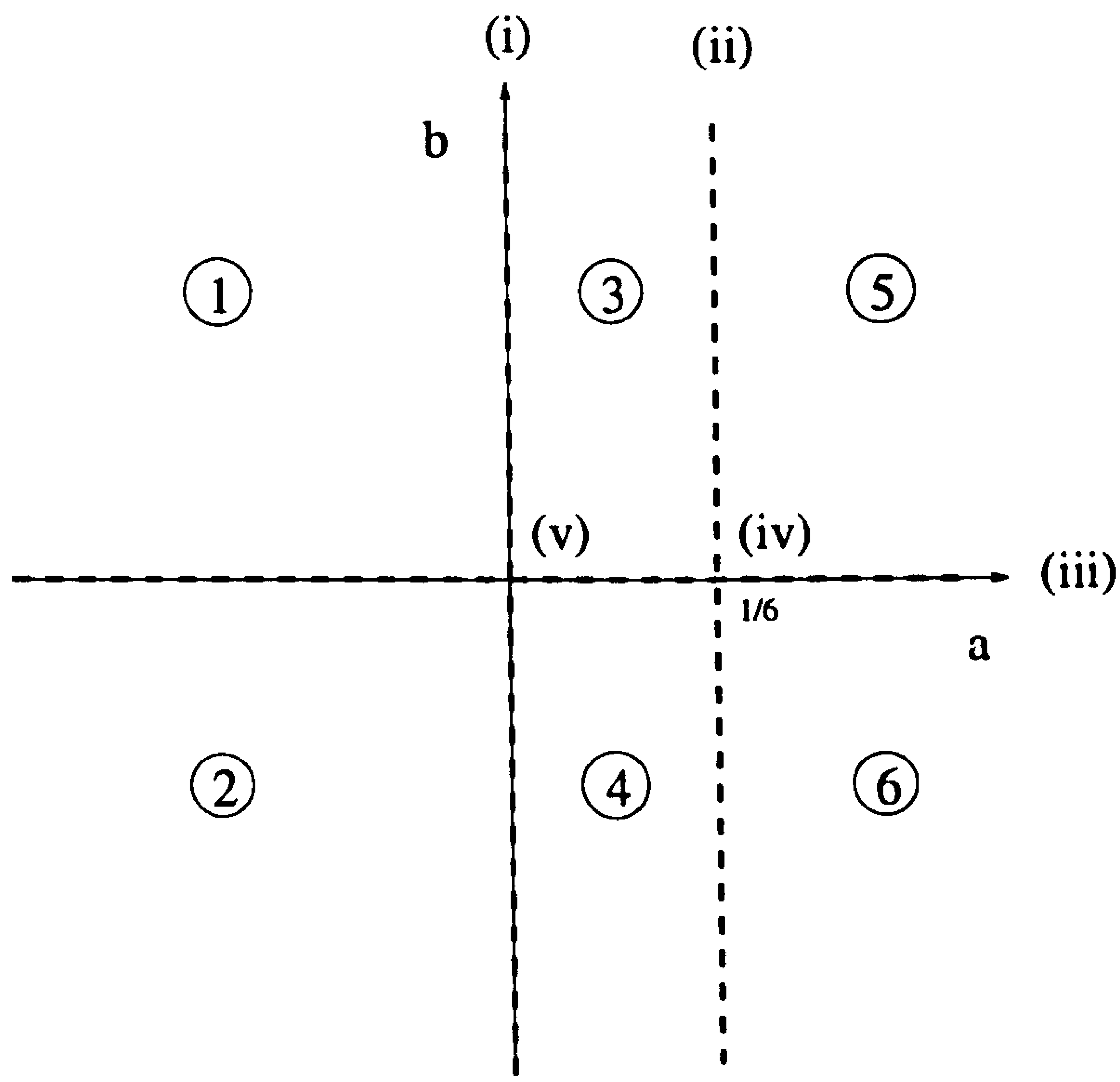
The $(5,1)$–transversal for (G) is $\{(0,0,z^5)\}$ giving us the orbit $(x,y,yz^2+x^2z+xz^3+az^5)$ to consider. This has $(5,3)$–transversal $\{(0,0,xz^4)\}$ which gives us three $J\mathcal{A}^5$–orbits over $(x,y,yz^2+x^2z+xz^3)$:

$$\begin{array}{ll} (x,y,yz^2+x^2z+xz^3+az^5+xz^4) & \text{(G)}, \\ (x,y,yz^2+x^2z+xz^3+az^5) & (\sharp), \\ (x,y,yz^2+x^2z+xz^3+az^5-xz^4) & \text{(G}'). \end{array}$$

The $(5,1)$–transversal for (H) is $\{(0,0,z^5)\}$ giving us the 2–parameter family

$$f_{a,b}=(x,y,yz^2+z^4+xz^3+ax^3z+bz^5).$$

The tangent space contains the vectors $(0,0,x^3z)$ and $(0,0,z^5)$ except when $-\frac{3}{2}ba+9ba^2=0$. This gives us the following regions:



So we have eight regions of the (a,b)–plane to consider (as 1 is $\mathcal{A}$–equivalent to 2 via the change of coordinates $(x,y,z) \longmapsto (-x,y,-z)$. Similarly $3 \sim_{\mathcal{A}} 4$ and $5 \sim_{\mathcal{A}} 6$.

- 1. This is the region defined by $a < 0,\ b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + z^4 + xz^3 - x^3z + z^5)$.

- 3. This is the region defined by $0 < a < \frac{1}{6},\ b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + z^4 + xz^3 + \frac{1}{10}x^3z + z^5)$.

- 5. This is the region defined by $a > \frac{1}{6},\ b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + z^4 + xz^3 + x^3z + z^5)$.

- (i). This is the line $a = 0$. Along this line we have constant codimension 8. For our representative we take $(x, y, yz^2 + z^4 + xz^3 + z^5)$.

- (ii). This is the line $a = \frac{1}{6}$. Along this line we have constant codimension 8. For our representative we take $(x, y, yz^2 + z^4 + xz^3 + \frac{1}{6}x^3z + z^5)$.

- (iii). This is the line $b = 0$. Along this line we have constant codimension 8. For our representative we take $(x, y, yz^2 + z^4 + xz^3 + x^3z)$.

- (iv). This is the pont $(a,b) = (\frac{1}{6}, 0)$. At this point we have codimension 8.

- (v). This is the pont $(a,b) = (0,0)$. At this point we have codimension 8.

So we have the following eight orbits to consider:

$$
\begin{array}{ll}
(x, y, yz^2 + z^4 + xz^3 - x^3z + z^5) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + \frac{1}{10}x^3z + z^5) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + x^3z + z^5) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + z^5) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + \frac{1}{6}x^3z + z^5) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + x^3z) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3 + \frac{1}{6}x^3z) & (\natural), \\
(x, y, yz^2 + z^4 + xz^3) & (\natural).
\end{array}
$$

All the above have exceeded the codimension limit so we need not consider any higher $(5, s)$–transversals.

6.3.6 The 6–Jets

The $(6,1)$–transversal for (A) is $\{(0,0,z^6)\}$ giving two orbits to consider (since $(x, y, yz + xz^2 + z^6) \sim_{\mathcal{A}} (x, y, yz + xz^2 - z^6)$ via the change of coordinates $(x, y, z) \longmapsto (-x, -y, z)$ and $(u, v, w) \longmapsto (-u, -v, -w)$);

- $(x, y, yz + xz^2 + z^6)$; all higher $(6, s)$–transversals are empty.
- $(x, yz+xz^2)$ has $(6,3)$–transversal $\{(0,0,xz^5)\}$; all higher $(6, s)$–transversals being empty.

This gives us three $J\mathcal{A}^6$–orbits (as $(x, y, yz + xz^2 + xz^5) \sim_{\mathcal{A}} (x, y, yz + xz^2 - xz^5)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, -z)$) over $(x, y, yz + xz^2)$:

$$\begin{array}{ll} (x, y, yz + xz^2 + z^6) & \text{(A)}, \\ (x, y, yz + xz^2 + xz^5) & \text{(B)}, \\ (x, y, yz + xz^2) & (\sharp). \end{array}$$

The $(6,9)$–transversal for (B) is $\{(0,0,x^4z^2)\}$ which gives us three orbits to consider;

$$\begin{array}{ll} (x, y, yz + z^4 + x^4z^2) & \text{(i)}, \\ (x, y, yz + z^4) & \text{(ii)}, \\ (x, y, yz + z^4 - x^4z^2) & \text{(iii)}. \end{array}$$

All higher $(6, s)$–transversals are empty in each case, giving three $J\mathcal{A}^6$–orbits over $(x, y, yz + z^4)$:

$$\begin{array}{ll} (x, y, yz + z^4 + x^4z^2) & \text{6–determined}, \\ (x, y, yz + z^4) & (\sharp), \\ (x, y, yz + z^4 - x^4z^2) & \text{6–determined}. \end{array}$$

The $(6,1)$–transversal over (C) is $\{(0,0,z^6)\}$ giving two orbits to consider as $(x, y, yz + xz^3 + z^5 + z^6) \sim_{\mathcal{A}} (x, y, yz + xz^3 + z^5 - z^6)$ via the change of coordinates $(x, y, z) \longmapsto (x, y, -z)$ and $(u, v, w) \longmapsto (u, v, -w)$;

- $(x, y, yz + xz^3 + z^5 + z^6)$; all higher $(6, s)$–transversals are empty.
- $(x, y, yz + xz^3 + z^5)$; all higher $(6, s)$–transversals are empty.

So we have two $J\mathcal{A}^6$–orbits over $(x, y, yz + xz^3 + z^5)$:

$$\begin{array}{ll} (x, y, yz + xz^3 + z^5 + z^6) & \text{6–determined,} \\ (x, y, yz + xz^3 + z^5) & \text{(C).} \end{array}$$

The $(6, 1)$–transversal over (D) is $\{(0, 0, z^6)\}$; this gives us two orbits to consider (since $(x, y, yz + xz^3 + z^6) \sim \mathcal{A}(x, y, yz + xz^3 - z^6)$ via the change of coordinates $(x, y, z) \longmapsto (x, y, -z)$);

- $(x, y, yz + xz^3 + z^6)$. All higher $(6, s)$–transversals are empty.
- $(x, y, yz+xz^3)$. This has $(6, 3)$–transversal $\{(0, 0, xz^5)\}$ giving the 1-parameter family

  $$(x, y, yz + xz^3 + axz^5).$$

  The tangent space contains the vector $(0, 0, xz^5)$ for all values of a and therefore is an $\mathcal{A}$–trivial family.

So we have two $J\mathcal{A}$–orbits over $(x, y, yz + xz^3)$:

$$\begin{array}{ll} (x, y, yz + xz^3 + z^6) & \text{(D),} \\ (x, y, yz + xz^3) & (\sharp). \end{array}$$

The $(6, 1)$–transversal over (E) is $\{(0, 0, z^6)\}$ giving three orbits to consider;

$$\begin{array}{ll} (x, y, yz + x^2z^2 + z^5 + z^6) & \text{(i),} \\ (x, y, yz + x^2z^2 + z^5) & \text{(ii),} \\ (x, y, yz + x^2z^2 + z^5 - z^6) & \text{(iii).} \end{array}$$

- $(x, y, yz + x^2z^2 + z^5 + z^6)$. All higher transversals are empty.
- $(x, y, yz + x^2z^2 + z^5)$. All higher $(6, s)$–transversals are empty.
- $(x, y, yz + x^2z^2 + z^5 - z^6)$. All higher $(6, s)$–transversals are empty.

This gives us three $J\mathcal{A}^6$–orbits over $(x, y, yz + x^2z^2 + z^5)$:

$$\begin{array}{ll} (x, y, yz + x^2z^2 + z^5 + z^6) & \text{(E)}, \\ (x, y, yz + x^2z^2 + z^5) & (\sharp), \\ (x, y, yz + x^2z^2 + z^5 - z^6) & \text{(E}'\text{)}. \end{array}$$

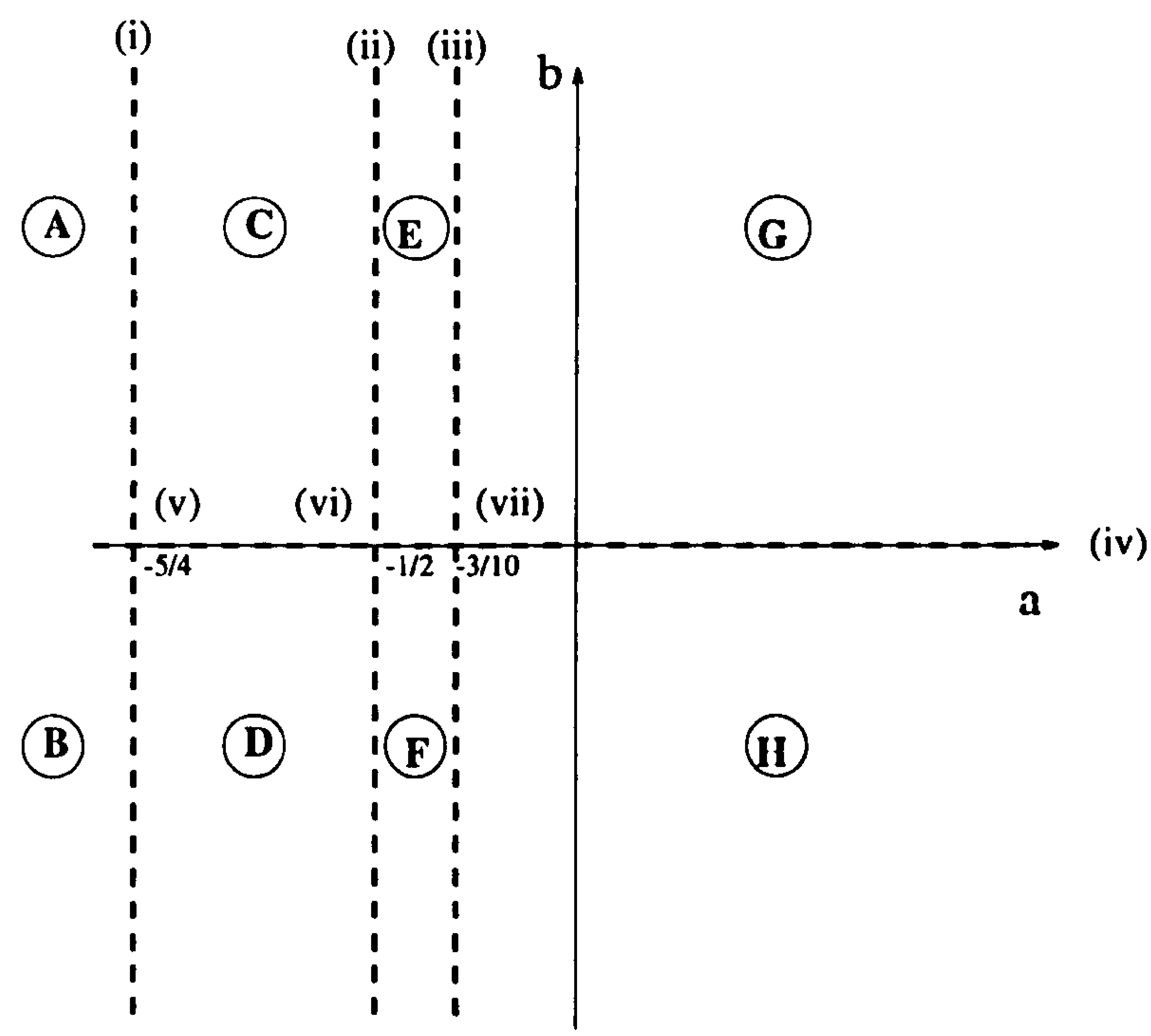
All $(6, s)$–transversals over (F) are empty. So we have one $J\mathcal{A}^6$–orbit over $(x, y, yz^2 + x^2z + z^4)$:

$$(x, y, yz^2 + x^2z + z^4) \quad \text{(F)}.$$

The $(6, 1)$–transversal for (G) is $\{(0, 0, z^6)\}$ giving the 2–parameter family

$$f_{a,b} = (x, y, yz^2 + x^2z + xz^3 + az^5 + xz^4 + bz^6).$$

The tangent space contains the vectors $(0, 0, z^5)$ and $(0, 0, z^6)$ except when either $b = 0$, $a = -\frac{5}{4}$, $a = -\frac{1}{2}$ or $a = -\frac{3}{10}$ giving the following regions:



So we have fifteen regions of the (a, b)–plane to consider

- **A**. This is the region defined by $a < -\frac{5}{4}$, $b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - 2z^5 + xz^4 + z^6)$.

- **B**. This is the region defined by $a < -\frac{5}{4}$, $b < 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - 2z^5 + xz^4 - z^6)$.

- **C**. This is the region defined by $-\frac{5}{4} < a < -\frac{1}{2}$, $b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - z^5 + xz^4 + z^6)$.

- **D**. This is the region defined by $-\frac{5}{4} < a < -\frac{1}{2}$, $b < 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - z^5 + xz^4 - z^6)$.

- **E**. This is the region defined by $-\frac{1}{2} < a < -\frac{3}{10}$, $b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 + xz^4 + z^6)$.

- **F**. This is the region defined by $-\frac{1}{2} < a < -\frac{3}{10}$, $b < 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 + xz^4 - z^6)$.

- **G**. This is the region defined by $a > -\frac{3}{10}$, $b > 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4 + z^6)$.

- **H**. This is the region defined by $a > -\frac{3}{10}$, $b < 0$. This is a connected set and has constant codimension 7. So by the Mather Lemma all these orbits are contained in a single $\mathcal{A}$–orbit. For its representative we take $(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4 - z^6)$.

- (i) This is the line $a = -\frac{5}{4}$. Along this line we have constant codimension 7. For our representative we take $(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 + xz^4 + z^6)$.
- (ii) This is the line $a = -\frac{1}{2}$. Along this line we have constant codimension 7. For our representative we take $(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 + xz^4 + z^6)$.
- (iii) This is the line $a = -\frac{3}{10}$. Along this line we have constant codimension 7. For our representative we take $(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 + xz^4 + z^6)$.
- (iv) This is the line $b = 0$. Along this line we have constant codimension 7. For our representative we take $(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4)$.
- (v). This is the pont $(a, b) = (-\frac{5}{4}, 0)$. At this point we have codimension 7.
- (vi). This is the pont $(a, b) = (-\frac{1}{2}, 0)$. At this point we have codimension 7.
- (vii). This is the pont $(a, b) = (-\frac{3}{10}, 0)$. At this point we have codimension 7.

So we have the following fifteen orbits to consider:

$$\begin{array}{ll}
(x, y, yz^2 + x^2z + xz^3 - 2z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - 2z^5 + xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - z^5 + xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 + xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 + xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 + xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 + xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 + xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 + xz^4) & (\sharp),
\end{array}$$

All the above have exceeded the codimension limit so we need not consider any higher $(6, s)$–transversals.

The $(6,1)$–transversal over (G ′) is $\{(0,0,z^6)\}$ giving the 2–parameter family

$$(x, y, yz^2 + x^2z + xz^3 + az^5 - xz^4 + bz^6).$$

This follows a similar pattern as (G) and so we have the following fifteen orbits:

$$
\begin{array}{ll}
(x, y, yz^2 + x^2z + xz^3 - 2z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - 2z^5 - xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - z^5 - xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{2}{5}z^5 - xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 - xz^4 - z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 - xz^4 + z^6) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 + z^5 - xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{5}{4}z^5 - xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{1}{2}z^5 - xz^4) & (\sharp), \\
(x, y, yz^2 + x^2z + xz^3 - \frac{3}{10}z^5 - xz^4) & (\sharp),
\end{array}
$$

All the above have exceeded the codimension limit so we need not consider any higher $(6,s)$–transversals.

6.3.7 The 7–Jets

All $(7,s)$–transversals over (A) are empty giving one $J\mathcal{A}^7$–orbit over $(x,y,yz+xz^2+z^6)$:

$$(x,y,yz+xz^2+z^6) \quad \text{(A)}.$$

The $(7,1)$–transversal over (B) is $\{(0,0,z^7)\}$ giving two orbits to consider as $(x,y,yz+xz^2+xz^5+z^7) \sim_{\mathcal{A}} (x,y,yz+xz^2+xz^5-z^7)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,z)$ and $(u,v,w) \longmapsto (-u,-v,-w)$

- $(x,y,yz+xz^2+xz^5+z^7)$. All higher $(7,s)$–transversals are empty.
- $(x,y,yz+xz^2+xz^5)$. All higher $(7,s)$–transversals are empty.

So we have two $J\mathcal{A}^7$–orbits over $(x,y,yz+xz^2 \pm xz^5)$:

$$\begin{array}{ll} (x,y,yz+xz^2+xz^5+z^7) & \text{(B)}, \\ (x,y,yz+xz^2+xz^5) & (\sharp). \end{array}$$

The $(7,1)$–transversal over (C) is $\{(0,0,z^7)\}$ giving us the 1–parameter family

$$f_a = (x,y,yz+xz^3+z^5+az^7).$$

The tangent space contains the vector $(0,0,z^7)$ for all values of a so we have one orbit:

$$(x,y,yz+xz^3+z^5) \quad \text{7–determined}.$$

The $(7,1)$–transversal over (D) is $\{(0,0,z^7)\}$ giving two orbits to consider as $(x,y,yz+xz^3+z^6+z^7) \sim_{\mathcal{A}} (x,y,yz+xz^3+z^6-z^7)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,-z)$;

- $(x,y,yz+xz^3+z^6+z^7)$. All higher $(7,s)$–transversals are empty.
- $(x,y,yz+xz^3+z^6)$. All higher $(7,s)$–transversals are empty.

So we have two $J\mathcal{A}^7$–orbits over $(x,y,yz+xz^3+z^6)$:

$$\begin{array}{ll} (x,y,yz+xz^3+z^6+z^7) & \text{(C)}, \\ (x,y,yz+xz^3+z^6) & (\sharp). \end{array}$$

The $(7,1)$–transversal over (E)((E$'$) respectively) is $\{(0,0,z^7)\}$ giving the 1–parameter family

$$(x,y,yz+x^2z^2+xz^5 \pm z^6+az^7).$$

All higher $(7,s)$–transversals are empty. This gives one $J\mathcal{A}^7$–orbit over $(x,y,yz+x^2z^2+z^5+z^6)$:

$$(x,y,yz+x^2z^2+z^5+z^6+az^7) \quad \text{7–determined.}$$

and one $J\mathcal{A}^7$–orbit over $(x,y,yz+x^2z^2+xz^5-z^6)$:

$$(x,y,yz+x^2z^2+z^5-z^6+az^7) \quad \text{7–determined.}$$

The $(7,1)$–transversal for (F) is $\{(0,0,z^7)\}$ giving three orbits to consider;

- $(x,y,yz^2+x^2z+z^4+z^7)$. All higher $(7,s)$–transversals are empty.
- $(x,y,yz^2+x^2z+z^4)$. All higher $(7,s)$–transversals are empty.
- $(x,y,yz^2+x^2z+z^4-z^7)$. All higher $(7,s)$–transversals are empty.

This gives three $J\mathcal{A}^7$–orbits over $(x, y, yz^2 + x^2z + z^4)$:

$$\begin{array}{ll} (x, y, yz^2 + x^2z + z^4 + z^7) & \text{7–determined,} \\ (x, y, yz^2 + x^2z + z^4) & (\sharp), \\ (x, y, yz^2 + x^2z + z^4 - z^7) & \text{7–determined.} \end{array}$$

6.3.8 The 8–Jets

The $(8,1)$–transversal for (A) is $\{(0, 0, z^8)\}$ giving us three orbits to consider

- $(x, y, yz + xz^2 + z^6 + z^8)$; all higher $(8, s)$–transversals are empty.
- $(x, y, yz + xz^2 + z^6)$; all higher $(8, s)$–transversals are empty.
- $(x, y, yz + xz^2 + z^6 - z^8)$; all higher $(8, s)$–transversals are empty.

So we have three $J\mathcal{A}^8$-orbits over $(x, y, yz + xz^2 + z^6)$:

$$\begin{array}{ll} (x, y, yz + xz^2 + z^6 + z^8) & \text{(A)}, \\ (x, y, yz + xz^2 + z^6) & \text{(B)}, \\ (x, y, yz + xz^2 + z^6 - z^8) & \text{(A ')}. \end{array}$$

All $(8, s)$–transversals over (B) are empty, giving us one $J\mathcal{A}^8$–orbit over $(x, y, yz + xz^2 + xz^5 + z^7)$:

$$(x, y, yz + xz^2 + xz^5 + z^7) \quad \text{(C)}.$$

The $(8,1)$–transversal over (C) is $\{(0, 0, z^8)\}$ giving the 1–parameter family

$$(x, y, yz + xz^3 + z^6 + z^7 + az^8).$$

All higher $(8, s)$–tranversals are empty. This gives one $J\mathcal{A}^8$–orbit over $(x, y, yz + xz^3 + z^6 + z^7)$:

$$(x, y, yz + xz^3 + z^6 + z^7 + az^8) \quad \text{(D)}.$$

6.3.9 The 9–Jets

The $(9,1)$–transversal over (A) ((A$'$) respectively) is $\{(0,0,z^9)\}$ giving the 1-parameter family

$$(x, y, yz + xz^2 + z^6 \pm z^8 + az^9).$$

All higher $(9,s)$–transversals are empty. This gives one $J\mathcal{A}^9$–orbit over $(x, y, yz + xz^2 + z^6 + z^8)$:

$$(x, y, yz + xz^2 + z^6 + z^8 + az^9) \quad \text{9–determined.}$$

and one $J\mathcal{A}^9$–orbit over $(x, y, yz + xz^2 + z^6 - z^8)$:

$$(x, y, yz + xz^2 + z^6 - z^8 + az^9) \quad \text{9–determined.}$$

The $(9,1)$–transversal over (B) is $\{(0,0,z^9)\}$ giving two orbits to consider since $(x, y, yz + xz^2 + z^6 + z^9) \sim_{\mathcal{A}} (x, y, yz + xz^2 + z^6 - z^9)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, -z)$;

- $(x, y, yz + xz^2 + z^6 + z^9)$. All higher $(9,s)$–transversals are empty.
- $(x, y, yz + xz^2 + z^6)$. All higher $(9,s)$–transversals are empty.

This gives two $J\mathcal{A}^9$–orbits over $(x, y, yz + xz^2 + z^6)$:

$$\begin{array}{ll} (x, y, yz + xz^2 + z^6 + z^9) & \text{9–determined,} \\ (x, y, yz + xz^2 + z^6) & (\sharp). \end{array}$$

The $(9,1)$–transversal over (C) is $\{(0,0,z^9)\}$ giving the 1–parameter family;

$$(x, y, yz + xz^2 + xz^5 + z^7 + az^9).$$

All higher $(9,s)$–transversals are empty. This gives one $J\mathcal{A}^9$–orbit over $(x, y, yz + xz^2 + xz^5 + z^7)$:

$$(x, y, yz + xz^2 + xz^5 + z^7 + az^9) \quad \text{(A)}.$$

All $(9,s)$–transversals over (D) are empty. This gives one $J\mathcal{A}^9$–orbit over $(x, y, yz + xz^3 + z^6 + z^7 + az^8)$:

$$(x, y, yz + xz^3 + z^6 + z^7 + az^8) \quad \text{(B)}.$$

6.3.10 The 10–Jets

The $(10,1)$–transversal over (A) is $\{(0,0,z^{10})\}$; all higher $(10,s)$–transversals are empty. This gives one $J\mathcal{A}^{10}$–orbit over $(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9)$:

$$(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9 + bz^{10}) \quad \text{(A)}.$$

All $(10,s)$–transversals over (B) are empty. This gives one $J\mathcal{A}^{10}$–orbit over $(x, y, yz + xz^3 \pm z^6 + z^7 + az^8)$:

$$(x, y, yz + xz^3 \pm z^6 + z^7 + az^8) \quad \text{(B)}.$$

6.3.11 The 11–Jets

The $(11,1)$–transversal over (A) is $\{(0,0,z^{11})\}$; all higher $(11,s)$–transversals are empty. This gives one $J\mathcal{A}^{11}$–orbit over $(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9 + bz^{10})$:

$$(x, y, yz + xz^2 \pm xz^5 + z^7 + az^9 + bz^{10} + cz^{11}) \quad \text{(b)}.$$

The $(11,1)$–transversal over (B) is $\{(0,0,z^{11})\}$; all higher $(11,s)$–transversals are empty. This gives one $J\mathcal{A}^{11}$–orbit over $(x,y,yz+xz^3 \pm z^6 + z^7 + az^8)$:

$$(x,y,yz+xz^3 \pm z^6 + z^7 + az^8 + bz^{11}) \qquad (\mathfrak{b}).$$

The 12–transversals for both of the above 11–jets are empty but to check the determinacy condition we would need to check transversals upto and including the 18–transversal. Unfortunately running these calculations we have found that we run out of CPU time even on an intensive machine.

We now consider the corank 2 case (B).

6.3.12 The 2–Jets

If we have a map-germ $f : \mathbb{R}^3, 0 \longrightarrow \mathbb{R}^3, 0$ with corank 2, i.e. f has rank 1 we can apply a change of coordinates so that $j^2 f$ has the following form

$$f(x,y,z) = (x, q_1(y,z) + u_1 xy + u_2 xz, q_2(y,z) + u_3 xy + u_4 xz),$$

where q_1, q_2 are quadratic forms. Using the technique in [B5] we consider the pencil of quadratic forms (q_1, q_2) under the action of $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$. This gives us 7 orbits in the space of such pencils:.

$$\begin{array}{c} (yz, y^2+z^2) \\ (yz, y^2-z^2) \\ (yz, y^2) \\ (y^2+z^2, 0) \\ (yz, 0) \\ (y^2, 0) \\ (0,0) \end{array}$$

We will consider each of these as a separate cases.

- (yz, y^2+z^2)

  With this as our pencil we have the following 2–jet:

  $$(x, yz + axy + bxz, y^2 \pm z^2 + cxy + dxz).$$

  Using the change of coordinates

  $$\begin{array}{ccc} y & \longmapsto & y + \alpha x \\ z & \longmapsto & z + \beta x \end{array}$$

  we get the following:

  $$\begin{aligned} (x, yz + x(\alpha z + \beta y) + \alpha\beta x^2 + axy + a\alpha x^2 + bxz + b\beta x^2, \\ y^2 + 2\alpha xy + \alpha^2 x^2 + z^2 + 2\beta xz + \beta^2 x^2 + cxy + c\alpha x^2 + dxz + d\beta x^2) \end{aligned}$$

  Now killing the x^2–terms by a change of coordinates in the target and putting $\beta = -a$ and $\alpha = -b$ we get the following normal form

$$j^2 f = (x, yz, y^2 + z^2 + Axy + Bxz).$$

Firstly, consider the case $A \neq 0$. In this case we can scale A to 1, giving us the 1–parameter family, f_B:

$$f_B = (x, yz, y^2 + z^2 + xy + Bxz).$$

In $J^2(3,3)$ the tangent space to the $\mathcal{A}^2$–orbit of f_B all have codimension 4 and contain the vector $(0, 0, xz)$ except for $B^2 = 1$. This gives us five $J\mathcal{A}^2$–orbits to consider:

$$\begin{array}{ll}
(x, yz, y^2 + z^2 + xy) & \text{2–determined,} \\
(x, yz, y^2 + z^2 + xy + 2xz) & \text{2–determined,} \\
(x, yz, y^2 + z^2 + xy - 2xz) & \text{2–determined,} \\
(x, yz, y^2 + z^2 + xy + xz) & \text{(C),} \\
(x, yz, y^2 + z^2 + xy - xz) & \text{(C').}
\end{array}$$

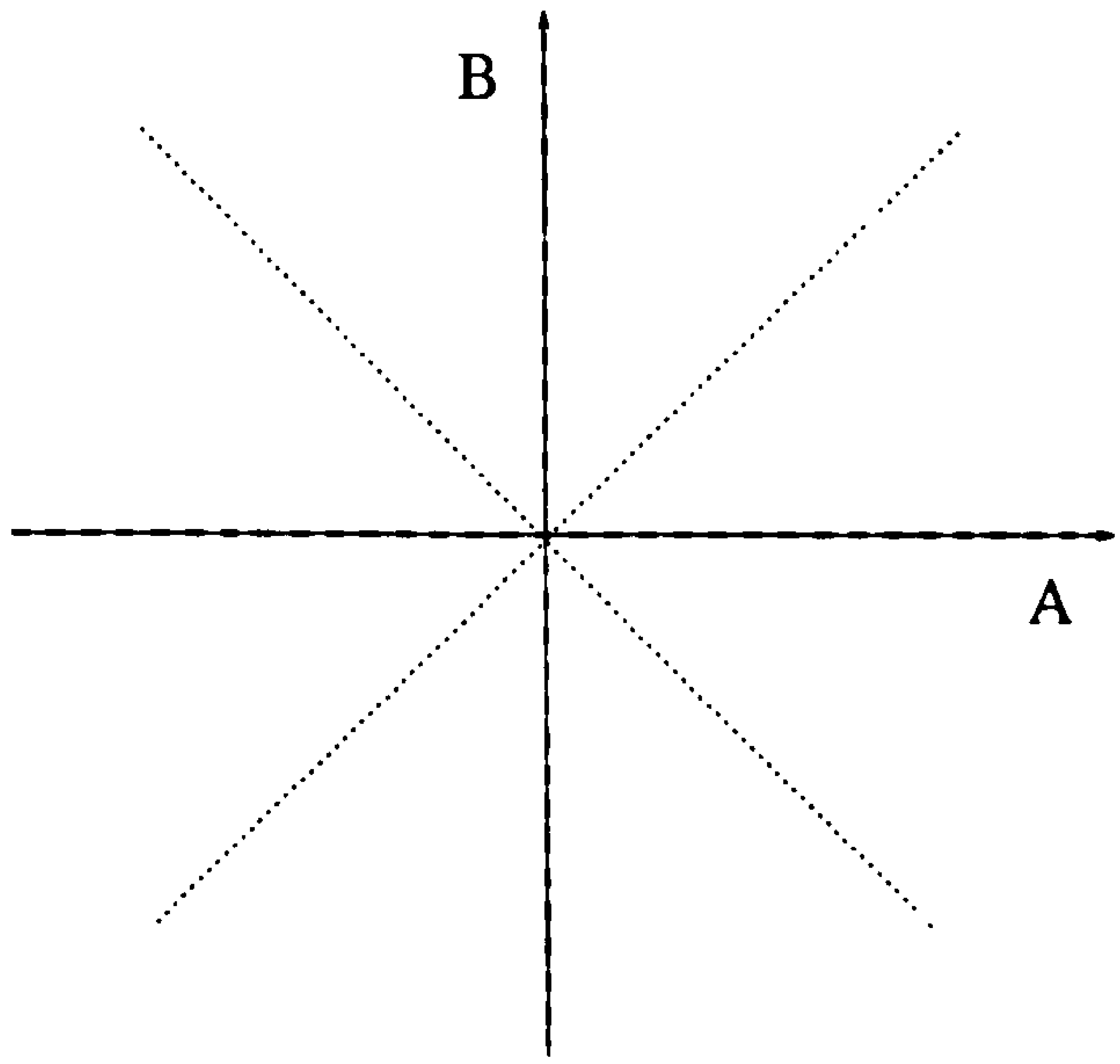
Now (C) and (C') are $\mathcal{A}$–equivalent by the change of coordinates $z \longmapsto -z$. Similarly, $(x, yz, y^2 + z^2 + xy + 2xz)$ is $\mathcal{A}$–equivalent to $(x, yz, y^2 + z^2 + xy - 2xz)$ for the same reason. So we have 3 $J\mathcal{A}^2$–orbits.

$$\begin{array}{ll}
(x, yz, y^2 + z^2 + xy) & \text{2–determined,} \\
(x, yz, y^2 + z^2 + xy + 2xz) & \text{2–determined,} \\
(x, yz, y^2 + z^2 + xy + xz) & \text{(C),}
\end{array}$$

We know that when $B^2 = 1$ the codimension increases by one but for $B = 0$ and $B > 1$ the codimension is the same and we wish to show that these two orbits are in fact the same. We do this by considering the 2–parameter family, $(x, yz, y^2 + z^2 + Axy + Bxz)$ in the (A, B)–plane. We can apply the following changes of coordinates:

1. scaling x: this is invariant in the (A, B)–plane;
2. $(x, y, z) \longmapsto (x, z, y)$: this gives a reflection in the line $A = B$;
3. $(x, y, z) \longmapsto (x, y, -z)$: this gives a reflection in the line $A = 0$;
4. $(x, y, z) \longmapsto (x, -y, z)$: this gives a reflection in the line $B = 0$;
5. $(x, y, z) \longmapsto (x, -z, -y)$: this gives a reflection in the line $A = -B$.

So we have the following diagram:



So using our changes of coordinates we find that all eight segments of the above are $\mathcal{A}$–equivalent and therefore

$$(x, yz, y^2 + z^2 + xy) \sim_{\mathcal{A}} (x, yz, y^2 + z^2 + xy + 2xz).$$

So we have two J^2–orbits:

$$\begin{array}{ll} (x, yz, y^2 + z^2 + xy) & \text{2–determined,} \\ (x, yz, y^2 + z^2 + xy + xz) & \text{(C).} \end{array}$$

If, on the other hand, $A = 0$ then we have

$$(x, yz, y^2 \pm z^2 + Bxz).$$

If $B \neq 0$ we have $(x, yz, y^2 + z^2 + xz)$ which is $\mathcal{A}$–equivalent to $(x, yz, y^2 + z^2 + xy)$; and if $B = 0$ then we have

$$(x, yz, y^2 + z^2) \quad \text{(D).}$$

- $(yz, y^2 - z^2)$.

  With this as our pencil we have the following 2–jet:

$$(x, yz + axy + bxz, y^2 - z^2 + cxy + dxz).$$

Using the change of coordinates

$$\begin{aligned} y &\longmapsto y + \alpha x \\ z &\longmapsto z + \beta x \end{aligned}$$

we get the following:

$$\begin{aligned}(x, yz + x(\alpha z + \beta y) + \alpha\beta x^2 + axy + a\alpha x^2 + bxz + b\beta x^2,\\ y^2 + 2\alpha xy + \alpha^2 x^2 - z^2 - 2\beta xz - \beta^2 x^2 + cxy + c\alpha x^2 + dxz + d\beta x^2)\end{aligned}$$

Now killing the x^2–terms by a change of coordinates in the target and putting $\beta = -a$ and $\alpha = -b$ we get the following normal form

$$j^2 f = (x, yz, y^2 - z^2 + Axy + Bxz).$$

Firstly, consider the case $A \neq 0$. In this case we can scale A to 1, giving us the 1–parameter family, f_B:

$$f_B = (x, yz, y^2 - z^2 + xy + Bxz).$$

In $J^2(3,3)$ the tangent space to the $\mathcal{A}^2$–orbit of f_B all have codimension 4 and contain the vector $(0, 0, xz)$.

This gives us one orbit:

$$(x, yz, y^2 - z^2 + xy) \quad \text{2–determined.}$$

If on the other hand, $A = 0$ then we have the 1–parameter family:

$$(x, yz, y^2 - z^2 + Bxz).$$

If $B \neq 0$ then we have $(x, yz, y^2 - z^2 + xz)$ (this is $\mathcal{A}$–equivalent to $(x, yz, y^2 - z^2 - xz)$ via the change of coordinates $x \longmapsto -x$), but this is $\mathcal{A}$–equivalent to $(x, yz, y^2 - z^2 + xy)$. If $B = 0$ then we have

$$(x, yz, y^2 - z^2) \qquad (\mathrm{D}\,').$$

- (yz, y^2)

With this as our pencil we have the following 2–jet:

$$(x, yz + axy + bxz, y^2 + cxy + dxz).$$

Using the change of coordinates

$$\begin{array}{lcl} y & \longmapsto & y + \alpha x \\ z & \longmapsto & z + \beta x \end{array}$$

we get the following:

$$\begin{array}{r} (x, yz + x(\alpha z + \beta y) + \alpha\beta x^2 + axy + a\alpha x^2 + bxz + b\beta x^2, \\ y^2 + 2\alpha xy + \alpha^2 x^2 + cxy + c\alpha x^2 + dxz + d\beta x^2) \end{array}$$

Now killing the x^2–terms via a change of coordinates in the target and putting $\beta = -a$ and $\alpha = -b$ we get the following normal form

$$j^2 f = (x, yz, y^2 + Axy + dxz).$$

Firstly, consider the case $A \neq 0$. In this case we can scale A to 1, giving us the 1–parameter family, f_d:

$$f_d = (x, yz, y^2 + xy + dxz).$$

In $J^2(3,3)$ the tangent space to the $\mathcal{A}^2$–orbit of f_d all have codimension 5 and contain the vector $(0, 0, xz)$ except for $d = 0$. This gives us two $J\mathcal{A}^2$–orbits since $(x, yz, y^2 + xy + xz) \sim_{\mathcal{A}} (x, yz, y^2 + xy - xz)$ via a change of coordinates :

$$\begin{array}{ll} (x, yz, y^2 + xy + xz) & \text{(E)}, \\ (x, yz, y^2 + xy) & \text{(F)}. \end{array}$$

If, on the other hand, $A = 0$ then we have

$$(x, yz, y^2 + dxz).$$

We can scale this to give us:

$$(x, yz, y^2 + xz) \quad \text{(G)}.$$

Finally if $d = 0$ we get:

$$(x, yz, y^2) \quad (\sharp).$$

- $(y^2 + z^2, 0)$

With this as our pencil we have the following 2–jet:

$$(x, y^2 + z^2 + axy + bxz, cxy + dxz).$$

Using the change of coordinates

$$\begin{array}{rcl} y & \longmapsto & y + \alpha x \\ z & \longmapsto & z + \beta x \end{array}$$

we get the following:

$$\begin{array}{l} (x, y^2 + 2\alpha xy + \alpha^2 x^2 + z^2 + 2\beta xz + \beta^2 x^2 \\ \qquad + axy + a\alpha x^2 + bxz + b\beta x^2, cxy + c\alpha x^2 + dxz + d\beta x^2) \end{array}$$

Now killing the x^2–terms via a change of coordinates in the target and putting $\beta = -a/2$ and $\alpha = -b/2$ we get the following form:

$$j^2 f = (x, y^2 + z^2, cxy + dxz).$$

Assuming $c \neq 0$ we get the 1–parameter family f_d:

$$f_d = (x, y^2 + z^2, xy + dxz).$$

In $J^2(3,3)$ the tangent space to the $\mathcal{A}^2$–orbit of f_d all have codimension 6 and contain the vector $(0, 0, xz)$ so we have one $J\mathcal{A}^2$–orbit:

$$(x, y^2 + z^2, xy) \quad \text{(H)}.$$

If $c = 0$ we get the 1–parameter family f_d:

$$f_d = (x, y^2 + z^2, dxz).$$

In $J^2(3,3)$ the tangent space th the $\mathcal{A}^2$–orbit of f_d have codimension 6 and contain the vector $(0, 0, xz)$ except when $d = 0$ giving two $J\mathcal{A}^2$–orbits as $(x, y^2 + z^2, xz) \sim_{\mathcal{A}} (x, y^2 + z^2, -xz)$:

$$\begin{array}{ll} (x, y^2 + z^2, xz) & \mathcal{A}\text{–equivalent to (H)}, \\ (x, y^2 + z^2, 0) & (\sharp). \end{array}$$

- $(yz, 0)$

  With this as our pencil we have the following 2–jet:

$$(x, yz + axy + bxz, cxy + dxz).$$

  Using the change of coordinates

$$\begin{array}{lcl} y & \longmapsto & y + \alpha x \\ z & \longmapsto & z + \beta x \end{array}$$

  we get the following:

$$(x, yz+x(\alpha z+\beta y)+\alpha\beta x^2+axy+a\alpha x^2+bxz+b\beta x^2, cxy+c\alpha x^2+dxz+d\beta x^2)$$

Now killing the x^2–terms via a change of coordinates in the target and putting $\beta = -a$ and $\alpha = -b$ we get the following normal form

$$j^2 f = (x, yz, cxy + dxz).$$

Assuming $c \neq 0$ we get the 1–parameter family f_d:

$$f_d = (x, yz, xy + dxz).$$

Again assuming $d \neq 0$ we can scale to get:

$$(x, yz, xy + xz) \quad \text{(I)}.$$

If $d = 0$ we have

$$(x, yz, xy) \quad (\sharp).$$

If $c = 0$ we have the 1–parameter family f_d:

$$(x, yz, dxz).$$

Assuming $d \neq 0$ we can scale to get:

$$(x, yz, xz) \quad \mathcal{A}\text{–equivalent to } (x, yz, xy).$$

Finally if $d = 0$ we have:

$$(x, yz, 0) \quad (\sharp).$$

So there are fourteen $J\mathcal{A}^2$–orbits over $(x, 0, 0)$.

$(x, yz, y^2 + z^2 + xy)$	2-determined,
$(x, yz, y^2 - z^2 + xy)$	2-determined,
$(x, yz, y^2 + z^2 + xy + xz)$	(C),
$(x, yz, y^2 + z^2)$	(D),
$(x, yz, y^2 - z^2)$	(D′),
$(x, yz, y^2 + xy + xz)$	(E),
$(x, yz, y^2 + xy)$	(F),
$(x, yz, y^2 + xz)$	(G),
(x, yz, y^2)	($\sharp$),
$(x, y^2 + z^2, xy)$	(H),
$(x, y^2 + z^2, 0)$	($\sharp$),
$(x, yz, xy + xz)$	(I),
(x, yz, xy)	($\sharp$),
$(x, yz, 0)$	($\sharp$).

6.3.13 The 3–Jets

A $(3,5)$–transversal for (C) is $\{(0,y^3,0)\}$ giving us two orbits to consider since $(x, yz+y^3, y^2+z^2+xy+xz) \sim_{\mathcal{A}} (x, yz-y^3, y^2+z^2+xy+xz)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,-z)$:

- $(x, yz+y^3, y^2+z^2+xy+xz)$

  This has $(3,7)$–transversal $\{(0,x^2y,0)\}$. The tangent space contains the vector $(0,x^2y,0)$ except when $a=-\frac{3}{16}$ giving three orbits:

$$\begin{array}{ll} (x, yz+y^3, y^2+z^2+xy+xz) & \text{(i)},\\ (x, yz+y^3-x^2y, y^2+z^2+xy+xz) & \text{(ii)},\\ (x, yz+y^3-\frac{3}{16}x^2y, y^2+z^2+xy+xz) & \text{(iii)}. \end{array}$$

  All higher $(3,s)$–transversals are empty in each case.

