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PROJECTIONS OF LINKS

Generalisations of some results on alternating diagrams

A thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

Peter Richard Cromwell.

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Abstract

PROJECTIONS OF LINKS

Generalisations of some results on alternating diagrams

P. R. Cromwell

In this thesis I investigate which 3-dimensional properties of links can be easily observed in, or constructed from, link diagrams. Much of the previous work on diagrams of links has concentrated on alternating diagrams. This work extends some well-known results on alternating links.

In chapter 1, the standard ideas of the classical theory of links are reviewed. Let D be a diagram of a link L, and let F be the orientable surface spanning L which is constructed from D by applying Seifert's algorithm.

Chapter 2 introduces a new class of links which I have named homogeneous links. This class contains the alternating links as a subclass, and also the positive (or standard) links. A link diagram is called homogeneous if it can be decomposed as a planar *-product of alternating links. A link which possesses such a diagram is called homogeneous. (Homogeneous links are also defined in terms of F). Properties of the new polynomial invariants are investigated which include bounds on the degrees of the variables. Some of these properties prove to be effective for deciding membership of this new class. A range of examples is given showing which kinds of links this class contains and which it excludes. The homogeneities of all but 5 of the 249 prime knots of orders up to 10 are determined, the classification being complete for knots up to order 9. Some of K. Murasugi's results on alternating links are shown to hold on this larger class. In particular, if D is a homogeneous diagram then the surface F is a minimal genus spanning surface for L. Also, if the leading coefficient of the Conway polynomial is 1 then L is a fibred link with fibre F.

Chapter 3 is devoted to proving two theorems which imply that positive braid diagrams represent split or non-prime links only in the obvious ways. W. W. Menasco has proved corresponding theorems for alternating diagrams. More explicitly: a projection $\pi(L)\subset\mathbb{R}^2$ is defined to be decomposable if

(a) there exists a 1-sphere $S^1\subset\mathbb{R}^2$ and two connected components U, V such that $U\cup V=\mathbb{R}^2$, $U\cap V=\partial U=\partial V=S^1$

(b) S^1 meets $\pi(L)$ in exactly two (non-double) points

(c) neither $U\cap\pi(L)$ nor $V\cap\pi(L)$ is a single embedded arc.

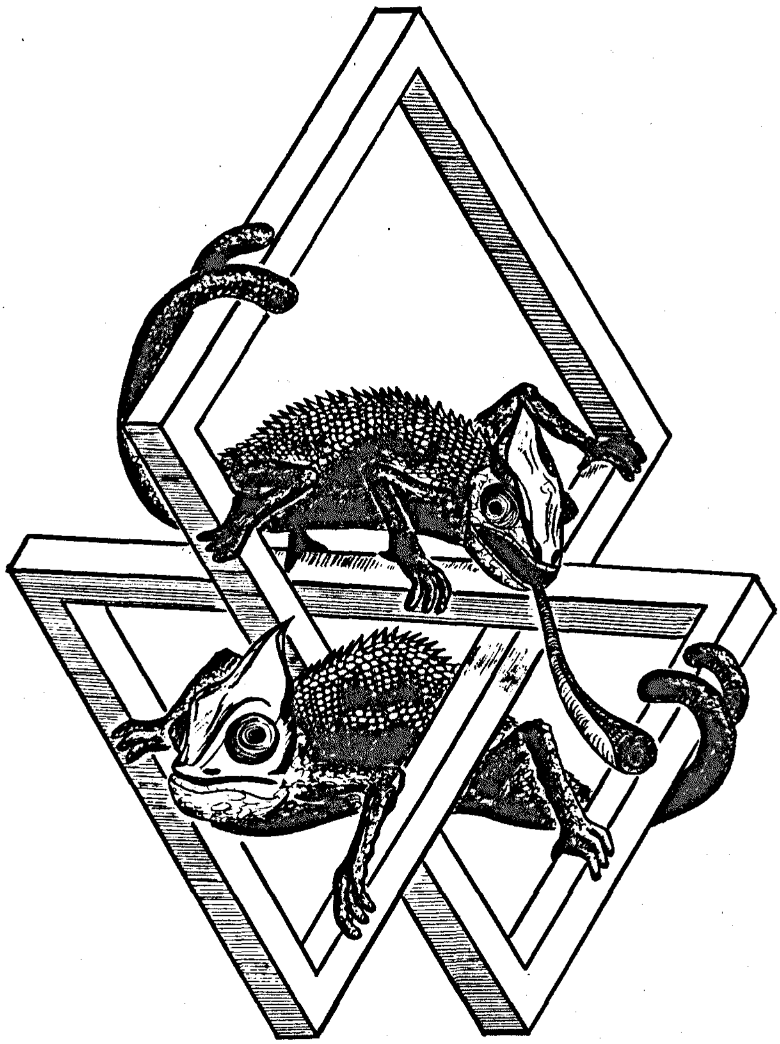
Suppose the diagram associated to $\pi(L)$ is a positive braid. Then

(1) L is split if and only if $\pi(L)$ is disconnected

(2) L is non-prime if and only if $\pi(L)$ is decomposable.

It is shown that (1) holds whenever the projection surface constructed from $\pi(L)$ has minimal genus, and it is conjectured that (2) also holds under these conditions.

PROJECTIONS OF LINKS



CONTENTS

FOREWORD

> "Our three-dimensional space is the only true reality we know. The two-dimensional is every bit as fictitious as the four-dimensional. ... And yet we stick to the convention that a wall or piece of paper *is* flat, and curiously enough, we still go on ... producing illusions of space on just such plane surfaces as these.
>
> [M. C. Escher]

This thesis is concerned with two-dimensional representations of three-dimensional objects. Any two-dimensional simulation of an object which of necessity requires three spatial dimensions to contain it always lacks the 'essence of three-dimensionality' which is intrinsic to the original, however cunning and ingenious the simulation may be.

Since the purpose of a simulation is to simplify a situation in some way, a model can never capture all the properties of the original. Conversely, an object in the model space may have properties which are inconsistent so that the model does not correspond to anything in the simulated space. For example, at first glance the sketch on the title page seems to represent a situation spatially extended in three directions: a knotted framework supporting two chameleon type creatures. These creatures add to the illusion by helping to disguise the fact that the framework is not realisable in ordinary three-dimensional space. The framework motif is repeated on the first page of each chapter as a reminder of the conflict between two and three dimensions, and that two-dimensional representations of three-dimensional objects are only simulations.

In more natural and more useful representations it is often possible to reconstruct the represented object. This is the case with the representations used here. Three-dimensionality is essential to the nature of phenomena like knots and links. This makes them difficult to describe, so they are usually represented by two-dimensional diagrams.

A vital connection between a two-dimensional diagram and the three-dimensional link which it represents is provided by Seifert's algorithm. Starting from a diagram, the algorithm constructs an orientable surface which spans the represented link. Such surfaces are used throughout the thesis and play an important role in the theory.

Much of the work on diagrams of links has concentrated on alternating diagrams (those where under and over-crossings alternate when following round the diagram). For many purposes these diagrams are the easiest ones to consider and are almost canonical representations of their object links. This thesis generalises some well-known results on alternating links.

Chapter 2 introduces a new class of links which I have named homogeneous links. This class contains the alternating links as a subclass, and also the positive (or standard) links. Some properties of the polynomial invariants prove to be effective for deciding membership of this class. A range of examples is given showing which kinds of links this class contains and which it excludes. The homogeneities of all but 5 of the 249 prime knots of orders up to 10 are given, the classification being complete for knots up to order 9. Some of K. Murasugi's results on alternating links are shown to hold on this larger class. In particular, the surface constructed from a homogeneous diagram by Seifert's algorithm has minimal genus; and also, if the leading coefficient of the Conway polynomial is 1 then the link is fibred. Most of this chapter is to appear in the *Proceedings of the London Mathematical Society*.

Chapter 3 is devoted to proving two theorems which imply that positive braid diagrams represent split or non-prime links only in the obvious ways. W. W. Menasco has proved corresponding theorems for alternating diagrams. These results show an interesting connection between two and three-dimensional space: a 2-sphere which partitions a link in 3-space in a particular way is simulated by a 1-sphere which partitions the link diagram in 2-space in an analogous way. Since the existence of a 2-sphere partition does not in general imply the existence of a 1-sphere partition, the fact that it does so in this case shows the special nature of the diagrams in question.

Finally, a few comments on the organisation of the thesis. Headings appear frequently and are numbered so that 2.1 denotes the first section of chapter 2; 2.1.4 denotes the fourth subsection of 2.1. The section number and section heading appear at the foot of each page, and are also listed in the table of contents. There are many illustrations and these are numbered consecutively in each chapter so that figure 2.5 is the fifth figure in chapter 2. Each chapter is selfcontained with any appendices or references forming its final sections. In the body of the chapter, references to these papers are written in bold type between square brackets. The symbol □ denotes the end of a proof, or, if it appears immediately after a statement, that no proof will be given. For other symbols there is a glossary which gives their meanings, and for the non-standard ones, a page reference is also given where the symbol is explained. There is also an index which gives page references to definitions. Many standard results and definitions have been included to make the thesis selfcontained to some extent, and these are collected together in chapter 1.

1

Standard Ideas and Definitions

1.1 3-DIMENSIONAL CONCEPTS

This chapter reproduces some of the standard definitions which are the foundations of classical knot theory. Good references which contain this material are [B-Z], [Ro]. Further definitions appear in later chapters.

A <u>link</u> of <u>multiplicity</u> μ is an embedding of μ disjoint (possibly oriented) 1-spheres into the oriented 3-sphere. If $\mu = 1$ then the link may also be called a <u>knot</u>. Two links are <u>equivalent</u> if they are ambient isotopic in the 3-sphere respecting any orientations. The definition of <u>link</u> can be extended to mean an equivalence class of embeddings as well as a representative element of that class. A link is <u>tame</u> if there is a polygonal (ie. piecewise linear) representative in its equivalence class. From here onwards, all links are assumed to be tame. A link is <u>trivial</u> if its components can be spanned by disjoint non-singular discs. A trivial link is also call an <u>unlink</u>.

1.1.1 Remark. Under the conditions of the theory of classical links (1-dimensional objects in 3-space), the topological, smooth, and piecewise linear categories are the same [Mo].

1.1.2 Split and Product links (decomposition by a sphere).

A link of multiplicity ⩾2 is <u>split</u> if its components can be separated by a 2-sphere embedded in S^3.

Let S^2 be a 2-sphere embedded in S^3 which meets a link L transversely in precisely two points {p,q}, and which separates S^3 into two 3-balls B_1, B_2. Choose an arc $\alpha \subset S^2$ joining p to q. For i=1,2 let $L_i = (B_i \cap L) \cup \alpha$. Then L is a <u>product</u> link or <u>connected sum</u> of links with <u>factors</u> L_1, L_2. This is denoted $L = L_1 \# L_2$. If the only factors of L are itself and the trivial knot then L is a <u>prime</u> link.

1.1.3 Theorem (Hashizume). Every non-trivial link is a product of finitely many prime links, and these factors are unique up to order [Ha]. □

This theorem was first proved for knots by Schubert [Sc].

1.1.4 Satellite links (decomposition by a torus).

Let $V = S^1 \times D^2$ be an unknotted solid torus and suppose K_p is a link contained in V so that no 3-ball in V contains K_p. Let $W \subset S^3$ be a solid tubular neighbourhood of a knot K_c. Let $h: V \to W$ be a homeomorphism and let K_s denote the image $h(K_p)$. Then K_s is a <u>satellite</u> link with <u>companion</u> K_c and <u>pattern</u> (V, K_p). For example, both factors are companions of a product knot: let W be the swallow-follow torus shown in figure 1.1 which follows one factor and swallows the other.

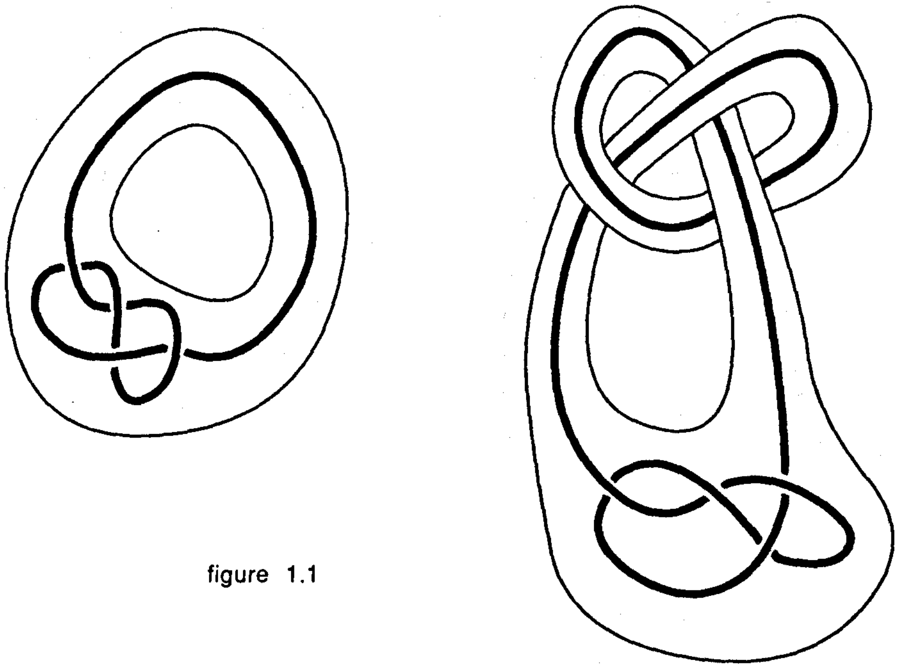


figure 1.1

If K_s is ambient isotopic in W to the curve in ∂W which is generated by q preferred meridians and r preferred longitudes of W then K_s is called a (q,r) cable link, and if K_c is trivial it is called a torus link. If the pattern is as shown in figure 1.2 then the satellite, K_s, is a double knot, and if h maps the preferred longitude and meridian of V to the preferred longitude and meridian of W then K_s is an untwisted double knot.

1.1.5 Surfaces spanning links.

A spanning surface for a link L is an orientable compact 2-manifold with boundary L.

1.1.6 Theorem (Seifert). Every link has a spanning surface [Se]. □

In the proof of this theorem, Seifert gives a method for constructing a spanning surface from a link diagram. This construction

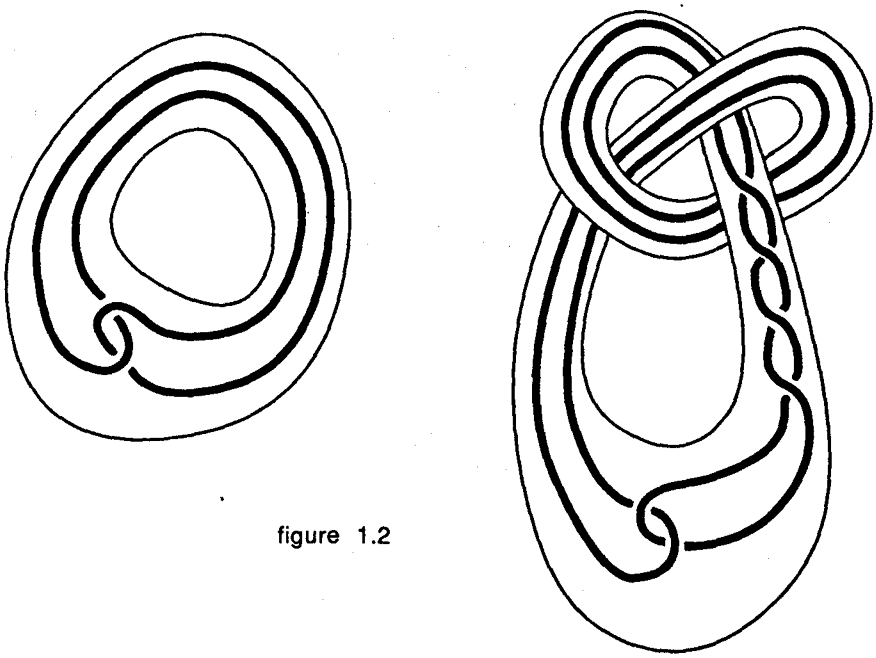


figure 1.2

is known as Seifert's algorithm. It is outlined in the appendix (§1.3).

The genus of a link L is the minimum genus of all surfaces spanning L. This is clearly an invariant of L, and it is denoted g(L). Only the trivial links have genus zero. The Euler characteristic $\chi(L)$ of a link L is defined as the maximum Euler characteristic over all spanning surfaces for L.

A non-split link whose components bound disjoint spanning surfaces is a boundary link.

1.1.7 Murasugi sum of surfaces.

Let F be a surface in S^3 which spans a link L. Let S^2 be a 2-sphere embedded in S^3 which separates S^3 into two 3-balls B_1, B_2 so that $B_1 \cup B_2 = S^3$, and $B_1 \cap B_2 = \partial B_i = S^2$. Suppose $F \cap S^2$ is a disc, D. Let $F_i = F \cap B_i$ for i=1,2. Then $F = F_1 \cup_D F_2$. Say that the surface F is a

Murasugi sum of the surfaces F_1, F_2. The example in figure 1.3 shows a Murasugi sum of two Hopf bands which form a surface spanning the figure-8 knot. This operation was first used by Murasugi to compute the genera of alternating knots [Mu].

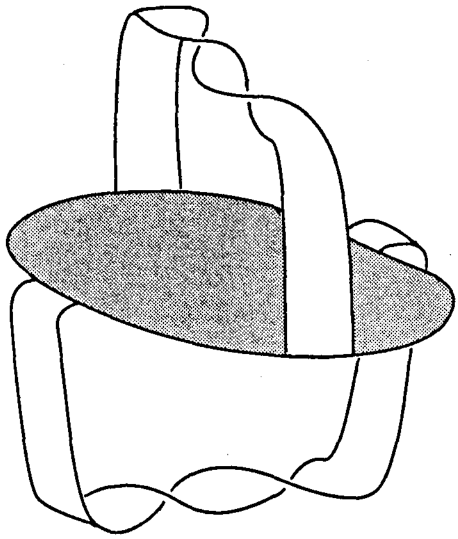


figure 1.3

1.1.8 Fibrations.

Let F be a surface in S^3 which spans a link L. Suppose there exists a map $M:(S^3-L)\to S^1$ such that for all $x\in S^1$, there is a neighbourhood N(x) so that $M^{-1}(N(x))$ is homeomorphic to a bicollar on F. Then the link complement S^3-L is fibred over S^1 with fibre F, and L is a fibred link.

1.2 2-DIMENSIONAL CONCEPTS

1.2.1 Projections.

Let $L \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ be a link in the 3-sphere, and let $\pi:\mathbb{R}^3 \to \mathbb{R}^2$ be a projection. Assume that L lies entirely on one side of $\mathbb{R}^2$. The projection $\pi(L)$ is regular if every self-intersection point is a transverse double point, and there are finitely many such double points. All of the projections referred to in this thesis are assumed to be regular. (This is consistent with the assumption that the links are tame.) The minimum number of double points over all possible projections of a link is called the crossing number or order. It is a link invariant and is denoted c(L). Only the trivial links have order zero.

A projection $\pi(L)$ is irreducible if there is no double point p such that $\pi(L)$-p is disconnected. If a projection is not irreducible then it does not have the minimum number of double points.

In a projection of an oriented link, a neighbourhood of each double point can be altered as shown in figure 1.4. This operation is called smoothing. If smoothing is applied to every double point in the projection, it is transformed into a set of disjoint simple closed curves. These are called Seifert circles. The minimum number of Seifert circles over all possible projections of a link is called the Seifert circle index (or braid index, see later) and is denoted s(L).

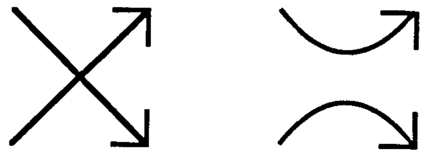


figure 1.4

1.2.2 Diagrams.

A projection of a link may be marked at the double points to indicate which arc has preimage nearest to the projection plane. It is common practice to break the image of this undercrossing arc, and the figures in this thesis follow this convention. A projection annotated in this way is called a diagram. A link can be reconstructed from a diagram but not from a projection. Diagrams of prime knots up to order 10 and prime links up to order 9 can be seen in [Ro](pp391-429). A subset of these is reproduced in appendix D of [B-Z]. The notation m_n refers to the n^{th} prime knot of order m as listed in these tables. For example 3_1 and 4_1 denote the trefoil and figure-8 knot respectively, and 8_{19} is the (3,4) torus knot.

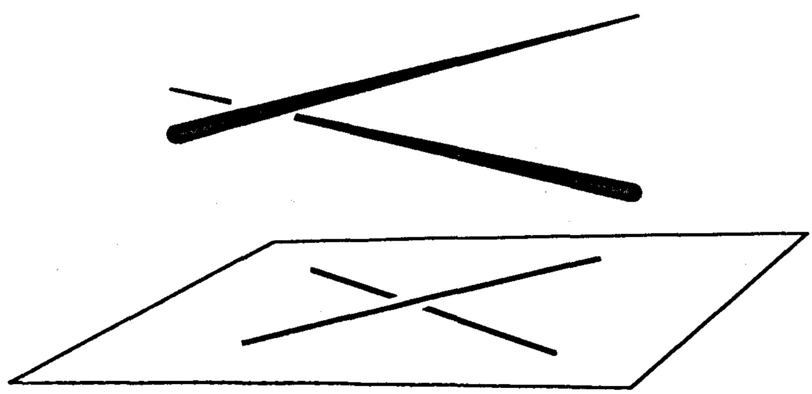


figure 1.5

A diagram is connected if its underlying projection is connected. Similarly, a diagram is irreducible if its underlying projection is irreducible.

An operation on a diagram effects a local change in a neighbourhood of a few crossings leaving the rest of the diagram unaltered. The three operations shown in figure 1.6 are called Reidemeister moves. Two diagrams are equivalent if one can be obtained from the other by performing a finite sequence of Reidemeister moves.

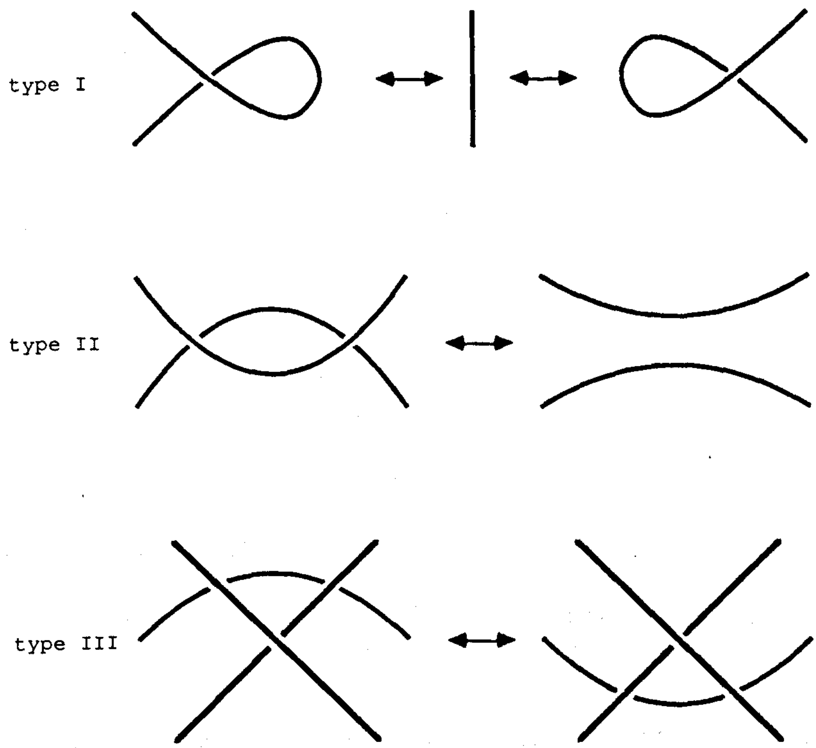


figure 1.6

1.2.3 Theorem (Reidemeister). Two links are equivalent if and only if all their diagrams are equivalent **[Re]**. □

A diagram is alternating if when following round each component, the crossings are encountered alternately as under and over-crossings. A link which possesses an alternating diagram is called an alternating link.

In an oriented diagram each crossing is one of two possible types which are shown in figure 1.7. If all of the crossings in an oriented diagram are of the same type then the diagram is positive or standard. A link which possesses a positive diagram is called a positive (or standard) link.

1.2.4 Remark. Non-alternating and non-positive links do exist (see chapter 2).

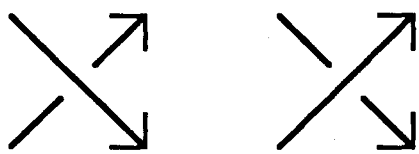


figure 1.7

1.2.5 Braids and tangles.

Let B_n be the group with the following presentation

$$< \sigma_1, \cdots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|\geq 2; \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1\leq i\leq n-2 >$$

This group is called the <u>n-string braid group</u>. It can be interpreted geometrically: represent the generators σ_i and $(\sigma_i)^{-1}$ by the <u>elementary braids</u> shown in figure 1.8 where each braid contains precisely one crossing between the i^{th} and $(i+1)^{st}$ strings.

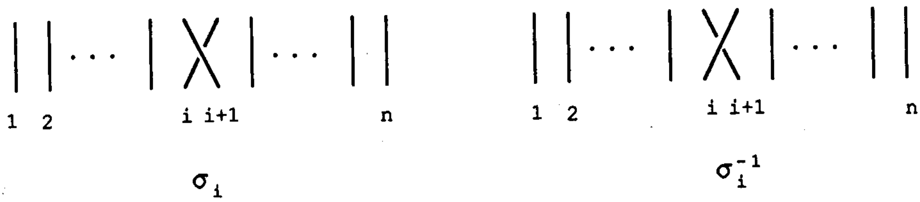
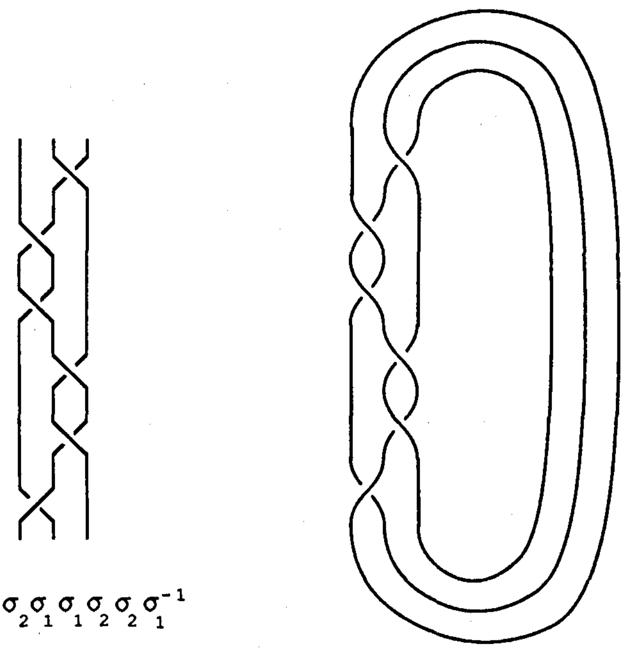


figure 1.8

A <u>braid</u> on n strings (or n-braid) is a word in B_n. It can be represented by <u>composing</u> the elementary braids writing them below one another. For example, figure 1.9(a) represents the braid $\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2(\sigma_1)^{-1}$ in B_3. A braid is <u>positive</u> if there are no occurrences of $(\sigma_i)^{-1}$ for any i. Following [St], a braid is <u>homogeneous</u> if for each i, the exponents of all occurrences of σ_i are the same.



$\beta = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1^{-1}$

(a) (b)

figure 1.9

A representation of a braid $\beta \in B_n$ can be closed to form a link diagram by joining the top and bottom of the braid. This diagram is denoted $\hat{\beta}$. The closure of the braid in figure 1.9(a) is shown in figure 1.9(b) and is a diagram of the tweeny knot, 5_2.

The relations in B_n correspond to equivalences of diagrams. So braid words which are equal in B_n have closures which are equivalent as diagrams.

1.2.6 Theorem (Alexander). Every link has a diagram which is the closure of a braid [Al]. □

The minimum number n such that a link L is the closure of a braid in B_n is called the braid index of L. This is the same as the Seifert circle index [Ya], and can therefore be denoted s(L).

An n-tangle is a subset of a diagram $D \subset \mathbb{R}^2$ which is contained in a rectangle $R \subset \mathbb{R}^2$ such that two opposite sides of R do not meet D, and each of the other two sides meets D transversely in exactly n points. All n-braids are examples of n-tangles. Tangles can be composed and closed in a manner similar to braids.

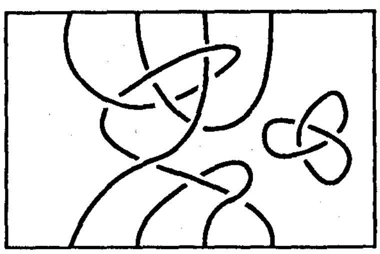


figure 1.10

1.3 APPENDIX : SEIFERT'S ALGORITHM

This appendix outlines the proof of theorem 1.1.6 by giving a construction known as Seifert's algorithm. It also appears in **[Ro]**(p120), and **[B-Z]**(p17).

Let L be a link in $\mathbb{R}^3$, and let $D \subset \mathbb{R}^2$ be a diagram of L. If L is unoriented then choose an orientation for each component of L. The projection underlying D can be transformed into a set of Seifert circles. These simple closed curves in $\mathbb{R}^2$ can be spanned by a set of discs. Although the Seifert circles may be nested, the discs can be made disjoint by lifting them out of $\mathbb{R}^2$. The parts of D which are not parts of these Seifert circles are in neighbourhoods of the crossings. A twisted rectangle (or band) may be added at each crossing of D as shown in figure 1.11. This forms an orientable surface with boundary L. □

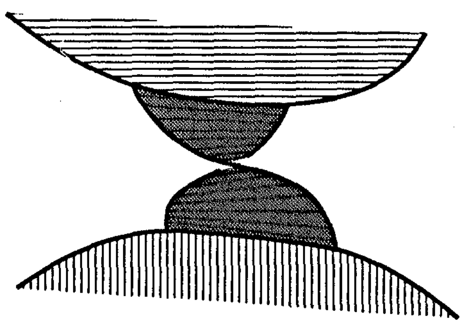


figure 1.11

An application of this algorithm is indicated in figure 1.12. On the left is a diagram of the figure-8 knot; in the centre, the discs spanning the Seifert circles are shown shaded; on the right the surface is completed by adding bands. The resulting surface is also shown in figure 1.3.

