Value-at-Risk under Measurement Error*

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ABSTRACT

We propose a method for estimating Value-at-Risk that corrects for the effect of measurement errors in stock prices. We show that the presence of measurement errors might pose serious problems for estimating risk measures. In particular, when stock prices are contaminated, existing estimators of Value-at-Risk are inconsistent and might lead to an underestimation of risk, which can result in extreme leverage ratios within the held portfolios. Using a Fourier transform and a deconvolution kernel estimator of the probability distribution function of actual latent prices, we derive a robust estimator of Value-at-Risk in the presence of measurement errors. Monte Carlo simulations and real data analysis illustrate satisfactory performance of the proposed method.

Keywords: Deconvolution kernel, Fourier transform, measurement error, market microstructure noise, optimization, Value-at-Risk.

JEL Classification Number: G11, G19, C14, C61, C63.

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We propose a method for estimating Value-at-Risk that corrects for the effect of measurement errors in stock prices. We show that the presence of measurement errors might pose serious problems for estimating risk measures. In particular, when stock prices are contaminated, existing estimators of Value-at-Risk are inconsistent and might lead to an underestimation of risk, which can result in extreme leverage ratios within the held portfolios. Using a Fourier transform and a deconvolution kernel estimator of the probability distribution function of actual latent prices, we derive a robust estimator of Value-at-Risk in the presence of measurement errors. Monte Carlo simulations and real data analysis illustrate satisfactory performance of the proposed method.

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1 Introduction

Since its first appearance in the 1980s, Value-at-Risk (VaR) has become the most widely used risk management tool in financial services industry. Indeed, VaR has gained ground because it is a relatively simple measure to estimate and it was established by the Basel II regulatory framework as a benchmark method for market risk capital requirements calculation; see Basel Committee on Banking Supervision (1996, 2006). In addition to its use as a risk measure, VaR can be used as a basis for portfolio optimization; see, e.g., Lwin et al. (2017), Yiu (2004), and Alexander and Baptista (2008). Due to its popularity as a tool for controlling risk, financial managers are rightfully concerned about the accuracy of VaR estimation. One problem that might affect this estimation is effectively the presence of measurement errors in assets' prices, which can be caused by non-synchronous trading, rounding errors, infrequent trading, market microstructure noise, manipulations (smoothing, extra revenues, fraudulent exchanges, informationless trading), etc. Finance literature has investigated the impact of contaminated prices on the estimation of volatility and developed robust methods to identify the variance of actual stock prices; see, e.g., Zhou (1996), Andersen et al. (2001), Zhang et al. (2005), Bandi et al. (2006), Barndorff-Nielsen et al. (2011), Hansen and Lunde (2006), and Mancino and Sanfelici (2008). However, no attention was paid to the effect of this contamination on VaR estimation, and thus no robust estimation technique is available under measurement errors.

The presence of measurement errors in high-frequency prices (also known as market microstructure noise) has been well established in the literature; see Madhavan (2000) for a survey on market microstructure noise. Many papers have also investigated the presence of measurement errors in low-frequency prices/returns. For instance, by collecting specific information on risk management from the annual reports of the 200 largest US and international commercial banks for the period 2005-2008, Frésard et al. (2011) find that only a very small fraction of banks (less than 6%) uses uncontaminated returns to estimate their VaR. Most of these banks use contaminated data which include intraday revenues, fees, or commissions. They also show that all available back-testing procedures are highly sensitive to data contamination. As such, data contamination has undesirable implications for model validation and can lead to the acceptance of misspecified VaR models, and therefore significantly reduced regulatory capital. Furthermore, Pérignon et al. (2008) assess the accuracy of banks' risk management systems based on daily VaR and profit-and-loss data. They find evidence supporting the idea that banks exhibit a systematic bias in their VaR estimates, which they attribute to several factors including measurement errors.

None of the previous papers, however, was interested in developing robust estimation methods to correct for the effect of measurement errors on VaR estimation. In this paper, we propose a novel semiparametric approach for the estimation of VaR in the presence of measurement errors in stock prices. It is worth mentioning that, both theoretically and computationally, it is more straightforward to deal with the measurement errors in the context of variance than in the context of VaR, because it is generally easier to work with moments than with quantiles. Moreover, working with an additive measurement error model makes the derivation of robust estimators much easier for variance than for quantiles (VaR). We remind the reader that an additive measurement error model defines a measurement error as the difference between the observed stock price and the actual (latent) stock price. For a general discussion of additive measurement error models, the reader is referred to Schennach (2016).

To derive a robust estimator of VaR, we use a different approach than the one used by the finance literature to develop robust estimators of variance under market microstructure noise. To deal with the measurement errors, we use a deconvolution kernel estimator for the density function of the actual latent portfolio returns.[1] There is a rich literature on using density deconvolution for estimating probability density functions. Adusumilli et al. (2020) have studied inference on the cumulative distribution function (CDF) in the context of classical measurement error problem using density deconvolution. They applied their results to construct confidence bands of CDFs and quantiles, to test goodness-of-fit for parametric probability density function (PDF) models, and to test stochastic dominance. For a review on deconvolution methods, the readers can consult Meister (2009), Fan (1991), Hall and Lahiri (2008), and Dattner et al. (2011) among others. Thereafter, we use Fourier inversion to compute the probability distribution function of the actual portfolio returns. We apply power series representations of sine and exponential functions to approximate the integral in the inversion formula and define an optimization problem that makes the calculation of VaR under measurement errors computationally feasible. Roughly speaking, power series representations are about representing common functions as polynomials with infinitely many terms, thus integrating a power series is as easy as integrating a polynomial.

[1] We remind the reader that the concept of deconvolution corresponds to computing the inverse of the convolution operation of two functions. Convolution operation on two functions, say f and g, produces a third function $(f * g)$ that expresses how the shape of one is modified by the other. It is defined as the integral of the product of the two functions after one is reversed and shifted. In the context of a measurement error problem, kernel deconvolution density estimation consists in estimating the density of a variable of interest (here the density of the true stock price) that is observable only with some measurement error.

The derivation of robust estimator of VaR is first made under the assumption that the density of the measurement error is known, but the distribution of the observed portfolio returns is always treated as unknown and estimated nonparametrically. Thereafter, we relax this assumption and propose a feasible way to deal with the measurement error's distribution. Specifically, we follow the literature and assume that the measurement error is normally distributed with an unknown variance that we estimate using high-frequency data. Indeed, we can obtain a consistent estimator for the variance of measurement error as shown in Zhang et al. (2005). We highlight that assuming measurement error is normally distributed does not contradict the fact that returns can be non-normally distributed.

We conduct a set of Monte Carlo simulations to examine the finite sample performance of our approach under the presence of measurement errors. We provide a comparison with a model-free estimator of VaR that does not adjust for the measurement errors. We investigate the performance of our method under different densities of the measurement error and the simulation results are very encouraging. Furthermore, we use our method and high-frequency data to estimate the VaRs of five international market indices. We compare our results with the unadjusted VaRs that we obtain using a model-free estimator that computes the sample quantiles of the five indices' returns. The empirical results suggest that ignoring measurement errors leads to an underestimation of risk.

The rest of the paper is organized as follows. In Section 2, we introduce the additive measurement error model for the asset prices and demonstrate how to estimate the characteristic function and the distribution function of latent portfolio returns. In Section 3, we derive an optimization-based estimator of VaR of latent portfolio returns. Section 4 analyzes the performance [bias, standard deviation, and root mean squared error] of our proposed method through Monte Carlo simulations. In Section 5, we use our approach and high-frequency data to estimate the adjusted VaR of five stock indices. Finally, Section 6 concludes. Mathematical proofs and tables of additional Monte Carlo simulation results can be found in the Online supplementary appendix.

2 Framework

The methodology that we develop in this paper works for both individual assets and portfolios. The exposition here is made for a portfolio of assets, but an individual asset is a special case by setting all portfolio weights equal to zero, except the weight of the asset in question. Formally, we assume that there are $n \geq 1$ risky assets in the economy. We denote by $\mathbf{P}_t = (p_{1,t}, \ldots, p_{n,t})'$ and

$\mathbf{P}_t^* = (p_{1,t}^*, \ldots, p_{n,t}^*)'$ the $n \times 1$ vectors of observed and actual latent log prices of the n assets at time t, respectively. We suppose that the observed log price of each asset j can be contaminated by a measurement error:

$$p_{j,t} = p_{j,t}^* + \epsilon_{j,t}, \text{ for } j = 1, \ldots, n \text{ and } t = 1, \ldots, T, \tag{1}$$

where $p_{j,t}$ (resp. $p_{j,t}^*$) is the observed (resp. actual latent) log price of asset j, and $\epsilon_{j,t}$ is the measurement error which we assume to be independent and identically distributed (i.i.d.) across time t with mean zero and variance $\sigma_{\epsilon_j}^2$ for each $j = 1, \ldots, n$. The additive measurement error model in (1) is defined for log-price and not for the actual price, which implicitly means that we allow for the multiplicative structure $P_{j,t} = P_{j,t}^* \varepsilon_{j,t}$, where $P_{j,t}$ and $P_{j,t}^*$ are respectively the levels of the observed and actual latent prices of asset j at time t.

The i.i.d. assumption for $\epsilon_{j,t}$ across time t is compelling and widely used in the literature; see for example the review paper by McAleer and Medeiros (2008). This assumption does not affect the dependence structure in returns. In other words, the additive measurement error model implies dependence in the observed returns when there is dependence in the actual latent returns even when the measurement errors are i.i.d. This means that the dependence structure in the process of returns is preserved under the assumption of i.i.d. measurement errors. Furthermore, the assumption of i.i.d measurement errors might be avoided, but this will be at the cost of complicating the calculation of our estimator of VaR. This might require the estimation of the distribution of the measurement error (which is very difficult in practical settings), or in the best case scenario (if we make an assumption on the distribution of the measurement error as we do in this paper), we will need to estimate the moments of the measurement errors (e.g. its variance as we do in this paper), but also its dependence/correlation structure. Indeed, the i.i.d. assumption is not as restrictive as it seems and is commonly imposed in the literature; see for example Zhang et al. (2005) and Griffin and Oomen (2011), Andersen et al. (2011) and Hounyo et al. (2017) among others. In addition, the vector error $\epsilon_t = (\epsilon_{1,t}, \ldots, \epsilon_{n,t})'$ is assumed to be independent of the vector of actual latent log prices P_t^*. Given the sources of the measurement errors that have been mentioned in the first paragraph of the introduction, we think that the assumption of independence between these errors and the actual price is plausible. In other words, we do not see how the actual price can, for example, be correlated with the errors that are due to non-synchronous trading, infrequent trading, market microstructure noise, manipulations, or rounding errors, etc. The elements of $\boldsymbol{\epsilon}_t$ are also assumed to be cross-sectionally independent and normally distributed; that is, $\boldsymbol{\epsilon}_t \sim N(0, \Sigma_\epsilon)$, a multivariate normal distribution with mean zero and covariance matrix $\Sigma_{\boldsymbol{\epsilon}} = diag\left(\sigma^2_{\epsilon_1}, \ldots, \sigma^2_{\epsilon_n}\right)$. In this section, we assume that the variance of the measurement errors $\sigma^2_{\epsilon_j}$, for $j = 1, \ldots, n$, is known, but later on we will relax this assumption and discuss how to estimate it.

For $j = 1, \ldots, n$, denote by $r_{j,t} = p_{j,t} - p_{j,t-1}$, $r^*_{j,t} = p^*_{j,t} - p^*_{j,t-1}$, and $u_{j,t} = \epsilon_{j,t} - \epsilon_{j,t-1}$. Then using equation (1), we have $r_{j,t} = r^*_{j,t} + u_{j,t}$ and therefore the observed and actual latent returns of a portfolio of n assets can be linked as follows:

$$r_{p,t} = r^*_{p,t} + e_t, \tag{2}$$

where $r_{p,t} = \sum_{j=1}^n \omega_j r_{j,t}$ is the observed portfolio return, $r^*_{p,t} = \sum_{j=1}^n \omega_j r^*_{j,t}$ is the actual latent portfolio return, and $e_t = \sum_{j=1}^n \omega_j u_{j,t}$ denotes the measurement error in the portfolio return, with $\boldsymbol{\omega} = (\omega_1, ..., \omega_n)'$ being a known vector of weights that are attributed to each risky asset in the portfolio. Here, $u_{j,t}$ for $j = 1, ..., n$, is a composite measurement error term that is normally distributed with mean zero and variance $\sigma^2_{u_j} = 2\sigma^2_{\epsilon_j}$. Note that the assets' returns (e.g., $r_{j,t}$ and $r^*_{j,t}$) correspond to continuously compounded returns (first differences of log assets' prices). It is also worth noting that the distributions of observed and actual latent portfolio returns are unknown. This fact implies that even when the composite measurement error $u_{j,t}$ is normally distributed, the observed and actual latent portfolio returns can be non-normally distributed.