- $(x, yz, y^2+z^2+xy+xz)$

  This has $(3,7)$–transversal $\{(0,x^2y,0)\}$ giving two orbits since $(x, yz+x^2y, y^2+z^2+xy+xz) \sim_{\mathcal{A}} (x, yz-x^2y, y^2+z^2+xy+xz)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,-z)$:

$$\begin{array}{ll} (x, yz+x^2y, y^2+z^2+xy+xz) & \text{(iv)},\\ (x, yz, y^2+z^2+xy+xz) & \text{(v)}. \end{array}$$

So we have five $J\mathcal{A}^3$-orbits over $(x, yz, y^2+z^2+xy+xz)$:

$$\begin{array}{ll} (x, yz+y^3, y^2+z^2+xy+xz) & \text{3-determined},\\ (x, yz+y^3-x^2y, y^2+z^2+xy+xz) & \text{3-determined},\\ (x, yz+y^3-\frac{3}{16}x^2y, y^2+z^2+xy+xz) & \text{(F)},\\ (x, yz+x^2y, y^2+z^2+xy+xz) & \text{(G)},\\ (x, yz, y^2+z^2+xy+xz) & (\sharp). \end{array}$$

A $(3,4)$–transversal for (D) is $\{(0,0,y^3)\}$ giving us two orbits to consider since $(x, yz, y^2+z^2+y^3) \sim_{\mathcal{A}} (x, yz, y^2+z^2-y^3)$ via the change of coordinates $(x,y,z) \longmapsto (x,-y,-z)$;

- $(x, yz, y^2+z^2+y^3)$. This has $(3,5)$–transversal $\{(0,y^3,0)\}$ giving the 1–parameter family

$$f_a = (x, yz + ay^3, y^2 + z^2 + y^3).$$

The tangent space contains the vector $(0, y^3, 0)$ except when $a = \pm\frac{1}{2}$. So we have five orbits;

$$\begin{array}{ll}(x, yz + y^3, y^2 + z^2 + y^3) & \text{(i)},\\ (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3) & \text{(ii)},\\ (x, yz, y^2 + z^2 + y^3) & \text{(iii)},\\ (x, yz - \frac{1}{2}y^3, y^2 + z^2 + y^3) & \text{(iv)},\\ (x, yz - y^3, y^2 + z^2 + y^3) & \text{(v)}.\end{array}$$

(i) has $(3,6)$–transversal $\{(0, 0, x^2y)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3, y^2 + z^2 + y^3 + ax^2y).$$

Now the tangent space contains the vector $(0, 0, x^2y)$ except when $a = 0$. So we have three orbits to consider;

$$\begin{array}{ll}(x, yz + y^3, y^2 + z^2 + y^3 + x^2y) & \text{(a)},\\ (x, yz + y^3, y^2 + z^2 + y^3) & \text{(b)},\\ (x, yz + y^3, y^2 + z^2 + y^3 - x^2y) & \text{(c)}.\end{array}$$

Now (a) has $(3,7)$–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3 + ax^2y, y^2 + z^2 + y^3 + x^2y).$$

(b) has $(3,7)$–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3 + ax^2y, y^2 + z^2 + y^3).$$

The tangent space contains the vector $(0, x^2y, 0)$ except when $a = 0$. So we have three orbits;

$$(x, yz + y^3 + x^2y, y^2 + z^2 + y^3),$$
$$(x, yz + y^3, y^2 + z^2 + y^3),$$

$$(x, yz + y^3 - x^2y, y^2 + z^2 + y^3).$$

(c) has (3, 7)–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3 + ax^2y, y^2 + z^2 + y^3 - x^2y).$$

(ii) has (3, 6)–transversal $\{(0, 0, x^2y)\}$ giving the 1–parameter family

$$f_a = (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3 + ax^2y).$$

The tangent space contains the vector $(0, 0, x^2y)$ except when $a = 0$ giving three orbits to consider:

$$\begin{array}{ll} (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3 + x^2y) & \text{(a)}, \\ (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3) & \text{(b)}, \\ (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3 - x^2y) & \text{(c)}. \end{array}$$

Now (a) has (3, 7)–transversal $\{(0, x^2y, 0)\}$ giving the following 1–parameter family

$$f_a = (x, yz + \frac{1}{2}y^3 + ax^2y, y^2 + z^2 + y^3 + x^2y).$$

The tangent space contains the vector $(0, x^2y, 0)$ except when $a = \pm\frac{1}{2}$ giving us five orbits:

$$\begin{array}{c} (x, yz + \frac{1}{2}y^3 + x^2y, y^2 + z^2 + y^3 + x^2y), \\ (x, yz + \frac{1}{2}y^3 + \frac{1}{2}x^2y, y^2 + z^2 + y^3 + x^2y), \\ (x, yz + \frac{1}{2}y^3, y^2 + z^2 + y^3 + x^2y), \\ (x, yz + \frac{1}{2}y^3 - \frac{1}{2}x^2y, y^2 + z^2 + y^3 + x^2y), \\ (x, yz + \frac{1}{2}y^3 - x^2y, y^2 + z^2 + y^3 + x^2y). \end{array}$$

(b) has (3, 7)–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + \frac{1}{2}y^3 + ax^2y, y^2 + z^2 + y^3).$$

The tangent space contains the vector $(0, x^2y, 0)$ except when $a = 0$ giving three orbits:

$$(x, yz + \tfrac{1}{2}y^3 + x^2y, y^2 + z^2 + y^3),$$

$$(x, yz + \tfrac{1}{2}y^3, y^2 + z^2 + y^3),$$
$$(x, yz + \tfrac{1}{2}y^3 - x^2y, y^2 + z^2 + y^3).$$

(c) has $(3,7)$–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + \frac{1}{2}y^3 + ax^2y, y^2 + z^2 + y^3 - x^2y).$$

Again the tangent space contains the vector $(0, x^2y, 0)$ except for when $a = \pm\frac{1}{2}$ giving five orbits:

$$(x, yz + \tfrac{1}{2}y^3 + x^2y, y^2 + z^2 + y^3 - x^2y),$$
$$(x, yz + \tfrac{1}{2}y^3 + \tfrac{1}{2}x^2y, y^2 + z^2 + y^3 - x^2y),$$
$$(x, yz + \tfrac{1}{2}y^3, y^2 + z^2 + y^3 - x^2y),$$
$$(x, yz + \tfrac{1}{2}y^3 - \tfrac{1}{2}x^2y, y^2 + z^2 + y^3 - x^2y),$$
$$(x, yz + \tfrac{1}{2}y^3 - x^2y, y^2 + z^2 + y^3 - x^2y).$$

Now (iii) gives similar results to (i), as does (v) and (iv) gives similar results to (ii).

- $(x, yz, y^2{+}z^2)$. This has $(3,5)$–transversal $\{(0, y^3, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + ay^3, y^2 + z^2).$$

The tangent space contains the vector $(0, y^3, 0)$ except when $a = 0$ giving two orbits to consider as $(x, yz + y^3, y^2 + z^2) \sim_{\mathcal{A}} (x, yz - y^3, y^2 + z^2)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, -z)$;

$$(x, yz + y^3, y^2 + z^2) \qquad \text{(i)},$$
$$(x, yz, y^2 + z^2) \qquad \text{(ii)}.$$

(i) has $(3,6)$–transversal $\{(0, 0, x^2y)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3, y^2 + z^2 + ax^2y).$$

The tangent space contains the vector $(0, 0, x^2y)$ except when $a = 0$ giving two orbits to consider as $(x, yz + y^3, y^2 + z^2 + x^2y) \sim_{\mathcal{A}} (x, yz + y^3, y^2 + z^2 - x^2y)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$ and $(u, v, w) \longmapsto (u, -v, w)$;

$$\begin{array}{ll} (x, yz + y^3, y^2 + z^2 + x^2y) & \text{(a)}, \\ (x, yz + y^3, y^2 + z^2) & \text{(b)}. \end{array}$$

(a) has $(3, 7)$–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_a = (x, yz + y^3 + ax^2y, y^2 + z^2 + x^2y).$$

The tangent space contains the vector $(0, x^2y, 0)$ except when $a = \frac{1}{2}$ giving three orbits:

$$\begin{gathered} (x, yz + y^3, y^2 + z^2 + x^2y), \\ (x, yz + y^3 + \tfrac{1}{2}x^2y, y^2 + z^2 + x^2y), \\ (x, yz + y^3 + x^2y, y^2 + z^2 + x^2y). \end{gathered}$$

All higher $(3, s)$–transversals are empty.

(b) has $(3, 7)$–transversal $\{(0, x^2y, 0)\}$ giving the 1–parameter family

$$f_1 = (x, yz + y^3 + ax^2y, y^2 + z^2).$$

The tangent space contains the vector $(0, x^2y, 0)$ except when $a = 0$ giving three orbits:

$$\begin{gathered} (x, yz + y^3 + x^2y, y^2 + z^2), \\ (x, yz + y^3, y^2 + z^2), \end{gathered}$$

$$(x, yz + y^3 - x^2y, y^2 + z^2).$$

The transversals for (D ′) are the same as above, giving us the following $J\mathcal{A}^3$-orbits over $(x, yz, y^2 \pm z^2)$:

$(x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$	3–determined,
$(x, yz \pm y^3 \pm x^2y, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz \pm y^3, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$	3–determined,
$(x, yz \pm \frac{1}{2}y^3 \pm x^2y, y^2 \pm z^2 + y^3 + x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}y^3 \pm \frac{1}{2}x^2y, y^2 \pm z^2 + y^3 + x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3 + x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}y^3 \pm x^2y, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz \pm \frac{1}{2}y^3 \pm x^2y, y^2 \pm z^2 + y^3 - x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}y^3 \pm \frac{1}{2}x^2y, y^2 \pm z^2 + y^3 - x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}y^3, y^2 \pm z^2 + y^3 - x^2y)$	(♯),
$(x, yz + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$	3–determined,
$(x, yz \pm x^2y, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz, y^2 \pm z^2 + y^3)$	(♯),
$(x, yz + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$	3–determined,
$(x, yz + y^3, y^2 \pm z^2 + x^2y)$	(♯),
$(x, yz + y^3 + \frac{1}{2}x^2y, y^2 + z^2 + x^2y)$	(♯),
$(x, yz + y^3 + x^2y, y^2 + z^2 + x^2y)$	(♯),
$(x, yz + y^3 + x^2y, y^2 \pm z^2)$	(♯),
$(x, yz + y^3, y^2 \pm z^2)$	(♯),
$(x, yz + y^3 - x^2y, y^2 \pm z^2)$	(♯),
$(x, yz \pm x^2y, y^2 \pm z^2 + x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}x^2y, y^2 \pm z^2 + x^2y)$	(♯),
$(x, yz, y^2 \pm z^2 + x^2y)$	(♯),
$(x, yz \pm x^2y, y^2 \pm z^2)$	(♯),
$(x, yz, y^2 \pm z^2)$	(♯),
$(x, yz \pm x^2y, y^2 \pm z^2 - x^2y)$	(♯),
$(x, yz \pm \frac{1}{2}x^2y, y^2 \pm z^2 - x^2y)$	(♯),
$(x, yz, y^2 \pm z^2 - x^2y)$	(♯).

A $(3,1)$–transversal for (E) is $\{(0,0,z^3)\}$ giving us two orbits to consider as $(x,yz,y^2+xy+xz+z^3) \sim_{\mathcal{A}} (x,yz,y^2+xy+xz-z^3)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,-z)$:

- $(x,yz,y^2+xy+xz+z^3)$. All higher $(3,s)$–transversals are empty.
- $(x,yz,y^2+xy+xz)$. All higher $(3,s)$–transversals are empty.

So we have two $J\mathcal{A}^3$–orbits over $(x,yz,y^2+xy+xz)$:

$$\begin{array}{ll} (x,yz,y^2+xy+xz+z^3) & \text{(H)}, \\ (x,yz,y^2+xy+xz) & \text{(I)}. \end{array}$$

A $(3,1)$–transversal for (F) is $\{(0,0,z^3)\}$ giving two orbits to consider as $(x,yz,y^2+xy+z^3) \sim_{\mathcal{A}} (x,yz,y^2+xy-z^3)$ via the change of coordinates $(x,y,z) \longmapsto (x,y,-z)$:

- (x,yz,y^2+xy+z^3); this has $(3,3)$–transversal $\{(0,0,xz^2)\}$ giving the 1-parameter family
$$(x,yz,y^2+xy+z^3+axz^2).$$
The tangent space contains the vector $(0,0,xz^2)$ except when $a=0$ giving us two orbits to consider as $(x,yz,y^2+xy+z^3+xz^2) \sim_{\mathcal{A}} (x,yz,y^2+xy+z^3-xz^2)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,z)$:
$$\begin{array}{ll} (x,yz,y^2+xy+z^3+xz^2) & \text{(i)}, \\ (x,yz,y^2+xy+z^3) & \text{(ii)}. \end{array}$$
(i) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving the 1–parameter family
$$(x,yz,y^2+xy+z^3+xz^2+ax^2z).$$
All higher $(3,s)$–transversals are empty.

(ii) has $(3,5)$–transversal $\{(0,0,x^2z)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xy + z^3 + ax^2z).$$

The tangent space contains the vector $(0, 0, x^2z)$ except when $a = 0$ giving three orbits:

$$(x, yz, y^2 + xy + z^3 + x^2z),$$
$$(x, yz, y^2 + xy + z^3),$$
$$(x, yz, y^2 + xy + z^3 - x^2z).$$

All higher $(3, s)$–transversals are empty.

- (x, yz, y^2+xy). This has $(3, 3)$–transversal $\{(0, 0, xz^2)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xy + axz^2).$$

The tangent space contains the vector $(0, 0, xz^2)$ except when $a = 0$ giving two orbits to consider as $(x, yz, y^2 + xy + xz^2) \sim_{\mathcal{A}} (x, yz, y^2 + xy - xz^2)$ via the change of coordinates $(x, y, z) \longmapsto (-x, -y, z)$:

$$(x, yz, y^2 + xy + xz^2) \qquad \text{(i)},$$
$$(x, yz, y^2 + xy) \qquad \text{(ii)}.$$

(i) has $(3, 5)$–transversal $\{(0, 0, x^2z)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xy + xz^2 + ax^2z).$$

This is an $\mathcal{A}$–trivial family. All higher $(3, s)$–transversals are empty.

(ii) has $(3, 5)$–transversal $\{(0, 0, x^2z)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xy + ax^2z).$$

The tangent space contains the vector $(0, 0, x^2z)$ except when $a = 0$ giving two orbits (as $(x, yz, y^2 + xy + x^2z) \sim_{\mathcal{A}} (x, yz, y^2 + xy - x^2z)$ by the change of coordinates $z \longmapsto -z$)):

$$(x, yz, y^2 + xy + x^2z),$$

$$(x, yz, y^2 + xy).$$

In both cases, all higher $(3, s)$–transversals are empty.

So we have seven $J\mathcal{A}^3$–orbits over $(x, yz, y^2 + xy)$:

$$\begin{array}{ll} (x, yz, y^2 + xy + z^3 + xz^2 + ax^2z) & \text{(J)}, \\ (x, yz, y^2 + xy + z^3 + x^2z) & (\sharp), \\ (x, yz, y^2 + xy + z^3) & (\sharp), \\ (x, yz, y^2 + xy + z^3 - x^2z) & (\sharp), \\ (x, yz, y^2 + xy + xz^2) & (\sharp), \\ (x, yz, y^2 + xy + x^2z) & (\sharp), \\ (x, yz, y^2 + xy) & (\sharp). \end{array}$$

A $(3, 1)$–transversal over (G) is $\{(0, 0, z^3)\}$ giving us two orbits to consider:

- (x, yz, y^2+xz+z^3) (this is $\mathcal{A}$–equivalent to (x, yz, y^2+xz-z^3) via the change of coordinates $(x, y, z) \longmapsto (-x, -y, -z)$); all higher $(3, s)$–transversals are empty.
- $(x, yz, y^2 + xz)$; all higher $(3, s)$–transversals are empty.

So we have two $J\mathcal{A}^3$–orbits over $(x, yz, y^2 + xz)$:

$$\begin{array}{ll} (x, yz, y^2 + xz + z^3) & \text{(K)}, \\ (x, yz, y^2 + xz) & \text{(L)}. \end{array}$$

A $(3, 1)$–transversal over (H) is $\{(0, 0, z^3)\}$ giving us two orbits to consider $((x, y^2 + z^2, xy + z^3) \sim_{\mathcal{A}} (x, y^2 + z^2, xy - z^3)$ via the change of coordinates $(x, y, z) \longmapsto (x, y, -z))$:

- $(x, y^2 + z^2, xy + z^3)$. This has $(3, 3)$–transversal $\{(0, 0, y^2z)\}$ giving the 1–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z).$$

This has $(3,4)$–transversal $\{(0,0,y^3)\}$ giving the 2–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3).$$

All higher $(3,s)$–transversals are empty.

- $(x, y^2 + z^2, xy)$. This has $(3,3)$–transversal $\{(0,0,y^2z)\}$ which gives the 1–parameter family

$$(x, y^2 + z^2, xy + ay^2z).$$

The tangent space contains the vector $(0,0,y^2z)$ except when $a = 0$ giving two orbits to consider as $(x, y^2 + z^2, xy + y^2z) \sim_{\mathcal{A}} (x, y^2 + z^2, xy - y^2z)$ via the change of coordinates $(x,y,z) \longmapsto (x,y,-z)$:

$$\begin{array}{ll} (x, y^2 + z^2, xy + y^2z) & \text{(i)}, \\ (x, y^2 + z^2, xy) & \text{(ii)}. \end{array}$$

(i) has $(3,4)$–transversal $\{(0,0,y^3)\}$ giving the 1–parameter family

$$(x, y^2 + z^2, xy + y^2z + ay^3).$$

All higher $(3,s)$–transversals are empty.

(ii) has $(3,4)$–transversal $\{(0,0,y^3)\}$ giving the 1–parameter family

$$(x, y^2 + z^2, xy + ay^3).$$

The tangent space contains the vector $(0,0,y^3)$ except when $a = 0$ giving two orbits (as $(x, y^2 + z^2, xy + y^3) \sim_{\mathcal{A}} (x, y^2 + z^2, xy - y^3)$ via the change of coordinates $(x,y,z) \longmapsto (-x,-y,z)$):

$$(x, y^2 + z^2, xy + y^3),$$

$$(x, y^2 + z^2, xy).$$

This gives us four $J\mathcal{A}^3$–orbits over $(x, y^2 + z^2, xy)$:

$$\begin{array}{ll} (x, y^2 + z^2, xy + z^3 + ay^2z + by^3) & \text{(M)}, \\ (x, y^2 + z^2, xy + y^2z + ay^3) & (\sharp), \\ (x, y^2 + z^2, xy + y^3) & (\sharp), \\ (x, y^2 + z^2, xy) & (\sharp). \end{array}$$

A $(3,1)$–transversal over (I) is $\{(0, 0, z^3)\}$ giving us two orbits to consider as $(x, yz, xy + xz + z^3) \sim_{\mathcal{A}} (x, yz, xy + xz - z^3)$ via the change of coordinates $(x, yz) \longmapsto (-x, -y, -z)$:

- $(x, yz, xy + xz + z^3)$. This has $(3,3)$–transversal $\{(0, 0, y^2z)\}$ giving the 1–parameter family

  $$(x, yz, xy + xz + z^3 + ay^2z).$$

  This has $(3,4)$–transversal $\{(0, 0, y^3)\}$ giving the 2–parameter family

  $$(x, yz, xy + xz + z^3 + ay^2z + by^3).$$

  All higher $(3, s)$–transversals are empty.

- $(x, yz, xy + xz)$. This has $(3,3)$–transversal $\{(0, 0, y^2z)\}$ which gives us the 1–parameter family

  $$(x, yz, xy + xz + ay^2z).$$

  The tangent space contains the vector $(0, 0, y^2z)$ except when $a = 0$ giving two orbits to consider ($(x, yz, xy + xz + y^2z) \sim_{\mathcal{A}} (x, yz, xy + xz - y^2z)$ via the change of coordinates $(x, y, z) \longmapsto (-x, -y, -z)$):

  $$\begin{array}{ll} (x, yz, xy + xz + y^2z) & \text{(i)}, \\ (x, yz, xy + xz) & \text{(ii)}. \end{array}$$

  (i) has $(3,4)$–transversal $\{(0, 0, y^3)\}$ giving the 1–parameter family

$$(x, yz, xy + xz + y^2z + ay^3).$$

All higher $(3, s)$–transversals are empty.

(ii) has $(3, 4)$–transversal $\{(0, 0, y^3)\}$ giving the 1–parameter family

$$(x, yz, xy + xz + ay^3).$$

The tangent space contains the vector $(0, 0, y^3)$ except when $a = 0$ giving two orbits as $(x, yz, xy + xz + y^3) \sim_{\mathcal{A}} (x, yz, xy + xz - y^3)$ via the change of coordinates $(x, y, z) \longmapsto (-x, -y, -z)$:

$$(x, yz, xy + xz + y^3),$$
$$(x, yz, xy + xz).$$

All higher $(3, s)$–transversals are empty in both cases.

This gives us four $J\mathcal{A}^3$–orbits over $(x, yz, xy + xz)$:

$$\begin{array}{ll} (x, yz, xy + xz + z^3 + ay^2z + by^3) & (\mathrm{N}), \\ (x, yz, xy + xz + y^2z + ay^3) & (\sharp), \\ (x, yz, xy + xz + y^3) & (\sharp), \\ (x, yz, xy + xz) & (\sharp). \end{array}$$

6.3.14 The 4–Jets

A $(4, 9)$–transversal over (F) is $\{(0, x^3y, 0)\}$ giving the 1–parameter family

$$(x, yz + y^3 - \frac{3}{16}x^2y + ax^3y, y^2 + z^2 + xy + xz).$$

The tangent space contains the vector $(0, x^3y, 0)$ except when $a = 0$ giving three orbits:

$$(x, yz + y^3 - \tfrac{3}{16}x^2y + x^3y, y^2 + z^2 + xy + xz),$$

$$(x, yz + y^3 - \tfrac{3}{16}x^2y, y^2 + z^2 + xy + xz),$$
$$(x, yz + y^3 - \tfrac{3}{16}x^2y - x^3y, y^2 + z^2 + xy + xz).$$

All higher $(4, s)$–transversals are empty in each case. This gives us three $J\mathcal{A}^4$–orbits over $(x, yz + y^3 - \frac{3}{16}x^2y, y^2 + z^2 + xy + xz)$:

$(x, yz + y^3 - \frac{3}{16}x^2y + x^3y, y^2 + z^2 + xy + xz)$	4–determined,
$(x, yz + y^3 - \frac{3}{16}x^2y, y^2 + z^2 + xy + xz)$	($\sharp$),
$(x, yz + y^3 - \frac{3}{16}x^2y - x^3y, y^2 + z^2 + xy + xz)$	4–determined.

All $(4, s)$–transversal for (G) are empty giving one $J\mathcal{A}^4$–orbit over $(x, yz + x^2y, y^2 + z^2 + xy + xz)$:

$$(x, yz + x^2y, y^2 + z^2 + xy + xz) \qquad \text{(I)}.$$

A $(4, 1)$–transversal for (H) is $\{(0, 0, z^4)\}$; this gives us the 1–parameter family $f_a = (x, yz, y^2 + xy + xz + z^3 + az^4)$. This has $(4, 2)$–transversal $\{(0, 0, yz^3)\}$ which gives us the 2–parameter family

$$f_{a,b} = (x, yz, y^2 + xy + xz + z^3 + az^4 + byz^3)$$

The tangent space contains the vector $(0, 0, yz^3)$ if it contains the vector $(0, 0, z^4)$. This gives us a unimodular family with the following representative:

$$(x, yz, y^2 + xy + xz + z^3 + ayz^3) \qquad \text{4–determined,}$$

A $(4, 1)$–transversal over (I) is $\{(0, 0, z^4)\}$ giving us three orbits to consider;

- $(x, yz, y^2 + xy + xz + z^4)$; this has $(4, 2)$–transversal $\{(0, 0, yz^3)\}$ giving us three orbits:

$$(x, yz, y^2 + xy + xz + z^4 + yz^3),$$

$$(x, yz, y^2 + xy + xz + z^4),$$
$$(x, yz, y^2 + xy + xz + z^4 - yz^3).$$

All higher $(4, s)$–transversals are empty in each case.

- $(x, yz, y^2 + xy + xz)$. This has $(4, 2)$–transversal $\{(0, 0, yz^3)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xy + xz + ayz^3).$$

The tangent space contains the vector $(0, 0, yz^3)$ except when $a = 0$ giving three orbits:

$$(x, yz, y^2 + xy + xz + yz^3),$$
$$(x, yz, y^2 + xy + xz),$$
$$(x, yz, y^2 + xy + xz - yz^3).$$

All higher $(4, s)$–transversals are empty in each case.

So we have six $J\mathcal{A}^4$–orbits over $(x, yz, y^2 + xy + xz)$:

$$\begin{array}{ll} (x, yz, y^2 + xy + xz + z^4 + yz^3) & \text{(J)}, \\ (x, yz, y^2 + xy + xz + z^4) & (\sharp), \\ (x, yz, y^2 + xy + xz + z^4 - yz^3) & \text{(J}\,'), \\ (x, yz, y^2 + xy + xz + yz^3) & (\sharp), \\ (x, yz, y^2 + xy + xz) & (\sharp), \\ (x, yz, y^2 + xy + xz - yz^3) & (\sharp). \end{array}$$

All $(4, s)$–transversals over (J) are empty, giving us one J^4–orbit:

$$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z) \quad \text{(K)}.$$

A $(4, 1)$–transversal for (K) is $\{(0, 0, z^4)\}$ giving us three orbits to consider;

- $(x, yz, y^2 + xz + z^3 + z^4)$; this has $(4, 2)$–transversal $\{(0, 0, yz^3)\}$ giving the 1–parameter family

$$(x, yz, y^2, y^2 + xz + z^3 + z^4 + ayz^3).$$

All higher $(4, s)$–transversals are empty.

- (x, yz, y^2+xz+z^3); this has $(4,2)$–transversal $\{(0,0,yz^3)\}$ giving two orbits to consider as $(x, yz, y^2 + xz + z^3 + yz^3) \sim_{\mathcal{A}} (x, yz, y^2 + xz + z^3 - yz^3)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$:

$$\begin{array}{ll} (x, yz, y^2 + xz + z^3 + yz^3) & \text{(i)}, \\ (x, yz, y^2 + xz + z^3) & \text{(ii)}. \end{array}$$

All higher $(4, s)$–transversals are empty in each case.

So we have three $J\mathcal{A}^4$–orbits over $(x, yz, y^2 + xz + z^3)$:

$$\begin{array}{ll} (x, yz, y^2 + xz + z^3 + z^4 + ayz^3) & \text{4–determined}, \\ (x, yz, y^2 + xz + z^3 + yz^3) & \text{4–determined}, \\ (x, yz, y^2 + xz + z^3) & (\sharp). \end{array}$$

A $(4,1)$–transversal over (L) is $\{(0,0,z^4)\}$ giving only three orbits to consider;

- $(x, yz, y^2 + xz + z^4)$; this has $(4,2)$–transversal $\{(0,0,yz^3)\}$ which gives us two orbits as $(x, yz, y^2 + xz + z^4 + yz^3) \sim_{\mathcal{A}} (x, yz, y^2 + xz + z^4 - yz^3)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$:

$$\begin{array}{c} (x, yz, y^2 + xz + z^4 + yz^3), \\ (x, yz, y^2 + xz + z^4). \end{array}$$

All higher $(4, s)$–transversals are empty in each case.

- $(x, yz, y^2 + xz)$. This has $(4,2)$–transversal $\{(0,0,yz^3)\}$ which gives the 1–parameter family

$$(x, yz, y^2 + xz + ayz^3).$$

The tangent space contains the vector $(0, 0, yz^3)$ except when $a = 0$ giving two orbits as $(x, yz, y^2 + xz + yz^3) \sim_{\mathcal{A}} (x, yz, y^2 + xz - yz^3)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$;

$$(x, yz, y^2 + xz + yz^3),$$
$$(x, yz, y^2 + xz).$$

- $(x, yz, y^2 + xz - z^4)$; this has $(4, 2)$–transversal $\{(0, 0, yz^3)\}$ which gives us two orbits as $(x, yz, y^2 + xz - z^4 + yz^3) \sim_{\mathcal{A}} (x, yz, y^2 + xz - z^4 - yz^3)$ via the change of coordinates $(x, y, z) \longmapsto (x, -y, z)$:

$$(x, yz, y^2 + xz - z^4 + yz^3,$$
$$(x, yz, y^2 + xz - z^4).$$

All higher $(4, s)$–transversals are empty in each case.

So we have six $J\mathcal{A}^4$–orbits over $(x, yz, y^2 + xz)$:

$(x, yz, y^2 + xz + z^4 + yz^3)$	(L),
$(x, yz, y^2 + xz + z^4)$	($\sharp$),
$(x, yz, y^2 + xz + yz^3)$	($\sharp$),
$(x, yz, y^2 + xz)$	($\sharp$),
$(x, yz, y^2 + xz - z^4 + yz^3)$	(L$'$),
$(x, yz, y^2 + xz - z^4)$	($\sharp$).

A $(4, 4)$–transversal for (M) is $\{(0, 0, y^3z)\}$ giving the 3–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + cy^3z).$$

The tangent space contains the vector $(0, 0, y^3z)$ except when $c = 0$ giving three orbits to consider:

- $(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z)$. This has $(4, 5)$–transversal $\{(0, 0, y^4)\}$ which gives the 3–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4).$$

- $(x, y^2 + z^2, xy + z^3 + ay^2z + by^3)$. This has $(4,5)$–transversal $\{(0,0,y^4)\}$ which gives the 3–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + cy^4).$$

- $(x, y^2+z^2, xy+z^3+ay^2z+by^3-y^3z)$. This has $(4,5)$–transversal $\{(0,0,y^4)\}$ which gives the 3–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4).$$

So we have three $J\mathcal{A}^4$–orbits over $(x, y^2 + z^2, xy + z^3 + ay^2z + by^3)$:

$$\begin{array}{ll} (x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4) & (\text{M}), \\ (x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + cy^4) & (\natural), \\ (x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4) & (\text{M}'), \end{array}$$

A $(4,4)$–transversal for (N) is $\{(0,0,y^3z)\}$ giving the 3–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + cy^3z).$$

The tangent space contains the vector $(0,0,y^3z)$ except when $c = 0$ giving three orbits to consider:

- $(x, yz, xy+xz+z^3+ay^2z+by^3+y^3z)$. This has $(4,5)$–transversal $\{(0,0,y^4)\}$ which gives the 3–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4).$$

- $(x, yz, xy + xz + z^3 + ay^2z + by^3)$. This has $(4,5)$–transversal $\{(0,0,y^4)\}$ which gives the 3–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + cy^4).$$

- $(x, yz, xy+xz+z^3+ay^2z+by^3-y^3z)$. This has $(4,5)$–transversal $\{(0,0,y^4)\}$ which gives the 3–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4).$$

So we have three $J\mathcal{A}^4$–orbits over $(x, yz, xy + xz + z^3 + ay^2z + by^3)$:

$$\begin{array}{ll} (x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4) & \text{(N)}, \\ (x, yz, xy + xz + z^3 + ay^2z + by^3 + cy^4) & (\sharp), \\ (x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4) & \text{(N}'), \end{array}$$

6.3.15 The 5–Jets

A $(5,7)$–transversal for (I) is $\{(0, y^5, 0)\}$ giving three orbits to consider:

- $(x, yz+x^2y+y^5, y^2+z^2+xy+xz)$. All higher $(5,s)$–transversals are empty.
- $(x, yz + x^2y, y^2 + z^2 + xy + xz)$. All higher $(5,s)$–transversals are empty.
- $(x, yz+x^2y-y^5, y^2+z^2+xy+xz)$. All higher $(5,s)$–transversals are empty.

So this gives us three $J\mathcal{A}^5$–orbits over $(x, yz + x^2y, y^2 + z^2 + xy + xz)$:

$$\begin{array}{ll} (x, yz + x^2y + y^5, y^2 + z^2 + xy + xz) & \text{5–determined}, \\ (x, yz + x^2y, y^2 + z^2 + xy + xz) & (\sharp), \\ (x, yz + x^2y - y^5, y^2 + z^2 + xy + xz) & \text{5–determined}. \end{array}$$

A $(5,1)$–transversal over (J) is $\{(0,0,z^5)\}$ which gives the 1–parameter family

$$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5).$$

This has $(5,2)$–transversal $\{(0,0,yz^4)\}$ which gives us one $J\mathcal{A}^5$-orbit over $(x, yz, y^2 + xy + xz + z^4 + yz^3)$:

$$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4) \qquad \text{(H)}.$$

A $(5,1)$–transversal over (J ′) is $\{(0,0,z^5)\}$ which gives the 1–parameter family

$$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5).$$

This has $(5,2)$–transversal $\{(0,0,yz^4)\}$ which gives us one $J\mathcal{A}^5$–orbit over $(x, yz, y^2 + xy + xz + z^4 - yz^3)$:

$$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4) \qquad \text{(H ′)}.$$

A $(5,1)$–transversal over (K) is $\{(0,0,z^5)\}$ giving the 2–parameter family

$$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5).$$

All higher $(5,s)$–transversals are empty. This gives us one $J\mathcal{A}^5$–orbit over $(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z)$:

$$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5) \qquad \text{5–determined.}$$

A $(5,1)$–transversal over (L) is $\{(0,0,z^5)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xz + z^4 + yz^3 + az^5).$$

This has $(5,2)$–transversal $\{(0,0,yz^4)\}$ giving the 2–parameter family

$$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4).$$

All higher $(5, s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, yz, y^2 + xz + z^4 + yz^3)$:

$$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4) \qquad \text{(I)}.$$

A $(5, 1)$–transversal over (L ′) is $\{(0, 0, z^5)\}$ giving the 1–parameter family

$$(x, yz, y^2 + xz + z^4 - yz^3 + az^5).$$

This has $(5, 2)$–transversal $\{(0, 0, yz^4)\}$ giving the 2–parameter family

$$(x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4).$$

All higher $(5, s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, yz, y^2 + xz + z^4 - yz^3)$:

$$(x, yz, y^2 + xz + z^4 - yz^3 + Az^5 + byz^4) \qquad \text{(I ′)}.$$

A $(5, 3)$–transversal for (M) is $\{(0, 0, y^2z^3)\}$ giving the 4–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3).$$

This has $(5, 5)$–transversal $\{(0, 0, y^4z)\}$ which again gives a family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z).$$

Which in turn has $(5,6)$–transversal $\{(0,0,y^5)\}$ giving the 6–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5).$$

All higher $(5,s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4)$:

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5) \quad \text{(J)}.$$

A $(5,3)$–transversal for (M ′) is $\{(0,0,y^2z^3)\}$ giving the 4–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3).$$

This has $(5,5)$–transversal $\{(0,0,y^4z)\}$ which again gives a family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z).$$

Which in turn has $(5,6)$–transversal $\{(0,0,y^5)\}$ giving the 6–parameter family

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5).$$

All higher $(5,s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4)$:

$$(x, y^2 + z^2, xy + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5) \quad \text{(J)}.$$

A $(5,3)$–transversal for (N) is $\{(0,0,y^2z^3)\}$ giving the 4–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3).$$

This has $(5,5)$–transversal $\{(0,0,y^4z)\}$ which again gives a family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z).$$

Which in turn has $(5,6)$–transversal $\{(0,0,y^5)\}$ giving the 6–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5).$$

All higher $(5,s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4)$:

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 + y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5) \quad \text{(K)}.$$

A $(5,3)$–transversal for (N ′) is $\{(0,0,y^2z^3)\}$ giving the 4–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3).$$

This has $(5,5)$–transversal $\{(0,0,y^4z)\}$ which again gives a family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z).$$

Which in turn has $(5,6)$–transversal $\{(0,0,y^5)\}$ giving the 6–parameter family

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5).$$

All higher $(5,s)$–transversals are empty.

So we have one $J\mathcal{A}^5$–orbit over $(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4)$:

$$(x, yz, xy + xz + z^3 + ay^2z + by^3 - y^3z + cy^4 + dy^2z^3 + ey^4z + fy^5) \quad \text{(J)}.$$

6.3.16 The 6–Jets

A $(6,1)$–transversal for (H) is $\{(0,0,z^6)\}$ giving the 3–parameter family

$$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6).$$

All higher $(6,s)$–transversals are empty.

This gives one $J\mathcal{A}^6$–orbit over $(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4)$:

$$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6) \quad \text{6–determined.}$$

A $(6,1)$–transversal for (H$'$) is $\{(0,0,z^6)\}$ giving the 3–parameter family

$$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6).$$

All higher $(6,s)$–transversals are empty.

This gives one $J\mathcal{A}^6$–orbit over $(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4)$:

$$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6) \quad \text{6–determined.}$$

A $(6,1)$–transversal for (I) is $\{(0,0,z^6)\}$ giving the 3–parameter family

$$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6).$$

All higher $(6,s)$–transversals are empty.

This gives one $J\mathcal{A}^6$–orbit over $(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4)$:

$$(x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6) \quad \text{6–determined.}$$

A $(6,1)$–transversal for (I ′) is $\{(0,0,z^6)\}$ giving the 3–parameter family

$$(x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6).$$

All higher $(6,s)$–transversals are empty.

This gives one $J\mathcal{A}^6$–orbit over $(x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4)$:

$$(x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6) \quad \text{6–determined.}$$

The $(6,s)$–transversals for (J), (J ′), (K) and (K ′) produce more moduli which make it impossible for us to extend the analysis of these branches any further.

6.3.17 Example of Computer Output

In this section we will take one branch of our classification and follow it through until we find a determined jet giving the computer output at each level.

We use the 'unipotent' group $\mathcal{G}$ described in Section 1.3. We will consider the 1–jet $(x, y, 0)$. The following command sets up the program to work with correct dimensions and Lie algebra and Maple responds by printing out values of various parameters.

```
> setup_classn(3,3,[0,x3]);
```

```
                    liealg = stdjacobian
                        equiv = A
                    compltrans = true
                     source_dim = 3
                    source_power = 2
                    target_power = 2
                    nilp = true_order
                         R_nilp:
                 [[x1,2],[x1,3],[x2,3]]
                         L_nilp:
                  [[2,1],[3,1],[3,2]]

>
```

We now input our 1–jet $(x, 0, 0)$ which we want to study.

```
> f := [x1,x2,0];
> jetcalc(f,2);
```

Maple eventually replies Ready and we now use one of the 'print' routines to view the complete transversal.

```
> pcomp();
```

$$[0,0,x3^2]$$

```
[0,0,x2 x3]
[0,0,x1 x3]

>
```

We now need to work out what $(2, s)$–levels these belong to, we to this using a different 'print' routine;

```
> pmons(2);

level:  (2,1)
  [0,0,x3^2]
level:  (2,2)
  [0,x3^2,0]
  [0,0,x2 x3]
level:  (2,3)
  [x3^2,0,0]
  [0,x2 x3,0]
  [0,0,x2^2]
```

and this continues through all the possible $(?, s)$–levels. From this we find that we get three $J\mathcal{A}^2$–orbits over $(x, y, 0)$:

$$\begin{array}{ll} (x, y, z^2) & \text{(i)} \\ (x, y, yz) & \text{(ii)} \\ (x, y, 0) & \text{(iii)} \end{array}$$

(i) is 2–determined, but this calculation is very simple so we will carry on until we reach one for which we need to use the determinacy criterion in the program. If we look at (ii) and we go through the same routine to get the three $J\mathcal{A}^3$–orbits:

$$\begin{array}{ll} (x, y, yz + z^3) & \text{(a)} \\ (x, y, yz + xz^2) & \text{(b)} \\ (x, y, yz) & \text{(c)} \end{array}$$

We would like to prove that (a) is 3–determined. To do this we use the `Aclassify` routine;

```
> f := [x1,x2,x2*x3+x3^3]:
> Aclassify(f,3);
```

Maple responds by calculating the 4– ,5– ,6– and 7–transversals; the result is

```
[x1, x2, x2 x3 + x3^3]
germ is 3--A--determined
```

>

Now lets look at the 3–jet of corank 2: $(x, yz, y^2 + xy + xz + z^3)$, say. Firstly we want to find its 4–transversal, we do this as follows:

```
> f:=[x1,x1*x2,x2^2+x1*x2+x1*x3+x3^3];

f := [x1,x2 x3,x2^2 +x1 x2+x1 x3+x3^3]

> jetcalc(f,4);
```

Maple will then take a couple of seconds working this out and finally come out with the response Ready so we then ask it to print its answer using pcomp:

```
Ready.

> pcomp();

[0,0,x3^4]
```

```
[0,0,x2 x3^3]
```

We can then find out which $(4,s)$–level each of these belong to using the **pmons** command. When we do this we find that $(0,0,z^4)$ belongs to the $(4,1)$–level and $(0,0,yz^3)$ to the $(4,2)$–level, so we can consider them individually. $\{(0,0,z^4)\}$ gives us the 1–parameter family $(x, yz, y^2 + xy + xz + z^3 + az^4)$. Now we cannot 'scale' a using obvious techniques so we wish to check if a is a modulus. To do this we need to use the group $\mathcal{A}$ and work in the 4–jet–space to show that

$$(0,0,z^4) \notin L\mathcal{A} \cdot f \quad \text{modulo} \quad \mathcal{M}_3^5.\mathcal{E}(3,3);$$

To begin with we need to set up the global variables to define the $\mathcal{A}$ group. Within the program this is done as follows:

```
> setup_unf(2);

                    liealg = stdjacobian
                         equiv = A
                   compltrans = false
                     source_dim = 3
                    source_power = 0
                    target_power = 0
                  nilp = true_order
                         R_nilp:
                         R_nilp:
                         L_nilp:
                         L_nilp:
```

This gives us the $\mathcal{A}_e$ group, to change this to the $\mathcal{A}$ group we need to change the source and target powers:

```
> source_power:=1; target_power:=1;
```

We now calculate the tangent space $T\mathcal{A} \cdot f$ in $J^5(3,3)$ and determine whether the vector $(0,0,z^4)$ belongs to this tangent space.

```
> f :=[x1,x2*x3,x2^+x1*x2+x1*x3+x3^+a*x3^4];
> jetcalc(f,4);
> intangent([0,0,x3^4]);
```

```
                              false
```

```
>
```

The Maple response `false` indicates that a is a modulus. We now find that the $(4,2)$–transversal for this 1–parameter family is given by $\{(0,0,yz^3)\}$ giving the 2–parameter family $f_{a,b} = (x, yz, y^2+xy+xz+z^3+az^4+byz^3)$ which contains the vector $(0,0,yz^3)$ for all values of b giving us one $J\mathcal{A}^4$–orbit.

Chapter 7

Unfoldings and Bifurcations

We have our list of map-germs :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ and we wish to study the geometry of their unfoldings.

Theorem 7.0.2 *For our map-germs* :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ *the versal unfoldings and the invariants associated to those with smooth critical set are given in the following table.*

R	Versal unfolding	d	c	S	T
1	(x, y, z)	-	-	-	-
2	(x, y, z^2)	-	-	-	-
3	$(x, y, yz + z^3)$	-	*0*	-	*0*
4	$(x, y, yz + xz^2 + z^4)$	*0*	*0*	*1*	*1*
5	$(x, yz + ay, y^2 + z^2 + xy)$				
6	$(x, yz + ay, y^2 - z^2 + xy)$				
7	$(x, y, z^3 + (y^2 + x^2)z + az)$				
$8 \sim 9$	$(x, y, z^3 + (y^2 - x^2)z + az)$				
10	$(x, y, z^3 - (y^2 + x^2)z + az)$				
11	$(x, y, yz + z^4 + x^2z^2 + az^2)$	*1*	*1*	*2*	*2*
12	$(x, y, yz + z^4 - x^2z^2 + az^2)$	*1*	*1*	*2*	*2*
13	$(x, y, yz + xz^2 + z^5 + az^3)$	*5*	*0*	*2*	*4*

Table 7.1: Versal unfoldings of map-germs :$(\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$

R	Versal unfolding	d	c	S	T
14	$(x, yz + y^3, y^2 + z^2 + xy + xz + ay + bxy)$				
15	$(x, yz + y^3 - x^2y, y^2 + z^2 + xy + xz + ay + bxy)$				
16	$(x, y, z^3 + (y^2 + x^3)z + az + bxz)$				
17	$(x, yz, y^2 + xy + xz + z^3 + ayz^3 + by + cz^2)$				
18	$(x, yz, y^2 + xz + z^3 + z^4 + ayz^3 + by + cz^2)$				
19	$(x, y, yz + z^4 + x^3z^2 + az^2 + bxz^2)$	*2*	*2*	*3*	*3*
20	$(x, y, yz^2 + x^2z + z^4 + z^5 + az^2 + bz^4)$				
21	$(x, y, yz^2 + x^2z + z^4 - z^5 + az^2 + bz^4)$				
22	$(x, y, yz + xz^3 + z^5 + z^6 + az^2 + bz^4)$	*22*	*1*	*3*	*6*
23	$(x, y, yz + xz^2 + z^6 + z^8 + az^9 + bz^3 + cz^4)$	*22*	*0*	*3*	*10*
24	$(x, y, yz + xz^2 + z^6 - z^8 + az^9 + bz^3 + cz^4)$	*22*	*0*	*3*	*10*
25	$(x, yz \pm y^3 + ax^2y + by, y^2 \pm z^2 + y^3 + x^2y + cy + dxy),\ a \neq \frac{1}{2}$				
26	$(x, yz \pm y^3 + ax^2y + by, y^2 \pm z^2 + y^3 - x^2y + cy + dxy),\ a \neq \frac{1}{2}$				
27	$(x, yz + ax^2y, y^2 \pm z^2 + y^3 + x^2y + by + cxy + dx^2y),\ a \neq \frac{1}{2}$				
28	$(x, yz + ax^2y, y^2 \pm z^2 + y^3 - x^2y + by + cxy + dx^2y),\ a \neq \frac{1}{2}$				
29	$(x, yz + y^3 - \frac{3}{16}x^2y + x^3y, y^2 + z^2 + xy + xz + ay + bz^2 + cyz^3)$				
30	$(x, yz + y^3 - \frac{3}{16}x^2y - x^3y, y^2 + z^2 + xy + xz + ay + bz^2 + cyz^3)$				
31	$(x, yz, y^2 + xz + z^3 + yz^3 + ay + bz^2 + cz^4)$				
32	$(x, y, z^3 + (y^2 + x^4)z + az + bxz + cx^2z)$				
33	$(x, y, z^3 + (y^2 - x^4)z + az + bxz + cx^2z)$				
34	$(x, y, z^3 - (x^2 - y^4)z + az + byz + cy^2z)$				
35	$(x, y, z^3 - (x^2 + y^4)z + az + byz + cy^2z)$				
36	$(x, yz + x^2y + y^5, y^2 + z^2 + xy + xz + ay + bxy + cy^3)$				

Table 7.2: Table 7.1 continued

R	Versal unfolding	d	c	S	T
37	$(x, yz + x^2y - y^5, y^2 + z^2 + xy + xz$ $+ay + bxy + cy^3)$				
38	$(x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5$ $+cz + dz^2 + exz) \quad a \neq \frac{3}{8}$				
39	$(x, y, yz + z^4 + x^4z^2 + az^2 + bxz^2 + cx^2z^2)$	3	3	4	4
40	$(x, y, yz + z^4 - x^4z^2 + az^2 + bxz^2 + cx^2z^2)$	3	3	4	4
41	$(x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5$ $+byz^4 + cz^6 + dy + ez^2 + fz^3)$				
42	$(x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5$ $+byz^4 + cz^6 + dy + ez^2 + fz^3)$				
43	$(x, yz, y^2 + xz + z^4 + yz^3 + az^5$ $+byz^4 + cz^6 + dy + ez^2 + fz^3)$				
44	$(x, yz, y^2 + xz + z^4 - yz^3 + az^5$ $+byz^4 + cz^6 + dy + ez^2 + fz^3)$				
45	$(x, y, yz + xz^3 + z^5 + az^2 + bz^4 + cz^6)$	10	1	3	6
46	$(x, y, yz + x^2z^2 + z^5 + z^6 + az^7 + bz^2$ $+cz^3 + dxz^3), \quad a \neq 0$	15	2	4	8
47	$(x, y, yz + x^2z^2 + z^5 - z^6 + az^7 + bz^2$ $+cz^3 + dxz^3), \quad a \neq 0$	15	2	4	8
48	$(x, y, yz^2 + x^2z + z^4 + z^7 + az + bz^3 + cz^5)$				
49	$(x, y, yz^2 + x^2z + z^4 - z^7 + az + bz^3 + cz^5)$				
50	$(x, y, yz + xz^2 + z^6 + z^9 + az^3 + bz^4 + cz^8)$	22	0	3	10

Table 7.3: Table 7.1 continued

where **R** *is the place at which the normal form appears in Table 6.16 of Chapter 6,* **d** *is the Milnor number of the double point curve of the discriminant* $f(\sum f)$, **c** *is the Milnor number of the cuspidaledge curve of the discriminant* $f(\sum f)$, **S** *is the maximum number of swallowtails in the unfolding and* **T** *is the maximum number of triplepoints in the unfolding.*

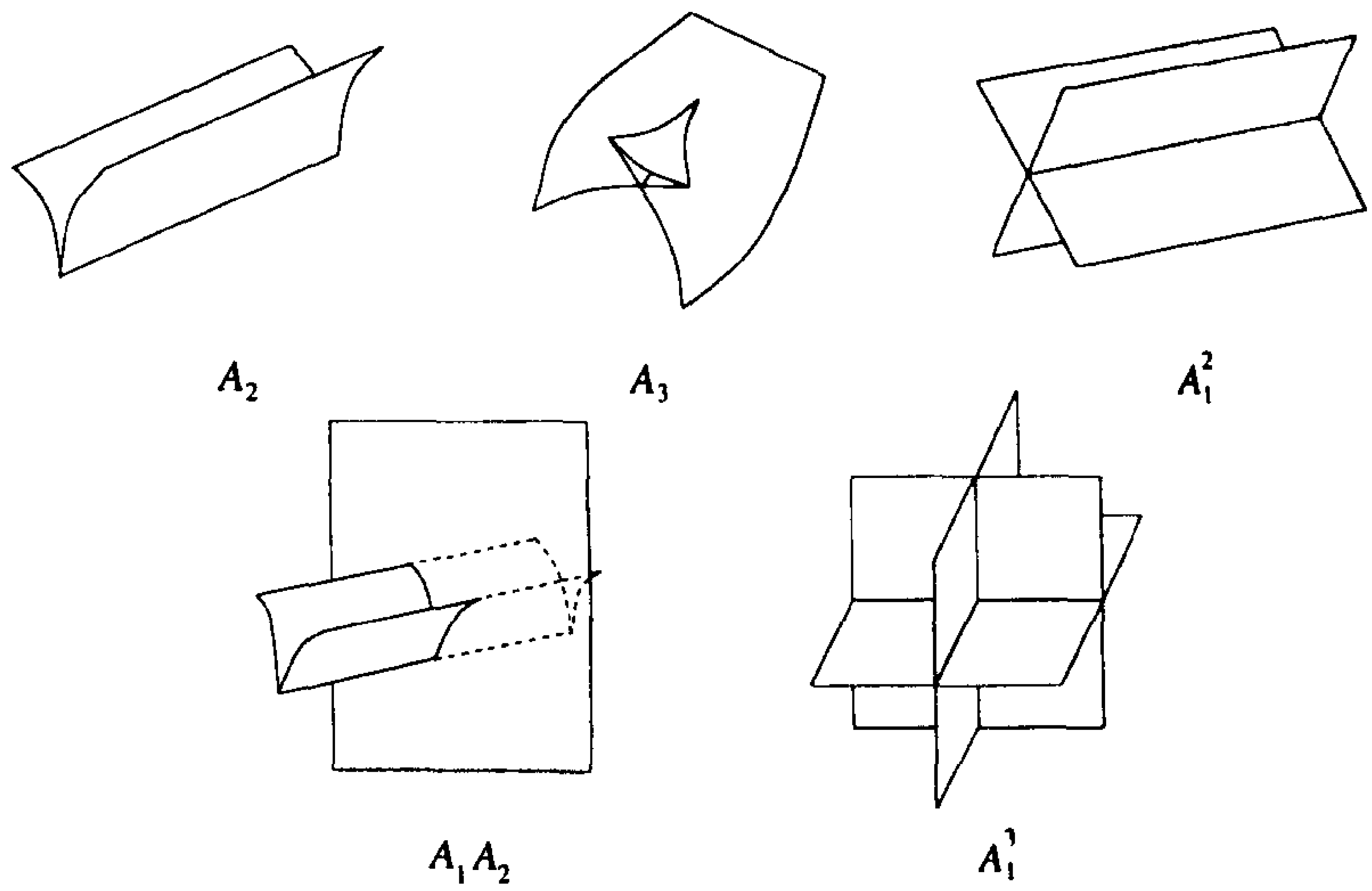


Figure 7.1: $A_2, A_3, A_1^2, A_1A_2, A_1^3$ singularities

7.1 The Invariants

Consider a map–germ $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$. We denote the critical set of f by $\sum$, and the discriminant of f by Δ, i.e. $\Delta = f(\sum)$. When $\sum$ is smooth, the restriction of f to $\sum$ can be considered as a map–germ $F : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ parametrizing Δ. The map–germ F is generally not finitely $\mathcal{A}$-determined due to the presence of cuspidal edges. Arnol'd, [A2] showed that F are as in Fig. 7.1.

