Rumors with Changing Credibility

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Abstract

Randomized rumor spreading processes diffuse information on an undirected graph and have been widely studied. In this work, we present a generic framework for analyzing a broad class of such processes on regular graphs. Our analysis is protocol-agnostic, as it only requires the expected proportion of newly informed vertices in each round to be bounded, and a natural negative correlation property.

This framework allows us to analyze various protocols, including PUSH, PULL, and PUSH-PULL, thereby extending prior research. Unlike previous work, our framework accommodates message failures at any time $t \geq 0$ with a probability of $1 - q(t)$, where the *credibility* $q(t)$ is any function of time. This enables us to model real-world scenarios in which the transmissibility of rumors may fluctuate, as seen in the spread of "fake news" and viruses. Additionally, our framework is sufficiently broad to cover dynamic graphs.

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1 Introduction

The rise of online social networks has facilitated a way for network users to rapidly obtain information, express their opinion, and stay in touch with friends and family. However, at the same time the large scale information cascades enabled by these new social technologies provide fertile ground for the spread of misinformation, rumors and hoaxes. This in turn can have severe consequences such as public panic, growing polarization, the manipulation of political events, and also economic damage. For instance, in 2013 a rumor that President Obama was injured in two explosions at the White House led to $90 billion USD being temporarily wiped off the value of United States stock market [30]. In the same year the World Economic Forum report [21] listed "massive digital misinformation" as one of the main risks for the modern society. More recently we have seen the spread of misinformation surrounding the Covid-19 pandemic [4]. Consequently, there has been a growing body of work aiming to gain insights into the rumor spreading dynamics [12, 26, 31, 35].

For a long time, randomized rumor spreading protocols such as the PUSH, PULL and PUSH-PULL protocols have been used to model the dissemination of information on graphs, e.g., [2, 13, 23]. Both by mathematical analysis on "scale free" graphs in addition to experimental results on real-world social networks, it has been demonstrated that these protocols (in particular, PUSH-PULL) spread a rumor to a large fraction of vertices in a very short time (e.g., [14]).

However, one shortcoming of the previous works that analyze these protocols is the assumption that the probability with which an individual believes the rumor, when receiving it, is constant over time – in fact, in many studies it is assumed that this *credibility* is equal to one in all rounds. In real world settings, one can imagine that the occurrence of emergent events (such as an earthquake or a new possibly lethal decease) can intensify the formation and propagation of rumors due to their suddenness and urgency, followed by a decrease in credibility once more information has become available. A related example is the spread of viruses, where counter-measures such as vaccination or social distancing, but also seasonal effects may affect the transmissibility over time, potentially even periodically/non-monotonically.

Moreover, it is often assumed that the graph is fixed throughout the execution of randomized rumor spreading protocols, which is rather restrictive since many networks, e.g., social networks, P2P networks or communication networks, are subject to frequent changes.

To address these issues, we introduce a new methodology for analyzing randomized rumor spreading protocols that allows us to study PUSH, PULL, and PUSH-PULL processes under the presence of a time-changing credibility (or transmissibility) function $q(t)$ and dynamic graphs $(G_t)_{t\geq 0}$. However, our method is more general and allows us to study a broader class of spreading processes on dynamic graphs. To show the effectiveness of our analysis, we recover known results for the PUSH, PULL, and PUSH-PULL protocols in the context of a constant credibility function q, and provide analysis for specific time-dependent credibility functions $q(t)$.

1.1 Our Contribution

In this work, we present a general framework for analyzing a large class of randomized rumor spreading models. Our main results give concentration for the number of vertices informed after a certain stopping time. These results are very general however we show in detail how they can be applied to several models.

- **Broad Class of Spreading Processes.** Instead of using protocol specific characteristics, our framework only requires some mild conditions on the spreading process (i.e., bounded expected growth and a natural negative correlation property; see Definition 1). This allows our setting to cover many models of randomized rumor spreading, beyond the standard PULL and PUSH models (see Lemma 8, the final bullet point below, and Section 2.5).
- **Credibility Function $q(t)$.** Our model allows for a time-dependent credibility function $q(t) \in [0, 1]$, which specifies how transmissible the rumor is in each step. This can be seen as a major generalization of the prevalent notion of "robustness" in the literature, which usually refers to the uniform fault model with q fixed over t. Unlike in previous models, our credibility functions can be arbitrary, in particular they do not need to be monotone.
- **Stopping time Criterion.** We introduce a new technical tool based on a stopping time criterion. Roughly, for some desired number of vertices B to be informed, the stopping time triggers when a sum of expected growth factors of the process exceeds a threshold depending on B. The aforementioned growth factors are conditional expectations of the proportion of new vertices informed in the next step. We show that if this stopping criterion is met, then B vertices are informed with high probability (see Theorem 9). This is complemented by Theorem 15 with a dual statement on the shrinking of the uninformed vertices. Both results are significantly more general than previous analyses,

which usually rely on a growth factor "target" that is independent of t and the set of informed vertices.