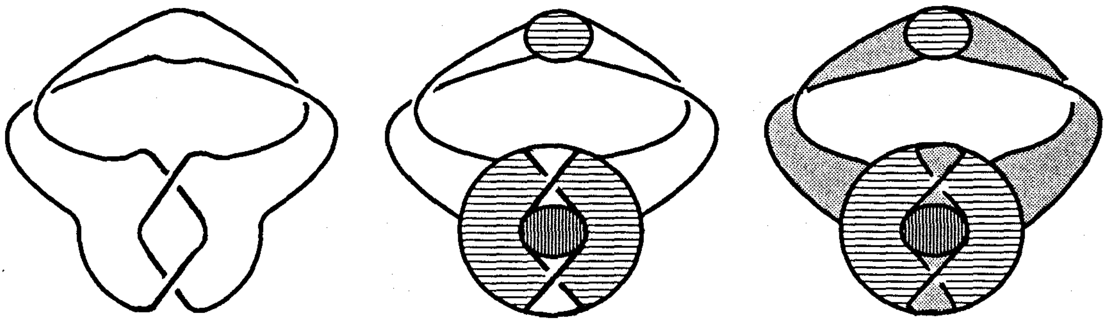


figure 1.12

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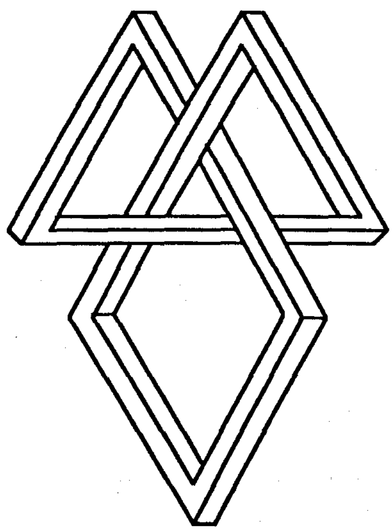
2

Homogeneous Links



2.1 INTRODUCTION

This chapter introduces the class of homogeneous links. The class contains the alternating and positive links as extreme cases. Some results which are known to hold on at least one of these subclasses are extended to the new larger class.

In theorem 2.3.1, the technique which Lickorish and Millett use to prove the existence of the two variable polynomial P(v,z) is refined to show that a resolution can be completely determined by the choice of ordered basepoints on a diagram. This result, and a graph associated with the surface constructed by Seifert's algorithm, are used to prove the two main theorems in §2.4 (which are 2.4.4 and 2.4.10). As their corollaries show, these theorems concern the genus of a link and the possible fibration of the link complement.

In §2.5 various techniques for determining the homogeneity of a link are given with many kinds of links being used as examples. The homogeneity of each of the prime knots of orders up to 10 is given in §2.7. (It is undetermined in only five cases.) The question of whether other properties of alternating links can be generalised to homogeneous links is raised in §2.6 where the problem of minimal order diagrams is examined. The information about link polynomials which is quoted in the text of the chapter is collected in §2.8.

2.2 PRELIMINARIES

Let D be an oriented diagram of a link L. An orientable surface F spanning L can be constructed from D using Seifert's algorithm. A surface constructed in this way is named a projection surface associated to the diagram.

The spine of this surface is a graph, Γ. The vertices of Γ correspond to the discs in F which span the Seifert circles of D; the edges of Γ correspond to the twisted rectangles in F, and hence to the crossings in D. Thus, two vertices of Γ are joined by an edge if and only if their associated Seifert discs are connected by a rectangle. Since Γ is a deformation retract of F, $H_1(\Gamma) = H_1(F)$. Let $rk(\Gamma) = \text{rank } H_1(\Gamma)$.

Each edge in Γ can be given a sign according to the sense of its associated crossing using the convention shown in figure 2.1.

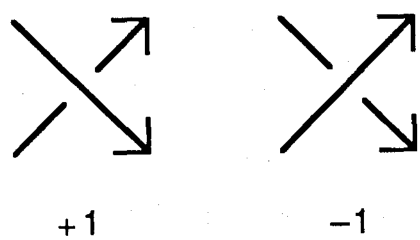


figure 2.1

A signed graph constructed from a diagram in this manner is named a Seifert graph. I shall assume that links are non-split. This implies that the Seifert graphs are connected.

Let Γ be any connected graph. An edge e in Γ is an isthmus if Γ-e is disconnected. A vertex v in Γ is a cut vertex if Γ-v is disconnected. Suppose Γ contains a cut vertex, v, and let $\Gamma_1,\cdots,\Gamma_n$ be the connected components of Γ-v. Then the n subgraphs $\Gamma_1 \cup v,\cdots,\Gamma_n \cup v$ are obtained from Γ by cutting Γ at v. Cutting Γ at each of its cut vertices produces a set of connected components, each one being a subgraph of Γ containing no cut vertices. Such a component is called a block.

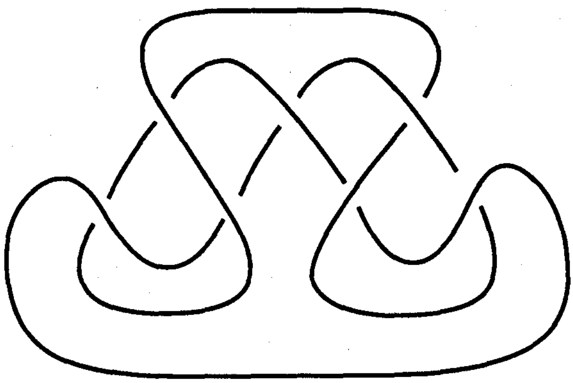
A block of a Seifert graph is homogeneous if all its edges have the same sign. A Seifert graph is homogeneous if each of its blocks is homogeneous. A diagram is homogeneous if its Seifert graph is homogeneous. A link is homogeneous if there is some diagram of the link which is homogeneous.

Suppose that a diagram D is a presentation of a link as the closure of a braid, β. If β is a homogeneous braid in the sense of [St] then D is a homogeneous diagram. (This is the origin of the name.) The converse is not true, however. There are homogeneous links which cannot be presented as homogeneous braids, just as there are alternating links which cannot be presented as alternating braids.

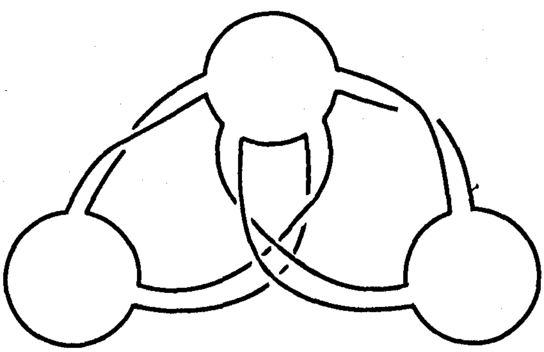
Let **H** denote the class of homogeneous links; let **A** denote the class of alternative links in the sense of Kauffman **[Ka1]**; let **P** denote the class of pseudo-alternating links in the sense of Murasugi and Mayland **[M-M]**. Then $\mathbf{A} \subseteq \mathbf{H} \subseteq \mathbf{P}$. Kauffman conjectures in **[Ka1]**(p125) that $\mathbf{A} \equiv \mathbf{P}$ implying that all three classes are identical. This is not obvious. The two diagrams in figure 2.2 show

(a) a homogeneous diagram which is not an alternative diagram

(b) a pseudo-alternating diagram which is not a homogeneous diagram.



(a)



(b)

figure 2.2

The Seifert circles of a diagram can be separated into two kinds: a circle is of type I if it does not contain any other Seifert circles, otherwise it is of type II. Let $D \subset \mathbb{R}^2$ be a link diagram, and suppose that C is one of its type II Seifert circles. Then C separates $\mathbb{R}^2$ into two components U, V such that $U \cup V = \mathbb{R}^2$, $U \cap V = \partial U = \partial V = C$. Let $D_1 = D \cap U$ and $D_2 = D \cap V$. If both $(U-C) \cap D \neq \Phi$ and $(V-C) \cap D \neq \Phi$ then the type II circle C decomposes D as a *-product of the two diagrams D_1 and D_2. This is written $D = D_1 * D_2$ [Mul]. Let Γ, Γ_1, Γ_2 be the Seifert graphs of D, D_1, D_2 respectively. Then $\Gamma = \Gamma_1 \cup \Gamma_2$, and the single vertex $v = \Gamma_1 \cap \Gamma_2$ is a cut certex of Γ and is associated to the disc spanned by C. Thus, type II Seifert circles in D which decompose it as a *-product are associated to cut vertices in Γ.

A diagram which contains no decomposing type II Seifert circles is called a special diagram. A special diagram has at most one type II Seifert circle which (if it exists) contains all the other Seifert circles.

Each block of a Seifert graph is associated to a special diagram. If the block is homogeneous then all the crossings in the diagram have the same sign. A special positive diagram is alternating. These observations show the following.

2.2.1 Theorem. A homogeneous link is a *-product of special alternating links. □

2.3 RESOLUTIONS

Let D_+, D_-, D_0 denote three diagrams which are identical except within a small neighbourhood where they differ as in figure 2.3.

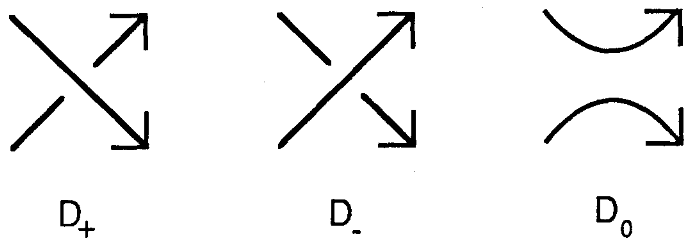


figure 2.3

The two variable link polynomial, P, [F-Y-H-L-M-O], [P-T] can be defined using a recursive relation between diagrams differing in this way, together with a normalising relation:

$$v^{-1} P(D_+) - v\, P(D_-) = z\, P(D_0)$$

$$P(\text{unknot}) = 1.$$

This polynomial depends on a link L and has variables v and z. It is variously denoted $P(L)(v,z)$, $P_L(v,z)$, $P(L)$ according to context.

A <u>resolution</u> of a diagram, D, is a parsing tree with a diagram associated to each node such that

(1) D is at the root

(2) there is a trivial link at each terminal node

(3) each triple (parent, leftchild, rightchild) is of the form (D_+, D_-, D_0) or (D_-, D_+, D_0).

Each edge of the parsing tree can be labelled with a monomial in v and z as shown in figure 2.4.

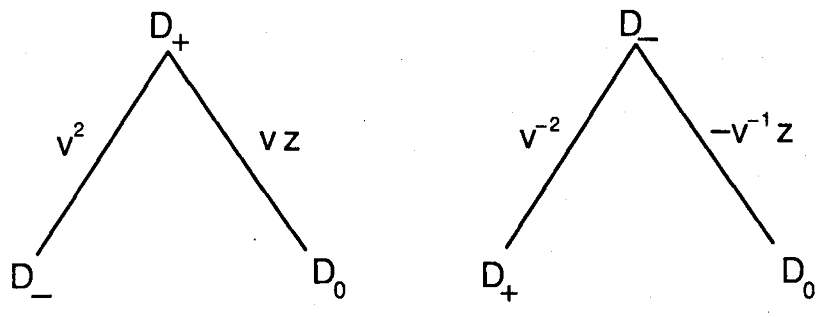


figure 2.4

Let π_i denote the product of the edge labels for the edges on the (unique) path between a terminal node T_i and the root of the parsing tree. Let $|T_i|$ denote the number of components in the trivial link associated to T_i, and let $\delta = (v^{-1}-v)/z$. Then

$$P(D) = \sum \pi_i \, \delta^{|T_i|-1}.$$

In their proof of the existence of P(v,z), Lickorish and Millett [L-M] use based ordered diagrams of oriented links to construct ascending diagrams and then induct on the number of crossings. This induction introduces ambiguity into the resolution. The following theorem refines their technique to show that a complete resolution is determined by the choice of basepoints and ordering of components. Furthermore, no crossing need be altered more than once.

2.3.1 Theorem. It is possible to construct a resolution for a diagram so that in a path from a terminal node of the parsing tree to the root no crossing is changed more than once.

Proof. Let $L = \bigcup L_i$ be a link with μ components. Let D be a diagram of L and let D_i be subdiagrams of D so that each D_i is a diagram of L_i.

Orient the link and choose a basepoint on each D_i distinct from any crossing in D. For each i in sequence follow round the diagram in the direction of the orientation until a crossing, c, is reached which is first encountered as an over-crossing. Let A(D) denote the subset of D traversed before reaching c. This is the ascending set. Let N(D) = D - A(D) denote the non-ascending set which contains both over and under-crossing arcs at c. Let |A(D)| denote the number of crossings of D through which A(D) passes, each crossing counted at most once; and let |N(D)| denote the number of crossings where neither arc is in A(D). Then |A(D)| + |N(D)| = c(D), the number of crossings in D. The theorem is proved by induction on |N(D)|.

If |N(D)| = 0 then c(D) = |A(D)| and the diagram is ascending, hence trivial.

If |N(D)| $\neq$ 0 then N(D) contains a crossing, c, which separates D into A(D) and N(D). Let D' be the diagram obtained from D by switching the sense of c. The orientation, basepoints and ordering on the components can be taken to be the same as those for D since the underlying projections of D and D' are identical. Then $A(D') \supset A(D)$ and $|A(D')| > |A(D)|$. Also $c(D') = c(D)$. So

$$|N(D')| = c(D') - |A(D')| < c(D) - |A(D)| = |N(D)|.$$

Let D^0 be the diagram obtained from D by removing crossing c. The orientation of D^0 is induced from that of D. The number of components in D^0 is one more or one less than in D depending on whether the arcs at c are (i) from the same subdiagram or (ii) from different subdiagrams.

To put an ordering on the subdiagrams of D^0 give the unaltered subdiagrams the same index in the ordering and the same basepoints as they have in D. For the altered subdiagrams, consider cases (i), (ii) separately.

case (i): Let D_r be the subdiagram which has been disconnected. One connected component will contain the basepoint of D_r. Let this component be given r^{th} place in the ordering, and give it the same basepoint as D_r. Place the other component $(\mu+1)^{th}$ in the ordering and place a basepoint on it where c has been removed.

case (ii): Let D_r, D_s be the two subdiagrams which have been joined and suppose $r<s$. Place this new subdiagram r^{th} in the ordering and give it the same basepoint as D_r. (Note that the ordered set of subdiagrams now has no element of index s.)

Now $A(D^0) \supseteq A(D)$ and $|A(D^0)| \geq |A(D)|$. Also $c(D^0) = c(D)-1$. So,

$$|N(D^0)| = c(D^0)-|A(D^0)| < c(D)-|A(D)| = |N(D)|.$$

If $\varepsilon(c)$ denotes the sign of crossing c then

$$P(D) = v^{2\varepsilon(c)} P(D') + \varepsilon(c)\, v^{\varepsilon(c)} z\, P(D^0).$$

The diagram has been partially resolved into two diagrams D' and D^0 with $|N(D')|$ and $|N(D^0)|$ less than $|N(D)|$ without changing the sense of any crossing in A(D), and such that both $A(D')$ and $A(D^0)$ contain A(D). This step can be repeated inductively to complete the required resolution. □

A resolution constructed according to this algorithm is called a <u>based diagram resolution</u>. Once the basepoints have been chosen, the resolution is completely determined.

A resolution constructed in this manner may contain some redundant operations. Let c be a crossing in a diagram D which is associated to an isthmus in the Seifert graph of D. If an ascending diagram is constructed from D, switching crossing c does not alter the link type of D. When constructing a resolution for D all such operations can be ignored. A based diagram resolution reduced in this way is named a <u>standard resolution</u>.

It is known [vB] that positive braids have positive Conway polynomial, that is all the coefficients are non-negative. (An introduction to the Conway polynomial which contains the results used below is [Ka2]. It is denoted ∇ and is obtained from P by setting $v = 1$. Thus $\nabla(z) = P(1,z)$.)

2.3.2 Corollary. A positive link has positive Conway polynomial.

Proof. A positive diagram has a based diagram resolution in which each triple (parent, leftchild, rightchild) is of the form (D_+, D_-, D_0). Hence, each stage in the parsing tree has the form

$$\nabla(D_+) = \nabla(D_-) + z\,\nabla(D_0).$$

At every terminal node there will be a trivial knot or a split link which have $\nabla = +1$ or 0 respectively. Therefore, ∇ for a positive link is a sum of positively signed monomials in z. □

The following corollary does not have much intrinsic interest, but is of use later.

2.3.3 Corollary. If a link L has a diagram in which every crossing except one is positive then $\nabla(L)$ is positive.

Proof. Suppose D is a diagram of L with subdiagrams D_i and let c be the only negative crossing in D. Choose a basepoint for D on the undercrossing arc at c so that when following round this component of the diagram in the given orientation, the basepoint is reached just before c. Choose a basepoint for each of the other subdiagrams distinct from any crossing of D. Put an ordering on the components so that the undercrossing arc at c is in the subdiagram of index 1. Then the undercrossing arc at c is contained in the ascending set, A(D), and c need not be altered. The result now follows as in 2.3.2. □

2.4 PROPERTIES OF HOMOGENEOUS LINKS

Seifert graphs are a useful tool when considering resolutions. This is because smoothing a crossing in a diagram (and hence removing it from the diagram) corresponds to deleting its associated edge from the Seifert graph. The 'simplicity' of the graph can have some resemblance to the 'simplicity' of the link.

2.4.1 Theorem. Let D be a homogeneous diagram with Seifert graph Γ. Then D represents the trivial knot if and only if Γ is a tree.

Proof($\Leftarrow$). If Γ is a tree then the surface spanning the link is a disc; hence the link is a trivial knot.

Proof($\Rightarrow$). Suppose that Γ is a homogeneous graph of a diagram D representing the trivial knot. Choose a basepoint so that D is not an ascending diagram with respect to this basepoint. Construct a standard resolution of D. The diagrams of the trivial links associated to all but the leftmost terminal node will have fewer crossings than D. If any of these diagrams has a Seifert graph which is connected and is not a tree then repeat this procedure.

Label the resolution as before, then set v=1. This is equivalent to considering the Conway polynomial, $\nabla(D)$, of D [Ka2]. Consider the terms of degree $rk(\Gamma)$ in z in $\nabla(D)$. These occur only at the terminal nodes where the Seifert graph is a tree since deleting rk(Γ) edges from Γ leaves either a tree or a disconnected graph. Where the Seifert graph has become disconnected the contribution is zero since ∇(split link)=0; where the Seifert graph is a tree the diagram is of the trivial knot, hence ∇=1.

Write Γ as a union of its blocks $\Gamma = \bigcup B_i$, and let $\varepsilon(B_i)$ denote the sign of the edges in the (homogeneous) block B_i. When Γ is reduced to a tree, so is each block. So the sign of any term of degree rk(Γ) in z is $\prod \varepsilon(B_i)^{rk(B_i)}$. Hence all terms of degree rk(Γ) have the same sign and do not cancel. Thus, there is a term of degree rk(Γ) in z in ∇(D). But D is a diagram of the trivial knot so ∇(D)=1; hence rk(Γ)=0 and Γ is a tree. □

The following result is known for alternating links [Au], and for positive links [Mu5].

2.4.2 Corollary. A link with a connected homogeneous diagram is non-split.

Proof. If D is a connected homogeneous diagram of a link L with Seifert graph Γ then ∇(L) contains a term of degree rk(Γ) in z. Hence, $\nabla(L) \neq 0$ and L is non-split. (If D is disconnected then clearly L is split.) □

2.4.3 Notation. Let maxdeg_z P denote the highest degree of z in the polynomial P, and mindeg_v P denote the lowest degree of v in P. Other combinations are defined similarly.

2.4.4 Theorem. Let L be a homogeneous link and let $\chi(L)$ denote the maximal Euler characteristic over all orientable surfaces spanning L. Then

(a) $\text{maxdeg}_z P(L) = 1-\chi(L)$

(b) $\text{mindeg}_v P(L) \leqslant 1-\chi(L)$ with equality if and only if L is positive.

2.4.5 Remark. In [Mo1], Morton conjectures for all links L that

$$\text{mindeg}_v P(L) \leqslant 1-\chi(L).$$

Proof. Let D be a homogeneous diagram of L with Seifert graph Γ. If F is the projection surface constructed from D then $\text{rk}(\Gamma) = 1-\chi(F)$.

Construct a standard resolution of D labelled to give P(L), and consider the terms of degree $\text{rk}(\Gamma)$ in z. The diagram associated to each of the terminal nodes has a connected Seifert graph (since isthmuses are never deleted when constructing a standard resolution). If $\text{rk}(\Gamma)$ non-isthmus edges are deleted from Γ then the result is a tree. Hence, only terminal nodes where the Seifert graph is a tree contribute a term of degree $\text{rk}(\Gamma)$ in z to P(L).

Let T be such a terminal node and let π be the path in the resolution from T to the root. Since the Seifert graph of the diagram associated to T is a tree, the diagram represents the trivial knot (theorem

2.4.1). Hence, the term contributed to P(L) by T is the product of the monomials labelling the edges of π. Let this term be denoted $P|_\pi$.

When Γ is converted into a tree, so is each block B_i of Γ. Let $\varepsilon(B_i)$ denote the sign of the edges in block B_i. Deleting $rk(B_i)$ edges from B_i corresponds to removing $rk(B_i)$ crossings from D. These changes correspond to the monomials labelling π whose product is the following factor of $P|_\pi$:

$$\varepsilon(B_i)^{rk(B_i)} \cdot v^{rk(B_i)\varepsilon(B_i)} \cdot z^{rk(B_i)}.$$

This factor does not depend on the choice of edge deletion. Each block contributes a factor, so $P|_\pi$ can be written

$$m(v) \cdot \prod\varepsilon(B_i)^{rk(B_i)} \cdot v^{\sum rk(B_i)\varepsilon(B_i)} \cdot z^{rk(\Gamma)}.$$

The monomial m(v) is derived from the leftchild edge labels in π which correspond to switching the sense of crossings in D. Thus, m(v) is a product of v^2 and v^{-2}, and hence is positive.

All terms of degree $rk(\Gamma)$ in z have the same sign and do not cancel. Hence, setting v=1 in P(L) shows that $\nabla(L)$ has degree $rk(\Gamma)$. The degree of $\nabla(L) \leqslant 1-\chi(L)$ where $\chi(L)$ runs over all orientable surfaces spanning L [Ka1], [B-Z]. Hence (a) follows.

Furthermore, there is at least one term with m(v) = 1 obtained from the rightmost terminal node of the resolution. So there is a term in

$P(L)$ with degree $\sum rk(B_i)\varepsilon(B_i)$ in v. Now $\sum rk(B_i)\varepsilon(B_i) \leq rk(\Gamma)$ with equality if and only if all $\varepsilon(B_i) = +1$ for all i, that is if and only if D is a positive diagram. So (b) follows. □

2.4.6 Corollary. Let L be a homogeneous link with μ components, and let g(L) denote its genus. Then

$$g(L) = \tfrac{1}{2}\,[\ \mathrm{maxdeg}_z\, P(L) - \mu + 1\ ].$$

The projection surface associated to a homogeneous diagram is a spanning surface of minimal genus.

Proof. Let F be the projection surface associated to a homogeneous diagram of L. Then

$$2g(F) = 2-(\chi(F)+\mu) = 1+(1-\chi(F))-\mu = \mathrm{maxdeg}_z\, P(L) - \mu + 1.$$ □

2.4.7 Corollary. Let T(p,q) denote the (p,q) torus link. Then

$$g(T(p,q)) = \tfrac{1}{2}\,[\ (p-1)(q-1)-\mu+1\ ].$$

Proof. Let β be a positive braid presentation of the (p,q) torus link, and let Γ be the Seifert graph constructed from the closure of β. Then $rk(\Gamma) = (p-1)(q-1)$. □

2.4.8 Remark. Let L be any link. If L possesses a positive diagram then $\mathrm{maxdeg}_z\, P(L) = \mathrm{mindeg}_v\, P(L)$. This provides another method of deciding whether a link can be presented as a positive braid, and completes the classification of prime knots of orders ≤10 given in [vB]. None of the four undecided cases have positive braid presentations.

For any link L of multiplicity μ, $\text{mindeg}_z P(L) = 1-\mu$ [L-M]. Suppose that the polynomial P(v,z) is written in the following form:

$$P(v,z) = \sum_{i=1-\mu}^{r} \alpha_i(v)\, z^i$$

where μ is the number of components in the link, each $\alpha_i(v)$ is a polynomial in v, and $r = \text{maxdeg}_z P$. Define h(P)(v) to be the polynomial $\alpha_r(v)$.

2.4.9 Corollary (Traczyk). It follows from the construction used in proving theorem 2.4.4 that, for homogeneous links, the coefficients of h(P) are all non-negative or all non-positive. □

Recall that $\nabla(z)$ denotes the Conway polynomial. Murasugi has shown that an alternating link where the leading coefficient of ∇ is ±1 is a *-product of (p,2) torus links [Mu1].

2.4.10 Theorem. Let L be a homogeneous link. Then the leading coefficient of $\nabla(L)$ is ±1 if and only if L is a *-product of (p,2) torus links.

Proof. By theorem 2.2.1, L is a *-product of special alternating links, L_i. Let $h(\nabla)$ denote the term of highest degree in the polynomial ∇. It is sufficient to show that each L_i has leading coefficient of $\nabla(L_i) = \pm 1$ if and only if it is a (p,2) torus link since $h(\nabla(L_1 * L_2)) = h(\nabla(L_1)) \cdot h(\nabla(L_2))$ [Mu3]. (Murasugi and Przytycki have used theorem 2.3.1 to generalise this result to $P(v,z)$. Thus $h(P(L_1 * L_2)) = h(P(L_1)) \cdot h(P(L_2))$ [M-P].)

(proof ⇐) Claim: if L is a (p,2) torus link then

$$h(\nabla(L)) = \begin{cases} z^{p-1} & ;p>0 \\ (-z)^{|p|-1} & ;p<0 \end{cases}$$

Proof of claim: Suppose $p>0$. The (1,2) torus link is trivial and so has $\nabla=1$. The (2,2) torus link is a Hopf link and has $\nabla = z$. Now, for $p>2$

$$\nabla(p,2) = \nabla(p-2,2) + z\,\nabla(p-1,2).$$

By the inductive hypothesis the two polynomials on the right hand side have highest terms z^{p-3} and z^{p-1} respectively, so $h(\nabla(p,2)) = z^{p-1}$. Similarly for $p<0$: replace z by -z.

(proof ⇒) Let D be a special projection of L_i so that Γ, the Seifert graph of D, is a 1-block graph. If L_i is a (p,2) torus link then Γ consists of two vertices joined by p edges.

Let C be a circuit in Γ such that $|C|$, the number of edges in C, is maximal. Suppose that D is not a (p,2) torus link then $|C|>2$. Since Γ is homogeneous, all the edges in C have the same sign. Assume, without loss of generality, that this is +1.

Choose an edge e in Γ-C such that $rk(\Gamma-e) < rk(\Gamma)$. Let c be a crossing in D which is associated to e and let D_0, D_- be the diagrams obtained by removing c from D, and by reversing the sense of c in D respectively. Then

$$\nabla(D) = \nabla(D_-) + z\,\nabla(D_0).$$

Now D_- is a diagram with all crossings except one positive, so $\nabla(D_-)$ is a positive polynomial (by 2.3.3).

This process can be repeated $rk(\Gamma)-1$ times to reduce Γ to a graph Γ' which spans Γ and contains C as its only circuit. The isthmuses of Γ' correspond to twists in the diagram which can be undone by applying type I Reidemeister moves. Hence, the diagram D' associated to Γ' is equivalent to the (p,2) torus link with oppositely oriented strings which has C as its Seifert graph. It can be seen inductively that $\nabla(D') = \frac{1}{2}|C|\ z$. Now

$$\begin{aligned}\nabla(D) &= z^{rk(\Gamma)-1}\,\nabla(D') + \text{positive polynomials in } z\\ &= \tfrac{1}{2}|C|\cdot z^{rk(\Gamma)} + \text{positive polynomials in } z.\end{aligned}$$

The contribution of $\nabla(D')$ is not cancelled and is a term of highest degree. Also $\frac{1}{2}|C| \neq 1$ since $|C|>2$. Hence, the leading coefficient of $\nabla(L) \neq \pm 1$.

If D! denotes the diagram obtained from D by switching the sense of every crossing then $\nabla(D!)(z) = \nabla(D)(-z)$, so the result holds for diagrams in which every crossing has sign -1. □

The first two of the following corollaries are generalisations of some results in [Mu1].

2.4.11 Corollary. If L is a homogeneous link and the leading coefficient of $\nabla(L)$ is ±1 then L has order at most $2\cdot\text{maxdeg}\ \nabla(L)$.

Proof. The link L possesses a homogeneous diagram D with Seifert graph Γ such that each circuit has exactly 2 edges. By theorem 2.4.4, $rk(\Gamma) = \text{maxdeg}_z\ P(L)$ and the maximum number of edges in Γ (hence the maximum number of crossings in D) is $2rk(\Gamma)$. □

2.4.12 Corollary. Let L be a homogeneous link and let D be a special homogeneous diagram of L. Then the leading coefficient of $\nabla(L)$ is ±1 if and only if L is a connected sum of (p,2) torus links.

Proof. Let F be the projection surface associated to D and let Γ be its Seifert graph.

A diagram can be chessboard shaded with two colours so that two regions of the same colour meet only at crossings. A graph can be formed from this shading as shown in figure 2.5. If the edges of the graph are signed according to the sense of the crossings then the diagram and shading can be reconstructed from the graph. The Seifert graph obtained from a special diagram is the same as its chessboard graph. Hence, for special projections, the diagram can be reconstructed from its Seifert graph.