We can study a lot of the geometry of F by studying a plane curve singularity, namely the double point curve of F. The double point curve is the locus of points $p \in \sum f$ such that the number of pre–images of p is two, i.e. $\#F^{-1}(F(p)) = 2$, together with the points where F fails to be immersive. The defining equation $h : (\mathbb{R}^2) \longrightarrow (\mathbb{R}, 0)$ of the double point curve can be obtained using the following result from Bruce & Marar, [BM] and Marar & Tari, [MT]:

Proposition 7.1.1 *Let $F : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ be an analytic map–germ with $F(x,y) = (a(x,y), b(x,y), c(x,y))$. Let $G : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}, 0)$ be an analytic function such that $G(X,Y,Z) = 0$ is the defining equation of the image of F. Then the defining equation, h of the double point curve of F is given by*

$$h(x,y) = \frac{\frac{\partial G}{\partial Z}(a(x,y), b(x,y), c(x,y)).}{\left|\frac{\partial(a,b)}{\partial(x,y)}\right|}$$

If our F is corank 1 at the origin we can simplify the above by choosing a suitable system of coordinates. The simplified formula being:

$$h(x,y) = \frac{\frac{\partial G}{\partial Z}(x, \alpha(x,y), \beta(x,y)).}{\frac{\partial \alpha}{\partial y}(x,y)}$$

Now in practice, the equation $G(X,Y,Z) = 0$ is obtained by eliminating the variable y from the equations $Y - \alpha(x,y) = 0$ and $Z - \beta(x,y) = 0$. This is done using MAPLE.

From [BM] we know that h factors as

$$h(x,y) = d(x,y)(c(x,y))^2,$$

where $d(x,y) = 0$ defines the closure of ordinary double points of F and $c(x,y) = 0$ defines the curve in the source which is mapped into the cuspidaledges of F. We call $d(x,y) = 0$ the *double point curve* and $c(x,y) = 0$ the *cuspidaledge curve* of Δ. From [MT] we know that the Milnor numbers $\mu(d)$ and $\mu(c)$ of these plane curve singularities are analytic invariants of the map–germ f.

The number of swallowtails of Δ is also an invariant for f. We denote this by $\#A_3$. To calculate $\#A_3$ for corank 1 map–germs we use the following result from Marar, Montaldi & Ruas, [MMR]:

Proposition 7.1.2 *Let* $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0)$ *be a corank 1 map-germ with* $f(x,y,z) = (x, y, g(x,y,z))$. *Then*

$$\#A_3 = dim_{\mathbb{C}} \mathcal{E}_3 / (g_z, g_{zz}, g_{zzz}) \mathcal{E}_3.$$

In particular, if g *is quasi–homogeneous of weights* w_1, w_2 *and* w_3 *and degree* d*, then*

$$\#A_3 = \frac{(d-w_3)(d-2w_3)(d-3w_3).}{w_1w_2w_3}$$

We also wish to calculate the number of triplepoints which occur in the unfolding of f. To do this we introduce the work of Mond [Mo1, Mo2]. We assume that our map–germ f has smooth critical set and the restriction of the map to this critical set can be parametrized to give a map–germ $g : \mathbf{C}^2 \longrightarrow \mathbf{C}^3$. If we want to look at double points then we consider two sets of coordinates in the source,(x, y) and (x', y') say, i.e. we look at $\mathbf{C}^2 \times \mathbf{C}^2$. Now g is going to have the form $(x, g_1(x, y), g_2(x, y))$. We need the following to be true.

$$\begin{aligned} x - x' &= 0, \\ g_1(x,y) - g_1(x,y') &= 0, \\ g_2(x,y) - g_2(x,y') &= 0. \end{aligned}$$

So we have the ideal I_2 given by

$$I_2 = \langle x - x', \frac{g_1(x,y) - g_1(x,y')}{y - y'}, \frac{g_2(x,y) - g_2(x,y')}{y - y'} \rangle \mathcal{O}_4.$$

We can embed this is $\mathbf{C} \times \mathbf{C}^2$ giving

$$I_2 = \langle x, \frac{g_1(x,y) - g_1(x,y')}{y - y'}, \frac{g_2(x,y) - g_2(x,y')}{y - y'} \rangle \mathcal{O}_3.$$

Now triplepoints live in $\mathbf{C}^2 \times \mathbf{C}^2 \times \mathbf{C}^2$, and similarly we can embed this in $\mathbf{C} \times \mathbf{C}^3$ $((x, y, y', y''))$.

So

$$I_3 = \langle x, h_1, h_2, k_1, k_2 \rangle \mathcal{O}_4$$

where $h_i = \frac{g_i(x,y) - g_i(x,y')}{y - y'}$ and $k_i = \frac{h_i(x,y,y') - h_i(x,y,y'')}{y' - y''}$.

Theorem 7.1.3 (Mond) *A generic deformation of a germ* $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ *with rank 1 at 0, has* $\frac{1}{6} dim \mathcal{O}_4 / I_3(f)_0$ *triple points.*

Call this the number of triple points of a map–germ $F : (\mathbf{R}^3, 0) \longrightarrow (\mathbf{R}^3, 0)$ (where $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^3, 0)$ is the restriction of F to the critical set. So using MAPLE and SINGULAR we can calculate the codimension of these ideals.

7.2 Calculating the Invariants

- $f(x, y, z) = (x, y, yz + z^3)$.

  The critical set for f is given by

  $$\sum f = \left\{ (x, y, z) : y + 3z^2 = 0 \right\}.$$

  Now we need to look at where the critical set is singular, i.e. when $\frac{\partial \sum}{\partial x} = \frac{\partial \sum}{\partial y} = \frac{\partial \sum}{\partial z} = 0$. This is never the case, so $\sum f$ is smooth. This means that we can parametrize $\sum f$ by $(x, -3z^2, z)$, and our F is given by $(x, -3z^2, -2z^3)$. In this case our defining equation G can be found by inspection to be

  $$G(X, Y, Z) = 27Z^2 + 4Y^3 = 0.$$

  So using Prop. 7.1.1 we have

  $$h(x, z) = \frac{54(-2z^3)}{\begin{vmatrix} 1 & 0 \\ 0 & -6z \end{vmatrix}} = \frac{-108z^3}{-6z} = 18z^2.$$

  So f has no double point curve but has cuspidal edge curve $z = 0$ giving $\mu(c) = 0$.

  Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2 \rangle$ with

  $$\begin{aligned} h_1 &= -3z - 3a, \\ h_2 &= -2z^2 - 2az - 2a^2, \\ k_1 &= -3, \\ k_2 &= -2a - 2z - 2b. \end{aligned}$$

  This ideal has codimension 0.

- $f(x, y, z) = (x, y, yz + xz^2 + z^4)$.

The critical set for f is given by

$$\sum f = \left\{(x, y, z) : y + 2xz + 4z^3 = 0\right\}.$$

Again $\sum f$ is smooth and we can parametrize it by $(x, -2xz - 4z^3, z)$, and F is given by $(x, -2xz - 4z^3, -xz^2 - 3z^4)$. So we have the equations

$$\begin{aligned} Y + 2Xz + 4z^3 &= 0 \\ Z + Xz^2 + 3z^4 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X, Y, Z) = 4X^3Y^2 + 27Y^4 + 16X^4Z + 128Z^2X^2 + 144XY^2Z + 256Z^3 = 0.$$

So using Prop. 7.1.1 we have

$$h(x, z) = (x + 6z^2)^2(x + 2z^2).$$

So f has double point curve given by $d(x, z) = x + 2z^2$ and cuspidal edge curve given by $c(x, z) = x + 6z^2$. So we have $\mu(d) = \mu(c) = 0$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 2, y has weight 3 and z has weight 1, and the degree of g is 4 $(f(x, y, z) = (x, y, g(x, y, z))$):

$$\begin{aligned} \#A_3 &= \frac{(4-1)(4-2)(4-3)}{2.3.1} \\ &= 1. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$$\begin{aligned} h_1 &= -4z^2 - 4az - 2x - 4a^2, \\ h_2 &= -3z^3 - 3az^2 - xz - 3za^2 - xa - 3a^3, \\ k_1 &= -4a - 4z - 4b, \\ k_2 &= -3a^2 - 3az - 3ab - 3z^2 - x - 3bz - 3b^2. \end{aligned}$$

This ideal has codimension 1.

- $f(x, y, z) = (x, yz, y^2 + z^2 + xy)$.

  The critical set for f is given by

$$\begin{aligned}\Sigma f &= \left\{(x,y,z) : rk\begin{pmatrix} 1 & 0 & 0 \\ 0 & z & y \\ y & 2y+x & 2z \end{pmatrix} < 3\right\} \\ &= \{(x,y,z) : 2z^2 - 2y^2 - xy = 0\}.\end{aligned}$$

Now this curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, yz, y^2 - z^2 + xy)$.

  The critical set for f is given by

$$\begin{aligned}\Sigma f &= \left\{(x,y,z) : rk\begin{pmatrix} 1 & 0 & 0 \\ 0 & z & y \\ y & 2y+x & -2z \end{pmatrix} < 3\right\} \\ &= \{(x,y,z) : -2z^2 - 2y^2 - xy = 0\}.\end{aligned}$$

  Now this curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (y^2 + x^2)z)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + y^2 + x^2 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (y^2 - x^2)z)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + y^2 - x^2 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 - (y^2 - x^2)z)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 - y^2 + x^2 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 - (y^2 + x^2)z)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 - y^2 - x^2 = 0\right\}.$$

This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x,y,yz + z^4 + x^2z^2)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 4z^3 + 2x^2z = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -4z^3 - 2x^2z, z)$, and F is given by $(x, -4z^3 - 2x^2z, -3z^4 - x^2z^2)$. So we have the equations

$$\begin{aligned} Y + 2X^2z + 4z^3 &= 0 \\ Z + X^2z^2 + 3z^4 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X,Y,Z) = 4X^6Y^2 + 27Y^4 + 16X^8Z + 128Z^2X^4 + 144X^2Y^2Z + 256Z^3 = 0.$$

So using Prop. 7.1.1 we have

$$h(x,z) = (x^2 + 6z^2)^2(x^2 + 2z^2).$$

So f has double point curve given by $d(x,z) = x^2 + 2z^2$ and cuspidal edge curve given by $c(x,z) = x^2 + 6z^2$. So we have $\mu(d) = \mu(c) = 1$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 1, y has weight 3 and z has weight 1, and the degree of g is 4 ($f(x,y,z) = (x,y,g(x,y,z))$):

$$\begin{aligned} \#A_3 &= \tfrac{(4-1)(4-2)(4-3)}{1.3.1} \\ &= 2. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2 \rangle$ with

$$\begin{aligned} h_1 &= -4z^2 - 4az - 2x^2 - 4a^2, \\ h_2 &= -3z^3 - 3az^2 - x^2z - 3za^2 - x^2a - 3a^3, \\ k_1 &= -4a - 4z - 4b, \\ k_2 &= -3a^2 - 3az - 3ab - 3z^2 - x^2 - 3bz - 3b^2. \end{aligned}$$

This ideal has codimension 2.

- $f(x,y,z) = (x,y,yz + z^4 - x^2z^2)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 4z^3 - 2x^2z = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -4z^3 + 2x^2z, z)$, and F is given by $(x, -4z^3 + 2x^2z, -3z^4 + x^2z^2)$. So we have the equations

$$\begin{aligned} Y - 2X^2z + 4z^3 &= 0 \\ Z - X^2z^2 + 3z^4 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X,Y,Z) = 4X^6Y^2 + 27Y^4 + 16X^8Z + 128Z^2X^4 + 144X^2Y^2Z + 256Z^3 = 0.$$

So using Prop. 7.1.1 we have

$$h(x,z) = (x^2 - 6z^2)^2(x^2 - 2z^2).$$

So f has double point curve given by $d(x,z) = x^2 - 2z^2$ and cuspidal edge curve given by $c(x,z) = x^2 - 6z^2$. So we have $\mu(d) = \mu(c) = 1$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 1, y has weight 3 and z has weight 1, and the degree of g is 4 ($f(x,y,z) = (x,y,g(x,y,z))$):

$$\begin{aligned} \#A_3 &= \tfrac{(4-1)(4-2)(4-3)}{1.3.1} \\ &= 2. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$$\begin{aligned} h_1 &= -4z^2 - 4az + 2x^2 - 4a^2, \\ h_2 &= -3z^3 - 3az^2 + x^2z - 3za^2 + x^2a - 3a^3, \\ k_1 &= -4a - 4z - 4b, \\ k_2 &= -3a^2 - 3az - 3ab - 3z^2 + x^2 - 3bz - 3b^2. \end{aligned}$$

This ideal has codimension 2.

- $f(x,y,z) = (x, y, yz + xz^2 + z^5)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 2xz + 5z^4 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -2xz - 5z^4, z)$, and F is given by $(x, -2xz - 5z^4, -xz^2 - 4z^5)$. So we have the equations

$$\begin{aligned} Y + 2Xz + 5z^4 &= 0 \\ Z + Xz^2 + 4z^5 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X,Y,Z) = -27X^4Y^2 + 2250X^2YZ^2 + 1600XY^3Z + 3125Z^4 + 256Y^5 - 108X^5Z.$$

So using Prop. 7.1.1 we have

$$h(x,z) = (27x^2 + 140xz^3 + 200z^6)(x + 10z^3)^2).$$

So f has double point curve given by $d(x,z) = 27x^2 + 140xz^3 + 200z^6$ and cuspidal edge curve given by $c(x,z) = x + 10z^3$. So we have $\mu(d) = 5$ and $\mu(c) = 0$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 3, y has weight 4 and z has weight 1, and the degree of g is 5 ($f(x,y,z) = (x,y,g(x,y,z))$:

$$\begin{aligned} \#A_3 &= \tfrac{(5-1)(5-2)(5-3)}{3.4.1} \\ &= 2. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$h_1 = -5z^3 - 5az^2 - 5za^2 - 2x - 5a^3$,
$h_2 = -4z^4 - 4az^3 - 4a^2z^2 - xz - 4za^3 - xa - 4a^4$,
$k_1 = -5a^2 - 5az - 5ab - 5z^2 - 5bz - 5b^2$,
$k_2 = -4a^3 - 4za^2 - 4a^2b - 4az^2 - 4abz - 4ab^2 - 4z^3 - x - 4bz^2 - 4zb^2 - 4b^3$.

This ideal has codimension 4.

- $f(x,y,z) = (x, yz + y^3, y^2 + z^2 + xy + xz)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : (z+3y^2)(2z+x) - y(2y+x) = 0\right\}.$$

Now $\frac{\partial \sum}{\partial x} = z + 3y^2 - y$, $\frac{\partial \sum}{\partial y} = 6y(2z+x) - 4y - x$ and $\frac{\partial \sum}{\partial z} = 4z + x + 6y^2$. The only solution for these all to be identically 0 is $x = y = z = 0$. So $\sum f$ is singular at the origin.

- $f(x,y,z) = (x, yz + y^3 - x^2y, y^2 + z^2 + xy + xz)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : (z+3y^2-x^2)(2z+x) - y(2y+x) = 0\right\}.$$

  Now $\frac{\partial \sum}{\partial x} = -2x(2z+x) + z + 3y^2 - x^2 - y$, $\frac{\partial \sum}{\partial y} = 6y(2z+x) - 4y - x$ and $\frac{\partial \sum}{\partial z} = 4z + x + 6y^2 - 2x^2$. The only solutions for these all to be identically 0 are $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (y^2 + x^3)z)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + y^2 + x^3 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (x^2 - y^3)z)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + x^2 - y^3 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, yz, y^2 + xy + xz + z^3 + ayz^3)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : z(x + 3z^2 + 3ayz^2) - y(2y+x) = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, yz + z^4 + x^3z^2)$.

The critical set for f is given by

$$\sum f = \left\{(x, y, z) : y + 4z^3 + 2x^3 z = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -4z^3 - 2x^3 z, z)$, and F is given by $(x, -4z^3 - 2x^3 z, -3z^4 - x^3 z^2)$. So we have the equations

$$\begin{aligned} Y + 2X^3 z + 4z^3 &= 0 \\ Z + X^3 z^2 + 3z^4 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X, Y, Z) = 4X^9Y^2 + 27Y^4 + 16X^{12}Z + 128Z^2X^6 + 144X^3Y^2Z + 256Z^3 = 0.$$

So using Prop. 7.1.1 we have

$$h(x, z) = (x^3 + 6z^2)^2(x^3 + 2z^2).$$

So f has double point curve given by $d(x, z) = x^3 + 2z^2$ and cuspidal edge curve given by $c(x, z) = x^3 + 6z^2$. So we have $\mu(d) = \mu(c) = 2$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 2, y has weight 9 and z has weight 3, and the aegree of g is 12 $(f(x, y, z) = (x, y, g(x, y, z))$:

$$\begin{aligned} \#A_3 &= \frac{(12-3)(12-6)(12-9)}{2.9.3} \\ &= 3. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$$\begin{aligned} h_1 &= -4z^2 - 4az - 2x^3 - 4a^2, \\ h_2 &= -3z^3 - 3az^2 - x^3 z - 3za^2 - x^3 a - 3a^3, \\ k_1 &= -4a - 4z - 4b, \\ k_2 &= -3a^2 - 3az - 3ab - 3z^2 - x^3 - 3bz - 3b^2. \end{aligned}$$

This ideal has codimension 3.

- $f(x, y, z) = (x, y, yz^2 + x^2 z + z^4 + z^5)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 2yz + x^2 + 4z^3 + 5z^4 = 0\right\}.$$

This is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, yz^2 + x^2 z + z^4 - z^5)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 2yz + x^2 + 4z^3 - 5z^4 = 0\right\}.$$

  This is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, yz + xz^3 + z^5 + z^6)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 3xz^2 + 5z^4 + 6z^5 = 0\right\}.$$

  So $\sum f$ is smooth and we can parametrize it by $(x, -3xz^2 - 5z^4 - 6z^5, z)$, and F is given by $(x, -3xz^2 - 5z^4 - 6z^5, -2xz^3 - 4z^5 - 5z^6)$. So we have the equations

$$\begin{aligned} Y + 3Xz^2 + 5z^4 + 6z^5 &= 0 \\ Z + 2Xz^3 + 4z^5 + 5z^6 &= 0. \end{aligned}$$

  Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

  $G(X,Y,Z) = 3125Y^6 + (2000X + 256)Y^5 + (-900X^3 - 128X^2 - 320Z - 2250ZX)Y^4 + (1350X^3Z + 410Z^2 + 108X^5 + 208ZX^2 + 16X^4 + 27000Z^2X)Y^3 + (-540Z^3 + 2000Z^2X + 15417Z^2X^2)Y^2 + (-2250Z^3X + 38880Z^4 - 15552Z^3X^2 - 6318X^4Z^2 - 900Z^2X^3)Y + 108X^5Z^2 + 46656Z^5 + 3125Z^4 + 27000Z^4X + 34992Z^4X^2 + 1350X^3Z^3 + 8748X^4Z^3 + 729X^6Z^2$.

  So using Prop. 7.1.1 we have

  $h(x,z) = -2(27x^2 + 4x + 81xz^2 + 8z^2 + 108xz^3 + 12z^3 + 63z^4 + 162z^5 + 108z^6)(x^2 + 15xz^2 - 40xz^3 + 25z^4 - 50z^5 - 100z^6)(3x + 10z^2 + 15z^3)^2z^2$.

  So f has double point curve given by $d(x,z) = (27x^2 + 4x + 81xz^2 + 8z^2 + 108xz^3 + 12z^3 + 63z^4 + 162z^5 + 108z^6)(x^2 + 15xz^2 - 40xz^3 + 25z^4 - 50z^5 - 100z^6)$ and cuspidal edge curve given by $c(x,z) = (3x + 10z^2 + 15z^3)z$. So we have $\mu(d) = 22$ and $\mu(c) = 1$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 3.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with
$h_1 = -6z^4 - 5z^3 - 6az^3 - 5az^2 - 6a^2z^2 - 3xz - 5za^2 - 6za^3 - 3xa - 5a^3 - 6a^4$,
$h_2 = -5z^5 - 4z^4 - 5z^4a - 4az^3 - 5z^3a^2 - 2xz^2 - 4a^2z^2 - 5z^2a^3 - 2zxa - 4za^3 - 5za^4 - 2xa^2 - 4a^4 - 5a^5$,
$k_1 = -6a^3 - 6za^2 - 5a^2 - 6a^2b - 5az - 6az^2 - 6abz - 5ab - 6ab^2 - 3x - 5z^2 - 6z^3 - 5bz - 6bz^2 - 6zb^2 - 5b^2 - 6b^3$,
$k_2 = -4z^3 - 4a^3 - 5z^4 - 2xz - 2xa - 4az^2 - 4za^2 - 4bz^2 - 4zb^2 - 2xb - 5a^4 - 4b^3 - 4abz - 5a^2bz - 5az^3 - 5a^2z^2 - 5za^3 - 5b^4 - 5bz^3 - 5b^2z^2 - 5zb^3 - 4a^2b - 4ab^2 - 5abz^2 - 5azb^2 - 5a^3b - 5a^2b^2 - 5ab^3$.

This ideal has codimension 6.

- $f(x, y, z) = (x, y, yz + xz^2 + z^6 + z^8 + az^9)$.

The critical set for f is given by

$$\sum f = \left\{(x, y, z) : y + 2xz + 6z^5 + 8z^7 + 9az^8 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -2xz - 6z^5 - 8z^7 - 9az^8, z)$, and F is given by $(x, -2xz - 6z^5 - 8z^7 - 9az^8, -xz^2 - 5z^6 - 7z^8 - 8az^9)$. So we have the equations

$$\begin{aligned} Y + 2Xz + 6z^5 + 8z^7 + 9az^8 &= 0 \\ Z + Xz^2 + 5z^6 + 7z^8 + 8az^9 &= 0 \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation.

his gives $\mu(d) = 22$ and $\mu(c) = 0$.

To calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 3.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$h_1 = -9z^7 - 8z^6 - 9z^6a - 8z^5a - 9z^5a^2 - 6z^4 - 8z^4a^2 - 9z^4a^3 - 6az^3 - 8z^3a^3 - 9z^3a^4 - 6a^2z^2 - 8z^2a^4 - 9z^2a^5 - 6za^3 - 8za^5 - 9za^6 - 2x - 6a^4 - 8a^6 - 9a^7$,
$h_2 = -xz - xa - 5z^5 - 5a^5 - 5z^4a - 5z^3a^2 - 5z^2a^3 - 5za^4 - 8z^8 - 7z^7 - 7a^7 - 8a^8 - 7z^6a - 7z^5a^2 - 7z^4a^3 - 7z^3a^4 - 7z^2a^5 - 7za^6 - 8z^7a - 8z^6a^2 - 8z^5a^3 - 8z^4a^4 - 8z^3a^5 - 8z^2a^6 - 8za^7$,
$k_1 = -9az^4b - 8a^3bz - 6z^3 - 9a^3zb^2 - 9a^4bz - 6a^3 - 6az^2 - 6za^2 - 6bz^2 - 6zb^2 - 9a^3bz^2 - 9a^2bz^3 - 6b^3 - 6abz - 8z^5 - 8a^5 - 6a^2b - 6ab^2 - 9z^6 - 9a^6 - 8z^4a - 8z^3a^2 - 8z^2a^3 - 8za^4 - 8b^5 - 8z^4b - 8z^3b^2 - 8z^2b^3 - 8zb^4 - 9z^5a - 9z^4a^2 - 9z^3a^3 - 9z^2a^4 - 9za^5 - 8a^2bz^2 - 9a^2b^2z^2 - 8a^2zb^2 - 9a^2zb^3 - 8abz^3 - 9az^3b^2 - 8ab^2z^2 - 9az^2b^3 - 8azb^3 - 9azb^4 - 9b^6 - 9z^5b - 9z^4b^2 - 9z^3b^3 - 9z^2b^4 - 9zb^5 - 9a^5b - 8a^4b - 9a^4b^2 - 8a^3b^2 - 9a^3b^3 - 8a^2b^3 - 9a^2b^4 - 8ab^4 - 9ab^5$,
$k_2 = -7az^4b - 8a^3bz^3 - x - 8a^5bz - 7a^3zb^2 - 7a^4bz - 5z^4 - 5a^4 - 7a^3bz^2 - 7a^2bz^3 - 5a^2bz - 5az^3 - 5a^2z^2 - 5za^3 - 5b^4 - 5bz^3 - 5b^2z^2 - 5zb^3 - 8a^4bz^2 - 7z^6 - 7a^6 - 5abz^2 - 5azb^2 - 5a^3b - 5a^2b^2 - 5ab^3 - 8z^7 - 8a^7 - 8z^6a - 7z^5a - 8z^5a^2 - 7z^4a^2 - 8z^4a^3 - 7z^3a^3 - 8z^3a^4 - 7z^2a^4 - 8z^2a^5 - 7za^5 - 8za^6 - 7a^2b^2z^2 - 7a^2zb^3 - 7az^3b^2 - 7az^2b^3 - 7azb^4 - 8a^4zb^2 - 8a^3b^2z^2 - 8a^3zb^3 - 8a^2z^4b - 8a^2z^3b^2 - 8a^2z^2b^3 - 8a^2zb^4 - 8az^5b - 8az^4b^2 - 8az^3b^3 - 7b^6 - 8b^7 - 8z^6b - 7z^5b - 8z^5b^2 - 7z^4b^2 - 8z^4b^3 - 7z^3b^3 - 8z^3b^4 - 7z^2b^4 - 8z^2b^5 - 7zb^5 - 8zb^6 - 7a^5b - 7a^4b^2 - 7a^3b^3 - 7a^2b^4 - 7ab^5 - 8a^6b - 8a^5b^2 - 8a^4b^3 - 8a^3b^4 - 8a^2b^5 - 8ab^6 - 8az^2b^4 - 8azb^5$.

This ideal has codiimension 10.

- $f(x, y, z) = (x, y, yz + xz^2 + z^6 - z^8 + az^9)$.

The critical set for f is given by

$$\sum f = \left\{(x, y, z) : y + 2xz + 6z^5 - 8z^7 + 9az^8 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -2xz - 6z^5 + 8z^7 - 9az^8, z)$, and F is given by $(x, -2xz - 6z^5 + 8z^7 - 9az^8, -xz^2 - 5z^6 + 7z^8 - 8az^9)$. So we have the equations

$$\begin{aligned} Y + 2Xz + 6z^5 - 8z^7 + 9az^8 &= 0 \\ Z + Xz^2 + 5z^6 - 7z^8 + 8az^9 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation.

This gives $\mu(d) = 22$ and $\mu(c) = 0$.

To calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 3.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with
$h_1 = -9z^7+8z^6-9z^6a+8z^5a-9z^5a^2-6z^4+8z^4a^2-9z^4a^3-6az^3+8z^3a^3-9z^3a^4-6a^2z^2+8z^2a^4-9z^2a^5-6za^3+8za^5-9za^6-2x-6a^4+8a^6-9a^7$,
$h_2 = -xz-xa-5z^5-5a^5-5z^4a-5z^3a^2-5z^2a^3-5za^4-8z^8+7z^7+7a^7-8a^8+7z^6a+7z^5a^2+7z^4a^3+7z^3a^4+7z^2a^5+7za^6-8z^7a-8z^6a^2-8z^5a^3-8z^4a^4-8z^3a^5-8z^2a^6-8za^7$,
$k_1 = -9az^4b+8a^3bz-6z^3-9a^3zb^2-9a^4bz-6a^3-6az^2-6za^2-6bz^2-6zb^2-9a^3bz^2-9a^2bz^3-6b^3-6abz+8z^5+8a^5-6a^2b-6ab^2-9z^6-9a^6+8z^4a+8z^3a^2+8z^2a^3+8za^4+8b^5+8z^4b+8z^3b^2+8z^2b^3+8zb^4-9z^5a-9z^4a^2-9z^3a^3-9z^2a^4-9za^5+8a^2bz^2-9a^2b^2z^2+8a^2zb^2-9a^2zb^3+8abz^3-9az^3b^2+8ab^2z^2-9az^2b^3+8azb^3-9azb^4-9b^6-9z^5b-9z^4b^2-9z^3b^3-9z^2b^4-9zb^5-9a^5b+8a^4b-9a^4b^2+8a^3b^2-9a^3b^3+8a^2b^3-9a^2b^4+8ab^4-9ab^5$,
$k_2 = 7az^4b-8a^3bz^3-x-8a^5bz+7a^3zb^2+7a^4bz-5z^4-5a^4+7a^3bz^2+7a^2bz^3-5a^2bz-5az^3-5a^2z^2-5za^3-5b^4-5bz^3-5b^2z^2-5zb^3-8a^4bz^2+7z^6+7a^6-5abz^2-5azb^2-5a^3b-5a^2b^2-5ab^3-8z^7-8a^7-8z^6a+7z^5a-8z^5a^2+7z^4a^2-8z^4a^3+7z^3a^3-8z^3a^4+7z^2a^4-8z^2a^5+7za^5-8za^6+7a^2b^2z^2+7a^2zb^3+7az^3b^2+7az^2b^3+7azb^4-8a^4zb^2-8a^3b^2z^2-8a^3zb^3-8a^2z^4b-8a^2z^3b^2-8a^2z^2b^3-8a^2zb^4-8az^5b-8az^4b^2-8az^3b^3+7b^6-8b^7-8z^6b+7z^5b-8z^5b^2+7z^4b^2-8z^4b^3+7z^3b^3-8z^3b^4+7z^2b^4-8z^2b^5+7zb^5-8zb^6+7a^5b+7a^4b^2+7a^3b^3+7a^2b^4+7ab^5-8a^6b-8a^5b^2-8a^4b^3-8a^3b^4-8a^2b^5-8ab^6-8az^2b^4-8azb^5$.

This ideal has codimension 3.

- $f(x,y,z) = (x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$.
  The critical set for f is given by
  $$\sum f = \left\{(x,y,z) : \pm 2z(z \pm 3y^2 + ax^2) - y(2y + 3y^2 + x^2) = 0\right\}.$$
  Singular when $x = y = z = 0$ and at 3 other points.

- $f(x,y,z) = (x, yz \pm y^3 + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$.
  The critical set for f is given by
  $$\sum f = \left\{(x,y,z) : \pm 2z(z \pm 3y^2 + ax^2) - y(2y + 3y^2 - x^2) = 0\right\}.$$
  Singular when $x = y = z = 0$ and at 3 other points.

- $f(x,y,z) = (x, yz + ax^2y, y^2 \pm z^2 + y^3 + x^2y)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : \pm 2z(z+ax^2) - y(2y+3y^2+x^2) = 0\right\}.$$

Singular when $x = y = z = 0$ and at 3 other points.

- $f(x,y,z) = (x, yz + ax^2y, y^2 \pm z^2 + y^3 - x^2y)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : \pm 2z(z+ax^2) - y(2y+3y^2-x^2) = 0\right\}.$$

  Singular when $x = y = z = 0$ and at 3 other points.

- $f(x,y,z) = (x, yz + y^3 - \frac{3}{16}x^2y + x^3y, y^2 + z^2 + xy + xz)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : (2z+x)(z+3y^2 - \frac{3}{16}x^2 + x^3) - y(2y+x) = 0\right\}.$$

  Singular when $x = y = z = 0$ and at 1 other point.

- $f(x,y,z) = (x, yz + y^3 - \frac{3}{16}x^2y - x^3y, y^2 + z^2 + xy + xz)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : (2z+x)(z+3y^2 - \frac{3}{16}x^2 - x^3) - y(2y+x) = 0\right\}.$$

  Singular when $x = y = z = 0$ and at 1 other point.

- $f(x,y,z) = (x, yz, y^2 + xz + z^3 + z^4 + ayz^3)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : z(x+3z^2+4z^3+3ayz^2) - y(2y+az^3) = 0\right\}.$$

  Singular when $x = y = z = 0$.

- $f(x,y,z) = (x, yz, y^2 + xz + z^3 + yz^3)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : z(x + 3z^2 + 3yz^2) - y(2y + z^3) = 0\right\}.$$

Singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (y^2 + x^4)z)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + y^2 + x^4 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 + (y^2 - x^4)z)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 + y^2 - x^4 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 - (x^2 - y^4)z)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 - x^2 + y^4 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, z^3 - (x^2 + y^4)z)$.
  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : 3z^2 - x^2 - y^4 = 0\right\}.$$

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, yz + x^2y + y^5, y^2 + z^2 + xy + xz)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : (z + x^2 + 5y^4)(2z + x) - y(2y + x) = 0\right\}.$$

This curve is singular when $x = y = z = 0$ at at one other point.

- $f(x, y, z) = (x, yz + x^2y - y^5, y^2 + z^2 + xy + xz)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : (z + x^2 - 5y^4)(2z + x) - y(2y + x) = 0\right\}.$$

  This curve is singular when $x = y = z = 0$ at at one other point.

- $f(x, y, z) = (x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : z(3z^2 + 2xz + ax^2 + 5bz^4) - y(2y + x) = 0\right\}.$$

  This curve is singular when $x = y = z = 0$ at at one other point.

- $f(x, y, z) = (x, y, yz + z^4 + x^4z^2)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : y + 4z^3 + 2x^4z = 0\right\}.$$

  So $\sum f$ is smooth and we can parametrize it by $(x, -4z^3 - 2x^4z, z)$, and F is given by $(x, -4z^3 - 2x^4z, -3z^4 - x^4z^2)$. So we have the equations

$$\begin{aligned} Y + 2X^4z + 4z^3 &= 0 \\ Z + X^4z^2 + 3z^4 &= 0. \end{aligned}$$

  Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X, Y, Z) = 4X^{12}Y^2 + 27Y^4 + 16X^{16}Z + 128Z^2X^8 + 144X^4Y^2Z + 256Z^3 = 0.$$

  So using Prop. 7.1.1 we have

$$h(x,z) = (x^4+6z^2)^2(x^4+2z^2).$$

So f has double point curve given by $d(x,z) = x^4+2z^2$ and cuspidal edge curve given by $c(x,z) = x^4+6z^2$. So we have $\mu(d) = \mu(c) = 3$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 1, y has weight 6 and z has weight 2, and the degree of g is 8 where $(f(x,y,z) = (x,y,g(x,y,z))$:

$$\begin{aligned} \#A_3 &= \tfrac{(8-2)(8-4)(8-6)}{1.6.2} \\ &= 4. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$$h_1 = -4z^2 - 4az - 2x^4 - 4a^2, \tag{7.1}$$
$$h_2 = -3z^3 - 3az^2 - x^4z - 3za^2 - x^4a - 3a^3, \tag{7.2}$$
$$k_1 = -4a - 4z - 4b, \tag{7.3}$$
$$k_2 = -3a^2 - 3az - 3ab - 3z^2 - x^4 - 3bz - 3b^2. \tag{7.4}$$

This ideal has codimension 4.

- $f(x,y,z) = (x,y,yz+z^4-x^4z^2)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 4z^3 - 2x^4z = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -4z^3+2x^4z, z)$, and F is given by $(x, -4z^3+2x^4z, -3z^4+x^4z^2)$. So we have the equations

$$\begin{aligned} Y - 2X^4z + 4z^3 &= 0 \\ Z - X^4z^2 + 3z^4 &= 0 \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$$G(X,Y,Z) = 4X^{12}Y^2+27Y^4+16X^{16}Z+128Z^2X^8+144X^4Y^2Z+256Z^3 = 0.$$

So using Prop. 7.1.1 we have

$$h(x,z) = (x^4 - 6z^2)^2(x^4 - 2z^2).$$

So f has double point curve given by $d(x,z) = x^4 - 2z^2$ and cuspidal edge curve given by $c(x,z) = x^4 - 6z^2$. So we have $\mu(d) = \mu(c) = 3$. Finally to calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 1, y has weight 6 and z has weight 2, and the degree of g is 8 where $(f(x,y,z) = (x,y,g(x,y,z))$:

$$\begin{aligned} \#A_3 &= \tfrac{(8-2)(8-4)(8-6)}{1.6.2} \\ &= 4. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2 \rangle$ with

$$h_1 = -4z^2 - 4az + 2x^4 - 4a^2, \tag{7.5}$$

$$h_2 = -3z^3 - 3az^2 + x^4 z - 3za^2 + x^4 a - 3a^3, \tag{7.6}$$

$$k_1 = -4a - 4z - 4b, \tag{7.7}$$

$$k_2 = -3a^2 - 3az - 3ab - 3z^2 + x^4 - 3bz - 3b^2. \tag{7.8}$$

This ideal has codimension 4.

- $f(x,y,z) = (x, yz, y^2 + xy + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6)$.

  The critical set for f is given by

  $$\Sigma f = \{(x,y,z) : z(x + 4z^3 + 3yz^2 + 5az^4 + 4byz^3 + 6cz^5) - y(2y + x + z^3 + bz^4) = 0\}.$$

  This curve is singular when $x = y = z = 0$ at at one other point.

- $f(x,y,z) = (x, yz, y^2 + xy + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6)$.

  The critical set for f is given by

  $$\Sigma f = \{(x,y,z) : z(x + 4z^3 - 3yz^2 + 5az^4 + 4byz^3 + 6cz^5) - y(2y + x - z^3 + bz^4) = 0\}.$$

  This curve is singular when $x = y = z = 0$ at at one other point.

- $f(x,y,z) = (x, yz, y^2 + xz + z^4 + yz^3 + az^5 + byz^4 + cz^6)$.

  The critical set for f is given by

$\sum f$
$= \{(x,y,z) : z(x + 4z^3 + 3yz^2 + 5az^4 + 4byz^3 + 6cz^5) - y(2y + z^3 + bz^4) = 0\}$.

This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, yz, y^2 + xz + z^4 - yz^3 + az^5 + byz^4 + cz^6)$.

  The critical set for f is given by

  $\sum f = \{(x,y,z) : z(x + 4z^3 - 3yz^2 + 5az^4 + 4byz^3 + 6cz^5) - y(2y - z^3 + bz^4) = 0\}$.

  This curve is singular when $x = y = z = 0$.

- $f(x,y,z) = (x, y, yz + xz^3 + z^5)$.

  The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 3xz^2 + 5z^4 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -3xz^2 - 5z^4, z)$, and F is given by $(x, -3xz^2 - 5z^4, -2xz^3 - 4z^5)$. So we have the equations

$$\begin{aligned} Y + 3Xz^2 + 5z^4 &= 0 \\ Z + 2Xz^3 + 4z^5 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

$G(X,Y,Z) = 16X^4Y^3 - 128X^2Y^4 - 900X^3YZ^2 + 2000XY^2Z^2 + 256Y^5 + 3125Z^4 + 108X^5Z^2$.

So using Prop. 7.1.1 we have

$$h(x,z) = (x + 2z^2)(25z^4 + 15xz^2 + x^2)(3x + 10z^2)^2 z^2.$$

So f has double point curve given by $d(x,z) = (x+2z^2)(25z^4 + 15xz^2 + x^2)$ and cuspidal edge curve given by $c(x,z) = z(3x + 10z^2)$. So we have $\mu(d) = 10,\ \mu(c) = 1$.

To calculate the number of swallowtails we use Prop. 7.1.2.

So for f we have x has weight 2, y has weight 4 and z has weight 1, and the degree of g is 5 where $(f(x,y,z) = (x,y,g(x,y,z))$:

$$\begin{aligned} \#A_3 &= \tfrac{(5-1)(5-2)(5-3)}{2.4.1} \\ &= 3. \end{aligned}$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$h_1 = -5z^3 - 5az^2 - 3xz - 5za^2 - 3xa - 5a^3$,
$h_2 = -4z^4 - 4az^3 - 2xz^2 - 4a^2z^2 - 2zxa - 4za^3 - 2xa^2 - 4a^4$,
$k_1 = -5a^2 - 5az - 5ab - 5z^2 - 3x - 5bz - 5b^2$,
$k_2 = -4a^3 - 4za^2 - 4a^2b - 4az^2 - 2xa - 4abz - 4ab^2 - 4z^3 - 2xz - 4bz^2 - 2xb - 4zb^2 - 4b^3$.

This ideal has codimension 6.

- $f(x,y,z) = (x, y, yz + x^2z^2 \pm z^5 + z^6 + az^7)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 2x^2z \pm 5z^4 + 6z^5 + 7az^6 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -2x^2z \mp 5z^4 - 6z^5 - 7az^6, z)$, and F is given by $(x, -2x^2z \mp 5z^4 - 6z^5 - 7az^6, -x^2z^2 \mp 4z^5 - 5z^6 - 6az^7)$. So we have the equations

$$\begin{aligned} Y + 2X^2z \pm 5z^4 + 6z^5 + 7az^6 &= 0 \\ Z + X^2z^2 \pm 4z^5 + 5z^6 + 6az^7 &= 0. \end{aligned}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation.

This gives $\mu(d) = 15$, $\mu(c) = 2$. To calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 4.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with
$h_1 = -7z^5 - 6z^4 - 7z^4a - 5z^3 - 6az^3 - 7z^3a^2 - 5az^2 - 6a^2z^2 - 7z^2a^3 - 5za^2 - 6za^3 - 7za^4 - 2x^2 - 5a^3 - 6a^4 - 7a^5$,
$h_2 = -6z^6 - 5z^5 - 6z^5a - 4z^4 - 5z^4a - 6z^4a^2 - 4az^3 - 5z^3a^2 - 6z^3a^3 - 4a^2z^2 - 5z^2a^3 - 6z^2a^4 - x^2z - 4za^3 - 5za^4 - 6za^5 - x^2a - 4a^4 - 5a^5 - 6a^6$,
$k_1 = -5z^2 - 5az - 5bz - 6z^3 - 5a^2 - 6a^3 - 5b^2 - 7z^4 - 6az^2 - 6za^2 - 6bz^2 - 6zb^2 - 7a^4 - 6b^3 - 5ab - 6abz - 7a^2bz - 7az^3 - 7a^2z^2 - 7za^3 - 7b^4 - 7bz^3 - 7b^2z^2 - 7zb^3 - 6a^2b - 6ab^2 - 7abz^2 - 7azb^2 - 7a^3b - 7a^2b^2 - 7ab^3$,
$k_2 = -6a^3bz - 4z^3 - 4a^3 - 5z^4 - 4az^2 - 4za^2 - 4bz^2 - 4zb^2 - 5a^4 - 4b^3 - x^2 - 4abz - 5a^2bz - 6z^5 - 5az^3 - 5a^2z^2 - 5za^3 - 6a^5 - 5b^4 - 5bz^3 - 5b^2z^2 - 5zb^3 - 4a^2b - 4ab^2 - 6z^4a - 6z^3a^2 - 6z^2a^3 - 6za^4 - 5abz^2 - 5azb^2 - 6b^5 - 6z^4b - 6z^3b^2 - 6z^2b^3 - 6zb^4 - 5a^3b - 5a^2b^2 - 5ab^3 - 6a^2bz^2 - 6a^2zb^2 - 6abz^3 - 6ab^2z^2 - 6azb^3 - 6a^4b - 6a^3b^2 - 6a^2b^3 - 6ab^4$.

This ideal has codimension 8.

- $f(x,y,z) = (x, y, yz + x^2z^2 \pm z^5 - z^6 + az^7)$.