Since $\mathbf{P}^*_t$ and $\boldsymbol{\epsilon}_t$ are independent processes (and thus $r^*_{p,t}$ and e_t are independent processes), using equation (2) and the Fourier transform, we can obtain:

$$\phi_{r_p}(s) = \phi_{r^*_p}(s)\, \phi_e(s), \tag{3}$$

where $\phi_{r_p}(s) = E[\exp(\mathrm{i}s r_{p,t})]$, $\phi_{r^*_p}(s) = E[\exp(\mathrm{i}s r^*_{p,t})]$, and $\phi_e(s) = E[\exp(\mathrm{i}s e_t)]$ represent the characteristic functions of $r_{p,t}$, $r^*_{p,t}$ and e_t, respectively, with $\mathrm{i} = \sqrt{-1}$ denoting the imaginary unit.

On the one hand, under the i.i.d. assumption $\epsilon_{j,t} \sim N(0, \sigma^2_{\epsilon_j})$ across t for $j = 1, \ldots, n$, the characteristic function $\phi_e(s)$ of the portfolio measurement error e_t is given by:

$$\begin{aligned}\phi_e(s) =& E\left[\exp\left(\mathrm{i}s\sum_{j=1}^n \omega_j u_{j,t}\right)\right] = E\left[\exp\left(\mathrm{i}s\sum_{j=1}^n \omega_j \epsilon_{j,t}\right)\right] E\left[\exp\left(-\mathrm{i}s\sum_{j=1}^n \omega_j \epsilon_{j,t-1}\right)\right] \\ =& \prod_{j=1}^n E\left[\exp\left(\mathrm{i}s\omega_j\epsilon_{j,t}\right)\right] \prod_{j=1}^n E\left[\exp\left(-\mathrm{i}s\omega_j\epsilon_{j,t-1}\right)\right] = \exp\left(-s^2\sum_{j=1}^n \sigma^2_{\epsilon_j}\omega^2_j\right).\end{aligned} \tag{4}$$

On the other hand, the characteristic function $\phi_{r_p}(s)$ of the observed portfolio return $r_{p,t}$ can be estimated using its empirical analogue:

$$\hat{\phi}_{r_p}(s) = \frac{1}{T}\sum_{t=1}^T \exp\left(\mathrm{i}s r_{p,t}\right), \tag{5}$$

where T is the number of observations. Observe that the empirical characteristic function such as $\hat{\phi}_{r_p}(s)$ in (5) has been widely used in time series analysis; see Carrasco and Florens (2002), Knight and Yu (2002), Yu (2004) and references therein. If a time series is strictly stationary and ergodic, which we impose on $r_{p,t}$ throughout this paper, then the empirical characteristic function is a consistent estimator of the characteristic function; see Theorem 2.1 of Feuerverger (1990).

Combining equations (3), (4) and (5), an estimator for the characteristic function $\phi_{r_p^*}(s)$ of the actual latent portfolio returns $r_{p,t}^*$ is given by:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} = \frac{1}{T}\sum_{t=1}^{T} \exp\left(\mathrm{i} s r_{p,t} + s^2 \sum_{j=1}^{n} \sigma_{\epsilon_j}^2 \omega_j^2 \right).$$

Nevertheless, later we will be interested in the estimation of the probability distribution function of $r_{p,t}^*$, which is defined (by Fourier transform) as the integral of $\exp(-\mathrm{i} s r_p^*)\hat{\phi}_{r_p^*}(s)$. However, this integral is not well-defined as $\hat{\phi}_{r_p^*}(s)$ is neither integrable nor square integrable over $\mathbb{R}$. In this case, $\hat{\phi}_{r_p^*}(s)$ will not be a good estimator of $\phi_{r_p^*}(s)$ for large values of s. One way to overcome this issue is by regularizing $\hat{\phi}_{r_p^*}(s)$ as follows:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} K^{ft}(sb), \tag{6}$$

where $K^{ft}(sb)$ is a Fourier transform of a kernel function with an appropriate bandwidth $b \in R^+$; see, e.g., Adusumilli et al. (2020) and Otsu and Taylor (2021). Hereafter, we assume $\phi_e(s) \neq 0$ for all $s \in R$, and $K^{ft}(s) = I(-1 \leq s \leq 1)$, with $I(A)$ designating an indicator function for the event A. These assumptions are common in the measurement error literature; see for example Otsu and Taylor (2021). In particular, using $K^{ft}(s) = I(-1 \leq s \leq 1)$, the function $\hat{\phi}_{r_p^*}(s)$ in equation (6) is supported on $[-1/b, 1/b]$ and bounded whenever $\phi_e(s) \neq 0$ for all $s \in R$. Therefore, the regularized estimator $\hat{\phi}_{r_p^*}(s)$ is well-defined. Note that $K^{ft}(s)$ is the Fourier transform of the sinc kernel $K(x) = \sin(x)/(\pi x)$ popularly adopted in the deconvolution method. Using $K^{ft}(s)$ also helps to facilitate the computation of the estimator of $\Pr(r_{p,t}^* < r_p^*)$ in equation (8), and thus the subsequent optimization that is based on it. Although other forms of $K^{ft}(s)$, such as $K^{ft}(s) = (1-s^2)^3 I(-1 \leq s \leq 1)$ used in Delaigle et al. (2008), may also be considered, they generally lead to more computationally demanding estimators. With proper regularization, we obtain the following semiparametric type estimator of the characteristic function of the actual latent portfolio return:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} K^{ft}(sb) = \frac{1}{T}\sum_{t=1}^{T} \exp\left(\mathrm{i} s r_{p,t} + s^2 \sum_{j=1}^{n} \sigma_{\epsilon_j}^2 \omega_j^2 \right) K^{ft}(sb). \tag{7}$$

We would like to point out here that similar calculations to those made above under normality assumption can be followed to obtain the expression of characteristic function of $\phi_{r_p^*}(s)$ and its estimator $\hat{\phi}_{r_p^*}(s)$ in (7) when the measurement error follows another distribution (other than the normal distribution) with characteristic function that has a closed-form expression. See the Online supplementary appendix A.2 for an example when the measurement errors follows a stable distribution.

We can now use $\hat{\phi}_{r_p^*}(s)$ to derive an estimator for the distribution function of the actual latent portfolio return $r_{p,t}^*$. A standard Fourier-inversion formula [see Gil-Pelaez (1951)] implies

$$\Pr\left(r_{p,t}^* < r_p^*\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left[\phi_{r_p^*}(s)\exp(-\mathrm{i}sr_p^*)\right]}{s}ds, \text{ for all } r_p^* \in \mathbb{R}, \tag{8}$$

where $\operatorname{Im}[\cdot]$ stands for the imaginary part of a complex number. After replacing $\phi_{r_p^*}(s)$ in equation (8) by its estimator in (7) and using the trigonometric form of a complex number, the distribution function $\Pr\left(r_{p,t}^* < r_p^*\right)$ can be estimated as follows:

$$\widehat{\Pr}\left(r_{p,t}^* < r_p^*\right) = \frac{1}{2} - \frac{1}{\pi}\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sum_{j=1}^n \sigma_{\epsilon_j}^2\omega_j^2\right)\sin\left(s\left(\frac{r_{p,t}-r_p^*}{b}\right)\right)}{s}ds.$$

Alternatively, the above expression can be rewritten as

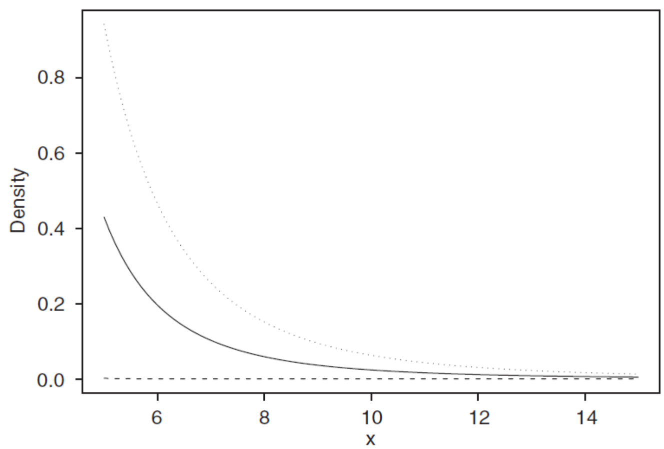
$$\widehat{\Pr}\left(r_{p,t}^* < r_p^*\right) = \frac{1}{2} - \frac{1}{T}\sum_{t=1}^{T} L\left(\frac{r_{p,t}-r_p^*}{b}\right), \tag{9}$$

where $L(u) = \frac{1}{\pi}\int_0^1 \frac{\sin(su)}{s}\frac{1}{\phi_e(\frac{s}{b})}ds$ is the so-called deconvolution kernel, $\phi_e(\cdot)$ is the Fourier transform of the density of the portfolio measurement error e_t, that is, $\phi_e(\frac{s}{b}) = \exp\left(-\frac{s^2}{b^2}\sum_{j=1}^n \sigma_{\epsilon_j}^2\omega_j^2\right)$, and $b = b(T) \in R^+$ is the sequence of bandwidth parameters converging to zero at a suitable rate as the sample size T increases.

In the next section, we use the result in equation (9) to derive an optimization-based estimator of VaR which is robust to the presence of measurement errors that affect stock prices. We shall first consider Figure 1 [see Nason (2006)] to illustrate the undesirable impact that measurement errors can have on risk estimation. The figure shows the right tail parts of standard normal distribution (dashed line), here e_t, Student's t distribution with 3 degrees of freedom (dotted line), here r_t^*, and their sum $r_t = r_t^* + e_t$ (solid line). The latter can be viewed as an additive measurement error model, where r_t represents the observed stock or portfolio return, r_t^* is the actual latent stock or portfolio return, and e_t is a measurement error. The density of r_t can be represented as the convolution of the density functions of e_t and r_t^*. The figure shows that the right tail of the distribution of

r_t^* (actual latent return) dominates the right tail of the distribution of r_t (observed return), and similar situation occurs at the left tail since the distributions of r_t, r_t^*, and e_t are all symmetric. Therefore, the existing estimators of VaR that ignore the measurement errors are inconsistent and lead to an underestimation of risk. Underestimating risk might have disastrous effects with the financial crisis of 2007-2008 being an example. We all know that this crisis was in part due to underestimating the risk magnitude of portfolios of subprime mortgages, which resulted in extreme leverage ratios within these portfolios and left institutions unable to cover billions of dollars in losses as subprime mortgage values collapsed. Consequently, providing robust estimation methods for VaR under measurement errors should be extremely valuable for risk analysis and management.

Figure 1: Tail of convoluting Normal and Student distributions



Note [Source Nason (2006)]: Tail part of PDF of $r_t = r_t^* + e_t$ (solid line); standard normal density for e_t (dashed line); sphered Student's t distribution of 3 degrees of freedom for r_t^* (dotted line).

3 VaR under measurement errors

Essentially, VaR is a quantile measure that quantifies the worst expected loss over a given horizon (typically a day or a week) at a given statistical confidence level $\alpha \in (0,1)$ (typically 1%, 5% or 10%). Formally, portfolio's VaR can be defined as follows:

$$\Pr\left(r_{p,t}^* < VaR\right) = \alpha, \tag{10}$$

where $r_{p,t}^*$ is the portfolio return. Equation (10) states that a loss equal to or larger than the VaR occurs with probability α. Conversely, the VaR can be written as a function of the probability:

$$VaR = F^{-1}(\alpha);$$

where $F(\cdot)$ is the distribution function of $r^*_{p,t}$; i.e., $F(r^*_p) = \Pr(r^*_{p,t} < r^*_p)$ for $r^*_p \in R$.

