On the impact of insurance on households susceptible to random proportional losses: An analysis of poverty trapping

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Abstract

In this paper, we consider a risk process with deterministic growth and multiplicative jumps to model the capital of a low-income household. Reflecting the high-risk nature of the low-income environment, capital losses are assumed to be proportional to the level of accumulated capital at the jump time. Our aim is to derive the probability that a household falls below the poverty line, i.e. the trapping probability, where "trapping" occurs when the level of capital of a household holds falls below the poverty line, to an area from which it is difficult to escape without external help. Considering the remaining proportion of capital to be distributed as a special case of the beta distribution, closed-form expressions for the trapping probability are obtained via analysis of the Laplace transform of the infinitesimal generator of the process. To study the impact of insurance on this probability, introduction of an insurance product offering proportional coverage is presented. The infinitesimal generator of the insured process gives rise to non-local differential equations. To overcome this, we propose a recursive method for deriving a closed-form solution of the integro-differential equation associated with the infinitesimal generator of the insured process and provide a numerical estimation method for obtaining the trapping probability. Constraints on the rate parameters of the process that prevent certain trapping are derived in both the uninsured and insured cases using classical results from risk theory.

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1 Introduction

Low-income households living close to, but above, the poverty line are extremely susceptible to entering extreme poverty, particularly in the event of a financial loss. This problem, and the true nature of low-income loss experience, must be studied in order to increase rates of poverty reduction. One indicator that can be used to assess financial stability is capital. In the low-income setting, where monetary wealth is often limited, the concept of capital should reflect all forms of capital that enable production, whether for trade or selfsustaining purposes. This may include land, property, physical and human capital, with health a form of

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capital in extreme cases where sufficient health services and food accessibility are not guaranteed [Dasgupta, 1997]. With agricultural work often prevalent in low-income economies, the threat of catastrophic loss events, including floods, droughts, earthquakes and disease, is of great concern, particularly under this broad definition of capital. In contrast to losses relating to health, life or death, agricultural losses can immediately eliminate a high proportion of a household's ability to produce through loss of land and livestock, irrespective of their level of capital.

In this paper, we study the behaviour of the capital of a low-income household under the assumption of proportional capital loss experience. Proportionality in loss experience captures the exposure of households of all capital levels to both catastrophic and low severity loss events. This is particularly relevant in the low-income setting, where, in addition to low frequency, high severity events such as natural disasters, commonly occurring events, such as hospital admissions and household deaths, can be detrimental. To do this, we adopt the ruin-theoretic approach proposed in Kovacevic and Pflug [2011], using a ruin-type model with deterministic growth and multiplicative losses to represent household-level capital. At loss events, accumulated capital is reduced by a random proportion of itself, rather than by an amount of random value, as in Flores-Contró et al. [2022]. Processes of this structure are typically referred to as a growth-fragmentation or growth-collapse processes, characterised by their growth in between the random collapse times at which downwards jumps occur. The randomly occurring jumps have random size dependent on the state of the process immediately before the jump.

Our aim in adopting this model is to derive the probability that a household falls below the poverty line, where this probability mimics an insurer's ruin probability. To the best of our knowledge, only Kovacevic and Pflug [2011] and Flores-Contró et al. [2022] have, so far, studied this problem in the ruin-theoretic setting. As in this earlier work, in this paper, we consider the probability in two cases, one in which the household has no insurance coverage, and the other in which they are proportionally insured. We introduce insurance to assess its effectiveness as a measure of poverty reduction. Aligning with the low-income setting, proportional coverage is assumed to be provided through an inclusive insurance product, specifically designed to cater for those excluded from traditional insurance services or without access to alternative effective risk management strategies. This type of product, targeted towards low-income populations, is commonly referred to as microinsurance. In Flores-Contró et al. [2022], the risk process with deterministic growth and random-value losses is instead used to assess the impact of government premium subsidy schemes on the probability of falling below the poverty line.

Although important, we do not consider the behaviour of a household below the poverty line. Households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Poverty trapping is a well-studied topic in development economics (the interested reader may refer to Azariadis and Stachurski [2005], Bowles et al. [2006], Kraay and McKenzie [2014], Barrett et al. [2016] and references therein for further discussion; see Matsuyama [2008] for a detailed description of the mechanics of poverty traps), however, for the purpose of this study, we use the term "trapping" only to describe the event that a household falls into poverty, focusing our interest on low-income behaviours above this critical line.

In Kovacevic and Pflug [2011], estimates of the infinite-time trapping probability of a discretised version of the capital process adopted in this paper are obtained through numerical simulation. Azaïs and Genadot [2015] perform further numerical analysis on the same model, discussing applications to the capital setting of Kovacevic and Pflug [2011] and to population dynamics, where the critical level denotes extinction. In both cases, derivation of analytical solutions of infinitesimal generator equations is not attempted. Our main contribution is therefore in the derivation of closed-form solutions of the infinitesimal generator equations associated with risk processes of this type and, in the case of proportional insurance, in the proposition of a novel approach to deriving the trapping probability recursively.

Due to the proportionality of losses, generators of the capital process no longer directly align with those of classical models used to describe the surplus process of an insurer. Obtaining the solution of the infinitesimal generator equation is therefore non-trivial. Traditionally a sum of independent random variables, random absolute losses are correlated with one another, and with the inter-arrival times of loss events. In addition, only the surplus of a household's capital above the critical capital grows exponentially. To ensure that the Lundberg equation is well-defined, and thus mitigate certain trapping, constraints on the parameters of the capital growth processes are derived. Laplace transform and derivative operators are then used to obtain the associated trapping probabilities, under no insurance coverage and proportional insurance coverage, respectively.

Research on growth-collapse processes with applications outside the field of actuarial mathematics includes Altman et al. [2002] and Löpker and Van Leeuwaarden [2008] for congestion control in data networks, Eliazar and Klafter [2004] and Eliazar and Klafter [2006] for phenomena in physical systems, Derfel et al. [2009] for cell growth and division and Peckham et al. [2018] in a model of persistence of populations subject to random shock. Aligning with the Laplace transform approach adopted in the case of no insurance, Löpker and Van Leeuwaarden [2008] obtain the Laplace transform of the transient moments of a growth-collapse process, while Eliazar and Klafter [2004] consider the state of a growth-collapse process at equilibrium, computing Laplace transforms of the system and of the high- and low-levels of the growth-collapse cycle.

Previous research on the impact of microinsurance mechanisms on the probability of falling below the poverty line from a non-ruin perspective has been undertaken through application of multi-equilibrium models and dynamic stochastic programming [Ikegami et al., 2017, Chantarat et al., 2017, Carter and Janzen, 2018, Liao et al., 2020, Janzen et al., 2021, Kovacevic and Semmler, 2021]. With the exception of the latter, each of these studies considers the impact of subsidisation and the associated cost to the subsidy provider. Will et al. [2021] and Henshaw et al. [2023] extend the problem to the group-setting, assessing the impact of risk-sharing on the probability. Will et al. [2021] undertake a simulation-based study and Henshaw et al. [2023] propose a Markov modulated stochastic dissemination model of group wealth interactions, using a bivariate normal approximation to calculate the trapping probability.

Notably, Kovacevic and Pflug [2011], Liao et al. [2020] and Flores-Contró et al. [2022] suggest that purchase of insurance and the associated need for premium payment increases the risk of falling below the poverty line for the most vulnerable. Barriers to microinsurance penetration that exist due to constraints on product affordability resulting from fundamental features of the microinsurance environment likely contribute to such observations. Limited consumer financial literacy and experience, product accessibility and data availability are examples of the unique characteristics that must be accounted for when designing effective and affordable microinsurance products. Through our analysis, we further investigate the case of proportional loss experience to assess the associated implications on the affordability of insurance.

Janzen et al. [2021] optimise the level of insurance coverage across the population, observing that those in the neighbourhood of the poverty line do not optimally purchase insurance (without subsidies), instead suppressing their consumption and mitigating the probability of falling into poverty. This aligns with the increase in probability observed in the aforementioned studies, when those closest to the poverty line purchase insurance. Similarly, Kovacevic and Semmler [2021] derive the retention rate process that maximises the expected discounted capital, by allowing adjustments in the retention rate of the policyholder after each capital loss throughout the lifetime of the insurance contract. In this paper, however, the proportion of insurance coverage and the choice to insure is fixed across the population, as in Kovacevic and Pflug [2011], Chantarat et al. [2017] and Flores-Contró et al. [2022].

An outline of the remainder of the paper is as follows. Section 2 introduces the capital growth model and its alignment with the classical Crámer-Lundberg model. This connection enables derivation of constraints on the parameters of the model that ensure the Lundberg equation is well-defined, thus preventing certain trapping. Derivation of the trapping probability for uninsured losses and $\text{Beta}(\alpha, 1)$ distributed remaining proportions of capital is presented in Section 3. The trapping probability for households covered by proportional insurance coverage is derived in Section 4 for Beta(1, 1) distributed remaining proportions of capital. The non-locality of the differential equations associated with the infinitesimal generator of the insured process is highlighted and the recursive method for deriving the trapping probability proposed. Uninsured and insured trapping probabilities are compared in Section 5 and are presented alongside additional findings of interest. Concluding remarks are provided in Section 6.

Throughout the paper, we use the term "insurance" to refer to any form of microinsurance product. Our analysis does not consider a specific type of product but can be tailored through the selection of appropriate parameters.