The critical set for f is given by

$$\sum f = \left\{(x,y,z) : y + 2x^2z \pm 5z^4 - 6z^5 + 7az^6 = 0\right\}.$$

So $\sum f$ is smooth and we can parametrize it by $(x, -2x^2z \mp 5z^4 + 6z^5 - 7az^6, z)$, and F is given by $(x, -2x^2z \mp 5z^4 + 6z^5 - 7az^6, -x^2z^2 \mp 4z^5 + 5z^6 - 6az^7)$. So we have the equations

$$\begin{array}{rcl} Y + 2X^2z \pm 5z^4 - 6z^5 + 7az^6 & = & 0 \\ Z + X^2z^2 \pm 4z^5 - 5z^6 + 6az^7 & = & 0. \end{array}$$

Using MAPLE to calculate the resultant of these two equations we obtain our defining equation.

This gives $\mu(d) = 15,\ \mu(c) = 2$.

To calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}} \mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 4.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2 \rangle$ with
$h_1 = -7z^5 + 6z^4 - 7z^4a - 5z^3 + 6az^3 - 7z^3a^2 - 5az^2 + 6a^2z^2 - 7z^2a^3 - 5za^2 + 6za^3 - 7za^4 - 2x^2 - 5a^3 + 6a^4 - 7a^5$,
$h_2 = -6z^6 + 5z^5 - 6z^5a - 4z^4 + 5z^4a - 6z^4a^2 - 4az^3 + 5z^3a^2 - 6z^3a^3 - 4a^2z^2 + 5z^2a^3 - 6z^2a^4 - x^2z - 4za^3 + 5za^4 - 6za^5 - x^2a - 4a^4 + 5a^5 - 6a^6$,
$k_1 = -5z^2 - 5az - 5bz + 6z^3 - 5a^2 + 6a^3 - 5b^2 - 7z^4 + 6az^2 + 6za^2 + 6bz^2 + 6zb^2 - 7a^4 + 6b^3 - 5ab + 6abz - 7a^2bz - 7az^3 - 7a^2z^2 - 7za^3 - 7b^4 - 7bz^3 - 7b^2z^2 - 7zb^3 + 6a^2b + 6ab^2 - 7abz^2 - 7azb^2 - 7a^3b - 7a^2b^2 - 7ab^3$,
$k_2 = -6a^3bz - 4z^3 - 4a^3 + 5z^4 - 4az^2 - 4za^2 - 4bz^2 - 4zb^2 + 5a^4 - 4b^3 - x^2 - 4abz + 5a^2bz - 6z^5 + 5az^3 + 5a^2z^2 + 5za^3 - 6a^5 + 5b^4 + 5bz^3 + 5b^2z^2 + 5zb^3 - 4a^2b - 4ab^2 - 6z^4a - 6z^3a^2 - 6z^2a^3 - 6za^4 + 5abz^2 + 5azb^2 - 6b^5 - 6z^4b - 6z^3b^2 - 6z^2b^3 - 6zb^4 + 5a^3b + 5a^2b^2 + 5ab^3 - 6a^2bz^2 - 6a^2zb^2 - 6abz^3 - 6ab^2z^2 - 6azb^3 - 6a^4b - 6a^3b^2 - 6a^2b^3 - 6ab^4$.

This ideal has codimension 8.

- $f(x,y,z) = (x, y, yz^2 + x^2z + z^4 + z^7)$.

The critical set for f is given by

$$\sum f = \left\{(x, y, z) : 2yz + x^2 + 4z^3 + 7z^6 = 0\right\}.$$

This is singular when $x = y = z = 0$.

- $f(x, y, z) = (x, y, yz^2 + x^2 z + z^4 - z^7)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : 2yz + x^2 + 4z^3 - 7z^6 = 0\right\}.$$

  This is singular when $x = y = z = 0$.

- $f(x, y, z) = (x, y, yz + xz^2 + z^6 + z^9)$.
  The critical set for f is given by

$$\sum f = \left\{(x, y, z) : z + 2xz + 6z^5 + 9z^8 = 0\right\}.$$

  So $\sum f$ is smooth and we can parametrize it by $(x, -2xz - 6z^5 - 9z^8, z)$, and F is given by $(x, -2xz - 6z^5 - 9z^8, -xz^2 - 5z^6 - 8z^9)$. So we have the equations

$$\begin{aligned} Y + 2Xz + 6z^5 + 9z^8 &= 0 \\ Z + Xz^2 + 5z^6 + 8z^9 &= 0. \end{aligned}$$

  Using MAPLE to calculate the resultant of these two equations we obtain our defining equation:

  $G(X, Y, Z) = 145800Z^3Y^3 + 2646X^4Y^4 - 13500X^2Y^5 + 7873200Z^4X^3 - 208324872Z^5X^3 - 3031182Z^4Y^3 - 6912X^5Y^2 + 592704X^6Z^2 - 27648X^6Z + 1728000ZY^6 - 11239424X^3Y^6 - 7372800XY^7 - 99837360Z^2X^3Y^3 - 123294312Z^5XY - 154028952Z^2X^5Y^2 - 114082668Z^4X^2Y^2 - 80607744ZX^4Y^4 - 52780032ZX^2Y^5 + 283553298Z^4X^4Y - 73903104Z^3XY^4 + 768144384X^3Y^3Z^3 - 1259712Z^5 - 84375Y^6 + 25509168Z^6 - 2420208X^7YZ + 169869312XY^7Z + 272097792Z^5Y^3 - 49009212X^6Z^3 - 11943936Z^2Y^6 - 605052X^6Y^3 - 823543X^8Y^2 - 3294172X^9Z + 16777216Y^9 + 387420489Z^8 + 803538792Z^6XY + 585252864Z^2X^2Y^5 - 172186884Z^7 + 24065748Z^3X^2Y^2 - 29148336Z^3X^4Y + 12470760Z^2XY^4 + 629856Z^4XY - 607500ZXY^4 - 1166400Z^2X^2Y^2 - 92880ZX^3Y^3 - 155520Z^2X^4Y - 373248Z^3X^3 + 158760X^5Y^2Z$

  So using Prop. 7.1.1 we have

$$h(x,z) = -2(-58320z^{15}+30664656z^{18}+14070546x^5z^4+291600z^{12}+15448740552z^{27}+71464091616z^{30}+162976499712z^{33}+39053988z^{21}+1111237299z^{24}-198288x^2z^7+12042108x^2z^{10}-373248xz^{11}+30909600xz^{14}-157636044xz^{17}+201715163136z^{36}+130606940160z^{39}+34828517376z^{42}+99164528640xz^{32}+31926140928xz^{35}+341156448x^4z^{14}+4880030976x^3z^{18}+2693357568x^3z^{21}+30703380480x^2z^{25}+12496343040x^2z^{28}+823543x^6+233280xz^8+2131596x^3z^6-191996244x^2z^{13}+1111055778xz^{20}+92990241x^4z^8+315727092x^3z^{12}+2721137328x^3z^{15}+671204637x^2z^{16}+9721419120x^2z^{19}+27275704128x^2z^{22}+18701456904xz^{23}+70812610560xz^{26}+120851730432xz^{29}+402634152z^{11}x^4-15788952x^4z^5-81685476x^3z^9+137592x^4z^2-1210104x^5z-30240z^3x^3+24589224z^7x^5+66096z^4x^2+6912x^3)(x+15z^4+36z^7)^2$$

So f has double point curve given by $d(x,z) = -2(-58320z^{15}+30664656z^{18}+14070546x^5z^4+291600z^{12}+15448740552z^{27}+71464091616z^{30}+162976499712z^{33}+39053988z^{21}+1111237299z^{24}-198288x^2z^7+12042108x^2z^{10}-373248xz^{11}+30909600xz^{14}-157636044xz^{17}+201715163136z^{36}+130606940160z^{39}+34828517376z^{42}\ 99164528640xz^{32}+31926140928xz^{35}+341156448x^4z^{14}+4880030976x^3z^{18}+2693357568x^3z^{21}+30703380480x^2z^{25}+12496343040x^2z^{28}+823543x^6+233280xz^8+2131596x^3z^6-191996244x^2z^{13}+1111055778xz^{20}+92990241x^4z^8+315727092x^3z^{12}+2721137328x^3z^{15}+671204637x^2z^{16}+9721419120x^2z^{19}+27275704128x^2z^{22}+18701456904xz^{23}+70812610560xz^{26}+120851730432xz^{29}+402634152z^{11}x^4-15788952x^4z^5-81685476x^3z^9+137592x^4z^2-1210104x^5z-30240z^3x^3+24589224z^7x^5+66096z^4x^2+6912x^3)$ and cuspidal edge curve given by $c(x,z) = x+15z^4+36z^7$. So we have $\mu(d) = 22,\ \mu(c) = 0$.

To calculate the number of swallowtails we use Prop. 7.1.2.

$$dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3 = 3.$$

Now for the triplepoints we have the following ideal $\langle h_1, h_2, k_1, k_2\rangle$ with

$h_1 = -9z^7 - 9z^6a - 9z^5a^2 - 6z^4 - 9z^4a^3 - 6az^3 - 9z^3a^4 - 6a^2z^2 - 9z^2a^5 - 6za^3 - 9za^6 - 2x - 6a^4 - 9a^7$,
$h_2 = -8z^8 - 8z^7a - 8z^6a^2 - 5z^5 - 8z^5a^3 - 5z^4a - 8z^4a^4 - 5z^3a^2 - 8z^3a^5 - 5z^2a^3 - 8z^2a^6 - xz - 5za^4 - 8za^7 - xa - 5a^5 - 8a^8$,
$k_1 = -9az^4b - 6z^3 - 9a^3zb^2 - 9a^4bz - 6a^3 - 6az^2 - 6za^2 - 6bz^2 - 6zb^2 - 9a^3bz^2 - 9a^2bz^3 - 6b^3 - 6abz - 6a^2b - 6ab^2 - 9z^6 - 9a^6 - 9z^5a - 9z^4a^2 - 9z^3a^3 - 9z^2a^4 - 9za^5 - 9a^2b^2z^2 - 9a^2zb^3 - 9az^3b^2 - 9az^2b^3 - 9azb^4 - 9b^6 - 9z^5b - 9z^4b^2 - 9z^3b^3 - 9z^2b^4 - 9zb^5 - 9a^5b - 9a^4b^2 - 9a^3b^3 - 9a^2b^4 - 9ab^5$,
$k_2 = -8a^3bz^3 - x - 8a^5bz - 5z^4 - 5a^4 - 5a^2bz - 5az^3 - 5a^2z^2 - 5za^3 - 5b^4 - 5bz^3 - 5b^2z^2 - 5zb^3 - 8a^4bz^2 - 5abz^2 - 5azb^2 - 5a^3b - 5a^2b^2 - 5ab^3 - 8z^7 - 8a^7 - 8z^6a - 8z^5a^2 - 8z^4a^3 - 8z^3a^4 - 8z^2a^5 - 8za^6 - 8a^4zb^2 - 8a^3b^2z^2 - 8a^3zb^3 - 8a^2z^4b - 8a^2z^3b^2 - 8a^2z^2b^3 - 8a^2zb^4 - 8az^5b - 8az^4b^2 - 8az^3b^3 - 8b^7 - 8z^6b - 8z^5b^2 - 8z^4b^3 - 8z^3b^4 - 8z^2b^5 - 8zb^6 - 8a^6b - 8a^5b^2 - 8a^4b^3 - 8a^3b^4 - 8a^2b^5 - 8ab^6 - 8az^2b^4 - 8azb^5$.

This ideal has codimension 10.

7.3 Bifurcations

We have the following pictures of the images of the critical sets obtained from GEOMVIEW on a SILICON GRAPHICS WORKSTATION.

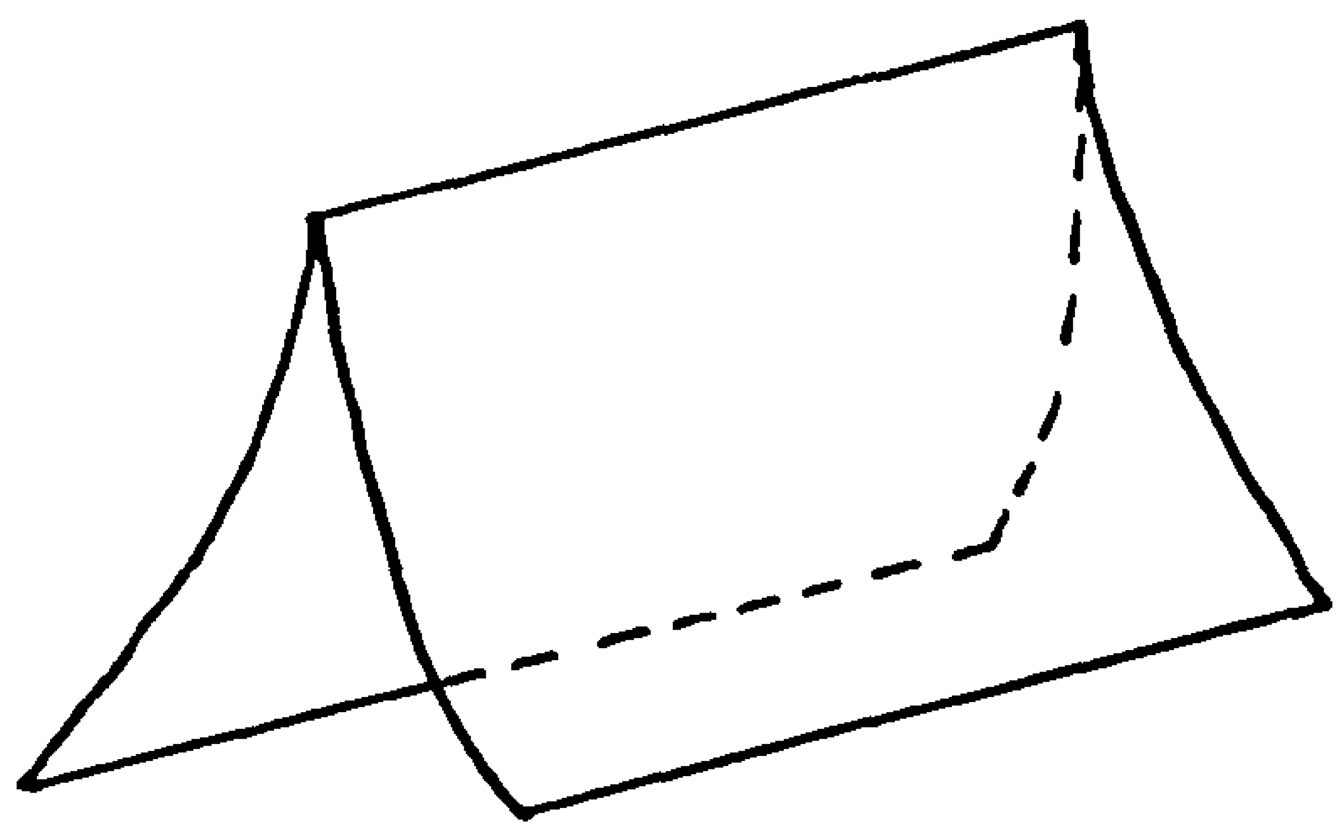


Figure 7.2: $f(x, y, z) = (x, y, yz + z^3)$

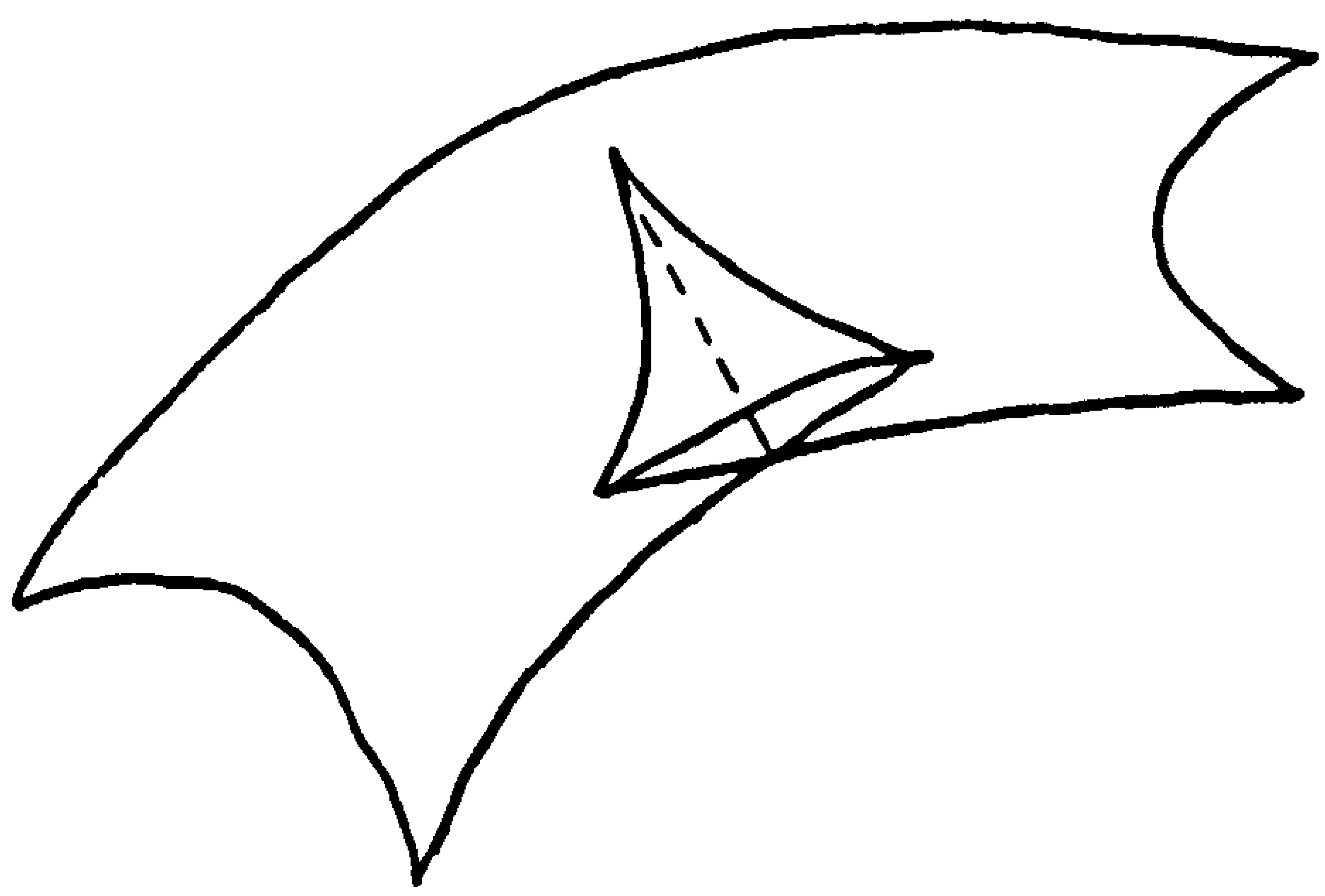
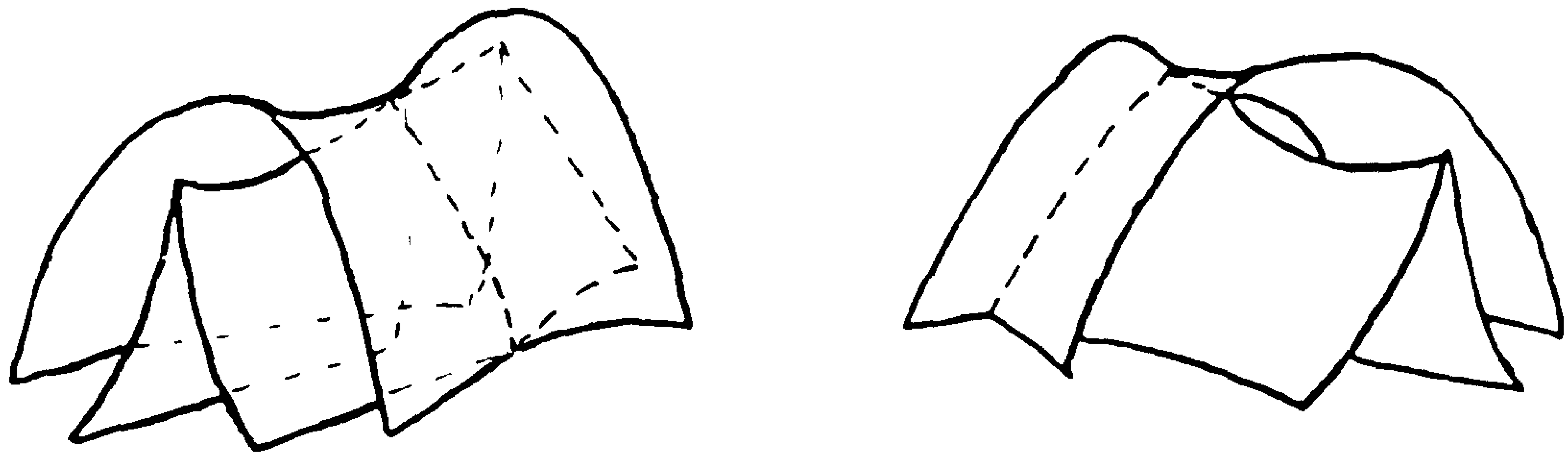
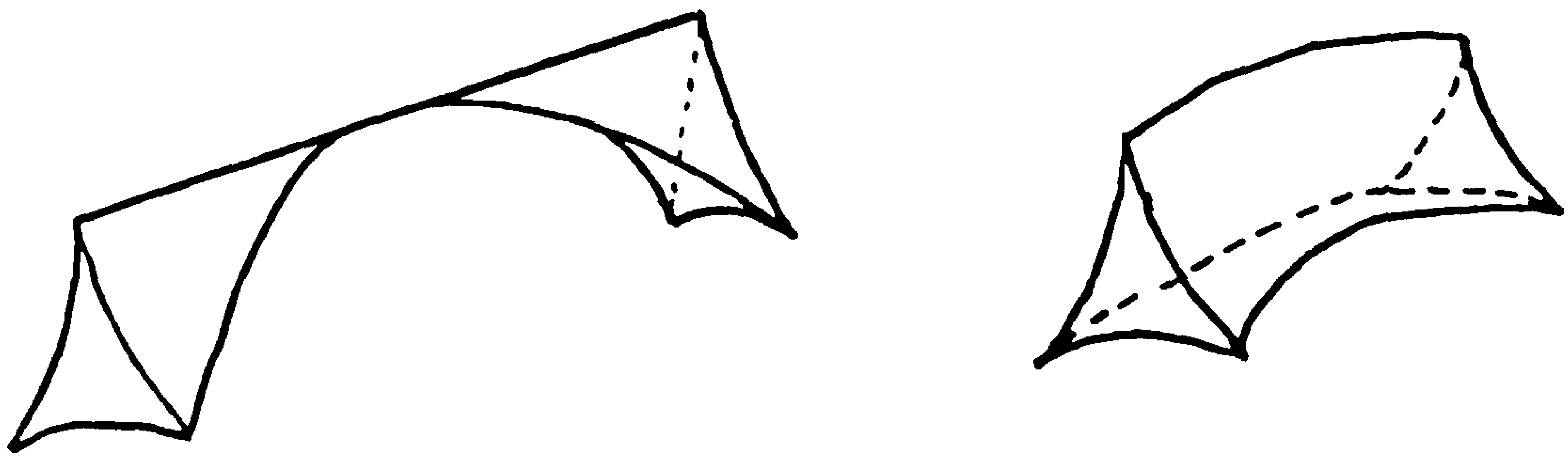