If D is a connected sum of diagrams then Γ must contain a cut vertex and hence, more than one block. Conversely, if Γ contains a cut vertex then D is a connected sum of diagrams.

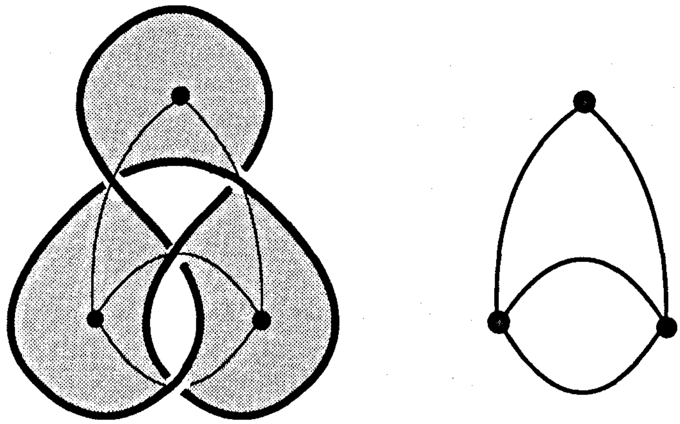


figure 2.5

The leading coefficient of $\nabla(L)$ is ±1 if and only if each block reconstructs a (p,2) torus link. □

It is known that the Conway polynomial of a fibred knot has leading coefficient ±1 [Ra], and that the converse holds for alternating knots [Mu3]. The following corollary shows that this result extends to homogeneous links.

2.4.13 Corollary. Let L be a homogeneous link. Then L is fibred if and only if the leading coefficient of $\nabla(L)$ is ±1.

Proof. The leading coefficient of $\nabla(L)$ is ±1 if and only if $L = L_1 * \ldots * L_n$ where each L_i is a (p,2) torus link. L is fibred if and only if each L_i is fibred [Ga1]. The (p,2) torus links are fibred [Mu3]. □

2.4.14 Corollary. Let L be a homogeneous link. If L can be presented as a braid whose closure is a homogeneous diagram (ie: as a homogeneous braid in the sense of [St]) then the leading coefficient of $\nabla(L)$ is ±1. □

It was remarked in §2.2 that there are homogeneous links which do not have homogeneous braid presentations. This corollary shows that the prime knot 5_2 is one example since the leading coefficient of $\nabla(5_2)$ is 2.

2.4.15 Corollary. Let L be an alternating link. If L can be presented as a braid whose closure is an irreducible alternating diagram, and hence as a braid with the minimum number of crossings [Ka3], [Mu2], [Th], then the leading coefficient of $\nabla(L)$ is ±1. □

2.5 EXAMPLES

The simplest examples of homogeneous links are those which possess a diagram in which all the crossings have the same sign. These are the standard or positive links. They include the torus links, Lorenz links, and the links associated with complex algebraic singularities [Mi]. The alternating links are homogeneous since they are alternative [Ka1] and all alternative links are homogeneous. The knot 9_{43} is a homogeneous knot with the least number of crossings which is both non-alternating and non-positive.

The homogeneity of a link is orientation dependent. For example, of the two links shown in figure 2.6, (a) is homogeneous since the diagram is positive, but (b) is non-homogeneous (this is a corollary of theorem 2.5.7 - see also appendix II, §2.8).

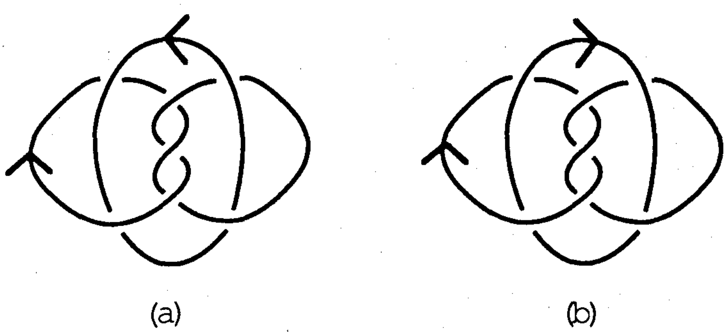


figure 2.6

The rest of this section shows various techniques for determining the homogeneity of links. Examples are taken from the arborescent, double, boundary and pretzel links. The homogeneity of the prime knots of orders ≤10 are given in appendix I, §2.7.

2.5.1 Arborescent links.

Let T be a tree whose vertices are weighted with elements of $\mathbb{Z}$. Choose a root vertex, $v \in T$ of weight λ and let $v_1, \cdots, v_n$ denote the vertices adjacent to v in cyclic order. Associate to v the 2-tangle of figure 2.7 where λ is the signed number of half-twists (shown for positive λ), and each v_i represents a 2-tangle of similar form.

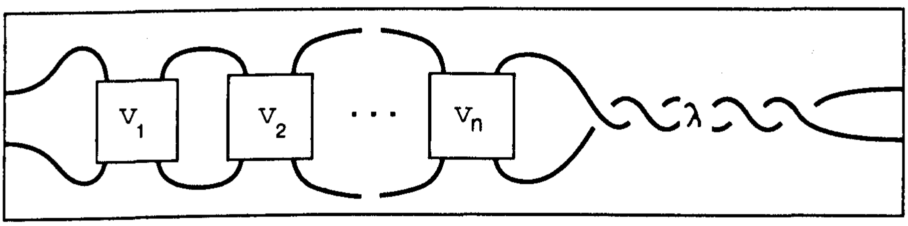
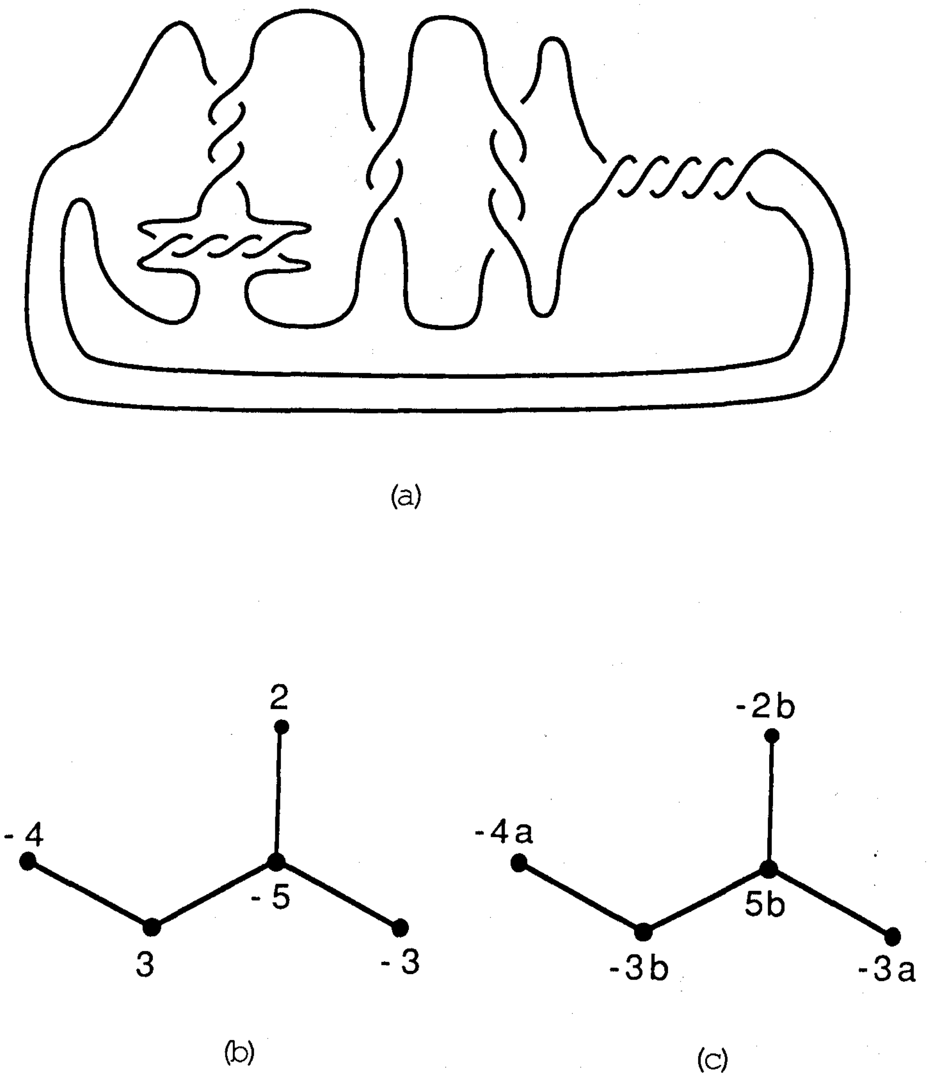


figure 2.7

This can be repeated recursively associating a 2-tangle to each vertex, with the leaves of the tree contributing twists only. The closure of the resulting 2-tangle is an arborescent link. This construction is best illustrated with an example: see figure 2.8(a) and (b) (see also [Ga2]).



(a)

(b)

(c)

figure 2.8

When the link is oriented the vertex labels of T can be replaced by labels in $\mathbb{Z}\{a,b\}$, figure 2.8(c). Suppose a vertex $v \in T$ is weighted with $\lambda \in \mathbb{Z}$. The new label depends on sense of the half-twists, and on the the orientations of the strings in the 2-tangle as shown in figure 2.9.

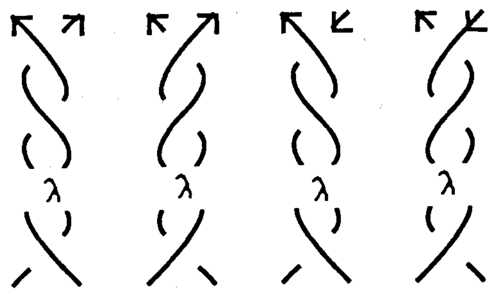


figure 2.9

Let Γ denote the Seifert graph of the oriented arborescent link diagram which is associated with a vertex-labelled tree, T. Two adjacent vertices of T cannot both be labelled a, otherwise an inconsistency in the orientation arises. Also, the 2-tangles associated to two adjacent vertices of T give rise to different blocks of Γ if and only if both of the vertices are labelled b. Therefore, to ensure that an arborescent link constructed from a tree T is homogeneous, it is sufficient that, for all pairs of adjacent vertices in T with labels λa, μb; $\lambda,\mu\in\mathbb{Z}$, the coefficients λ, μ have the same sign.

2.5.2 Double knots and boundary links.

If L is a homogeneous link then the coefficients of $h(P_L)(v)$ have constant sign (by corollary 2.4.9), hence $h(P_L)(1)\neq 0$ and $\text{maxdeg}_z\, P_L = \text{maxdeg}\, \nabla_L$.

If K is a homogeneous knot such that $P_K=1$ then K has a rank zero Seifert graph, and (by theorem 2.4.1) K is the trivial knot. Thus, non-trivial knots which have $\nabla=1$ such as all the Kinoshita-Terasaka knots [K-T] are non-homogeneous.

2.5.3 Theorem. An untwisted double knot with non-trivial companion is non-homogeneous.

Proof. Such a knot has $\nabla=1$ [Mo2], and all satellite links are non-trivial [B-Z](p37). □

2.5.4 Theorem. A double knot with non-trivial non-cable companion is non-homogeneous.

Proof. Let K be a double knot with non-trivial non-cable companion. A minimal genus surface F spanning K is obtained by plumbing a Hopf band to a knotted annulus. The surface F is knotted (that is $\Pi_1(S^3-F)$ is not free). By [Wh], K possesses a unique isotopy type of minimal genus spanning surface. A projection surface is unknotted. Hence, F is not isotopic to a projection surface. The projection surface associated to a homogeneous diagram has minimal genus. So, the existence of a homogeneous diagram of K would contradict the uniqueness of F. □

2.5.5 Theorem. A (non-split) boundary link is non-homogeneous.

Proof. Let L be a boundary link. Then $\nabla(L) = 0$, so $h(P_L)$ contains both positive and negative terms. Hence, L is non-homogeneous. □

2.5.6 Classification of prime knots up to order 9.

Let D be a homogeneous diagram of a link L. Construct a based diagram resolution of D, and consider the rightmost terminal node. Let r denote the total number of crossings which have been removed, and let n denote the number of these that have negative signs. Then $r = \text{maxdeg}_z\, P(L)$, and the term contributed by the rightmost terminal node is

$$(-v^{-1})^n\ (v)^{r-n}\ z^r.$$

Let s=r-2n. Then the expression becomes

$$(-1)^{\frac{1}{2}(r-s)}\ v^s\ z^r.$$

Since D is homogeneous, this term is not cancelled. This observation proves the following.

2.5.7 Theorem. A link L is non-homogeneous if P(L) has no terms of the form

$$\lambda\ (-1)^{\frac{1}{2}(r-s)}\ v^s\ z^r \quad \text{for } \lambda\in\mathbb{N},\ r = \text{maxdeg}_z\, P(L),\ s\leqslant r.$$

□

2.5.8 Corollary. The prime knots 8_{20}, 8_{21}, 9_{42}, 9_{44}, 9_{45}, 9_{46} are non-homogeneous.

Proof. These knots do not have polynomials of the above form (see appendix II). □

This trick does not work for the prime knot 9_{48}. However, maxdeg $\nabla(9_{48}) = 4$ (see appendix II). So if 9_{48} were homogeneous then (by corollary 2.4.11) it would have order at most 8 - a contradiction.

2.5.9 Classification. The non-homogeneous prime knots of order at most 9 are 8_{20}, 8_{21}, 9_{42}, 9_{44}, 9_{45}, 9_{46}, 9_{48}. For all of the other knots the diagrams in appendix D of [B-Z] are homogeneous. □

These techniques can be applied to the prime knots of order 10 and are sufficient to determine the homogeneity in all but eleven cases. Of these, 10_{129}, 10_{130}, 10_{135}, 10_{146}, 10_{147}, 10_{164} are shown to be non-homogeneous by the following theorem of Murasugi [Mu4](p170).

2.5.10 Theorem (Murasugi). Let $D, D_1, \cdots, D_n$ be diagrams of links $L, L_1, \cdots, L_n$ where $D = D_1 * \cdots * D_n$ and each D_i is a standard (positive or negative) diagram. Let $f(L) = \text{breadth } V_L(t) - \mu(L)+1$ where V denotes the Jones polynomial **[Jo]** and μ the number of components. Then

$$f(L) \leqslant \sum f(L_i) .$$

□

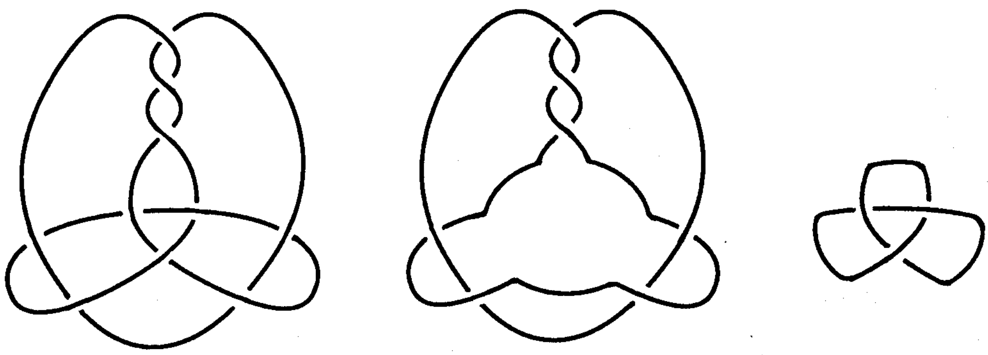
As an example, consider the knot 10_{129}. It can be decomposed as a *-product of a trefoil and a trivial knot (see figure 2.10). Now

$$f(\text{unknot}) = 0-1+1 = 0$$

$$f(\text{trefoil}) = 3-1+1 = 3$$

$$f(10_{129}) = 8-1+1 = 8.$$

So this decomposition of 10_{129} fails to satisfy theorem 2.5.10 and hence 10_{129} is not homogeneous.



10_{129} unknot trefoil

figure 2.10

2.5.11 Remark. The homogeneity of a link cannot be determined from $P(v,z)$ alone since 8_8 and 10_{129} have the same polynomial, and the former is homogeneous where-as the latter is not.

2.5.12 Pretzel knots.

Let $(a_1, \cdots, a_n)$ denote the pretzel knot comprised of 2-tangles each having a_i signed half-twists where the a_i are odd (see figure 2.11).

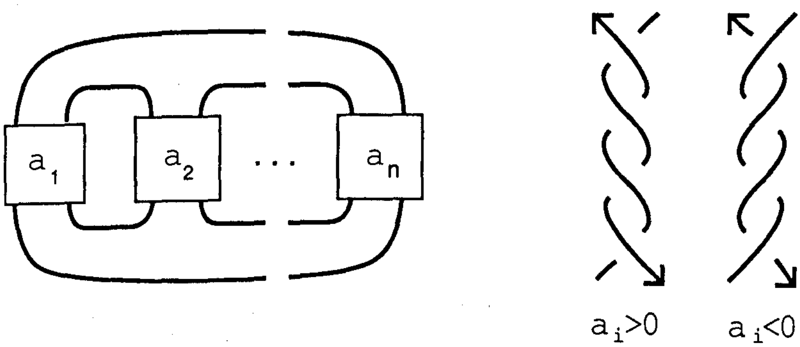


figure 2.11

Clearly, if all a_i have the same sign then the knot is homogeneous. For non-standard pretzel knots we have the following.

2.5.13 Theorem. Let $(-p,q,r)$ be a pretzel knot with $q,r \geqslant p \geqslant 3$. Then $(-p,q,r)$ is non-homogeneous.

The proof of this theorem consists of calculating $P(v,z)$ for the pretzel knots in question, and applying corollary 2.4.9. This is done in several steps.

Step 1.

$$P(p,1) = v^2 P(p-2,1) + vz$$

$$= vz \sum_{i=0}^{\frac{1}{2}(p-1)} v^{2i} + v^{p+1} P(-1,1)$$

$$= vz \sum_{i=0}^{\frac{1}{2}(p-1)} v^{2i} + v^{p+1} \delta.$$

$$P(-p,1) = v^{-2} P(-(p-2),1) - v^{-1}z$$

$$= -v^{-1}z \sum_{i=0}^{\frac{1}{2}(p-3)} v^{-2i} + v^{-(p-1)} P(-1,1)$$

$$= -v^{-1}z \sum_{i=0}^{\frac{1}{2}(p-3)} v^{-2i} + v^{-(p-1)} \delta.$$

□

Remark. Notice that $(p,1)$ is a $(p+1,2)$-torus link with the components having opposite orientations.

Step 2.

$$P(-p,1,p) = v^2\, P(-p,1,p-2) + vz\, P(-p,1)$$

$$= vz \sum_{j=0}^{\frac{1}{2}(p-1)} v^{2j}\, P(-p,1) \qquad + v^{p+1}\, P(-p,1,-1)\ .$$

Now $(-p,1,-1)$ is the trivial knot so $h(P)$ comprises the terms in z^2, and

$$h(P) = \left(-\sum_{j=0}^{\frac{1}{2}(p-1)} v^{2j} \right) \cdot \left(\sum_{i=0}^{\frac{1}{2}(p-3)} v^{-2i} \right)$$

$$= -\sum_{i=-\frac{1}{2}(p-3)}^{\frac{1}{2}(p-1)} v^{2i}\, \lambda_i$$

where the coefficients $\lambda_i = \frac{1}{2}(p - |2i-1|)$. □

Remarks. Note that $v^{-1}\cdot h(P)$ is a symmetric polynomial. For any odd $q,r \in \mathbb{Z}$ the knot $(q,1,r)$ is homogeneous, see figure 2.12. Hence, $(-p,1,p)$ is homogeneous.

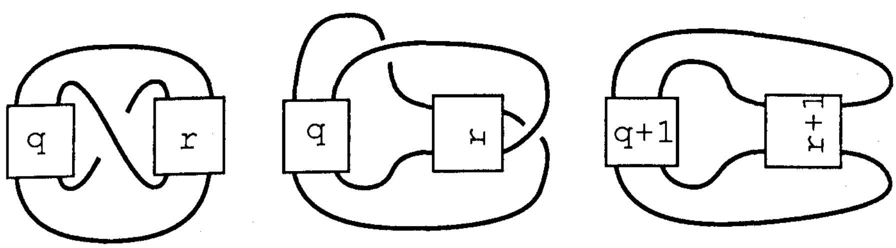


figure 2.12

Step 3.

$$P(-p,p,p) = v^2\, P(-p,p,p-2) + vz\, P(-p,p)$$

$$= vz \sum_{i=0}^{\frac{1}{2}(p-3)} v^{2i}\, \delta \qquad + v^{p-1}\, P(-p,p,1) \ .$$

The only terms in z^2 which are contributed to h(P) come from P(-p,p,1). So,

$$h(P) \;=\; -v^{p-1} \sum_{i=-\frac{1}{2}(p-3)}^{\frac{1}{2}(p-1)} v^{2i}\, \lambda_i$$

$$= \; - \sum_{i=1}^{(p-1)} v^{2i}\, \lambda_i$$

where the coefficients $\lambda_i = \frac{1}{2}(p - |2i-p|)$. □

Remarks. $v^{-p}\cdot h(P)$ is a symmetric polynomial. The term v^2z^2 is negative and there are no terms of lower degree in v in h(P). Therefore (by theorem 2.5.7), (-p,p,p) is non-homogeneous.

Step 4.

$$P(-p,p,r) = v^2\, P(-p,p,r-2) + vz\, P(-p,p)$$

$$= vz \sum_{i=0}^{\frac{1}{2}(r-p)-1} v^{2i}\, \delta \qquad + v^{r-p}\, P(-p,p,p) \ .$$

The only terms in z^2 which are contributed to h(P) come from P(-p,p,p). So,

$$h(P) = -v^{r-p} \sum_{i=1}^{p-1} v^{2i} \lambda_i$$

where the coefficients $\lambda_i = \frac{1}{2}(p - |2i-r|)$. □

Remarks. $v^{-r} \cdot h(P)$ is a symmetric polynomial. Since there are no terms of degree 2 or less in h(P), (-p,p,r) is non-homogeneous.

Proof (of theorem 2.5.13).

$$P(-p,q,r) = v^2 P(-p,q-2,r) + vz\, P(-p,r)$$

$$= vz \sum_{i=0}^{\frac{1}{2}(q-p)-1} v^{2i} P(-p,r) \quad + v^{q-p} P(-p,p,r) .$$

Now $P(-p,r) = P(r-p-1,1)$.

Let $$A(v) = v^2 \left(\sum_{i=0}^{\frac{1}{2}(q-p)-1} v^{2i} \right) \cdot \left(\sum_{j=0}^{\frac{1}{2}(r-p)-1} v^{2j} \right)$$

and $$B(v) = v^{q-p} v^{r-p} \sum_{i=1}^{p-1} v^{2i} \lambda_i \quad .$$

Then $h(P)(v) = A(v) - B(v)$, and $v^{-\alpha} \cdot A(v)$ and $v^{-\beta} \cdot B(v)$ are both symmetric polynomials in v with positive coefficients; $\alpha = \frac{1}{2}(q+r-2p)$, $\beta = r+q+p$. The degree bounds of these polynomials are

$$\text{mindeg } A(v) = 2$$

$$\text{maxdeg } A(v) = (q-p)-2 + (r-p)-2 + 2 = q+r-2p-2$$

$$\text{mindeg } B(v) = (q-p) + (r-p) + 2 = q+r-2p+2$$

$$\text{maxdeg } B(v) = (q-p) + (r-p) + 2p - 2 = q+r-2$$

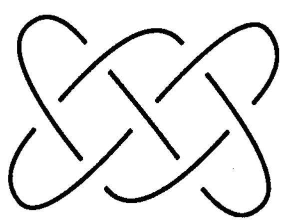
There is no term in h(P) of degree q+r-2p. All terms of a lower degree have positive sign, and all terms of higher degree have negative sign. Since h(P) contains terms of both signs, (-p,q,r) is non-homogeneous. □

2.5.14 Remark. The knot (-3,5,5) is the knot with the least number of crossings known such that h(P) contains both positive and negative terms (see appendix II).

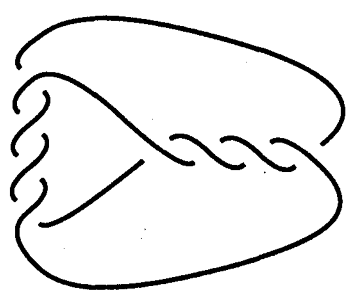
2.6 COMMENTS AND QUESTIONS

Many of the ideas in this chapter arose from trying to unify the known results about the genus of positive and alternating links, then trying to generalise other results about alternating links. To avoid giving the impression that all theorems generalise I conclude with some results on crossing number.

It is known [Ka3], [Mu2], [Th] that any two connected irreducible alternating diagrams of a link have the same crossing number, and that this is minimal over all diagrams of the link. This statement is no longer true if it is generalised to homogeneous diagrams. A link can possess homogeneous diagrams of different crossing numbers, which may even construct non-isotopic projection surfaces. The diagrams of 7_4 in figure 2.13 are an example [Ko].



(a)

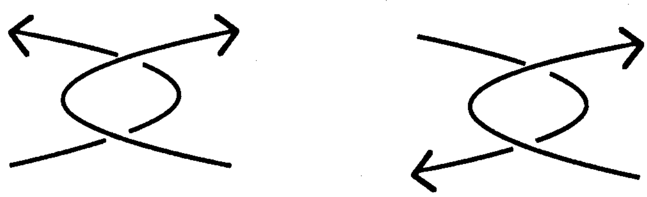


(b)

figure 2.13

However, if D is an irreducible homogeneous diagram then the Reidemeister moves which reduce the number of crossings cannot be performed on D.

Proof. It is not possible to perform a type I Reidemeister move since D is irreducible. If a type II move is possible then D contains one of the patterns in figure 2.14. In case (a) D is not homogeneous, and in case (b) either D is not homogeneous or a type I move is possible - a contradiction in both cases. □



(a) (b)

figure 2.14

Murasugi also remarks that a non-alternating projection of a prime alternating link cannot have minimal crossing number. To generalise this poses two questions.

Question 1. Is there a homogeneous link with a non-homogeneous diagram of minimal crossing number?

Question 2. Does every homogeneous link possess a homogeneous diagram of minimal crossing number?

2.7 APPENDIX I

8_{19}	h	10_{131}	nh	10_{149}	nh
8_{20}	nh	10_{132}	nh	10_{150}	nh
8_{21}	nh	10_{133}	nh	10_{151}	?
9_{42}	nh	10_{134}	h	10_{152}	h
9_{43}	h	10_{135}	nh	10_{153}	nh
9_{44}	nh	10_{136}	nh	10_{154}	h
9_{45}	nh	10_{137}	nh	10_{155}	nh
9_{46}	nh	10_{138}	h	10_{156}	h
9_{47}	h	10_{139}	h	10_{157}	nh
9_{48}	nh	10_{140}	nh	10_{158}	?
9_{49}	h	10_{141}	nh	10_{159}	nh
10_{124}	h	10_{142}	h	10_{160}	?
10_{125}	nh	10_{143}	nh	10_{161}	} h
10_{126}	nh	10_{144}	?	10_{162}	}
10_{127}	nh	10_{145}	nh	10_{163}	nh
10_{128}	h	10_{146}	nh	10_{164}	nh
10_{129}	nh	10_{147}	nh	10_{165}	nh
10_{130}	nh	10_{148}	nh	10_{166}	?

The table gives the homogeneity of the non-alternating prime knots of orders $\leqslant 10$ using the notation h : homogeneous; nh : non-homogeneous; ? : undetermined. All alternating links are homogeneous. The knots 10_{161} and 10_{162} are equivalent (they are the famous Perko pair).

2.8 APPENDIX II

The polynomials of links referred to in chapter 2 are given below.

the knot 5_2: $\nabla(z) = 1 + 2z^2$

the knot 8_{20}:

$$\begin{aligned} P(v,z) = &\quad (-1 + 4v^2 - 2v^4) \\ &+ z^2 (-1 + 4v^2 - v^4) \\ &+ z^4 (v^2) \end{aligned}$$

the knot 8_{21}:

$$\begin{aligned} P(v,z) = &\quad (3v^2 - 3v^4 + v^6) \\ &+ z^2 (2v^2 - 3v^4 + v^6) \\ &+ z^4 (- v^4) \end{aligned}$$

the knot 9_{42}:

$$\begin{aligned} P(v,z) = &\quad (2v^{-2} - 3 + 2v^2) \\ &+ z^2 (v^{-2} - 4 + v^2) \\ &+ z^4 (- 1) \end{aligned}$$

the knot 9_{44}:

$$\begin{aligned} P(v,z) = &\quad (v^{-2} - 2 + 3v^2 - v^4) \\ &+ z^2 (- 2 + 3v^2 - v^4) \\ &+ z^4 (v^2) \end{aligned}$$

the knot 9_{45}:

$$P(v,z) = (2v^2 - 2v^4 + 2v^6 - v^8) + z^2(2v^2 - 2v^4 + 2v^6) + z^4(v^4)$$

the knot 9_{46}:

$$P(v,z) = (2 - v^2 - v^4 + v^6) + z^2(-v^2 - v^4)$$

the knot 9_{48}:

$$P(v,z) = (+3v^4 - 2v^6) + z^2(1 - v^2 + 3v^4) + z^4(-v^2)$$

$$\nabla(z) = 1 + 3z^2 - z^4$$

the pretzel knot (-3,5,5):

$$P(v,z) = (2v^2 - v^6 - v^8 + v^{10}) + z^2(v^2 - v^6 - v^8)$$

the link in figure 2.6(b):

$$P(v,z) = z^{-1}(-v^{-1} + 3v - 2v^3) + z(-v^{-1} + 4v - v^3) + z^3(v)$$

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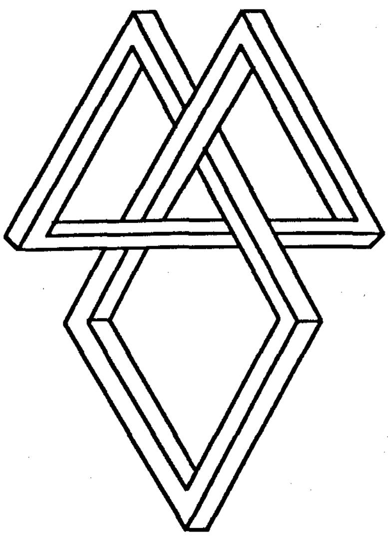
3

PROJECTIONS OF SPLIT AND NON-PRIME LINKS



3.1 INTRODUCTION

Let $L \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ be an oriented link in the 3-sphere. Let S^2 be a 2-sphere embedded in S^3 which meets L transversely and which separates S^3 into two 3-balls B_1, B_2 such that $B_1 \cup B_2 = S^3$, $B_1 \cap B_2 = \partial B_i = S^2$, and $B_i \cap L \neq \Phi$; for i=1,2. If $S^2 \cap L = \Phi$ then L is a <u>split link</u>. Suppose S^2 meets L in exactly two points: $S^2 \cap L = \{p,q\}$. Choose an arc $\alpha \subset S^2$ joining p to q, and let $L_i = (B_i \cap L) \cup \alpha$; for i=1,2. Then L is a <u>product</u> with <u>factors</u> L_1, L_2. The factorisation is <u>trivial</u> if at least one factor is a trivial knot. If every factorisation of L is trivial then L is a <u>prime</u> link.