- **Dynamic Graphs.** Due to the general nature of our framework and stopping criteria, our analysis "abstracts away" the graph and the specific spreading process. Hence, we can cover sequences of dynamic regular graphs $(G_t)_{t\geq 0}$ instead of a fixed graph G. This flexibility comes from the fact that the connectivity of each G_t is captured by the growth factor of the process at round t, which in turn determines the stopping criterion. In particular, we do not require the graph to be connected at each step (see Remark 10).
- **Applications.** We prove several new results for general and specific credibility functions. First, for general credibility functions, we combine our stopping time criterion with a simple lower bound based on sub-martingales. Together, they reveal a threshold phenomenon, very roughly saying that for expander graphs the quantity $\sum_{k=0}^{t} \log(1+q(k))$ approximates $\log(|I_t|)$, where I_t is the set of vertices informed by time t (see Section 4.1).
  After that, we turn to some specific credibility functions, including additive, multiplicative and Power-Law (see Sections 4.2–4.4 for the respective definitions and results). There, we prove several dichotomies in terms of the decay of $q(t)$.
  Despite the generality and abstract nature of our main results, we also recover some previous results for *static* graphs (and time-invariant $q(t)$) as a special case; however, our results for `PUSH`, `PULL` and `PUSH-PULL` additionally apply to *dynamic* graphs (see, e.g., the results in Section 4.5).

Due to space restrictions most proofs are deferred to the full version of this paper [27].

1.2 Related Work

Classical Protocols and Robustness.

Given a rumor spreading process on an n-vertex graph, define the spreading time by $T(n)$ as the first time all vertices are informed. The spreading time of `PUSH` was first investigated on complete graphs by Frieze and Grimmett [18]. Pittel [32] improved on this, showing that for `PUSH` on the complete graph, the spreading time is given by $T(n) = \log_2(n) + \log(n) \pm f(n)$ with probability (w.p.) $1 - o(1)$, for any $f(n) = \omega(1)$. Karp, Schindelhauer, Schenker and Vöcking [23] investigated the `PUSH-PULL` model (and variants) with a focus on the total number of messages sent. In particular, they exploit the phenomenon that once a constant fraction of vertices are informed, `PULL` manages to inform all vertices in just $O(\log \log n)$ rounds.

Doerr and Kostrygin [15] derived a bound on the expected spreading time $\mathbf{E}[T(n)]$ of `PUSH`, replicating the bound from [32] but only with an additive $O(1)$ error instead of $f(n)$. Furthermore, [15] also considered `PULL` and `PUSH-PULL` on complete graphs, and determined these spreading times up to and additive $O(1)$ error. They also presented a more general result for the uniform fault model, where the leading factors are delicate functions of the (time-invariant) credibility $q \in (0, 1]$. We are able to recover a with high probability version of the upper bounds from [15] for `PUSH`, `PULL` and `PUSH-PULL` (see Section 4.5).

Fountoulakis, Huber and Pangiotou [16] considered the uniform fault setting of `PUSH` on random graphs with n vertices where each edge is present w.p. $p = \omega(\log n/n)$. They proved that, up to lower-order terms, the same bound as for the complete graph holds. For the model without faults, Fountoulakis and Panagiotou [17] presented a tight analysis for `PUSH` on random d-regular graph for any constant $d \geq 3$. Panagiotou, Perez-Gimenez, Sauerwald and Sun [28] analyzed `PUSH` on almost-regular strong expanders, recovering the runtime bound for complete graphs up to low order terms (see Equation (1) for the definition of strong expander for regular graphs).

Finally, Daknama, Panagiotou and Reisser [10] greatly extended and unified these lines of works in terms of the graph classes considered, and the uniform fault model. Among other results, they proved that the aforementioned results from [15] (for `PUSH`, `PULL` and `PUSH-PULL`) also hold for almost-regular strong expanders, without any change in the leading factor. Our framework allows us to recover the upper bounds in [10] for regular graphs as well as dynamic sequences of regular graphs (see Section 4.5).

For general graphs (including highly non-regular ones), Chierichetti, Giakkoupis, Lattanzi and Panconesi [6] proved an upper bound of $O(\log n/\varphi)$ on the time to inform all vertices for `PUSH-PULL`, where φ is the conductance of the graph. A similar, but more complicated bound was shown by Giakkoupis [19] for the `PUSH-PULL` model, where the conductance is replaced by the vertex expansion. The results of both works also extend to `PUSH` and `PULL`, if the graph is (approximately) regular.

Dynamic Graphs.

Extending the aforementioned bounds for conductance and vertex expansion, Giakkoupis, Sauerwald and Stauffer [20] proved similar bounds for dynamic graphs in the `PUSH-PULL` model, where each graph $G_{t\geq 0} = (V, E_{t\geq 0})$ must be d_t-regular. In particular, they proved that if the sum of the conductances over rounds $0, 1, \ldots, T$ is $\Omega(\log n)$, then by round T all vertices are informed. Pourmiri and Mans [33] analyzed an asynchronous version of `PUSH-PULL`. While some of their positive results are similar to the ones in [20], they also established dichomoties between the synchronous and asynchronous version on dynamic graphs. Our approach can be seen as a refinement and generalization of the methods employed in these two works, since our stopping time aggregates over the (random) conductances of the sets I_t, for $t = 0, 1, \ldots, T$, and it works for arbitrary, so-called C_{grow}-growing and C_{shrink}-shrinking processes.