Figure 7.3: $f(x, y, z) = (x, y, yz + xz^2 + z^4)$



$a = 0$ $a \neq 0$

Figure 7.4: $f(x, y, z) = (x, yz + ay, y^2 + z^2 + xy)$



$a = 0$ $a \neq 0$

Figure 7.5: $f(x, y, z) = (x, yz + ay, y^2 - z^2 + xy)$

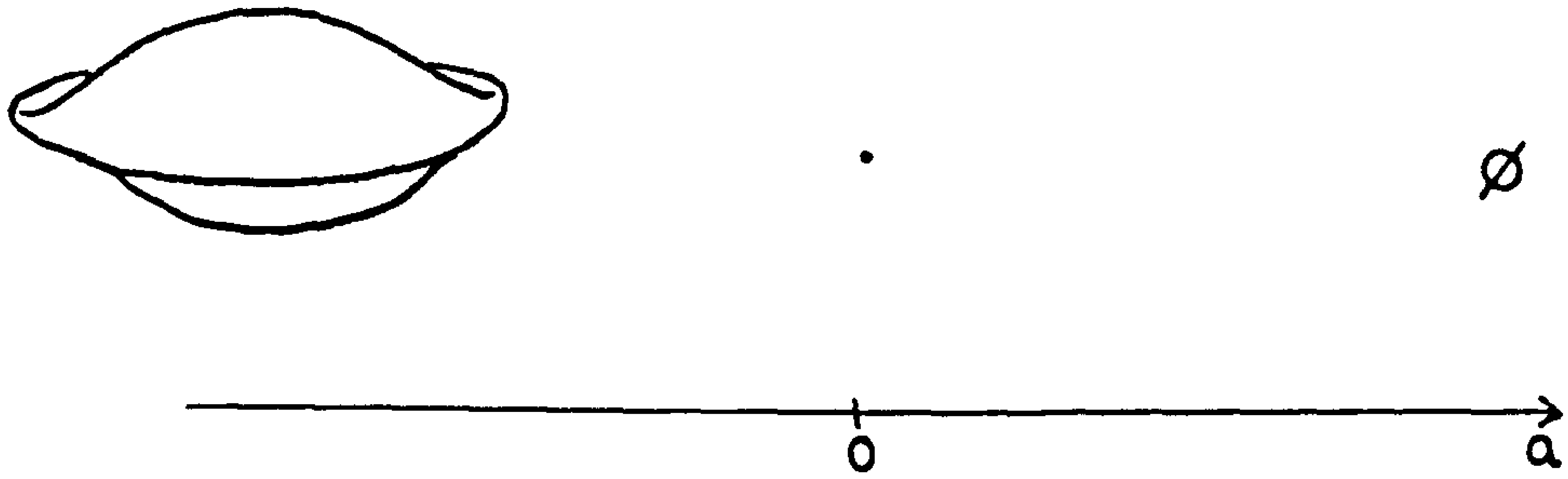


Figure 7.6: $f(x,y,z) = (x, y, z^3 + (y^2 + x^2)z + az)$

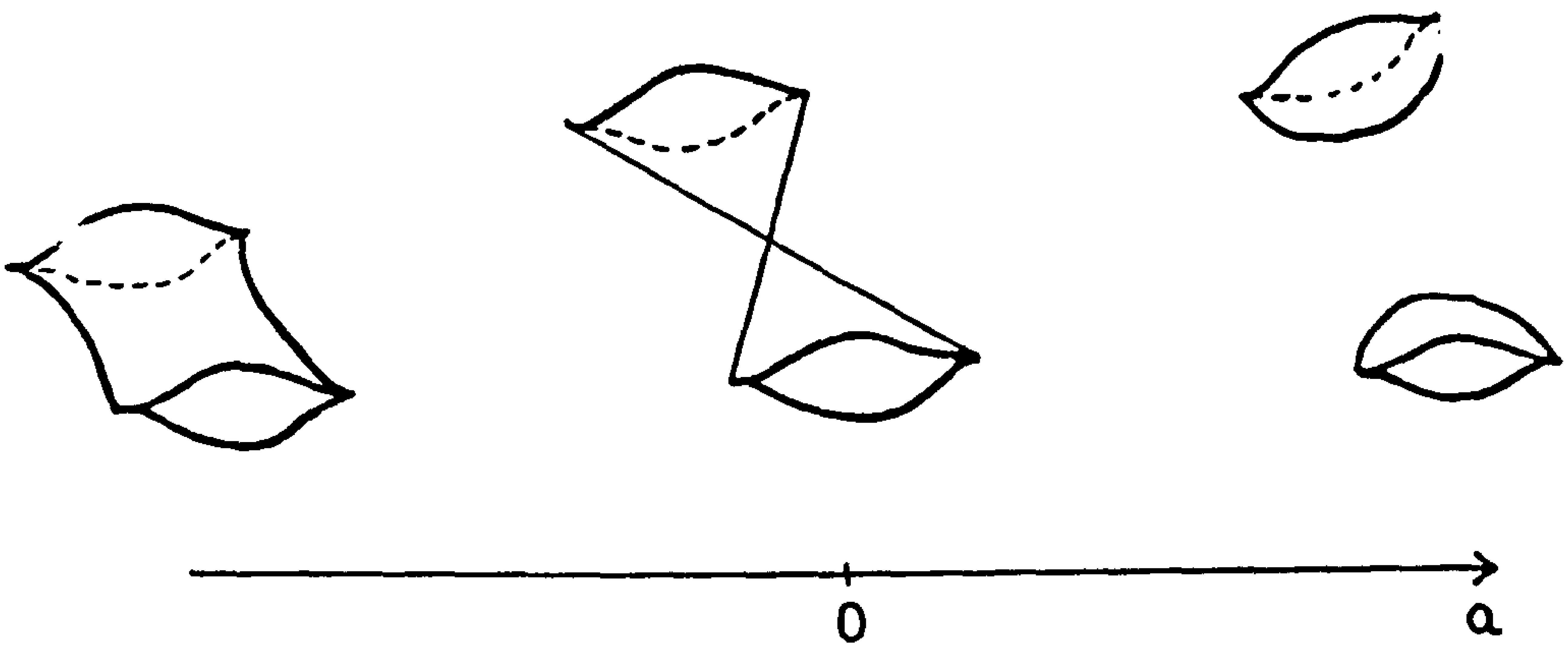


Figure 7.7: $f(x,y,z) = (x, y, z^3 + (y^2 - x^2)z + az)$

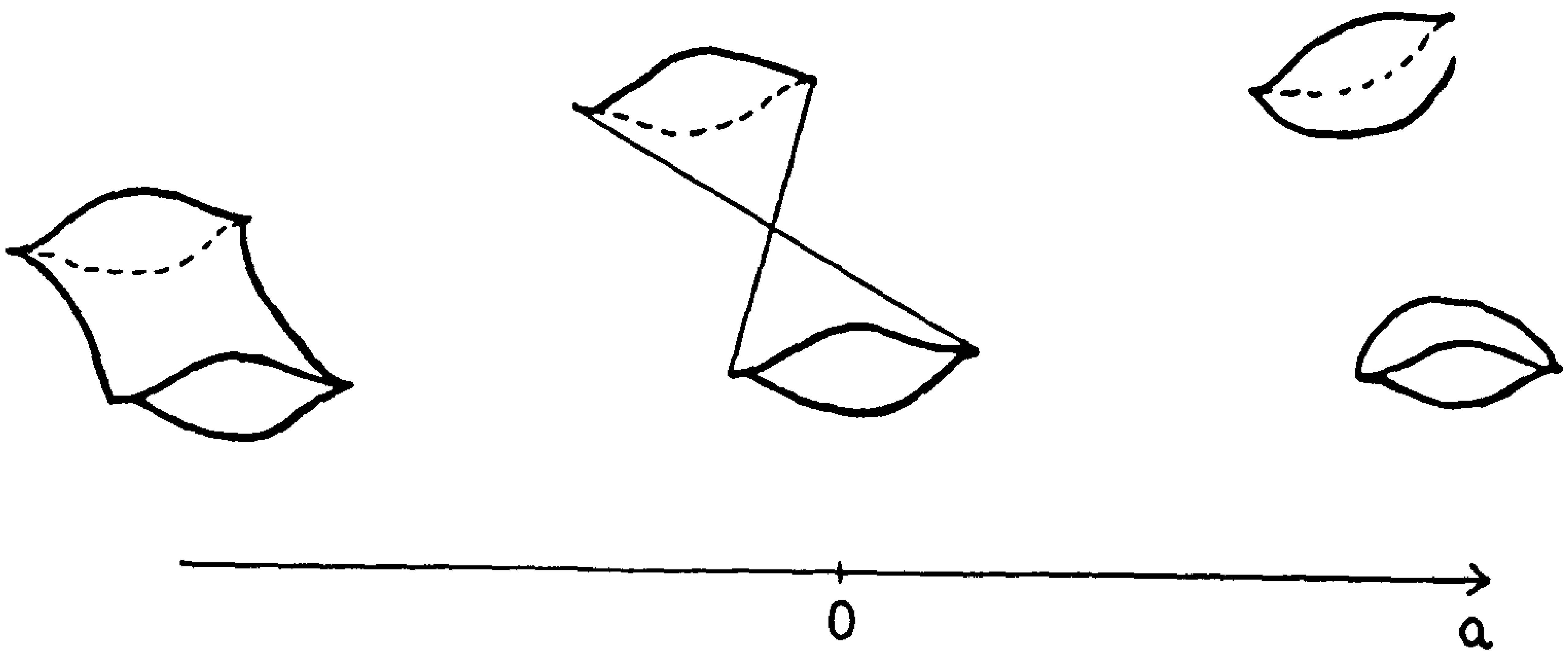


Figure 7.8: $f(x, y, z) = (x, y, z^3 - (y^2 - x^2)z + az)$

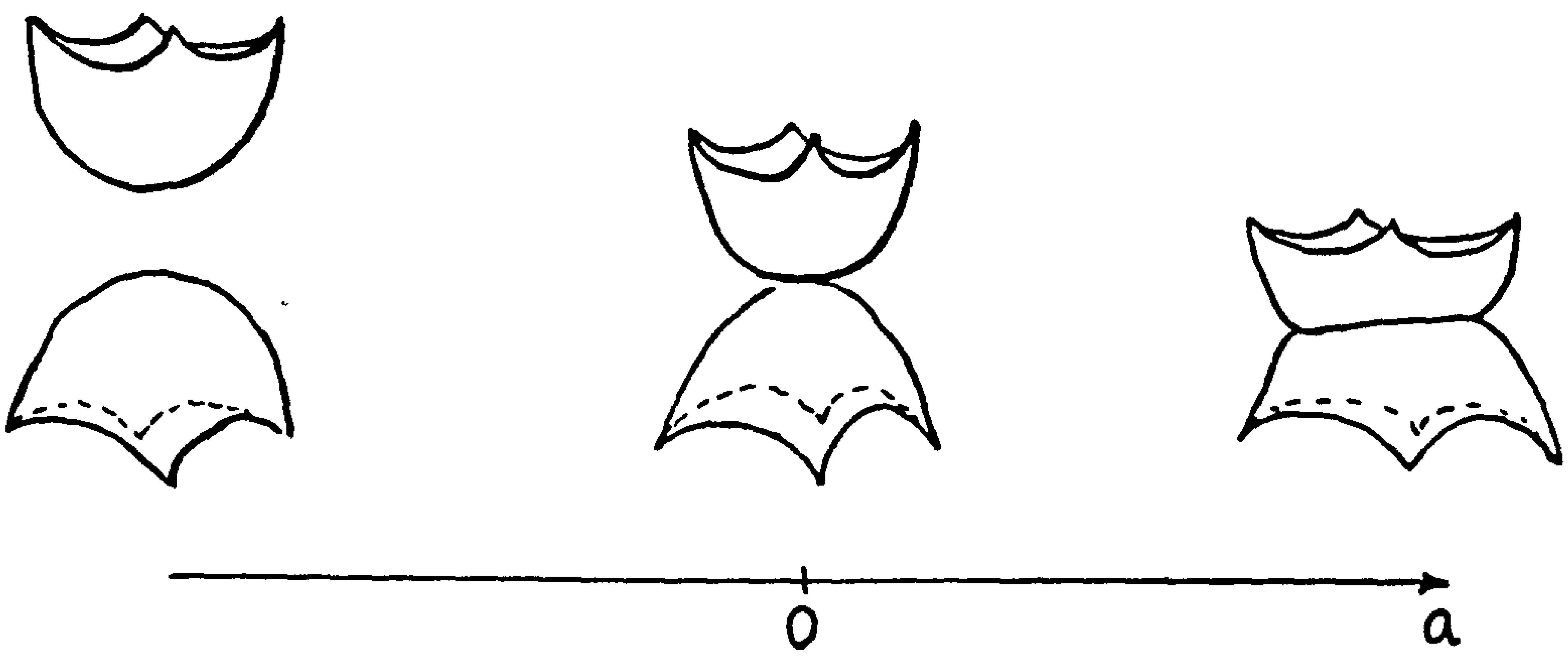


Figure 7.9: $f(x, y, z) = (x, y, z^3 - (y^2 + x^2)z + az)$

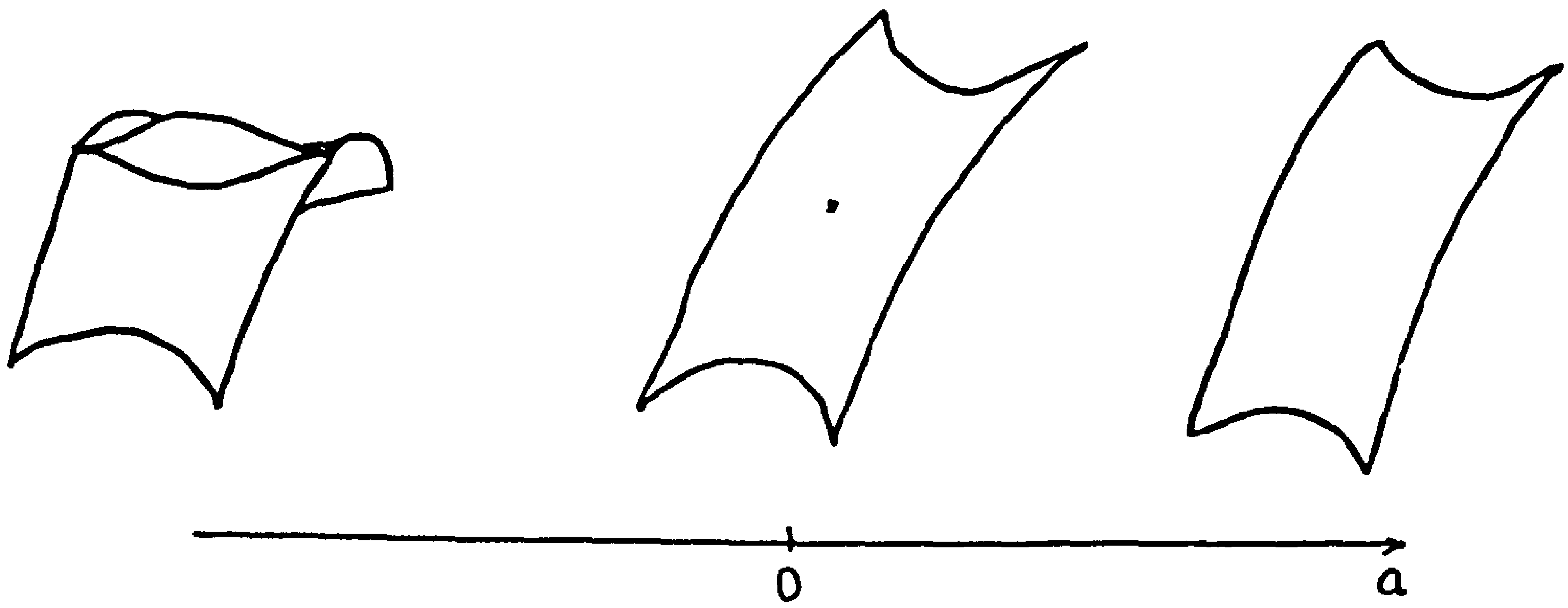


Figure 7.10: $f(x, y, z) = (x, y, yz + z^4 + x^2z^2 + az^2)$

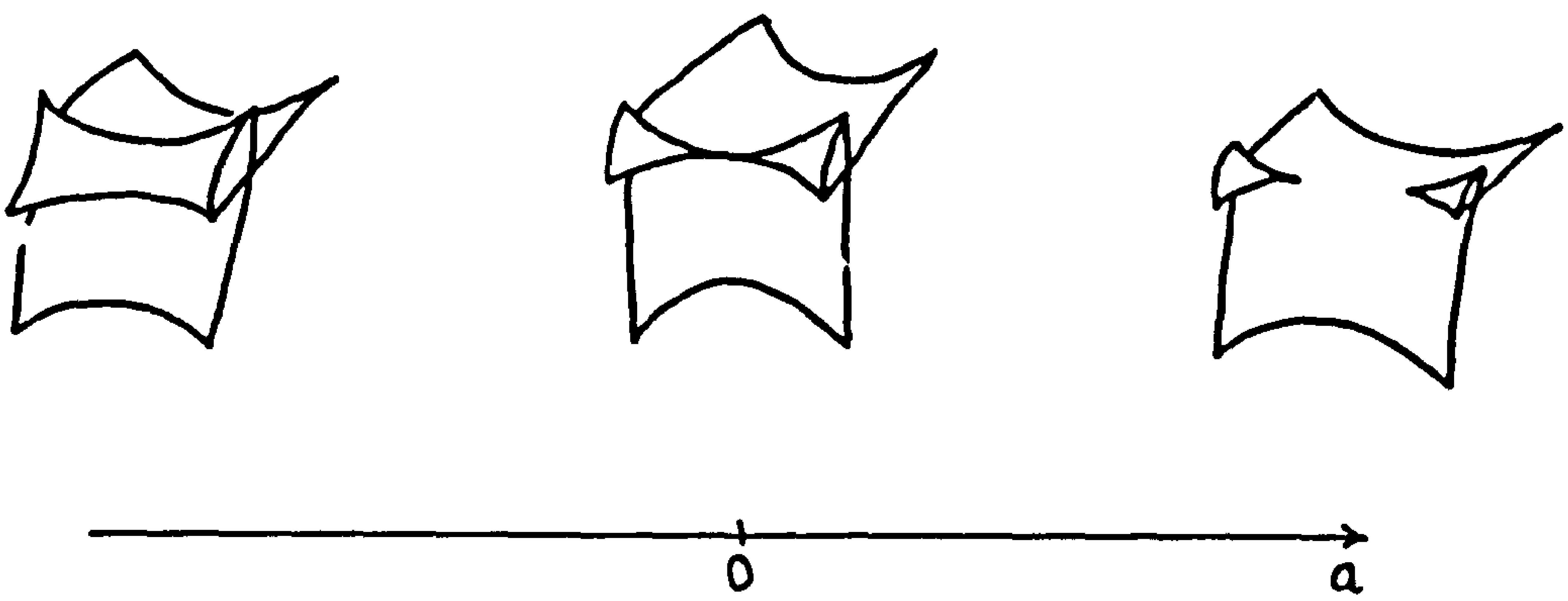


Figure 7.11: $f(x, y, z) = (x, y, yz + z^4 - x^2z^2 + az^2)$

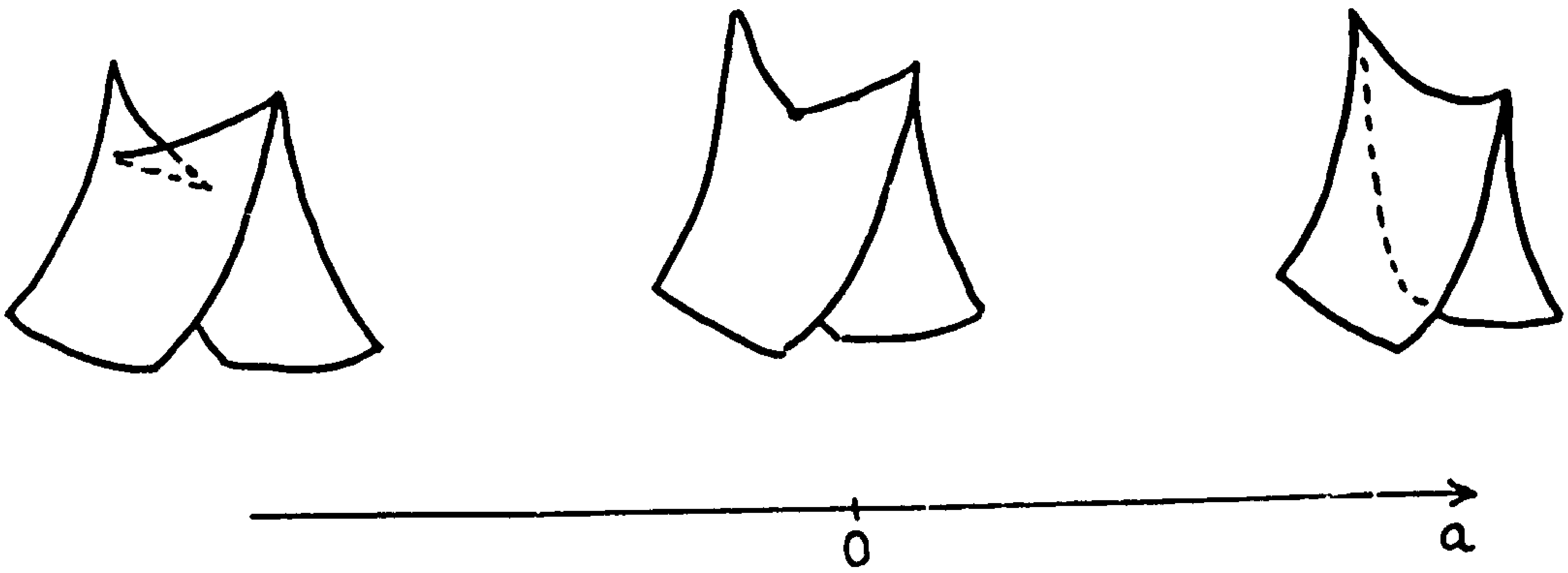


Figure 7.12: $f(x, y, z) = (x, y, yz + xz^2 + z^5 + az^3)$

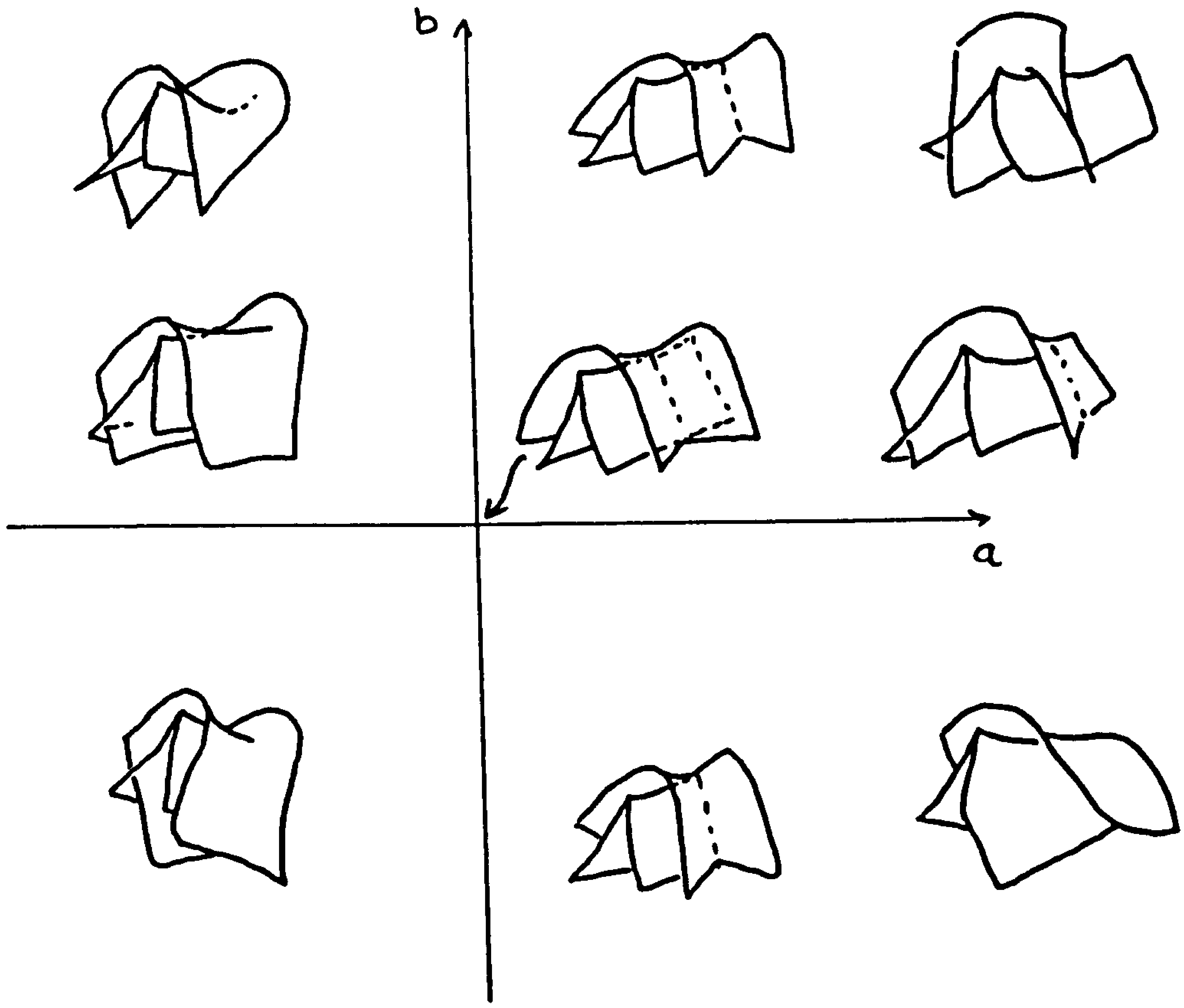


Figure 7.13: $f(x,y,z) = (x, yz + y^3, y^2 + z^2 + xy + xz + ay + bxy)$

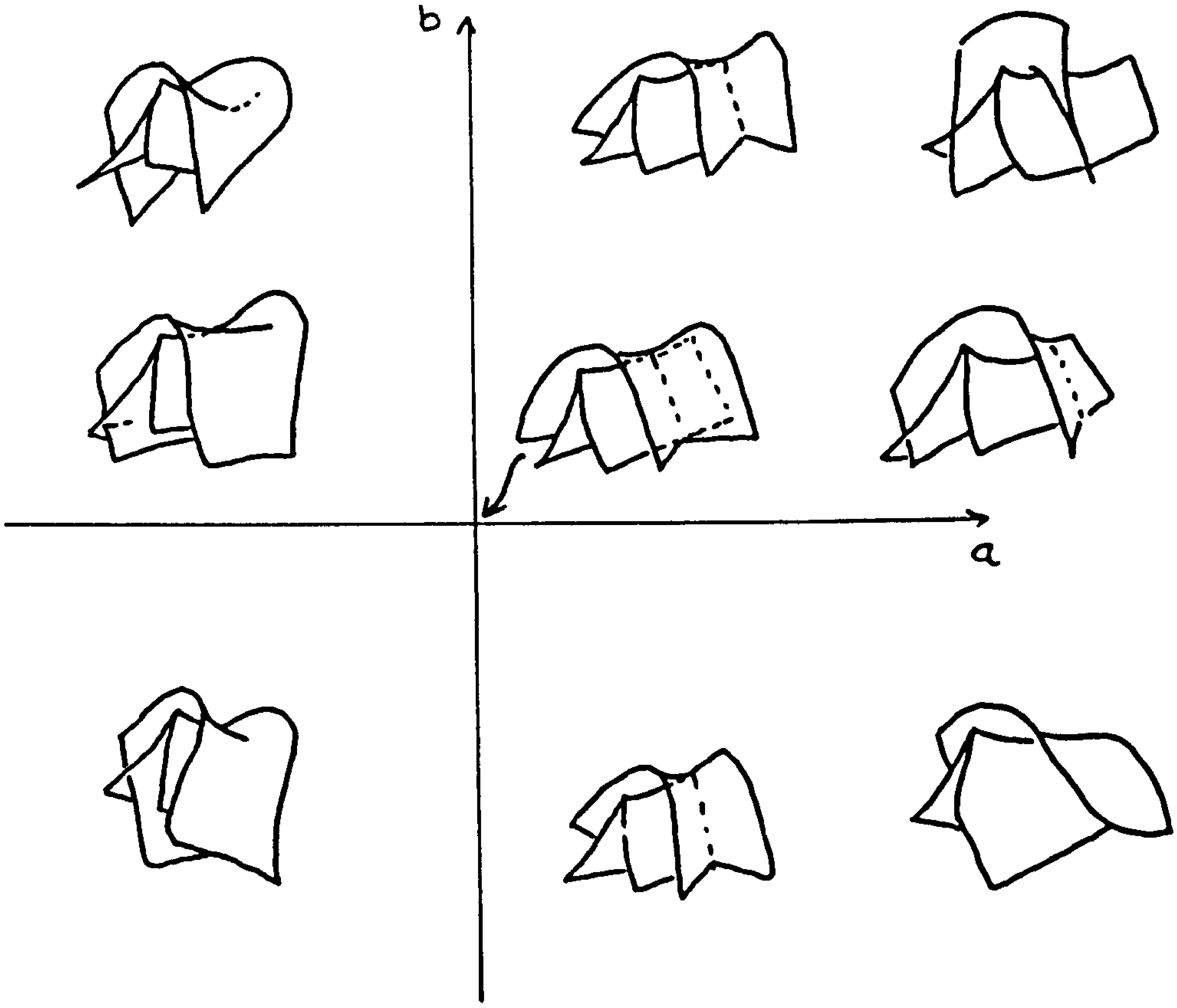


Figure 7.14: $f(x, y, z) = (x, yz + y^3 - x^2y, y^2 + z^2 + xy + xz + ay + bxy)$

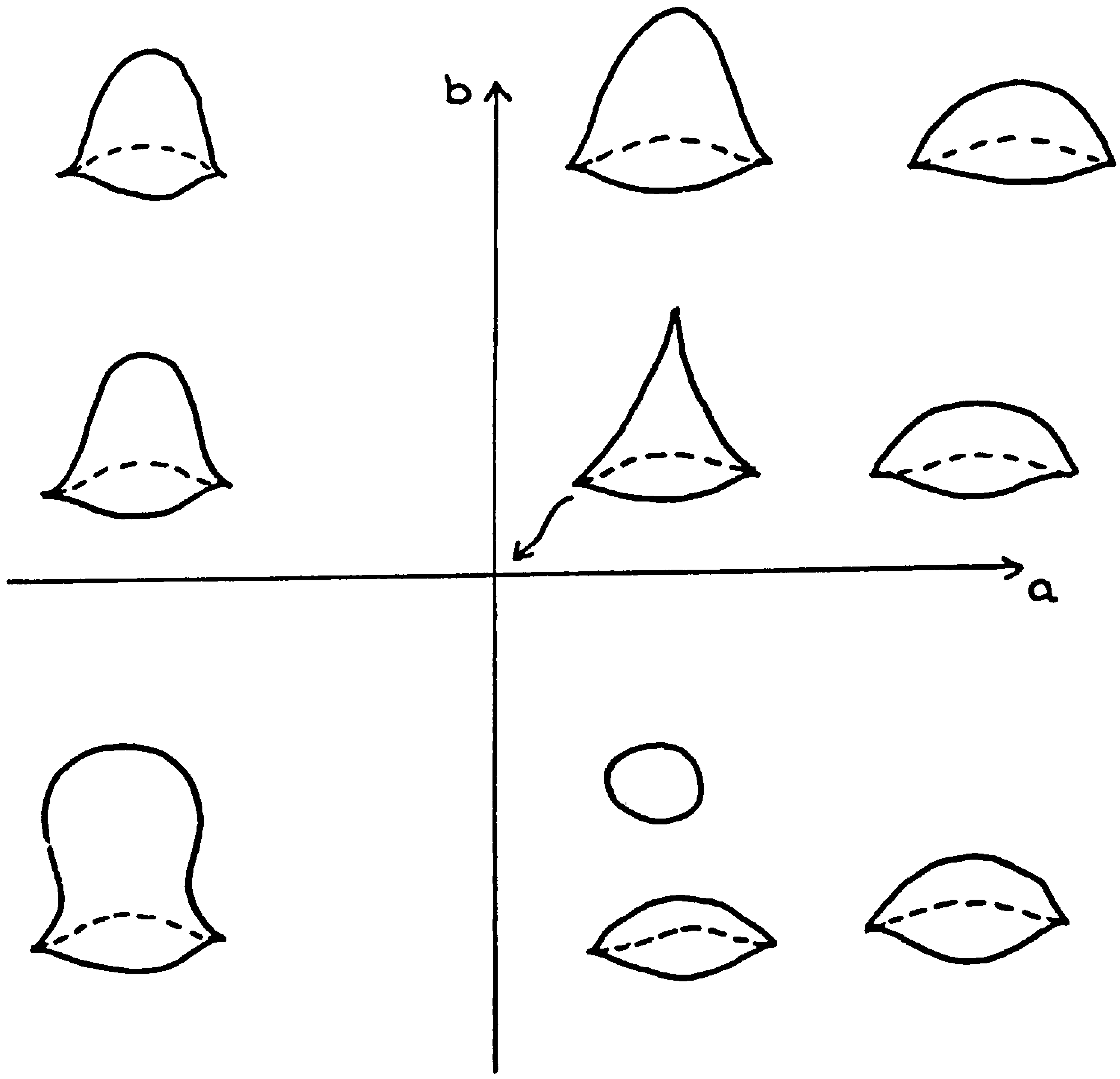


Figure 7.15: $f(x, y, z) = (x, y, z^3 + (y^2 + x^3)z + az + bxz)$

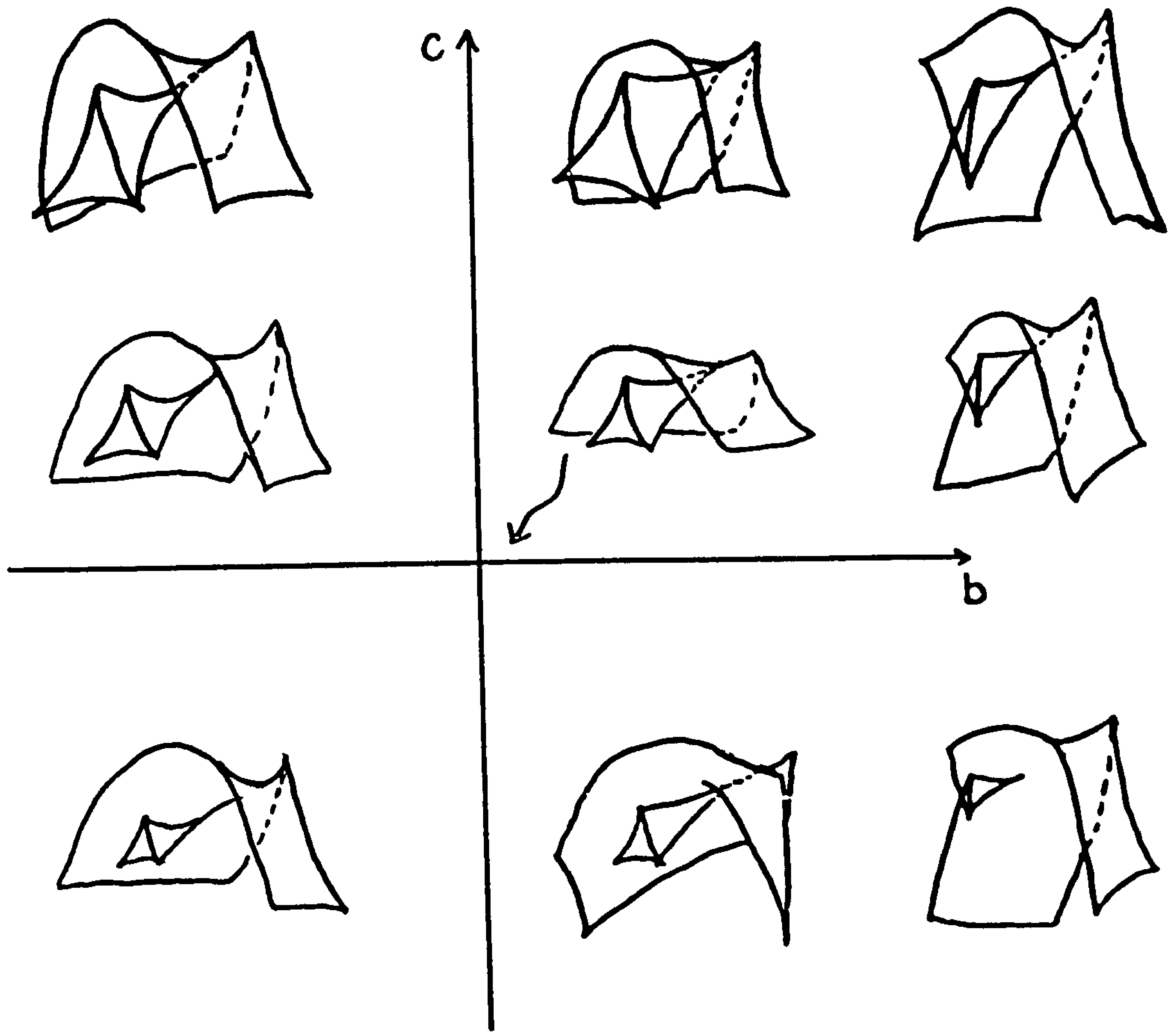


Figure 7.16: $f(x,y,z) = (x, yz, y^2 + xy + xz + z^3 + ayz^3 + by + cz^2)$, $a = 0$

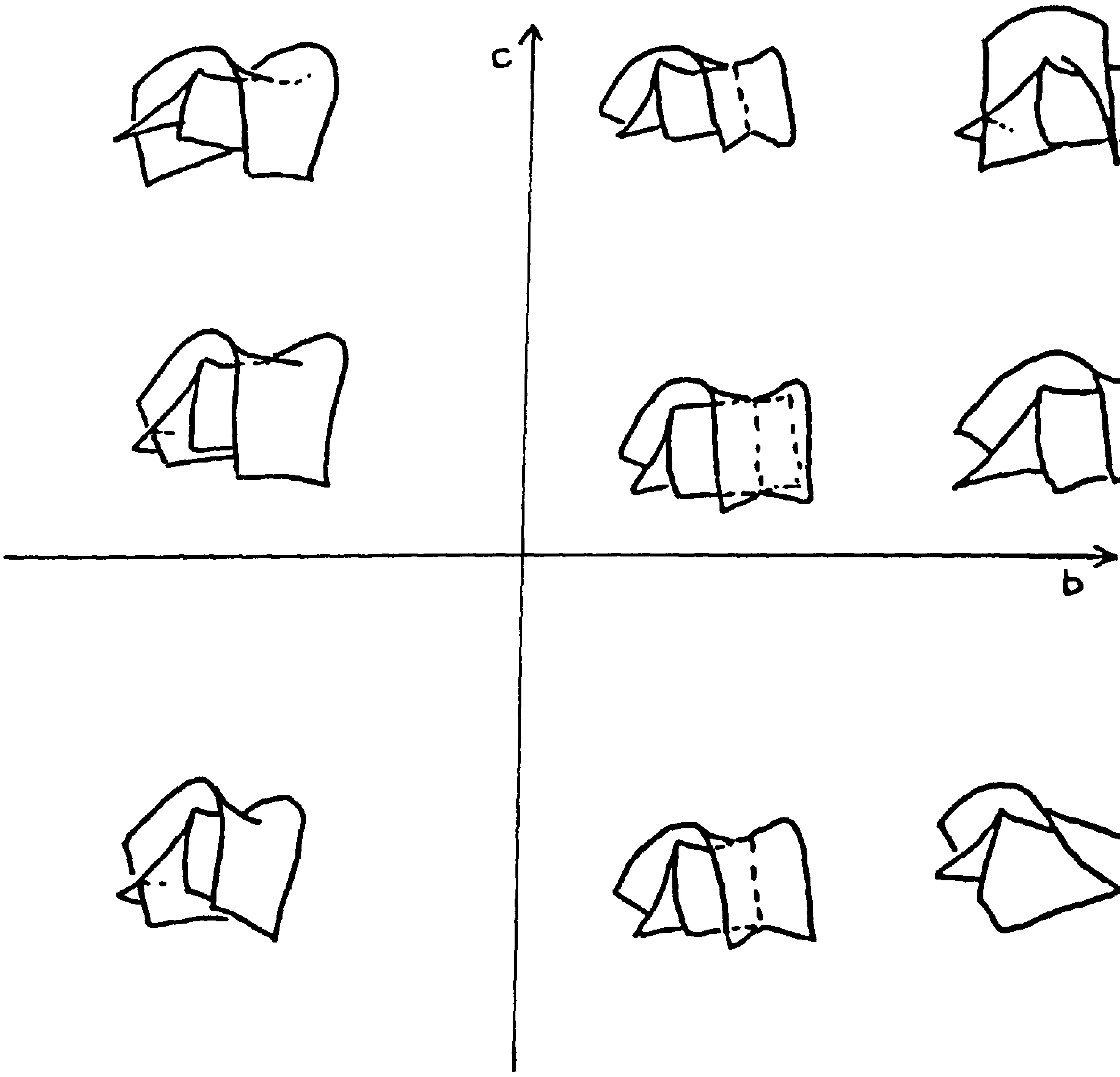


Figure 7.17: $f(x,y,z) = (x, yz, y^2 + xz + z^3 + z^4 + ayz^3 + by + cz^2)$, $a = 0$

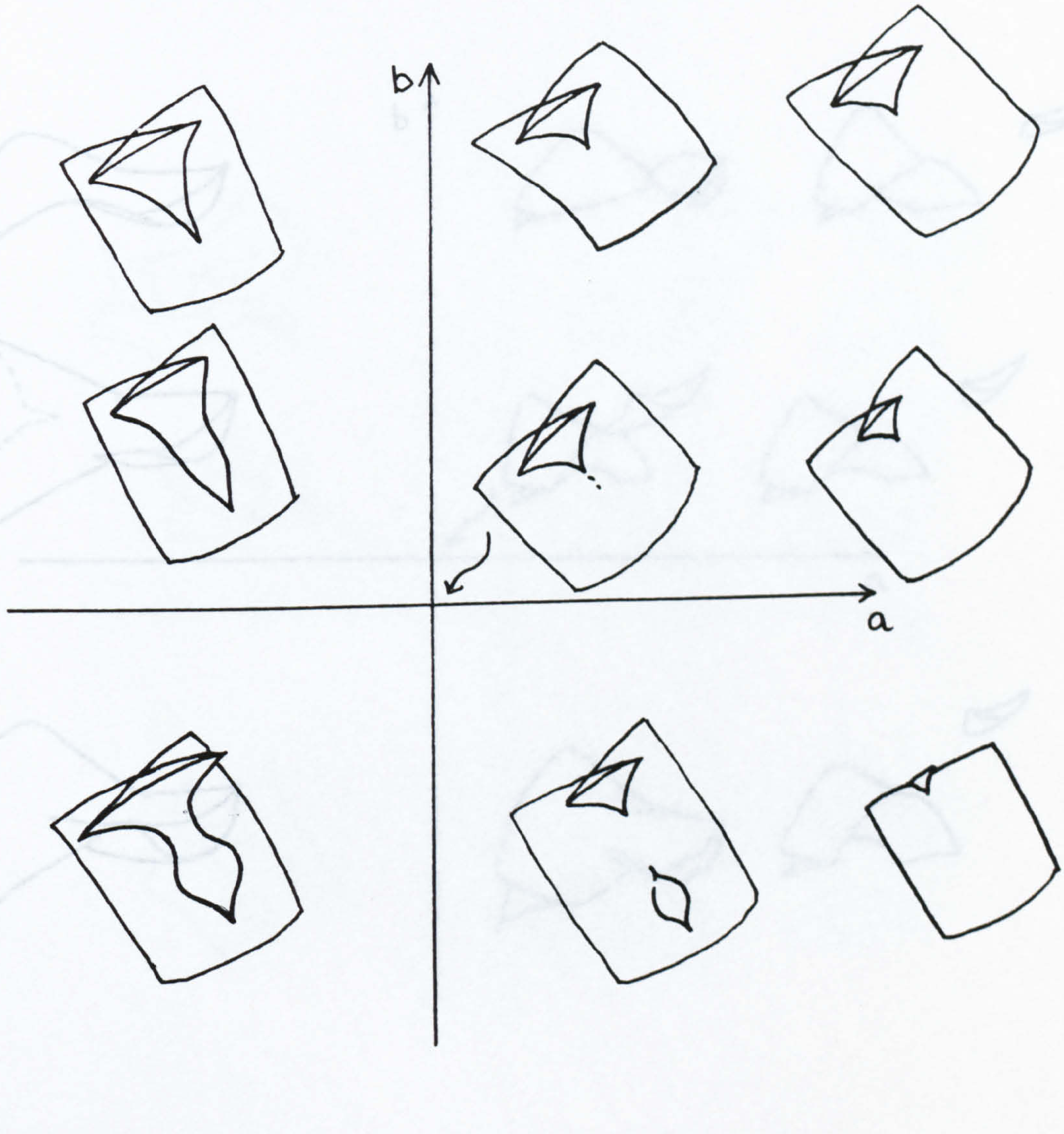


Figure 7.18: $f(x, y, z) = (x, y, yz + z^4 + x^3z^2 + az^2 + bxz^2)$

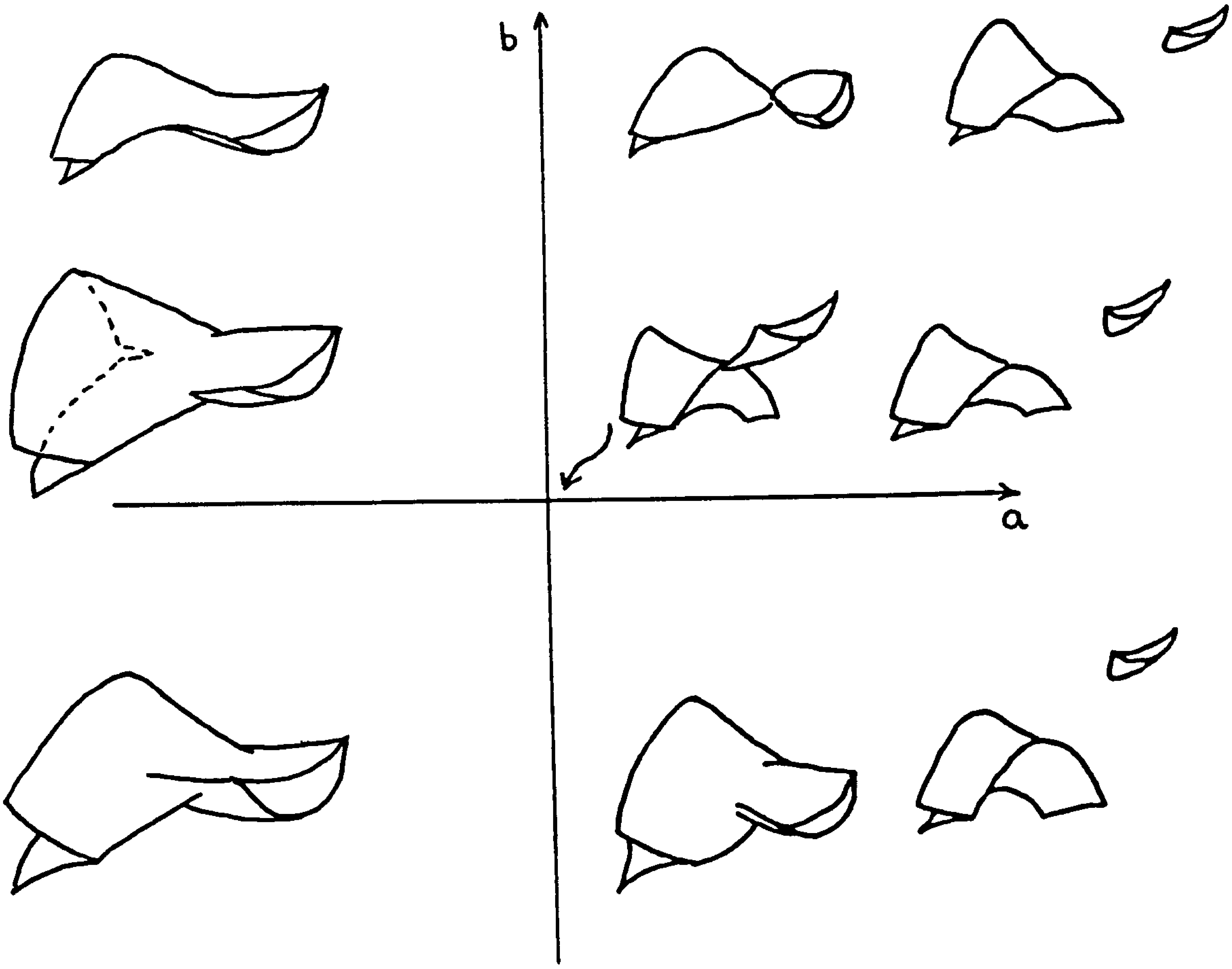


Figure 7.19: $f(x, y, z) = (x, y, yz^2 + x^2z + z^4 + z^5 + az^2 + bz^4)$

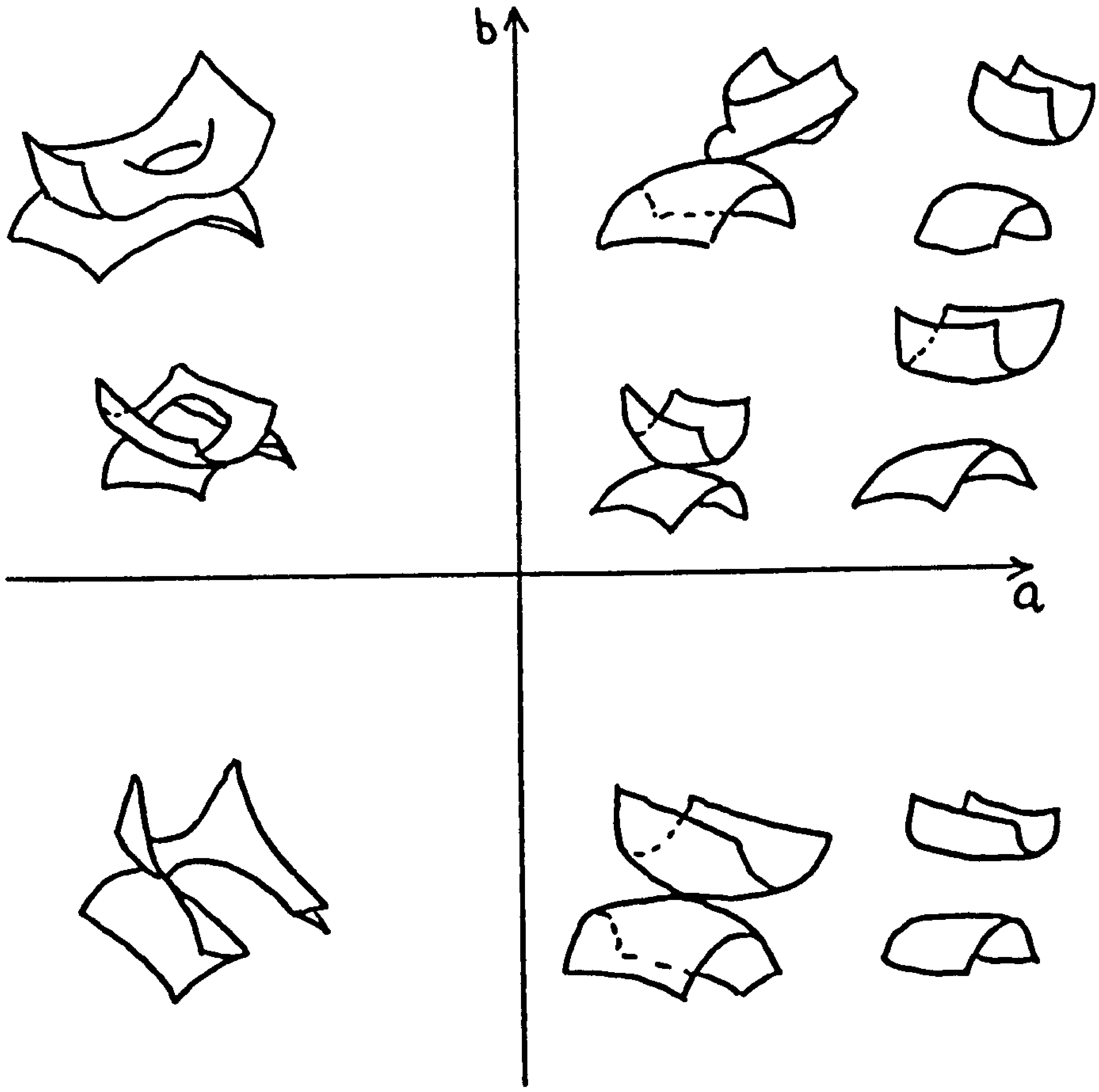


Figure 7.20: $f(x, y, z) = (x, y, yz^2 + x^2z + z^4 - z^5 + az^2 + bz^4)$

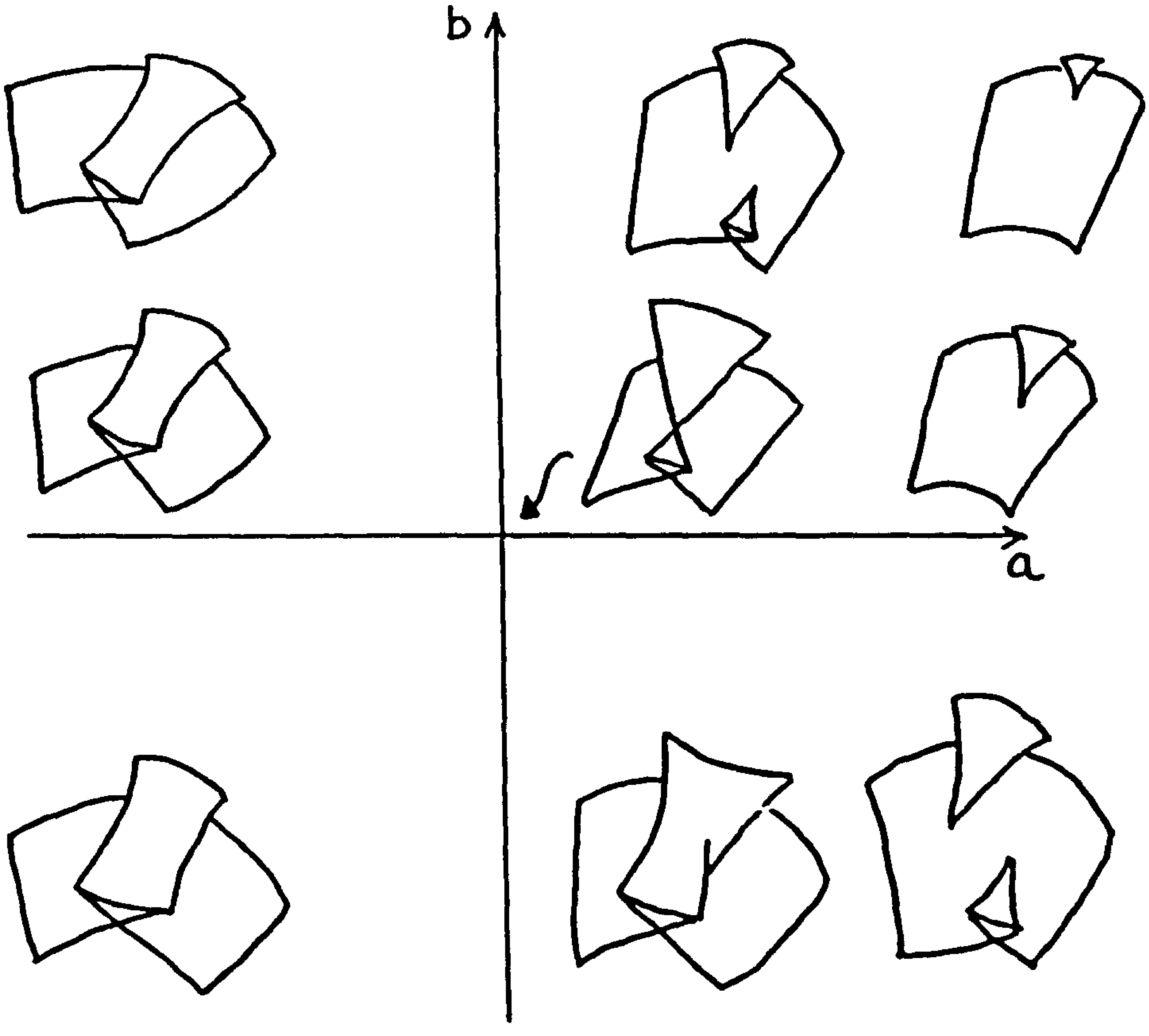


Figure 7.21: $f(x, y, z) = (x, y, yz + xz^3 + z^5 + z^6 + az^2 + bz^4)$

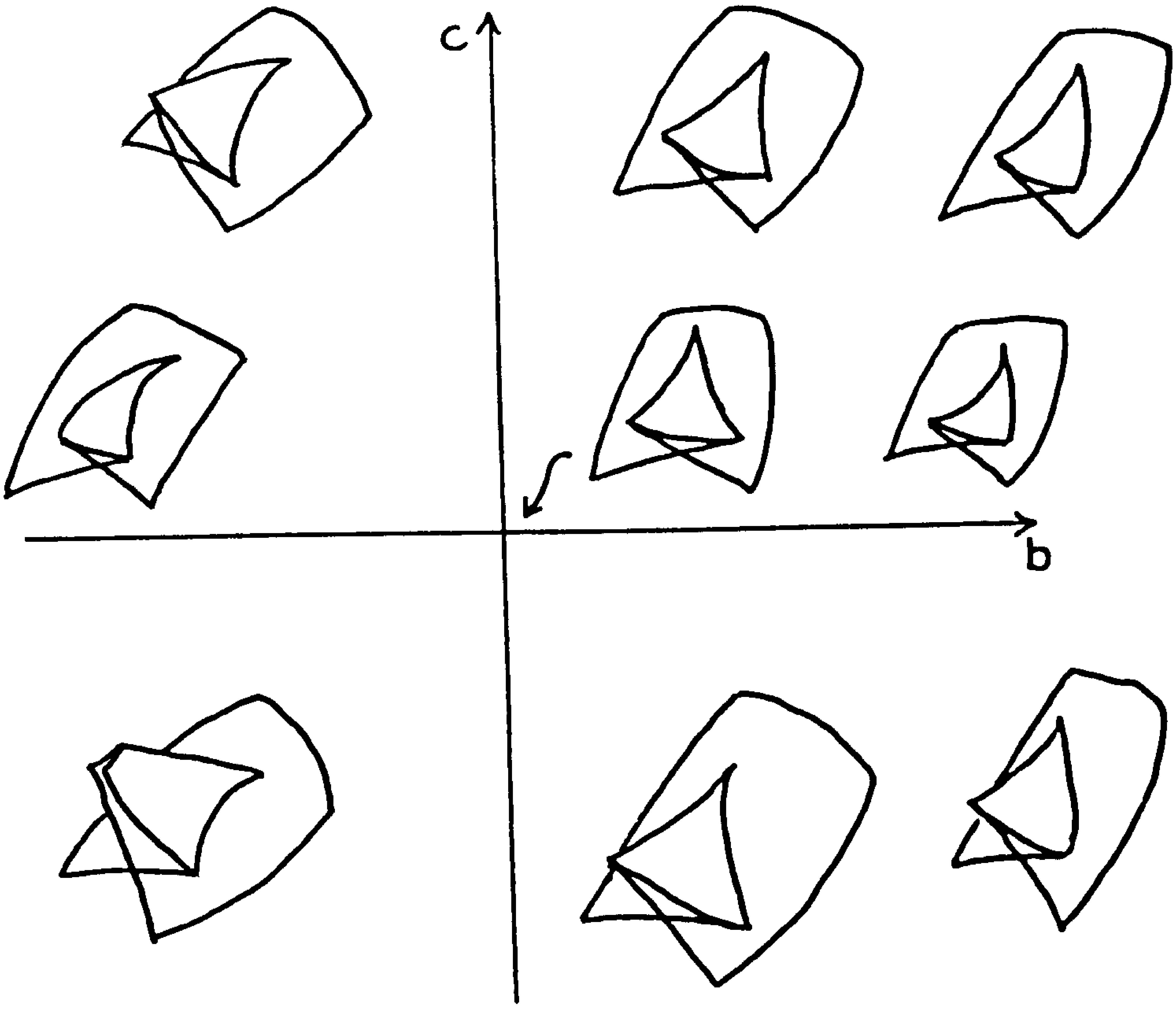


Figure 7.22: $f(x, y, z) = (x, y, yz + xz^2 + z^6 \pm z^8 + az^9 + bz^3 + cz^4)$, $a = 0$

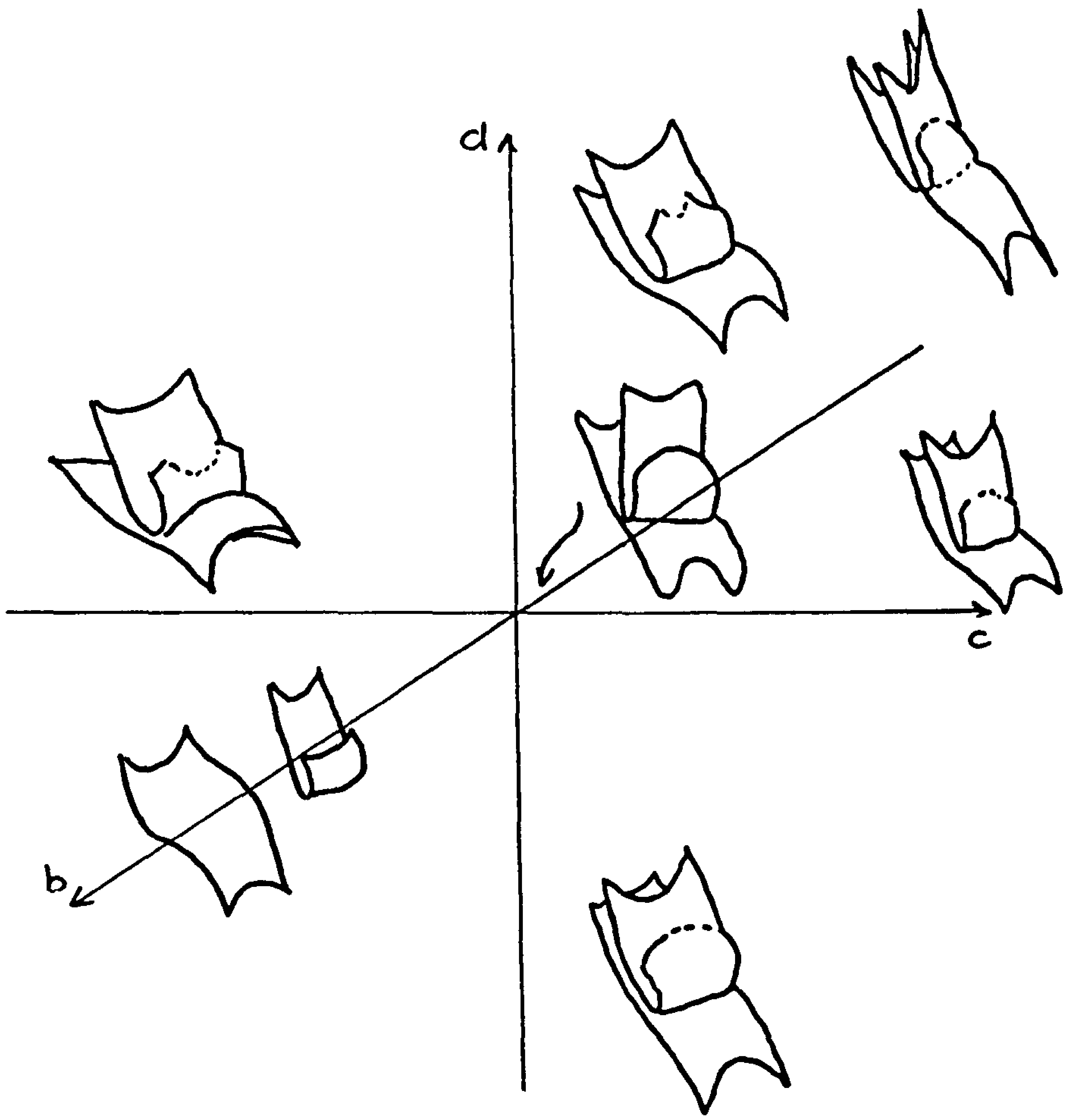


Figure 7.23: $f(x, y, z) = (x, yz \pm y^3 + ax^2y + by, y^2 \pm z^2 + y^3 \pm x^2y + cy + dxy)$,
$a = 1$

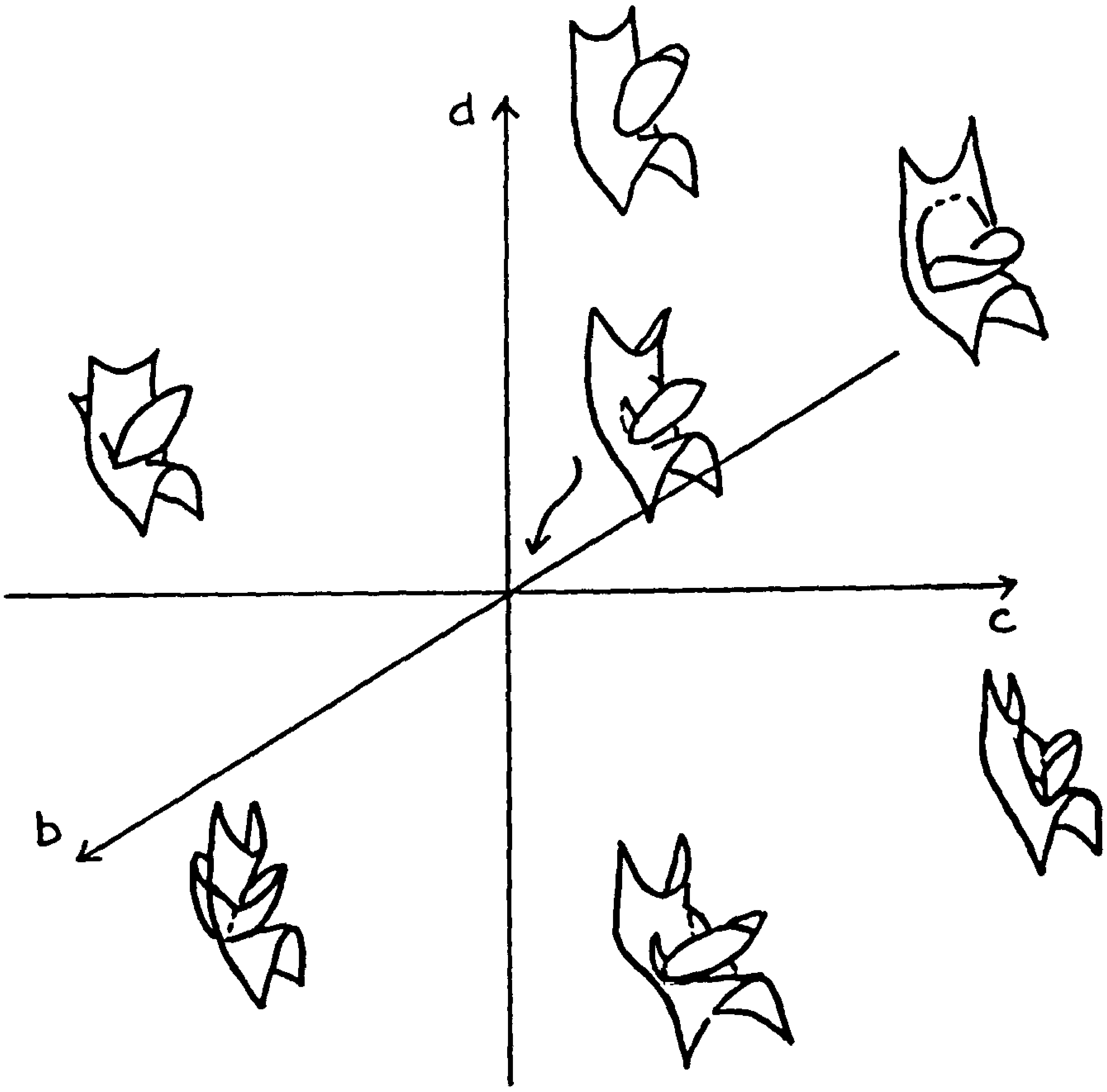


Figure 7.24: $f(x,y,z) = (x, yz + ax^2y, y^2 \pm z^2 + y^3 \pm x^2y + by + cxy + dx^2y)$, $a = 1$

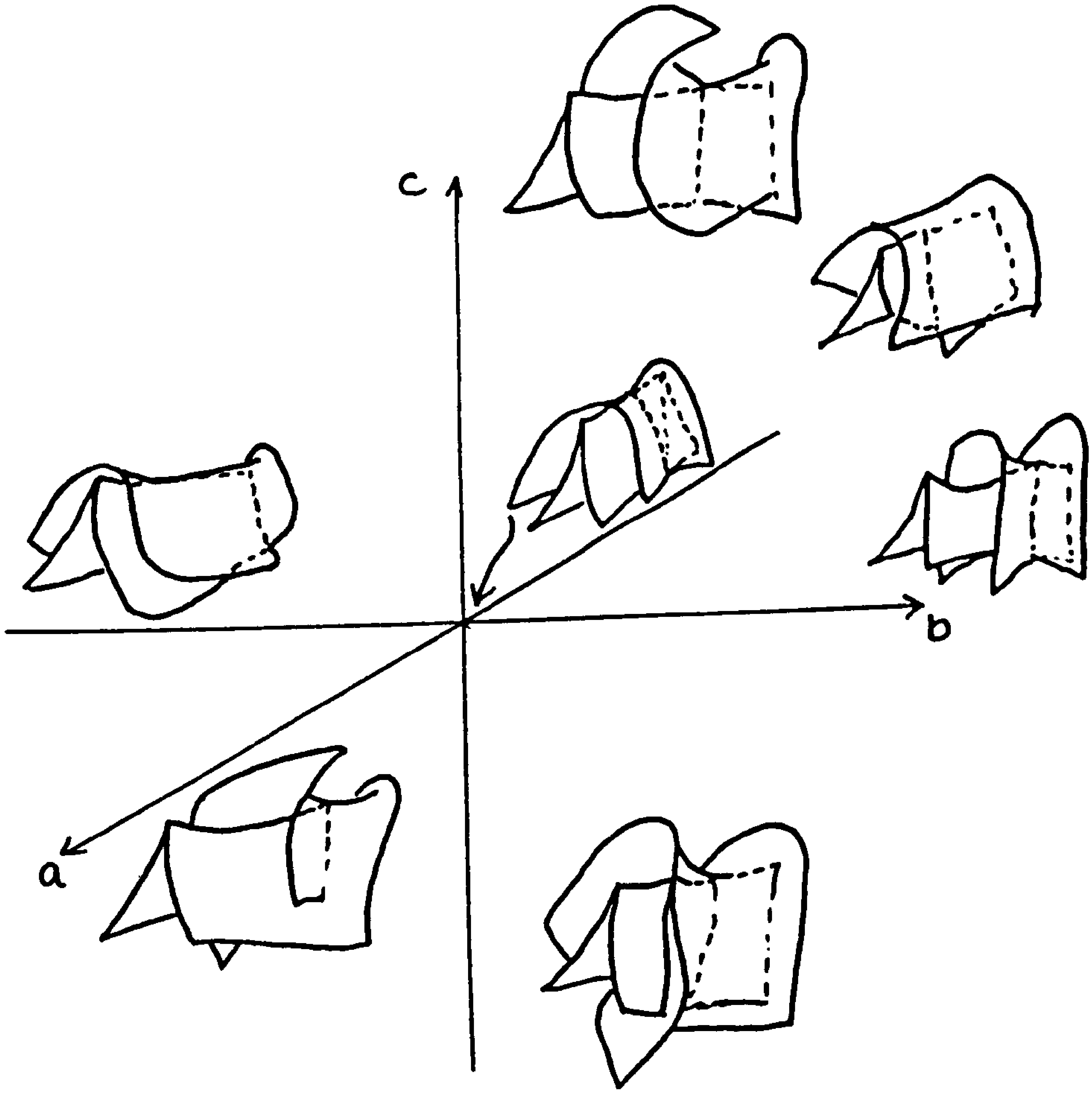


Figure 7.25: $f(x,y,z) = (x, yz+y^3-\frac{3}{16}x^2y \pm x^3y, y^2+z^2+xy+xz+ay+bz^2+cyz^3)$