Several parametric and nonparametric approaches have been used to derive estimators of VaR; for a review the reader can consult Abad, Benito, and López (2014). The level of difficulty of these approaches depends on the assumptions made about the underlying process of returns. As we allow for the latter to capture more stylized effects, the estimation approach becomes more complex. In addition, except when the returns follow elliptical conditional distribution, the estimation of VaR generally requires the use of either simulation or optimization methods. All these estimation techniques, however, are sensitive to the presence of measurement errors in assets' prices. In the following, we use the results from the previous section to derive an optimization-based semiparametric estimator of VaR when the assets' prices are contaminated.

We now follow the convention and let the VaR of the actual latent portfolio returns, say $VaR^\alpha(r^*_{p,t})$ for the confidence level α, be a positive quantity. Then, replacing r^*_p by $-VaR^\alpha(r^*_{p,t})$ in equation (9) leads to

$$\widehat{\Pr}\left(r^*_{p,t} < -VaR^\alpha(r^*_{p,t})\right) = \frac{1}{2} - \frac{1}{\pi}\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sum_{j=1}^{n}\sigma^2_{\epsilon_j}\omega^2_j\right)\sin\left(s\left(\frac{r_{p,t}+VaR^\alpha(r^*_{p,t})}{b}\right)\right)}{s}ds. \quad (11)$$

Following a similar approach to Duffie and Pan (2001) and Taamouti (2009), $VaR^\alpha(r^*_{p,t})$ can then be calculated by inverting the estimated distribution function in (11). However, for reasons we explain below, analytically inverting the function (11) is not feasible and a numerical solution is required. We have the following proposition which can be deduced immediately from equation (11).

Proposition 1 *The VaR of the actual latent portfolio return $r^*_{p,t}$ in (2), at a nominal coverage rate α, denoted by $VaR^\alpha(r^*_{p,t})$, is the solution of the following equation:*

$$\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sum_{j=1}^{n}\sigma^2_{\epsilon_j}\omega^2_j\right)\sin\left(s\left(\frac{r_{p,t}+VaR^\alpha(r^*_{p,t})}{b}\right)\right)}{s}ds - \left(\frac{1}{2}-\alpha\right)\pi = 0,$$

where $r_{p,t}$ is the observed portfolio return in (2), $\sigma^2_{\epsilon_j}$ for $j = 1, ..., n$ are the variances of the measurement errors $\epsilon_{j,t}$, ω_j for $j = 1, ..., n$ are the portfolio weights, and b is the bandwidth parameter.

From Proposition 1, $VaR^\alpha(r^*_{p,t})$ can be obtained by numerically solving the equation:

$$f\left(VaR^\alpha(r^*_{p,t})\right) = \frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sum_{j=1}^{n}\sigma^2_{\epsilon_j}\omega^2_j\right)\sin\left(s\left(\frac{r_{p,t}+VaR^\alpha(r^*_{p,t})}{b}\right)\right)}{s}ds - \left(\frac{1}{2}-\alpha\right)\pi = 0. \quad (12)$$

The function $f(VaR^{\alpha}(r^*_{p,t}))$ can be rewritten as follows:

$$f\left(VaR^{\alpha}(r^*_{p,t})\right) = -\pi\left[\widehat{\Pr}\left(r^*_{p,t} < -VaR^{\alpha}(r^*_{p,t})\right) - \alpha\right]. \tag{13}$$

From equation (13) and the properties of the distribution function [monotonically increasing, $\lim_{x\to-\infty}\widehat{\Pr}(r^*_{p,t} < x) = 0$ and $\lim_{x\to\infty}\widehat{\Pr}(r^*_{p,t} < x) = 1$], we can show that equation (12) admits a unique solution. A more convenient way to calculate $VaR^{\alpha}(r^*_{p,t})$ is to consider the following optimization problem:

$$\widehat{VaR}^{\alpha}(r^*_{p,t}) = \underset{VaR^{\alpha}(r^*_{p,t})}{argmin}\left[\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sum_{j=1}^{n}\sigma^2_{\epsilon_j}\omega_j^2\right)\sin\left(s\left(\frac{r_{p,t}+VaR^{\alpha}(r^*_{p,t})}{b}\right)\right)}{s}ds - \left(\frac{1}{2}-\alpha\right)\pi\right]^2. \tag{14}$$

In practice, an *exact* solution for the above minimization problem is not feasible, since the integral term in (14) is quite difficult to assess. This issue can be solved using a numerical integration based on equally spaced abscissas as in Davies (1973) and Davies (1980); see Duffie and Pan (2001) and Taamouti (2009). However, this approach introduces two types of errors: the discretization error and the truncation error. In this paper, we instead propose a closed-form expression for the Fourier inversion in (8) by regularizing the estimated characteristic function $\hat{\phi}_{r^*_p}(s)$ using the characteristic function $K^{ft}(sb) = I(-1 \leq sb \leq 1)$, which is defined on $s \in [-1/b, 1/b]$, and by using power series representations of the functions $\sin(\cdot)$ and $\exp(\cdot)$. Specifically, from the power series representations of $\sin(\cdot)$ and $\exp(\cdot)$, we obtain the following corollary [see the proof of Corollary 1 in the Online supplementary appendix].

Corollary 1 *The VaR of the actual latent portfolio return $r^*_{p,t}$ in (2), at a nominal coverage rate α, denoted by $VaR^{\alpha}(r^*_{p,t})$, is the solution of the following optimization problem:*

$$\widehat{VaR}^{\alpha}(r^*_{p,t}) = \underset{VaR^{\alpha}(r^*_{p,t})}{argmin}\left[\frac{1}{T}\sum_{t=1}^{T}\sum_{i,j=0}^{\infty}\frac{a^i}{i!}\frac{(-1)^j\left(\frac{r_{p,t}+VaR^{\alpha}(r^*_{p,t})}{b}\right)^{1+2j}}{(1+2j)!(2i+2j+1)} - \left(\frac{1}{2}-\alpha\right)\pi\right]^2, \tag{15}$$

where $a = \frac{\sum_{j=1}^{n}\sigma^2_{\epsilon_j}\omega_j^2}{b^2} > 0$, $r_{p,t}$ is the observed portfolio return in (2), $\sigma^2_{\epsilon_j}$ for $j = 1, ..., n$ are the variances of the measurement errors $\epsilon_{j,t}$, ω_j for $j = 1, ..., n$ are the portfolio weights, and b is the bandwidth parameter.

The optimization problem in (15) depends on the double infinite sum $\sum_{i,j=0}^{\infty}$, which has to be truncated for practical computation. Thus, the estimation of $VaR^{\alpha}(r^*_{p,t})$ will involve a truncation error that we need to control. Specifically, denoting $\kappa = \frac{r_{p,t}+VaR^{\alpha}(r^{*}_{p,t})}{b}$, the double infinite sum can be decomposed as follows:

$$S = \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = S_{l,k} + R_{l,k} = S_{l,k} + R^{(1)}_{l,k} + R^{(2)}_{l,k} + R^{(3)}_{l,k}, \qquad (16)$$

where the truncated term $S_{l,k}$ and the remaining term $R_{l,k}$ (i.e., the summation of $R^{(1)}_{l,k}$, $R^{(2)}_{l,k}$, and $R^{(3)}_{l,k}$) are defined in equation (21) of the Online supplementary appendix. Furthermore, we show that the remaining terms $R^{(1)}_{l,k}$, $R^{(2)}_{l,k}$, and $R^{(3)}_{l,k}$ are bounded [see equations (22), (23) and (24) of the Online supplementary appendix], and consequently $\lim_{l,k \longrightarrow \infty} \left\{ R^{(s)}_{l,k} \right\} = 0$, for $s = 1, 2, 3$ [see the proof in the Online supplementary appendix]. However, how large should l and k be in practice to make the remaining term sufficiently negligible is a question that we investigated extensively by simulations. For the data generating processes that we consider in Section 4 and a bandwidth b, which we select according to the rule of thumb $b = c\left(2\sigma^2_e / \log T\right)^{1/2}$, with $c > 1$ denoting an adjustment parameter and σ^2_e denoting the variance of the normal measurement error e_t [see, e.g., Section 3.7 of Delaigle et al. (2008)], we find that taking $l = k = 10$ in Formula (16) yields satisfactory results. Specifically, we find that there is no substantial improvement in terms of bias and root mean squared error when we increase l and k beyond 10.

The above calculation of VaR, however, depends on the unknown variance of the measurement errors $\sigma^2_{\epsilon_j}$. We next discuss how one can estimate this variance using the high-frequency data. A more general approach for the estimation of an unknown probability density function of the measurement error, say f_e, is also available but it requires the use of repeated measurements of the actual latent portfolio returns. In other words, if we further assume that the Fourier transform of the measurement errors $\phi_e(s)$ is real-valued, that is, the density f_e is symmetric around zero, then - if repeated measurements of the actual latent portfolio returns are available - we can estimate $\phi_e(s)$ using the estimator proposed by Delaigle et al. (2008). Unfortunately, in this paper we rule out the use of Delaigle et al.'s (2008) approach for the fully unknown case since in finance repeated measurements for asset prices are not available.

In the context of high-frequency data, a consistent estimator of the variance of the measurement error (market microstructure noise) can be obtained as a by-product of the results in Zhang et al. (2005). Formally, assuming that the full grid containing all of the observation points is given by $G = \{t_0, ..., t_m\}$ and using the consistent estimator of the variance of the market microstructure noise provided in Zhang et al. (2005, p. 1402), a consistent estimator of the variance of the market microstructure noise of each individual stock j over a time period $[0,t]$ can be obtained as follows:

$$\hat{\sigma}^2_{u_j} = 1/m \sum_{t_{i-1},t_i \in \mathcal{G}, t_i \leq t} (p_{j,t_i} - p_{j,t_{i-1}})^2, \text{ for } j = 1,...,n,$$

where $p_{j,t_{i-1}}$ (resp. p_{j,t_i}) is the price of stock j at the intraday time t_{i-1} (resp. t_i), and m is the number of sampling intervals over $[0,t]$. Consequently, a consistent estimator of the variance of the market microstructure noise of the portfolio over a time period $[0,t]$ can be obtained as follows:

$$\hat{\sigma}^2_e = 2\left(\sum_{j=1}^{n} \hat{\sigma}^2_{\epsilon_j}\omega^2_j\right) = \left(\sum_{j=1}^{n} \hat{\sigma}^2_{u_j}\omega^2_j\right) = 1/m \sum_{j=1}^{n} \omega^2_j \sum_{t_{i-1},t_i \in \mathcal{G}, t_i \leq t} (p_{j,t_i} - p_{j,t_{i-1}})^2. \tag{17}$$

4 Monte Carlo simulations

We conduct some Monte Carlo simulations to examine the finite sample performance of our robust VaR estimation technique that adjusts for the effect of measurement errors in the prices [hereafter adjusted VaR]. We also provide a comparison with a model-free estimator of VaR that does not adjust for the measurement errors [hereafter unadjusted VaR]. We assess the performance of our approach under two cases: when the measurement errors density is specified and when it is misspecified to some degree.