2 The capital model

Construction of the capital model follows that of Kovacevic and Pflug [2011]. Consider a household with accumulated capital $(X_t)_{t\geq 0}$. Under the basic assumption that the household has no loss experience, their growth in accumulated capital is given by

$$\frac{dX_t}{dt} = r \cdot \left[X_t - x^*\right]^+,\tag{1}$$

where $[x]^+ = \max(x, 0)$. The dynamics in (1) are built on the assumption that a household's income (I_t) is split into consumption (C_t) and savings or investments (S_t) , such that at time t,

$$I_t = C_t + S_t,\tag{2}$$

where consumption is an increasing function of income:

$$C_t = \begin{cases} I_t, & \text{if } I_t \le x^* \end{cases}$$
(3a)

$$I^* + a(I_t - I^*), \quad if \ I_t > x^*$$
 (3b)

for 0 < a < 1. The critical point below which a household consumes all of their income, with no facility for savings or investment, is denoted I^* . Accumulated capital is assumed to grow proportionally to the level of savings, such that

$$\frac{dX_t}{dt} = cS_t,\tag{4}$$

for 0 < c < 1, and income is generated through the accumulated capital, such that

$$I_t = bX_t,$$

for b > 0.

Combining (2), (3a), (3b) and (4) gives exactly the dynamics in (1), where the capital growth rate r = $(1-a) \cdot b \cdot c > 0$ incorporates household rates of consumption (a), income generation (b) and investment or savings (c), while $x^* = I^*/b > 0$ denotes the threshold below which a household lives in poverty. The notion of a household in this model setting may be extended for consideration of poverty trapping within economic units such as community groups, villages and tribes, in addition to the traditional household structure.

Reflecting the ability of a household to produce, the level of accumulated capital of a household X_t is composed of land, property, physical and human capital. The poverty threshold x^* represents the amount of capital required to forever attain a critical level of income below which a household would not be able to sustain their basic needs, facing elementary problems relating to health and food security. We refer to this threshold as the critical capital or the poverty line. Since (1) is positive for all levels of capital greater than the critical capital, all points less than or equal to x^* are stationary, the level of capital remains constant if the critical capital is not met. In this basic model, stationary points below the critical capital are not attractors of the system if the initial capital exceeds x^* , in which case the capital process grows exponentially with rate r.

In line with Kovacevic and Pflug [2011], we expand the dynamics of (1) under the assumption that households are susceptible to the occurrence of capital losses such as those highlighted in Section 1, including severe illness, the death of a household member or breadwinner and catastrophic events such as droughts, floods and earthquakes. The occurrence of loss events is assumed to follow a Poisson process with intensity λ , where the capital process follows the dynamics of (1) in between events. On the occurrence of the *i*-th loss, the capital process experiences a downwards jump to $X_{T_i} \cdot Z_i$, where $Z_i \in [0,1]$ is the random proportion determining the remaining capital after loss i and X_{T_i} the level of capital accumulated up to the loss time. The sequence $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common distribution function G(z), independent of the Poisson process. In this paper, the proportion of capital remaining after each loss event Z_i is assumed to follow a beta distribution with parameters $\alpha > 0$ and $\beta > 0$.

A household reaches the area of poverty if it suffers a loss large enough that the remaining capital is attracted into the poverty trap. Since a household's capital does not grow below the critical capital x^* , households that fall into the area of poverty will never escape without external help. Once below the critical capital, households are exposed to the risk of falling deeper into poverty. However, in contrast to Flores-Contró et al. [2022] where random-valued losses are considered, the dynamics of the model do not allow for the possibility of negative capital due to the proportionality of loss experience.

The structure of the process in-between loss events is derived through solution of the first order Ordinary Differential Equation (ODE) in (1). The stochastic capital process with deterministic exponential growth and multiplicative losses is then formally defined as follows:

Definition 2.1 (Kovacevic and Pflug [2011]). Let T_i be the i^{th} event time of a Poisson process $(N_t)_{t>0}$ with parameter λ , where $T_0 = 0$. Let $Z_i \ge 0$ be a sequence of independent and identically distributed random variables with distribution function G(z), independent of the process N_t . For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital X_t is defined as

$$X_{t} = \begin{cases} \left(X_{T_{i-1}} - x^{*}\right)e^{r\left(t - T_{i-1}\right)} + x^{*}, & \text{if } X_{T_{i-1}} > x^{*} \\ X_{T_{i-1}}, & \text{otherwise.} \end{cases}$$
(5a)
(5b)

$$otherwise.$$
 (5b)

At the jump times $t = T_i$, the process is given by

$$X_{T_{i}} = \begin{cases} \left[\left(X_{T_{i-1}} - x^{*} \right) e^{r \left(T_{i} - T_{i-1} \right)} + x^{*} \right] \cdot Z_{i}, & \text{if } X_{T_{i-1}} > x^{*} \\ X_{T_{i-1}} \cdot Z_{i}, & \text{otherwise.} \end{cases}$$

As in Kovacevic and Pflug [2011] and Flores-Contró et al. [2022], the aim of this paper is to study the probability that a household falls below the poverty line, i.e. the trapping probability. By Definition 2.1, the capital level of the household follows a piecewise deterministic Markov process [Davis, 1984, 2018] of compound Poisson-type, which is deterministic in-between the randomly occurring jump times at which large capital losses occur.

The infinite-time trapping probability describes the distribution of the time at which a household becomes trapped, referred to as the trapping time. Given a household has initial capital x, their trapping time, denoted τ_x , is given by

$$\tau_x := \inf \left\{ t \ge 0 : X_t < x^* | X_0 = x \right\},\$$

where τ_x is fixed at infinity if $X_t \ge x^* \ \forall t$. It then follows that the trapping probability f(x) is given by

$$f(x) = \mathbb{P}\left(\tau_x < \infty\right)$$

Analysis of the trapping probability can be undertaken through study of the infinitesimal generator. The infinitesimal generator \mathcal{A} of the stochastic process $(X_t)_{t\geq 0}$ as in Definition 2.1 is given by

$$\mathcal{A}f(x) = r(x - x^*)f'(x) + \lambda \int_0^1 [f(x \cdot z) - f(x)] dG(z),$$
(7)

for $x \ge x^*$. The remainder of the paper works towards solving $\mathcal{A}f = 0$, in line with the classical theorem of Paulsen and Gjessing [1997]. Intuitively, the boundary conditions of the trapping probability are as follows:

$$\lim_{x \to x^*} f(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0,$$
(8)

such that under the assumption that f(x) is a bounded and twice continuously differentiable function on $x \ge x^*$, with a bounded first derivative, and since we consider only what happens above the critical capital x^* , the theorem of Paulsen and Gjessing [1997] is applicable.

Closed-form expressions for Laplace transforms of ruin (trapping) probabilities are often more easily obtained than for the probability itself. However, multiplication of the initial capital by the random proportion in the integral function makes Laplace transform methods typically used in risk theory no longer straightforward. Solution of the integro-differential equation in (7) has so far only been undertaken numerically, see, for example, Kovacevic and Pflug [2011]. In this paper, closed-form trapping probabilities are obtained through solution of (7) for special cases of remaining proportions of capital.

First, note that there exists a relationship between the capital model of Definition 2.1 and the classical Crámer-Lundberg model. This enables specification of an upper bound on the trapping probability of the capital growth process X_t through Lundberg's inequality, derived in Lundberg [1926]. Consider an adjustment of the capital process that is discretised at loss event times such that $\tilde{X}_i = X_{T_i}$, i.e. the capital process studied in Kovacevic and Pflug [2011]. Taking the logarithm of the adjusted process with critical capital x^* fixed at 0 yields

$$L_{i} = L_{i-1} + r(T_{i} - T_{i-1}) + \log(Z_{i}) = \log x + rT_{i} + \sum_{i=1}^{N_{t}} \log(Z_{i}),$$
(9)

where L_i is the logarithm of the *i*-th step in the discretised process \tilde{X}_i and $\log(Z_i) < 0$. The model on the right-hand side of (9) is a version of the classical Crámer-Lundberg model introduced by Lundberg [1903] and Cramér [1930], which assumes an insurance company collects premiums continuously and pays claims of random size at random times. The corresponding surplus process is given by

$$U_t = u + ct - \sum_{k=1}^{N_t} X_k,$$

where u is the initial capital, c the constant premium rate, $X_1, X_2, ..., X_{N_t}$ the random claim sizes and N_t the number of claims in the interval [0, t]. Claim sizes are assumed to be independent and identically distributed, N_t a homogeneous Poisson process and the sequence of claim sizes $\{X_k\}_{k\in\mathbb{N}^+}$ and N_t independent.

The net profit condition is a constraint that ensures, on average, that the capital gains of a household are superior to their losses. If this condition is not satisfied then trapping is certain. It is well-known in ruin theory that if the net profit condition holds, the process U_t converges to infinity almost surely as $t \to \infty$ and there is a positive probability that $U_t \ge 0$ for all t. As a consequence of the net profit condition, it also holds that $\lim_{u\to\infty} \psi(u) = 0$, where $\psi(u)$ is the ruin probability under the classical model. However, derivation of the net profit condition from the drift of U_t to infinity is not always straightforward. The Lundberg equation provides an alternative method for deriving the net profit condition. Assume that there exists a constant R > 0 such that the process $\{e^{-RL_i}\}_{i\ge 0}$ is a martingale. The resulting equation is the Lundberg equation, and is given by

$$\mathbb{E}[e^{-R\log(Z_i)}]\mathbb{E}[e^{-Rr\tilde{T}_i}] = \mathbb{E}[e^{-R(\log(Z_i) + r\tilde{T}_i)}] = 1,$$

where $\tilde{T}_i = T_i - T_{i-1}$ and the unique solution R is the adjustment coefficient. Thus, for R to exist, it must hold that $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$. In fact, for R to exist the net profit condition must hold. As such, the existence of R ensures that $\lim_{u\to\infty} \psi(u) = 0$.