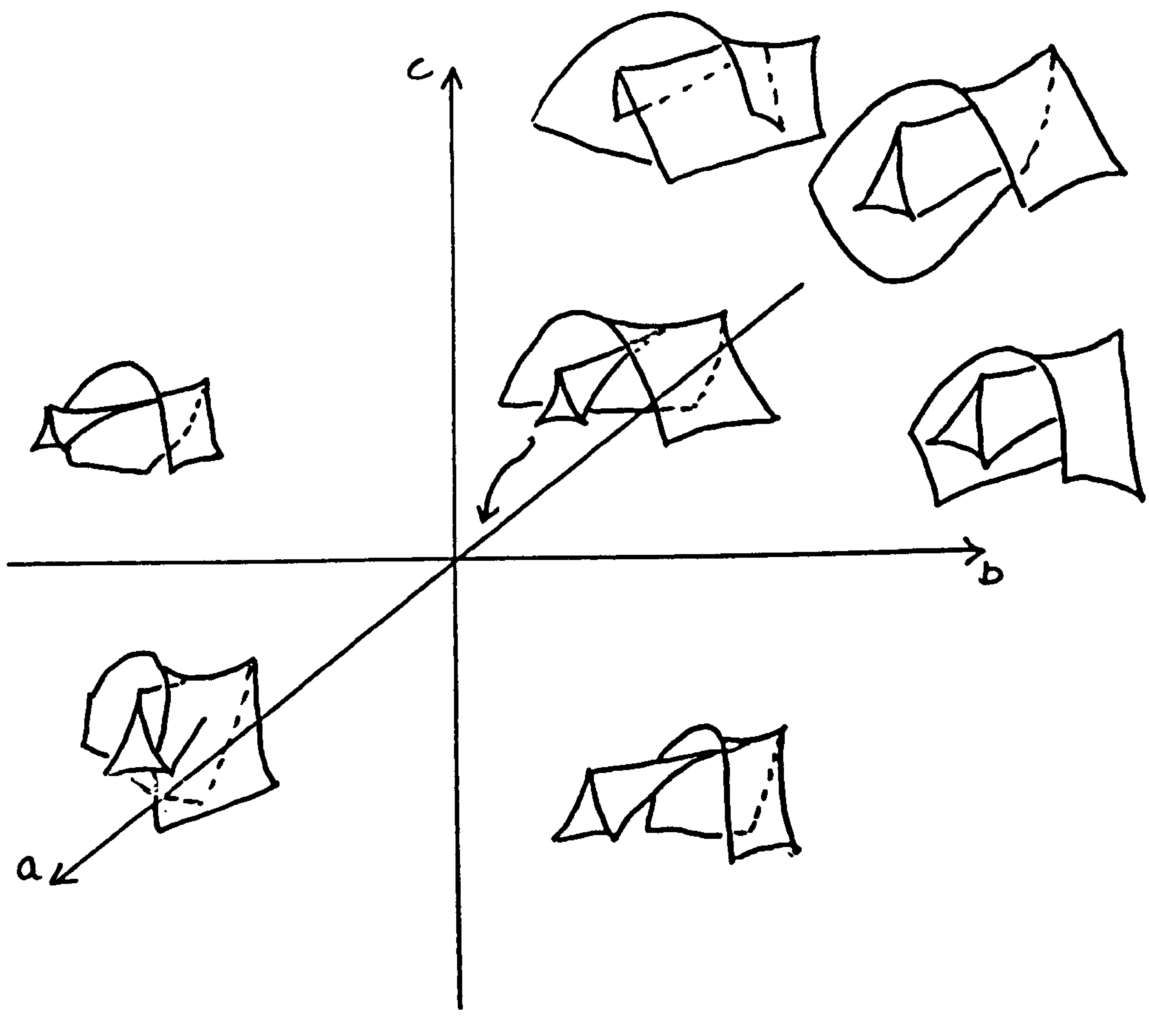


Figure 7.26: $f(x,y,z) = (x, yz, y^2 + xz + z^3 + yz^3 + ay + bz^2 + cz^4)$

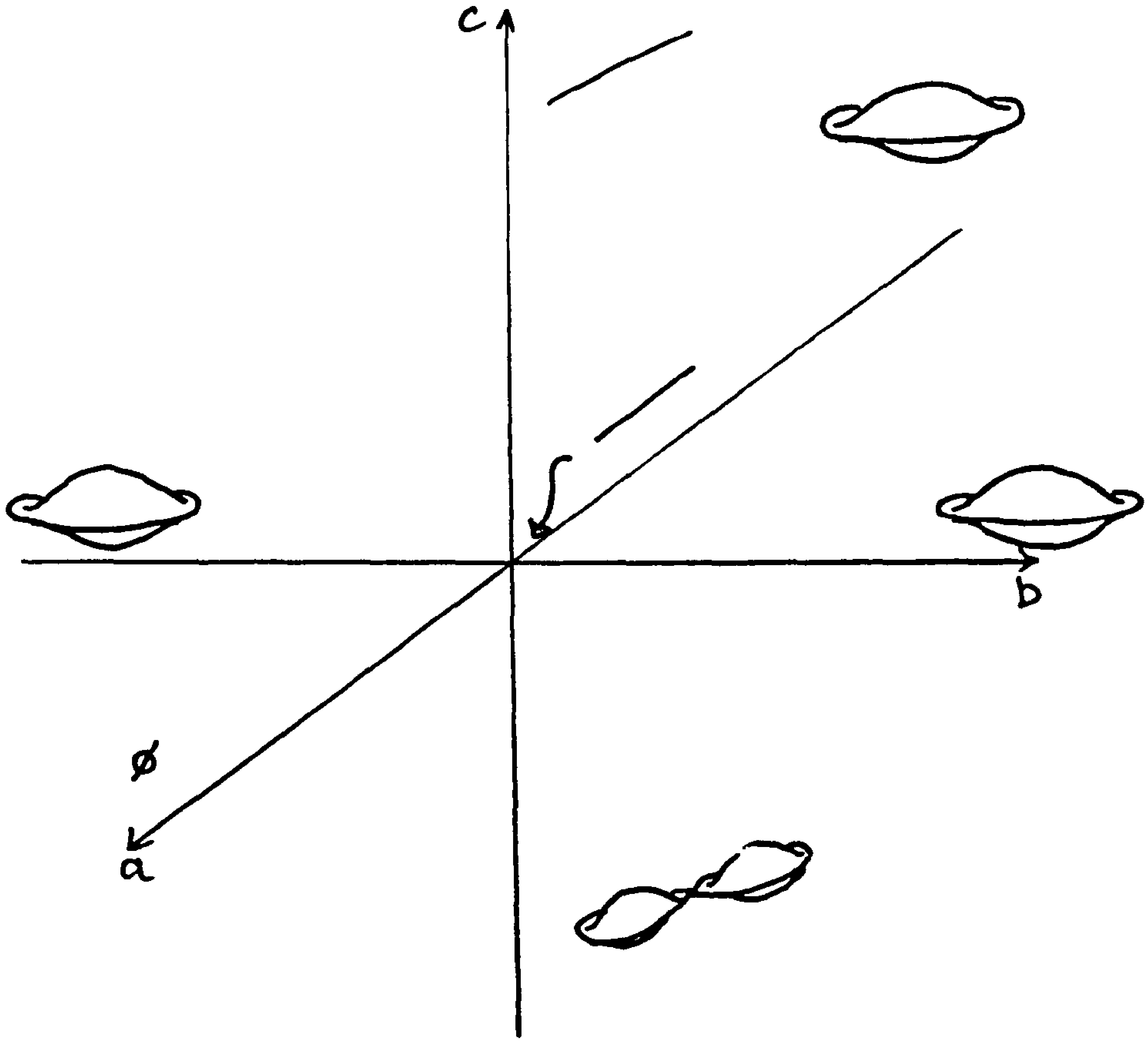


Figure 7.27: $f(x, y, z) = (x, y, z^3 + (y^2 + x^4)z + az + bxz + cx^2z)$

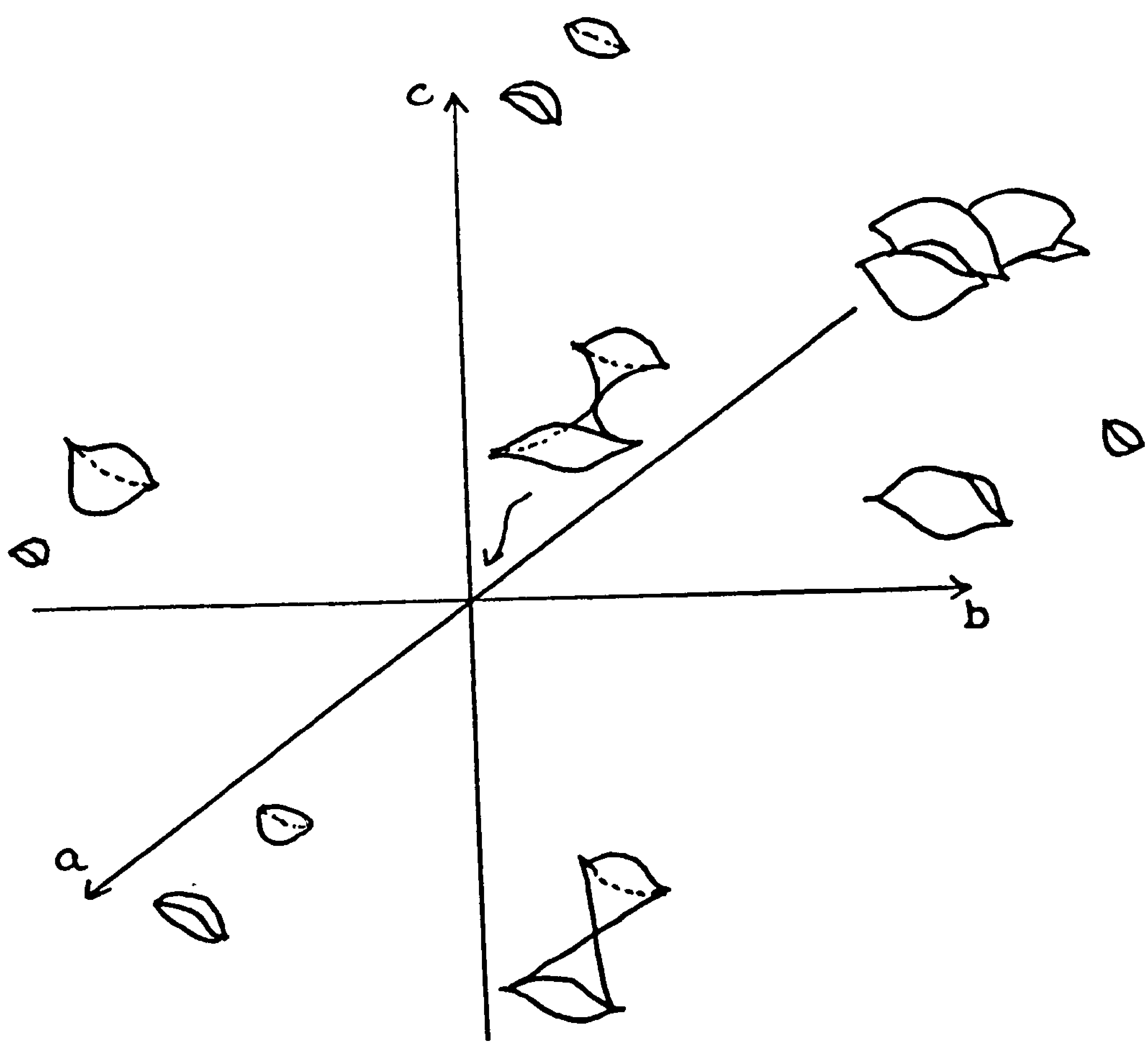


Figure 7.28: $f(x, y, z) = (x, y, z^3 + (y^2 - x^4)z + az + bxz + cx^2z)$

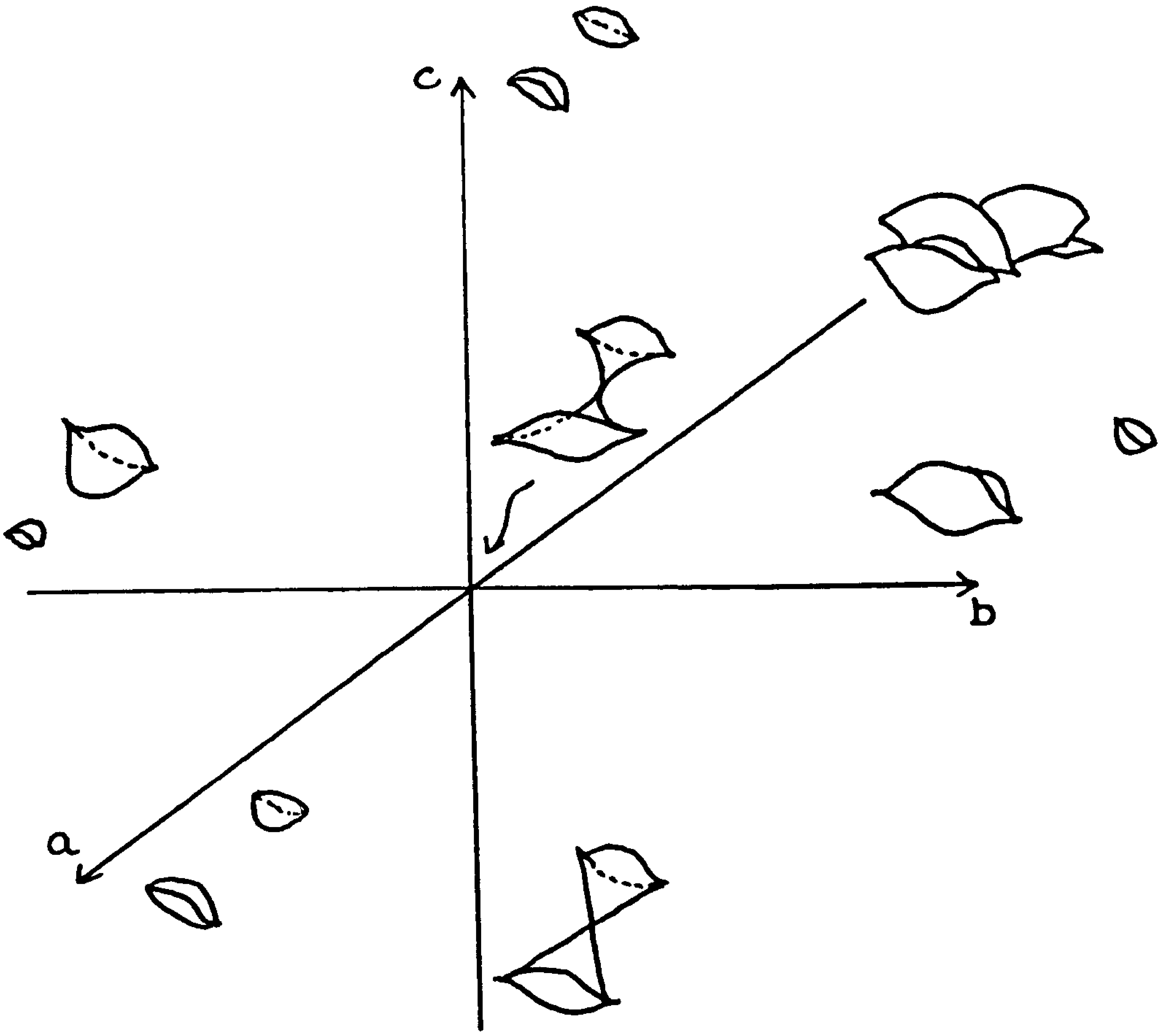


Figure 7.29: $f(x, y, z) = (x, y, z^3 - (x^2 - y^4)z + az + byz + cy^2z)$

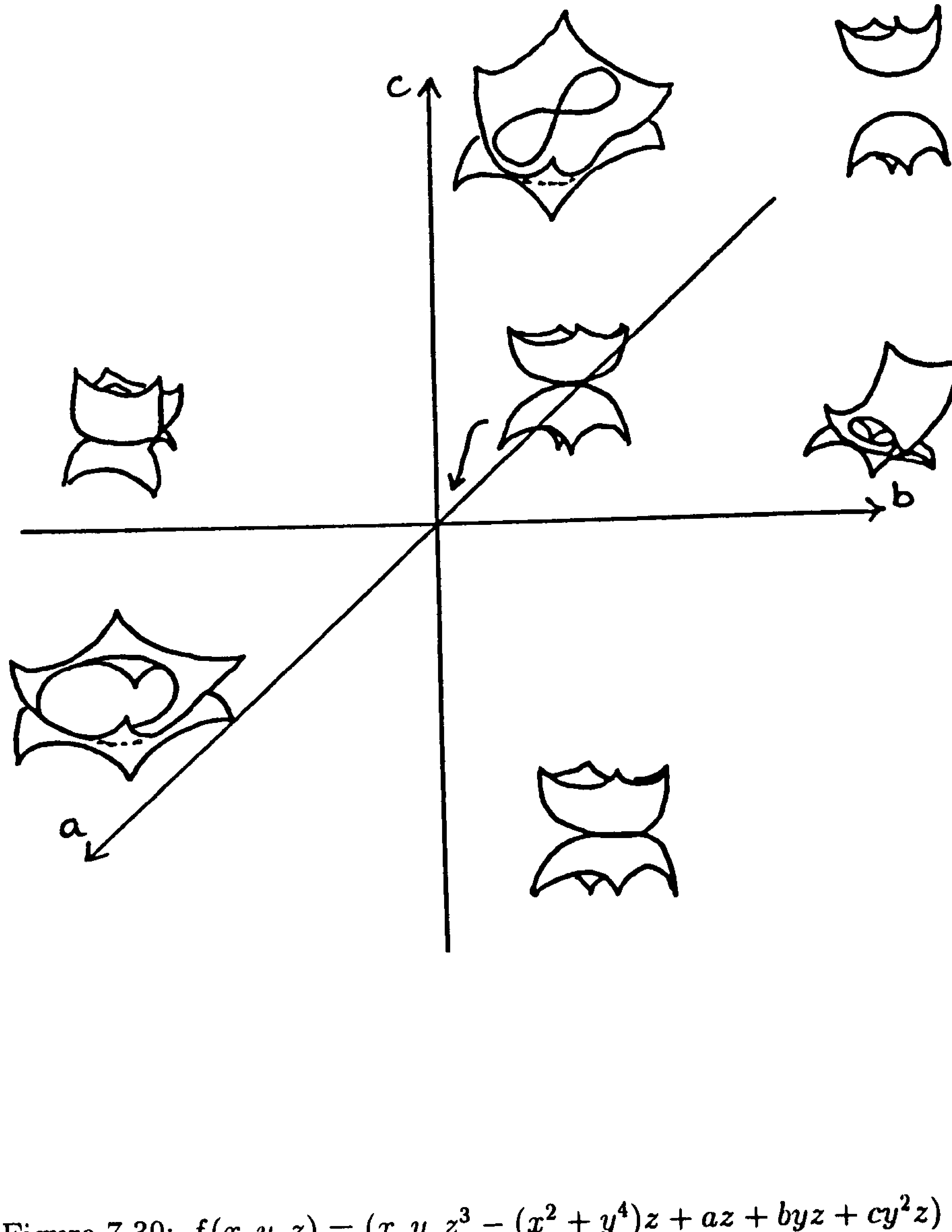


Figure 7.30: $f(x, y, z) = (x, y, z^3 - (x^2 + y^4)z + az + byz + cy^2z)$

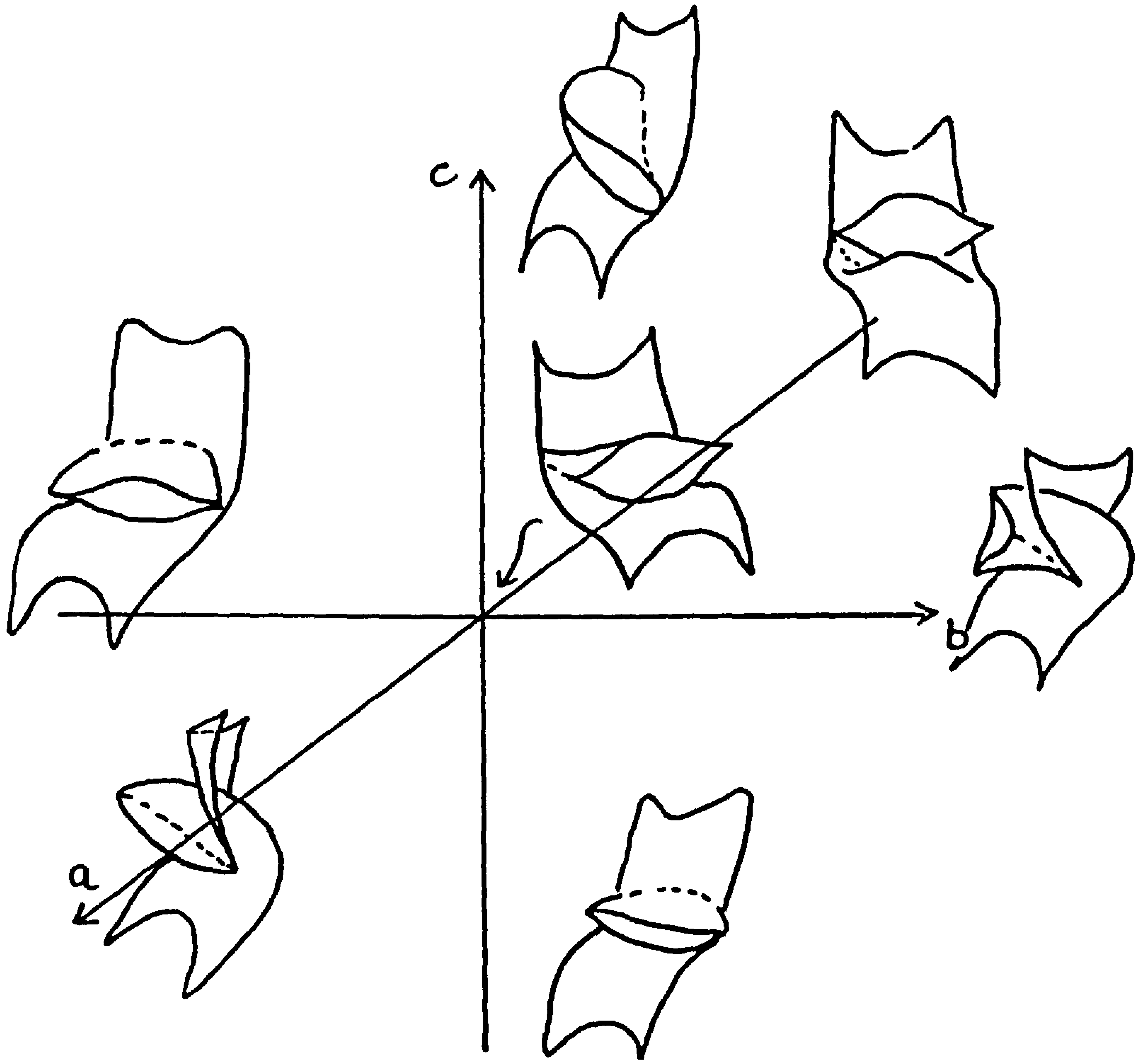


Figure 7.31: $f(x, y, z) = (x, yz + x^2y + y^5, y^2 + z^2 + xy + xz + ay + bxy + cy^3)$

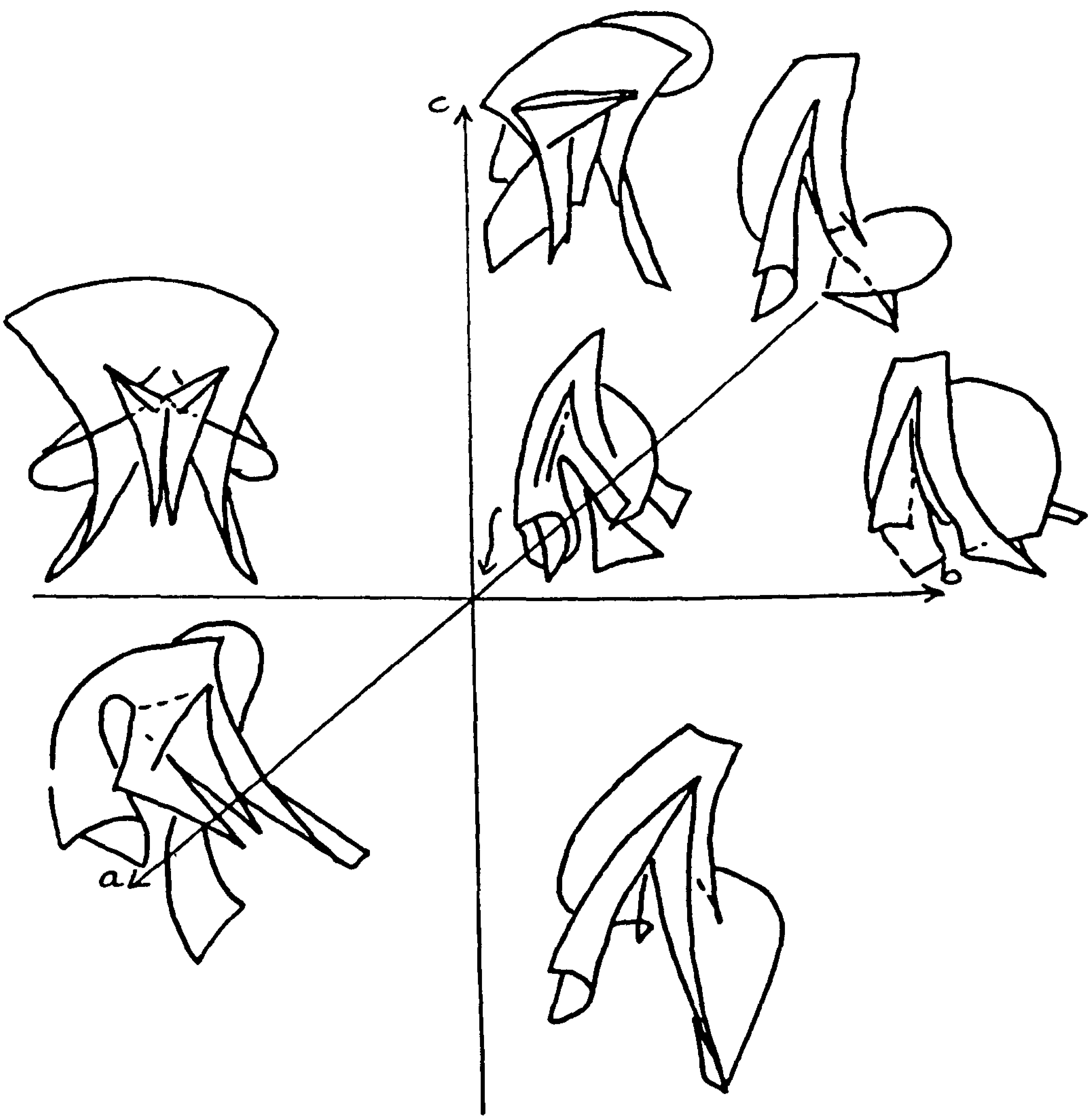


Figure 7.32: $f(x, y, z) = (x, yz + x^2y - y^5, y^2 + z^2 + xy + xz + ay + bxy + cy^3)$

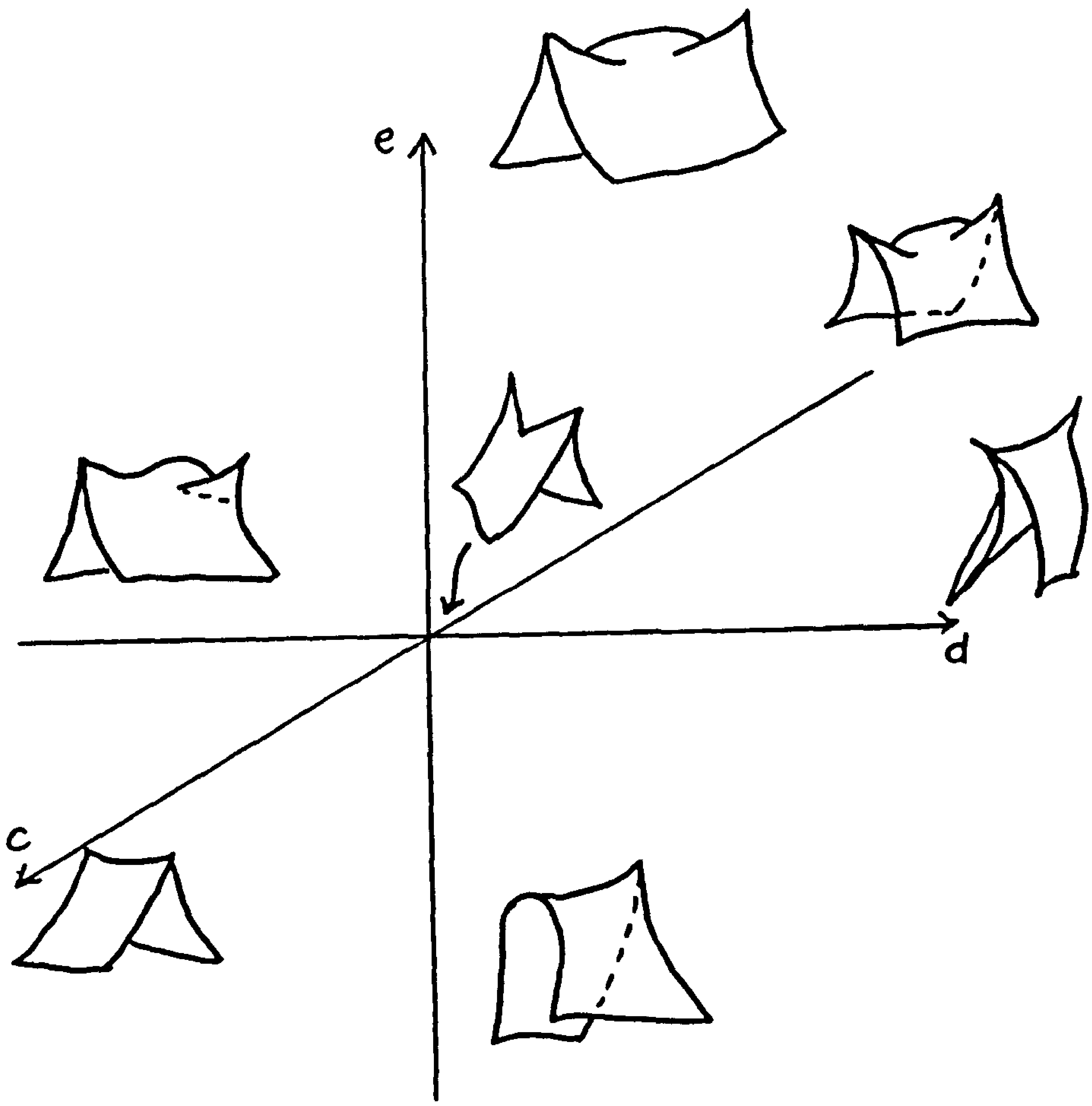


Figure 7.33: $f(x,y,z) = (x, yz, y^2 + xy + z^3 + xz^2 + ax^2z + bz^5 + cz + dz^2 + exz)$,
$a = b = 1$

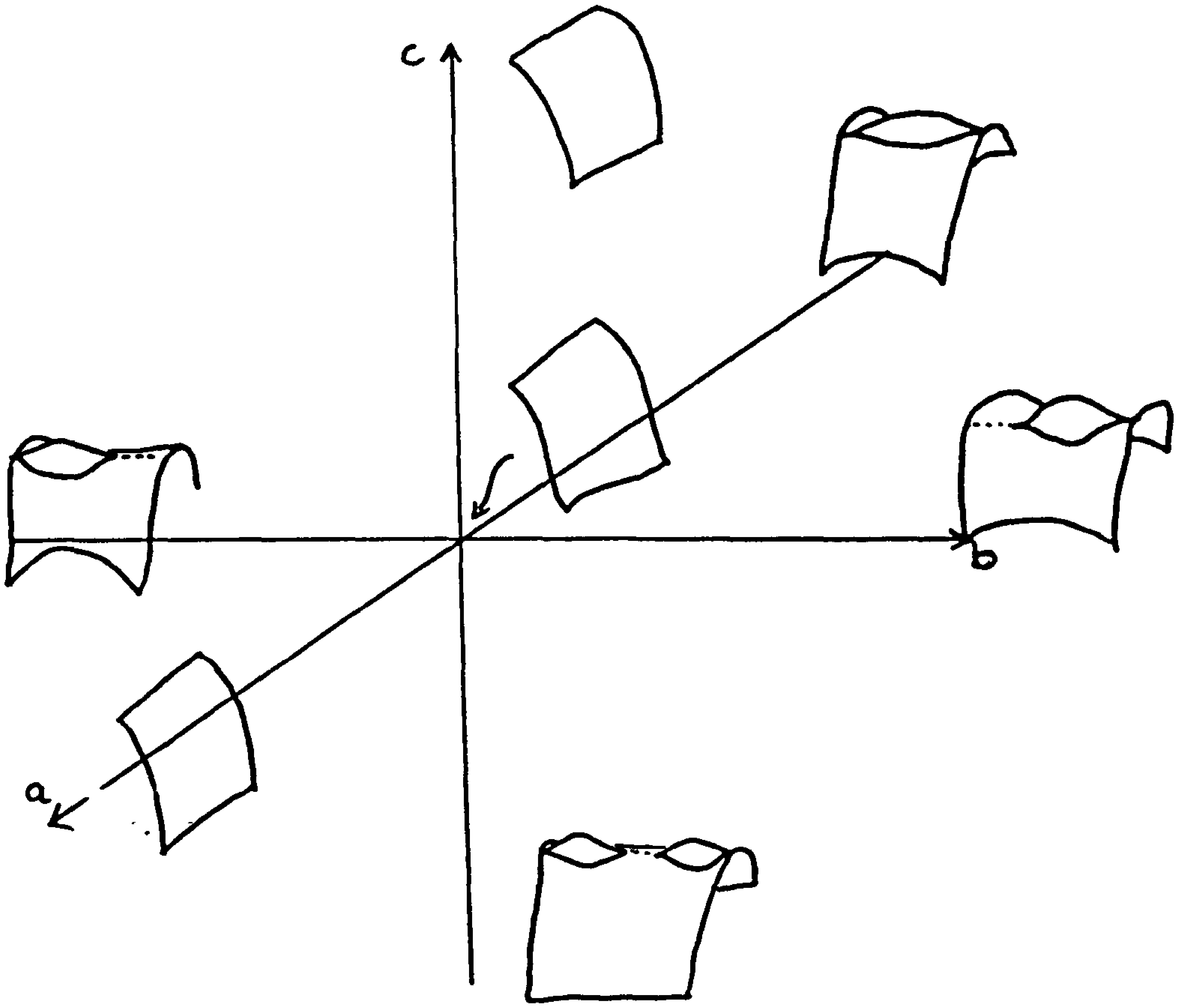


Figure 7.34: $f(x, y, z) = (x, y, yz + z^4 + x^2z^2 + az^2 + bxz^2 + cx^2z^2)$

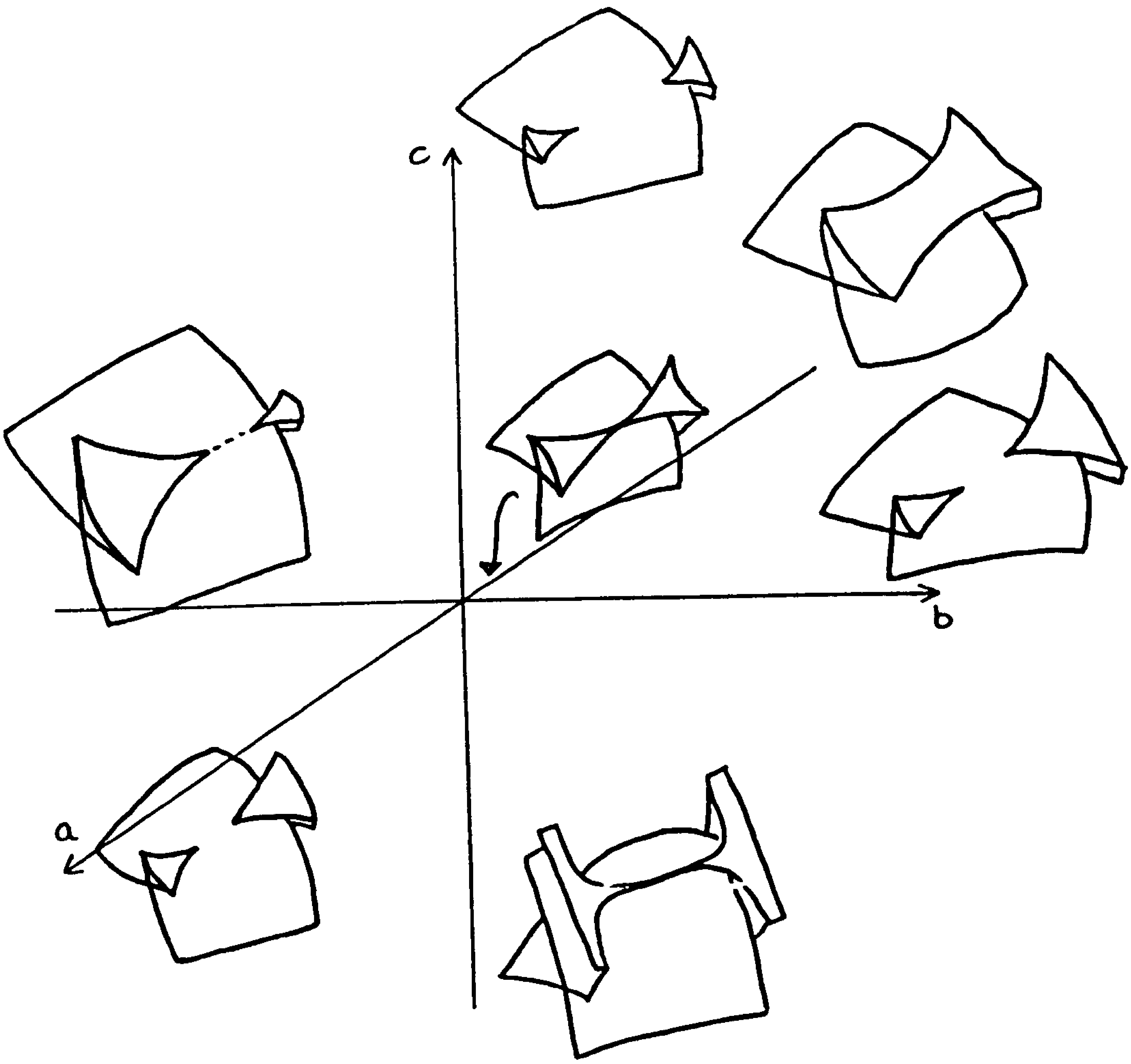


Figure 7.35: $f(x, y, z) = (x, y, yz + z^4 - x^2z^2 + az^2 + bxz^2 + cx^2z^2)$

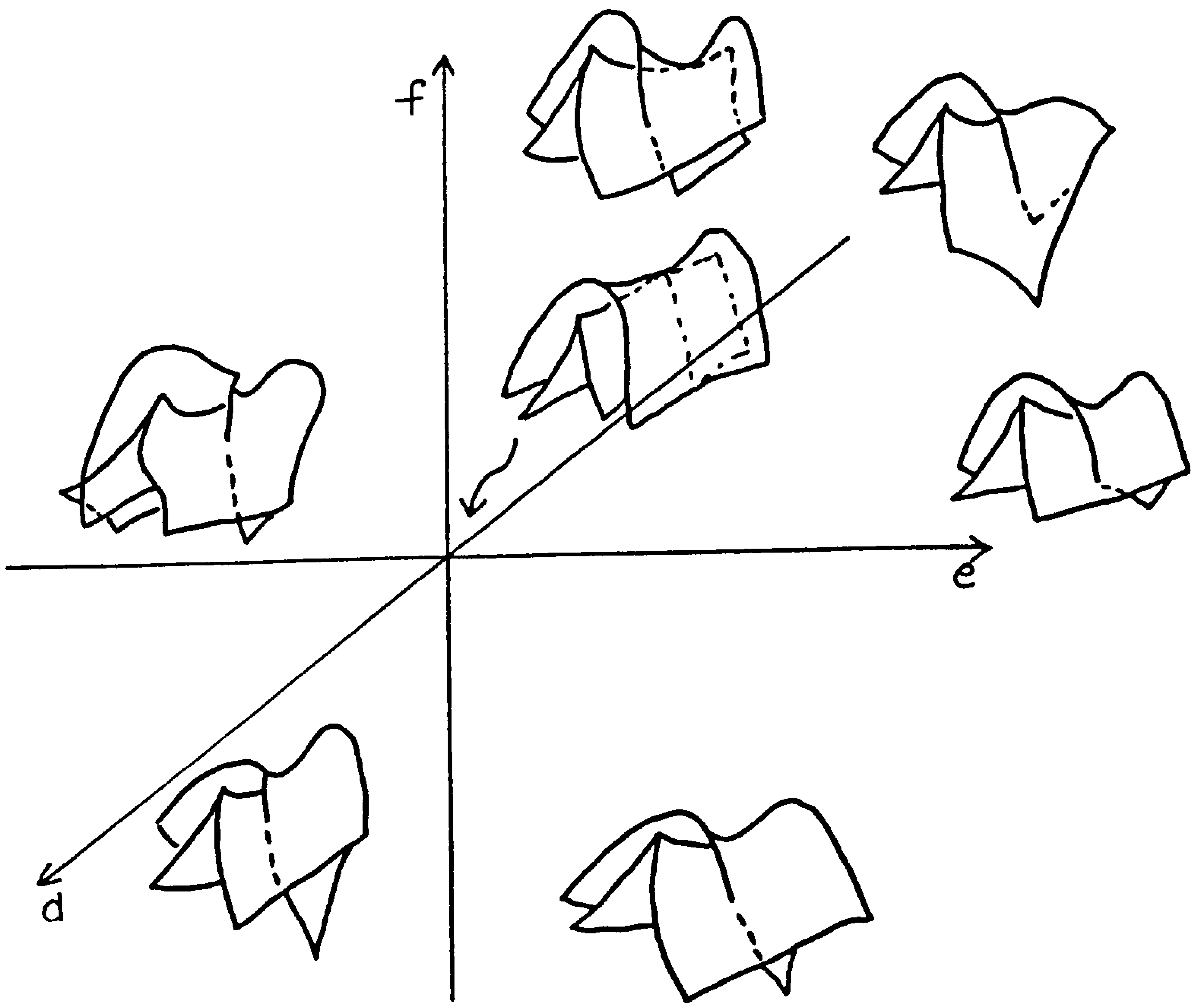


Figure 7.36: $f(x, y, z) = (x, yz, y^2 + xy + xz + z^4 \pm yz^3 + az^5 + byz^4 + cz^6 + dy + ez^2 + fz^3)$, $a = b = c = 1$

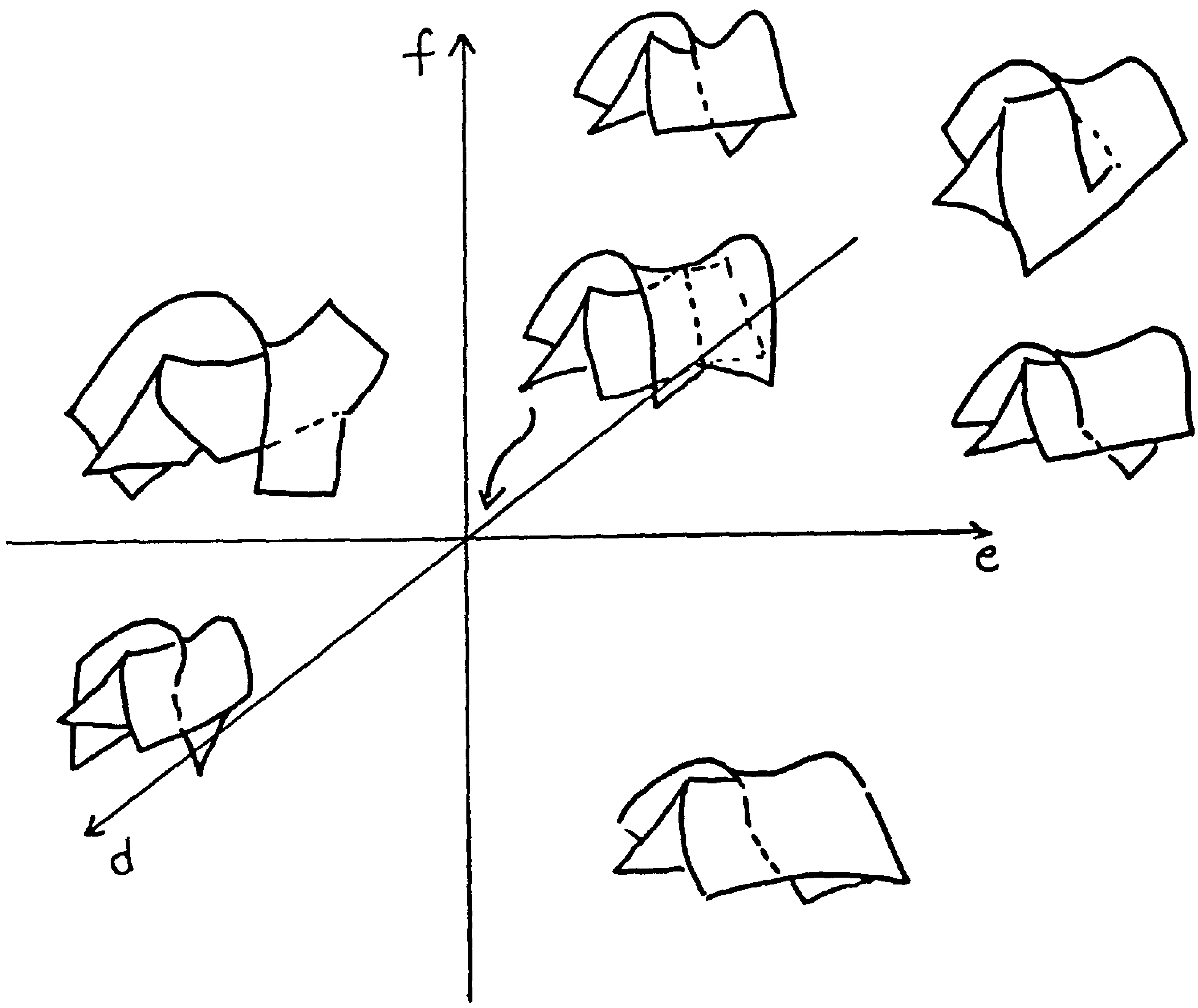


Figure 7.37: $f(x,y,z) = (x, yz, y^2 + xz + z^4 \pm yz^3 + az^5 + byz^4 + cz^6 + dy + ez^2 + fz^3)$, $a = b = c = 1$