Finally, Clementi, Crescenzi, C. Doerr, Fraigniaud, Pasquale and Silvestri [9] analyzed `PUSH` on a random dynamic graph model called Edge Markovian Evolving Graph, and proved a runtime bound of $O(\log n)$ for certain parameter ranges of their model. Ideas and techniques related to rumor spreading have also been employed in the analysis of components in a temporal random graph model [1, 5].

Other Models with Time Dependent Credibility Functions.

The inclusion of a local time dependent forgetting rate in the SIR model [25] was empirically investigated by Zhao, Xie, Gao, Qiu, Wang, and Zhang [37], leading to $q(t) := \mu - e^{\beta\cdot t}$, for $0 \leq \mu - e^{\beta\cdot t} \leq 1$, for μ and β parameters indicating the initial credibility and the speed with which the credibility decreases. Very recently, Zehmakan, Out and Khelejan [36] studied a version of the Independent Cascade model [24] where $q(t)$ is a variant of the multiplicative credibility function (with $\alpha = 1/2$, see Definition 28), but additionally is edge dependent (i.e. a function $q(t, uv)$, $uv \in E(G)$) and depends on the Jaccard similarity between two vertices u and v.

2 Models and Notation

We will cover some basic notation before introducing the models studied in this paper.

2.1 Notation

We let $\mathbb{N}$ denote the natural numbers (starting from 0) and let $\mathbb{R}$ denote the reals.

Graph Notation.

Throughout this paper, all considered graphs $G = (V, E)$ will be simple and undirected. We denote $n := |V|$ and $m := |E|$. For a node $v \in V$, $N(v) := \{w \in V : \{w, v\} \in E\}$ is the *neighborhood* of v, and $\deg(v) := |N(v)|$ is called the *degree* of v. We say a graph is *regular* if every vertex has the same degree. For $U \subseteq V$ we let $N_U(v) := \{w \in U : \{v, w\} \in E\} = N(v) \cap U$, and denote $\deg_U(v) := |N_U(v)|$. We will also consider *dynamic graphs*, which can be thought of as a sequence of graphs $(G_t)_{t\geq 0}$ where each graph $G_t = (V, E_t)$ is on the same vertex set, however the edge sets E_t may change over time.

For any two sets $U, W \subseteq V$, we let $e(U, W) := |\{\{u, w\} \in E : u \in U, w \in W\}|$ denote the number of edges between U and W. The *volume* of a set $U \subseteq V$ is the sum of the degrees of the vertices in U, $\mathrm{vol}(U) := \sum_{u \in U} \deg(u)$. We let A be the adjacency matrix of G and denote the degree matrix by $D := \mathrm{diag}(\mathbf{d})$, where $\mathbf{d}(u) = \deg(u)$, which is the matrix with the degrees of the vertices on the diagonal and the rest of the entries equal to 0. Lastly, we let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the normalized adjacency matrix $D^{-1/2}AD^{-1/2}$ and let $\lambda := \max\{|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|\} \geq 0$.

We say that a regular graph G of degree d is a *strong expander* if,

$$\lim_{n\to\infty} \lambda \to 0. \tag{1}$$

Note that a necessary requirement for that is $d \to \infty$. As noted in other works on rumor spreading, the class of random d-regular graphs with $d = \omega(1)$ forms an example of strong expander graphs with w.p. $1 - o(1)$ [3, 34]. We refer to [10, 28] for the exact definition of strong expander graphs when G is almost-regular.

The *conductance* [22] of any vertex set $\emptyset \subsetneq S \subsetneq V$ in a graph $G = (V, E)$ is

$$\varphi_G(S) := \frac{e(S, V \setminus S)}{\min\left(\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\right)}.$$

If the graph G or graph sequence $(G_t)_{t\geq 0}$ is clear from the context, we drop the subscript. The conductance of G is in turn defined as,

$$\varphi(G) := \min_{\emptyset \subsetneq S \subsetneq V} \frac{e(S, V \setminus S)}{\min\left(\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\right)}.$$

Model Notation.

As mentioned, we will consider random processes on a sequence of d_t-regular graphs, $(G_t)_{t\geq 0}$ where each G_t has a common vertex set V. We always assume that $d_t > 0$ (i.e., we do not consider the empty graph). These processes produce a sequence of sets $(I_t)_{t\geq 0}$ where I_t is the set of *informed* vertices at time t (i.e., after t rounds are completed) and $I_t \subseteq I_{t+1} \subseteq V$ for all $t \geq 0$. Similarly, we let $U_t := V \setminus I_t$ denote the set of *uninformed* vertices at time $t \geq 0$. Lastly, we define $\Delta_t := I_t \setminus I_{t-1}$ to be the set of vertices that get informed in round t. Further notation relating to such process is given in Section 2.3.