4.1 Case of specified density of measurement errors

We suppose that the observed and actual latent returns are related according to equation (2). We first consider that the density f_e of the measurement error e_t in equation (2) is correctly specified and given by a standard normal distribution, i.e., $e_t \sim$ i.i.d. $N(0, \sigma^2_e)$, with $\sigma^2_e = 1$, and we simulate the actual latent returns from the following three data generating processes (DGPs) that represent different contexts encountered in practice:

Model 1: The actual return $r^*_{p,t}$ follows an AR(1) process:

$$r^*_{p,t} = 0.5 r^*_{p,t-1} + \eta_t, \text{ with } \eta_t \sim \text{i.i.d.}\, N(0,1). \tag{18}$$

Model 2: The actual return $r^*_{p,t}$ follows an MA(2) process:

$$r^*_{p,t} = \eta_t + 0.65\eta_{t-1} + 0.24\eta_{t-2}, \text{ with } \eta_t \sim \text{i.i.d.}\, N(0,1). \tag{19}$$

Model 3: The actual return $r^*_{p,t}$ is generated from a GARCH (1,1) model:

$$r^*_{p,t} = \sigma_t \eta_t, \text{ with } \eta_t \sim \text{i.i.d.}\, N(0,1) \text{ and } \sigma^2_t = 0.05 + 0.85\sigma^2_{t-1} + 0.1 r^{*2}_{p,t-1}. \tag{20}$$

Table 1: True VaRs for AR(1), MA(2) and GARCH(1,1) models

VaR of actual return $r^*_{p,t}$			
	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
Model 1: AR(1)	-2.6861	-1.8992	-1.4797
Model 2: MA(2)	-2.8301	-2.0012	-1.5592
Model 3: GARCH(1,1)	-2.4320	-1.6235	-1.2399

Note: This table reports the actual VaR using Model 1 [equation (18)], Model 2 [equation (19)], and Model 3 [equation (20)]. The results are obtained using $T = 100000$ and 10000 replications.

We then use the simulated actual return $r^*_{p,t}$ and the standard normal measurement errors to simulate the observed return $r_{p,t}$ using equation (2). Note that the above DGPs and the corresponding parameters are only selected to reflect the commonly used financial time series models in the simulation design, which offers a wide range of dependent structures. For example, similar choices are also considered in Chen and Tang (2015).

For each of the above models, we analytically calculate the 1%, 5% and 10% VaR of actual return $r^*_{p,t}$. The values are reported in Table 1 and will be used to assess the bias (Bias), standard deviation (Std.), and root mean squared error (RMSE) of the adjusted and unadjusted estimates of VaR calculated using our semiparametric estimation technique and a model-free estimator (see below) that does not take into account the measurement errors, respectively. The sample sizes range from $T = 125$ to $T = 500$, which corresponds to data ranging from 6 months to 2 years. From each model, we generate $T + 1000$ observations and then discard the first 1000 observations to minimize the effect of the initial values. All the results are based on 1000 replications, except that for the calculation of the actual VaRs in Table 1 we use 10000 replications.

To calculate the unadjusted VaR, we simply use the sample quantile estimator based on the observed returns $\{r_{p,t}\}_{t=1}^T$ that are contaminated by the measurement errors, i.e.,

$$\widehat{VaR}^{\alpha}(r_{pt}) = \inf\left\{u : F_{r_p,T}(u) \geq \alpha\right\}, \text{ for } \alpha = 1\%,\ 5\%,\ \text{and } 10\%,$$

where $F_{r_p,T}(u) = T^{-1}\sum_{t=1}^T I(r_{p,t} \leq u)$ is the standard empirical cumulative distribution function based on the contaminated return $\{r_{p,t}\}_{t=1}^T$.

We next use our approach to compute the adjusted VaR under Models (18), (19) and (20). To control for the truncation errors, in Formula (16) we take $l = k = 10$. In addition, our approach requires the use of regularization via the bandwidth parameter b. In this simulation study, we follow

Delaigle et al. (2008) to select the bandwidth b according to the rule of thumb $b = c\left(2\sigma_e^2/\log T\right)^{1/2}$ with $c > 1$, which is motivated from the theoretically optimal choice for estimating the density of actual latent return.[2] We do not provide a theory to guide the choice of a data-driven bandwidth (e.g., searching for an optimal c) for our semiparametric estimation approach. However, in order to help select this important parameter, we consider a battery of simulations for different values of c, which varies in the grid $\{0.5, 1, 1.5, 2, 2.5, 3, 3.5\}$. This allows us to assess the sensitivity of our adjusted VaR estimates to different values of the bandwidth parameter b. Note that $c = 0.5$ violates the requirement $c > 1$ and it yields poor results as expected. As we see in the tables below, we can identify the optimal values of c that work for almost all DGPs under consideration. More precisely, if we choose $c = 2$ or $c = 2.5$, the gains in terms of bias, standard deviation and RMSE over the unadjusted approach are generally very significant.

Tables 2 to 4 report the simulation results for the 1% VaR using Model 1 to Model 3. The results for the 5% and 10% VaRs using the same DGPs can be found in Tables A.1 to A.6 in the Online supplementary appendix. Considering first the bias of the VaR estimates, the tables show that our estimation approach dominates the unadjusted VaR estimation for suitable c. These results are encouraging and seem to be consistent with the theory. The performance of our method shows some variation across the different bandwidth choices. However, it is not surprising to see that our semiparametric approach depends on the bandwidth parameter when faced with the measurement errors. Considering a battery of simulations for different values of c, we find that the performance of our technique is generally much better for $c = 2.0$ or 2.5. Furthermore, the unadjusted VaR always has a much larger negative bias compared to the adjusted VaR for $c = 2.0$ or 2.5. This implies that the unadjusted VaR may result in a larger underestimation of risk in the presence of measurement errors. Concerning the standard deviation (Std.) of the VaR estimates, we again find that the adjusted VaR dominates the unadjusted VaR in most of the cases. We also note that there is a clear trade-off between bias and variance, which is intuitive and well known in the classical nonparametric estimation literature. However, we see that the balance is reasonably achieved with $c = 2.0$ or 2.5, as is evident from the tables. Regarding the RMSE, a common point to all the results is that the estimated adjusted VaR has in general a smaller RMSE than the unadjusted

[2] Following Delaigle et al. (2008), for supersmooth measurement errors such as normal considered in this paper, the pointwise mean squared error of the conventional deconvolution kernel density estimator is of optimal order when using a bandwidth $b = D(\log T)^{-1/\alpha}$, where $D > (4\gamma)^{1/\alpha}$ denotes a constant and the constants $\alpha > 0$ and $\gamma > 0$ are defined in equation (3.17) of Delaigle et al. (2008). Under normal measurement error, we can set $\alpha = 2$ and $\gamma = \sigma_e^2/2$, leading to the rule of thumb $b = c\left(2\sigma_e^2/\log T\right)^{1/2}$ with $c > 1$.

Table 2: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with measurement error variance $\sigma_e^2 = 1$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 0.5$	2.6549	0.4544	2.6935			
$c = 1.0$	0.1989	0.7527	0.7786			
$c = 1.5$	0.7052	0.3191	0.7740			
$c = 2.0$	0.1743	0.2335	0.2913	-0.8366	0.4954	0.9723
$c = 2.5$	-0.3834	0.2140	0.4391			
$c = 3.0$	-0.9840	0.2162	1.0075			
$c = 3.5$	-1.7664	0.2157	1.7795			
			$T = 250$			
$c = 0.5$	2.6946	0.3889	2.7225			
$c = 1.0$	0.8214	0.6374	1.0397			
$c = 1.5$	0.8095	0.3763	0.8927			
$c = 2.0$	0.3264	0.1643	0.3654	-0.8356	0.3785	0.9173
$c = 2.5$	-0.2062	0.1619	0.2622			
$c = 3.0$	-0.7629	0.1544	0.7784			
$c = 3.5$	-1.5093	0.1533	1.5171			
			$T = 500$			
$c = 0.5$	2.6683	0.3824	2.6956			
$c = 1.0$	1.3505	0.3548	1.3963			
$c = 1.5$	0.8411	0.4680	0.9625			
$c = 2.0$	0.4481	0.1213	0.4642	-0.8585	0.2681	0.8994
$c = 2.5$	-0.0528	0.1107	0.1226			
$c = 3.0$	-0.5735	0.1059	0.5832			
$c = 3.5$	-1.3014	0.1056	1.3057			

Table 3: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with measurement error variance $\sigma_e^2 = 1$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 0.5$	2.7710	0.4667	2.8100			
$c = 1.0$	0.4626	0.7203	0.8561			
$c = 1.5$	0.8192	0.2760	0.8644			
$c = 2.0$	0.2806	0.2254	0.3599	-0.8084	0.5085	0.9550
$c = 2.5$	-0.2988	0.2150	0.3681			
$c = 3.0$	-0.8600	0.2031	0.8837			
$c = 3.5$	-1.6361	0.2015	1.6484			
			$T = 250$			
$c = 0.5$	2.8348	0.4132	2.8648			
$c = 1.0$	1.0920	0.5570	1.2258			
$c = 1.5$	0.9087	0.2380	0.9393			
$c = 2.0$	0.4129	0.1660	0.4450	-0.8038	0.3831	0.8905
$c = 2.5$	-0.1044	0.1532	0.1854			
$c = 3.0$	-0.6400	0.1358	0.6543			
$c = 3.5$	-1.3877	0.1480	1.3956			
			$T = 500$			
$c = 0.5$	2.8117	0.3867	2.8382			
$c = 1.0$	1.4877	0.3176	1.5213			
$c = 1.5$	0.9428	0.3809	1.0168			
$c = 2.0$	0.5354	0.1209	0.5489	-0.8179	0.2702	0.8613
$c = 2.5$	0.0495	0.1123	0.1228			
$c = 3.0$	-0.4565	0.1046	0.4683			
$c = 3.5$	-1.1887	0.1112	1.1939			

Table 4: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with measurement error variance $\sigma_e^2 = 1$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 0.5$	2.3103	0.4471	2.3531			
$c = 1.0$	-0.3953	0.6979	0.8021			
$c = 1.5$	0.6222	0.3182	0.6989			
$c = 2.0$	0.0384	0.1892	0.1930	-0.8670	0.5477	1.0255
$c = 2.5$	-0.5667	0.1683	0.5911			
$c = 3.0$	-1.1629	0.1540	1.1730			
$c = 3.5$	-1.9498	0.1591	1.9563			
			$T = 250$			
$c = 0.5$	2.4206	0.4015	2.4536			
$c = 1.0$	0.1343	0.7663	0.7780			
$c = 1.5$	0.6920	0.4332	0.8164			
$c = 2.0$	0.1967	0.1411	0.2421	-0.8647	0.4362	0.9685
$c = 2.5$	-0.3781	0.1274	0.3989			
$c = 3.0$	-0.9269	0.1141	0.9339			
$c = 3.5$	-1.6985	0.1096	1.7020			
			$T = 500$			
$c = 0.5$	2.4142	0.3915	2.4457			
$c = 1.0$	0.8847	0.7202	1.1407			
$c = 1.5$	0.3774	1.0178	1.0855			
$c = 2.0$	0.3233	0.1034	0.3394	-0.8816	0.3085	0.9340
$c = 2.5$	-0.2052	0.0906	0.2243			
$c = 3.0$	-0.7398	0.0823	0.7444			
$c = 3.5$	-1.4803	0.0842	1.4827			

Table 5: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with measurement error variance $\sigma_e^2 = 0.75$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	0.4564	0.4265	0.6247			
$c = 1.5$	0.3153	0.3438	0.4665			
$c = 2.0$	0.0744	0.2413	0.2525	-0.6722	0.4869	0.8300
$c = 2.5$	-0.3206	0.2150	0.3860			
$c = 3.0$	-0.7662	0.2126	0.7951			
$T = 250$						
$c = 1.0$	0.7582	0.3462	0.8335			
$c = 1.5$	0.3540	0.2456	0.4309			
$c = 2.0$	0.1520	0.1785	0.2345	-0.6361	0.3685	0.7351
$c = 2.5$	-0.1881	0.1543	0.2433			
$c = 3.0$	-0.5980	0.1493	0.6164			
$T = 500$						
$c = 1.0$	1.0322	0.3121	1.0784			
$c = 1.5$	0.3452	0.2002	0.3991			
$c = 2.0$	0.2173	0.1325	0.2545	-0.6716	0.2606	0.7204
$c = 2.5$	-0.0950	0.1115	0.1465			
$c = 3.0$	-0.4548	0.1087	0.4676			

Table 6: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with measurement error variance $\sigma_e^2 = 0.75$