Let $\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a projection such that $\pi(L)$ is regular, that is the self-intersection set is a finite number of transverse double points.

Menasco has shown that alternating diagrams represent split or non-prime links only in the obvious ways [Me]. This chapter extends these results to other classes of diagrams.

3.1.1 Theorem. Suppose that Seifert's algorithm constructs a minimal genus spanning surface for L when applied to the diagram associated with $\pi(L)$. Then

L is a split link if and only if $\pi(L)$ is disconnected. □

The projection $\pi(L)$ is <u>irreducible</u> if there is no double point p such that $\pi(L)$-p is disconnected. Suppose the 1-sphere S^1 meets $\pi(L)$ transversely in exactly two (non-double) points. Let U, V be the two connected components of $\mathbb{R}^2$ such that $U \cup V = \mathbb{R}^2$, $U \cap V = \partial U = \partial V = S^1$. If neither $U \cap \pi(L)$ nor $V \cap \pi(L)$ is a single embedded arc, then $\pi(L)$ is <u>decomposable</u>.

Let B_n denote the n-string braid group. A braid $\beta \in B_n$ which can be written

$$\beta_1(\sigma_1,\cdots,\sigma_{r-1})\ \beta_2(\sigma_r,\cdots,\sigma_{n-1})$$

in terms of the standard generators is called a <u>decomposable braid</u> (or, confusingly, a split braid in [Bi], [Mo]). Clearly, the closures $\hat{\beta}_1$, $\hat{\beta}_2$ of $\beta_1 \in B_r$ and $\beta_2 \in B_{n-r}$ are factors of $\hat{\beta}$, and the standard projection of $\hat{\beta}$ is decomposable.

Birman conjectured that a braid $\beta \in B_n$ with non-prime closure is conjugate in B_n to a decomposable braid [Bi](p99). In [Mo], Morton exhibits a counterexample in B_5, showing that the conjecture is false in general. However, the following theorem shows that the conjecture is true for the positive braids.

3.1.2 Theorem. Let β be a positive braid such that $\hat{\beta}$ is an irreducible projection of a non-split link L. Then

L is a non-prime link if and only if $\hat{\beta}$ is decomposable. □

The following corollary solves a problem raised by Williams in [Wil](number 18.1 in the appendix of problems).

3.1.3 Corollary. Positive braids with a full twist are prime. □

The Lorenz links are of this type. Hence, the following result (which is already known [Wi2], [B-W]).

3.1.4 Corollary. Lorenz links are prime. □

The positive braids belong to the class of diagrams from which Seifert's algorithm constructs a minimal genus spanning surface. So do the alternating diagrams. This observation motivates the following conjecture.

3.1.5 Conjecture. Let $\pi(L)$ be an irreducible projection of a non-split link L. Suppose that Seifert's algorithm constructs a minimal genus spanning surface for L when applied to the diagram associated with $\pi(L)$. Then

L is a non-prime link if and only if $\pi(L)$ is decomposable. □

This conjecture includes Menasco's result and theorem 3.1.2 above as special cases. The homogeneous diagrams also satisfy the conditions of the conjecture (see 2.4.6).

With this conjecture in mind the definitions and lemmas of the first few sections are stated in a general context in the hope that the technique used here may be extended to other cases.

3.1.6 The intuitive idea behind the proof.

Since the proof of theorem 3.1.2 is fairly long, a few words on its general progession will be given.

§3.2 This section contains preliminary definitions and lemmas: Let D be a diagram of a link L, and let F be an orientable surface spanning L which is constructed from D by applying Seifert's algorithm. Suppose S^2 is a 2-sphere which meets F transversely and factorises $\partial F = L$ as $L_1 \# L_2$. The sphere S^2 can be isotoped so that $S^2 \cap F$ is a single arc, α, properly embedded in F. (Also, theorem 3.1.1 is proved in this section.)

§3.3 An unusual presentation of $\Pi_1(S^3 - F) = G$ is derived. The group is actually free, but the added complication of the relations is offset by the increase in clarity in the sequel.

§3.4 The surface F is replaced by one which is isotopic to it, and which can be separated in a natural way into pieces of similar shape. By listing the type of intersection that α makes with each of these pieces, the embedding of α in F can be completely described. Such a description is called the film of α.

Let λ be the loop $(\alpha_+ \cup \alpha_-)$ in $S^3 - F$. The presentation of G enables a word, $w \in G$, represented by λ to be read off directly from the film.

§3.5 The connection between the embedding of α in F and the word w in G is made stronger with the statement of a fundamental lemma:

there is a unique shortest film for each homotopy class of embeddings.

The above sections apply when D is any diagram satisfying the conditions of the conjecture. In the remaining sections attention is restricted to the case when D is a positive braid diagram.

§3.6 Since λ is contractible in S^3-F, the word w collapses to 1 in G. The restrictions which this places on the film and on w are investigated in a series of lemmas.

§3.7 In this concluding section, the above ideas are brought together and are used determine the possible films of α. In some cases the fibration of the link complement is also needed. The restrictions on the way that α can be embedded in F show that the diagram is decomposable.

3.2 PRELIMINARIES

A height function $h:\mathbb{R}^3 \to \mathbb{R}$ can be defined by projection onto the line normal to $\mathbb{R}^2$. If $\mathbb{R}^2$ is translated along this normal then $\pi(L)$ is unchanged, so assume that $h(L) > 0$, that is assume $h(x) > 0$ for all $x \in L$. The projection $\pi(L)$ can be converted to a diagram by altering a neighbourhood of each double point in the projection so that the arc with lowest preimage is broken. The crossings in an oriented diagram are of two types indicated in figure 3.1. Unfortunately, the conventions chosen by many previous authors ([Bi] for example) lead to the situation where every crossing in a positive braid is of negative type. In this chapter, however, a generator σ_i of a braid group refers to an elementary braid in which the crossing is positive. With this definition (opposite to the usual one given in 1.2.5) a positive braid has positive crossings. This convention is becoming more common.

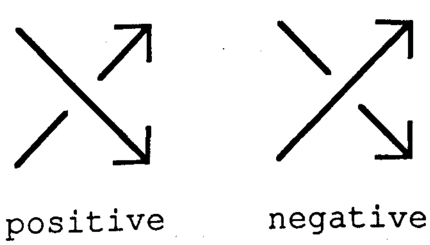


figure 3.1

Take a set of <u>Seifert discs</u> $\{\Delta_i\}$ in $\mathbb{R}^3$ such that $\pi(\partial\Delta_i)$ is one of the Seifert circles of $\pi(L)$ for each i, and so that h is constant on each disc $h(\Delta_i) = k_i > 0$, with $k_i > k_j$ whenever $\pi(\Delta_i) \subset \pi(\Delta_j)$. Complete the construction of a surface F by adding a set of <u>bands</u> joining the discs where each band is a rectangle twisted so that ∂F is equivalent to L, and $\pi(\partial F) = \pi(L)$. Call F the <u>projection surface</u> constructed from $\pi(L)$.

3.2.1 Remark. This notion of projection surface is stronger than the one used in chapter 2. It requires that the discs spanning nested Seifert circles are arranged so that all the discs are partially visible from above.

3.2.2 Lemma. Let F be a surface of minimal genus spanning a link L in S^3. If F is connected then L is non-split.

Proof: Suppose (for a contradiction) that L is split. Let S^2 be a 2-sphere which separates the components of ∂F and which meets F transversely so that $S^2 \cap F$ consists of disjoint simple loops. Isotop S^2 so that the number of such loops is minimal. S^2 separates S^3 into two 3-balls B_1, B_2 such that $B_i \cap \partial F \neq \phi$; $i=1,2$. If $S^2 \cap F$ is empty then, since F is connected, $F \subset B_1$ or $F \subset B_2$ contradicting that S^2 separates ∂F.

Let λ be a loop in $S^2 \cap F$ which bounds a disc $\Delta \subset S^2$ such that $\Delta \cap F = \partial\Delta$. If λ is compressible in F then S^2 can be isotoped to remove λ from $S^2 \cap F$ contradicting that the number of intersections is minimal. If λ is incompressible in F then let $b:\Delta\times[-1,1]\longrightarrow S^3$ be a bicollar of Δ in S^3 such that $b(\partial\Delta\times[-1,1]) \subset F$ and $b(\Delta\times 0) = \Delta$. Now

$$(F - b(\Delta\times[-1,1])) \cup b(\Delta\times\{-1,1\})$$

is a surface spanning L with genus less than F: a contradiction. □

3.2.3 Proof of theorem 3.1.1. Let L be a link in S^3 and let F be the projection surface for L constructed from $\pi(L)$ so that ∂F is equivalent to L. If $\pi(L)$ is connected then F is connected. By hypothesis, F is a minimal genus spanning surface, hence, L is non-split.

Clearly, if $\pi(L)$ is disconnected then L is split. □

(3.2.4) Suppose L is a link in S^3 and let F be a minimal genus surface spanning L. Let S^2 be a 2-sphere which factorises ∂F non-trivially and which meets F transversely. The set $S^2 \cap F$ consists of disjoint simple loops together with a single simple arc properly embedded in F. Let α denote the arc component of $S^2 \cap F$. Any loop component of $S^2 \cap F$ bounds two discs in S^2 only one of which contains α. Since F has minimal genus, the innermost loop argument of lemma 3.2.2 can still be applied. Therefore, S^2 can be isotoped in S^3 so that $S^2 \cap F$ is a single arc.

3.2.5 Lemma. Let F be the projection surface constructed from $\pi(L)$ and let S^2 be a 2-sphere which factorises ∂F. Suppose that $S^2 \cap F$ is a single arc which is contained in a Seifert disc $\Delta \subset F$. Then $\pi(L)$ is decomposable.

Proof. Let $S^2 \cap F = \alpha$, and let F_1 and F_2 denote the two connected components of $F-\alpha$. Since $\alpha \subset \Delta$, Δ is the only Seifert disc which meets both F_1 and F_2. All the other discs and all the bands lie entirely in either F_1 or F_2. It now follows from the construction of F that

$$\pi(\partial F_1-\alpha) \cap \pi(\partial F_2-\alpha) = \phi.$$

Hence, there is an $S^1 \subset \mathbb{R}^2\times 0$ which separates $\pi(\partial F_1-\alpha)$ from $\pi(\partial F_2-\alpha)$, and which meets $\pi(L)$ in exactly two points, namely $\pi(\partial\alpha)$. □

3.2.6 Definitions. Suppose F is an orientable surface spanning an oriented link. Let $b:F\times[-1,1]\to S^3$ be a bicollar on F with $b(F\times 0) = F$. For any subset $X \subseteq F$ let X_+ denote $b(X\times 1)$ and X_- denote $b(X\times -1)$. Let Δ be a Seifert disc of F. Choose b so that the orientation on $\partial\Delta$ (induced from ∂F) is as shown in figure 3.2. This notation will usually be applied to the whole surface F or to an arc α properly embedded in F, thus giving $F_\pm$ and $\alpha_\pm$.

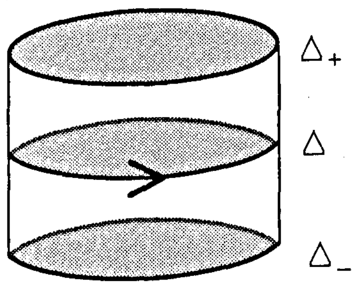


figure 3.2

Suppose α is an arc properly embedded in F. There is a disc R_α embedded in S^3 with

(1) $R_\alpha \cap F = \alpha$

(2) $b(\alpha\times[1,-1]) \subset R_\alpha$

(3) $(\alpha_+ \cup \alpha_-) \subset \partial(R_\alpha) \subset S^3-F$.

Call R_α the <u>region around α</u>, and call $\lambda_\alpha = \partial(R_\alpha)$ the <u>loop around α</u>.

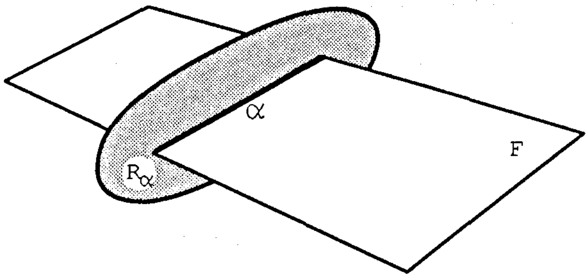


figure 3.3

3.2.7 Lemma. Suppose α is an arc properly embedded in F. Then α lies in a 2-sphere which factorises ∂F if and only if the loop around α is contractible in S^3-F.

Proof (⇒). Suppose that S^2 is a 2-sphere which factorises ∂F and that $\alpha \subset S^2$. Isotop S^2 in S^3 so that $R_\alpha \subset S^2$. The loop around α, λ_α, bounds the disc $(S^2\text{-}R_\alpha)$ in S^3. Hence, it is contractible.

Proof (⇐). Let N(F) be a regular neighbourhood of F in S^3 so that R_α is properly embedded in N(F). The loop around α, λ_α, is contractible in S^3-N(F). If λ_α is not essential in $\partial(S^3\text{-}N(F))$ then it bounds a disc Δ in $\partial(S^3\text{-}N(F))$. The 2-sphere $\Delta \cup R_\alpha$ bounds a ball in N(F), and factorises ∂F trivially. If λ_α is essential in $\partial(S^3\text{-}N(F))$ then, by Dehn's lemma, λ_α bounds a disc Δ properly embedded in S^3-N(F), and the 2-sphere $\Delta \cup R_\alpha$ factorises ∂F non-trivially. □

3.3 A PRESENTATION FOR $\Pi_1(S^3\text{-}F)$

Suppose that the projection $\pi(L)$ can be written as a $*$-product of n links $L_1,\cdots,L_n$

$$\pi(L) = \pi(L_1) * \cdots * \pi(L_n).$$

Let F be the projection surface constructed from $\pi(L)$, and let $F_i \subset F$ be the subsurface which is also the projection surface constructed from $\pi(L_i)$; $1 \leq i \leq n$, so that $\partial(F_i)$ is equivalent to L_i. Then F is a Murasugi sum of the surfaces F_i.

Let $G = \Pi_1(S^3\text{-}F)$ and $G_i = \Pi_1(S^3\text{-}F_i)$; $1 \leq i \leq n$. Then G is a free product with factors G_i

$$G = G_1 * \cdots * G_n.$$

In order to describe a presentation for $\Pi_1(S^3\text{-}F)$, it is sufficient to consider the groups $\Pi_1(S^3\text{-}F_i)$.

Suppose F_i is a projection surface with p_i bands and q_i discs. For each band B_j; $1 \leq j \leq p_i$, let λ_j denote the loop around the co-core of B_j oriented as shown in figure 3.4. Choose a basepoint x_0 in $S^3\text{-}F$ with $h(x_0) > h(F)$. Use the same basepoint for all F_i. Let a_j be an arc in $S^3\text{-}F$ which connects λ_j to a point in the plane $\mathbb{R}^2 \times h(x_0) \subset \mathbb{R}^3$, such that $\pi(a_j)$ is a single point. Extend a_j by an arc contained in $\mathbb{R}^2 \times h(x_0)$ to connect λ_j to x_0, denoting this extended arc also by a_j, and orienting it from x_0 to λ_j. The loop $g_j = a_j \cup \lambda_j \cup (-a_j)$ based at x_0 represents a <u>generator</u> of $\Pi_1(S^3\text{-}F_i)$.

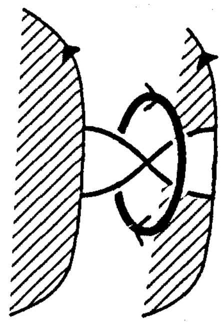


figure 3.4

For each Seifert disc $\Delta_k \subset F_i$; $1 \leq k \leq q_i$, follow the boundary of Δ_k in the direction of its orientation and order the bands attached to Δ_k consecutively $1 \cdots r_i$ as each is encountered, for some $r_i \leq p_i$. Then let ρ_k denote the <u>relator</u>

$$\rho_k = \prod_{j=1}^{r_i} g_j$$

where g_j is the generator associated to the band B_j.

3.3.1 Proposition. The group $\Pi_1(S^3-F_i)$ admits the presentation

$$G_i = < g_j;\ 1 \leq j \leq p_i \mid \rho_k;\ 1 \leq k \leq q_i >.$$

Proof. It remains to check that ρ_k; $1 \leq k \leq q_i$ are defining relations.

Let J_i denote the deformation retract of F_i formed by retracting the bands onto their respective cores, and the Seifert discs onto points. Then J_i is a planar graph which is the spine of F_i and $\Pi_1(S^3-F_i) = \Pi_1(\mathbb{R}^3-J_i)$.

Embed J_i in $\mathbb{R}^2 \times 0 \subset \mathbb{R}^3$, and recall that $h(x_0)>0$. Let w be a word in the generators of G_i such that w=1. (Each generator is now a loop circling an edge of J_i.) There is a loop λ in $\mathbb{R}^3 - J_i$ which represents w. Regard $\mathbb{R}^3$ as a simplicial complex Σ containing $J_i \times (-\infty, 0]$ as a subcomplex. Let Σ^* denote the dual complex. The loop λ can be approximated by a loop in the 1-skeleton of Σ^*.

Since λ is contractible in $\mathbb{R}^3 - J_i$, there is a sequence of cellular moves which deform λ so that $\lambda \cap (J_i \times (-\infty, 0]) = \phi$. Let one such cellular move be across a 2-cell $\sigma \in \Sigma^*$ which replaces $\lambda \cap \partial\sigma$ by $\partial\sigma - \lambda$. If $\sigma \cap J_i \times (-\infty, 0] = \phi$ then the deformation has no effect on the word w which λ represents. If $\sigma \cap J_i \times (-\infty, 0]$ is an arc then the deformation either inserts or deletes a word gg^{-1} or $g^{-1}g$ in w. This results in an element of G_i equivalent to w. If σ meets $J_i \times (-\infty, 0]$ in a wedge of lines then the deformation inserts or deletes a conjugate of ρ_k or $(\rho_k)^{-1}$ for some k. □

3.3.2 Remark. Although the chosen generators are not free generators, the group G is actually a free group.

In the special case of theorem 3.1.2, each of the links L_i is a $(p_i, 2)$ torus link for some p_i; $1 \leq i \leq n$. The projection surface constructed from $\pi(L_i)$ comprises two Seifert discs connected by p_i bands. Follow the boundary of one of the two discs in the direction of its orientation and order the bands consecutively $1 \cdots p_i$ as each is encountered. The ordering is independent of the choice of disc. The only relator in $\Pi_1(S^3 - F_i)$ is

(3.3.3)
$$\rho_i = \prod_{j=1}^{p_i} g_j \ .$$

Thus

$$G_i = < g_j;\ 1 \leqslant j \leqslant p_i \mid \rho_i > \ .$$

3.4 PICTURES, FILMS AND THE MAP Ω

Let F be a projection surface spanning a link L. If α is an arc properly embedded in F then the loop around α represents an element of $\Pi_1(S^3-F)$. A method of writing this element in terms of the above generators for G is now described. First F is repositioned.

Suppose Δ is a Seifert disc of F. Let $N(\Delta)$ denote an ε-neighbourhood in Δ of $\partial\Delta$. Choose ε small enough that $\pi(N(\Delta_i)) \cap \pi(N(\Delta_j)) = \phi$ for all $i \neq j$. Let

$$Z = F - \bigcup(\Delta_i - N(\Delta_i)) .$$

Thus Z is obtained from F by removing the interiors of all the Seifert discs.

Let $\eta \subset \bigcup N(\Delta_i) \subset Z$ be a simple arc connecting ∂F to $(\partial Z - \partial F)$ such that $\pi(\partial F) \cap \pi(\eta) \subset \partial\eta$. There exists a disjoint set E of such arcs which partition Z so that each component of $Z-(E \cup \partial Z)$ is an open topological disc which contains exactly one band of F. The closure of each of these components is called an <u>H-piece</u>, and the elements of E meeting an H-piece are its <u>ends</u>. Each arc in E is an end of two distinct H-pieces because $\pi(L)$ is irreducible. By noting the orientation on ∂F, each of the four ends of an H-piece can be given a unique label in {NW,NE,SW,SE} as shown in figure 3.5.

Note: In figures, the edges of an H-piece drawn in smooth lines are in ∂F and those in ragged lines are in $(\partial Z-\partial F)$.

The generators of G can be used to label the H-pieces in F since there is a one-one correspondence between the two sets.

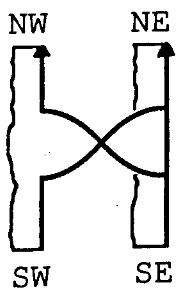


figure 3.5

Choose a subset of ends $E'\subset E$ so that every Seifert disc of F contains precisely one end in E'. Cut Z at each end in E'. This produces a surface $\tilde{F}$ which is isotopic to F and which is composed of H-pieces. In figure 3.6 this procedure is applied to a surface spanning the figure-8 knot, 4_1.

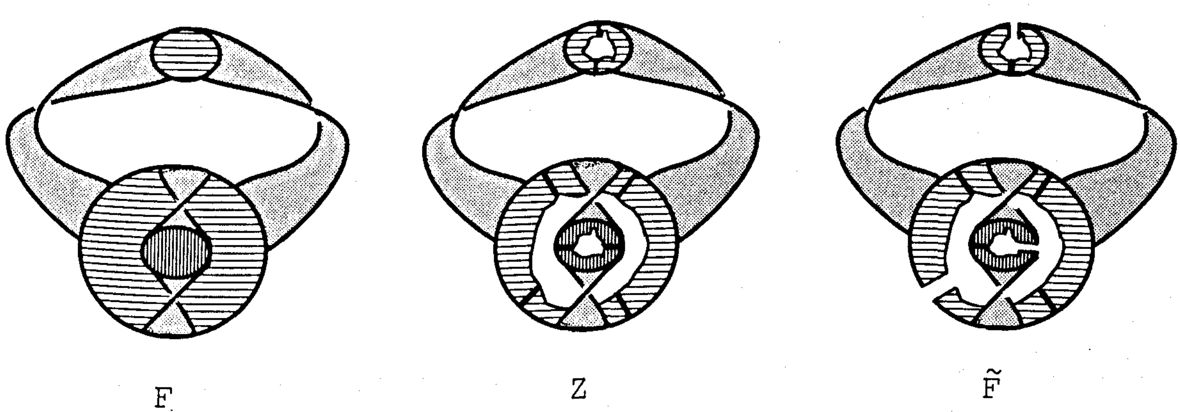


figure 3.6

An arc $\alpha \subset \tilde{F}$ is H-embedded in $\tilde{F}$ if, for each H-piece B in $\tilde{F}$,

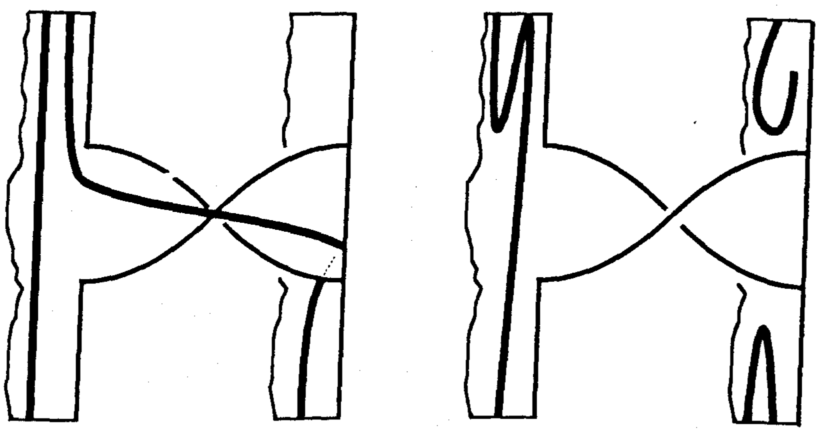
(1) $\alpha \cap B$ is a finite set of disjoint simple arcs

(2) $\partial(\alpha \cap B) = (E \cap B) \cap \alpha$

(3) No component of $\alpha \cap B$ has both its boundary components in the same end of B

(4) $\alpha \cap \partial\tilde{F} \subseteq \alpha \cap \partial F \subseteq \partial\alpha$.

Condition (1) ensures that α is simple. Condition (2) implies that each arc-component of $\alpha \cap B$ meets E exactly twice, only at its boundary points. Condition (4) implies that if $\partial\alpha$ meets $\partial\tilde{F}$ then it does so only at points of ∂F. If α is H-embedded in $\tilde{F}$ and $\partial\tilde{F} \cap \alpha = \partial\alpha$ then say that α is properly H-embedded in $\tilde{F}$.



H-embedded arcs non H-embedded arcs

figure 3.7

Any arc properly embedded in F is isotopic to an arc properly H-embedded in $\tilde{F}$.

A picture is a triple $(g;\eta_1,\eta_2)$; $\eta_1 \neq \eta_2$ which represents an oriented arc H-embedded in the H-piece associated to the generator g. The arc connects the ends $\eta_1,\eta_2 \in E$, and is oriented from η_1 to η_2. Examples are shown in figure 3.9.

3.4.1 Notation. A lower-case letter between square brackets will be used to denote pictures. If [p] denotes the picture $(g;\eta_1,\eta_2)$ then $[\hat{p}]$ denotes $(g;\eta_2,\eta_1)$, the arc having the reverse orientation. The symbol $[p_1,\cdots,p_r]$ is used to denote the picture [p] where $p \in \{p_1,\cdots,p_r\}$. For example [a,c,f] means one of [a], [c] or [f]. Also, $[p]^r$ denotes a sequence of r copies of the picture [p].

Using this notation, the symbols in column 1 of table 3.8 will be used throughout this chapter to denote the pictures in column 2, shown diagramatically in figure 3.9.

symbol	picture	$\bar{\Omega}([p];w)$	$\bar{\Omega}([\hat{p}];w)$
[a]	(g;NW,SW)	gw	$g^{-1}w$
[b]	(g;NE,SE)	wg^{-1}	wg
[c]	(g;NW,NE)	gw	$g^{-1}w$
[d]	(g;SW,SE)	wg^{-1}	wg
[e]	(g;SW,NE)	w	w
[f]	(g;NW,SE)	gwg^{-1}	$g^{-1}wg$

table 3.8

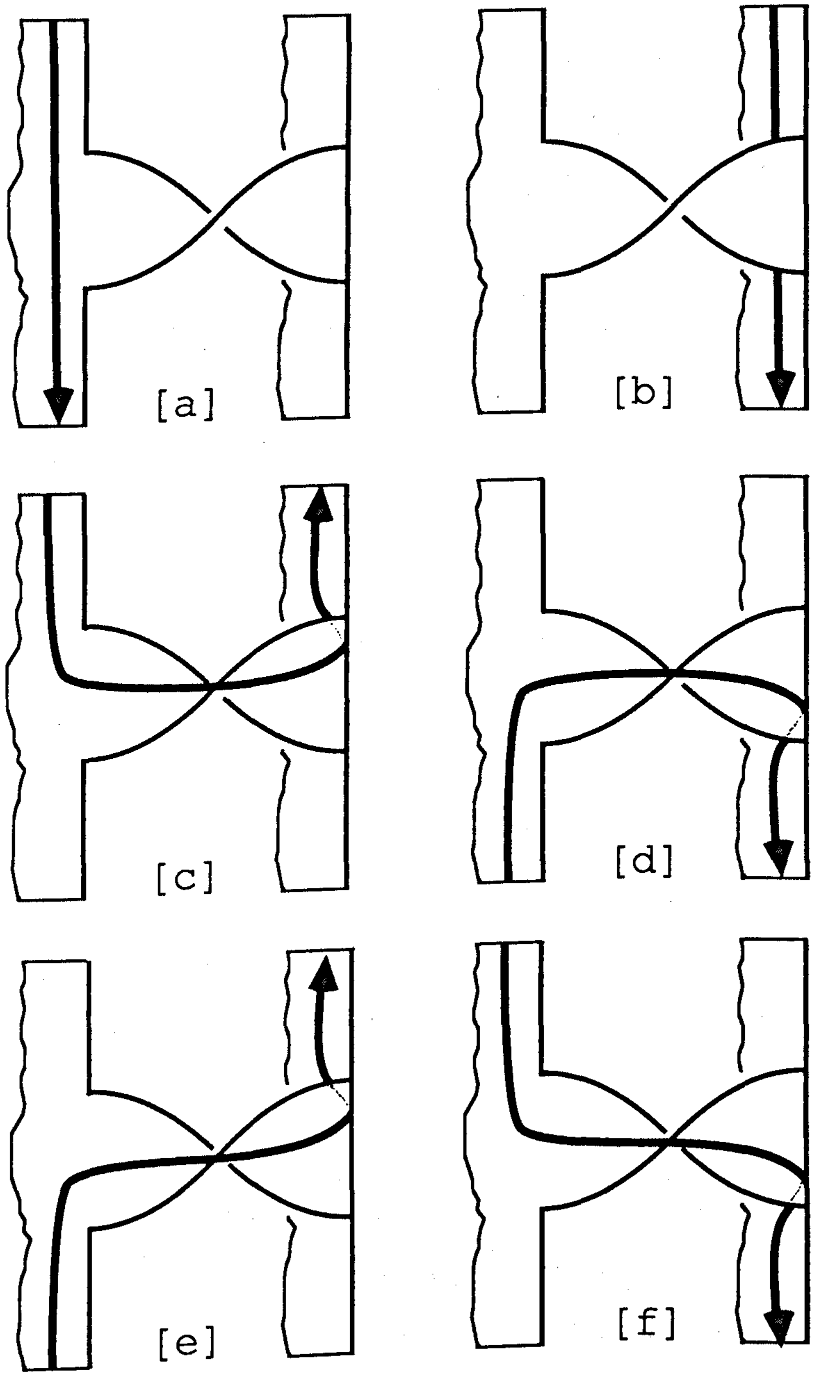


figure 3.9

Let α be an oriented arc H-embedded in $\tilde{F}$. There is a unique sequence of pictures which completely describes α. Such a sequence is called the film of α and is denoted ϕ_α. When the square bracket notation is used, the element $g \in G$ in the picture is suppressed. This loss of information means that the arc can no longer be constructed from the condensed form of the film. However, if the initial point of α is given, or the H-piece associated with the leftmost picture of ϕ_α is identified, then the square bracket notation is sufficient to enable the arc to be reconstructed. The length of the film, denoted $|\phi_\alpha|$, is the number of pictures in the sequence. The number of H-pieces in $\tilde{F}$ equals the number of double points in $\pi(L)$, and hence is finite. If α is properly H-embedded in $\tilde{F}$ then α meets each H-piece a finite number of times (from (1) in the definition of H-embedded). Therefore $1 \leqslant |\phi_\alpha| < \infty$.