Mathematical Notation and Assumptions.

We use asymptotic notation $\mathcal{O}(\cdot), o(\cdot), \Omega(\cdot), \omega(\cdot), \Theta(\cdot), \ldots$ throughout, this is always defined relative to the number of vertices n. All logarithms are to base e, unless indicated otherwise.

We let n tend to infinity and say an event $\mathcal{E}$ happens *with high probability* (w.h.p.) if it occurs w.p. $1 - o(1)$. For $f : X \to \mathbb{R}$ a non-negative real-valued function with domain X, we let $\mathrm{Supp}(f) := \{x \in X : f(x) \neq 0\}$. We define $\mathfrak{F}^t$ to be the filtration corresponding to the first t rounds of the process, in particular $\mathfrak{F}^t$ reveals $I_0, I_1, \ldots, I_t$. For brevity, we set

$$\mathbf{P}_t\left[\,\cdot\,\right] := \mathbf{P}\left[\,\cdot \mid \mathfrak{F}^t\,\right], \qquad \mathbf{E}_t\left[\,\cdot\,\right] := \mathbf{E}\left[\,\cdot \mid \mathfrak{F}^t\,\right], \quad \text{and} \quad \mathbf{Var}_t\left[\,\cdot\,\right] := \mathbf{E}\left[\left(\cdot - \mathbf{E}\left[\,\cdot \mid \mathfrak{F}^t\,\right]\right)^2 \mid \mathfrak{F}^t\right].$$

2.2 Standard Rumor Spreading Protocols and Credibility Function $q(t)$

Given any graph sequence, $G_{t\geq 0} = (V, E_{t\geq 0})$ initially one node v^* in graph G_0 is informed of the rumor, i.e., $I_0 = \{v^*\}$. We recall the definition of the `PULL`, `PUSH`, and `PUSH-PULL` protocols [18, 23]. In the `PULL` model, in every round $t = 0, 1, \ldots$, every *uninformed* vertex v chooses a neighbor u uniformly and independently at random. If u is informed, then as a response u transmits the rumor to v, so v becomes informed. In the `PUSH` protocol, in each round, every *informed* node v chooses a neighbor u uniformly at random, and transmits the rumor to u. Lastly, `PUSH-PULL` is the combination of both strategies: In each round, if the node knows the rumor, it chooses a random neighbor to send the rumor to. Otherwise, it chooses a random neighbor to request the rumor from.

We can extend the `PULL`, `PUSH` and `PUSH-PULL` models by including a credibility function $q(t)$ for $q(t) : \mathbb{N} \to [0, 1]$ and $t \geq 0$. In the `PULL`, `PUSH` and `PUSH-PULL` *with credibility* $q(t)$ models, at the beginning of each round $t = 0, 1, \ldots,$ for any uninformed node $v \notin I_{t-1}$ and for each transmission of the rumor to v (regardless of whether that was due to a `PUSH` or `PULL` transmission), it becomes informed with w.p. $q(t)$ independently, and remains uninformed otherwise[1]. This is depicted for the `PUSH-PULL` model in Algorithm 1. Notice that $q(t)$ may be time-dependent, and also that when $q(t) = q = 1$ we return to the standard `PULL`, `PUSH`, and `PUSH-PULL` models, whereas with $q(t) = q$ being a constant in $(0, 1)$ we recover the "uniform failure" model studied in [10, 15].

Algorithm 1 Round $t \in \mathbb{N}$ of `PUSH-PULL` with credibility function $q(t)$

Input: $G_t, I_t, q(t)$
Initialize: $\Delta_{t+1} \leftarrow \emptyset$
for each $v \in I_t$ **do** ▷ `PUSH`
  Sample a neighbor $v' \in N_{G_t}(v)$ chosen uniformly at random.
  if $v' \notin \Delta_{t+1}$ **then**
    With probability $q(t)$, $\Delta_{t+1} \leftarrow \Delta_{t+1} \cup \{v'\}$
for each $v \in V \setminus I_t$ **do** ▷ `PULL`
  Sample a neighbor $v' \in N_{G_t}(v)$ chosen uniformly at random.
  if $v' \in I_t$ **then**
    With probability $q(t)$, $\Delta_{t+1} \leftarrow \Delta_{t+1} \cup \{v\}$
$I_{t+1} \leftarrow I_t \cup \Delta_{t+1}$

[1] Hence if in a round, an uninformed vertex receives k transmissions (regardless of whether these are `PULL` or `PUSH` transmissions), then the probability it gets informed is $1 - (1 - q(t))^k$, i.e. each transmission is independent.

2.3 Our Class of Spreading Processes

We now introduce two general spreading processes, that are crucial to our framework. This is an abstraction of the aforementioned examples of PUSH, PULL and PUSH-PULL with credibility function $q(t)$, since we are now only considering the expected growth (or shrinking) factors. We point out that these may depend on several quantities such as the conductance of the informed set I_t (or uninformed set U_t, respectively), and $q(t)$ of course.