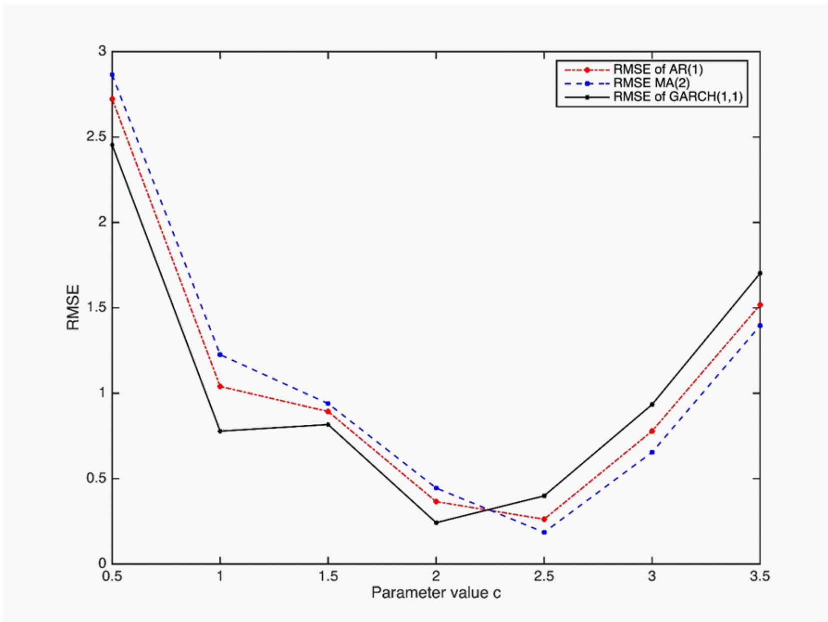
	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.5754	0.4161	0.7101			
$c = 1.5$	0.3268	0.4181	0.5307			
$c = 2.0$	0.1406	0.2411	0.2791	-0.6346	0.4854	0.7990
$c = 2.5$	-0.2384	0.2127	0.3195			
$c = 3.0$	-0.6681	0.1999	0.6973			
			$T = 250$			
$c = 1.0$	0.9228	0.3569	0.9894			
$c = 1.5$	0.3722	0.2910	0.4724			
$c = 2.0$	0.2121	0.1766	0.2760	-0.6331	0.3748	0.7357
$c = 2.5$	-0.1086	0.1621	0.1951			
$c = 3.0$	-0.4925	0.1510	0.5152			
			$T = 500$			
$c = 1.0$	1.2798	0.3519	1.3273			
$c = 1.5$	0.3750	0.2113	0.4305			
$c = 2.0$	0.2794	0.1325	0.3092	-0.6387	0.2576	0.6887
$c = 2.5$	-0.0056	0.1160	0.1162			
$c = 3.0$	-0.3626	0.1018	0.3766			

Table 7: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with measurement error variance $\sigma_e^2 = 0.75$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	0.1081	0.5394	0.5502			
$c = 1.5$	0.3078	0.3164	0.4414			
$c = 2.0$	-0.0396	0.2275	0.2309	-0.6473	0.5195	0.8300
$c = 2.5$	-0.4736	0.1784	0.5061			
$c = 3.0$	-0.9307	0.1599	0.9444			
$T = 250$						
$c = 1.0$	0.5405	0.4196	0.6842			
$c = 1.5$	0.3554	0.2617	0.4413			
$c = 2.0$	0.0722	0.1637	0.1789	-0.6865	0.4118	0.8006
$c = 2.5$	-0.3267	0.1349	0.3535			
$c = 3.0$	-0.7591	0.1170	0.7680			
$T = 500$						
$c = 1.0$	0.8568	0.3994	0.9453			
$c = 1.5$	0.3711	0.2969	0.4752			
$c = 2.0$	0.1448	0.1257	0.1918	-0.7064	0.3092	0.7711
$c = 2.5$	-0.2072	0.1008	0.2305			
$c = 3.0$	-0.6098	0.0906	0.6165			

VaR. Thus, from these simulations we conclude that accounting for measurement errors is indeed very important to draw correct conclusions and measurement errors must not simply be ignored. Finally, as shown in Figure 2, we plot the RMSEs of the adjusted estimator for the 1% VaR and $T = 250$ to examine the sensitivity of this estimator with respect to the adjustment parameter c. The result confirms again that $c = 2.0$ and $c = 2.5$ are good choices under measurement errors.

Figure 2: RMSE of the adjusted estimator of VaR versus bandwidth parameter



Note: This figure plots the Root Mean Squared Error (RMSE) of the adjusted estimator of VaR as a function of the parameter c in the bandwidth parameter when $T = 250$ and under different models AR, MA and GARCH of the simulated actual returns.

As $c = 0.5$ and $c = 3.5$ deliver poor results, in the following we only focus on $c \in \{1, 1.5, 2, 2.5, 3\}$ to save space. To investigate the effects of smaller measurement error variances on our robust estimator, we also consider the cases of $e_t \sim$ i.i.d. $N(0, \sigma_e^2)$ with $\sigma_e^2 = 0.75$ and $\sigma_e^2 = 0.5$. Simulation results for the 1% VaR under $\sigma_e^2 = 0.75$ and $\sigma_e^2 = 0.5$ are reported in Tables 5-7 and Tables A.7-A.9 of the Online supplementary appendix, respectively. For both cases, we still find that the performance of the adjusted VaR for $c = 2.0$ or $c = 2.5$ dominates that of the unadjusted VaR in terms of bias, standard deviation and RMSE. Nonetheless, it is noteworthy that it becomes more challenging to adjust for the presence of measurement errors as the variance of measurement errors gets smaller.

Finally, additional results (not reported, but available upon request) were obtained by considering alternative values (e.g. highly persistent AR process) for the coefficients of Model 1 to Model 3 and for larger variances of the measurement errors (e.g., $\sigma_e^2 = 1.5$ and $\sigma_e^2 = 2$). Regarding the changes in the coefficients of models 1 to 3, we find that higher persistence makes the variance of both adjusted and unadjusted VaRs significantly higher than in the less persistent case. Although these results show that the adjusted VaR does not perform well compared to the unadjusted VaR, we emphasize that this happens under the very high persistent level of the underlying time series process, and in this case any estimator should be used with care since high persistence is known to have a negative impact on the estimators obtained under the stationarity assumption. Furthermore, it is well established that stock returns are weakly persistent, see for example Ding et al. (1993). Concerning the effects of larger measurement error variances, the additional results show that a larger variance of the measurement error affects even more notoriously the performance of the unadjusted VaR, while our proposed adjusted VaR still works reasonably well and delivers much smaller biases and RMSEs. Indeed, the latter finding is not unexpected since a larger variance will further reduce the information in the contaminated returns, and as such, an (unadjusted) estimator that does not address the measurement errors will fail to uncover useful information from the contaminated returns.

4.2 Case of misspecified density of measurement errors

We run additional simulations to investigate the finite sample performance of our approach when the distribution of the measurement errors is misspecified. Specifically, we still estimate our adjusted VaR using formula (15) in Corollary 1, which we construct from the characteristic function of standard normal distribution, but the measurement errors are in fact generated from some non-normal distributions. We consider the same simulation setup as in Section 4.1, but we now generate the measurement errors according to: (i) a Student's t distribution with 5 degrees of freedom, say $t(5)$; and (ii) a mixture of normal distributions, say $0.5N(0,1) + 0.5N(5,1)$. In addition, the variances of $t(5)$ and $0.5N(0,1) + 0.5N(5,1)$ are standardized to one to be comparable to Tables 2 to 4 in Section 4.1. It is important to note that $t(5)$ has fatter tails than the standard normal distribution while $0.5N(0,1) + 0.5N(5,1)$ is bimodal.

The bias, standard deviation, and root mean squared error of the adjusted and unadjusted 1% VaR estimates under $t(5)$ and normal mixture distributions are reported in Tables 8-10 and Tables A.10-A.12 of the Online supplementary appendix, respectively. To save space, we do not report the results for the other coverage rates [i.e., 5% and 10%], but they are available upon

Table 8: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with Student's $t(5)$ measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.3671	0.9794	1.0460			
$c = 1.5$	0.7557	0.4717	0.8909			
$c = 2.0$	0.2247	0.3198	0.3909	-0.9909	0.6771	1.2001
$c = 2.5$	-0.3913	0.2715	0.4762			
$c = 3.0$	-0.9725	0.2211	0.9973			
			$T = 250$			
$c = 1.0$	1.0682	1.0054	1.4669			
$c = 1.5$	0.8367	0.5841	1.0204			
$c = 2.0$	0.3573	0.2788	0.4532	-0.9583	0.4680	1.0665
$c = 2.5$	-0.1845	0.1539	0.2403			
$c = 3.0$	-0.7459	0.1485	0.7606			
			$T = 500$			
$c = 1.0$	1.7290	0.9493	1.9725			
$c = 1.5$	0.8563	0.7922	1.1665			
$c = 2.0$	0.4818	0.4557	0.6631	-0.9557	0.3196	1.0077
$c = 2.5$	-0.0342	0.1077	0.1129			
$c = 3.0$	-0.5512	0.1059	0.5613			

request. For each of the misspecified cases, the above tables show similar patterns to those found for the correctly specified case [see Tables 2 to 4 in Section 4.1]. This indicates a good degree of robustness of our proposed methodology to various deviations to the misspecified measurement errors. In particular, we find that the estimates of the adjusted VaR perform better than the estimates of the unadjusted VaR. As in the specified measurement error density case, we find that the performance of our approach depends on the bandwidth parameter. But again, after providing a battery of simulations for different values of c, we see that the performance of our method is generally much better in terms of bias, standard deviation, and root mean squared error when $c = 2.0$ or 2.5.

Table 9: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with Student's $t(5)$ measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.6103	0.9902	1.1632			
$c = 1.5$	0.8831	0.5677	1.0499			
$c = 2.0$	0.2994	0.2492	0.3895	-0.9571	0.6736	1.1704
$c = 2.5$	-0.2848	0.2297	0.3659			
$c = 3.0$	-0.8506	0.2092	0.8759			
			$T = 250$			
$c = 1.0$	1.3156	0.9433	1.6188			
$c = 1.5$	0.9656	0.2983	1.0106			
$c = 2.0$	0.4392	0.2754	0.5184	-0.9018	0.4732	1.0184
$c = 2.5$	-0.0778	0.1736	0.1902			
$c = 3.0$	-0.6322	0.1476	0.6492			
			$T = 500$			
$c = 1.0$	1.8755	0.9434	2.0994			
$c = 1.5$	1.0148	0.6881	1.2261			
$c = 2.0$	0.5783	0.2252	0.6206	-0.9166	0.3273	0.9733
$c = 2.5$	0.0620	0.1121	0.1281			
$c = 3.0$	-0.4517	0.1301	0.4701			

Table 10: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with Student's $t(5)$ measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.0854	0.9749	0.9786			
$c = 1.5$	0.6637	0.3147	0.7345			
$c = 2.0$	0.0459	0.1913	0.1968	-1.0642	0.7198	1.2847
$c = 2.5$	-0.5472	0.1747	0.5744			
$c = 3.0$	-1.1578	0.1582	1.1685			
			$T = 250$			
$c = 1.0$	0.5376	1.1656	1.2836			
$c = 1.5$	0.6982	0.6209	0.9343			
$c = 2.0$	0.2183	0.3468	0.4098	-1.0061	0.5119	1.1288
$c = 2.5$	-0.3538	0.1218	0.3742			
$c = 3.0$	-0.9169	0.1164	0.9243			
			$T = 500$			
$c = 1.0$	1.3423	1.1259	1.7519			
$c = 1.5$	-0.0641	1.3668	1.3683			
$c = 2.0$	0.3640	0.1790	0.4056	-1.0216	0.3715	1.0871
$c = 2.5$	-0.1873	0.0930	0.2091			
$c = 3.0$	-0.7293	0.0818	0.7339			

Table 11: Time-span of the high-frequency data

Index	**From**	**To**	**No. of Observations**
CAC 40	02/01/2000	29/12/2017	458,353
DAX 30	31/05/2009	29/12/2017	227,508
FTSE 100	04/01/2000	29/12/2017	501,185
FTSE MIB	03/01/2000	29/12/2017	508,175
S&P 500	03/01/2000	29/12/2017	394,898

Note: This table reports the time-span and the number of observations on the sparse trade price series collected at a five minute sampling frequency for the indixes CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500.