Let P be the set of the twelve possible pictures. Columns 3 and 4 of table 3.8 define a map $\bar{\Omega}: P \times G \longrightarrow G$ whose image is written $\bar{\Omega}([p];w)$ where [p] is a picture $(g;\eta_1,\eta_2)$ and w is a word in G. The mapping generates words in G ignoring the group structure of G, regarding it as a set only.

Note. For clarity, the table gives the values of $\bar{\Omega}([p];w)$ only in the cases when [p] is associated to a positive crossing (the cases used later). The method of constructing the map is given below, and the description will enable the reader to provide the corresponding values for negative crossings.

Let $\bar{\Omega}^{(r)}$ denote the composition of r copies of $\bar{\Omega}$. Define $\bar{\Omega}$ on films inductively as follows: $\bar{\Omega}^{(r)}: P^r \times G \to G$

$$\bar{\Omega}^{(r)}([p_1][p_2]\cdots[p_r];\ w) = \bar{\Omega}^{(r-1)}([p_2]\cdots[p_r];\ \bar{\Omega}([p_1];w)\ ).$$

3.4.2 Definition: Ω. Let ϕ_α be a film of length r. Then

$$\Omega(\phi_\alpha) = \bar{\Omega}^{(r)}(\ \phi_\alpha;\ 1\ ).$$

3.4.3 A geometric interpretation of Ω.

Consider the picture $[p] = (g;\eta_1,\eta_2)$. Let α denote the arc in the H-piece associated to g oriented from end η_1 to end η_2. Extend α by an arc in each end so that α is properly H-embedded in $\tilde{F}$.

Connect each point of $\partial\alpha_+$ to the basepoint x_0 in $S^3-\tilde{F}$ in the same manner as were the loops λ_j in the construction of the generators. This forms a loop in $S^3-\tilde{F}$ based at x_0, and which contains α_+ as a subarc. Orient the loop so that α_+ has the same orientation as α. In $\Pi_1(S^3-\tilde{F})$ this loop represents g, g^{-1}, or 1. This is the element of G which is placed on the right of w in $\bar{\Omega}([p];w)$.

Similarly, construct a loop containing α_- oriented so that α_- has the opposite orientation to α. The element of G which this loop represents (also g, g^{-1}, or 1) is placed on the left of w in $\bar{\Omega}([p];w)$.

The loop around α is formed from α_+ and α_-, and represents the element $\bar{\Omega}([p];1)$ in G.

Now consider a sequence of two pictures $[p_1][p_2]$ each giving an arc α_1, α_2 respectively, and each arc extended in E to become properly embedded in $\tilde{F}$. Notice that $\partial\alpha_1 \cap \partial\alpha_2 \neq \Phi$. Construct four loops in $S^3-\tilde{F}$ each containing one of α_{1+}, α_{1-}, α_{2+}, α_{2-}.

Then

$$\bar{\Omega}^{(2)}([p_1][p_2];\ 1) \quad = \quad \bar{\Omega}([p_2];\ \bar{\Omega}([p_1];1))$$

which represents the loop $(\alpha_{2-})\cup(\alpha_{1-})\cup(\alpha_{1+})\cup(\alpha_{2+})$ in $S^3-\tilde{F}$ which is the loop around $(\alpha_1\cup\alpha_2)$. Similarly, for any sequence of pictures ϕ_α describing an arc α in $\tilde{F}$, $\Omega(\phi_\alpha)$ is the element of $\Pi_1(S^3-\tilde{F})$ represented by the loop around α.

3.4.4 Notation. The notation rhs $\Omega(\phi_\alpha)$ denotes the word in G formed on the right of the 1 in $\bar{\Omega}^{(r)}(\ \phi_\alpha;\ 1)$, represented in $S^3-\tilde{F}$ by the loop containing α_+. Similarly, lhs $\Omega(\phi_\alpha)$ denotes the word formed on the left represented by the loop containing α_-. Also the symbol r/lhs $\Omega(\phi_\alpha)$ means 'rhs $\Omega(\phi_\alpha)$ or lhs $\Omega(\phi_\alpha)$'.

The next lemma is a corollary of lemma 3.2.7.

3.4.5 Lemma. Suppose α is an arc properly H-embedded in $\tilde{F}$. Then α lies in a 2-sphere which factorises $\partial\tilde{F}$ if and only if $\Omega(\phi_\alpha) = 1$ in G.

□

3.5 A FUNDAMENTAL LEMMA

Suppose F is a minimal genus projection surface constructed from a projection $\pi(L)$ of a link L. Let $\tilde{F}$ be a surface composed of H-pieces (as constructed in §3.4) which is isotopic to F. The following lemma shows that there is a unique shortest film for each homotopy class of arcs in $\tilde{F}$.

3.5.1 Lemma. Let β, γ be two arcs each H-embedded in $\tilde{F}$ with $\partial\beta = \partial\gamma$, and suppose that $|\phi_\beta|$ and $|\phi_\gamma|$ are minimal. If β is homotopic in $\tilde{F}$ to γ keeping $\partial\beta$ fixed then $\phi_\beta = \phi_\gamma$. □

3.5.2 Corollary. Let β, γ be two arcs H-embedded in $\tilde{F}$ with $\partial\beta = \partial\gamma$, and suppose that $|\phi_\beta|$ and $|\phi_\gamma|$ are minimal.
If rhs $\Omega(\phi_\beta)$ = rhs $\Omega(\phi_\gamma)$, or if lhs $\Omega(\phi_\beta)$ = lhs $\Omega(\phi_\gamma)$, then $\phi_\beta = \phi_\gamma$.

Proof. Assume without loss of generality that rhs $\Omega(\phi_\beta)$ = rhs $\Omega(\phi_\gamma)$. Then the loop $\beta_+ \cup \gamma_+ = (\beta \cup \gamma)_+$ is contractible in $S^3 - \tilde{F}$. The surface $\tilde{F}$ has minimal genus (2.4.6), hence $\beta \cup \gamma$ is contractible in F **[Ne]**(p29). This implies β is homotopic in $\tilde{F}$ to γ keeping $\partial\beta$ fixed. □

A short digression passing through the realms of simplicial homotopy, graphs, and free groups which culminates in theorem 3.5.4 precedes the proof of lemma 3.5.1.

Let Σ be a simplicial complex with $\Sigma^0 \subset \Sigma$ the subset of 0-simplices or vertices. An edge path in Σ is a finite sequence of vertices $v_0, \cdots, v_n$ such that $v_i \in \Sigma^0$ for each i, and v_{i-1} and v_i span a 0-simplex or a 1-simplex of Σ for $1 \leq i \leq n$. Two edge paths are equivalent in Σ if one can be derived from the other by applying a finite sequence of the following operations:

(1) $vv \longleftrightarrow v$; where v is a 0-simplex of Σ

(2) $uvu \longleftrightarrow uu$; where u,v span a 1-simplex of Σ

(3) $uvw \longleftrightarrow uw$; where u,v,w span a 2-simplex of Σ

The double-headed arrow indicates that if the sequence of vertices on the left (or right) of the arrow appears in an edge path then it can be replaced by what is on the right (or left).

These operations preserve the first and last vertices of an edge path. Two edge paths are equivalent if and only if they are homotopic in Σ keeping their boundaries fixed. Thus, the equivalence classes of edge paths are in fact homotopy classes [C-V](p375).

Now consider the special case when Σ is a graph, Γ, which contains no loops. The above notation will be modified to take advantage of this restriction.

Let $\Sigma^1 \subset \Sigma$ denote the set of ordered pairs of vertices (v_i,v_j) which span 1-simplices or edges of Σ. If $e \in \Sigma^1$ denotes the edge (v_i,v_j) then let e^{-1} denote the reverse edge (v_j,v_i).

Let $v_0, \cdots, v_n$ be an edge path in Γ such that $v_{i-1} \neq v_i$ for $1 \leqslant i \leqslant n$. (This can be ensured by applying operation (1) if necessary.) Then, to each pair v_{i-1}, v_i of vertices there corresponds an edge $e_i \in \Sigma^1$, and the edge path can be written as a sequence of edges $e_1, \cdots, e_n$. Of the above operations (1) cannot be applied by assumption; (3) cannot be applied since a graph contains no 2-simplices; and (2) can be replaced by the following:

(2') insert or delete $e^{-1}e$ or ee^{-1} from the sequence for any $e \in \Sigma^1$.

Hence, two such edge paths in Γ are homotopic keeping their boundaries fixed if and only if they differ by a finite number of operations of the form in (2').

An edge path in Γ is <u>reduced</u> if it does not contain the patterns $e^{-1}e$ or ee^{-1}, for any $e \in \Sigma^1$.

This notation is isomorphic to that of [M-K-S] concerning free groups. In particular, a word in a free group G is <u>freely reduced</u> if it does not contain the patterns $g^{-1}g$ or gg^{-1} for any $g \in G$. Also, two words in G are <u>freely equal</u> if one can be obtained from the other by a finite sequence of insertions and deletions of the form $g^{-1}g$ or gg^{-1} for $g \in G$ [M-K-S](p34).

3.5.3 Theorem (Magnus, Karrass and Solitar). Every word in a free group is freely equal to a unique freely reduced word. □

Returning to the simplicial case, this theorem translates as follows.

3.5.4 Theorem. Every edge path in a graph is homotopic keeping the boundary fixed to a unique reduced edge path. □

Proof (of lemma 3.5.1).
An H-piece can be retracted onto a tree with five edges and two 3-valent vertices. The reduced edge paths which start and finish at the leaves of the tree are in one-one correspondence with the pictures of arcs in the H-piece (see figure 3.10). The surface $\tilde{F}$ can be retracted onto a graph Γ so that each H-piece is retracted as above. The film of an arc in $\tilde{F}$ corresponds to an edge path in Γ. The edge path is reduced if and only if the length of the film is minimal.

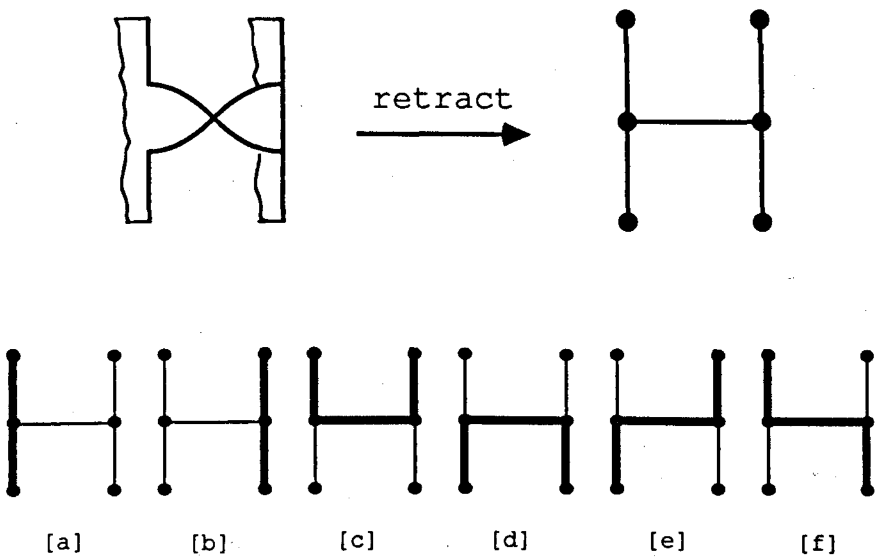


figure 3.10

The two arcs β, γ in $\tilde{F}$ correspond to two edge paths β', γ' in Γ. Since β is homotopic to γ in $\tilde{F}$ keeping $\partial\beta$ fixed, and since Γ is a deformation retract of $\tilde{F}$, β' is homotopic to γ' in Γ keeping $\partial\beta'$ fixed. Also, β', γ' are reduced since $|\phi_\beta|$, $|\phi_\gamma|$ are minimal. There is a unique reduced edge path in each homotopy class (theorem 3.5.4), hence $\beta' = \gamma'$ which implies $\phi_\beta = \phi_\gamma$. □

3.5.5 Notation. Let β, γ be two arcs each H-embedded in $\tilde{F}$ with $\partial\beta = \partial\gamma$. If β is homotopic to γ in $\tilde{F}$ keeping $\partial\beta$ fixed then the relation between their films is denoted $\phi_\beta \simeq \phi_\gamma$.

3.6 SPECIALISED LEMMAS AND DEFINITIONS

The preceding sections dealt with the most general situation regarding the conjecture (3.1.5). The results are valid for any projection which has a minimal genus projection surface. In this section, attention is restricted to the case where the projection $\pi(L)$ underlies a closed braid diagram in which every crossing is positive.

Let $\beta \in B_{n+1}$ be a positive braid such that the closure $\hat{\beta}$ is an irreducible diagram of a non-split non-prime link, L. Let F be the projection surface constructed from $\hat{\beta}$. Form a subsurface $Z \subset F$ composed of annuli and bands by removing the interiors of the Seifert discs as in §3.4. Each band in Z corresponds to a letter in the braid word β, and each annulus corresponds to a braid string which meets part of its boundary. Let A_i denote the annulus corresponding to the i^{th} string for $1 \leq i \leq n+1$. To form the surface $\tilde{F}$ read along the word β and, as each letter is encountered, locate the corresponding band in Z and the H-piece which contains it. Suppose the letter is σ_i. Then the H-piece will meet A_i which contains its NW and SW ends, and A_{i+1} containing the NE and SE ends. If A_i is still annular then cut it at the NW end of the H-piece; if A_{i+1} is still annular then cut it at the NE end. Proceed in this manner until each annulus has been cut exactly once. The resulting surface is denoted $\tilde{F}$ (cf. §3.4). Figure 3.11 shows the surface $\tilde{F}$ constructed from the braid $\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2$.

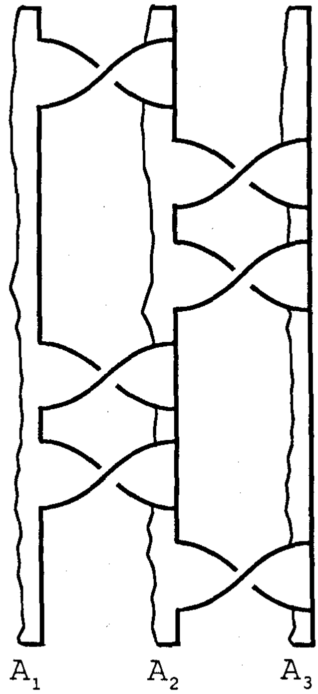


figure 3.11

Let $\tilde{F}_i$ be the subsurface of $\tilde{F}$ composed of the cut annuli A_i, A_{i+1} and all the bands corresponding to the letters σ_i in β. Then $\tilde{F}$ is a Murasugi sum of the surfaces $\tilde{F}_i$.

Let $G = \Pi_1(S^3-\tilde{F})$ and $G_i = \Pi_1(S^3-\tilde{F}_i)$. Then G is a free product with factors G_i

$$G = G_1 * \cdots * G_n.$$

Each of the $\tilde{F}_i$ contains p_i bands and $\partial\tilde{F}_i$ is a $(p_i,2)$ torus link. (Note that $p_i>1$ since $\hat{\beta}$ is irreducible.) Assume each G_i is presented as in §3.3, and label the generators of G_i from 1 to p_i consecutively so that the generator associated with the lowest band is labelled 1 (see figure 3.12). Two generators g_j, $g_k \in G_i$ are <u>adjacent</u> if $|k-j| = 1$. If $k-j = 1$ then say g_k <u>follows</u> g_j. Also g_1, g_{p_i} are called the <u>first</u>, and <u>last</u> generators in G_i respectively. They are considered to be <u>adjacent</u> generators.

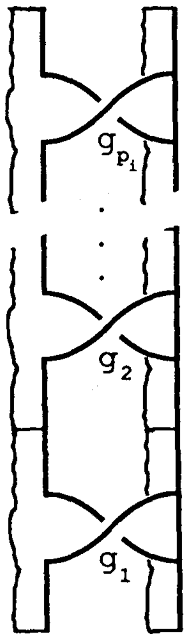


figure 3.12

Let S^2 be a 2-sphere which factorises $\partial\tilde{F}$ non-trivially and which meets $\tilde{F}$ transversely. Assume that S^2 is isotoped to remove all loop components of $S^2\cap\tilde{F}$ (3.2.4) leaving only a single arc which is denoted α.

Isotop S^2 such that α is properly H-embedded in $\tilde{F}$ (recall that this requires $\alpha\cap\partial\tilde{F} = \alpha\cap\partial F$), and so that $|\phi_\alpha|$ minimal. Furthermore, isotop S^2 without increasing $|\phi_\alpha|$ so that the leftmost and rightmost pictures in ϕ_α belong to $\{[a],[\hat{a}],[b],[\hat{b}]\}$.

Recall that all relations in G have the form in (3.3.3), and also from lemma 3.4.5 that $\Omega(\phi_\alpha)$ collapses to 1 in G. Trying to discover how these results restrict the embedding of α in $\tilde{F}$ is the motivation for the three following lemmas.

3.6.1 Lemma. Suppose e is a word in G with e=1, and g is a generator of G. Then the patterns geg^{-1} and $g^{-1}eg$ do not occur in r/lhs $\Omega(\phi_\alpha)$.

Proof. Suppose (for a contradiction) that the pattern geg^{-1} appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

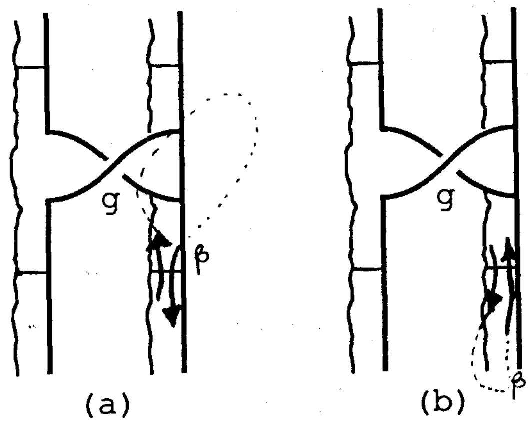
$$[\hat{b},\hat{d},\hat{f}] \cdots [b,d,f].$$

The leftmost picture contributes g to rhs $\Omega(\phi_\alpha)$, and the rightmost picture contributes g^{-1}. The sequence between represented as $\cdots$ contributes a trivial word, e.

This subfilm of ϕ_α is the film of a subarc $\beta \subset \alpha$ which is H-embedded in $\tilde{F}$. Each of the pictures $[\hat{b}]$, $[\hat{d}]$, $[\hat{f}]$ has the form $(g;SE,\cdot)$. Consequently, [b], [d], [f] all have the form $(g;\cdot,SE)$. Let η denote the SE end of the H-piece associated to the generator g. Then $\partial\beta \subset \eta$. Let $\gamma \subset \eta$ be an arc with $\partial\beta = \partial\gamma$ (see figure 3.13(a)). Then rhs $\Omega(\phi_\beta) = geg^{-1} = 1$ in G and rhs $\Omega(\phi_\gamma) = 1$. Hence, by corollary 3.5.2, $|\phi_\beta| = |\phi_\gamma| = 0$: a contradiction.

Now suppose that the pattern $g^{-1}eg$ appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

$$[b,d,f][p_1] \cdots [p_r][\hat{b},\hat{d},\hat{f}].$$



(a) (b)

figure 3.13

The subfilm $[p_1] \cdots [p_r]$ defines a subarc $\beta \subset \alpha$ H-embedded in $\tilde{F}$ as before, with $\partial\beta \subset \eta$ (figure 3.13(b)). Again corollary 3.5.2 implies that $|\phi_\beta| = 0$. Hence, $r = 0$ and ϕ_α contains the sequence $[b,d,f][\hat{b},\hat{d},\hat{f}]$. Now $|\phi_\alpha|$ can be reduced contrary to assumption.

For the two other proofs, note that lhs $\Omega(\phi_\alpha)$ is generated from right to left. Consequently, if geg^{-1} appears in lhs $\Omega(\phi_\alpha)$ then ϕ_α contains the subsequence

$$[\hat{a},\hat{c},\hat{f}][p_1] \cdots [p_r][a,c,f].$$

And if $g^{-1}eg$ appears in lhs $\Omega(\phi_\alpha)$ then ϕ_α contains the subsequence

$$[a,c,f] \cdots [\hat{a},\hat{c},\hat{f}].$$

The proofs now follow similarly to those above. □

3.6.2 Remark. This shows that both rhs $\Omega(\phi_\alpha)$ and lhs $\Omega(\phi_\alpha)$ are reduced words in G.

3.6.3 Lemma. Suppose e is a word in G with e = 1, and g_1, g_k are the first and last generators in some factor G_i of G. Then the patterns $g_k e g_1$ and $(g_1)^{-1}e(g_k)^{-1}$ do not occur in r/lhs $\Omega(\phi_\alpha)$.

Proof. Suppose (again, for a contradiction) that $g_k e g_1$ appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

$$[\hat{b},\hat{d},\hat{f}][p_1]\cdots[p_s][\hat{b},\hat{d},\hat{f}].$$

Each of the pictures $[\hat{b}]$, $[\hat{d}]$, $[\hat{f}]$ has the form (g;SE,·). Let η_1, $\eta_k \in E$ be the SE ends of the H-pieces associated with the generators g_1, g_k respectively. Then the subfilm $[\hat{b},\hat{d},\hat{f}][p_1]\cdots[p_s]$ is the film of a subarc $\beta \subset \alpha$ which is H-embedded in $\tilde{F}$ and which runs from η_k to η_1. There is an arc γ contained in $A_{i+1} \subset \tilde{F}$ with $\partial\gamma = \partial\beta$ and

$$\phi_\gamma = [a]^{r_1}[b][a]^{r_2}[b]\cdots[b][a]^{r_{k-1}}[b]$$

where $r_j \geqslant 0$; for $1 \leqslant j \leqslant k-1$ (figure 3.14 with γ oriented from η_k to η_1).

Now
$$\begin{aligned} \text{rhs } \Omega(\phi_\gamma) &= (g_{k-1})^{-1}\cdots(g_2)^{-1}(g_1)^{-1} \\ &= g_k \quad \text{in } G_i \\ &= \text{rhs } \Omega(\phi_\beta). \end{aligned}$$

Hence (by corollary 3.5.2) $\phi_\beta = \phi_\gamma$ which is a contradiction since their leftmost pictures are different.

Now suppose the pattern $(g_1)^{-1}e(g_k)^{-1}$ appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

$$[b,d,f][p_1]\cdots[p_s][b,d,f].$$

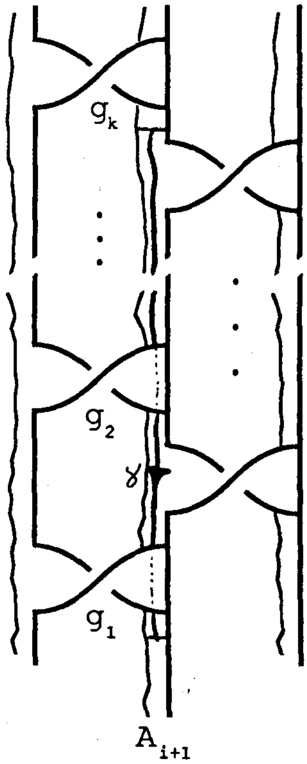


figure 3.14

Each of [b], [d], [f] has the form (g;·,SE). Let η_1, $\eta_k \in E$ be as above. The subfilm $[p_1]\cdots[p_s][b,d,f]$ defines a subarc $\beta \subset \alpha$ H-embedded in $\tilde{F}$ which runs from η_1 to η_k. There is an arc $\gamma \subset A_{i+1} \subset \tilde{F}$ with $\partial\gamma = \partial\beta$ such that

$$\phi_\gamma = [\hat{b}][\hat{a}]^{r_1}[\hat{b}][\hat{a}]^{r_2}[\hat{b}]\cdots[\hat{b}][\hat{a}]^{r_{k-1}}$$

where $r_j \geqslant 0$; for $1 \leqslant j \leqslant k-1$ (figure 3.14 with γ oriented from η_1 to η_k).

Now

$$\begin{aligned} \text{rhs } \Omega(\phi_\gamma) &= g_1 g_2 \cdots g_{k-1} \\ &= (g_k)^{-1} \text{ in } G_i \\ &= \text{rhs } \Omega(\phi_\beta). \end{aligned}$$

Again (by corollary 3.5.2) $\phi_\beta = \phi_\gamma$: a contradiction since the rightmost pictures differ.

The proofs for lhs $\Omega(\phi_\alpha)$ are similar. □

3.6.4 Lemma. Suppose e is a word in G with e = 1, and g_j, g_{j+1} are adjacent generators in some factor G_i of G with g_{j+1} following g_j. If one of the patterns $g_j e g_{j+1}$ or $(g_{j+1})^{-1} e (g_j)^{-1}$ occurs in r/lhs $\Omega(\phi_\alpha)$ then $e = \Phi$, and furthermore ϕ_α contains one of the following sequences as a subfilm:

(1) $[\hat{b}][\hat{a}]^r[\hat{b},\hat{d},\hat{f}]$; if $g_j e g_{j+1}$ appears in rhs $\Omega(\phi_\alpha)$

(2) $[b,d,f][a]^r[b]$; if $(g_{j+1})^{-1} e (g_j)^{-1}$ appears in rhs $\Omega(\phi_\alpha)$

(3) $[a][b]^r[a,c,f]$; if $g_j e g_{j+1}$ appears in lhs $\Omega(\phi_\alpha)$

(4) $[\hat{a},\hat{c},\hat{f}][\hat{b}]^r[\hat{a}]$; if $(g_{j+1})^{-1} e (g_j)^{-1}$ appears in lhs $\Omega(\phi_\alpha)$.

Proof. Suppose $g_j e g_{j+1}$ appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

$$[\hat{b},\hat{d},\hat{f}][p_1]\cdots[p_s][\hat{b},\hat{d},\hat{f}].$$

Each of the pictures $[\hat{b}]$, $[\hat{d}]$, $[\hat{f}]$ has the form $(g;SE,\cdot)$. Let $\eta_j, \eta_{j+1} \in E$ be the SE ends of the H-pieces associated with the generators g_j, g_{j+1} respectively. Then the subfilm $[\hat{b},\hat{d},\hat{f}][p_1]\cdots[p_s]$ is the film of a subarc $\beta \subset \alpha$ which is H-embedded in $\tilde{F}$ and which runs from η_j to η_{j+1}. There is an arc γ contained in $A_{i+1} \subset \tilde{F}$ running from η_j to η_{j+1} such that $\phi_\gamma = [\hat{b}][\hat{a}]^r$; $r \geq 0$ (figure 3.15 with γ oriented from η_j to η_{j+1}).

Now rhs $\Omega(\phi_\beta) = g_j e$ and rhs $\Omega(\phi_\gamma) = g_j$. Hence, by corollary 3.5.2, $\phi_\beta = \phi_\gamma$.

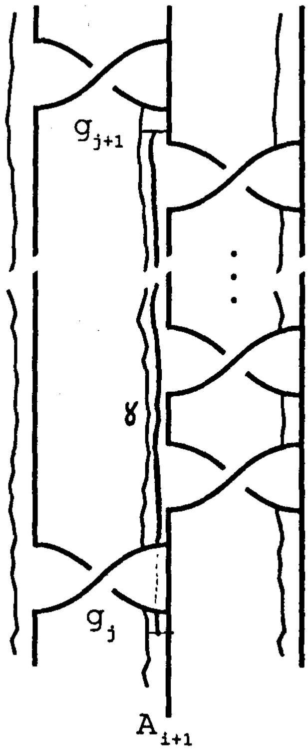


figure 3.15

Now suppose the pattern $(g_{j+1})^{-1}e(g_j)^{-1}$ appears in rhs $\Omega(\phi_\alpha)$. Then ϕ_α contains the subsequence

$$[b,d,f][p_1]\cdots[p_s][b,d,f].$$

Each of [b], [d], [f] has the form $(g;\cdot,SE)$. Let $\eta_j, \eta_{j+1} \in E$ be as above. The subfilm $[p_1]\cdots[p_s][b,d,f]$ defines a subarc $\beta \subset \alpha$ H-embedded in $\tilde{F}$ which runs from η_{j+1} to η_j. There is an arc $\gamma \subset A_{i+1} \subset \tilde{F}$ with $\partial\gamma = \partial\beta$ such that $\phi_\gamma = [a]^r[b]$; $r \geq 0$ (figure 3.15 with γ oriented from η_{j+1} to η_j).