Then, if $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$, the logarithmic process in (9) converges to infinity almost surely, and

$$\lim_{\log x \to \infty} \mathbb{P}(L_i < 0 | L_0 = \log x) = 0.$$

Since $\log x \to \infty$ implies $x \to \infty$ it holds that

$$\lim_{x \to \infty} f(x) \sim \lim_{x \to \infty} f(x|x^* = 0) \le \lim_{x \to \infty} \mathbb{P}(X_t < 1|X_0 = x) = \lim_{x \to \infty} \mathbb{P}(L_i < 0|L_0 = \log x) = 0,$$

where we have applied the equivalence of \tilde{X}_i and X_t at loss event times and the fact that asymptotically, the behaviour of the trapping probability f(x) remains unchanged for any x^* . The upper boundary condition in (8) therefore holds if $\mathbb{E}[\log(Z_i) + r\tilde{T}_i] > 0$.

In Sections 3 and 4 we use the net profit condition to derive constraints on the parameters of the capital model for uninsured and proportionally insured households, respectively. The closed-form trapping probabilities are then derived through consideration of the associated infinitesimal generators for uninsured losses with Beta(α , 1) distributed remaining proportions of capital (Section 3) and proportionally insured losses with Beta(1, 1) distributed remaining proportions of capital (Section 4). Laplace transform methods are applied in Section 3 and a derivative approach in Section 4, where a solution of the infinitesimal generator equation is derived recursively.

3 Derivation of trapping probability under no insurance coverage

Under the assumption of remaining proportions of capital with distribution $Z_i \sim \text{Beta}(\alpha, 1)$, letting $u = x \cdot z$ reduces the infinitesimal generator of the capital growth process in (7) to

$$\mathcal{A}f(x) = r(x - x^*)f'(x) - \lambda f(x) + \frac{\lambda\alpha}{x^{\alpha}} \int_0^x f(u)u^{\alpha - 1}du,$$
(10)

for $x \ge x^*$.

Proposition 3.1. Consider a household capital process as proposed in Definition 2.1 with initial capital $x \ge x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$. The adjustment coefficient of the corresponding Lundberg equation exists if

$$\frac{\lambda}{r} < \alpha. \tag{11}$$

Proof. For remaining proportions of capital with distribution $\text{Beta}(\alpha, 1)$, given that Z_i and \tilde{T}_i are independent and since $\mathbb{E}[\log(Z_i)] = \alpha \int_0^1 \log(z) z^{\alpha-1} dz$, $\mathbb{E}[\log(Z_i) + r\tilde{T}_i]$ holds if and only if (11) is satisfied, as required. \Box

As λ specifies the number of claims per unit time, accounting for the fact that the mean loss size under Beta $(\alpha, 1)$ distributed remaining proportions of capital is $1 - (\alpha + 1)^{-1}$, the ratio of capital loss to capital growth is $\lambda \alpha/(r(\alpha + 1))$.

We now derive the trapping probability through solution of $\mathcal{A}f(x) = 0$ in line with the discussion of Section 2. Since households face certain trapping if the net profit condition is violated, our analysis focuses only on the region for which (11) holds.

Proposition 3.2. Consider a household capital process as proposed in Definition 2.1 with initial capital $x \ge x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital with distribution $Beta(\alpha, 1)$. The closed-form trapping probability is given by

$$f(x) = \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\lambda}{r}\right)\Gamma\left(\alpha - \frac{\lambda}{r} + 1\right)} \left(\frac{x}{x^*}\right)^{\frac{\lambda}{r} - \alpha} {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; \alpha - \frac{\lambda}{r} + 1; \frac{x^*}{x}\right)$$
(12)

for $\frac{\lambda}{r} < \alpha$, where $_2F_1(\cdot)$ is the Gauss hypergeometric function.

Proof. Fix $\mathcal{A}f(x) = 0$ and take the Laplace transform, where the infinitesimal generator of the process for $x \leq x^*$ is zero. Then, denoting $F(s) := \int_0^\infty f(x) e^{-sx} ds$,

$$s^{2}F^{(\alpha+1)}(s) + s\left(\left(\alpha+1+\frac{\lambda}{r}\right) + x^{*}s\right)F^{(\alpha)}(s) + \alpha\left(x^{*}s+\frac{\lambda}{r}\right)F^{(\alpha-1)}(s) = 0,$$
(13)

where $F^{(n)}$ denotes the *n*-th derivative of *F*. Letting $y(s) = F^{(\alpha-1)}(s)$, such that $y'(s) = F^{(\alpha)}(s)$ and $y''(s) = F^{(\alpha+1)}(s)$, and substituting $y(s) = s^{-\alpha}w(s)$ reduces (13) to the second-order ODE

$$sw''(s) + \left(\left(1 + \frac{\lambda}{r} - \alpha\right) + x^*s\right)w'(s) = 0,$$

which solves to give

$$F^{(\alpha-1)}(s) = C_1 x^{*\left(\frac{\lambda}{r}-\alpha\right)} s^{-\alpha} \gamma\left(\alpha - \frac{\lambda}{r}, x^*s\right) + C_2 s^{-\alpha},\tag{14}$$

where $\frac{\lambda}{r} < \alpha$.

Since $F'(s) = -\mathcal{L}(xf(x))$ it is possible to prove by induction that $F^{(n)}(s) = (-1)^n \mathcal{L}(x^n f(x))$. As such, application of the inverse Laplace transform to (14), see, for example, Section (3.10) of Prudnikov et al. [1992]), gives that the general solution of $\mathcal{A}f(x) = 0$ for $\mathcal{A}f(x)$ in (10) is

$$f(x) = \begin{cases} C_2 \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} + C_1 x^{*\left(\frac{\lambda}{r} - \alpha\right)} \frac{\Gamma\left(\alpha - \frac{\lambda}{r}\right)}{\Gamma(\alpha)} (-1)^{1-\alpha}, & 0 < x < x^* \\ C_2 \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} + C_1 \frac{\left(\alpha - \frac{\lambda}{r}\right)^{-1}}{\Gamma\left(\frac{\lambda}{r}\right)} (-1)^{1-\alpha} x^{\frac{\lambda}{r} - \alpha} {}_2F_1\left(\alpha - \frac{\lambda}{r}, 1 - \frac{\lambda}{r}; \alpha - \frac{\lambda}{r} + 1; \frac{x^*}{x}\right), & x^* < x, \end{cases}$$

for $\operatorname{Re}(-\lambda/r) < 1$ and $\operatorname{Re}(\alpha, x^*)$, $\operatorname{Re}(s) > 0$.

Applying the boundary conditions on f(x) in (8) yields

$$C_2 = 0$$
 and $C_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \frac{\lambda}{r})} (-1)^{\alpha - 1} x^{*(\alpha - \frac{\lambda}{r})},$

such that the closed-form trapping probability is given by (12), as required. The hypergeometric series corresponding to the solution in (12) has domain of convergence $|x^*/x| < 1$, such that the solution converges for all levels of capital in the domain of f(x).

Corollary 3.1. The closed-form trapping probability in (12) is equivalent to

$$f(x) = 1 - \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\lambda}{r}+1\right)\Gamma\left(\alpha-\frac{\lambda}{r}\right)} \left(1 - \frac{x^*}{x}\right)^{\frac{\lambda}{r}} {}_2F_1\left(\frac{\lambda}{r}, 1 + \frac{\lambda}{r} - \alpha; 1 + \frac{\lambda}{r}; 1 - \frac{x^*}{x}\right)$$
(16)

for $\frac{\lambda}{r} < \alpha$, where $_2F_1(\cdot)$ is the Gauss hypergeometric function.

Proof. Apply the hypergeometric transform:

$${}_{2}F_{1}(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} {}_{2}F_{1}\left(a,a-c+1;a+b-c+1;1-\frac{1}{z}\right) \\ + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} z^{a-c} {}_{2}F_{1}\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right),$$

which holds for $|\arg z| < \pi$ and $|\arg(1-z)| < \pi$, to (12), where we extend the gamma function to negative non-integer values by the relation

$$\Gamma(x) := \frac{1}{x}\Gamma(x+1),$$

for $x < 0, x \notin \mathbb{Z}$. The two series corresponding to the resulting hypergeometric solutions have domain of convergence $|1 - x/x^*| < 1$, such that the solutions diverge where $x > 2x^*$. Applying the relation

$$_{2}F_{1}(a,b;c;z) = (1-z)^{c-a-b} {}_{2}F_{1}(c-a,c-b;c;z)$$

and transforming via the formula

$$_{2}F_{1}(a,b;c;z) = (1-z)^{-a} {}_{2}F_{1}\left(a,c-b;c;\frac{z}{z-1}\right)$$

adjusts the domain of convergence of the corresponding hypergeometric series to $|1 - x^*/x| < 1$, such that the solution converges over all $x > x^*$, and gives (16), as required.

For details of the hypergeometric solutions, their relation and transformation formulas and domains of convergence that are used throughout this paper, see Abramowitz and Stegun [1972] and Kristensson [2010].

Remark 3.1. Substitution of $\alpha = 1$ into (12), or equivalently (16), yields the closed-form trapping probability under uniformly distributed remaining proportions of capital, i.e. the case $Z_i \sim Beta(1, 1)$.

The closed-form trapping probability for households susceptible to proportional losses with $\text{Beta}(\alpha, 1)$ distributed remaining proportions of capital, as derived in Proposition 3.2, is presented in Figure 1a for varying initial capital x and shape parameter α . Note that the trapping probability tends to 1 as λ/r tends to α in line with the constraint of Proposition 3.1. The low value of the rate parameter λ reflects the vulnerability of low-income households to both high and low frequency loss events, while aligning with the constraint in Proposition 3.1. Increasing α increases the mean of the distribution of the remaining proportion of capital. Observation of a decreasing trapping probability with increasing α is therefore intuitive and aligns with the reduction in loss. Figure 1b presents the same trapping probability for varying loss frequency λ and fixed $\alpha = 1$. In this case, remaining proportions of capital are uniformly distributed as in Section 4. Increasing the frequency of loss events increases the trapping probability, as is to be expected. Parameters a, b and c are selected to correspond with those in Flores-Contró et al. [2022].