▶ **Definition 1** (Growing and Shrinking Processes). *Let $(G_t)_{t\geq 0}$ be a sequence of graphs. Let $\mathcal{P}$ be a stochastic process on $(G_t)_{t\geq 0}$ with a sequence of informed vertices $(I_t)_{t\geq 0} \subseteq V(G_t)$ and uninformed vertices $U_i = V(G_t) \setminus I_t$ for all $t \geq 0$. We begin by defining the following property of such a process*

- $\mathcal{P}_1$ *(**Negative Correlation**): For any round $t \geq 0$ and any subset $S \subseteq U_t$,*

$$\mathbf{P}_t\left[\bigcap_{u\in S}\{u \in I_{t+1}\}\right] \leq \prod_{u\in S} \mathbf{P}_t\left[\, u \in I_{t+1}\,\right].$$

For some time-independent value $C_{\text{grow}} > 0$ we say that $\mathcal{P}$ is a C_{grow}-growing process if it satisfies $\mathcal{P}_1$ and

- $\mathcal{P}_2$ *(**Monotonicity**): For any round $t \geq 0$, it holds deterministically that $I_t \subseteq I_{t+1}$ (and $|I_0| \geq 1$),*
- $\mathcal{P}_3$ *(**Bounded Expected Growth**): For any round $t \geq 0$ the* expected growth factor satisfies,

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] \leq C_{\text{grow}}.$$

Similarly, for some time-independent $C_{\text{shrink}} < 1$, $\mathcal{P}$ is a C_{shrink}-shrinking process if it satisfies $\mathcal{P}_1$ and

- $\widetilde{\mathcal{P}}_2$ *(**Monotonicity**): For any round $t \geq 0$, it holds deterministically that $U_t \supseteq U_{t+1}$ (and $|U_0| \leq n/2$),*
- $\widetilde{\mathcal{P}}_3$ *(**Bounded Expected Shrinking**): For any round $t \geq 0$ the* expected shrinking factor satisfies,

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right] \leq C_{\text{shrink}}.$$

For convenience, we also define for all rounds $t \geq 0$ a "combined" growth/shrinking factor as

$$\delta_t := \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{\min\left(|I_t|, |U_t|\right)}\right] = \max\left(\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right], \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right]\right).$$

We now prove that the negative correlation property immediately implies a strong upper bound on the variance of the growth (shrinking) factor. The same result was derived in [10] for PUSH, PULL and PUSH-PULL using the concept of self-bounding functions.

▶ **Lemma 2.** *Consider any stochastic process with sequence of informed vertices $(I_t)_{t\geq 0}$ satisfying $\mathcal{P}_1$. Then, also the following property also holds:*

- $\mathcal{P}_4$ *(**Bounded Variance**): For any round $t \geq 0$, $\mathbf{Var}_t\left[\,|\Delta_{t+1}|\,\right] \leq \mathbf{E}_t\left[\,|\Delta_{t+1}|\,\right]$.*

	δ_t	
	Lower Bound	Upper Bound
PULL	$q(t) \cdot \varphi(I_t)$	
PUSH	$q(t) \cdot \left(1 - \frac{q(t)}{2}\right) \cdot \varphi(I_t)$	$q(t) \cdot \varphi(I_t)$
PUSH-PULL	$\frac{3}{2} \cdot q(t) \cdot \left(1 - \frac{1}{2}q(t)\right) \cdot \varphi(I_t)$	$2 \cdot q(t) \cdot \varphi(I_t)$

Table 1 Basic lower and upper bounds on the expected growth factor δ_t for PUSH, PULL and PUSH-PULL in terms of $q(t)$ and the conductance $\varphi(I_t)$ on regular graphs.

2.4 Specific Protocols and Growth Factors

In this subsection, we analyze specific protocols (in particular, PUSH, PULL and PUSH-PULL with credibility function $q(t)$) and verify that they are C_{grow}-growing and C_{shrink}-shrinking processes in the sense of Definition 1. Let $(G_t)_{t\geq 0}$ be a sequence of regular graphs. Recall that in our setting $|I_0| = 1$ and $\Delta_{t+1} = I_{t+1} \setminus I_t$. In order to capture the progress of the rumor spreading process between the rounds t_1 and t_2, we observe the following identities,

$$\frac{|I_{t_2}|}{|I_{t_1}|} = \prod_{t=t_1}^{t_2-1} \frac{|I_{t+1}|}{|I_t|} = \prod_{t=t_1}^{t_2-1} \frac{|I_t| + |\Delta_{t+1}|}{|I_t|} = \prod_{t=t_1}^{t_2-1} \left(1 + \frac{|\Delta_{t+1}|}{|I_t|}\right)$$
$$\frac{|U_{t_2}|}{|U_{t_1}|} = \prod_{t=t_1}^{t_2-1} \frac{|U_{t+1}|}{|U_t|} = \prod_{t=t_1}^{t_2-1} \frac{|U_t| + |\Delta_{t+1}|}{|U_t|} = \prod_{t=t_1}^{t_2-1} \left(1 - \frac{|\Delta_{t+1}|}{|U_t|}\right).$$

As such, we prove upper and lower bounds on the expectation of the growth factor, $\frac{|\Delta_{t+1}|}{\min(|I_t|,|U_t|)}$ of the PUSH, PULL and PUSH-PULL protocols.