Now rhs $\Omega(\phi_\beta) = e(g_j)^{-1}$ and rhs $\Omega(\phi_\gamma) = (g_j)^{-1}$. Hence, (by corollary 3.5.2) $\phi_\beta = \phi_\gamma$.

The proofs for lhs $\Omega(\phi_\alpha)$ are similar. □

3.6.5 Definition (factor word). A word $w \in G = G_1 * \cdots * G_n$ can be written as a product of subwords

$$w = w_1 w_2 \cdots w_m$$

such that there is a map between sets $\zeta: \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ with $w_j \in G_{\zeta(j)}$; $1 \leqslant j \leqslant m$, and $\zeta(j) \neq \zeta(j+1)$; $1 \leqslant j \leqslant m-1$. Call each w_j a <u>factor word</u> of w. Thus, a factor word is a word in a factor of G and adjacent factor words are in different factors.

3.6.6 Lemma-Definition (antiword). For every non-trivial factor word in r/lhs $\Omega(\phi_\alpha)$ there is a (non-trivial) factor word in l/rhs $\Omega(\phi_\alpha)$ which annihilates it.

Proof. Let $w = \Omega(\phi_\alpha)$, and let w_s be a factor word of rhs $\Omega(\phi_\alpha)$ such that $w_s \neq 1$ in G.

Suppose w_s is a factor word of w. (This may not be the case if w_s is the leftmost factor word of rhs $\Omega(\phi_\alpha)$.) Since w=1 in G, there is at least one factor word of w which cancels to 1. Let w' denote the word which remains when all such trivial factor words are deleted from w. Notice that w'=1 in G, and that the factor words of w' are products of the factor words of w. Let w_s' denote the factor word of w' which contains w_s as a factor.

If $w_s' \neq 1$ in G then repeat the above procedure on w'. So, assume that $w_s' = 1$. Let u, v be products of non-trivial factor words of w such that $w_s' = uw_sv$. Let w_t denote the leftmost factor word in w of v, and let w_r denote the rightmost factor word in w of u. Now $\zeta(r) = \zeta(s) = \zeta(t)$, and from the definition of factor word $\zeta(s-1) \neq \zeta(s) \neq \zeta(s+1)$. Hence,

$$r < s-1 \text{ and } s+1 < t$$
$$\Rightarrow \quad r+1 < s < t-1 .$$

Let g' denote the rightmost letter of w_s and g" the leftmost letter of w_t. Since $w_s'=1$, either $g' = (g")^{-1}$, or g' and g" are adjacent. Hence, the subword of rhs $\Omega(\phi_\alpha)$ given by $g'\cdots g"$ has one of the following forms:

$$geg^{-1},\ g^{-1}eg,\ g_keg_1,\ (g_1)^{-1}e(g_k)^{-1},\ g_jeg_{j+1},\ (g_{j+1})^{-1}e(g_j)^{-1}$$

where e=1 in G, and k is the number of generators in $G_{\zeta(s)}$. The first two options are excluded by lemma 3.6.1, the second two by lemma 3.6.3. For the last two options, lemma 3.6.4 implies that $e = \Phi$, hence g' and g" belong to the same factor word of w which is a contradiction. Therefore, $v = \Phi$.

If the rightmost letter of w_r is in rhs $\Omega(\phi_\alpha)$ then a similar contradiction is obtained. So,

$$w_r \subseteq \text{lhs } \Omega(\phi_\alpha)$$
$$\Rightarrow \quad u \subseteq \text{lhs } \Omega(\phi_\alpha).$$

If u is a product of factor words of w then similar contradictions are obtained. So, $u = w_r$.

Thus, w_s and w_r are two factor words of w such that

(1) $w_s \subseteq \text{rhs } \Omega(\phi_\alpha)$

(2) $w_r \subseteq \text{lhs } \Omega(\phi_\alpha)$

(3) $w_r w_s = 1$ in G

(4) $\prod_{k=r+1}^{s-1} w_k = 1$ in G.

Define w_r to be the <u>antiword</u> of w_s, denoted $\text{anti}(w_s)$. A similar definition follows if w_s is a factor word of w, and $w_s \subseteq \text{lhs } \Omega(\phi_\alpha)$.

Now suppose that $w_s \subseteq \text{rhs } \Omega(\phi_\alpha)$ is not a factor word of w. Let w_t be the factor word of w which contains w_s as a subword, and let $w_r = w_t \cap \text{lhs } \Omega(\phi_\alpha)$. Note that w_r is a factor word of $\text{lhs } \Omega(\phi_\alpha)$ and $w_t = w_r w_s$.

If $w_t = 1$ then w_r, w_s are two subwords of w which satisfy (1), (2), (3) above (in this case (4) has become an empty product). So define $w_r = \text{anti}(w_s)$.

If $w_t \neq 1$ then form w' from w by deleting all the trivial factor words of w as above. There are subwords u, v of w such that $u w_t v = 1$, and a similar analysis to above yields a contradiction. □

3.6.7 Definition (co-word). Let w_s be a factor word of $w = \Omega(\phi_\alpha)$ with $w_s = 1$ in G, and suppose $w_s \subseteq$ rhs $\Omega(\phi_\alpha)$. Let $[p_1]$ be the picture in ϕ_α which generates the leftmost letter of w_s, and let $[p_r]$ be the picture which generates the rightmost letter of w_s. The subfilm $[p_1]\cdots[p_r]$ of ϕ_α defines a subarc $\beta \subset \alpha$ H-embedded in $\tilde{F}$ with rhs $\Omega(\phi_\beta) = w_s$. The complementary word lhs $\Omega(\phi_\beta)$ is the <u>co-word</u> of w_s, and is denoted $co(w_s)$. A co-word is a subword of w. Since $\pi(L)$ is a braid, a co-word is a subword of a factor word of w, but may not itself be a factor word. A similar definition is made when $w_s \subseteq$ lhs $\Omega(\phi_\alpha)$.

Though the main theorem of this chapter was stated in the introduction, its statement is recalled here.

Theorem 3.1.2. Let β be a positive braid such that $\hat{\beta}$ is an irreducible projection of a non-split link L. Then

L is a non-prime link if and only if $\hat{\beta}$ is decomposable.

Proof.

Let w_s be the rightmost factor word of lhs $\Omega(\phi_\alpha)$ and let w_t be the leftmost factor word of rhs $\Omega(\phi_\alpha)$. (For definitions see 3.6.5, 3.4.4, and 3.4.3.) In the proof, the following four cases are considered:

(A) $w_s = 1$ and $co(w_s) = \Phi$,

or $w_t = 1$ and $co(w_t) = \Phi$;

(B) $w_s = 1$ and $co(w_s) = 1$,

or $w_t = 1$ and $co(w_t) = 1$;

(C) $w_s = 1$ and $w_t = 1$, (cases A, B excluded);

(D) $w_s \neq 1$ or $w_t \neq 1$.

Case (A) is the one which leads to a decomposition of the diagram. Each of the other cases is shown to lead to a contradiction, or to a situation where case (A) can be applied.

In the first three cases, at least one of w_s and w_t is a trivial word. Without loss of generality, suppose $w_s = 1$ in G. No subword of w_s is trivial in G otherwise w_s would contain one of the following patterns

$$geg^{-1},\ g^{-1}eg,\ g_k e g_1,\ (g_1)^{-1}e(g_k)^{-1},\ g_j e g_{j+1},\ (g_{j+1})^{-1}e(g_j)^{-1}$$

where e = 1 in G, and $k = p_{\zeta(s)}$, the number of generators in $G_{\zeta(s)}$ (or equivalently, the number of bands in $F_{\zeta(s)}$). Each of these possibilities is excluded by lemmas 3.6.1, 3.6.3, and 3.6.4.

Recall from 3.3.3 that the only relation in $G_{\zeta(s)}$ is

$$\prod_{j=1}^{k} g_j = 1 \qquad \text{where } k = p_{\zeta(s)}.$$

So, from lemma 3.6.3,

$$w_s = g_1 g_2 \cdots g_k$$
$$\text{or} \quad w_s = (g_k)^{-1} \cdots (g_2)^{-1}(g_1)^{-1}.$$

By lemma 3.6.4, ϕ_α contains one of the following subsequences as a subfilm:

$$[a][b]^{r_1}[a][b]^{r_2}[a]\cdots[a][b]^{r_{k-1}}[a,c,f]$$
$$\text{or} \quad [\hat{a},\hat{c},\hat{f}][\hat{b}]^{r_1}[\hat{a}][\hat{b}]^{r_2}[\hat{a}]\cdots[\hat{a}][\hat{b}]^{r_{k-1}}[\hat{a}]$$

where $r_j \geq 0$; for $1 \leq j \leq k-1$.

case (A): $w_s = 1$ and $co(w_s) = \phi$.

Suppose $co(w_s) = \phi$. Then $r_j = 0$; $1 \leq j \leq k-1$, and ϕ_α contains $[a]^{k-1}[a,c,f]$ or $[\hat{a},\hat{c},\hat{f}][\hat{a}]^{k-1}$ as a subfilm. These two films are descriptions of the same arc, which is indicated in figure 3.16(a). The films differ because the initial point of the arc is taken at the different ends of the arc.

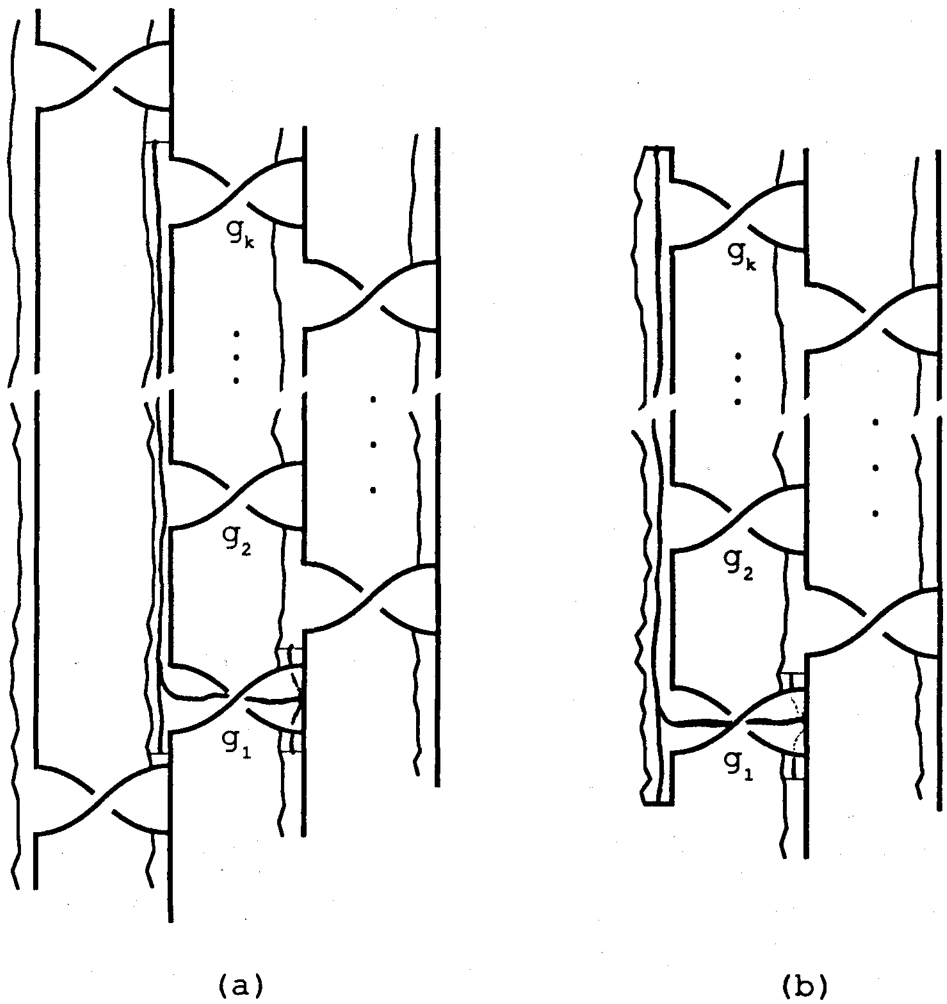


figure 3.16

Suppose that $\zeta(s) = 1$. This means that there are no bands or discs on the left of α (figure 3.16(b)). If ϕ_α contains $[a]^{k-1}[c,f]$ then the factorising 2-sphere S^2 can be isotoped in S^3 inducing an isotopy of α in $\tilde{F}$ which leaves α properly H-embedded in $\tilde{F}$ (although in the intermediate stages, it is not since $\partial\alpha$ moves along $(\partial\tilde{F}-\partial F)$) and which changes ϕ_α in the following way:

$$[a]^{k-1}[c] \text{ is replaced by } [e]$$
$$[a]^{k-1}[f] \text{ is replaced by } [d].$$

Since $k>1$, both of these isotopies reduce $|\phi_\alpha|$, which is a contradiction. In the case when ϕ_α contains $[a]^{k-1}[a]$ there is equality: $\phi_\alpha = [a]^k$ because there are no H-pieces into which α can continue, and since $|\phi_\alpha|$ is minimal, α cannot reverse. This contradicts the fact that the original choice of factorisation is non-trivial. Therefore $\zeta(s) \neq 1$.

The case when α is oppositely oriented similarly gives $\zeta(s) > 1$.

The projection $\pi(L)$ is decomposable, and β can be conjugated in B_{n+1} to be in the form

$$\beta_1(\sigma_1,\cdots,\sigma_{\zeta(s)-1})\ \beta_2(\sigma_{\zeta(s)},\cdots,\sigma_n).$$

The projection surface for $\hat{\beta}_1$ is a Murasugi sum of the surfaces $F_1, \cdots, F_{\zeta(s)-1}$, and since β_1 is a positive braid, it is a spanning surface of minimal genus (2.4.6). Since $\pi(L)$ is irreducible, the genus is non-zero. Therefore, $\hat{\beta}_1$ is a non-trivial link. Similarly, $\hat{\beta}_2$ is non-trivial. Thus, the factorisation of L as $\hat{\beta}_1 \# \hat{\beta}_2$ is non-trivial.

case (B): $w_s = 1$ and $co(w_s) = 1$.

In this case

$$\sum_{j=1}^{k-1} r_j = p_{\zeta(s)-1} .$$

The corresponding situation is shown in figure 3.17. The argument used above to show that $\zeta(s) \neq 1$ can be used here to show that either $|\phi_\alpha|$ can be reduced, or else the original factorisation is trivial, both of which are contradictions.

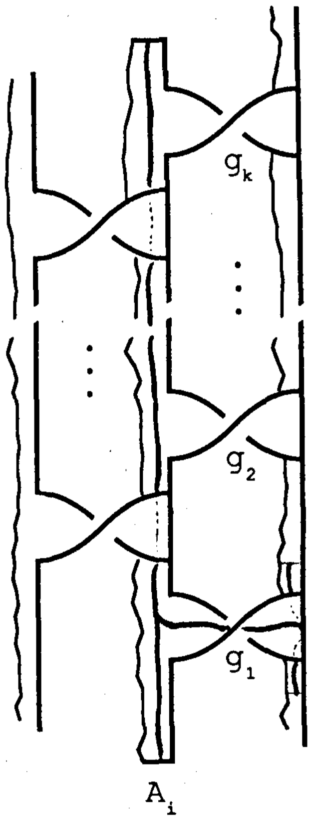


figure 3.17

The cases when $w_t = 1$ and $co(w_t)$ is Φ or 1 are similar.

case (C): $w_s = 1$ and $w_t = 1$.

Assume that none of the above cases hold.

Since $w_s = 1$, ϕ_α contains one of the following subsequences as a subfilm:

$$(1) \quad [a][b]^{r_1}[a][b]^{r_2}[a]\cdots[a][b]^{r_{k-1}}[a,c,f]$$

$$\text{or} \quad (2) \quad [\hat{a},\hat{c},\hat{f}][\hat{b}]^{r_1}[\hat{a}][\hat{b}]^{r_2}[\hat{a}]\cdots[\hat{a}][\hat{b}]^{r_{k-1}}[\hat{a}]$$

where $r_j \geqslant 0$; for $1 \leqslant j \leqslant k-1$, and $k = p_{\zeta(s)}$.

And since $w_t = 1$, ϕ_α also contains one of the following subfilms:

$$(3) \quad [b,d,f][a]^{q_1}[b][a]^{q_2}[b]\cdots[b][a]^{q_{k-1}}[b]$$

$$\text{or} \quad (4) \quad [\hat{b}][\hat{a}]^{q_1}[\hat{b}][\hat{a}]^{q_2}[\hat{b}]\cdots[\hat{b}][\hat{a}]^{q_{k-1}}[\hat{b},\hat{d},\hat{f}]$$

where $q_j \geqslant 0$; for $1 \leqslant j \leqslant k'-1$, and $k' = p_{\zeta(t)}$.

If $co(w_s) \cap w_t = \Phi$ then either $co(w_s) = \Phi$ or w_t is generated before w_s, implying that $co(w_t) = \Phi$. Both of these possibilities are dealt with above and are excluded here. Therefore,

$$co(w_s) \cap w_t \neq \Phi$$

$$\text{and also } co(w_t) \cap w_s \neq \Phi.$$

This implies that the above subfilms of ϕ_α must be interlaced. They can be combined in two ways.

Combining (1) and (3) gives

$$[b,d,f][a]^{q'_1}[b]^{r'_1}[a]^{q'_2}[b]^{r'_2}\cdots[a]^{q'_?}[b]^{r'_?}[a,c,f]$$

where

$$\sum r'_j = p_{\zeta(s)} - 1$$

$$\sum q'_j = p_{\zeta(t)} - 1$$

The picture [b,d,f] is leftmost in ϕ_α and hence must be [b]. This situation is shown in figure 3.18(a).

Combining (2) and (4) gives

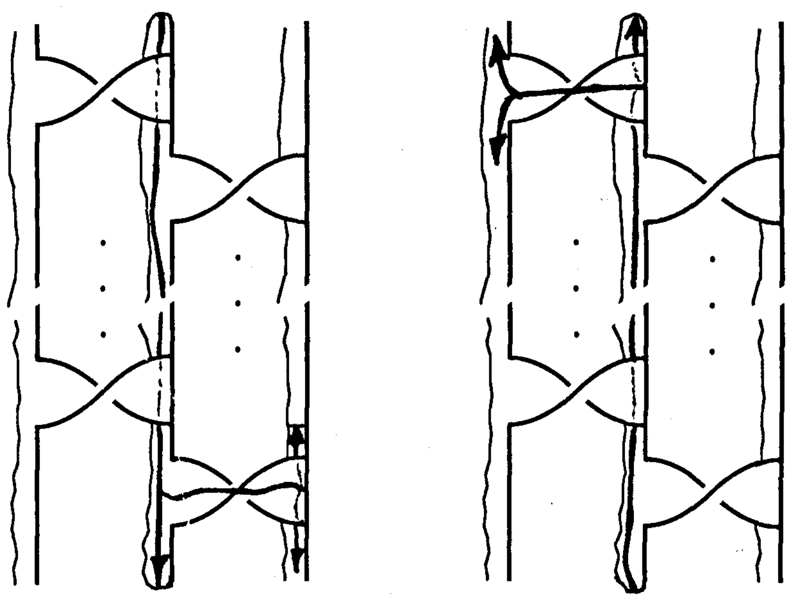
$$[\hat{a},\hat{c},\hat{f}][\hat{b}]^{r'_1}[\hat{a}]^{q'_1}[\hat{b}]^{r'_2}[\hat{a}]^{q'_2}\cdots[\hat{b}]^{r'_?}[\hat{a}]^{q'_?}[\hat{b},\hat{d},\hat{f}]$$

where (again)

$$\sum r'_j = p_{\zeta(s)} - 1$$

$$\sum q'_j = p_{\zeta(t)} - 1$$

The picture $[\hat{a},\hat{c},\hat{f}]$ is leftmost in ϕ_α and hence must be $[\hat{a}]$. This situation is shown in figure 3.18(b).



(a) (b)

figure 3.18

Now the above argument can be used again to show that either the factorisation is trivial, or else $|\phi_\alpha|$ can be reduced. So this case also leads to a contradiction.

case (D): $w_s \neq 1$ or $w_t \neq 1$.

Suppose, without loss of generality, that $w_s \neq 1$ in G. Then anti(w_s) is a factor word of rhs $\Omega(\phi_\alpha)$. Either w_t = anti(w_s) or else $w_t = 1$ in G since it lies between w_s and anti(w_s).

Let g' denote the rightmost letter of w_s, and let g" denote the leftmost letter of anti(w_s). Notice that if $w_t \neq 1$ then g" denotes the leftmost letter of w_t.

The subword of $\Omega(\phi_\alpha)$ given by g'···g" has one of the following forms:

$$geg^{-1},\ g^{-1}eg,\ g_k e g_1,\ (g_1)^{-1}e(g_k)^{-1},\ g_j e g_{j+1},\ (g_{j+1})^{-1}e(g_j)^{-1}$$

where e=1 in G, and $k = p_{\zeta(s)}$. (In the case that anti(w_s) = w_t, $e = \phi$.) Therefore, the leftmost end of ϕ_α is one of the following subsequences:

(1)	$\cdots[\hat{a},\hat{c},\hat{f}]\cdots[\hat{b},\hat{d},\hat{f}]$	$g^{-1}eg$
(2)	$\cdots[\hat{b},\hat{d},\hat{f}]\cdots[\hat{a},\hat{c},\hat{f}]$	
(3)	$\cdots[a,c,f]\cdots[b,d,f]$	geg^{-1}
(4)	$\cdots[b,d,f]\cdots[a,c,f]$	
(5)	$\cdots[a,c,f]\cdots[\hat{b},\hat{d},\hat{f}]$	$g_k e g_1$
(6)	$\cdots[\hat{b},\hat{d},\hat{f}]\cdots[a,c,f]$	
(7)	$\cdots[\hat{a},\hat{c},\hat{f}]\cdots[b,d,f]$	$(g_1)^{-1}e(g_k)^{-1}$
(8)	$\cdots[b,d,f]\cdots[\hat{a},\hat{c},\hat{f}]$	
(9)	$\cdots[a,c,f]\cdots[\hat{b},\hat{d},\hat{f}]$	$g_j e g_{j+1}$
(10)	$\cdots[\hat{b},\hat{d},\hat{f}]\cdots[a,c,f]$	
(11)	$\cdots[\hat{a},\hat{c},\hat{f}]\cdots[b,d,f]$	$(g_{j+1})^{-1}e(g_j)^{-1}$.
(12)	$\cdots[b,d,f]\cdots[\hat{a},\hat{c},\hat{f}]$	

The odd numbered cases are those where g' is generated before g", the even numbered ones conversely. It will be shown that each of these cases leads to a contradiction or to a situation where case (A) can be applied. The cases are not dealt with in the order in which they appear above, but are grouped according to the method of proof, and in a rough order of increasing complexity.

Consider first the even numbered cases. Let [p'], [p"] denote the pictures which generate g' and g" respectively. Then the leftmost subsequence of ϕ_α can be written

$$[p_1]\cdots[p_r][p'']\cdots[p'].$$

Let $\beta \subset \alpha$ be the subarc H-embedded in $\tilde{F}$ defined by

$$\phi_\beta = [p_1]\cdots[p_r].$$

Suppose that r>0. The first contribution to anti(w_s) is generated by [p"], so $w_t \neq$ anti(w_s) and $w_t \subset$ rhs $\Omega(\phi_\beta)$. Each of the factor words between w_s and anti(w_s) is trivial, therefore $w_t = 1$ in G. Also, since the first contribution to lhs $\Omega(\phi_\alpha)$ is generated by [p'], lhs $\Omega(\phi_\beta) = \Phi$ and hence co(w_t) = Φ. Therefore case (A) applies and the diagram is decomposable.

So we can assume that r=0 and hence that [p"] is the leftmost picture in ϕ_α implying that [p"] $\in$ {[a],[â],[b],[b̂]}.

<u>case (2)</u>: [b̂]···[â,ĉ,f̂].

The picture [b̂] is of the form (g;SE,·), and each of [â], [ĉ], [f̂] has the form (g;·,NW). Let $\beta \subset \alpha$ be the subarc H-embedded in $\tilde{F}$ defined by

$$\phi_\beta = [\hat{b}]\cdots[\hat{a},\hat{c},\hat{f}].$$

Thus, β connects the SE end of the H-piece associated to g to its NW end. Also lhs $\Omega(\phi_\beta) = g^{-1}$. Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [\hat{f}]$ (see figure 3.19). Then lhs $\Omega(\phi_\gamma) = g^{-1}$, and hence, by corollary 3.5.2, $\phi_\beta = \phi_\gamma$: a contradiction.

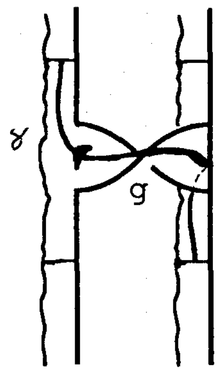


figure 3.19

<u>case (10)</u>: $[\hat{b}][p_y]\cdots[p_z][a,c,f]$

Let $\beta \subset \alpha$ be the subarc H-embedded in $\tilde{F}$ defined by

$$\phi_\beta = [\hat{b}][p_y]\cdots[p_z].$$

Then lhs $\Omega(\phi_\beta) = \Phi$. Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [\hat{d}][b]^r$ for some $r \geqslant 0$ (see figure 3.20). Then lhs $\Omega(\phi_\gamma) = \Phi$, and therefore (corollary 3.5.2) $\phi_\beta = \phi_\gamma$: a contradiction.

<u>case (6)</u>: $[\hat{b}][p_y]\cdots[p_z][a,c,f]$

As in case (10), let $\beta \subset \alpha$ be the subarc H-embedded in $\tilde{F}$ defined by

$$\phi_\beta = [\hat{b}][p_y]\cdots[p_z].$$

Then lhs $\Omega(\phi_\beta) = \Phi$. Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ such that $\phi_\gamma = [\hat{f}]$ followed by a sequence of $[\hat{a}]$'s and $[\hat{b}]$'s so that the last picture is $[\hat{a}]$ and there are (k-1) $[\hat{a}]$'s in total. Figure 3.21

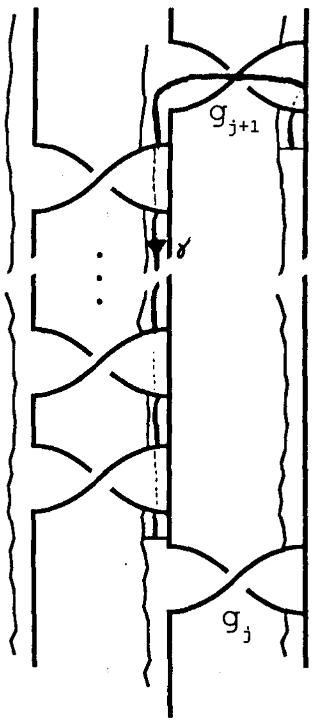


figure 3.20

depicts this situation. The $[\hat{b}]$ pictures do not contribute anything to lhs $\Omega(\phi_\gamma)$. The other pictures do contribute. Recall that lhs $\Omega(\phi_\gamma)$ is generated from right to left. So

$$\begin{aligned} \text{lhs } \Omega(\phi_\gamma) &= (g_k)^{-1} \cdots (g_2)^{-1}(g_1)^{-1} \\ &= 1 \quad \text{in } G \\ &= \text{lhs } \Omega(\phi_\beta). \end{aligned}$$

Therefore (by corollary 3.5.2) $\phi_\beta = \phi_\gamma$.

So ϕ_α contains the pattern $[\hat{a}][a,c,f]$ which implies $|\phi_\alpha|$ can be reduced: a contradiction. (Also, the leftmost picture of ϕ_α cannot be $[\hat{f}]$.)

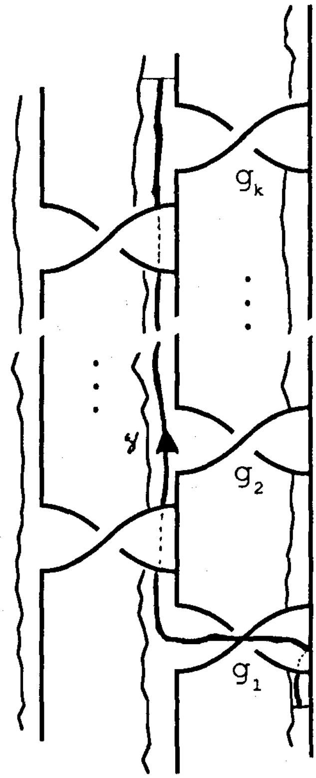


figure 3.21

The consideration of cases (4), (8) and (12) requires a different technique. In all three cases, the leftmost subfilm of ϕ_α can be represented

$$[p''][p_y]\cdots[p_z][p']$$

where $[p''] = [b]$. Let $\beta \subset \alpha$ be the subarc H-embedded in $\tilde{F}$ defined by

$$\phi_\beta = [b][p_y]\cdots[p_z].$$

Then lhs $\Omega(\phi_\beta) = \Phi$. Therefore, every picture in ϕ_β must belong to $\{[b],[\hat{b}],[d],[\hat{d}],[e],[\hat{e}]\}$.

Consider the automaton shown in figure 3.22. It takes a film ϕ as input and checks whether lhs $\Omega(\phi) = \Phi$ or not. The four states have the same labels as the ends of an H-piece, and the transition arrows are labelled with pictures.