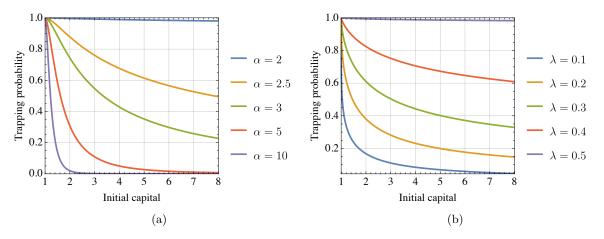


Figure 1: Trapping probability f(x) in (12) for $x^* = 1$ and $Z_i \sim \text{Beta}(\alpha, 1)$, considering: (a) $\lambda = 1$ and different values of α , (b) $\alpha = 1$ and different values of λ . The value of the capital growth rate r = 0.504 is computed with a = 0.1, b = 1.4, c = 0.4.

Particularly high levels of accumulated capital are not relevant in the microinsurance and poverty trapping context. However, the asymptotic behaviour of the analytic trapping probability at infinity is interesting for understanding the behaviour of the function. Since $\lim_{z\to 0} {}_2F_1(a,b;c;z) = 1$, (12) behaves asymptotically like the power function

$$\frac{\Gamma\left(\alpha\right)}{\Gamma\left(\frac{\lambda}{r}\right)\Gamma\left(\alpha-\frac{\lambda}{r}+1\right)}\left(\frac{x}{x^{*}}\right)^{\frac{\lambda}{r}-\alpha},\tag{17}$$

such that the uninsured trapping probability has power-law asymptotic decay as $x \to \infty$.

We now compare the decay of the household-level trapping probability under proportional losses and no insurance coverage with that of the exponentially distributed random-valued loss case of Flores-Contró et al. [2022]. The equivalent uninsured trapping probability under random-valued losses for $x \ge x^*$ is given by

$$f(x) = \frac{\Gamma\left(\frac{\lambda}{r}; \mu(x - x^*)\right)}{\Gamma\left(\frac{\lambda}{r}\right)},\tag{18}$$

where $\Gamma(a;z)$ is the upper incomplete gamma function: $\Gamma(a;z) := \int_{z}^{\infty} e^{-t} t^{a-1} dt$. The probability in (18) follows

$$\mu^{\frac{\lambda}{r}-1}(x-x^*)^{\frac{\lambda}{r}-1}e^{-\mu(x-x^*)}(1+\mathcal{O}(|\mu(x-x^*)|^{-1}))$$
(19)

asymptotically, where μ is the exponential loss parameter. The limiting behaviour of the ratio of (19) to (17) is

$$Cx^{\alpha-1}e^{-\mu(x-x^*)}(1+\mathcal{O}(|\mu(x-x^*)|^{-1})),$$

for constant $C = x^{*\lambda/r-\alpha} \mu^{\lambda/r-1} \Gamma(\lambda/r) \Gamma(\alpha - \lambda/r + 1) \Gamma(\alpha)^{-1}$. The trapping probability in the randomvalued case therefore decays at a faster rate than when a household experiences proportional losses, with the severity of this difference dependent on the parameters of the loss distributions. This result is intuitive, since proportional losses are more risky than random-valued losses at high capital levels due to the non-zero probability of a household losing all (or a high proportion) of its wealth. This is particularly severe in the uniform case of the following section, where high and low levels of proportional losses are equally likely. When $\alpha = 1$, the trapping probability in the random-valued case decays exponentially faster than in the proportional case. A comparison of the decay of the trapping probability under proportional losses against that of randomvalued losses is provided in the inset of Figure 2, where the probabilities are plotted on the logarithmic scale. Here, the slower rate of decay under proportional losses is clearly observable.

Figure 2 compares trapping probabilities under proportional (12) and random-valued (18) losses for a given set of parameters. Trapping probabilities for a number of exponential claim size distributions are compared with the trapping probability under proportional losses with an expected value of approximately 16.7% of accumulated capital. For random-valued claim sizes with an expected value of 0.5 ($\mu = 2$) the trapping probability is greater than for proportional losses for the most vulnerable, however, as capital increases the trapping probability under proportional losses exceeds the random-valued case. If the expected claim size increases to 1 ($\mu = 1$) the trapping probability for proportional losses is significantly lower than in the randomvalued case at all levels of initial capital. Compared to the mean loss associated with beta distributed remaining proportions with $\alpha = 5$, an expected claim size of 1 is low with respect to high levels of initial capital. For x = 6 the two loss rates coincide. This therefore suggests that for equivalent loss size, the trapping probability for proportional losses is reduced in comparison to random-valued losses. However, for capital levels below this point random-valued losses account for a greater proportion of capital than the proportional loss case selected for comparison and thus the increased trapping probability is intuitive. Further analysis would be needed to validate the consistency in the reduction of the probability for equivalent losses.

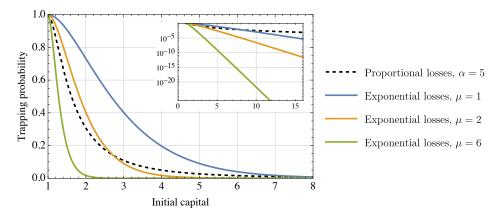


Figure 2: Comparison between the trapping probability f(x) in (18) for random-valued losses with distribution $\text{Exp}(\mu)$ for different values of μ and the trapping probability f(x) in (12) for proportional losses with distribution Beta(5,1), with parameters $x^* = 1$, $\lambda = 1$ and r = 0.504 computed with a = 0.1, b = 1.4, c = 0.4. The inset presents the same curves on the logarithmic scale and on a wider domain.

4 Derivation of trapping probability under proportional insurance coverage

In line with Kovacevic and Pflug [2011] and Flores-Contró et al. [2022], in this section, we extend the model under the assumption that capital losses are covered by a proportional insurance product. Consider the presence of a fixed premium insurance product that covers $100 \cdot (1 - \kappa)$ percent of all household losses, where $1 - \kappa$ for $\kappa \in (0, 1]$ is the proportionality factor. Assume that coverage is purchased by all households. Under proportional insurance coverage, the critical capital (or poverty line) and capital growth rate associated with an insured household must account for the need for premium payments. As such, define

$$r(\kappa,\lambda,\theta) = (1-a) \cdot (b - \pi(\kappa,\lambda,\theta)) \cdot c \quad \text{and} \quad x^*(\kappa,\lambda,\theta) = \frac{I^*}{b - \pi(\kappa,\lambda,\theta)},$$
(20)

where $\pi(\kappa, \lambda, \theta)$ is the premium rate and is calculated according to the expected value principle:

$$\pi(\kappa, \lambda, \theta) = (1 + \theta) \cdot (1 - \kappa) \cdot \lambda \cdot \mathbb{E}[1 - Z_i].$$

For ease of presentation, throughout the remainder of the paper we denote the capital growth rate $r := r(\kappa, \lambda, \theta)$ and the critical capital $x^* := x^*(\kappa, \lambda, \theta)$.

Parameters a, b and c are household rates of consumption, income generation and investment or savings as defined in Section 2 and the parameter θ is the loading factor specified by the insurer. We assume that these parameters, and the critical income I^* , are not changed by the introduction of insurance. However, due to the need for premium payments, the critical capital in the insured case is greater than that of an uninsured household, while the capital growth rate is reduced.

The associated capital growth process has an analogous structure to that of Definition 2.1, with the remaining proportion of capital after each loss event instead denoted Y_i , where $Y_i = 1 - \kappa(1 - Z_i) \in [1 - \kappa, 1]$. As such, in between loss events, where $T_{i-1} \leq t < T_i$, the capital growth process follows (5a) and (5b). At event times $t = T_i$, the process is given by

$$X_{T_{i}} = \begin{cases} \left[\left(X_{T_{i-1}} - x^{*} \right) e^{r \left(T_{i} - T_{i-1} \right)} + x^{*} \right] \cdot Y_{i}, & \text{if } X_{T_{i-1}} > x^{*} \\ X_{T_{i-1}} \cdot Y_{i}, & \text{otherwise.} \end{cases}$$
(21a)
(21b)

Note that for $\kappa = 1$, the capital model in (21a) and (21b) and the parameters r and x^* exactly correspond to those of an uninsured household, as discussed in Section 3.

Proposition 4.1. Consider a household capital process defined by (5a) and (5b) in between loss events and by (21a) and (21b) at loss event times, with coverage proportionality factor $1 - \kappa \in (0, 1]$. For initial capital $x \ge x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital Z_i with distribution $Beta(\alpha, 1)$, the adjustment coefficient of the corresponding Lundberg equation exists if

$$\frac{r}{\lambda} > \frac{\kappa}{\alpha(\alpha+1)(1-\kappa)} {}_2F_1\left(1,\alpha+1;\alpha+2;-\frac{\kappa}{1-\kappa}\right),\tag{22}$$

where $_2F_1(\cdot)$ is the Gauss hypergeometric function.

Proof. The condition that must hold for the adjustment coefficient R to exist under proportional insurance coverage, and thus for the net profit condition to be satisfied, is

$$\mathbb{E}[r\tilde{T}_i + \log(1 - \kappa(1 - Z_i))] > 0 \iff \mathbb{E}[\log(1 - \kappa(1 - Z_i))] > -\frac{r}{\lambda}.$$

For $Z_i \sim \text{Beta}(\alpha, 1)$, using integration by parts,

$$\mathbb{E}[\log(1-\kappa(1-Z_i))] = -\frac{\kappa}{\alpha} \int_0^1 (1-\kappa+\kappa z)^{-1} z^\alpha dz,$$

the right-hand side of which is the integral representation of a Gauss hypergeometric function, giving exactly (22), as required.