Finally, we consider another simulation exercise where we compare the performance of the adjusted and unadjusted approaches when measurement errors are not present in the data (i.e., $\sigma_e^2 = 0$). We use the same simulation setup as in Section 4.1. The simulation results are reported in Tables A.13-A.15 of the Online supplementary appendix. One should expect that the unadjusted approach will perform better than our approach, however when we examine the above tables we see that the adjusted approach is doing well compared to the unadjusted one. It is true that the unadjusted estimator generally leads to a smaller bias, but when the "optimal" bandwidth parameter [$c = 2.5$ or $c = 2.0$] is used, the adjusted estimator still has much smaller variance and consequently a smaller mean squared error for all the DGPs under consideration.

5 Empirical application

In this section, we apply the semiparametric approach we proposed in the previous sections to high-frequency data to estimate the VaR of five international stock market indices over one day horizon. We compare our results to the unadjusted approach that estimates VaR by simply computing the sample quantiles of the five indices' returns.

Our data consist of high-frequency tick-by-tick trade prices on the stock market indices CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500, which we obtained from the Thomson Reuter's Tick History (TRTH) database, over the period January 2000 to December 2017. Our interest specifically lies in the sparse trade prices with a five minutes sampling frequency. Table 11 reports the time-span and the number of observations corresponding to the sparse trade price series collected at a

Table 12: The adjusted and unadjusted estimates of VaR for high-frequency financial returns

	Adjusted VaR		**Unadjusted VaR**	
	5%	10%	5%	10%
CAC 40	-2.2623	-1.9142	-1.6491	-1.1448
DAX 30	-2.3325	-1.9741	-1.6446	-1.1624
FTSE 100	-2.2281	-1.8855	-1.5763	-1.0888
FTSE MIB	-2.2415	-1.8969	-1.6105	-1.1255
S&P 500	-2.2373	-1.8936	-1.6038	-1.0849

Note: This table reports the estimated 5% and 10% VaR of standardized returns of CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500, using the approach introduced in Section 3 [adjusted VaR] and the unadjusted approach that does not adjust for measurement errors [unadjusted VaR]. Here we use $c = 2.5$.

five minutes sampling frequency. Evidently, the price series for each index expands to the desired time-span, with the exception of the DAX 30 index, which has trade prices only available from the 31$^{\text{st}}$ May 2009. We then use the above data to calculate the continuously compounded returns over each five minutes interval by taking the difference between the logarithm of the two tick prices immediately preceding each five minutes mark.

As we have seen before, the implementation of the semiparametric approach to estimate VaR requires the knowledge of the variance of the measurement errors. This variance, however, is unknown but it can be estimated using high-frequency data as shown at the end of Section 3. The semiparametric approach introduced in Section 3 is applied to estimate the 5% and 10% VaRs of each standardized stock index return [hereafter adjusted VaR]. We standardize the returns (returns divided by their standard deviations) to fairly compare the VaRs of the five stock indices. In addition, for comparison, we estimate the unadjusted 5% and 10% VaRs of these stock indices by simply calculating the sample quantiles using order statistics [hereafter unadjusted VaR].

The results are reported in Table 12. As expected, for both adjusted and unadjusted estimates of VaR, we see that the loss is higher at 5% than 10% statistical confidence levels. Interestingly, for all stock indices and confidence levels, we see that the adjusted estimates of VaRs are much bigger - in absolute value - than the unadjusted estimates. This suggests that ignoring measurement errors might lead to an underestimation of risk. If we take the example of S&P 500 index, the adjusted estimate of 5% VaR is 0.6335 (2.2373-1.6038) higher than the unadjusted one. Thus, an investor who invests, for example, $100 million in the S&P 500 index and uses unadjusted VaR will think that the magnitude of risk at 5% confidence level is equal to \$208,494 a day [\$100 million $\times$ 1.6038$\times$0.0013 (standard deviation of S&P 500 index)], whereas the true magnitude of risk (according to the adjusted estimate of VaR) is \$290,849 a day [\$100 million $\times$ 2.2373$\times$0.0013]. Hence, this investor will face an unexpected additional loss of \$82,355 a day [\$100 million $\times$ 0.6335$\times$0.0013]. Finally, Table 12 shows that the values of adjusted and unadjusted VaRs are similar across the five stock indices, which might indicate that international stock markets are driven by some common factors.

6 Conclusions

We have proposed a semiparametric approach for estimating the VaR of a portfolio of contaminated stock returns. We have shown that measurement errors cause serious problems for estimating risk, and unfortunately the existing methods are inconsistent in the presence of measurement errors. Using Fourier transform, we derived a robust estimator of VaR that takes into account measurement errors. We first used a deconvolution kernel estimator for the density function of the actual latent portfolio returns to deal with measurement errors. Second, we used Fourier inversion to calculate the probability distribution function of the latent portfolio returns. Thereafter, we used power series representations of sine and exponential functions to approximate the integral in the inversion formula and made the calculation of VaR feasible.

The derivation of robust estimator of VaR was first made under the assumption of known measurement errors' density, but the distribution of the observed portfolio returns was always treated as unknown and estimated nonparametrically. Thereafter, we relaxed this assumption and suggested a feasible way to deal with the measurement errors' distribution. We followed the literature and considered a parameterized measurement error's distribution. In particular, we assumed that the measurement error is normally distributed but with unknown variance that we estimated using high-frequency data and a consistent estimator of variance of measurement errors from Zhang et al. (2005).

Furthermore, we conducted a set of Monte Carlo simulations to examine the performance of our approach. We also provided a comparison with a model-free estimator of VaR that does not take into account measurement errors. We investigated the performance of our approach under different densities of the measurement errors and the simulation results were encouraging. Finally, we used our approach and high-frequency data to estimate the adjusted VaR of five US and European stock indices. We compared our results to the unadjusted VaR, which we estimated using a model-free approach that simply computes the sample quantiles based on the five indices' historical returns. The empirical results showed that ignoring measurement errors generally leads to an underestimation of risk.

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A Online supplementary appendix

In this online supplementary appendix, we provide the proofs of the main theoretical results as well as tables of the additional simulation results that are omitted from the main text.

A.1 Proofs of theoretical results

Proof of Corollary 1. We first calculate the integral:

$$\int_0^1 \frac{\exp\left(as^2\right)\sin(\kappa s)}{s} ds,$$

where $a = \frac{\sum_{j=1}^n \sigma_{\epsilon_j}^2 \omega_j^2}{b^2} > 0$ and $\kappa = \frac{r_{p,t} + VaR^\alpha(r_{p,t}^*)}{b}$. To do that, we use power series representation of $\sin(\kappa s)$ function:

$$\sin(\kappa s) = \sum_{i=0}^{\infty} \frac{(-1)^i \kappa^{1+2i} s^{1+2i}}{(1+2i)!}.$$

Furthermore, the power series representation of $\exp\left(as^2\right)$ is given by: $\exp(as^2) = \sum_{l=0}^{\infty} \frac{a^l s^{2l}}{l!}$. Thus,

$$\frac{\exp(as^2)\sin(\kappa s)}{s} = \sum_{i,j=0}^{\infty} \frac{a^i (-1)^j \kappa^{1+2j} s^{2i+2j}}{i!(1+2j)!}.$$

Consequently,

$$\int_0^1 \frac{\exp(as^2)\sin(\kappa s)}{s} ds = \sum_{i,j=0}^{\infty} \frac{a^i (-1)^j \kappa^{1+2j}}{i!(1+2j)!} \int_0^1 s^{2i+2j} ds = \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}.$$

Hence, the VaR of the latent portfolio's return $r_{p,t}^*$ with coverage probability α, denoted by $VaR^\alpha(r_{p,t}^*)$, is the solution of the following optimization problem:

$$\begin{aligned}
\widehat{VaR}^\alpha(r_{p,t}^*) &= \underset{VaR^\alpha(r_{p,t}^*)}{Argmin} \left[\frac{1}{T} \sum_{t=1}^{T} \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{2i+2j+1} - \left(\frac{1}{2} - \alpha\right)\pi \right]^2 \\
&= \underset{VaR^\alpha(r_{p,t}^*)}{Argmin} \left[\sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \frac{1}{T}\sum_{t=1}^{T} \left(\frac{r_{p,t}+VaR^\alpha(r_{p,t}^*)}{b}\right)^{1+2j}}{(1+2j)!\,(2i+2j+1)} - \left(\frac{1}{2} - \alpha\right)\pi \right]^2,
\end{aligned}$$

with $a = \frac{\sum_{j=1}^n \sigma_{\epsilon_j}^2 \omega_j^2}{b^2} > 0$. ■

Lemma 1: *A series of the form $S = \sum_n^{\infty} (-1)^n \varepsilon_n$ where either all ε_n are positive or all ε_n are negative is called an alternating series. Then says: if $|\varepsilon_n|$ decreases monotonically and $\lim_{n\to\infty} \varepsilon_n = 0$ then the partial sum $S_q = \sum_n^q (-1)^n a_n$ approximates S with error bounded by the next omitted term:*

$$R_q = \sum_{n=q}^{\infty} (-1)^n a_n \le |\varepsilon_{q+1}|.$$

Proof of Lemma 1. First of all, we know that when an alternating series converges to its limit S, this means the partial sum of this alternating series also "alternates" above and below the final limit, i.e., $S_{2q} < S < S_{2q+1}$.

We now show $|S_k - S| \leq \varepsilon_{k+1}$ by considering two cases:

1. When $k = 2q+1$, i.e., k is odd, then we have:

$$|S_{2q+1} - S| = S_{2q+1} - S \leq S_{2q+1} - S_{2q+2} = \varepsilon_{(2q+1)+1}.$$

2. When $k = 2q$, i.e., k is even, then we have:

$$|S_{2q} - S| = S - S_{2q} \leq S_{2q+1} - S_{2q} = \varepsilon_{2q+1}.$$

Both cases rely essentially on the inequality $S_{2q} < S < S_{2q+1}$. ■

Proof: Bounds of the remaining terms for the truncation of the double infinite sum in equation (15). First of all, note that the double infinite sum $\sum_{i,j=0}^{\infty}$ in the optimization problem in equation (15) can be decomposed as follows:

$$S = \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = S_{l,k} + R_{l,k},$$

where the truncated and remaining terms $S_{l,k}$ and $R_{l,k}$ are respectively given by:

$$S_{l,k} = \textstyle\sum_{i=0}^{l} \sum_{j=0}^{k} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)},$$

$$\begin{aligned} R_{l,k} = & \underbrace{\sum_{i=l+1}^{\infty} \sum_{j=0}^{k} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(1)}} + \underbrace{\sum_{i=0}^{l} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(2)}} \\ & + \underbrace{\sum_{i=l+1}^{\infty} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(3)}}, \end{aligned} \tag{21}$$

with $a = \frac{\sum_{j=1}^{n} \sigma_{\epsilon_j}^2 \omega_j^2}{b^2}$ and $\kappa = \frac{r_{p,t} + VaR^{\alpha}(r_{p,t}^*)}{b}$. We next derive a bound for each remaining term $R_{l,k}^{(1)}$, $R_{l,k}^{(2)}$ and $R_{l,k}^{(3)}$.

1). The bound for $R_{l,k}^{(1)}$: First, observe that

$$R_{l,k}^{(1)} = \sum_{i=l+1}^{\infty} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = \sum_{j=k+1}^{\infty} (-1)^j \varepsilon_j,$$

is an alternating series, where the term $\varepsilon_j = \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{\kappa^{2j+1}}{(1+2j)!} \frac{1}{(2i+2j+1)}$ is decreasing to zero since $\lim_{j\to\infty} \frac{\kappa^{2j+1}}{(1+2j)!} = 0$. From Lemma 1 of this Appendix, we obtain:

$$\begin{aligned} |R_{l,k}^{(1)}| &\leq |\varepsilon_{k+1}| = \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{1}{(2i+2k+3)} \\ &\leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{1}{(2i+2k+3)} \leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{\exp{(a)}}{(2l+2k+5)}, \end{aligned} \tag{22}$$

since $\sum_{i=l+1}^{\infty} \frac{a^i}{i!} \leq \sum_{i=0}^{\infty} \frac{a^i}{i!} = \exp(a)$ and a is a positive number defined before. Thus, $\lim\limits_{l,k\longrightarrow\infty} \left\{R_{l,k}^{(1)}\right\} = 0$.