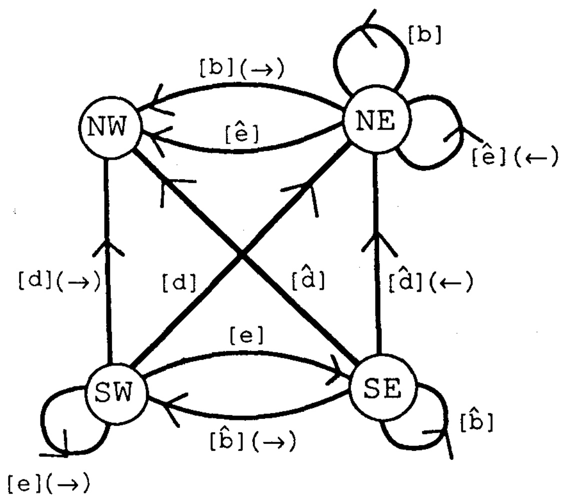


figure 3.22

Suppose the automaton is given the film $\phi = [p_1][p_2]\cdots[p_r]$ as input where $[p_i] = (g_i;\eta_{i,1},\eta_{i,2})$ for $1 \leq i \leq r$, and the g_i are not necessarily in the same factor of G. The automaton starts in the state labelled $\eta_{1,1}$. The picture $[p_1]$ should label an arrow oriented away from this state and towards the state labelled $\eta_{2,1}$. The picture $[p_i]$ should label an arrow oriented away from the state labelled $\eta_{i,1}$ and towards the state labelled $\eta_{i+1,1}$ for $1 \leq i < r$. If the input film ϕ passes this test then lhs $\Omega(\phi) = \Phi$; the automaton recognises all films with this property. The transition arrows occur in pairs since only the N or S part of the next state can be determined from the current picture, the E or W part depends on the following picture. The pictures $[p_i]$ and $[p_{i+1}]$ refer to the H-pieces associated to g_i and g_{i+1} respectively. If g_i and g_{i+1} belong to the same factor of G then the transition arrow is labelled with the picture $[p_i]$ only. If they are in adjacent factors however, then the symbol (←) or (→) is appended to the label indicating that g_{i+1} is in a factor of G which has lower or higher index (respectively) than the factor containing g_i.

In all three of the cases being considered, the leftmost picture is [b] so the initial state of the automaton is NE. The states which are accessible from this state are indicated in figure 3.23. Since there are no arrows oriented away from the NW state, the film ϕ_β must be recognised by this subautomaton.

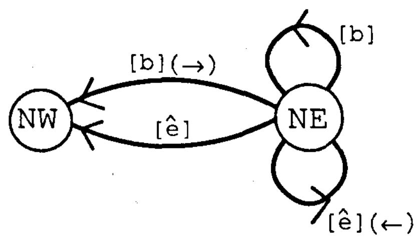


figure 3.23

In cases (8) and (12) the picture [p'] has the form (g;·,NW). Since there is no picture (g;NW,NW), the only possibility which can follow ϕ_β is (g;NE,NW) = [ĉ]. Thus the leftmost subsequence of ϕ_α is ϕ_β[ĉ], and it must be recognised by the automaton shown in figure 3.24(a). In case (4), [p'] has the form (g;NW,·), and ϕ_β[p'] must be recognised by the automaton shown in figure 3.24(b).

<u>case (12)</u>: [b][p_y]···[p_z][â,ĉ,f̂]

In this case, the pictures [p"] and [p'] are (g_j;NE,SE) and (g_{j+1};NE,NW) respectively. Each of the pictures between must be [b]

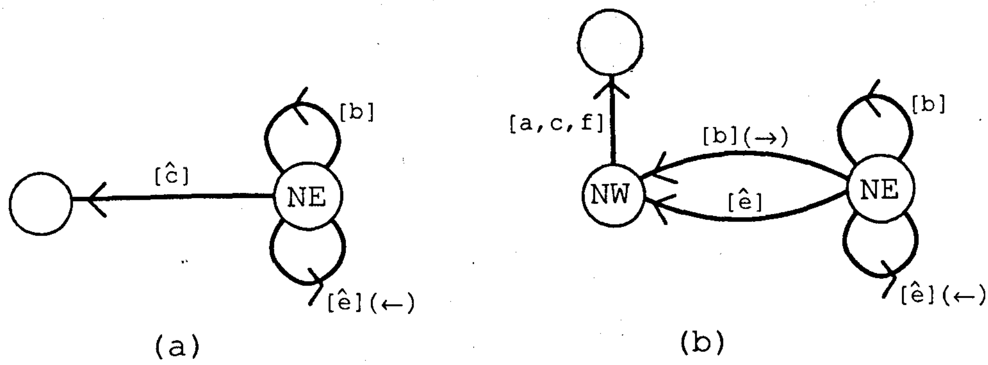


figure 3.24

or [ê], so β always moves 'vertically' S-wards or 'diagonally' SW-ward. Since β starts in the H-piece associated to g_j it cannot finish in the H-piece associated to g_{j+1} which is in the N-ward direction (see figure 3.25). Thus, ϕ_β has properties which are inconsistent.

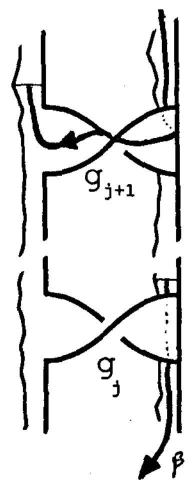


figure 3.25

<u>case (8)</u>: $[b][p_y]\cdots[p_z][\hat{a},\hat{c},\hat{f}]$

In this case, the pictures [p"] and [p'] are (g_k;NE,SE) and (g_1;NE,NW) respectively. Again, each of the pictures between must be [b] or [ê], implying that β always moves S-ward or SW-ward. The arc β starts at the NE end of the H-piece associated to g_k and finishes at the NE end of the H-piece associated to g_1. The only possibility for ϕ_β is

$[b]^{k-1}$ (see figure 3.26). If g_1, $g_k \notin G_n$ then the diagram is decomposable, and if g_1, $g_k \in G_n$ then $|\phi_\alpha|$ can be reduced contrary to assumption.

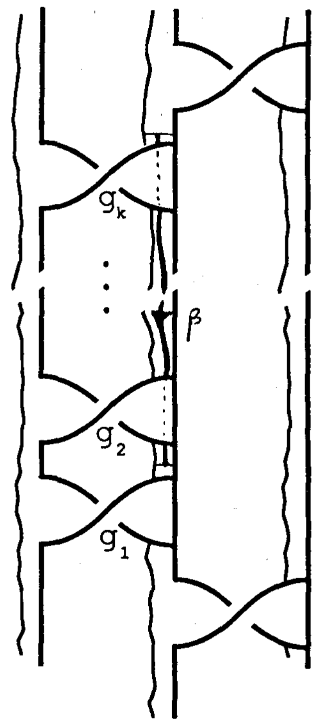


figure 3.26

<u>case (4)</u>: $[b][p_y]\cdots[p_z][a,c,f]$

The pictures [p"] and [p'] have the forms (g;·,SE), and (g;NW,·) respectively. As in case (12), the arc β starts in the H-piece associated to g and proceeds strictly S-ward implying that it cannot finish at the NW end of the H-piece associated to g.

All the even numbered cases are now dealt with, and attention is directed at the odd numbered ones.

At some stage in many of the foregoing arguments, the fundamental lemma of §3.5 was used. The applications of this lemma relied on the existence of contractible loops in the surface complement $S^3-\tilde{F}$ which arise as 'pushoffs' of loops in $\tilde{F}$. An embedded loop pushed off from $\tilde{F}$ can can be laid flat onto $\tilde{F}_+$ or $\tilde{F}_-$. In the remaining cases, such convenient loops are not easily found. However, there are other contractible loops in $S^3-\tilde{F}$ which arise from curves in $\tilde{F}$ and which prove to be useful.

The required trick is provided by noting that the positive braids are fibred (2.4.10 and 2.4.13). The link complement S^3-L is fibred over S^1 with fibre $\tilde{F}$ if there exists a map $M:(S^3-L)\rightarrow S^1$ such that for all $x\in S^1$, there is a neighbourhood $N(x)$ so that $M^{-1}(N(x))$ is homeomorphic to a bicollar on $\tilde{F}$. The translation of the fibre surface $\tilde{F}$ around the base space S^1 determines a homeomorphism $m:\tilde{F}_-\rightarrow\tilde{F}_+$ which is well defined up to isotopy. The map m is called the characteristic homeomorphism of the fibration, or the <u>holonomy</u> map.

Suppose that λ is an embedded loop in $\tilde{F}$ such that neither λ_+ nor λ_- is contractible in $S^3-\tilde{F}$. Suppose there exists an arc δ embedded in $\tilde{F}$ with $\partial\delta\cap\lambda \neq \phi$, and such that $(\delta_+)\cup(\lambda_+)\cup(\delta_-)$ is contractible in $S^3-\tilde{F}$. The fibration supplies an isotopy of $S^3-\tilde{F}$ which carries δ_- onto $(m(\delta))_+$ in $\tilde{F}_+$. Hence, $(\delta\cup\lambda\cup m(\delta))_+$ is contractible in $S^3-\tilde{F}$ and can be laid flat onto $\tilde{F}_+$.

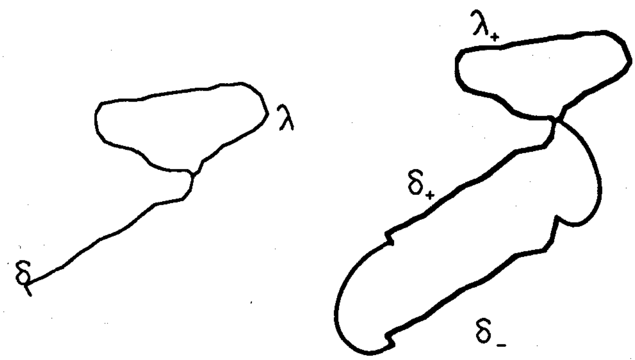


figure 3.27

(3.7.1) As an example of laying δ_- onto $\tilde{F}_+$, consider the case depicted in figure 3.28(a) where

$$\phi_\delta = [b] = (g;NE,SE)$$

$$\text{or} \quad \phi_\delta = [\hat{b}] = (g;SE,NE).$$

The arc δ_-, shown in 3.28(b), is laid flat onto $\tilde{F}_+$ in 3.28(c) where it is identified with $(m(\delta))_+$. Depending on the orientation of δ,

$$\phi_{m(\delta)} = [\hat{e}][c]$$

$$\text{or} \quad \phi_{m(\delta)} = [\hat{c}][e]$$

As an abuse of notation, write $m([b]) = [\hat{e}][c]$. For the other pictures, $m([p])$ is described below.

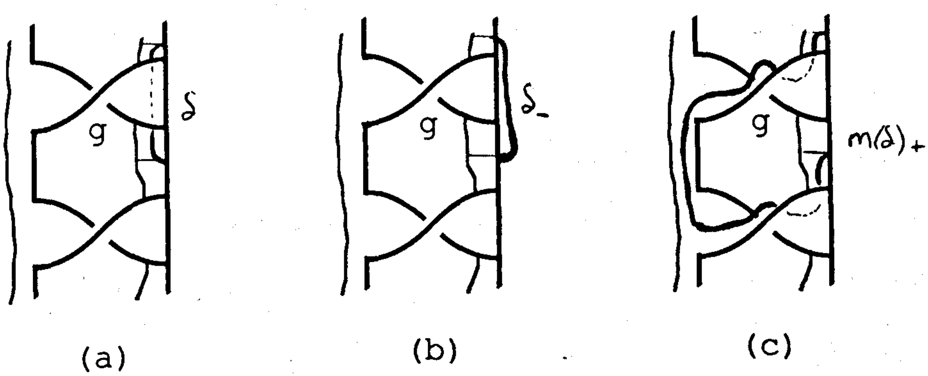
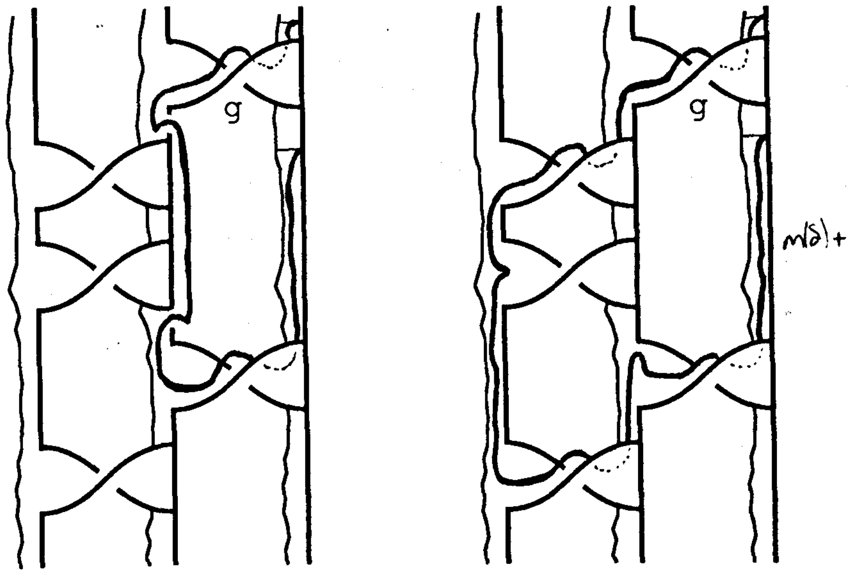


figure 3.28

The situation may not be as simple as depicted above. There may be bands in the adjacent column as shown in 3.29(a) which prevent the arc δ_- from being moved flat onto $\tilde{F}_+$. In this case, the above procedure can be used recursively, since there are finitely many columns (see figure 3.29).



(a) (b)

figure 3.29

The above figures do not apply when g is the first generator of a factor of G. In that case, m(δ) is as shown in figure 3.30.

To construct m(δ) from ϕ_δ in general, it is helpful to know how m behaves on each picture. Figure 3.31 shows the elementary cases of m([p]) (as in figure 3.28), where there are neither obstructions (as in figure 3.29) nor problems with the first and last generators of the factors of G (as in figure 3.30). These problems can be dealt with in a way similar to that of example 3.7.1. In the following figures, the arc m(δ) is built up from these basic units, then reduced by deleting parts of the arc where it retraces itself.

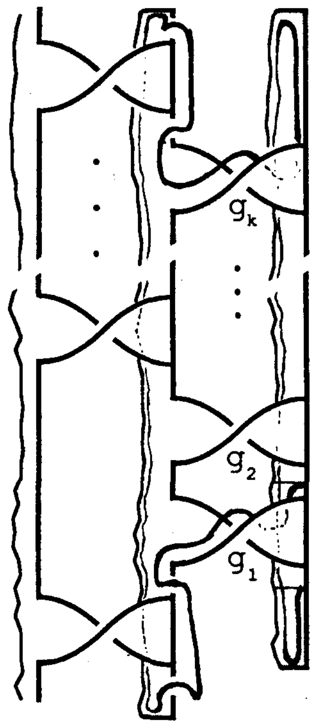


figure 3.30

Corollary 3.5.2 is central to the proofs of the above cases. The following lemma holds a similar position in the remaining cases.

3.7.2 Lemma. Let δ, β be two arcs each H-embedded in $\tilde{F}$ such that $\delta\cup\beta$ is a simple arc and $|\phi_{(\delta\cup\beta)}|$ is minimal. Suppose there exists an end $\eta\in E$ such that $\partial\beta\subset\eta$. Let $m:\tilde{F}_-\to\tilde{F}_+$ denote the holonomy map.

If $\Omega(\phi_\delta)\cdot\text{rhs }\Omega(\phi_\beta) = 1$ in G then $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta)}$.

(ie. $\delta\cup\beta$ is homotopic in $\tilde{F}$ to $m(\delta)$ keeping the boundary fixed.)

Proof. The word $\Omega(\phi_\delta)\cdot\text{rhs }\Omega(\phi_\beta)$

$= \text{lhs }\Omega(\phi_\delta)\cdot\text{rhs }\Omega(\phi_\delta)\cdot\text{rhs }\Omega(\phi_\beta).$

$= \text{lhs }\Omega(\phi_\delta)\cdot\text{rhs }\Omega(\phi_{(\delta\cup\beta)}).$

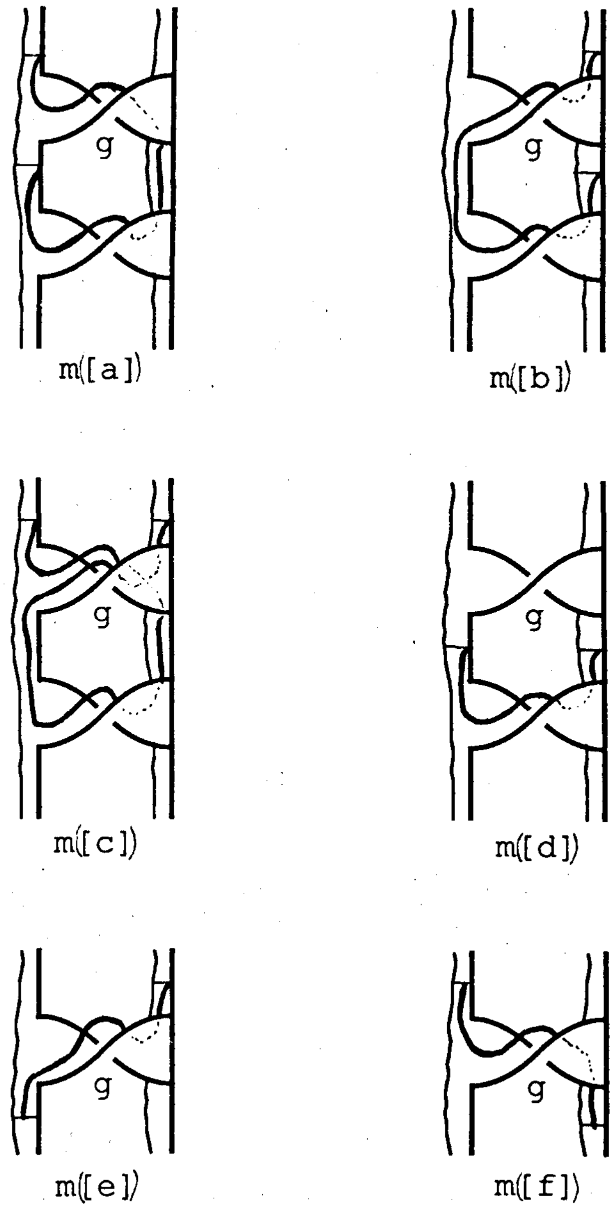


figure 3.31

Recall from 3.4.3 that the word rhs $\Omega(\phi_{(\delta\cup\beta)})$ in $\Pi_1(S^3\text{-}\tilde{F})$ represents the loop in $S^3\text{-}\tilde{F}$ based at x_0 which contains $(\delta\cup\beta)_+$ and which is oriented so that $(\delta\cup\beta)_+$ has the same orientation as $(\delta\cup\beta)$. Let λ_δ be the loop based at x_0 which contains δ_- and is oriented so that δ_- has the opposite orientation to δ. Then λ_δ represents lhs $\Omega(\phi_\delta)$. Let $m(\delta)_+$ denote $(m(\delta))_+$, and let $\lambda_{m(\delta)}$ be the loop based at x_0 containing

$m(\delta)_+$ which is oriented so that $m(\delta)_+$ has the same orientation as $m(\delta)$. Thus $\lambda_{m(\delta)}$ represents rhs $\Omega(\phi_{m(\delta)})$.

When the orientations are ignored, the loops λ_δ and $\lambda_{m(\delta)}$ are isotopic in S^3-$\tilde{F}$ via the fibration. Since the loops are oppositely oriented

$$\begin{aligned} &\text{rhs } \Omega(\phi_{m(\delta)}) \\ &= (\text{lhs } \Omega(\phi_\delta))^{-1} \\ &= \text{rhs } \Omega(\phi_{(\delta\cup\beta)}). \end{aligned}$$

As in the proof of 3.5.2, $\delta\cup\beta\cup m(\delta)$ is contractible in $\tilde{F}$ and $\phi_{m(\delta)} \approx \phi_{(\delta\cup\beta)}$. □

(3.7.3) In all of the odd numbered cases, g' is generated before g". Let $[p_1]\cdots[p_r][p']$ denote the leftmost subsequence of ϕ_α where [p'] is the picture which generates g'. Assume r>0 and let ϕ denote the sequence $[p_1]\cdots[p_r]$. Now lhs $\Omega(\phi) = \Phi$ and rhs $\Omega(\phi)$ is part of the trivial word denoted e in g'eg". Note that rhs $\Omega(\phi)$ must be an incomplete factor word since if it is trivial or contains a trivial subword then there exists a trivial factor word of $\Omega(\phi_\alpha)$ which has empty co-word, and so case (A) applies and the diagram is decomposable. Therefore,

$$\text{rhs } \Omega(\phi) = g_1\cdots g_r$$

$$\text{or} \quad \text{rhs } \Omega(\phi) = (g_k)^{-1}\cdots(g_j)^{-1} \quad \text{where } k\text{-}r\text{+}1 = j$$

where $g_1, \cdots, g_k$ generate some factor of G.

Since lhs $\Omega(\phi) = \Phi$, each $[p_i]$ in ϕ belongs to $\{[b],[\hat{b}],[d],[\hat{d}],[e],[\hat{e}]\}$. Lemma 3.6.4 implies that

$$\phi = [\hat{b}]^{r-1}[\hat{b},\hat{d},\hat{f}]$$

$$\text{or} \quad \phi = [b,d,f][b]^{r-1}.$$

The pictures $[f]$ and $[\hat{f}]$ cannot appear in ϕ, and by assumption $[p_1] \in \{[a],[\hat{a}],[b],[\hat{b}]\}$. So there are three possibilities for ϕ:

$$\phi = [\hat{b}]^r$$

$$\text{or} \quad \phi = [\hat{b}]^{r-1}[\hat{d}]$$

$$\text{or} \quad \phi = [b]^r.$$

The odd numbered cases fall naturally into two sets: those for which $[p']$ is $[a,c,f]$ which are (3), (5) and (9); and those for which $[p']$ is $[\hat{a},\hat{c},\hat{f}]$ which are (1), (7) and (11).

<u>case (3)</u>: $[p_1]\cdots[p_x][a,c,f]\cdots[b,d,f]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[p_x]$$

$$\phi_\beta = [a,c,f]\cdots[b,d,f]$$

If $\phi_\beta \simeq [f]$ then since $|\phi_\beta|$ is minimal, $\phi_\beta = [f]$ (3.5.2). In this case the leftmost subsequence of ϕ_α would be ϕ_δ, and rhs $\Omega(\phi_\alpha)$ would contain a trivial subword with empty co-word generated by ϕ_δ. So case (A) would apply.

Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [\hat{f}]$. Assume that γ is not homotopic in $\tilde{F}$ to β. The film $\phi_{(\beta\cup\gamma)}$ does not have minimal length since $[b,d,f][\hat{f}]$ can be reduced to $[\hat{c}]$, $[\hat{a}]$ or the null film. Assume that the arc $\beta\cup\gamma$ is isotoped in $\tilde{F}$ to effect this change. After this reduction, the resulting film, which will also be denoted $\phi_{\beta\cup\gamma}$, has minimal length, otherwise if further reduction were possible, $\phi_\beta \simeq [f]$. Now, $\phi_{(\delta\cup\beta\cup\gamma)}$, which is $\phi_\delta\phi_{(\beta\cup\gamma)}$, is a film of minimal length, and

$$
\begin{aligned}
&\text{lhs } \Omega(\phi_\delta)\cdot\text{rhs } \Omega(\phi_{(\delta\cup\beta)})\cdot\text{rhs } \Omega(\phi_\gamma) \\
&= 1\cdot\dot{e}g^{-1}\cdot g \\
&= 1 \quad \text{in } G.
\end{aligned}
$$

Hence, (by lemma 3.7.2) $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta\cup\gamma)}$.

The possible forms of ϕ_δ (which were listed in 3.7.3) are

$$\phi_\delta = [\hat{b}]^r$$

$$\text{or} \quad \phi_\delta = [\hat{b}]^{r-1}[\hat{a}]$$

$$\text{or} \quad \phi_\delta = [b]^r.$$

In the first case, r=0 and ϕ_δ is the null film otherwise there would be an inconsistency in ϕ_α. The leftmost picture of ϕ_β thus becomes the leftmost picture of ϕ_α and hence must be [a]. Applying the fundamental lemma to $\beta\cup\gamma$ gives

$$
\begin{aligned}
&\text{rhs } \Omega(\phi_{(\beta\cup\gamma)}) \\
&= eg^{-1}g \\
&= 1 \quad \text{in } G.
\end{aligned}
$$

Hence, by 3.5.2, $\phi_\beta \simeq \phi_\gamma$: a contradiction.

The cases in the last two rows are illustrated in figures 3.32(i) and (ii).

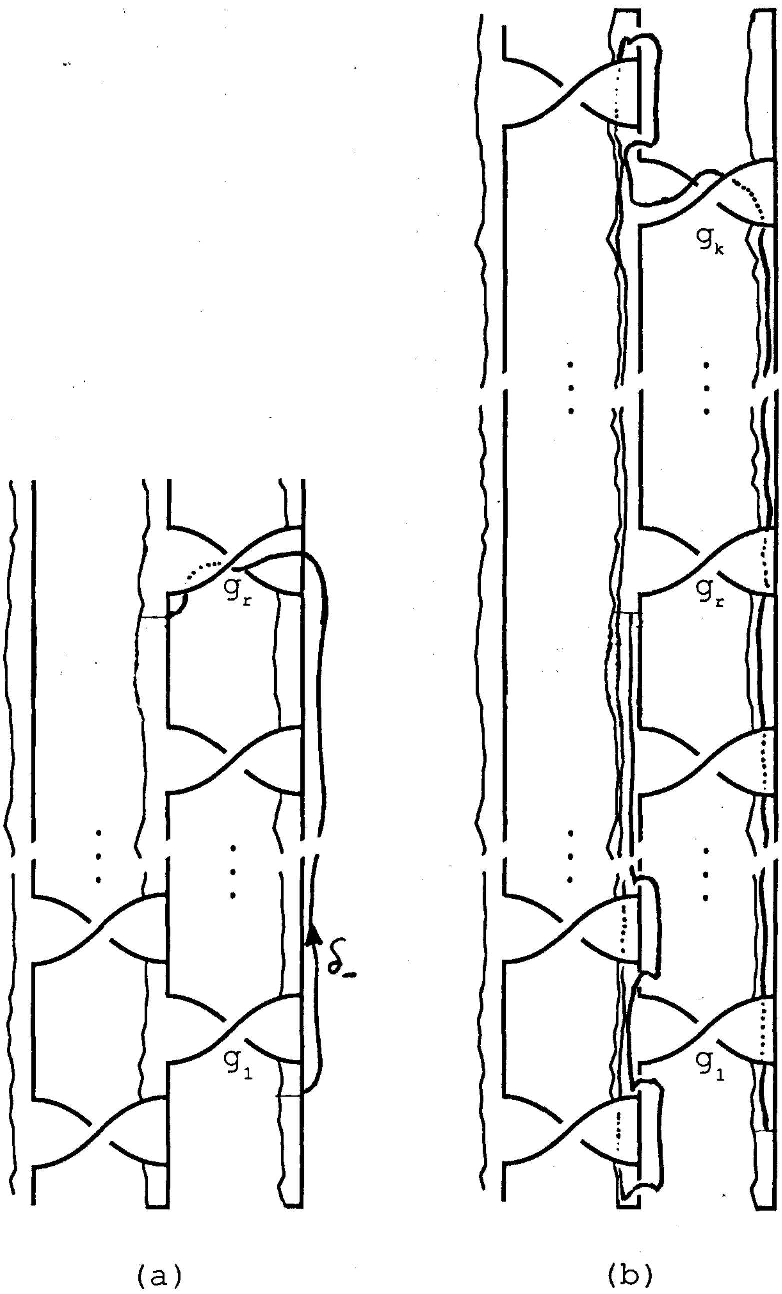
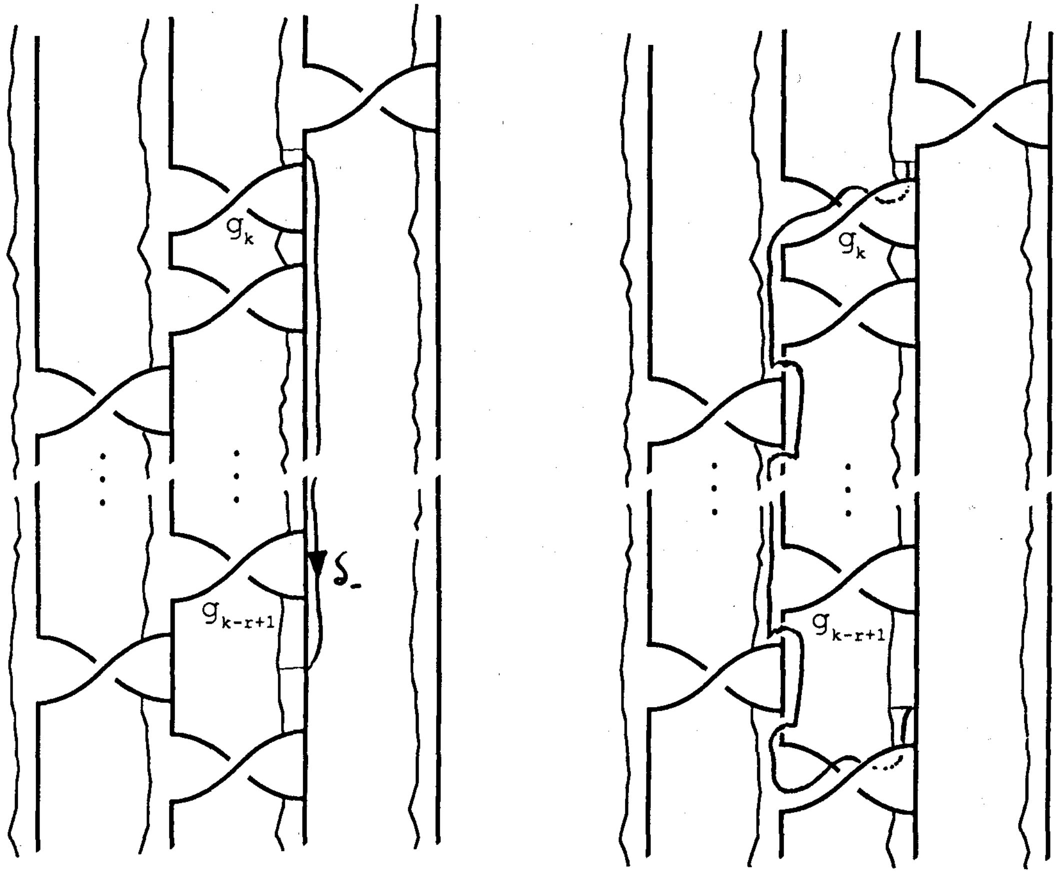


figure 3.32(i)



(a) (b)

figure 3.32(ii)

Figure 3.32(i). Part (a) shows the arc δ corresponding to the case in the second row. The generator g_r cannot be the last generator in a factor of G otherwise lhs $\Omega(\phi_\delta) = e = 1$ and rhs $\Omega(\phi_\delta) = \Phi$, and so case (A) would apply. In part (b) the corresponding m(δ) is indicated. There are places where m(δ) is left on $\tilde{F}_-$ and not laid flat onto $\tilde{F}_+$. All of these possible obstructions are like the arc in the picture [b]. Example 3.7.1 shows how such arcs can be laid flat onto $\tilde{F}_+$: m(δ) moves in the W-ward direction only (ie: into columns on the left).