Remark 4.1. For $Z_i \sim Beta(1,1)$, the constraint for existence of the adjustment coefficient reduces to

$$\frac{r}{\lambda} > 1 + \frac{1-\kappa}{\kappa} \ln(1-\kappa).$$
(23)

The constraint on λ in (23) is presented in Figure 3a for varying θ and Figure 3b for varying α . Note that the sensitivity of the constraint to the loading factor θ increases for decreasing κ and thus increasing insurance coverage. In the experiments considered in Figure 3b, the constraint is bounded above by the uniform case, where $\alpha = 1$. This indicates that the parameter region in which certain trapping is prevented is greater for uniformly distributed remaining proportions of capital. In a similar manner, 3a implies that lowering the loading factor θ increases the region in which certain trapping is prevented when remaining proportions are uniformly distributed.

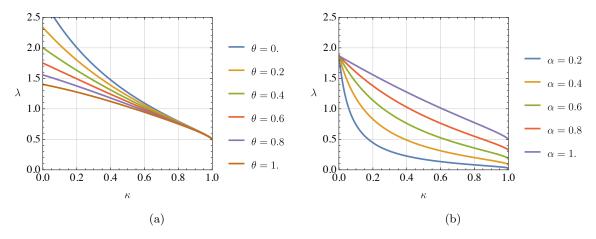


Figure 3: Upper boundary of the region defined by the constraint on λ/r in (23) for a = 0.1, b = 1.4, c = 0.4 with (a) fixed $\alpha = 1$ and different values of θ and (b) fixed $\theta = 0.5$ and different values of α .

Remark 4.2. For $\kappa = 1$, since ${}_2F_1(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)/(\Gamma(c-a)\Gamma(c-b))$, applying the identity

$$_{2}F_{1}(a,b;c;z) = (1-z)^{-a} {}_{2}F_{1}\left(a,c-b;c;\frac{z}{z-1}\right),$$

(22) reduces to the uninsured constraint in (11).

We approach the derivation of the trapping probability of the insured process in a manner analogous to that described in Section 2, noting the adjustment in the domain of the random variable capturing the remaining proportion of capital. The infinitesimal generator corresponding to the capital process in (5a), (5b), (21a) and (21b) is given by

$$\mathcal{A}f(x) = r(x - x^{*})f'(x) + \lambda \int_{1-\kappa}^{1} [f(x \cdot y) - f(x)] d\tilde{G}(y),$$
(24)

where $\tilde{G}(y) = G(1 - (1 - y)/\kappa)$ is the distribution function of Y_i . To derive the trapping probability under proportional insurance coverage we consider only the case $\alpha = 1$, i.e. $Z_i \sim \text{Beta}(1,1)$, where remaining proportions of capital are uniformly distributed and $d\tilde{G}(y) = dy/\kappa$.

Solution of $\mathcal{A}f(x) = 0$ is again sought to obtain the trapping probability of the insured process, where f(x) is assumed to be a bounded and twice continuously differentiable function on $x \ge x^*$ with a bounded first derivative and boundary conditions as in (8). Using equivalent arguments to those presented in the discussion of the net profit condition in Section 2, if (23) is satisfied the boundary condition $\lim_{x\to\infty} f(x) = 0$ holds. Households face certain trapping if the net profit condition is violated, therefore our analysis focuses only on the region in which (23) holds.

Taking the derivative of the infinitesimal generator equation $\mathcal{A}f(x) = 0$ with respect to x for $\mathcal{A}f(x)$ in (24) with $d\hat{G}(y) = dy/\kappa$, yields

$$(x - x^*) f''(x) + \left(1 - \frac{\lambda}{r}\right) f'(x) + \frac{\lambda}{\kappa} f(x) = \frac{\lambda(1 - \kappa)}{\kappa} f\left((1 - \kappa)x\right)$$

As such, even in the simple case of uniformly distributed remaining proportions of capital, application of the differential operator induces a non-local term in the resulting differential equation. When taking the Laplace transform of $\mathcal{A}f(x) = 0$, as in Section 3, a non-local differential equation is also obtained. Derivation of the trapping probability is therefore highly intractable when adopting classical approaches.

The non-locality is caused by the lower integral limit in (24). To overcome this, consider the following. If y is such that $x \cdot y \leq x^*$ then $f(x \cdot y)$ is known. In fact, for all $y \in [1 - \kappa, x^*/x]$ trapping occurs with the first loss, such that $f(x \cdot y) = 1$. For y in this interval, the integral in (24) is trivial. Exploiting this observation, we redefine the infinitesimal generator as a piecewise function with boundary at $x = x^*/(1 - \kappa)$, where $1 - \kappa$ is the lower bound of Y_i . In this way, for $x > x^*(1 - \kappa)$ a household cannot become trapped by the first loss for any realisation of Y_i . We therefore obtain a piecewise IDE that can be solved in a standard manner for $x < x^*(1 - \kappa)$, but for $x > x^*(1 - \kappa)$ the problem of non-locality remains. Our approach, as described below, partitions the domain of f(x) into subintervals such that the solution of $\mathcal{A}f(x) = 0$ for x in any given subinterval is informed by the solution in the previous subinterval. We begin by considering the two fundamental subintervals, divided where $x = x^*/(1 - \kappa)$.

The behaviour of the capital process above the critical capital x^* determines a household's trapping probability, with only surplus capital above the critical capital growing exponentially. Thus, additionally consider the change of variable $h(x) = f(x + x^*)$ for x > 0. Then, for $\tilde{x} = x - x^* > 0$, the piecewise infinitesimal generator $\mathcal{A}h(\tilde{x})$ is given by

$$\begin{cases} r(\tilde{x} + x^*)\tilde{x}h'(\tilde{x}) - \lambda(\tilde{x} + x^*)h(\tilde{x}) + \frac{\lambda}{\kappa} \int_{(\tilde{x} + x^*)(1-\kappa)}^{\tilde{x} + x^*} h(u - x^*)du, & \tilde{x} > \frac{x^*\kappa}{1-\kappa} \end{cases}$$
(25a)

$$\left(r(\tilde{x}+x^*)\tilde{x}h'(\tilde{x})-\lambda(\tilde{x}+x^*)h(\tilde{x})+\frac{\lambda}{\kappa}\int_{x^*}^{\tilde{x}+x^*}h(u-x^*)du+\lambda x^*-\frac{\lambda\tilde{x}(1-\kappa)}{\kappa},\quad \tilde{x}<\frac{x^*\kappa}{1-\kappa},\quad (25b)\right)$$

where the subintervals on the domain of \tilde{x} have interface at $x = x^*(1 - \kappa)$. Under this change of variable and assuming r/λ satisfies (23), the trapping probability satisfies $\mathcal{A}h(\tilde{x}) = 0$, with boundary conditions:

$$\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1 \quad \text{and} \quad \lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0.$$

For this purpose, we consider the derivative of the piecewise IDE in (25a) and (25b). Fixing $Ah(\tilde{x}) = 0$ and taking the derivative with respect to \tilde{x} gives

$$\tilde{x}(\tilde{x}+x^*)h''(\tilde{x}) + \left(\left(2-\frac{\lambda}{r}\right)\tilde{x}+x^*\left(1-\frac{\lambda}{r}\right)\right)h'(\tilde{x}) + \frac{\lambda(1-\kappa)}{r\kappa}h(\tilde{x}) = \frac{\lambda(1-\kappa)}{r\kappa}h((1-\kappa)\tilde{x}-x^*\kappa), \quad (26)$$

for $\tilde{x} > x^* \kappa / (1 - \kappa)$ and

$$\tilde{x}(\tilde{x}+x^*)h''(\tilde{x}) + \left(\left(2-\frac{\lambda}{r}\right)\tilde{x}+x^*\left(1-\frac{\lambda}{r}\right)\right)h'(\tilde{x}) + \frac{\lambda(1-\kappa)}{r\kappa}h(\tilde{x}) = \frac{\lambda(1-\kappa)}{r\kappa}$$
(27)

for $\tilde{x} < x^* \kappa / (1 - \kappa)$, where, as mentioned, we observe the non-local term $h((1 - \kappa)\tilde{x} - x^*\kappa)$ for $\tilde{x} > x^*\kappa / (1 - \kappa)$.

First consider the homogeneous parts of (26) and (27), noting their equivalence. Under the change of variable $m(z) := h(\tilde{x})$, where $z = -\tilde{x}/x^*$, this homogeneous differential equation is exactly Gauss' hypergeometric differential equation:

$$z(z-1)m''(z) + \left(\left(2-\frac{\lambda}{r}\right)z-1+\frac{\lambda}{r}\right)m'(z) + \frac{\lambda(1-\kappa)}{r\kappa}m(z) = 0,$$
(28)

with known solutions. We construct a general solution of (28) of the following form:

$$m(z) = C_1 (1-z)^{-a_1} {}_2F_1\left(a_1, c_1 - b_1, c_1; \frac{z}{z-1}\right) + C_2 z^{1-c_1} (1-z)^{c_1-a_1-1} {}_2F_1\left(1+a_1-c_1, 1-b_1, 2-c_1; \frac{z}{z-1}\right),$$

where $_{2}F_{1}(\cdot)$ is the Gauss hypergeometric function and

$$a_1 + b_1 = c_1$$
, $a_1 \cdot b_1 = \frac{\lambda(1 - \kappa)}{r\kappa}$ and $c_1 = 1 - \frac{\lambda}{r}$.

The parameters a_1 and b_1 are complex conjugates:

$$a_1 = \frac{1}{2}\left(1 - \frac{\lambda}{r}\right) \pm \frac{1}{2}\sqrt{\left(1 + \frac{\lambda}{r}\right)^2 - \frac{4\lambda}{r\kappa}} \quad \text{and} \quad b_1 = \frac{1}{2}\left(1 - \frac{\lambda}{r}\right) \mp \frac{1}{2}\sqrt{\left(1 + \frac{\lambda}{r}\right)^2 - \frac{4\lambda}{r\kappa}}$$

with positive real part where $\lambda/r < 1$.