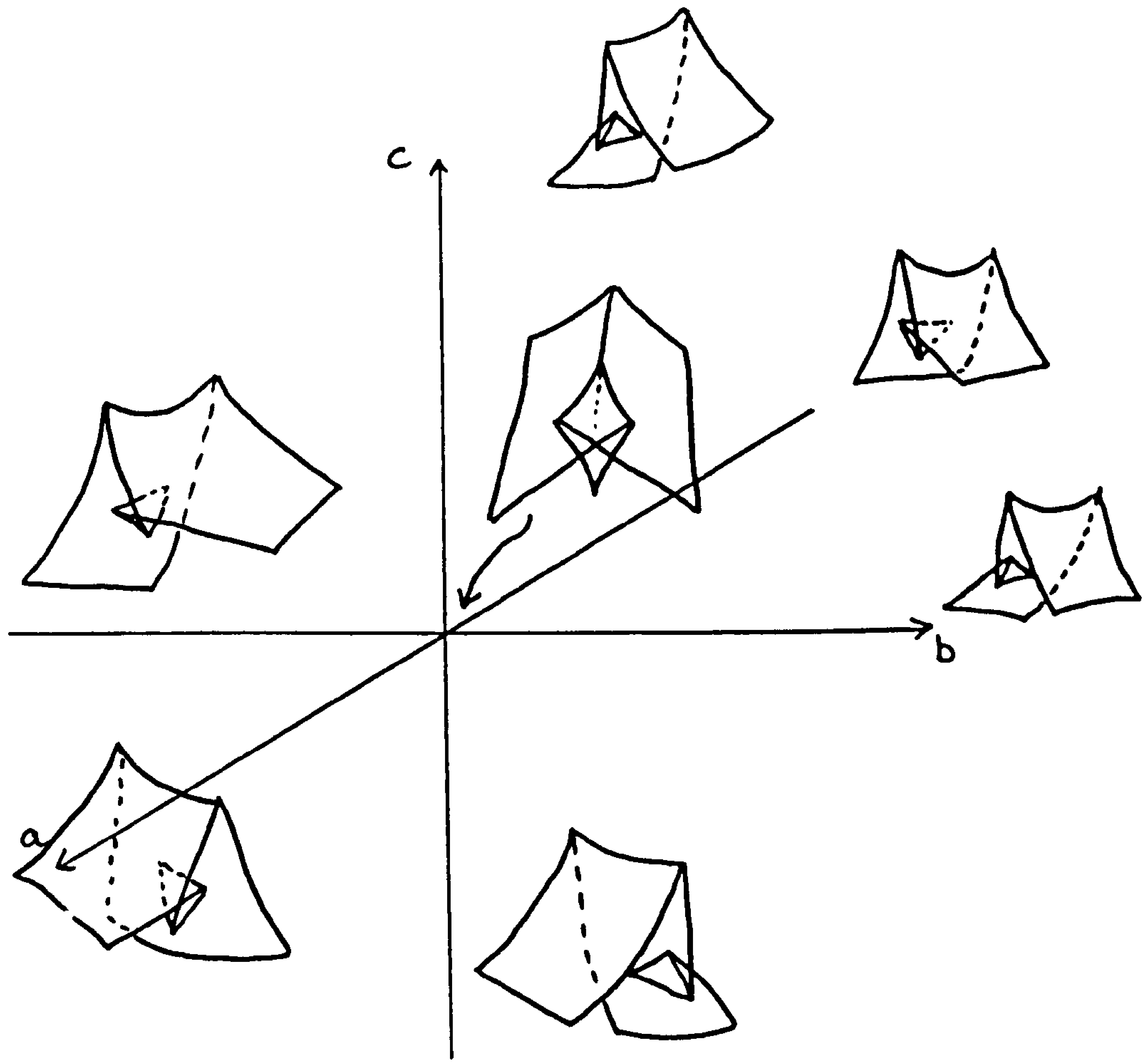


Figure 7.38: $f(x, y, z) = (x, y, yz + xz^3 + z^5 + az^2 + bz^4 + cz^6)$

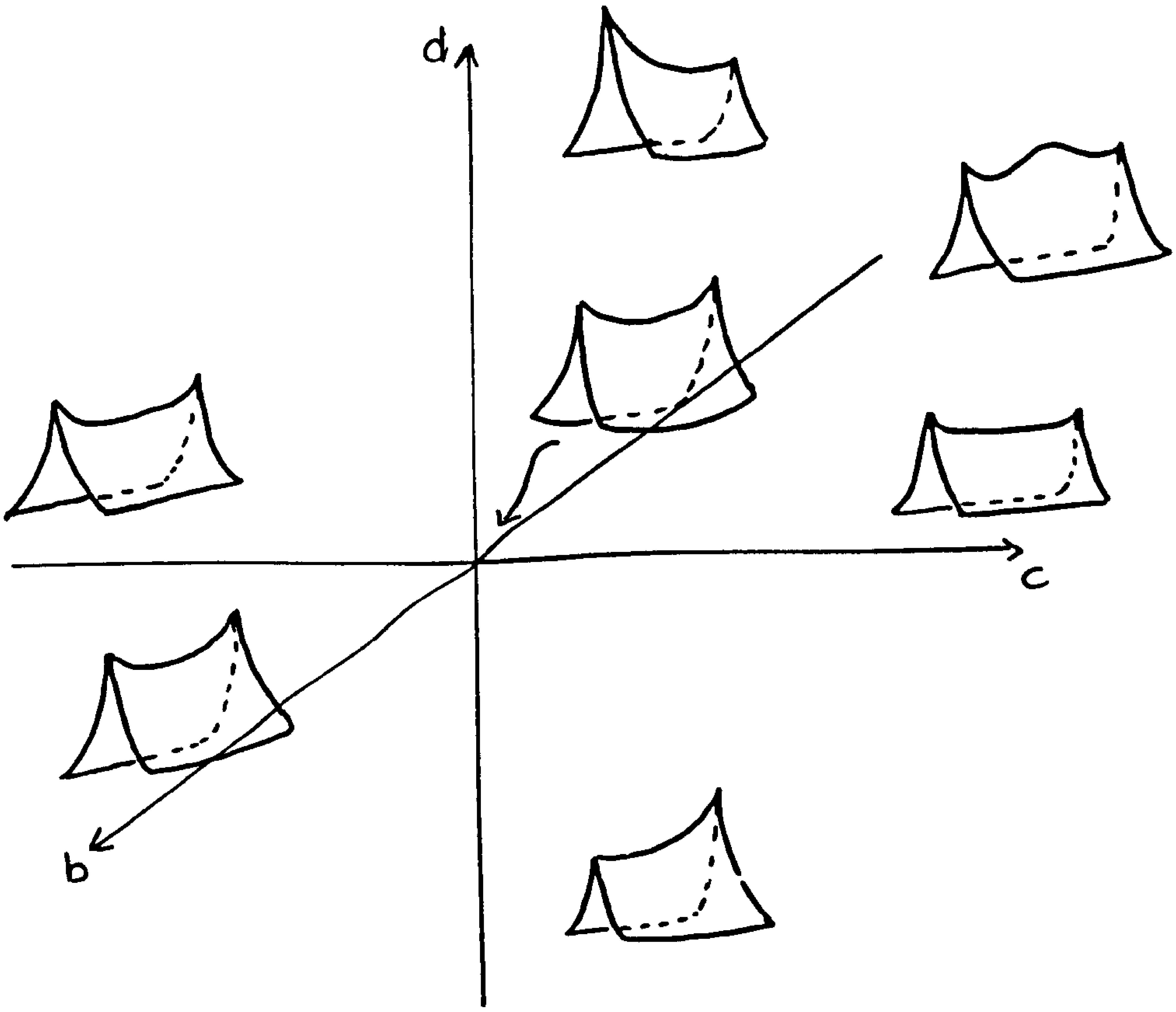


Figure 7.39: $f(x, y, z) = (x, y, yz + x^2z^2 + z^5 \pm z^6 + az^7 + bz^2 + cz^3 + dxz^3)$, $a = 1$

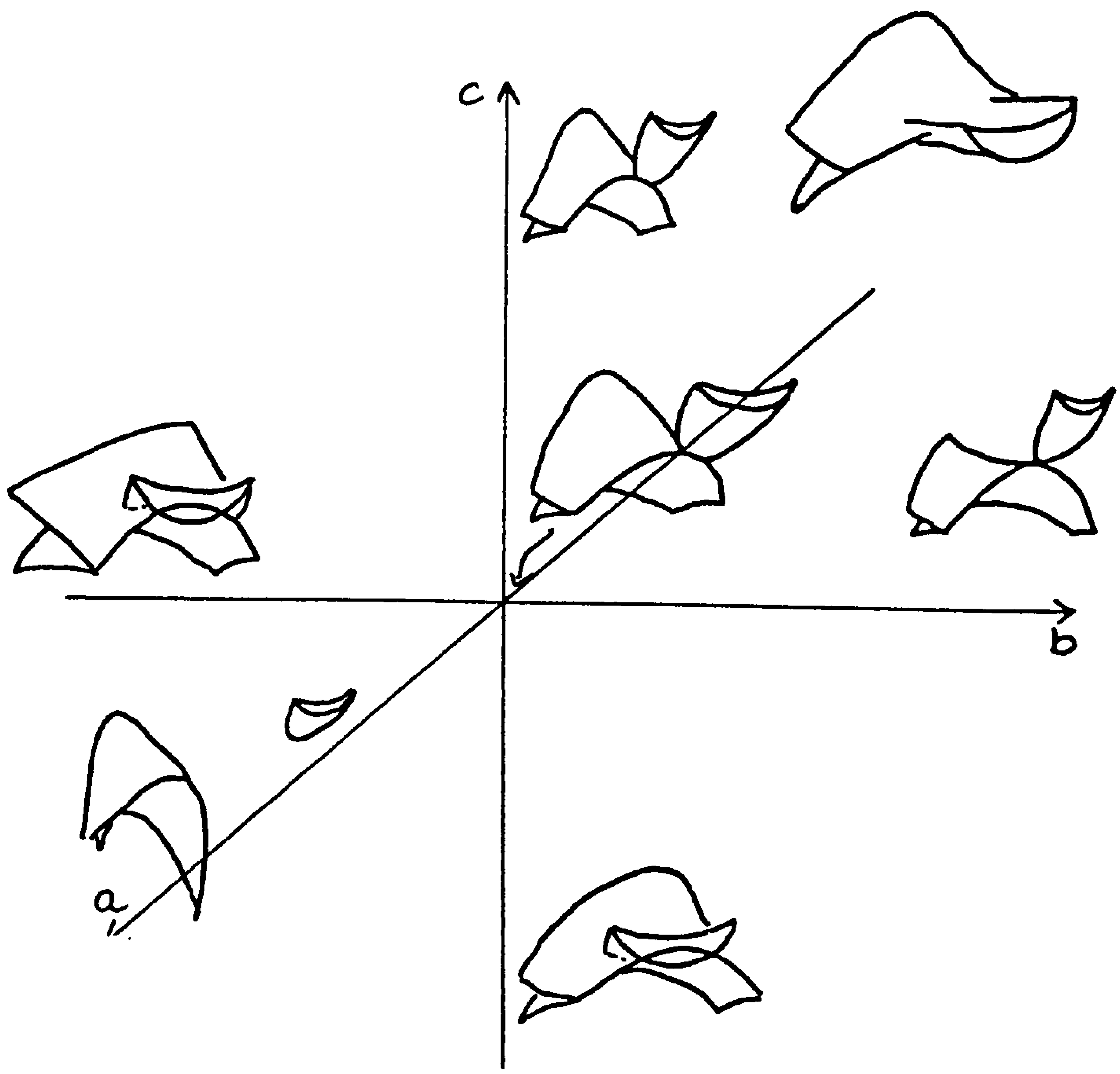


Figure 7.40: $f(x, y, z) = (x, y, yz^2 + x^2z + z^4 + z^7 + az + bz^3 + cz^5)$

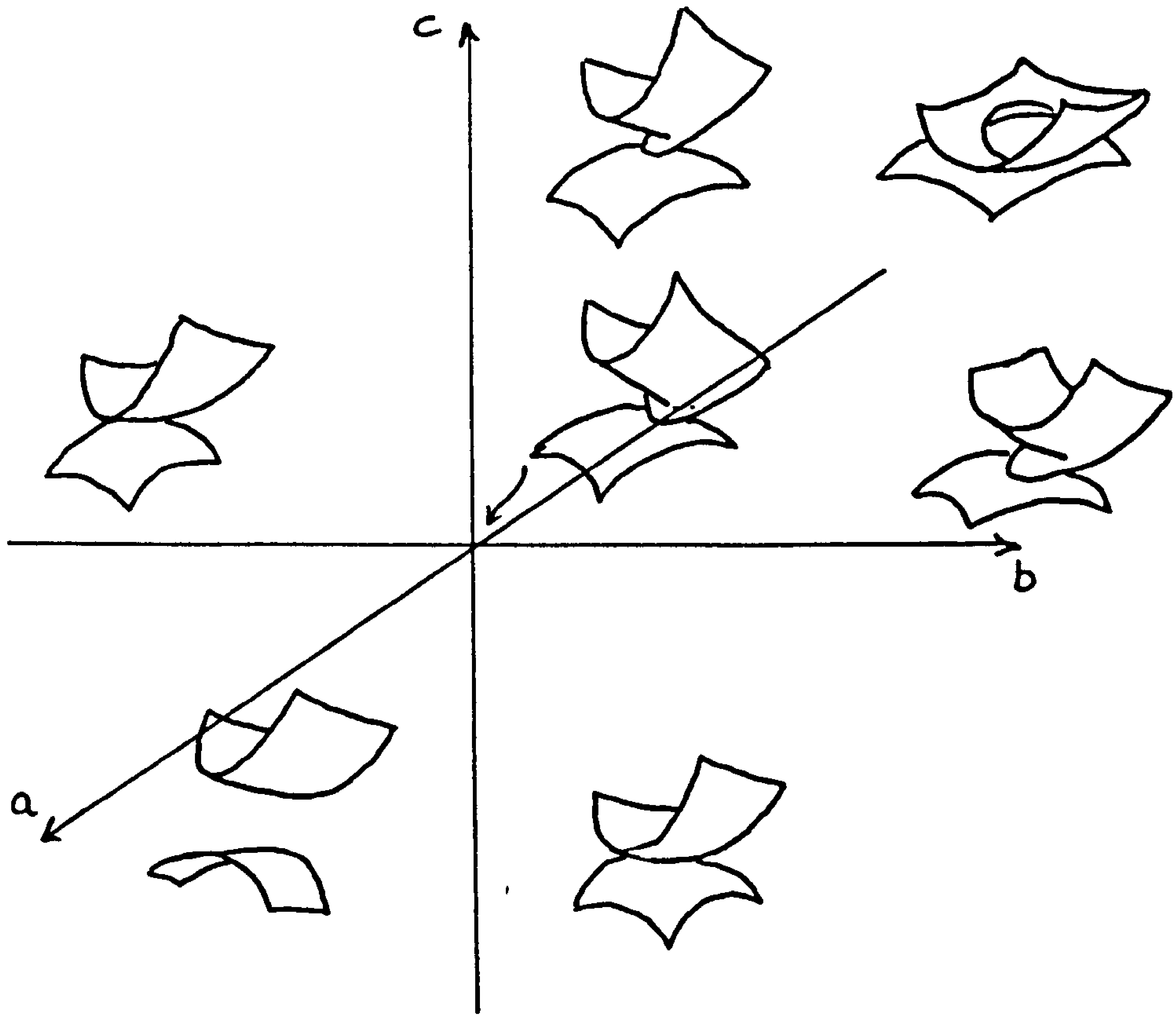


Figure 7.41: $f(x,y,z) = (x, y, yz^2 + x^2z + z^4 - z^7 + az + bz^3 + cz^5)$

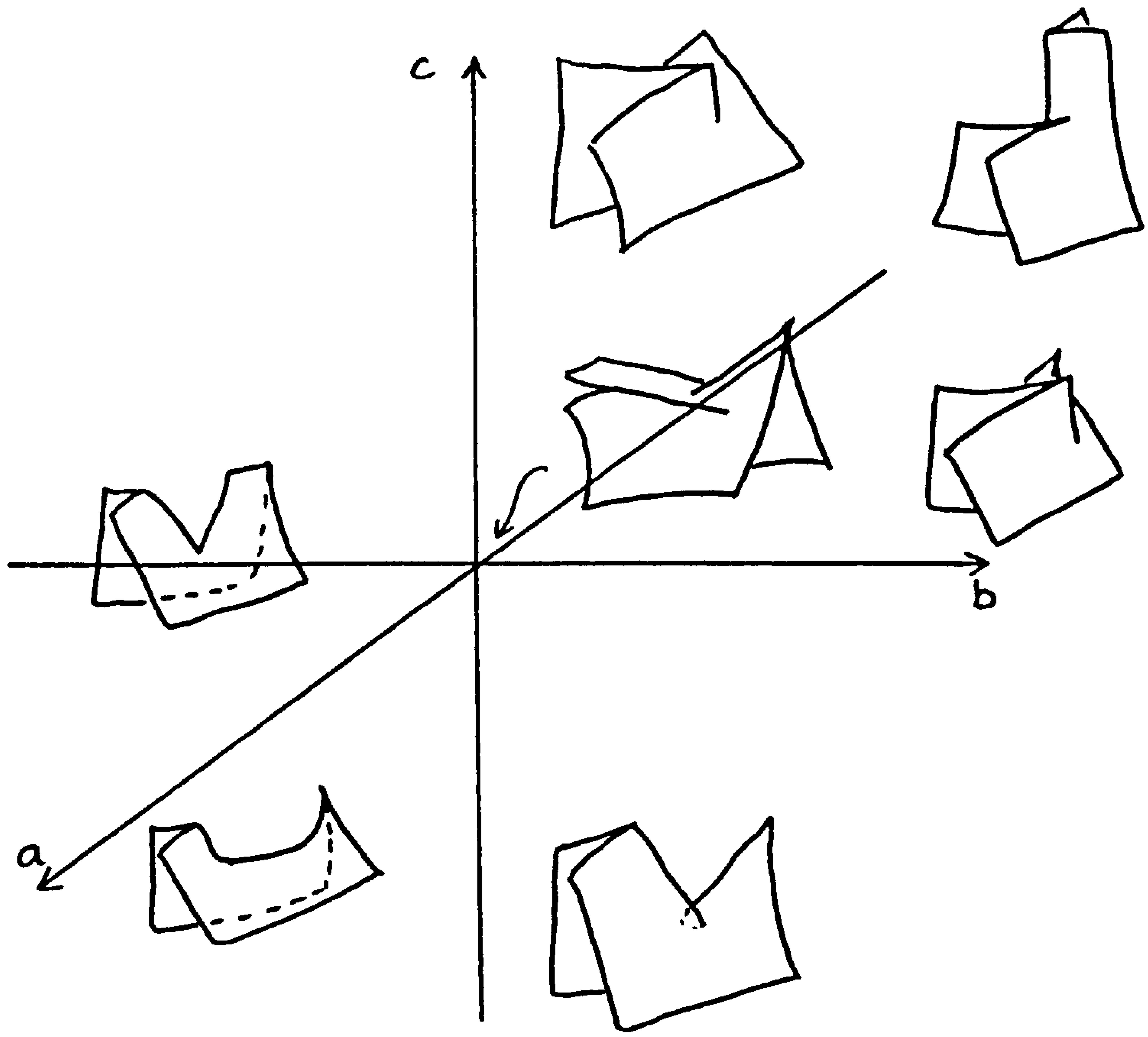


Figure 7.42: $f(x, y, z) = (x, y, yz + xz^2 + z^6 + z^9 + az^3 + bz^4 + cz^8)$

Appendix A

Quasihomogeneous functions

First we need some definitions.

Definition A.0.1 *1. If we consider the space* $\mathbf{C}^n$ *with fixed co-ordinates* $x_1, ..., x_n$. *A function* $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ *is said to be quasihomogeneous of degree* d *with exponents* $\alpha_1, ..., \alpha_n$ *if*

$$f(\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, ..., x_n) \text{ for all } \lambda.$$

2. *A quasihomogeneous function* f *is said to be non-degenerate if 0 is an isolated critical point.*

3. *We say that the monomial* $\mathbf{x}^k = x_1^{k_1} ... x_n^{k_n}$ *has degree* d *if* $\langle \alpha, \mathbf{k} \rangle = \alpha_1 k_1 + ... + \alpha_n k_n = d$.

4. *A polynomial has filtration* d *if all its monomials are of degree* d *or higher.*

5. *A polynomial is said to be semiquasihomogeneous of degree* d *with exponents* $\alpha_1, ..., \alpha_n$ *if it is of the form* $f = f_0 + f'$, *where* f_0 *is a non-degenerate quasihomogeneous polynomial of degree* d *with exponents* α, *and* f' *is a polynomial of filtration* $> d$.

Theorem A.0.2 *[A1]*

The multiplicity of the critical point 0 of a semiquasihomogeneous function f is that of its quasihomogeneous part: $\mu(f) = \mu(f_0)$.

Corollary A.0.3 *Assume that the system of monomials $e_1, ..., e_\mu$ is a basis of the local ring of the quasihomogeneous part f_0 of a semiquasihomogeneous function f. Then this same system of monomials is a basis of the local ring of f.*

Remark A.0.4 *The number of basis monomials of the local ring of a quasihomogeneous or semiquasihomogeneous function f having given degree d does not depend on the choice of the basis in the local ring.*

If we consider the local ring of a quasihomogeneous or semiquasihomogeneous function f of degree d and fix a system of monomials forming a basis for this ring.

Definition A.0.5 *A monomial is said to be upper or to lie above the diagonal (or lower, or diagonal) if it has degree greater than d (or less than d, or equal to d) for given quasihomogeneous exponents.*

Let $e_1, ..., e_s$ be the system of all upper basis monomials in a fixed basis of the local ring of the function f_0.

Lemma A.0.6 *Let f_0 be a quasihomogeneous function of degree d and $e_1, ..., e_r$ the set of all basis monomials of fixed degree $d' > d$ in the local ring of f_0. Then every series of the form $f_0 + f_1$, where the filtration of f_1 is greater than d, can be brought by a formal diffeomorphism to the form $f_0 + f_1'$, where the terms in f_1' of degree less than d' are the same as in f_1, and the terms of degree d' reduce to $c_1e_1 + ... + c_re_r$.*

Theorem A.0.7 *Every semiquasihomogeneous function with quasihomogeneous part f_0 is equivalent to a function of the form $f_0 + \sum c_k e_k$, where the c_k are constants.*

Proof. Using the above lemma.

□

Let $\alpha_1, ..., \alpha_p$ be a fixed collection of p quasihomogeneous types. We recall that the monomial $\mathbf{x}^{\mathbf{k}}$ is of degree $\langle \alpha_i, \mathbf{k} \rangle = \phi_i(\mathbf{k})$ in the i-th filtration. We define the piecewise degree of $\mathbf{x}^{\mathbf{k}}$ to be $\phi(\mathbf{k}) = \min[\phi_i(\mathbf{k}), ..., \phi_p(\mathbf{k})]$.

Definition A.0.8 *A power series has piecewise filtration d if all its monomials have piecewise filtration d or higher.*

The sum of the terms of lowest (piecewise) degree in a given power series is called the principal part of the series. A (piecewise) homogeneous function of degree d is a polynomial whose monomials all have (piecewise) degree d.

Definition A.0.9 *A piecewise homogeneous function f_0 of degree d satisfies condition A if for every function g of filtration $d + \delta > d$ in the ideal spanned by the derivatives of f_0 there is a decomposition*

$$g = \sum \frac{\delta f_0}{\delta x_i} v_i + g',$$

where the vector field $\mathbf{v}$ *has filtration δ, and the function g' has filtration greater than $d + \delta$.*

Note that a quasihomogeneous function always satisfies condition A.

We consider a basis of the local ring of a piecewise homogeneous function f_0 of finite multiplicity μ.

Definition A.0.10 *A basis $e_1, ..., e_\mu$ of homogeneous elements is said to be regular if, for each D, the elements of the basis of degree D are independent modulo the sum of the ideal $I = (\delta f / \delta x_0)$ and the space $E_{>D}$ of functions of filtrations greater than D.*

Proposition A.0.11 *There always exists a regular basis, in fact, one consisting entirely of monomials.*

Theorem A.0.12 *If the principal part f_0 of a function f satisfies condition A and has finite multiplicity μ, then f can be reduced by a diffeomorphism to the form $f_0 + c_1 e_1 + ... + c_s e_s$, where $e_1, ..., e_s$ are the superdiagonal monomials of the regular basis.*

Appendix B

Determinacy Results

- (x, y).

  This is clearly 1-$\mathcal{A}_1$-determined.

- $(x, y^2 \pm z^2)$.

$$TA_1 \cdot f = \mathcal{M}_3^2\langle(1,0),(0,y),(0,z)\rangle + f^*\mathcal{M}_2^2\{e_1,e_2\}.$$

We want to check to see if this germ is 2-$\mathcal{A}_1$-determined, so we want

$$\mathcal{M}_3^3 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^3(f^*\mathcal{M}_2 + \mathcal{M}_3^3).\mathcal{E}(3,2).$$

Clearly this is the case, so $(x, y^2 \pm z^2)$ is 2-$\mathcal{A}_1$-determined.

- $(x, xy + y^3 \pm z^2)$.

$$TA_1 \cdot f = \mathcal{M}_3^2\langle(1,y),(0,x+3y^2),(0,z)\rangle + f^*\mathcal{M}_2^2\{e_1,e_2\}.$$

We want to check to see if this germ is 3-$\mathcal{A}_1$-determined, so we want

$$\mathcal{M}_3^4 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^4(f^*\mathcal{M}_2 + \mathcal{M}_3^4).\mathcal{E}(3,2).$$

So we have

$$\begin{gathered}\phi e_i \text{ with } \phi \in \mathcal{M}_3^8\\ x\phi e_i \text{ with } \phi \in \mathcal{M}_3^4\\ y^3\phi e_i \text{ with } \phi \in \mathcal{M}_3^4\end{gathered}$$

$$z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^4	$\surd$	$\surd$
x^3y	$\surd$	$\surd$
x^2y^2	$\surd$	$\surd$
xy^3	$\surd$	$\surd$
y^4	$\surd$	$\surd$
y^5	$\surd$	$\surd$
y^6	$\surd$	$\surd$

We have $(x\phi, 0)$ with $\phi \in \mathcal{M}_3^3$.

$(0, x^3y) \longleftrightarrow (x^3, 0)$ which we have from $(x^4, 0)$.

$(0, x^2y^2) \longleftrightarrow (0, x^2)$ which we have from $(0, x^4)$.

$(0, xy^3) \longleftrightarrow (0, x^2y)$ which we have from (x^2, x^2y).

$(y^6, 0) \longleftrightarrow (0, y^7)$ which we have from $y^3(0, y^4)$.

$(y^5, 0) \longleftrightarrow (0, y^6) \longleftrightarrow (0, xy^4)$ which we have from $x(0, y^4)$.

$(y^4,) \longleftrightarrow (0, y^5) \longleftrightarrow (0, xy^3)$.

$(0, y^4) \longleftrightarrow (0, xy^2)$. If we use the nilpotent vector field $x\frac{\partial}{\partial y}$ we have $x(0, x + 3y^2)$.

So we find that $(x, xy + y^3 \pm z^2)$ is not 3-$\mathcal{A}_1$-determined, but it is 3-$\mathcal{A}$-determined.

- $(x, y^3 \pm x^2y \pm z^2)$.

$$T\mathcal{A}_1 \cdot f = \mathcal{M}_3^2\langle(1, \pm 2xy), (0, 3y^2 \pm x^2), (0, z)\rangle + f^*\mathcal{M}_2^2\{e_1, e_2\}.$$

We want to check to see if this germ is 3-$\mathcal{A}_1$-determined, so we want

$$\mathcal{M}_3^4 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^4(f^*\mathcal{M}_2 + \mathcal{M}_3^4).\mathcal{E}(3, 2).$$

So we have

$$\phi e_i \text{ with } \phi \in \mathcal{M}_3^8$$
$$x\phi e_i \text{ with } \phi \in \mathcal{M}_3^4$$
$$y^3\phi e_i \text{ with } \phi \in \mathcal{M}_3^4$$

$$z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2.$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^4	$\surd$	$\surd$
x^3y	$\surd$	$\surd$
x^2y^2	$\surd$	$\surd$
xy^3	$\surd$	$\surd$
y^4	$\surd$	$\surd$
y^5	$\surd$	$\surd$
y^6	$\surd$	$\surd$

We have $(x\phi, 0)$ with $\phi \in \mathcal{M}_3^3$.

$(0, x^3y) \longleftrightarrow (x^2, 0)$ which we have from $(x^i, 0)$.

$(0, x^2y^2) \longleftrightarrow (0, x^4)$ which we have from $(0, x^i)$.

$(0, xy^3) \longleftrightarrow (0, x^3y)$.

$(y^6, 0) \longleftrightarrow (0, y^7)$ which we have from $y^3(0, y^4)$.

$(y^5, 0) \longleftrightarrow (0, y^6) \longleftrightarrow (0, x^2y^4)$.

$(y^4, 0) \longleftrightarrow (0, y^5) \longleftrightarrow (0, x^2y^3)$.

$(0, y^4) \longleftrightarrow (0, x^2y^2)$.

So we see that $(x, y^3 \pm x^2y \pm z^2)$ is 3-$\mathcal{A}_1$-determined.

- $(x, xy + y^4 \pm z^2)$.

$$T\mathcal{A}_1 \cdot f = \mathcal{M}_3^2\langle(1, y), (0, x + 4y^3), (0, z)\rangle + f^*\mathcal{M}_2^2\{e_1, e_2\}.$$

We want to check to see if this germ is 4-$\mathcal{A}_1$-determined, so we want

$$\mathcal{M}_3^5 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^5(f^*\mathcal{M}_2 + \mathcal{M}_3^5).\mathcal{E}(3,2).$$

So we have

$$\phi e_i \text{ with } \phi \in \mathcal{M}_3^10$$
$$x\phi e_i \text{ with } \phi \in \mathcal{M}_3^5$$
$$y^4\phi e_i \text{ with } \phi \in \mathcal{M}_3^5$$

$$z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2 .$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^5	$\surd$	$\surd$
x^4y	$\surd$	$\surd$
x^3y^2	$\surd$	$\surd$
x^2y^3	$\surd$	$\surd$
xy^4	$\surd$	$\surd$
y^5	$\surd$	$\surd$
y^6	$\surd$	$\surd$
y^7	$\surd$	$\surd$
y^8	$\surd$	$\surd$

We have $(x\phi, 0)$ with $\phi \in \mathcal{M}_3^4$.

$(0, x^4y) \longleftrightarrow (x^4, 0)$ which we have from $(x^i, 0)$.

$(0, x^3y^2) \longleftrightarrow (0, x^2y^5)$.

$(0, x^2y^3) \longleftrightarrow (0, x^3)$ which we have from $(0, x^i)$.

$(0, xy^4) \longleftrightarrow (0, x^2y) \longleftrightarrow (x^2, 0)$.

$(y^8, 0) \longleftrightarrow (0, y^9)$ which we have from $y^4(0, y^5)$.

$(y^7, 0) \longleftrightarrow (0, y^8) \longleftrightarrow (0, xy^5)$.

$(y^6, 0) \longleftrightarrow (0, y^7) \longleftrightarrow (0, xy^4)$.

$(y^5, 0) \longleftrightarrow (0, y^6) \longleftrightarrow (0, xy^3)$ which we have if we use the nilpotent vector field $x\frac{\partial}{\partial y}$ in the source , i.e. we have $(0, x^2 + 4xy^3)$.

$(0, y^5) \longleftrightarrow (0, xy^2)$. So if we use the nilpotent vector field $(\frac{\partial}{\partial y}, 0)$ in the target we have $(xy + y^4, 0)$ which gives us $(y^4, y^5) - (xy + y^4, 0) + (xy, xy^2) - (0, xy^2 + 4y^5) = (0, -3y^5)$.

So our germ $(x, xy + y^4 \pm z^2)$ is 4-$\mathcal{A}$-determined but not 4-$\mathcal{A}_1$-determined.

- $(x, y^3 \pm x^3y \pm z^2)$.

$$T\mathcal{A}_1 \cdot f = \mathcal{M}_3^2\langle(1, \pm 3x^2y), (0, 3y^2 \pm x^3), (0, z)\rangle + f^*\mathcal{M}_2^2\{e_1, e_2\}.$$

We want to check to see if this germ is 4-$\mathcal{A}_1$-determined, so we want

$$\mathcal{M}_3^5 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^5(f^*\mathcal{M}_2 + \mathcal{M}_3^5).\mathcal{E}(3,2).$$

So we have

$$\begin{gathered} \phi e_i \text{ with } \phi \in \mathcal{M}_3^1 0 \\ x\phi e_i \text{ with } \phi \in \mathcal{M}_3^5 \\ y^3\phi e_i \text{ with } \phi \in \mathcal{M}_3^5 \\ z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2 \ . \end{gathered}$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^5	$\checkmark$	$\checkmark$
x^4y	$\checkmark$	$\checkmark$
x^3y^2	$\checkmark$	$\checkmark$
x^2y^3	$\checkmark$	$\checkmark$
xy^4	$\checkmark$	$\checkmark$
y^5	$\checkmark$	$\checkmark$
y^6	$\checkmark$	$\checkmark$
y^7	$\checkmark$	$\checkmark$

We have $(x\phi, 0)$ with $\phi \in \mathcal{M}_3^4$.

$(0, x^4y) \longleftrightarrow (x^2, 0)$ which we have from $(x^i, 0)$.

$(0, x^3y^2) \longleftrightarrow (0, x^6)$.

$(0, x^2y^3) \longleftrightarrow (0, x^5y)$ which we have from $x(0, x^4y)$.

$(0, xy^4) \longleftrightarrow (0, x^4y^2)$ which we have from $x(0, x^3y^2)$.

$(y^7, 0) \longleftrightarrow (0, y^8)$ which we have from $y^3(0, y^5)$.

$(y^6, 0) \longleftrightarrow (0, y^7) \longleftrightarrow (0, x^3y^5)$.

$(y^5, 0) \longleftrightarrow (0, y^6) \longleftrightarrow (0, x^3y^4)$.

$(0, y^5) \longleftrightarrow (0, x^3y^3)$.

So our germ $(x, y^3 \pm x^3y \pm z^2)$ is 4-$\mathcal{A}_1$-determined .

- $(x, xy^2 + y^4 + y^5 \pm z^2)$.

$$T\mathcal{A}_1 \cdot f = \mathcal{M}_3^2\langle(1, y^2), (0, 2xy + 4y^3 + 5y^4), (0, z)\rangle + f^*\mathcal{M}_2^2\{e_1, e_2\}.$$

We want to check to see if this germ is 5-$\mathcal{A}_1$-determined, and so want

$$\mathcal{M}_3^6 \subset T\mathcal{A}_1 \cdot f + \mathcal{M}_3^6(f^*\mathcal{M}_2 + \mathcal{M}_3^6).\mathcal{E}(3,2).$$

So we have

$$\begin{gathered}
\phi e_i \text{ with } \phi \in \mathcal{M}_3^1 2 \\
x\phi e_i \text{ with } \phi \in \mathcal{M}_3^6 \\
(y^4 + y^5)\phi e_i \text{ with } \phi \in \mathcal{M}_3^6 \\
z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2.
\end{gathered}$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^6	√	√
x^5y	√	√
x^4y^2	√	√
x^3y^3	√	√
x^2y^4	√	√
xy^5	√	√
y^6	√	√
y^7	√	√
y^8	√	√
y^9	√	√
y^{10}	√	√
y^{11}	√	√

We have $(\phi, 0)$ with $\phi \in \mathcal{M}_3^6$.

$(0, x^5y) \longleftrightarrow (0, x^4y^3)$.

$(0, x^4y^2) \longleftrightarrow (0, x^3y^4)$.

$(0, x^3y^3) \longleftrightarrow (0, x^2y^5)$.

$(0, x^2y^4) \longleftrightarrow (0, xy^6)$.

$(0, xy^5) \longleftrightarrow (0, x^2y^3) \longleftrightarrow (0, x^3y) \longleftrightarrow (x^2, 0)$.

$(0, y^{11}) \longleftrightarrow (0, y^12)$ from $(y^4 + y^5)(0, y^7)$.

$(0, y^{10}) \longleftrightarrow (0, y^{11})$.

$(0, y^9) \longleftrightarrow (0, xy^7)$.

$(0, y^8) \longleftrightarrow (0, xy^6)$.

$(0, y^7) \longleftrightarrow (0, xy^5)$.

$(0, y^6) \longleftrightarrow (0, xy^4) \longleftrightarrow (0, x^2y^2)$ which we get from $(0, x^2y^2 + 4xy^4)$ and $(0, x^2y^2 + xy^4)$.

So our germ $(x, xy^2 + y^4 + y^5 \pm z^2)$ is 5-$\mathcal{A}_1$-determined.

- $(x, xy + y^5 \pm y^7 \pm z^2)$.

$$TA_1 \cdot f = \mathcal{M}_3^2\langle(1, y), (0, x + 5y^4 \pm 7y^6), (0, z)\rangle + f^*\mathcal{M}_2^2\{e_1, e_2\}.$$

We want to check to see if this germ is 7-$\mathcal{A}_1$-determined, and so want

$$\mathcal{M}_3^8 \subset TA_1 \cdot f + \mathcal{M}_3^8(f^*\mathcal{M}_2 + \mathcal{M}_3^8).\mathcal{E}(3, 2).$$

So we have

$$\begin{gathered} \phi e_i \text{ with } \phi \in \mathcal{M}_3^{1}6 \\ x\phi e_i \text{ with } \phi \in \mathcal{M}_3^8 \\ (y^5 \pm y^7)\phi e_i \text{ with } \phi \in \mathcal{M}_3^8 \\ z\phi e_i \text{ with } \phi \in \mathcal{M}_3^2. \end{gathered}$$

So we are looking for

	$\mathbf{e_1}$	$\mathbf{e_2}$
x^8	√	√
x^7y	√	√
x^6y^2	√	√
x^5y^3	√	√
x^4y^4	√	√
x^3y^5	√	√
x^2y^6	√	√
xy^7	√	√
y^8	√	√
y^9	√	√
y^{10}	√	√
y^{11}	√	√
y^{12}	√	√
y^{13}	√	√
y^{14}	√	√
y^{15}	√	√

We have $(x\phi, 0)$ with $\phi \in \mathcal{M}_3^7$.

$(0,x^7y) \longleftrightarrow (x^7,0)$ which we have from $(x^7,0)$.

$(0,x^6y^2) \longleftrightarrow (0,x^5y^6)$.

$(0,x^5y^3) \longleftrightarrow (0,x^4y^7)$.

$(0,x^4y^4) \longleftrightarrow (0,x^5)$.

$(0,x^3y^5) \longleftrightarrow (0,x^2y^9)$.

$(0,x^2y^6) \longleftrightarrow (0,xy^10)$.

$(0,xy^7) \longleftrightarrow (0,x^2y^3) \longleftrightarrow (x^2y^2,0) \longleftrightarrow (xy^6,0)$ and $(y^{10},0) \longleftrightarrow (0,y^{11})$. So we have $(0,xy^7+y^{11}\pm y^{13})$, $(0,x^2y^3+xy^7)$, (x^2y^2,x^2y^3), $(x^2y^2+2xy^6+y^{10},0)$, (xy^6,xy^7) and (y^{10},y^{11}). Now we have $(y^{15},0)$ and $(0,y^{15})$ comes from $(y^5\pm y^7)(0,y^{10})$.Similarly for $(0,y^{14})$ and $(0,y^{13})$. This gives us $(0,xy^7+5y^{11})-5(y^{10},y^{11})+5(x^2y^2+2xy^6+y^{10},0)-5(x^2y^2,x^2y^3)+5(0,x^2y^3+5xy^7)-10(xy^6,xy^7)=(0,-4xy^7)$.

$(0,y^{12}) \longleftrightarrow (0,xy^8)$.

$(0,y^{10}) \longleftrightarrow (0,xy^6)$ and $(0,x^2y^2)$. We have $(0,xy^6+5y^{10})$, $(0,x^2y^2+5xy^6)$ and $(0,x^2y^2+2xy^6+y^{10})$.

$(0,y^9) \longleftrightarrow (0,xy^5) \longleftrightarrow (0,x^2y)$.

$(0,y^8) \longleftrightarrow (0,xy^4)$. Using the nilpotent vector field $x\frac{\partial}{\partial y}$ in the source we have $(0,x^2+5xy^4)$.

So our germ $(x,xy^2+y^4+y^5\pm z^2)$ is 7-$\mathcal{A}$-determined, but not 7-$\mathcal{A}_1$-determined..

- $(x,xy+z^3\pm y^2z+ay^3+y^5)$.

$$T\mathcal{A}_1\cdot f=\mathcal{M}_3^2\langle(1,y),(0,x\pm 2yz+3ay^2+5y^4),(0,3z^2\pm y^2)\rangle+f^*\mathcal{M}_2^2\{e_1,e_2\}.$$

We want to check to see if this germ is 5-$\mathcal{A}_1$-determined, and so want

$$\mathcal{M}_3^6\subset T\mathcal{A}_1\cdot f+\mathcal{M}_3^6(f^*\mathcal{M}_2+\mathcal{M}_3^6).\mathcal{E}(3,2).$$

This is too complex to complete by hand but the program TRANSVERSAL gave the map–germ to be 5–$\mathcal{A}$–determined.

Bibliography

[A1] V.I. Arnol'd, Normal Forms of Functions in Neighbourhoods of Degenerate Critical Points, *Russ. Math. Surv.* **29**:2, (1974), 10–50.

[A2] V.I. Arnol'd, Wave Front Evolution and Equivariant Morse Lemma, *Comm. Pure. App. Math.* **29**, (1976), 557–582.

[Bo] O. Bottema & B. Roth, Theoretical Kinematics, Dover Publications, New York, (1990).

[BL] Th. Bröcker & L. Lander, Differentiable Germs and Catastrophes, London Math. Soc. Lecture Note Series 17, C.U.P., (1975).

[B1] J.W. Bruce, Simple Singularities of Mappings $(\mathbf{C},0) \longrightarrow (\mathbf{C}^2,0)$, *Preprint*, University of Liverpool, (1978).

[B2] J.W. Bruce, Lines, Circles, Focal and Symmetry Sets, *Preprint*, University of Liverpool, (1992).

[B3] J.W. Bruce, On the Degree of Determinacy of Smooth Functions, *Bull. Lond. Math. Soc.* **13**, (1981), 55–58.

[B4] J.W. Bruce, Wavefront and Parallels in Euclidean Space, *Math. Proc. Camb. Phil. Soc.* **93**, (1983), 323–333.

[B5] J.W. Bruce, A classification of 1–parameter families of map–germs $(\mathbf{R}^3,0) \longrightarrow (\mathbf{R}^3,0)$ with application to condensation problems, *Jour. Lond. Math. Soc.* **33** (2), (1986), 375–384.

[BG] J.W. Bruce & T.J. Gaffney, Simple Singularities of Mappings $(\mathbf{C},0) \longrightarrow (\mathbf{C}^2,0)$, *Jour. Lond. Math. Soc.* **26**, (1982), 465–474.

[BH] J.W. Bruce & C.A. Hobbs, On Multi–germs of Mappings, *Preprint*, University of Liverpool, (1992).

[BduP] J.W. Bruce & A.A. du Plessis, Complete Transversals, *in preparation*, (1994).

[BduPW] J.W. Bruce, A.A. du Plessis & C.T.C. Wall, Determinacy and Unipotency, *Invent. Math.* **88**, (1987), 521–554.

[BduPWi] J.W. Bruce, A.A. du Plessis & L.C. Wilson, Discriminants and Liftable Vector Fields, *Preprint*, (1993).

[BM] J.W. Bruce & W.L. Marar, Images of Varieties, *Preprint*, University of Liverpool, (1993).

[D] A. Dimca, Topics on Real and Complex Singularities, Vieweg Advanced Lectures in Mathematics, (1987).

[DG] A. Dimca & C.G. Gibson, Contact germs from the plane to the plane, *Singularities, Proc. Symp. Pure. Math.*, Amer. Math. Soc., Providence, R. I. **40** (part 1), (1983), 277–282.

[Do1] P.S. Donelan, Generic Properties in Euclidean Kinematics, Ph.D Thesis, University of Southampton, (1984).

[Do2] P.S. Donelan, Generic Properties in Euclidean Kinematics, *Acta Applicandae Mathematicae* **12**, (1988), 265–296.

[duP] A.A. du Plessis, On the determinacy of smooth map-germs, *Inventiones Math* **58**, (1980), 101–160.

[duPGW] A.A. du Plessis, T.J. Gaffney & L.C. Wilson, Map-germs determined by their discriminants, *Preprint*, (1991).

[duPT] A.A. du Plessis & F. Tari, $\mathcal{A}$-classification of map-germs, *in preparation.*

[AWi] A.A. du Plessis & L.C. Wilson, On Right Equivalence, *Math. Zeit.* **190**, (1985), 163–205.

[GG] M. Golubitsky & V. Guillemin, Stable Mappings and their singularities, Springer–Verlag, New York, (1973).

[G] C.G. Gibson, Singular Points of Smooth Mappings, Research Notes in Math. **25**, Pitman, London, (1979).

[GHH] C.G. Gibson, W. Hawes & C.A. Hobbs, Local Pictures for General Two–Parameter Planar Motions, *Proceedings of the 4th Advances in Robot Kinematics, Slovenia*, (1994).

[GH1] C.G. Gibson & W. Hawes, Local Models for General Multi–Parameter Motions of the Plane, *in preparation.*

[GH2] C.G. Gibson & W. Hawes, Local Models for Three–Parameter Motions of Space, *in preparation.*

[GHo1] C.G. Gibson & C.A. Hobbs, Singularities of General One–Dimensional Motions of the Plane & Space, *Preprint*, University of Liverpool, (1992).

[GHo2] C.G. Gibson & C.A. Hobbs, Singularity and Bifurcation for General Two–dimensional Planar Motions, *Preprint*, University of Liverpool, (1994).

[GN] C.G. Gibson & P.E. Newstead, On the geometry of the Planar 4–bar Mechanism, *Acta Applicandae Mathematicae* **7**, (1986), 113–135.

[Hai] K. Hain, Applied Kinematics, McGraw–Hill, New York, (1967).

[Hau] E.J. Haug, Computer Aided Kinematics and Dynamics of Mechanical Systems, Volume 1, Allyn & Bacon, (1989).

[HD] R.S. Hartenberg & J. Denavit, Kinematic Synthesis of Linkages, McGraw–Hill, New York, (1964).

[Hob] C.A. Hobbs, On Kinematic Singularities of Low Dimension, Ph.D. Thesis, University of Liverpool, (1993).

[Hun] K. Hunt, Kinematic Geometry of Mechanisms, O.U.P., (1978).

[K] N.P. Kirk, Computational Aspects of Singualrity Theory, Ph.D. Thesis, University of Liverpool, (1993).

[MT] W.L. Marar & F. Tari, On the Geometry of simple germs of corank 1 maps from $\mathbf{R}^3$ to $\mathbf{R}^3$, *Preprint*, University of Liverpool, (1994).

[MMR] W.L. Marar, J. Montaldi & M.A.S. Ruas, Multiplicities of zero schemes in quasihomogeneous singularities, *in preparation.*

[Mart] J. Martinet, Singularities of Smooth Functions and Maps, Lond. Math. Soc. Lecture Notes **58**, (1982).

[MaIII] J.N. Mather, Stability of C^∞–mappings III: Finitely Determined Map germs, *Publ. Math. I. H. E. S.* **35**, (1968), 127–156.

[MaIV] J.N. Mather, Stability of C^∞–mappings IV: Classification of Stable germs by $\mathbb{R}$–algebras, *Publ. Math. I. H. E. S.* **37**, (1969), 223–248.

[MaV] J.N. Mather, Stability of C^∞–mappings V: Transversality, *Adv. Math.* **4**, (1970), 301–336.

[MaVI] J.N. Mather, Stability of C^∞–mappings VI: The Nice Dimensions, *Proc. Liverpool Singularities–Symp. 1*, (1970), 207–253.