2). The bound for $R_{l,k}^{(2)}$: Following the same argument as the one for the bound of $R_{l,k}^{(1)}$, we have

$$R_{l,k}^{(2)} = \sum_{i=0}^{l} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = \sum_{j=k+1}^{\infty} (-1)^j \varepsilon_j, \tag{23}$$

is an alternating series, where the term $\varepsilon_j = \sum_{i=0}^{l} \frac{a^i}{i!} \frac{\kappa^{2j+1}}{(1+2j)!} \frac{1}{(2i+2j+1)}$ is decreasing to zero since $\lim_{j\to\infty} \frac{\kappa^{2j+1}}{(1+2j)!} = 0$. From Lemma 1 of this Appendix, we obtain:

$$|R_{l,k}^{(2)}| \leq |\varepsilon_{k+1}| = \sum_{i=0}^{l} \frac{a^i}{i!} \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{1}{(2i+2k+3)} \leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \sum_{i=0}^{l} \frac{a^i}{i!} \frac{1}{(2i+2k+3)} \leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{\exp{(a)}}{(2k+3)}.$$

Thus, as for bound of $R_{l,k}^{(1)}$, we have $\lim\limits_{l,k\longrightarrow\infty} \left\{R_{l,k}^{(2)}\right\} = 0$.

3). The bound for $R_{l,k}^{(3)}$: Observe that:

$$\begin{aligned} |R_{l,k}^{(3)}| &= \left| \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \sum_{j=0}^{k} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} \right| \leq \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{1}{(2i+1)} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!} \\ &\leq \frac{1}{(2l+3)} \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!} \leq \frac{\exp{(a)}}{(2l+3)} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!}. \end{aligned} \tag{24}$$

Now, if we define $\nu_j = \frac{|\kappa|^{1+2j}}{(1+2j)!}$, we obtain:

$$\frac{\nu_{j+1}}{\nu_j} = (\frac{|\kappa|^{3+2j}}{(3+2j)!})/(\frac{|\kappa|^{1+2j}}{(1+2j)!}) = \frac{\kappa^2}{(2j+3)(2j+2)}.$$

From the above equation and based on D'Alembert criterion, the series $\sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!}$ converges for $k \longrightarrow \infty$, and consequently $\lim\limits_{l,k\longrightarrow\infty} \left\{R_{l,k}^{(3)}\right\} = 0$. ■

A.2 Characteristic function of the actual portfolio return under non-normal measurement errors

Hereafter, we provide the expression of the characteristic function of the actual portfolio return when the measurement error follows a non-normal distribution. We focus on the case of stable distributions, but the results can be extended to any other distribution for which the characteristic function of measurement errors has a closed-form expression.

Recall that from equation (3) of the main text, we have:

$$\phi_{r_p^*}(s) = \frac{\phi_{r_p}(s)}{\phi_e(s)}.$$

Now, if in the above formula we replace $\phi_{r_p}(s)$ by its nonparametric estimator [see equation (5) of the main text], we obtain:

$$\hat{\phi}_{r_p^*}(s) = \frac{1}{T}\sum_{t=1}^{T}\frac{\exp\left(\mathrm{i}sr_{p,t}\right)}{\phi_e(s)}. \tag{25}$$

Using equation (25), to compute the characteristic function of the actual latent portfolio return r_p^* we only need an expression for the characteristic function of measurement errors $\phi_e(s)$. The normality of the measurement errors is not essential for obtaining an expression for the characteristic function $\hat{\phi}_{r_p^*}(s)$. For example, if instead of normal distributions we assume that the measurement errors u_j follow stable distributions $u_j \sim \mathit{Stable}\left(\mu_j = 0, c_j, \beta_j, \alpha_j\right)$ - with $\mu_j, c_j, \beta_j, \alpha_j$ are the location, scale, skewness, and stability parameters, respectively -, then a linear combination of independent measurement errors with stable distributions ($e_t = \sum_{j=1}^{n}\omega_j u_{j,t}$) follow stable distribution, up to location and scale parameters. Formally, observe that

$$\phi_e(s) = E\left[\exp\left(\mathrm{i}se_t\right)\right] = E\left[\exp\left(\mathrm{i}s\sum_{j=1}^{n}\omega_j u_{j,t}\right)\right] = \prod_{j=1}^{n}E\left[\exp\left(\mathrm{i}s\omega_j u_{j,t}\right)\right].$$

Since the characteristic function of stable distribution $u_j \sim \mathit{Stable}\left(\mu_j = 0, c_j, \beta_j, \alpha_j\right)$ is given by:

$$\varphi\left(s, 0, c_j, \beta_j, \alpha_j\right) = \exp\left(-\left|c_j s\right|^{\alpha_j}\left(1 - \beta_j sgn\left(s\right)\Phi_j\right)\right),$$

where $sgn\left(s\right)$ is the sign of s and

$$\Phi_j = \begin{cases} \tan\left(\frac{\pi\alpha_j}{2}\right) \text{ if } \alpha_j \neq 1 \\ -\frac{2}{\pi}\log|s| \text{ if } \alpha_j = 1 \end{cases}.$$

we obtain:

$$\phi_e(s) = \prod_{j=1}^{n}\exp\left(-\left|c_j s\omega_j\right|^{\alpha_j}\left(1 - \beta_j sgn\left(s\omega_j\right)\Phi_j\right)\right) = \exp\left(-\sum_{j=1}^{n}\left|c_j s\omega_j\right|^{\alpha_j}\left(1 - \beta_j sgn\left(s\omega_j\right)\Phi_j\right)\right),$$

where

$$\Phi_j = \begin{cases} \tan\left(\frac{\pi\alpha_j}{2}\right) \text{ if } \alpha_j \neq 1 \\ -\frac{2}{\pi}\log|s\omega_j| \text{ if } \alpha_j = 1 \end{cases}.$$

Consequently,

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} = \frac{1}{T}\sum_{t=1}^{T}\exp\left(\mathrm{i}sr_{p,t} + \sum_{j=1}^{n}|c_j s\omega_j|^{\alpha_j}\left(1 - \beta_j sgn\left(s\omega_j\right)\Phi_j\right)\right),$$

which we can invert [using Fourier-inversion formula in Gil-Pelaez (1951)] to obtain the distribution function of actual latent portfolio return r_p^*, and then the VaR of r_p^* as we describe in the main text.

A.3 Additional Monte Carlo simulation results

In this section, we report some additional simulation results that are discussed but omitted from the main text. The simulation setup is described in Section 4 of the main text. Given the fact that $c = 0.5$ violates the condition $c > 1$ and our simulation shows that this case delivers poor results, to save space we do not report results for $c = 0.5$ and concentrate on $c \in \{1.0, 1.5, 2.0, 2.5, 3.0\}$ in the following tables.

Specifically speaking, Tables A.1-A.3 in the appendix report results for the 5% VaR under Model 1 to Model 3, while Tables A.4-A.6 report results for the 10% VaR. Tables A.7-A.9 report results for 1% VaR when the measurement error variance is $\sigma_e^2 = 0.5$. Tables A.10-A.12 report results for 1% VaR when the measurement error follows a normal mixture distribution (standardized to have variance one). Finally, Tables A.13-A.15 report results for 1% VaR when there does not exist measurement error (i.e., when $\sigma_e^2 = 0$).

Table A.1: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the AR(1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.5183	0.8712	1.0137			
$c = 1.5$	0.1771	0.4417	0.4759			
$c = 2.0$	-0.2947	0.2129	0.3636	-0.5917	0.3217	0.6735
$c = 2.5$	-0.8063	0.1977	0.8301			
$c = 3.0$	-1.3169	0.2108	1.3337			
			$T = 250$			
$c = 1.0$	0.1543	0.7206	0.7369			
$c = 1.5$	0.2503	0.3918	0.4649			
$c = 2.0$	-0.1672	0.1570	0.2293	-0.6069	0.2318	0.6497
$c = 2.5$	-0.6449	0.1500	0.6621			
$c = 3.0$	-1.1200	0.1472	1.1296			
			$T = 500$			
$c = 1.0$	0.7285	0.3469	0.8069			
$c = 1.5$	0.2865	0.5020	0.5781			
$c = 2.0$	-0.0668	0.1176	0.1353	-0.6102	0.1680	0.6329
$c = 2.5$	-0.5034	0.1073	0.5147			
$c = 3.0$	-0.9610	0.1038	0.9666			

Table A.2: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the MA(2) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.2425	0.8210	0.8561			
$c = 1.5$	0.2376	0.3270	0.4042			
$c = 2.0$	-0.2231	0.2139	0.3091	-0.5709	0.3280	0.6585
$c = 2.5$	-0.7270	0.2029	0.7548			
$c = 3.0$	-1.2461	0.2011	1.2622			
$T = 250$						
$c = 1.0$	0.4182	0.5876	0.7212			
$c = 1.5$	0.3095	0.3115	0.4391			
$c = 2.0$	-0.1084	0.1546	0.1888	-0.5815	0.2363	0.6276
$c = 2.5$	-0.5666	0.1447	0.5848			
$c = 3.0$	-1.0555	0.1397	1.0648			
$T = 500$						
$c = 1.0$	0.8183	0.2813	0.8653			
$c = 1.5$	0.3712	0.3382	0.5022			
$c = 2.0$	-0.0060	0.1119	0.1121	-0.5844	0.1665	0.6077
$c = 2.5$	-0.4353	0.1081	0.4485			
$c = 3.0$	-0.8856	0.1019	0.8914			

Table A.3: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the GARCH(1,1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-1.3376	0.7844	1.5507			
$c = 1.5$	0.0393	0.1983	0.2021			
$c = 2.0$	-0.4766	0.1742	0.5074	-0.6867	0.3290	0.7615
$c = 2.5$	-1.0081	0.1662	1.0217			
$c = 3.0$	-1.5350	0.1512	1.5424			
			$T = 250$			
$c = 1.0$	-0.6805	0.8463	1.0860			
$c = 1.5$	0.0782	0.5265	0.5323			
$c = 2.0$	-0.3426	0.1213	0.3635	-0.6912	0.2329	0.7294
$c = 2.5$	-0.8387	0.1095	0.8458			
$c = 3.0$	-1.3359	0.1101	1.3404			
			$T = 500$			
$c = 1.0$	0.1386	0.7616	0.7741			
$c = 1.5$	-0.4823	1.4075	1.4878			
$c = 2.0$	-0.2274	0.0931	0.2457	-0.6961	0.1716	0.7170
$c = 2.5$	-0.7022	0.0846	0.7073			
$c = 3.0$	-1.1662	0.0791	1.1689			

Table A.4: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the AR(1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.8049	0.9783	1.2668			
$c = 1.5$	0.0087	0.3893	0.3894			
$c = 2.0$	-0.3910	0.2157	0.4466	-0.4674	0.2782	0.5439
$c = 2.5$	-0.8328	0.2104	0.8589			
$c = 3.0$	-1.2632	0.2023	1.2793			
			$T = 250$			
$c = 1.0$	-0.0657	0.7661	0.7689			
$c = 1.5$	0.0727	0.4323	0.4384			
$c = 2.0$	-0.2895	0.1511	0.3265	-0.4697	0.1988	0.5101
$c = 2.5$	-0.6933	0.1486	0.7090			
$c = 3.0$	-1.0974	0.1466	1.1072			
			$T = 500$			
$c = 1.0$	0.4559	0.3727	0.5888			
$c = 1.5$	0.0738	0.5994	0.6039			
$c = 2.0$	-0.1918	0.1104	0.2213	-0.4723	0.1378	0.4920
$c = 2.5$	-0.5699	0.1074	0.5800			
$c = 3.0$	-0.9528	0.1064	0.9587			