Returning to the inhomogeneous differential equations in (26) and (27), let

$$\mathcal{L} = r(\tilde{x})\frac{d^2}{d\tilde{x}^2} + p(\tilde{x})\frac{d}{d\tilde{x}} + q(\tilde{x}),$$

where $r(\tilde{x}) = \tilde{x}(\tilde{x} + x^*)$, $p(\tilde{x}) = (2 - \lambda/r)\tilde{x} + x^*(1 - \lambda/r)$ and $q(\tilde{x}) = \lambda(1 - \kappa)/r\kappa$, denote the linear, second order operator for which

$$u(\tilde{x}) = \left(1 + \frac{\tilde{x}}{x^*}\right)^{-a_1} {}_2F_1\left(a_1, a_1, 1 - \frac{\lambda}{r}; \frac{\tilde{x}}{\tilde{x} + x^*}\right),$$

$$v(\tilde{x}) = \left(\frac{\tilde{x}}{\tilde{x} + x^*}\right)^{\frac{\lambda}{r}} \left(1 + \frac{\tilde{x}}{x^*}\right)^{-a_1} {}_2F_1\left(\frac{\lambda}{r} + a_1, \frac{\lambda}{r} + a_1, 1 + \frac{\lambda}{r}; \frac{\tilde{x}}{\tilde{x} + x^*}\right),$$
(29)

forms the fundamental solution set, and let

$$G(\tilde{x}, x') = \frac{u(x')v(\tilde{x}) - u(\tilde{x})v(x')}{r(x')W(x')}$$
(30)

be the Green's function corresponding to \mathcal{L} , where W(x) = u(x)v'(x) - u'(x)v(x) is the Wronskian of u and v. The hypergeometric series corresponding to the functions $u(\tilde{x})$ and $v(\tilde{x})$ are well-defined on the domain $|\tilde{x}/(\tilde{x}+x^*)| < 1|$. Since this holds for all $\tilde{x} > 0$, the functions are well-defined over the whole domain. The system in (25a) and (25b) that is to be solved can therefore be characterised as follows:

$$\mathcal{L}[h](\tilde{x}) = \begin{cases} \frac{\lambda(1-\kappa)}{r\kappa} h((1-\kappa)\tilde{x} - x^*\kappa), & \tilde{x} > \frac{x^*\kappa}{1-\kappa} \\ \frac{\lambda(1-\kappa)}{r\kappa}, & \tilde{x} < \frac{x^*\kappa}{1-\kappa}. \end{cases}$$
(31a)

Now, let the surplus of capital above the critical capital $\tilde{x} \in [0, \infty)$ be separated into subintervals $I_j = [\tilde{x}_j, \tilde{x}_{j+1}]$, where $\{\tilde{x}_j\}_{j \in \mathbb{N}_0}$ is an increasing sequence and $\tilde{x}_0 = 0$. Define a set of kernels recursively by

$$\begin{cases} g_1(x,s_1) = G(x,s_1) \\ g_{j+1}(x,s_1,...,s_{j+1}) = G(x,s_{j+1})g_j(l(s_{j+1}),s_1,...,s_j), \end{cases}$$

for $j \ge 1$. Then, the following theorem holds, where the proposition of a solution of the type (33) is informed by the solution of (27). **Theorem 4.1.** Consider a household capital process defined by (5a) and (5b) in between loss events and by (21a) and (21b) at loss event times, with coverage proportionality factor $1 - \kappa \in (0, 1]$. Assume initial capital x such that $\tilde{x} \ge 0$, capital growth rate r and loss intensity $\lambda > 0$ such that λ/r satisfies (23), and remaining proportions of capital with distribution Beta(1, 1). Then, a solution of $Ah(\tilde{x}) = 0$ for the infinitesimal generator $Ah(\tilde{x})$ in (25a) and (25b), that satisfies $\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1$, is given by the piecewise function

$$h(\tilde{x}) = 1 + Ay_j(\tilde{x}), \quad x \in I_j \tag{33}$$

for any constant A, where the functions $y_j(\tilde{x})$ are defined for $\tilde{x} \geq \tilde{x}_j$ and are given by the recursion:

$$y_0(\tilde{x}) = v(\tilde{x})$$
(34a)
(34a)

$$\begin{cases} y_{j+1}(\tilde{x}) = y_j(\tilde{x}) + c^{j+1} \int_{\tilde{x}_{j+1}}^x \int_{\tilde{x}_j}^{\iota(s_{j+1})} \cdots \int_{\tilde{x}_1}^{\iota(s_2)} g_{j+1}(\tilde{x}, s_1, \dots, s_{j+1}) v(l(s_1)) ds_1 \cdots ds_{j+1}, \tag{34b} \end{cases}$$

where $c = \lambda(1-\kappa)/r\kappa$, $\tilde{x}_{j+1} = (\tilde{x}_j + x^*\kappa)/(1-\kappa)$ and $l(x) = (1-\kappa)x - x^*\kappa$.

Proof. First consider the integro-differential equation for the solution in the first interval $I_0 = [\tilde{x}_0, \tilde{x}_1]$ given in (25b), where we define \tilde{x}_0 and \tilde{x}_1 to be the lower and upper limits of the first interval, namely 0 and $x^* \kappa/(1-\kappa)$, respectively. Proposing an Ansatz $h_p(\tilde{x}) = C$ for the particular solution yields C = 1, such that the general solution of $h(\tilde{x})$ for $\tilde{x} \in I_0$ is exactly

$$h(\tilde{x}) = C_1 u(\tilde{x}) + C_2 v(\tilde{x}) + 1$$

The lower boundary condition for $h(\tilde{x})$ in this interval: $\lim_{\tilde{x}\to 0} h(\tilde{x}) = 1$, then holds if and only if $C_1 = 0$. Letting $A = C_2$ and $y_0(\tilde{x}) = v(\tilde{x})$, $h(\tilde{x}) = 1 + Ay_0(\tilde{x})$ for $\tilde{x} \in I_0$, as required.

To solve in the upper part of the infinitesimal generator IDE, i.e. for intervals $I_j = [\tilde{x}_j, \tilde{x}_{j+1}]$ where $j \ge 1$, consider (25a). By the solution in the interval I_0 , $h((1 - \kappa)\tilde{x} - x^*\kappa)$ is known where

$$\tilde{x}_0 < (1-\kappa)\tilde{x} - x^*\kappa < \tilde{x}_1 \iff \tilde{x}_1 < \tilde{x} < \frac{\tilde{x}_1 + x^*\kappa}{1-\kappa}.$$

As such, letting $\tilde{x}_2 := (\tilde{x}_1 + x^* \kappa)/(1-\kappa)$, a solution for (25a) can be obtained in the interval $I_1 = [\tilde{x}_1, \tilde{x}_2]$. In fact, for any interval I_{j+1} , a solution can be determined by observing the value of the function in the previous interval, since $h((1-\kappa)\tilde{x} - x^*\kappa)$ for $\tilde{x} > \tilde{x}_{j+1}$ is known, up to a point, by the solution in I_j . It is simple to prove by induction that the upper limit of the *j*-th interval is given by

$$\tilde{x}_{j+1} = \frac{\tilde{x}_j + x^* \kappa}{1 - \kappa}.$$
(35)

Suppose that $\forall \tilde{x} \in I_j$ for $j \ge 1$, $\tilde{y}_j(\tilde{x}) = h(\tilde{x}) = 1 + Ay_j(\tilde{x})$. Then, by (31a), it must hold that

$$\mathcal{L}[\tilde{y}_{j+1}](\tilde{x}) = \frac{\lambda(1-\kappa)}{r\kappa} \tilde{y}_j((1-\kappa)\tilde{x} - x^*\kappa) \iff \mathcal{L}[y_{j+1}](\tilde{x}) = cy_j(l(\tilde{x}))$$
(36)

 $\forall \tilde{x} \geq \tilde{x}_{j+1}$, denoting $c = \lambda(1-\kappa)/r\kappa$ and $l(x) = (1-\kappa)x - x^*\kappa$. It therefore remains to prove that (36) holds when $y_{j+1}(\tilde{x})$ is given by the recursion in (34b). To prove by induction, consider the case j = 0:

$$\mathcal{L}[y_1](\tilde{x}) = \mathcal{L}\left[y_0(\tilde{x}) + c \int_{\tilde{x}_1}^{\tilde{x}} G(\tilde{x}, s_1) v(l(s_1)) ds_1\right].$$

By definition, $\mathcal{L}[y_0](\tilde{x}) = 0$ when y_0 is in the solution set and $\mathcal{L}[\int^{\tilde{x}} G(\tilde{x},s)\phi(s)ds] = \phi(x)$. As such,

$$\mathcal{L}[y_1](\tilde{x}) = cv(l(\tilde{x})) = cy_0(l(\tilde{x})),$$

as required. Assume (36) holds for j = k - 1. Then, $\mathcal{L}[y_k](\tilde{x}) = cy_{k-1}(l(\tilde{x}))$ for $\tilde{x} \ge \tilde{x}_k$. Finally, consider the case j = k. By (34b),

$$\mathcal{L}[y_{k+1}](\tilde{x}) = cy_{k-1}(l(\tilde{x})) + c^{k+1}\mathcal{L}\left[\int_{\tilde{x}_{k+1}}^{\tilde{x}} G(\tilde{x}, s_{k+1}) \int_{\tilde{x}_{k}}^{l(s_{k+1})} \cdots \int_{\tilde{x}_{1}}^{l(s_{2})} g_{k}(l(s_{k+1}), s_{1}, .., s_{k})v(l(s_{1}))ds_{1} \cdots ds_{k+1}\right],$$

which, by definition of the Green's function, is equivalent to

$$cy_{k-1}(l(\tilde{x})) + c^{k+1} \int_{\tilde{x}_k}^{l(\tilde{x})} \cdots \int_{\tilde{x}_1}^{l(s_2)} g_k(l(\tilde{x}), s_1, \dots, s_k) v(l(s_1)) ds_1 \cdots ds_{k+1} = cy_k(l(\tilde{x})),$$

as required.