When $|\phi_{m(\delta)}|$ is minimised, the leftmost subfilm of $\phi_{m(\delta)}$ is $[\hat{b}]^r\cdots$, and since $\phi_{(\delta\cup\beta\cup\gamma)} = [\hat{b}]^{r-1}[\hat{a}][\hat{c}]\cdots$, the two films are different. This is a contradiction.

Figure 3.32(ii). Part (a) shows the arc δ corresponding to the third case. The generator g_{k-r+1} is not the first generator in a factor of G otherwise lhs $\Omega(\phi_\delta) = 1$ and rhs $\Omega(\phi_\delta) = \Phi$, and again case (A) would apply. The corresponding m(δ) is indicated in part (b). The places where possible [b]-type obstructions may occur are indicated and again these are dealt with as in example 3.7.1. When $|\phi_{m(\delta)}|$ is minimised, the leftmost picture of $\phi_{m(\delta)}$ is $[\hat{e}]$, and the leftmost picture of $\phi_{(\delta\cup\beta\cup\gamma)} = [b]$. This is a contradiction since the two films must be equal.

case (9): $[p_1]\cdots[p_x][a,c,f][p_y]\cdots[p_z][\hat{b},\hat{d},\hat{f}]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[p_x]$$

$$\phi_\beta = [a,c,f][p_y]\cdots[p_z].$$

Then, β connects the NW end of the H-piece associated to g_j to the SE end of the H-piece associated to g_{j+1}. If $\phi_\beta \simeq [c][\hat{a}]^r$ then ϕ_δ generates the whole of e in $g_j e g_{j+1}$. In this case rhs $\Omega(\phi_\delta) = 1$ and lhs $\Omega(\phi_\delta) = \Phi$, so case (A) applies.

Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [a]^r[\hat{c}]$ for some $r \geq 0$ (see figure 3.33). Assume $\beta \cup \gamma$ is not contractible in $\tilde{F}$ otherwise $\phi_\beta \simeq [c][\hat{a}]^r$. Isotop $\beta\cup\gamma$ so that $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal.

Now lhs $\Omega(\phi_\delta)\cdot$rhs $\Omega(\phi_{\delta\cup\beta})\cdot$rhs $\Omega(\phi_\gamma)$

$= 1\cdot e\cdot 1$

$= 1$ in G.

Hence $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta\cup\gamma)}$. The result now follows as in case (3).

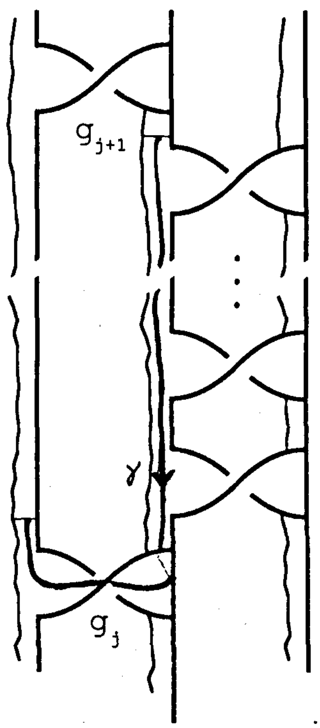


figure 3.33

case (5): $[p_1]\cdots[p_x][a,c,f][p_y]\cdots[p_z][\hat{b},\hat{d},\hat{f}]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[p_x]$$

$$\phi_\beta = [a,c,f][p_y]\cdots[p_z].$$

Then, β connects the NW end of the H-piece associated to g_k to the SE end of the H-piece associated to g_1. Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and so that ϕ_γ is a sequence of $[\hat{a}]$'s and $[\hat{b}]$'s so that the first picture is $[\hat{b}]$ and there are (k-1) $[\hat{b}]$'s in total, all of which are followed by $[\hat{f}]$. This situation is shown in figure 3.34.

If $\beta\cup\gamma$ is contracible in $\tilde{F}$ then rhs $\Omega(\phi_\delta) = 1$ and lhs $\Omega(\phi_\delta) = \Phi$, so case (A) applies. So assume that $\beta\cup\gamma$ is not contractible in $\tilde{F}$.

Isotop $\beta\cup\gamma$ so that $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal.

Now $\quad$ lhs $\Omega(\phi_\delta)\cdot$rhs $\Omega(\phi_{\delta\cup\beta})\cdot$rhs $\Omega(\phi_\gamma)$

$= 1\cdot e\cdot(g_1 g_2\cdots g_k)$

$= 1 \quad$ in G.

Hence $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta\cup\gamma)}$. The result now follows as in case (3).

case (1): $[p_1]\cdots[\hat{a},\hat{c},\hat{f}][p_y]\cdots[p_z][\hat{b},\hat{d},\hat{f}]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[\hat{a},\hat{c},\hat{f}]$$

$$\phi_\beta = [p_y]\cdots[p_z]$$

Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [\hat{f}]$. Then $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal, and

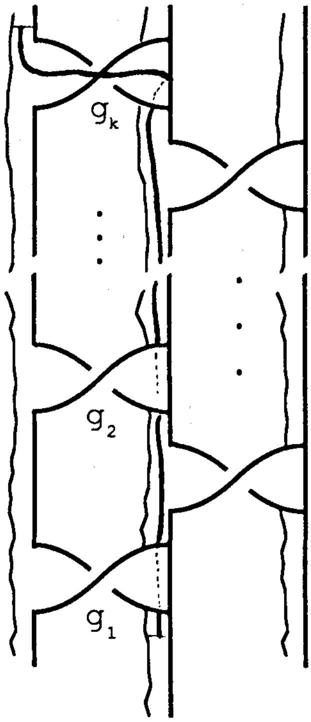


figure 3.34

$$\Omega(\phi_\delta)\cdot \text{rhs } \Omega(\phi_{\beta\cup\gamma})$$
$$= g^{-1}eg$$
$$= 1 \text{ in } G.$$

Therefore (by lemma 3.7.2) $\phi_{m(\delta)} \approx \phi_{(\delta\cup\beta\cup\gamma)}$.

From the above argument (3.7.3), there are nine possibilities for ϕ_δ:

$$\phi_\delta = [\hat{b}]^r[\hat{a},\hat{c},\hat{f}]$$
$$\text{or } \phi_\delta = [\hat{b}]^{r-1}[\hat{d}][\hat{a},\hat{c},\hat{f}]$$
$$\text{or } \phi_\delta = [b]^r[\hat{a},\hat{c},\hat{f}].$$

These reduce to four after eliminating those which are inconsistent (such as $[\hat{d}][\hat{f}]$) and those which contradict the minimality of $|\phi_\alpha|$. The possibilities which remain are

(i) $\phi_\delta = [\hat{b}]^r[\hat{a}]$

(ii) $\phi_\delta = [\hat{b}]^r[\hat{f}]$

(iii) $\phi_\delta = [\hat{b}]^{r-1}[\hat{d}][\hat{c}]$

(iv) $\phi_\delta = [b]^r[\hat{c}]$.

The figures on the following pages show each of these cases. The part of each figure labelled (a) shows the arc δ_-. The other parts of the figures show $(m(\delta))_+$ in the various circumstances described below.

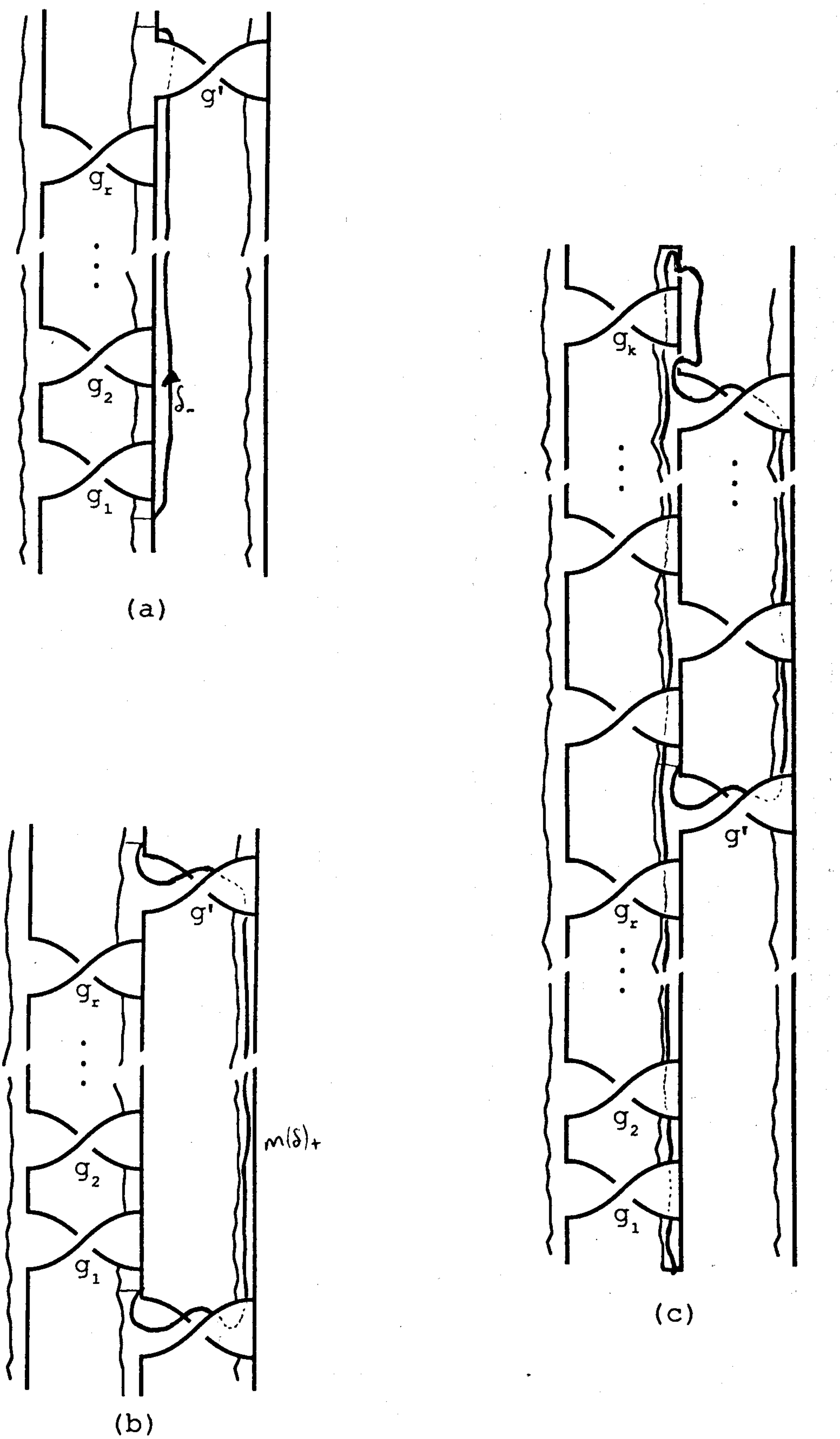


figure 3.35(i)

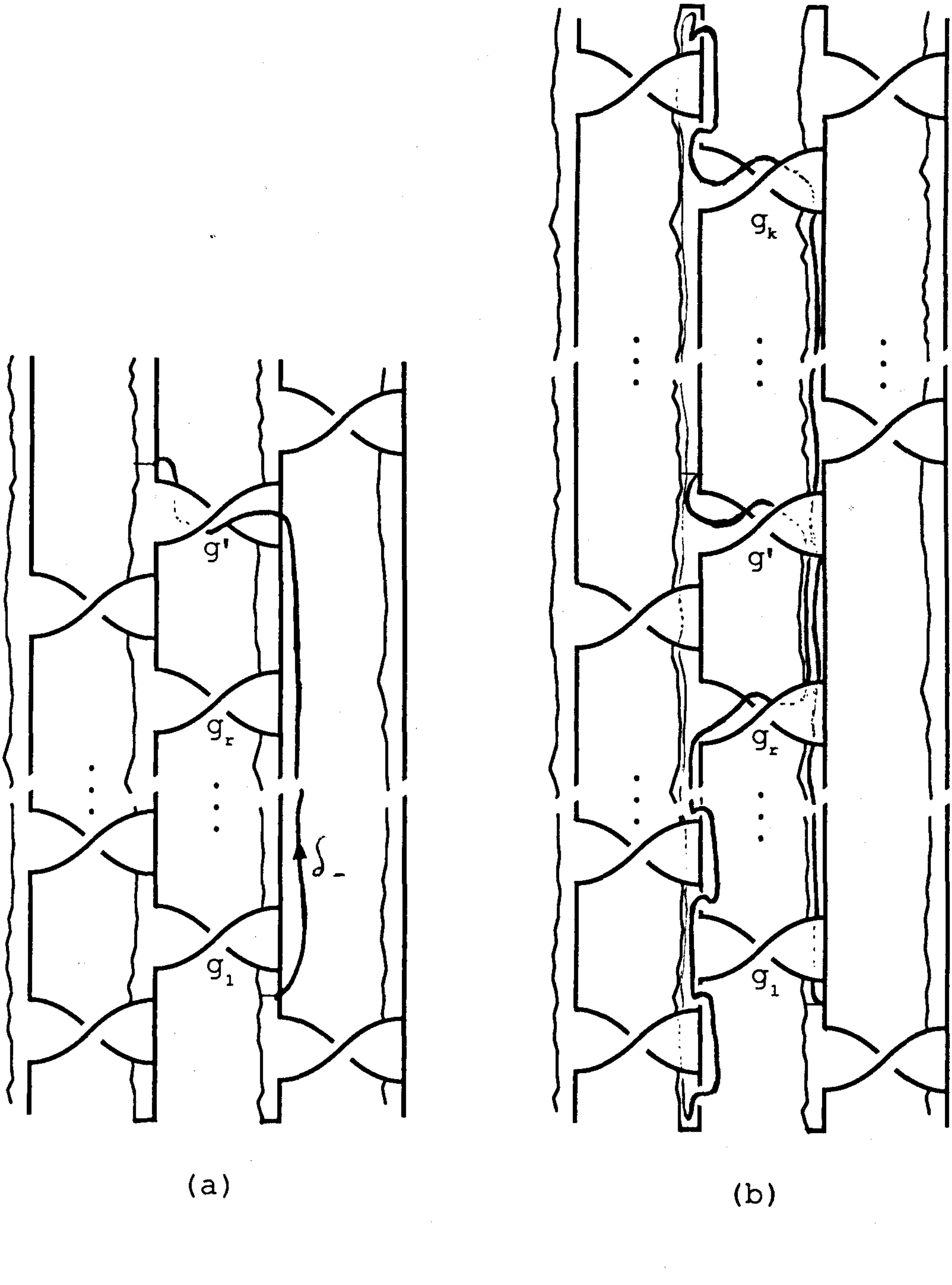
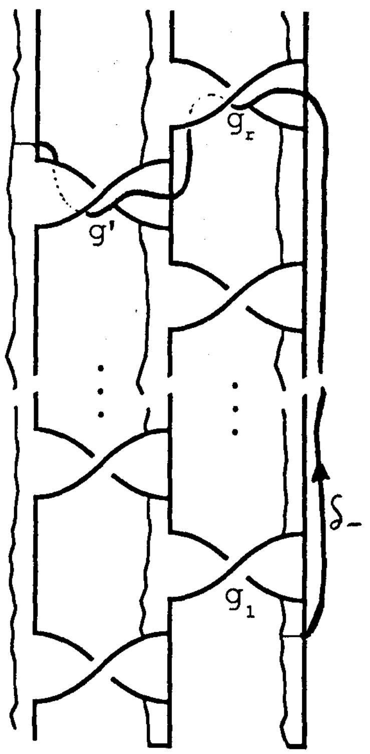
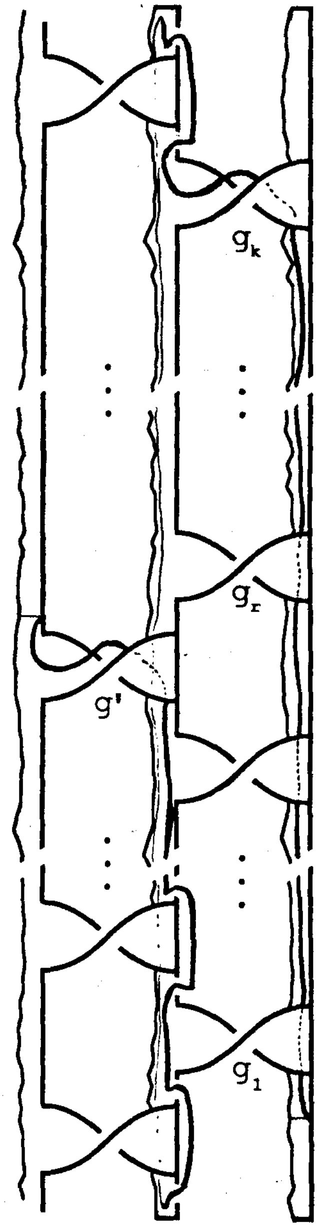


figure 3.35(ii)



(a)



(b)

figure 3.35(iii)

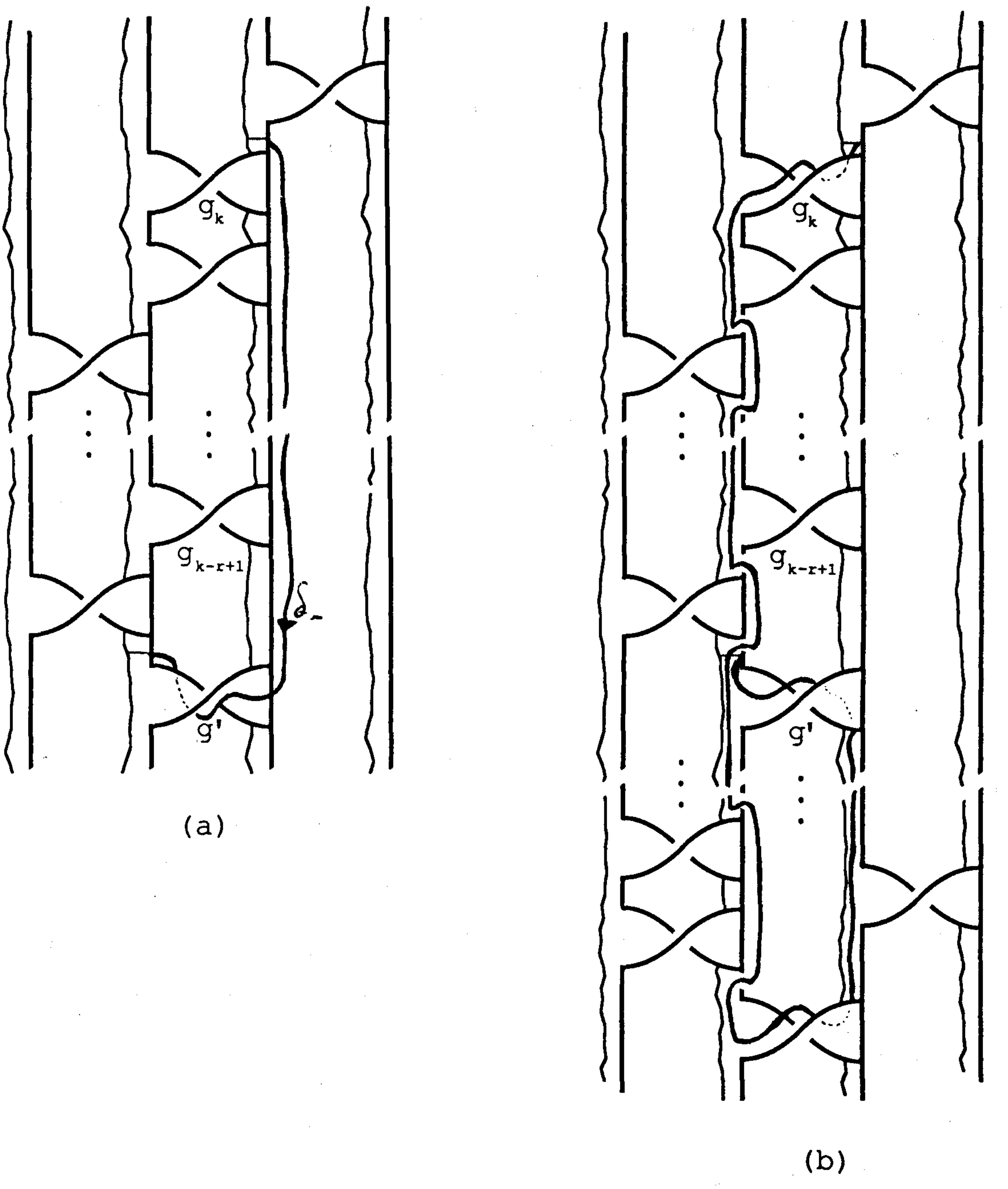


figure 3.35(iv)

Figure 3.35(i). Part (a) shows the arc δ corresponding to case (i). If g' is the first generator in a factor of G then part (c) shows $m(\delta)$, otherwise $m(\delta)$ is as shown in part (b). If $g_1,\cdots,g_r \in G_i$ then $g' \in G_{i+1}$. There are at least two generators in G_{i+1} so g' cannot be both the first and last generators, and either (b) or (c) must apply. Recall that $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal.

For the situation depicted in (b), $|\phi_{m(\delta)}|$ is minimal. Since there is a unique shortest film for each homotopy class of arcs in $\tilde{F}$ (3.5.1), these two films must be the same. But their leftmost pictures differ:

$$\phi_{(\delta\cup\beta\cup\gamma)} = [\hat{b}]\cdots,$$
$$\phi_{m(\delta)} = [c]\cdots.$$

This gives a contradiction.

For the situation shown in (c) there are places where $m(\delta)$ is left on $\tilde{F}_-$ and not laid onto $\tilde{F}_+$. These possible obstructions to $m(\delta)$ are all like the arc in the picture [b]. The example above (3.7.1) shows how such arcs can be laid flat onto $\tilde{F}_+$: $m(\delta)$ moves into columns on the left (ie. in the W-ward direction) only. When $\phi_{m(\delta)}$ is isotoped to minimise $|\phi_{m(\delta)}|$, the rightmost picture remains $[\hat{c}]$. The rightmost picture of $\phi_{(\delta\cup\beta\cup\gamma)}$ is $[\hat{f}]$. Hence, the two films are different: a contradiction.

Figure 3.35(ii). Part (a) shows the arc δ corresponding to case (ii), and part (b) shows the corresponding $m(\delta)$. If $g_1,\cdots,g_r \in G_i$ then $g' = (g_{r+1})^{-1}$, and g' is not the last generator of G_i otherwise $g_1\cdots g_{r+1}$ is a trivial subword of w_t. As in (i) part (c), there are possible [b]-type obstructions to laying m(δ) flat on $\tilde{F}_+$. When these are dealt with, and $|\phi_{m(\delta)}|$ is minimised, the leftmost subfilm of $\phi_{m(\delta)}$ is $[\hat{b}]^{r+1}\cdots$. Since $\phi_{(\delta\cup\beta\cup\gamma)} = [\hat{b}]^r[\hat{f}]\cdots$ the two films differ: a contradiction.

Figure 3.35(iii). Part (a) shows the arc δ corresponding to case (iii), and part (b) shows the corresponding m(δ). If $g_1,\cdots,g_r \in G_i$ then $g' \in G_{i-1}$. Also, g_r is not the last generator in G_i since w_t has no trivial subwords, and $co(w_t) \neq \Phi$. As before, possible [b]-type obstructions are indicated, and these can be dealt with as in 3.7.1. When $|\phi_{m(\delta)}|$ is minimised, the leftmost subfilm of $\phi_{m(\delta)}$ is $[\hat{b}]^r\cdots$, and since $\phi_{(\delta\cup\beta\cup\gamma)} = [\hat{b}]^{r-1}[\hat{d}][\hat{c}]\cdots$ the two films differ: a contradiction.

Figure 3.35(iv). Part (a) shows the arc δ corresponding to case (iv), and part (b) shows the corresponding m(δ). If $g_{k-r+1},\cdots,g_k \in G_i$ then $g' = g_{k-r}$, and g' is not the first generator of G_i otherwise there would be a trivial subword of w_t which is impossible. The places where possible [b]-type obstructions may occur are indicated and these can be dealt with as before. When $|\phi_{m(\delta)}|$ is minimised, the leftmost picture of $\phi_{m(\delta)}$ is [ê], and the leftmost picture of $\phi_{(\delta\cup\beta\cup\gamma)}$ is [b]. This contradicts the fact that the two films are equal.

It was assumed above that $r = |\phi_\delta| > 0$. If r=0 then [â,ĉ,f̂] is the leftmost picture in ϕ_α and so must be [â]. The argument in case (i) can be applied without modification.

<u>case (7)</u>: $[p_1]\cdots[\hat{a},\hat{c},\hat{f}][p_y]\cdots[p_z][b,d,f]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[\hat{a},\hat{c},\hat{f}],$$
$$\phi_\beta = [p_y]\cdots[p_z][b,d,f].$$

Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and where ϕ_γ is a sequence of [a]'s and (k-2) [b]'s which terminates with $[\hat{c}]$ as the rightmost picture. (Recall that k is the number of generators in the factor of G concerned). Then $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal, and

$$\text{rhs } \Omega(\phi_\gamma) = (g_{k-1})^{-1}\cdots(g_3)^{-1}(g_2)^{-1}.$$

Now

$$\begin{aligned}&\Omega(\phi_\delta)\cdot\text{rhs } \Omega(\phi_{\beta\cup\gamma})\\ &= (g_1)^{-1}e(g_k)^{-1}(g_{k-1})^{-1}\cdots(g_3)^{-1}(g_2)^{-1}.\\ &= 1 \quad \text{in G.}\end{aligned}$$

Hence, (by lemma 3.7.2) $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta\cup\gamma)}$. The argument now follows in the same way as for case (1).

<u>case (11)</u>: $[p_1]\cdots[\hat{a},\hat{c},\hat{f}][p_y]\cdots[p_z][b,d,f]$

Let δ, $\beta \subset \alpha$ be the subarcs H-embedded in $\tilde{F}$ defined by

$$\phi_\delta = [p_1]\cdots[\hat{a},\hat{c},\hat{f}],$$
$$\phi_\beta = [p_y]\cdots[p_z][b,d,f].$$

Let γ be the arc H-embedded in $\tilde{F}$ with $\partial\gamma = \partial\beta$ and $\phi_\gamma = [\hat{b}][\hat{a}]^r[\hat{f}]$; for some $r\geq0$. Then rhs $\Omega(\phi_\gamma) = g_j g_{j+1}$.

The film $\phi_{(\beta\cup\gamma)}$ does not have minimal length. This is because the subfilm formed from the rightmost picture of ϕ_β followed by the leftmost picture of ϕ_γ is $[b,d,f][\hat{b}]$ which can be replaced by the null film, [e] or [c]. Assume that $\beta\cup\gamma$ is isotoped in $\tilde{F}$ to achieve this replacement. Then $|\phi_{(\delta\cup\beta\cup\gamma)}|$ is minimal. Now

$$\begin{aligned}
&\Omega(\phi_{\delta})\cdot \mathrm{rhs}\ \Omega(\phi_{\beta\cup\gamma}) \\
&= (g_{j+1})^{-1}e(g_j)^{-1}g_j g_{j+1} \quad \text{(before isotopy of } \beta\cup\gamma\text{)} \\
&= (g_{j+1})^{-1}e g_{j+1} \quad \text{(after isotopy of } \beta\cup\gamma\text{)} \\
&= 1 \ \text{ in } G.
\end{aligned}$$

Hence, (by lemma 3.7.2) $\phi_{m(\delta)} \simeq \phi_{(\delta\cup\beta\cup\gamma)}$. The argument now follows in the same way as for case (1).

The cases when $w_t \neq 1$ are analogous. This exhausts all the possibilities. The proof of theorem 3.1.2 is completed by noting that the converse follows directly from the definition of decomposable. □

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GLOSSARY OF SYMBOLS

General Symbols

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$	sets of natural, integer, real numbers
$\mathbb{Z}_n$	set of integers mod n
$\mathbb{R}^n$	n-dimensional Euclidean space
S^n	topological n-sphere
ϕ	empty set
B_n	n-string braid group
β, $\hat{\beta}$	a braid and its closure
$\chi(F)$	Euler characteristic of the surface F
∂F	boundary of the surface F
$\Pi_1(X)$	fundamental group of X
$H_1(X)$	first homology group of X
rk(X)	rank of first homology group of X
$\mu(L)$	multiplicity of the link L
c(L)	order of the link L
s(L)	braid index (or Seifert circle index) of the link L
g(L)	genus of the link L
$\chi(L)$	Euler characteristic of the link L
$\pi(L)$	regular projection of the link L
$\nabla_L(z)$	Conway polynomial of the link L
$V_L(t)$	Jones polynomial of the link L
$P_L(v,z)$	two variable polynomial of the link L
$L_1 \# L_2$	product or connected sum of two links

$c(D)$	number of crossings in the diagram D
$s(D)$	number of Seifert circles in the diagram D
$D_1 * D_2$	*-product of two diagrams
$G_1 * G_2$	free product of two groups
□	end of proof

Special Symbols

INDEX OF DEFINITIONS