[Mo1] D.M.Q. Mond, On the classification of maps from $(\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^3, 0)$, *Proc. Lond. Math. Soc.* **50**, (1985), 333–369.

[Mo2] D.M.Q. Mond, Some remarks on the geometry and classification of germs of maps from surfaces to 3–space, *Topology* **26**, No. 3, (1987), 361–383.

[Por] I.R. Porteous, Simple Singularities of Maps, *Proc. Liverpool Singularities–Symp. 1*, (1970), 286–307.

[Ri1] J.H. Rieger, Apparent Contours and their singularities, Ph.D Thesis, University of London, (1988).

[Ri2] J.H. Rieger, Families of Maps from the Plane to the Plane, *Jour. Lond. Math. Soc.* **36**, No. 2, (1987), 351–369.

[RR] J.H. Rieger & M.A.S. Ruas, Classification of $\mathcal{A}$–simple germs from k^n to k^2, *Comp. Math.* **79**, (1991), 99–108.

[T] F. Tari, Geometric criteria for recognition of simple singularities of map–germs of the plane, *Preprint*, University of Liverpool, (1992).

[Wa1] C.T.C. Wall, Finite Determinacy of Smooth Map–germs, *Lond. Math. Soc. Bull.* **13**, (1981), 481–539.

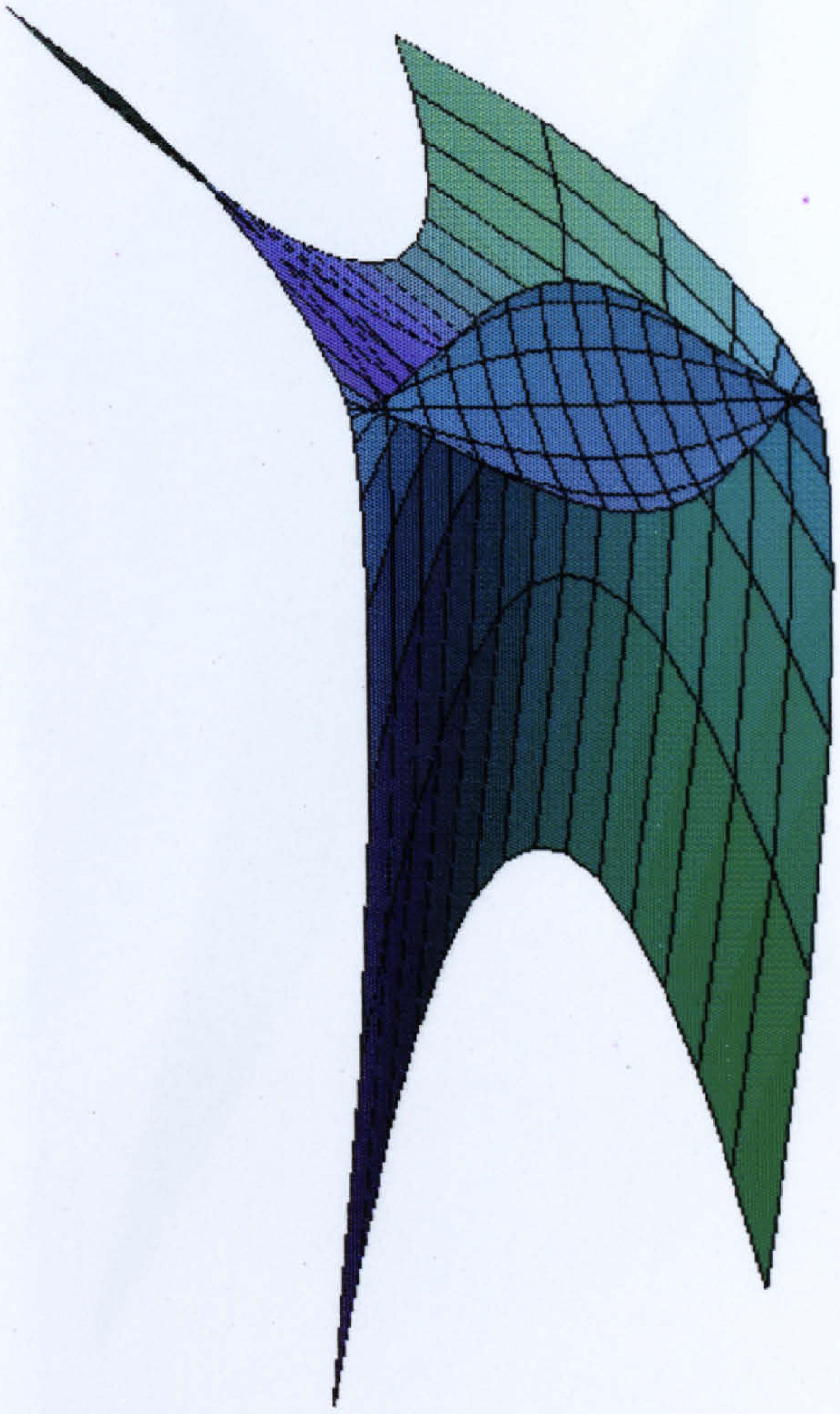
[Wa2] C.T.C. Wall, Notes on the Classification of Singularities, *Proc. Lond. Math. Soc.* **48**, (1984), 461–513.

[Wi] L.C. Wilson, Mappings of Finite Codimension, unpublished notes.

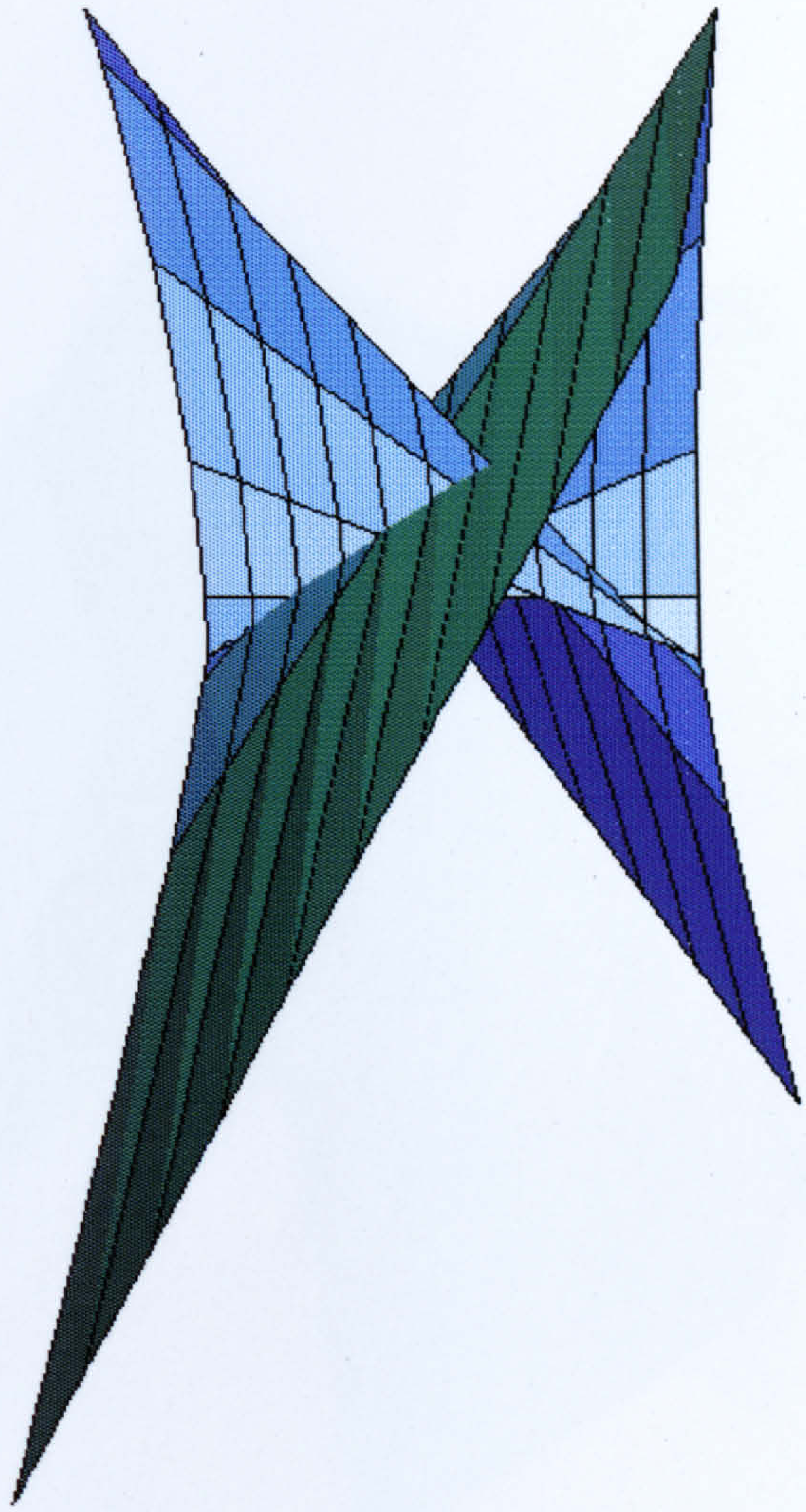
Appendix C

Surfaces

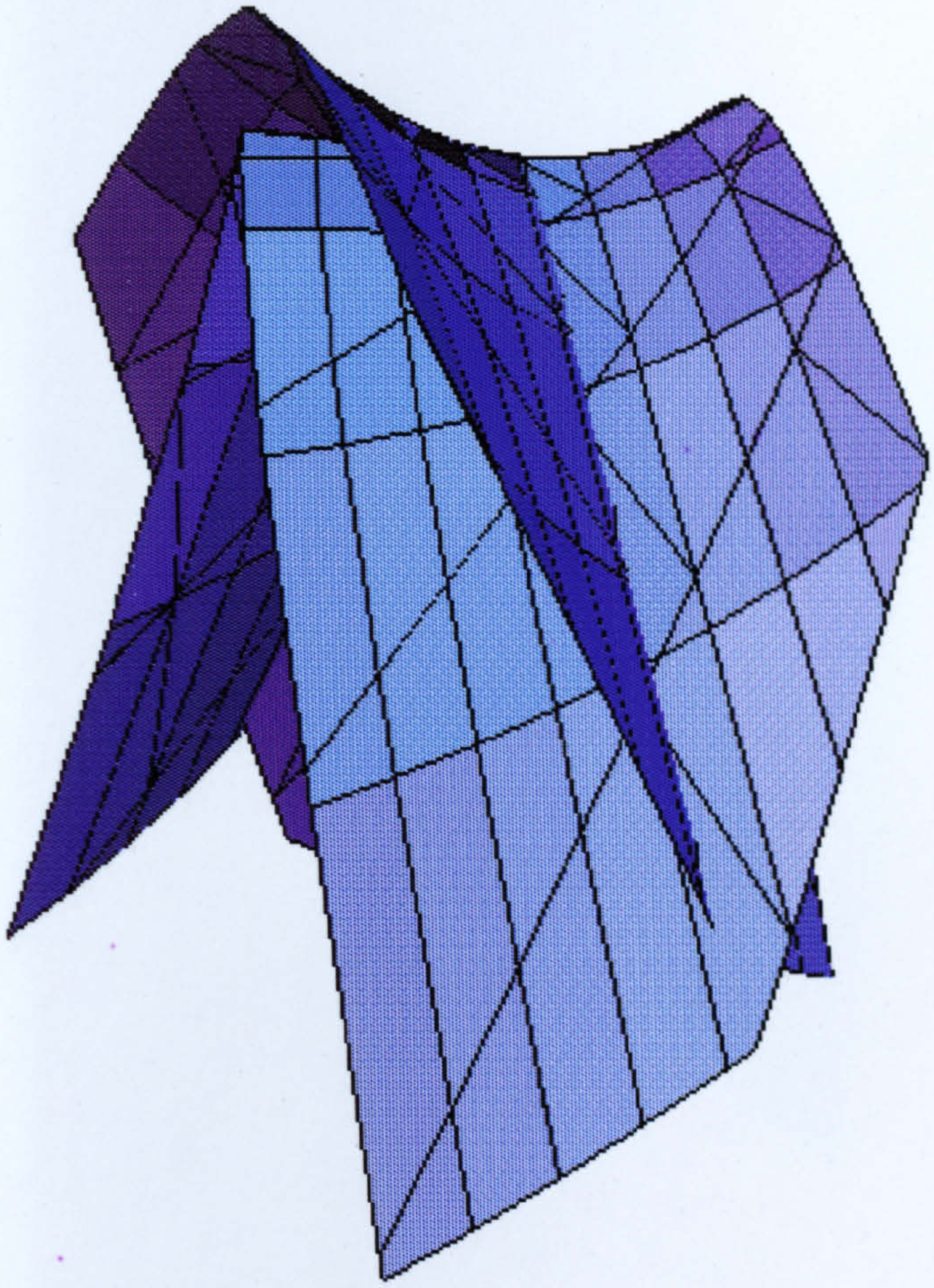
The following pictures represent the critical image of the given map germ. In each case the values of the unfolding parameters are chosen to give a characteristic surface which best describes the unfolding.



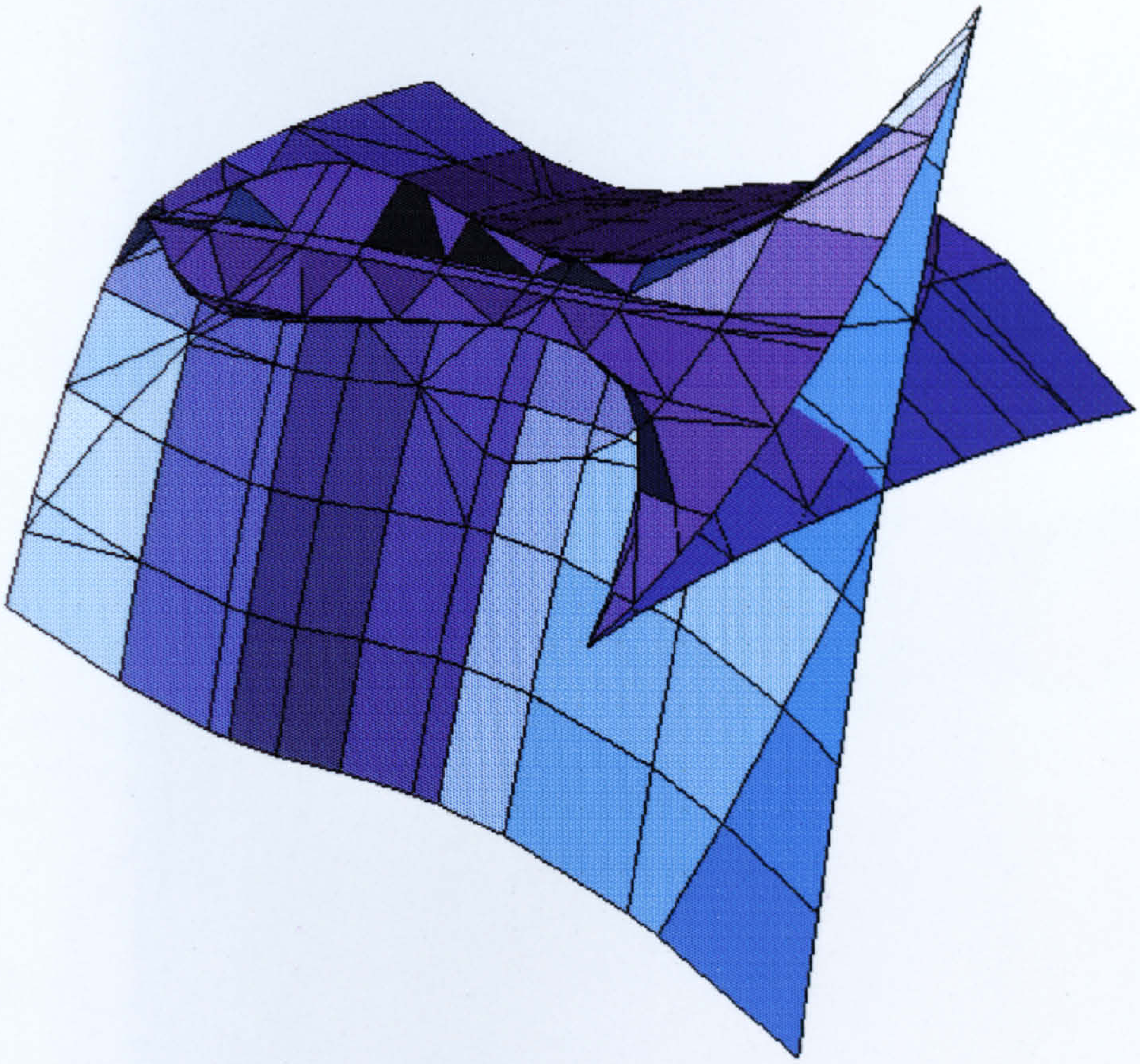
$f(x, y, z) = (x, y, yz + z^4 + x^2z^2 + az^2)$ for $a = -0.9$



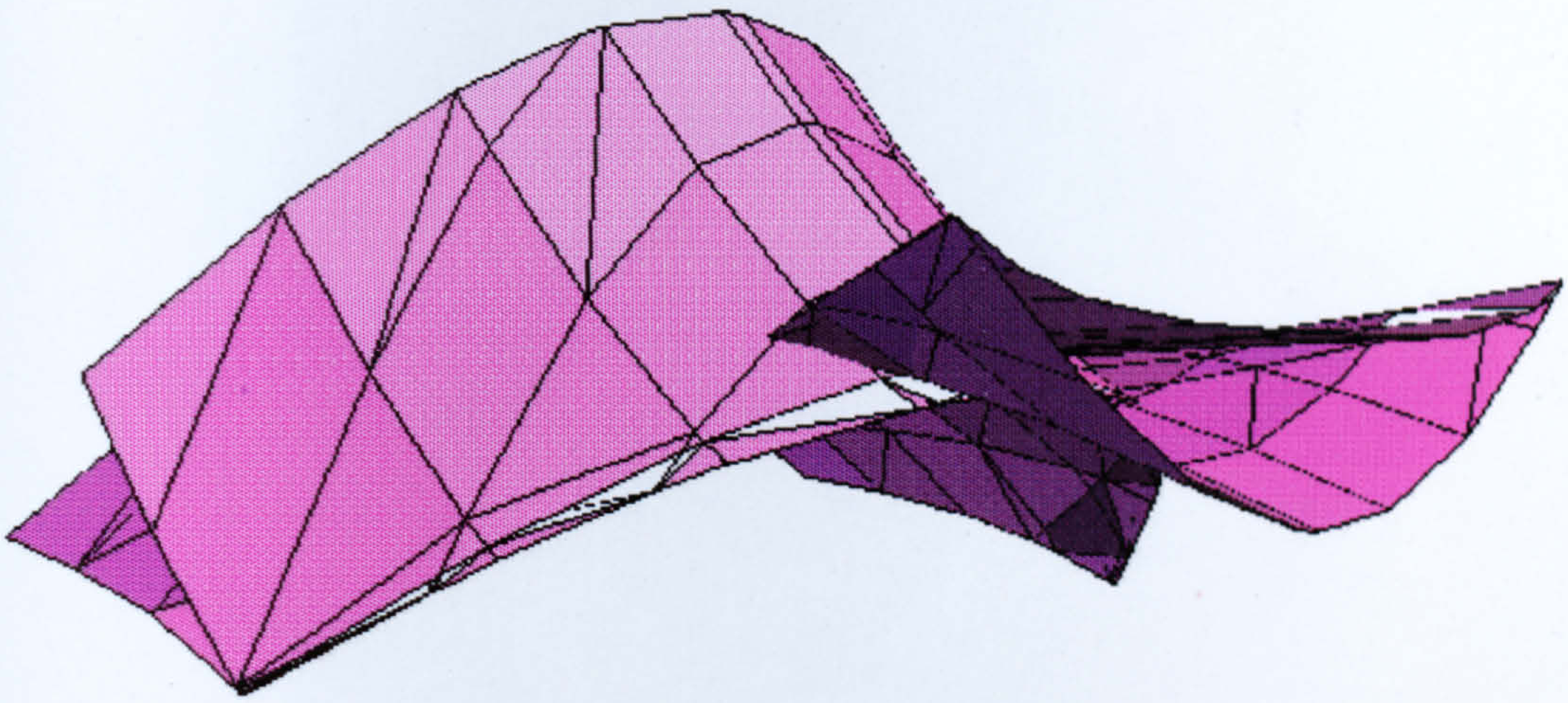
$f(x, y, z) = (x, y, yz + xz^2 + z^5 + az^3)$ for $a = -1.2$



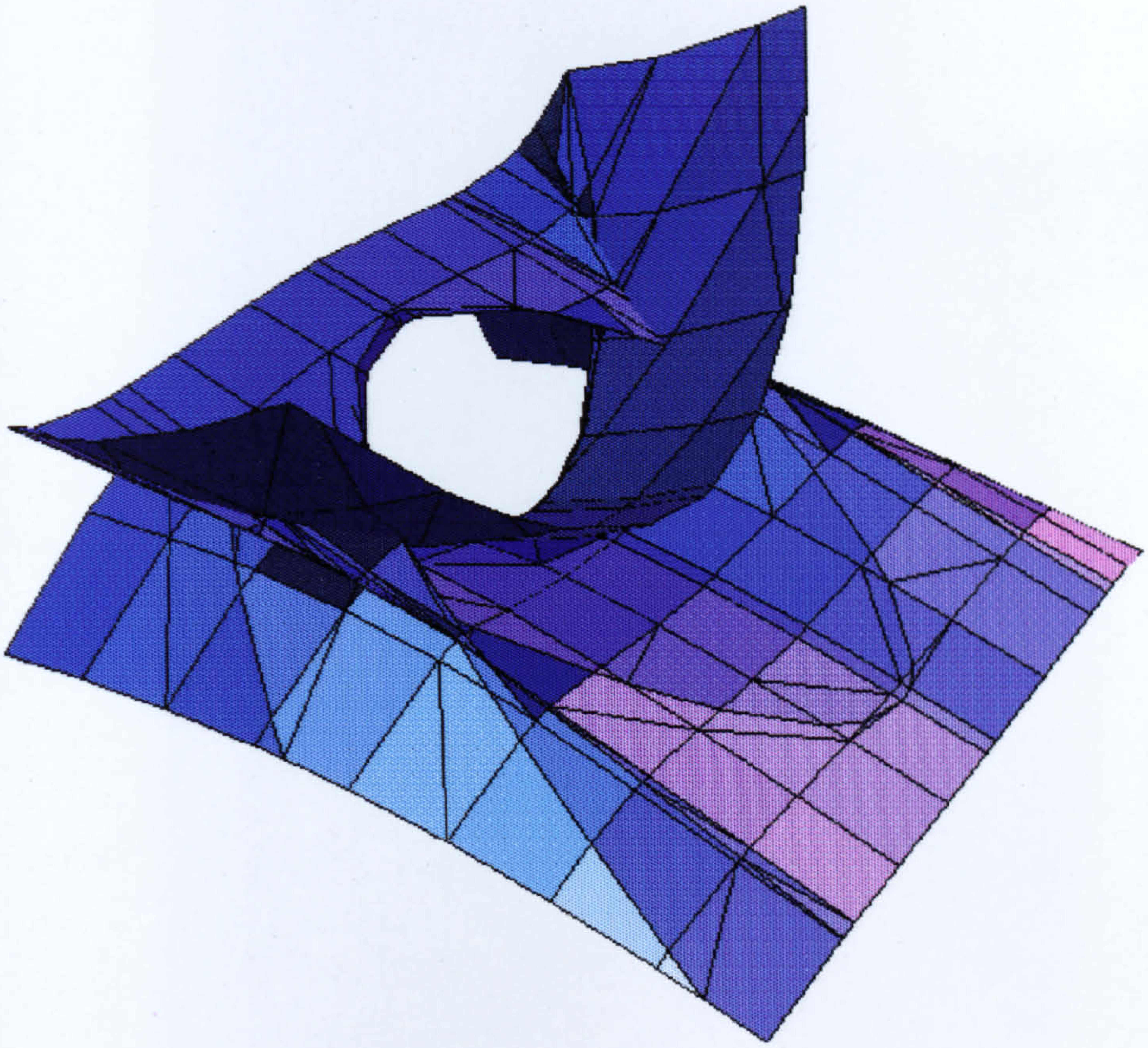
$f(x, y, z) = (x, yz + ay, y^2 + z^2 + xy)$ for $a = 0$



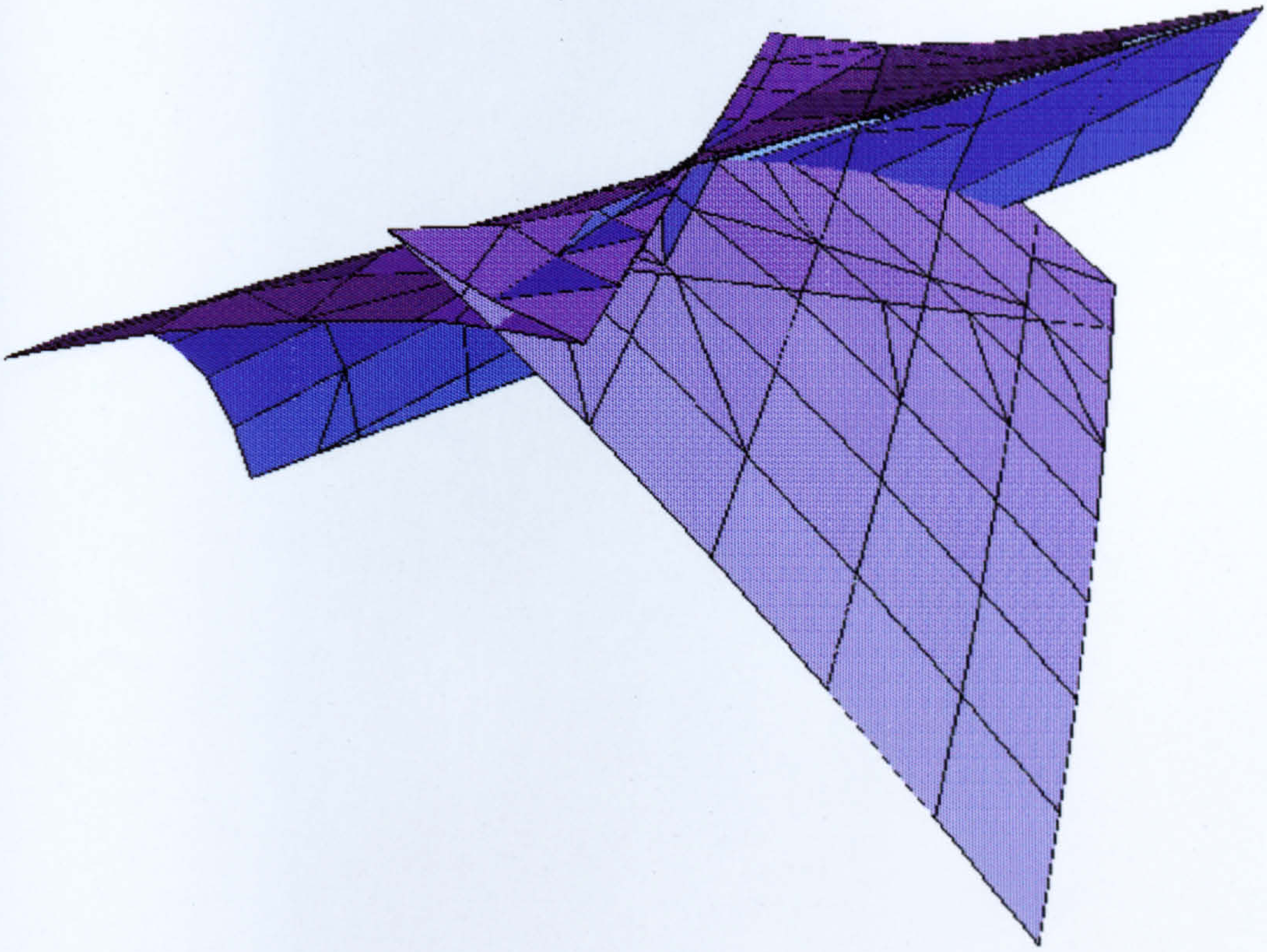
$f(x,y,z) = (x, y, yz + z^4 + x^3z^2 + az^2 + bxz^2)$ for $a = b = -0.5$



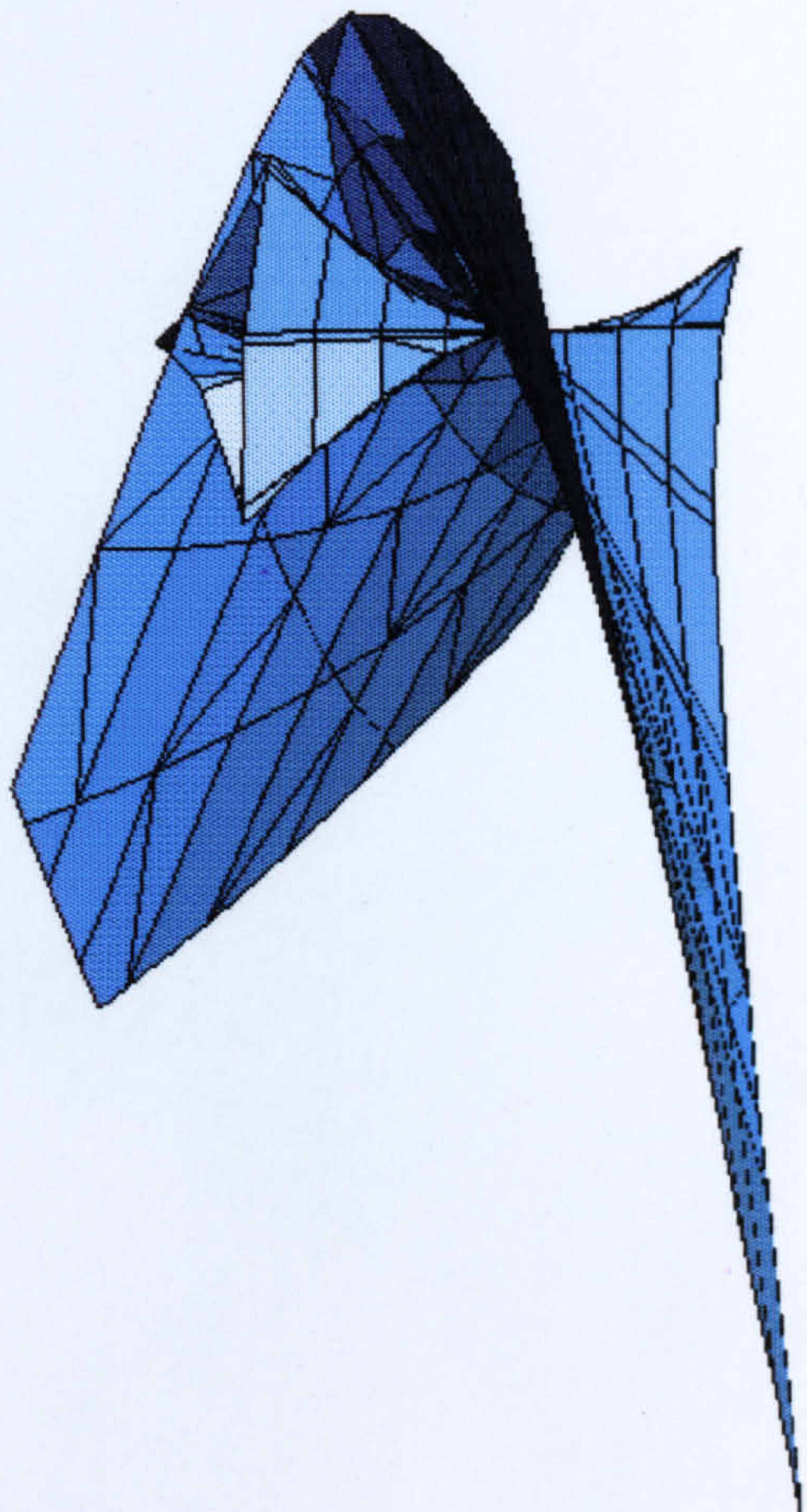
$f(x,y,z) = (x, y, yz^2 + x^2z + z^4 + z^5 + az + bz^3)$ for $a = 0,\ b = -0.5$



$f(x, y, z) = (x, y, yz^2 + x^2 z + z^4 - z^5 + az + bz^3)$ for $a = -0.5,\ b = 0.5$

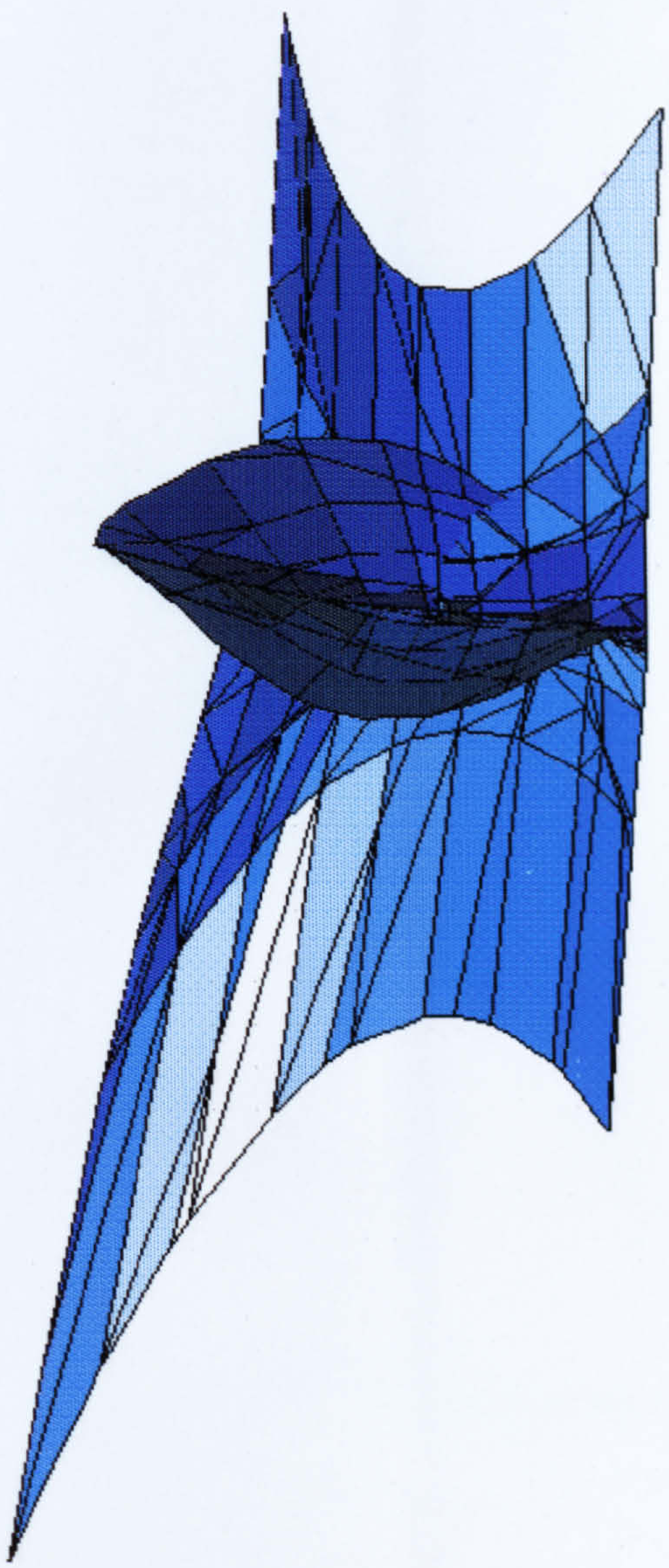


$f(x, y, z) = (x, y, yz + xz^3 + z^5 + z^6 + az^2 + bz^4)$ for $a = b = 0$

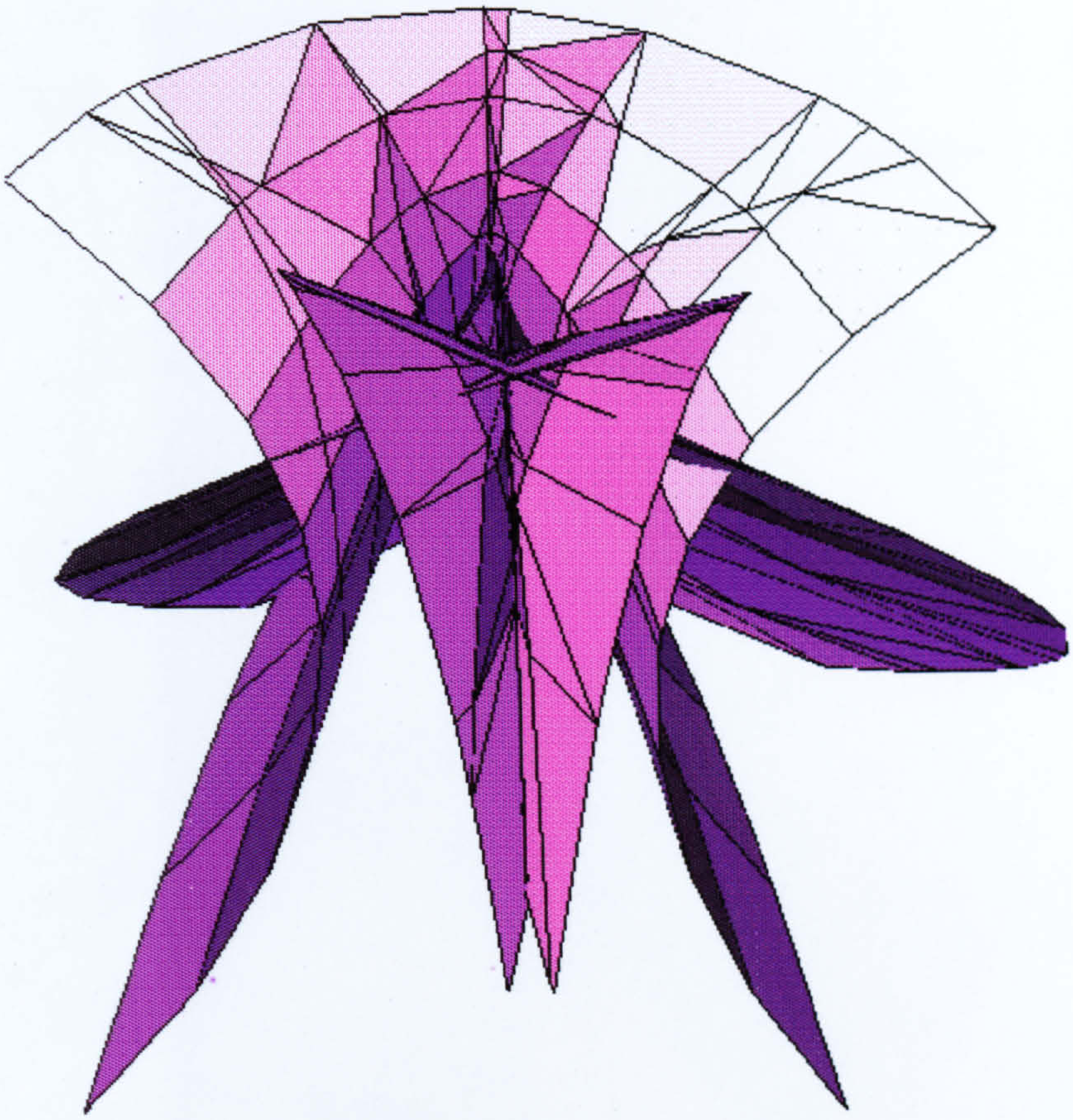


$f(x, y, z) = (x, yz, y^2 + xy + xz + z^3 + ayz^3 + bz + cz^2)$ for $a = b = c = 0$

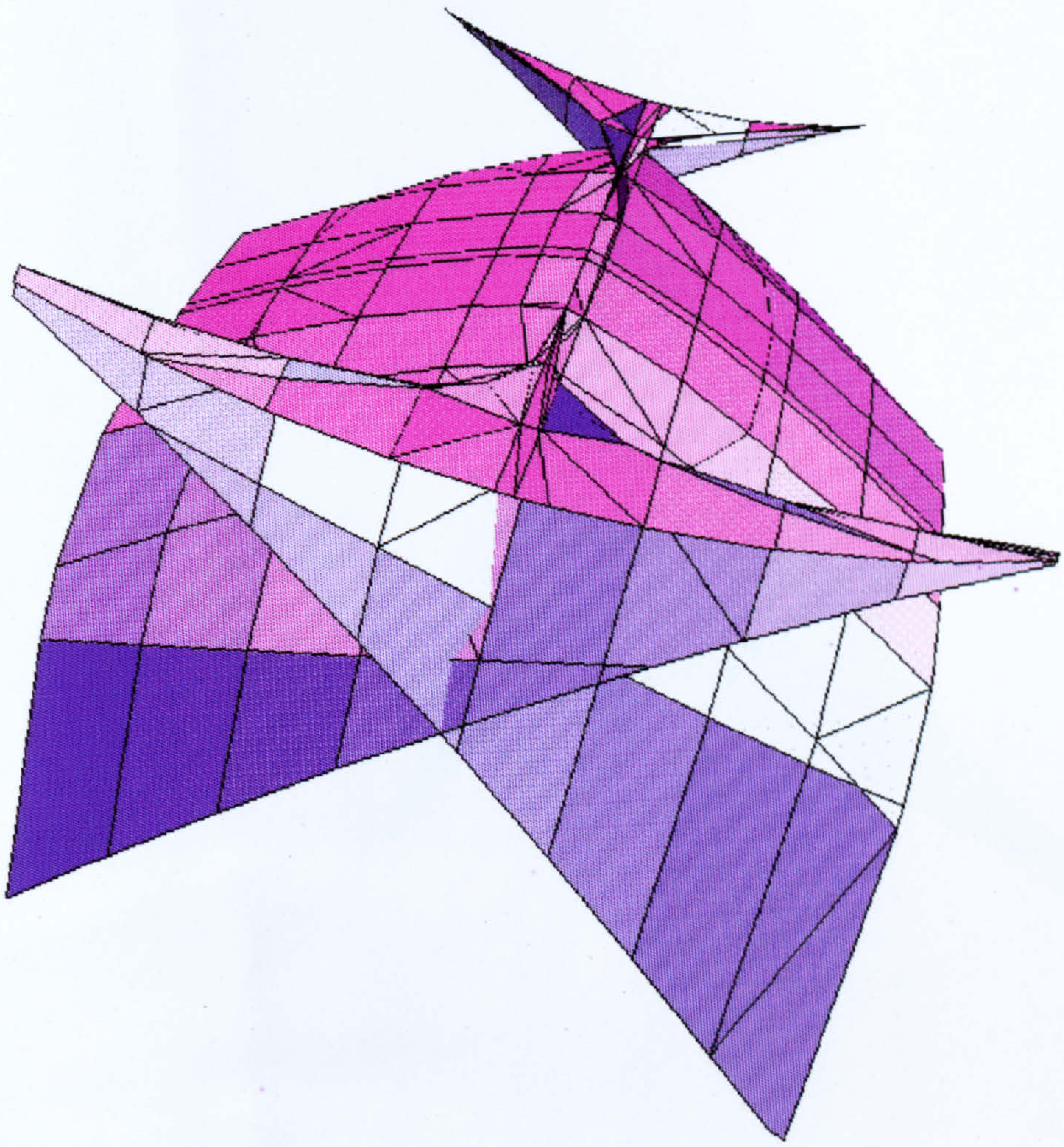
$f(x, y, z) = (x, y, z^3 - y^2 z - x^4 z + az + bxz + cx^2 z)$ for $a = b = 0,\ c = 0.5$



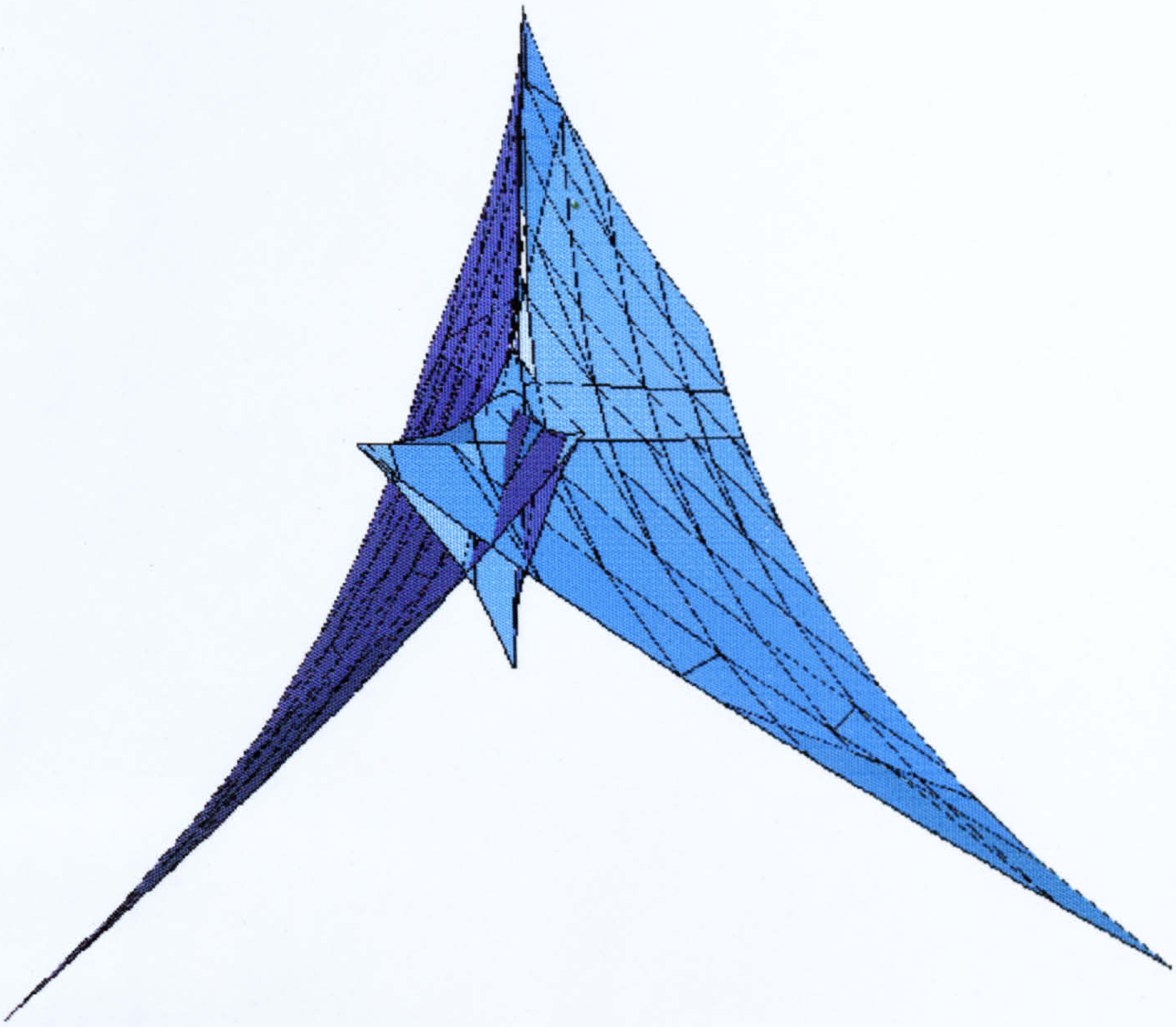
$f(x,y,z) = (x, yz + x^2y + y^5, y^2 + z^2 + xy + xz + ay + bxy + cy^3)$ for $a = b = c = 0$



$f(x, y, z) = (x, yz + x^2y - y^5, y^2 + z^2 + xy + xz + ay + bxy + cy^3)$ for $a = b = c = 0$



$f(x, y, z) = (x, y, yz + z^4 - x^4z^2 + az^2 + bxz^2 + cx^2z^2)$ for $a = b = c = 0$



$f(x, y, z) = (x, y, yz + xz^3 + z^5 + az^2 + bz^4 + cz^6)$ for $a = b = c = 0$