▶ **Lemma 3.** *Let $t \geq 0$ be any round, G_t a d_t-regular graph with n vertices and $d_t \geq 1$, and $q(t)$ an arbitrary credibility. Then,*

(i) *for the* PUSH *protocol,*

$$q(t) \cdot \left(1 - \frac{q(t)}{2}\right) \cdot \varphi(I_t) \leq \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{\min(|I_t|, |U_t|)}\right] \leq q(t) \cdot \varphi(I_t),$$

(ii) *for the* PULL *protocol,*

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{\min\left(|I_t|, |U_t|\right)}\right] = q(t) \cdot \varphi(I_t),$$

(iii) *and for the* PUSH-PULL *protocol,*

$$\frac{3}{2} \cdot q(t) \cdot \left(1 - \frac{q(t)}{2}\right) \cdot \varphi(I_t) \leq \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{\min\left(|I_t|, |U_t|\right)}\right] \leq 2 \cdot q(t) \cdot \varphi(I_t).$$

Next we prove tighter bounds for the PUSH and PUSH-PULL protocol if the graph is a strong expander.

▶ **Lemma 4.** *Consider the* PUSH *protocol, and let $t \geq 0$ be any round where with $|I_t| \leq n/2$ and G_t a d_t-regular graph with n vertices. Then, for $q(t)$ an arbitrary credibility and $\beta := \lambda + \frac{|I_t|}{n}$,*

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] \geq q(t) \cdot \left(1 - 7\sqrt{\beta}\right).$$

For the same setting in the PUSH-PULL *protocol,*

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] \geq q(t) \cdot \left(2 - 12\sqrt{\beta}\right).$$

The next lemma improves over the lower and upper bound in Lemma 3 (i) if $|I_t| \geq n/2$. Concerning the lower bound, we have $q(t) \cdot (1 - \frac{q(t)}{2}) \leq 1 - e^{-q(t)}$ since $e^{-z} \leq 1 - z + \frac{1}{2}z^2 = 1 - z \cdot \left(1 - \frac{z}{2}\right)$ for $z \in [0, 1]$. Further, if $d_t = \omega(1)$ and $q(t) \cdot \varphi(I_t)$ is bounded below by a constant, then the upper bound below is tighter as $1 - \exp(-x) \leq x$ for any $x \in \mathbb{R}$.

▶ **Lemma 5.** *Consider the* PUSH *protocol, and let $t \geq 0$ be any round, G_t is a d_t-regular graph with n vertices and $q(t)$ an arbitrary credibility. Then,*

(i) $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right] \geq \left(1 - e^{-q(t)}\right) \cdot \varphi(I_t)$.

(ii) *If G_t is connected, then,*

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right] \leq 1 - e^{-\varphi(I_t) \cdot q(t)} \cdot \left(1 - \frac{\varphi(I_t) \cdot (q(t))^2}{d_t}\right).$$

The next lemma improves the result for the PUSH-PULL protocol in Lemma 3 (iii).

▶ **Lemma 6.** *Consider the* PUSH-PULL *protocol, and let $t \geq 0$ be any round, G_t is a d_t-regular graph with n vertices and $q(t)$ an arbitrary credibility. Then,*

(i) $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right] \geq \left(1 - e^{-q(t)} \cdot (1 - q(t))\right) \cdot \varphi(I_t)$.

(ii) $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|U_t|}\right] \leq 1 - (1 - q(t))^{\varphi(I_t)} \cdot (1 - q(t) \cdot \varphi(I_t))$.

A summary of these tighter bounds for PUSH, PULL and PUSH-PULL is given in Table 2, and the more simple bounds are summarized in Table 1. For strong expanders, similar bounds have been derived in [10, 29].

Next, we state a simple but crucial fact:

▶ **Lemma 7.** *Let $(G_t)_{t \geq 0}$ be a sequence of d_t-regular graphs with n vertices and let $q(t)$ be an arbitrary credibility function . Then, the* PUSH, PULL *and* PUSH-PULL *protocol satisfy the negative correlation property (see Definition 1).*

Finally, we close this section by verifying that PUSH, PULL and PUSH-PULL satisfy the condition in Definition 1 for certain C_{grow} and C_{shrink}. Note that even for static graphs, PULL and PUSH-PULL require a restriction on $q(t)$; this is since if $q(t) = 1$, then on certain graphs (like the complete graph), PULL and PUSH-PULL would only need $O(\log \log n)$ steps in the shrinking phase. However, for dynamic graphs, even for PUSH we require a restriction on $q(t)$; this is because otherwise G_t could be a 1-regular graph, i.e., a perfect matching so that each vertex in U_t is matched to a vertex in I_t.