Remark 4.3. For $\kappa = 1$, since $\lim_{\kappa \to 1} x^* \kappa / (1 - \kappa) = \infty$, the upper limit of the first subinterval $\tilde{x}_1 = \infty$. The integro-differential equation in (25b) therefore holds over the whole domain $\tilde{x} > 0$ and the solution in Theorem 4.1 reduces to $h(\tilde{x}) = 1 + Av(\tilde{x})$, the solution in the first interval I_0 . In this case, the constant A can be derived analytically such that the upper boundary condition on the trapping probability: $\lim_{\tilde{x}\to\infty} h(\tilde{x}) = 0$, holds. The resulting trapping probability is exactly that of the uninsured case in (16) of Corollary 3.1.

The characterisation of the trapping probability f(x) satisfying (24) in the case of uniformly distributed proportional losses will follow from Theorem 4.1 if it can be shown that a solution of the form (33) tends to zero as $\tilde{x} \to \infty$, in line with the upper boundary condition. Specifically, we define the piecewise function

$$y(x) = y_j(x - x^*), \quad x - x^* \in I_j$$
(37)

with y_j and I_j as in Theorem 4.1, and pose the following:

Conjecture 4.1. The limit $L := \lim_{x \to \infty} y(x)$ exists and is different than zero.

If Conjecture 4.1 holds, then (33) yields that

$$f(x) = 1 + Ay(x), \quad A = -\frac{1}{L}$$
 (38)

is the unique solution to $\mathcal{A}f(x) = 0$, $f(x^*) = 1$, and $\lim_{x\to\infty} f(x) = 0$, as desired. Numerical computation of y(x) in (37) for large x is not a trivial matter, as the functions v and G in (29) and (30), respectively, are highly oscillatory for large values of \tilde{x} . Nevertheless, our numerical experiments appear to indicate that Conjecture 4.1 holds. Moreover, if Conjecture 4.1 is assumed to hold, there exists a practical method for estimating the true value of A in (33) and for obtaining a very good approximation to f(x).

Note that, by (34a)

$$f(x) = 1 + Av(x + x^*), \quad x \in [x^*, x^*/(1 - \kappa)]$$
(39)

which is easily computed for any value of A. In addition, the process $(X_t)_{t\geq 0}$ can be simulated to obtain estimates of the trapping probability for any initial capital $x \in [x^*, x^*/(1-\kappa)]$. As such, an estimate \hat{A} for the conjectured value of A can be estimated by fitting f(x) to the simulated data. A comparison between the trapping probability estimated via f(x) in (38) and simulated data is presented in Figure 4 for a given set of parameters.

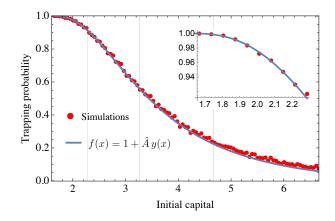


Figure 4: Comparison between the trapping probability estimated via f(x) in (38) and simulations of the capital process X_t . Each simulation point is obtained from an ensemble of 2000 realisations of $\{X_t : 0 \le t \le 500\}$ for different values of the initial capital $X_0 = x$. The vertical lines mark the subintervals $x^* + I_j$, $0 \le j \le 3$ used in the construction of y in Theorem 4.1. The estimate $\hat{A} = -3.556$ is obtained by fitting $1 + Av(x + x^*)$ to the simulated data for x in the first subinterval, as shown in the inset. Parameters used are $\lambda = 1$ and $\kappa = 0.3$. The values of $r(\kappa, \lambda, \theta)$ and $x^*(\kappa, \lambda, \theta)$ are computed via (20) with a = 0.1, b = 1.4, c = 0.4, $x^* = 1$ and $\theta = 0.5$.

The trapping probability for proportionally insured households susceptible to proportional losses with Beta(1, 1) distributed remaining proportions of capital, estimated via (38), is presented in Figure 5a for varying initial capital x and proportionality factor κ . For small values of κ and at higher subintervals, calculation of the trapping probability is highly computationally intensive. In Figures 5a and 5b, trapping probabilities are estimated for the first four subintervals, i.e. I_j for $0 \le j \le 3$. The limits of I_j in (35) are functions of κ . As such, changing the value of κ causes the trapping probability curves to terminate at different points, determined by the upper limit of I_3 , as can be observed in Figure 5a.

Note that, in Figure 5a, as κ tends to zero the trapping probability tends towards a step function. This is indicative of the fact that for $\kappa = 0$ households have full insurance coverage and do not experience loss events, inducing a trapping probability that is zero-valued for all levels of capital above the critical capital due to the restriction on the premium that ensures positive capital growth. Increasing κ and thus decreasing the level of insurance coverage intuitively causes an increase in the trapping probability. Figure 5b presents the same trapping probability for varying loss frequency λ and fixed κ , where half of every loss is insured. Increasing the frequency of loss events increases the trapping probability. For $\lambda = 0.5$, under the parameter set considered in this figure, λ/r is extremely close to one. Therefore, in the case of no insurance, households exhibiting this loss behaviour would be close to certain ruin. As presented in Figure 3, purchase of insurance eases this constraint, significantly reducing the probability of trapping. The fact that both figures presenting the estimated trapping probability are intuitive, provides further evidence for Conjecture 4.1.

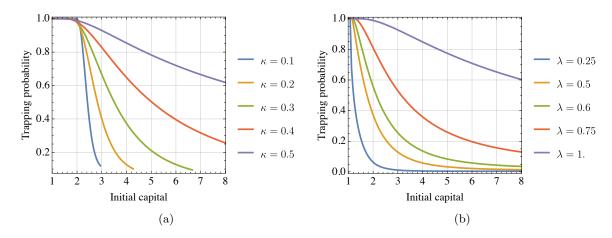


Figure 5: Estimation of the trapping probability f(x) via (38) assuming Conjecture 4.1 for (a) $\lambda = 1$ for different values of κ and (b) $\kappa = 0.5$ for different values of λ . Each curve is computed with the first three iterates of (34b) via numerical integration, the value of A is then estimated as explained in Figure 4. For each case, the values of $r(\kappa, \lambda, \theta)$ and $x^*(\kappa, \lambda, \theta)$ are computed via (20) with a = 0.1, b = 1.4, c = 0.4, $x^* = 1$ and $\theta = 0.5$ and λ is selected such that (23) holds.

5 Discussion

Figure 6 presents a comparison of trapping probabilities for the uninsured and insured capital processes as derived in (12) and (38), respectively, for two values of the parameter λ . For $\lambda = 0.25$, the insured trapping probability lies below the uninsured at almost all levels of initial capital, decaying at a much faster rate. Only for initial capital extremely close to the critical capital does the uninsured probability lie below the insured. At the higher loss frequency of $\lambda = 0.5$, the uninsured trapping probability lies close to 1 throughout the range of initial capital considered, significantly higher than the equivalent probability for insured losses at all capital levels. Note that in this case, λ/r lies close to the uninsured constraint preventing certain trapping in (11).

Sensitivity analysis on the trapping probabilities in (12) and (38) is presented in Figure 7 for low levels of initial capital and varying κ and λ . Specifically, trapping probabilities for households with capital between $x = x^*$, the uninsured poverty line, and $x = x^*/(1-\kappa)$, the upper limit of the first subinterval I_0 , corresponding to the trapping probability in I_0 given in (39), are presented. At this more granular level, the intersection point of the curves can be observed more clearly. This intersection point indicates when proportional insurance coverage is beneficial for reducing poverty trapping. In the estimation of the insured trapping probability, the increase in critical capital associated with the need for premium payment is accounted for through specification of $x^*(\kappa, \lambda, \theta)$, where an insured household is deemed to be trapped when their capital falls below $I^*/(b - \pi(\kappa, \lambda, \theta))$, where the critical income $I^* = b$ under the assumption of no change in the basic model parameters due to the purchase of insurance. Thus, in the insured case, households with initial capital slightly above x^* have already become trapped.

As in Kovacevic and Pflug [2011] and Flores-Contró et al. [2022] the increase in the trapping probabilities of the most vulnerable households when proportionally insured is observed in all cases considered. However, importantly, this increase occurs for a much smaller proportion of the low-income sample. Denoting the intersection point of the uninsured and insured trapping probabilities by x_c , the significance of the distance between the intersection point and the critical capital x^* is presented in Figure 8 for varying κ and λ . Considering three levels of the loading factor θ , the distance is positive under all sets of parameters tested. The depiction of $x_c - x^*$ in this figure highlights that the level of capital at which insurance becomes beneficial lies much closer to the poverty line than for more extreme [Kovacevic and Pflug, 2011] and random-valued losses [Flores-Contró et al., 2022], with only small distances between the intersection point and the critical capital observed. These results suggest that purchase of proportional insurance for proportional losses is beneficial for a larger proportion of those closest to the poverty line. In particular, proportional coverage appears to be more affordable than classical coverage for random-valued losses.