Table A.5: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the MA(2) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.5198	0.9414	1.0754			
$c = 1.5$	0.0610	0.3142	0.3201			
$c = 2.0$	-0.3433	0.2112	0.4030	-0.4449	0.2777	0.5245
$c = 2.5$	-0.7749	0.2008	0.8005			
$c = 3.0$	-1.2025	0.2012	1.2192			
$T = 250$						
$c = 1.0$	0.1582	0.6488	0.6678			
$c = 1.5$	0.1302	0.3182	0.3438			
$c = 2.0$	-0.2335	0.1390	0.2718	-0.4552	0.1971	0.4961
$c = 2.5$	-0.6397	0.1434	0.6555			
$c = 3.0$	-1.0391	0.1401	1.0485			
$T = 500$						
$c = 1.0$	0.5451	0.2662	0.6066			
$c = 1.5$	0.1761	0.3547	0.3960			
$c = 2.0$	-0.1444	0.1074	0.1800	-0.4589	0.1397	0.4797
$c = 2.5$	-0.5182	0.1004	0.5279			
$c = 3.0$	-0.9014	0.1006	0.9070			

Table A.6: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the GARCH(1,1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-1.7152	0.8601	1.9188			
$c = 1.5$	-0.1212	0.2618	0.2885			
$c = 2.0$	-0.5634	0.1592	0.5854	-0.5541	0.2652	0.6143
$c = 2.5$	-1.0119	0.1514	1.0231			
$c = 3.0$	-1.4607	0.1405	1.4675			
$T = 250$						
$c = 1.0$	-0.9770	0.9381	1.3545			
$c = 1.5$	-0.0774	0.4375	0.4443			
$c = 2.0$	-0.4398	0.1168	0.4551	-0.5554	0.1867	0.5860
$c = 2.5$	-0.8608	0.1095	0.8677			
$c = 3.0$	-1.2842	0.0972	1.2879			
$T = 500$						
$c = 1.0$	-0.1552	0.8065	0.8213			
$c = 1.5$	-0.6550	1.4626	1.6025			
$c = 2.0$	-0.3411	0.0821	0.3508	-0.5584	0.1346	0.5744
$c = 2.5$	-0.7417	0.0763	0.7456			
$c = 3.0$	-1.1405	0.0735	1.1429			

Table A.7: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with measurement error variance $\sigma_e^2 = 0.5$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.9525	0.3902	1.0294			
$c = 1.5$	0.2043	0.5018	0.5418			
$c = 2.0$	0.2879	0.2646	0.3911	-0.4362	0.4478	0.6251
$c = 2.5$	0.0424	0.2256	0.2296			
$c = 3.0$	-0.2824	0.2121	0.3532			
			$T = 250$			
$c = 1.0$	1.2890	0.3591	1.3381			
$c = 1.5$	0.2385	0.3546	0.4273			
$c = 2.0$	0.3279	0.1896	0.3788	-0.4568	0.3483	0.5745
$c = 2.5$	0.1337	0.1615	0.2097			
$c = 3.0$	-0.1622	0.1535	0.2233			
			$T = 500$			
$c = 1.0$	1.6524	0.3578	1.6906			
$c = 1.5$	0.3168	0.2443	0.4000			
$c = 2.0$	0.3525	0.1480	0.3823	-0.4763	0.2477	0.5368
$c = 2.5$	0.1930	0.1181	0.2263			
$c = 3.0$	-0.0521	0.1094	0.1212			

Table A.8: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with measurement error variance $\sigma_e^2 = 0.5$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	1.1177	0.4196	1.1938			
$c = 1.5$	0.2398	0.4938	0.5490			
$c = 2.0$	0.3417	0.2774	0.4401	-0.4341	0.4832	0.6496
$c = 2.5$	0.1245	0.2260	0.2580			
$c = 3.0$	-0.1956	0.2056	0.2838			
			$T = 250$			
$c = 1.0$	1.5102	0.3801	1.5573			
$c = 1.5$	0.3060	0.3373	0.4554			
$c = 2.0$	0.3772	0.2016	0.4277	-0.4181	0.3594	0.5512
$c = 2.5$	0.1935	0.1705	0.2579			
$c = 3.0$	-0.0736	0.1529	0.1696			
			$T = 500$			
$c = 1.0$	1.9124	0.3749	1.9488			
$c = 1.5$	0.3779	0.2478	0.4519			
$c = 2.0$	0.3779	0.1615	0.4110	-0.4351	0.2562	0.5049
$c = 2.5$	0.2539	0.1225	0.2819			
$c = 3.0$	0.0218	0.1110	0.1131			

Table A.9: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with measurement error variance $\sigma_e^2 = 0.5$

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.6636	0.4234	0.7871			
$c = 1.5$	-0.2524	0.8241	0.8618			
$c = 2.0$	0.2135	0.2737	0.3471	-0.4057	0.5827	0.7100
$c = 2.5$	-0.0694	0.2077	0.2190			
$c = 3.0$	-0.4385	0.1785	0.4734			
			$T = 250$			
$c = 1.0$	0.9993	0.3962	1.0750			
$c = 1.5$	-0.2939	0.7989	0.8513			
$c = 2.0$	0.3070	0.2090	0.3714	-0.4445	0.4379	0.6240
$c = 2.5$	0.0306	0.1538	0.1568			
$c = 3.0$	-0.2863	0.1255	0.3126			
			$T = 500$			
$c = 1.0$	1.3344	0.4577	1.4107			
$c = 1.5$	-0.0926	0.6538	0.6603			
$c = 2.0$	0.3586	0.1514	0.3892	-0.4706	0.3523	0.5879
$c = 2.5$	0.1146	0.1193	0.1654			
$c = 3.0$	-0.1729	0.0945	0.1971			

Table A.10: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with normal mixture measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.2242	0.7193	0.7534			
$c = 1.5$	0.6918	0.3010	0.7544			
$c = 2.0$	0.1664	0.2214	0.2770	-0.7016	0.4428	0.8297
$c = 2.5$	-0.3986	0.2046	0.4481			
$c = 3.0$	-0.9766	0.2054	0.9980			
			$T = 250$			
$c = 1.0$	0.7986	0.5333	0.9603			
$c = 1.5$	0.7782	0.2178	0.8081			
$c = 2.0$	0.3127	0.1629	0.3525	-0.7086	0.3347	0.7836
$c = 2.5$	-0.2172	0.1512	0.2646			
$c = 3.0$	-0.7580	0.1445	0.7716			
			$T = 500$			
$c = 1.0$	1.1445	0.2733	1.1767			
$c = 1.5$	0.8383	0.2956	0.8889			
$c = 2.0$	0.4314	0.1174	0.4470	-0.7259	0.2371	0.7636
$c = 2.5$	-0.0642	0.1052	0.1233			
$c = 3.0$	-0.5691	0.1018	0.5781			

Table A.11: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with normal mixture measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.4628	0.6784	0.8213			
$c = 1.5$	0.7629	0.3033	0.8210			
$c = 2.0$	0.2613	0.2240	0.3442	-0.6833	0.4551	0.8210
$c = 2.5$	-0.2938	0.2038	0.3576			
$c = 3.0$	-0.8780	0.2010	0.9007			
			$T = 250$			
$c = 1.0$	0.9930	0.4417	1.0868			
$c = 1.5$	0.8637	0.2246	0.8925			
$c = 2.0$	0.3937	0.1594	0.4247	-0.6957	0.3492	0.7785
$c = 2.5$	-0.1060	0.1481	0.1822			
$c = 3.0$	-0.6497	0.1418	0.6649			
			$T = 500$			
$c = 1.0$	1.2706	0.2701	1.2990			
$c = 1.5$	0.9337	0.1779	0.9505			
$c = 2.0$	0.5111	0.1101	0.5228	-0.7028	0.2426	0.7435
$c = 2.5$	0.0339	0.1051	0.1104			
$c = 3.0$	-0.4660	0.1048	0.4776			

Table A.12: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with normal mixture measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.4889	0.7829	0.9230			
$c = 1.5$	0.5925	0.2473	0.6420			
$c = 2.0$	0.0293	0.1824	0.1848	-0.6861	0.5253	0.8641
$c = 2.5$	-0.5666	0.1688	0.5912			
$c = 3.0$	-1.1610	0.1520	1.1709			
$T = 250$						
$c = 1.0$	0.0647	0.7885	0.7911			
$c = 1.5$	0.6880	0.2817	0.7434			
$c = 2.0$	0.1748	0.1350	0.2209	-0.7050	0.4365	0.8292
$c = 2.5$	-0.3801	0.1190	0.3983			
$c = 3.0$	-0.9383	0.1197	0.9459			
$T = 500$						
$c = 1.0$	0.8506	0.5781	1.0285			
$c = 1.5$	0.6384	0.6114	0.8839			
$c = 2.0$	0.3025	0.1013	0.3190	-0.7220	0.3082	0.7851
$c = 2.5$	-0.2127	0.0823	0.2281			
$c = 3.0$	-0.7445	0.0834	0.7491			

Table A.13: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with no measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.3574	0.3240	0.4824			
$c = 1.5$	0.8470	0.8722	1.2158			
$c = 2.0$	0.5068	0.1921	0.5420	0.0277	0.3983	0.3993
$c = 2.5$	-0.1303	0.1852	0.2264			
$c = 3.0$	-0.7713	0.1883	0.7939			
			$T = 250$			
$c = 1.0$	-0.1230	0.2436	0.2729			
$c = 1.5$	0.3654	1.2783	1.3295			
$c = 2.0$	0.6653	0.1323	0.6784	0.0466	0.2971	0.3008
$c = 2.5$	0.0637	0.1347	0.1490			
$c = 3.0$	-0.5226	0.1254	0.5374			
			$T = 500$			
$c = 1.0$	0.0718	0.2694	0.2788			
$c = 1.5$	-1.1150	0.8139	1.3804			
$c = 2.0$	0.8120	0.0985	0.8179	0.0055	0.2223	0.2224
$c = 2.5$	0.2365	0.0923	0.2539			
$c = 3.0$	-0.3254	0.0936	0.3386			

Table A.14: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with no measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.2503	0.3240	0.4258			
$c = 1.5$	1.0800	0.8722	1.2594			
$c = 2.0$	0.6093	0.1921	0.6355	0.0473	0.4139	0.4166
$c = 2.5$	-0.0138	0.1852	0.1801			
$c = 3.0$	-0.6562	0.1883	0.6789			
$T = 250$						
$c = 1.0$	-0.0079	0.2436	0.3222			
$c = 1.5$	0.6609	1.2783	1.3469			
$c = 2.0$	0.7626	0.1323	0.7751	0.0301	0.3212	0.3226
$c = 2.5$	0.1716	0.1347	0.2138			
$c = 3.0$	-0.4069	0.1254	0.4260			
$T = 500$						
$c = 1.0$	0.2630	0.2694	0.4857			
$c = 1.5$	-0.6736	0.8139	1.3169			
$c = 2.0$	0.9082	0.0985	0.9131	0.0007	0.2360	0.2360
$c = 2.5$	0.3309	0.0923	0.3434			
$c = 3.0$	-0.2166	0.0936	0.2340			

Table A.15: Finite-sample biases, standard deviations (Std.'s) and root mean squared errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with no measurement error

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.4236	0.2925	0.5148			
$c = 1.5$	0.8471	0.7002	1.0991			
$c = 2.0$	0.3449	0.1411	0.3726	0.0664	0.5345	0.5386
$c = 2.5$	-0.3159	0.1542	0.3515			
$c = 3.0$	-0.9509	0.1129	0.9576			
$T = 250$						
$c = 1.0$	-0.1976	0.2575	0.3246			
$c = 1.5$	-1.0542	1.2495	1.6348			
$c = 2.0$	0.5271	0.1018	0.5368	0.0455	0.4636	0.4658
$c = 2.5$	-0.1000	0.0999	0.1413			
$c = 3.0$	-0.7034	0.0846	0.7085			
$T = 500$						
$c = 1.0$	0.0330	0.3749	0.3764			
$c = 1.5$	-1.5633	0.1288	1.5686			
$c = 2.0$	0.6685	0.0762	0.6728	0.0106	0.3416	0.3417
$c = 2.5$	0.0773	0.0692	0.1038			
$c = 3.0$	-0.4977	0.0644	0.5019			