▶ **Lemma 8.** *Let $(G_t)_{t \geq 0}$ be any sequence of d_t-regular graphs and let $q(t)$ be an arbitrary credibility function.*

(i) *The* PUSH *protocol is a 1-growing process. Furthermore, if $q(t) \leq 1 - \varepsilon$, for $\varepsilon > 0$ (not necessarily constant), then the* PUSH *protocol is a $(1 - \varepsilon)$-shrinking process. Also, if all graphs in the sequence $(G_t)_{t \geq 0}$ are connected, then the* PUSH *protocol is a $(1 - e^{-1} \cdot \frac{1}{2})$-shrinking process.*

	δ_t, $1 \le \lvert I_t\rvert \le n/2$	δ_t, $n/2 \le \lvert I_t\rvert \le n$	
	Lower Bound	Lower Bound	Upper Bound
PULL	$q(t) \cdot \varphi(I_t)$	$q(t) \cdot \varphi(I_t)$	
PUSH	$q(t) \cdot \left(1 - 7\sqrt{\lambda + \frac{\lvert I_t\rvert}{n}}\right)$	$\left(1 - e^{-q(t)}\right) \cdot \varphi(I_t)$	$q(t) \cdot \varphi(I_t)$
P-P	$q(t) \cdot \left(2 - 12\sqrt{\lambda + \frac{\lvert I_t\rvert}{n}}\right)$	$\left(1 - e^{-q(t)} \cdot (1 - q(t))\right) \cdot \varphi(I_t)$	$1 - (1 - q(t))^{\varphi(I_t)} \cdot (1 - q(t) \cdot \varphi(I_t))$

Table 2 Refined bounds in terms of $q(t)$ and the spectral expansion λ on the expected growth factors of PUSH and PUSH-PULL on regular graphs. These bounds are tighter than the more basic ones (see Table 1), whenever $\lambda = o(d_t)$ (which also implies $\varphi(I_t) = 1 - o(1)$ if $|I_t| = o(n)$ as well as $\varphi(I_t) = 1 - o(1)$ if $|U_t| = o(n)$). The $1 - o(1)$ terms in the two upper bounds go to 1 if $d_t \to \infty$ or $\varphi \to 0$ or $q(t) \to 0$ for all $t \ge 0$.

(ii) *The* PULL *protocol is a* 1*-growing process. Furthermore, if* $q(t) \le 1 - \varepsilon$*, for* $\varepsilon > 0$ *(not necessarily constant), then the* PULL *protocol is a* $(1 - \varepsilon)$*-shrinking process.*

(iii) *The* PUSH-PULL *protocol is a* 2*-growing process. Furthermore, if* $q(t) \le 1 - \varepsilon$*, for* $\varepsilon > 0$ *(not necessarily constant), then the* PUSH-PULL *protocol is a* $(1 - \varepsilon^2)$*-shrinking process.*

2.5 Other Examples

We will briefly outline some other examples of $(C_{\text{grow}}, C_{\text{shrink}})$-spreading processes. We will not study these processes further in this paper, so for the sake of space we omit the proofs of membership.

- Variants of PUSH, PULL and PUSH-PULL where vertices accept all incoming messages w.p. $q(t)$, independent of the number of messages received, otherwise reject all. This is an alternative interpretation of the *credibility function* as being "belief-based", i.e. whenever a vertex receives at least one transmission (regardless of whether they are PUSH or PULL), it *believes* in the rumor w.p. $q(t)$. Hence, the "believed" versions of PUSH, PULL and PUSH-PULL are slower siblings of the "transmission-based" versions of PUSH, PULL and PUSH-PULL as defined in Section 2.2.
- A variant of PUSH where all vertices transmit to a random neighbor in each step (uninformed vertices transmit an "empty" message, informed vertices transmit the rumor). Each uninformed vertex chooses at most one received message (chosen uniformly at random from all received messages, ignoring all others). If they receive a message with the rumor they are informed; otherwise they are not. This process was introduced by Daum, Kuhn and Maus [11].
- The *multiple call* model, where each vertex pushes the opinion to k of random neighbors [29], for constant k (one could even consider k to be dependent on the node as in [29], or on the round t). This model can also be extended by using credibility functions.
- For any constant $\alpha \in [0, 1]$, in each round $t \ge 0$, each node performs a pull with w.p. α and a push w.p. $1 - \alpha$. This model can also support a credibility function.
- Variants of Broadcasting or Flooding models [8] where in each round each informed node sends the information to all its neighbors, however, edges may independently fail to transmit the message with some probability depending only on the edge.

3 Lower Bounds on the Number of Informed vertices

Our analysis will be split into two phases, a "growing" phase where $|I_t| \leq n/2$, and a "shrinking" phase where $|I_t| \geq n/2$.

3.1 Growing phase: $I_t \in [A, B]$

In this section, we prove a lower bound on the number of informed vertices after a stopping time τ_2, which aggregates over the expected growth factors between round 1 and $\tau_2 - 1$. In the following theorem (and throughout the rest of this paper) we use the convention that $\min\{\emptyset\} = \infty$.