Our consideration of a poverty line that varies with the level of insurance coverage accounts for the fact that premium payments limit a household's level of capital. We therefore consider "extreme poverty" at

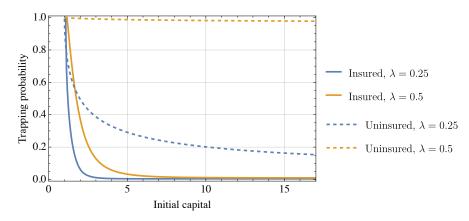


Figure 6: Comparison between the trapping probabilities of uninsured and insured households for $\kappa = 0.5$ and two different values of λ . Solid curves are computed via (38) assuming Conjecture 4.1 and dashed curves via (12). For each case, the values of $r(\kappa, \lambda, \theta)$ and $x^*(\kappa, \lambda, \theta)$ are computed via (20) with $a = 0.1, b = 1.4, c = 0.4, x^* = 1$ and $\theta = 0.5$. Recall that for uninsured losses, by (11) it must hold that $\lambda/r < 1$.

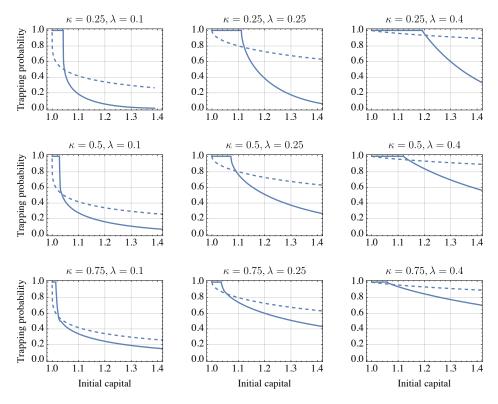


Figure 7: Comparison of the trapping probabilities of uninsured and insured households for small values of initial capital, $x \in [1, x^*/(1 - \kappa)]$ and different values of κ and λ , showing the existence of a level $x_c > x^*$ such that for $1 < x < x_c$ it is better for households not to insure. Solid curves are computed as in Figure 5b and dashed curves using expression (12). For each case, the values of $r(\kappa, \lambda, \theta)$ and $x^*(\kappa, \lambda, \theta)$ are computed via (20) with a = 0.1, b = 1.4, c = 0.4, $x^* = 1$ and $\theta = 0.5$.

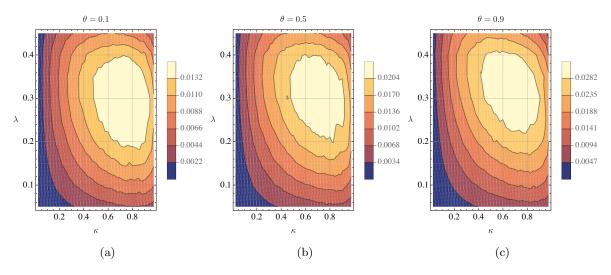


Figure 8: Estimated distance between x_c and x^* , i.e. $x_c - x^*$, for different values of λ and κ for which $\lambda/r < 1$ and (a) $\theta = 0.1$, (b) $\theta = 0.5$ and (c) $\theta = 0.9$, where x_c is the intersection point of the uninsured and insured trapping probabilities. For each case, the values of $r(\kappa, \lambda, \theta)$ and $x^*(\kappa, \lambda, \theta)$ are computed via (20) with a = 0.1, b = 1.4, c = 0.4 and $x^* = 1$.

an individualised level. In Kovacevic and Pflug [2011] and Flores-Contró et al. [2022] the uninsured trapping probability is instead compared with the insured trapping probability for a fixed critical capital x^* , irrespective of the parameters κ , λ and θ . Such a specification could be used to consider trapping with respect to an international poverty line, which is fixed for all households. Under this alternative assumption, the trapping probability under proportional insurance coverage of Section 4 lies below the uninsured probability of Section 3 at all capital levels. In this case, the purchase of insurance therefore does not increase the probability of trapping for any household above the poverty line.

Mathematical differences between the uninsured and insured capital processes and the associated parameter constraints may also provide indications of the impact of insurance. In Figure 3, the constraint that ensures existence of the Lundberg equation is presented. For uninsured losses with uniformly distributed remaining proportions of capital ($Z_i \sim \text{Beta}(1,1)$), by (11), an equivalent figure would display a horizontal line at $\lambda = r$. For the case considered in Figure 3, r = 0.504. As such, for all levels of θ , there exists a region in which the uninsured constraint in (11) is violated, while the insured constraint in (23) is not. This indicates that for households without insurance, the Lundberg equation fails to be well-defined in more cases. Increasing the level of insurance coverage therefore increases the loss frequency for which the net profit condition is satisfied. As a result, certain trapping is avoided in more cases.

Due to the increasing complexity of (34b) the constant A appears in an increasingly convoluted manner throughout the subintervals I_j . As we move through I_j for increasing \tilde{x} , estimation of the trapping probability under proportional insurance coverage becomes computationally intensive, particularly for small values of κ . However, analysis of the algebraic decay of the trapping probability can provide further insight into the behaviour of the function at high capital levels. Solution of the transcendental equation

$$r\gamma - \lambda + \frac{\lambda\alpha}{\kappa} \int_{1-\kappa}^{1} y^{\gamma} \left(1 - \frac{1-y}{\kappa}\right)^{\alpha-1} dy = 0, \tag{40}$$

derived from $\mathcal{A}f(x) = 0$ for $\mathcal{A}f(x)$ in (24) for $Z_i \sim \text{Beta}(\alpha, 1)$ under the assumption of polynomial asymptotic decay to zero at infinity: $f(x) \sim (x - x^*)^{\gamma}$ as $x \to \infty$ for constant γ , highlights that for Beta(1, 1) distributed remaining proportions of capital, as in Section 4, as κ increases and households maintain a higher risk level the trapping probability decays more slowly as initial capital x approaches infinity. The same observation can be found with less significance for fixed κ and decreasing λ/r . Solution of the transcendental equation in (40) for $\alpha > 0$ and $\kappa = 1$ yields that the trapping probability decays only if $\lambda/r < \alpha$, providing exactly the Lundberg condition in the case of no insurance coverage.

6 Concluding remarks

We have considered an adjustment of the capital process of Flores-Contró et al. [2022] in which low-income households are susceptible to losses proportional to their accumulated capital level, as in Kovacevic and Pflug [2011]. Under the assumption of proportional losses we capture the exposure of households of all capital levels to both catastrophic and low severity loss events, a feature particularly significant in the low-income setting. Typically considered to be protected from capital losses, households with higher levels of capital are still susceptible to large proportional losses on the occurrence extreme events, particularly in agriculturally rich areas. In addition to high severity loss events, low-income households closest to the poverty line experience large proportional losses due to events typically considered less severe in the high-income setting, such as hospital admissions and household deaths.

Focusing on the probability that a household falls below the poverty line, referred to as the trapping probability, in the analysis of this paper we have solved, for the first time analytically, infinitesimal generator equations associated with a capital process with exponential growth and multiplicative jumps. We have considered two cases: (i) households with no insurance coverage and (ii) households with proportional insurance coverage. In both cases, closed-form solutions of the infinitesimal generator equations associated with the trapping probability were derived alongside constraints on the parameters of the model that prevent certain trapping. Through the derivation of these probabilities we provide insights into the impact of proportional insurance for proportional losses. Comparison between the proportional assumption of this paper and the random-valued assumption of Flores-Contró et al. [2022] was additionally presented.

For households with no insurance coverage, explicit trapping probabilities for $Beta(\alpha, 1)$ distributed remaining proportions of capital were obtained using Laplace transform methods. In comparison to the corresponding trapping probability for random-valued losses, the proportional trapping probability exhibits a slower rate of decay, in line with the non-zero probability of high-income households losing a large proportion of their wealth.

Consideration of proportional insurance coverage requires redefinition of the infinitesimal generator of the process. Even under the assumption of uniformly distributed remaining proportions of capital the structure of the proportional insurance product induces non-local functional terms in the derivative and Laplace transform of the infinitesimal generator. Classical methods for solving the infinitesimal generator to derive the trapping probability were therefore not applicable. To overcome this, we propose a recursive method for deriving a solution of the IDE and estimate the unique solution numerically through the conjecture of the existence of a limit. Although only analytic up to a constant, the estimated trapping probability performs well when compared with simulations of the capital process and provides intuitive results under sensitivity analysis. Future work will involve deriving a mathematical proof that this conjecture holds.

Comparing trapping probabilities under no insurance coverage and proportional insurance coverage suggests that the increase in trapping probability observed under random-valued losses is less severe in this proportional case. This finding is in contrast to that of Kovacevic and Pflug [2011], where an increase in trapping probability similar to that of Flores-Contró et al. [2022] is observed under the same proportional model. However, this result is likely highly dependent on the specification of parameters. It should be noted that the distribution of the remaining proportion of capital considered in the numerical example of Kovacevic and Pflug [2011] is such that losses have an expected value of 88%, an extremely high proportion given a loss frequency parameter of 1. In turn, the associated premium rates are high and will constrain capital growth more significantly. The lower rate associated with the distribution selected for presentation in the analysis of this paper captures losses of varying severity, as is the experience of a low-income population, and will necessitate reduced premiums. Furthermore, when considering a critical capital that is fixed as in Kovacevic and Pflug [2011], irrespective of a household's insured status, the increase in trapping probability associated with purchase of insurance is not observed at any level of capital.

Ultimately, the findings of this paper suggest that insurance for proportional losses is more affordable than coverage for losses of random value. This aligns with the idea that premiums are normalised to wealth under the proportional loss structure, thus improving the variability in the affordability of premiums characteristic of insurance for random-valued losses. As such, if the assumption of proportionality is correct, in the context of subsidisation, the proportion of the low-income population requiring full government support may be narrower than anticipated. Under consideration of a universal poverty line, such as the international poverty line, insurance is beneficial at all capital levels. However, when considering the impact of insurance at a more granular level, where the critical level increases with increasing coverage, for those with capital just above the critical capital, as in the findings of existing studies, insurance and the associated need for premium payments increases their probability of falling below the poverty line.

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