▶ **Theorem 9.** *Let $(G_t)_{t\geq 0}$ be any sequence of d_t-regular n-vertex graphs and consider a C_{grow}-growing process $\mathcal{P}$ with expected growth factors δ_t. Let $t_1 \geq 0$ be any round, and let A, B be thresholds satisfying $1 \leq A < B \leq n/2$ and $\xi := 10^{-30}$. Define the stopping time $\tau_2 \in \mathbb{N} \cup \{\infty\}$ as*

$$\tau_2 := \min\left(s \geq t_1 \colon \sum_{t=t_1}^{s-1} \log(1+\delta_t) \geq \frac{\log\left(\frac{B}{A}\right) + \left(\log\left(\frac{B}{A}\right) + \log(1+C_{\text{grow}}) + 1\right)^{2/3}}{\left(1-(1-\xi)\cdot |I_t|^{-\xi}\right)^2} \right).$$

Then there is a constant $C_2 > 0$ such that

$$\mathbf{P}_{t_1}\left[|I_{\tau_2}| < B \;\middle|\; |I_{t_1}| \geq A \right] \leq \exp\left(-C_2 \cdot \left(\log\left(\frac{B}{A}\right)\right)^{1/3} \right) + \mathbf{P}_{t_1}\left[\tau_2 = \infty \;\middle|\; |I_{t_1}| \geq A \right].$$

Recall that the growth factors δ_t are conditional expectations given by $\delta_t = \mathbf{E}_t\left[\frac{|\Delta_t|}{|I_t|}\right]$ in the growing phase, where $|I_t| \leq n/2$. Intuitively, the stopping time τ_2 in Corollary 18 can be viewed as a partial observer who does not know the sequence I_t, but only gets to know the expected growth factors in each round.

▶ Remark 10. At first it might look challenging to apply Theorem 9, as one would need to control the probability that the stopping time is unbounded. However, in most applications we have a deterministic lower bound on the expected growth in each step and then, provided this bound is sufficiently large, this probability equals zero. We refer to Corollary 18 for a weaker but easier to apply variant of Theorem 9 which leverages this idea. The use of this stopping time also allows Theorem 9 to be very general. For instance, notice that G_t is not required to always be connected; this gives flexibility when handling dynamic graphs.

We will now give a brief overview of the proof of Theorem 9, followed by some helper lemmas and claims, and then complete the proof. The starting point is to analyze the growth rate of the number of informed vertices. To this end, we recall the following formula involving growth factors:

$$\frac{|I_{\tau_2}|}{|I_{t_1}|} = \prod_{t=t_1}^{\tau_2-1} \frac{|I_{t+1}|}{|I_t|} = \prod_{t=t_1}^{\tau_2-1}\left(1+\frac{|\Delta_{t+1}|}{|I_t|}\right). \tag{2}$$

In order to transform this product into a sum of random variables, we first define for any $t \geq 0$,

$$X_t := \log\left(1+\frac{|\Delta_{t+1}|}{|I_t|}\right).$$

Then, by taking logarithms in Equation (2) we obtain that

$$\log\left(\frac{|I_{\tau_2}|}{|I_{t_1}|}\right) = \sum_{t=t_1}^{\tau_2-1} X_t.$$

Our approach will be to lower bound the sum of these X_t's. Therefore, we will consider the expected (logarithmic) growth in each step (i.e. $\mathbf{E}_t\left[\, X_t \,\right]$) (note that due to the dependence on $\mathcal{F}_t$ it is also a random variable). We then show that $\sum_{t=t_1}^{\tau_2-1} X_t$ is tightly concentrated around $\sum_{t=t_1}^{\tau_2-1} \mathbf{E}_t\left[\, X_t \,\right]$, using a variant of Azuma's concentration inequality (Lemma 12). In doing so, we face the following difficulty of relating the expectation of X_t to the expected growth factor δ_t. Specifically, we would like to apply the following approximation:

$$\mathbf{E}_t\left[\log\left(1+\frac{|\Delta_{t+1}|}{|I_t|}\right)\right] \approx \log\left(1+\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]\right) = \log\left(1+\delta_t\right).$$

One direction in this approximation is immediate; since $\log(\cdot)$ is concave, Jensen's inequality gives us

$$\mathbf{E}_t\left[\log\left(1+\frac{|\Delta_{t+1}|}{|I_t|}\right)\right] \leq \log\left(1+\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]\right).$$

It thus remains to bound the other direction, which amounts to proving an "approximate reverse version" of Jensen's inequality. This is fairly involved, but we manage to establish the following general lemma:

▶ **Lemma 11.** *For a fixed round $t \geq 0$, let G_t be a regular n-vertex graph and consider a C_{grow}-growing process $\mathcal{P}$. If $|I_t| \in [A, n/2]$, then, for $\xi := 10^{-30}$, we have*

$$\mathbf{E}_t\left[\log\left(1+\frac{|\Delta_{t+1}|}{|I_t|}\right)\right] \geq \left(1-(1-\xi)\cdot|I_t|^{-\xi}\right)^2 \cdot \log\left(1+\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]\right).$$

Note that the first factor on the right-hand side of the inequality above is $(1-o(1))$ in the case $|I_t| = \omega(1)$ (i.e., a super-constant number of vertices are informed).

As mentioned above we will also need the following variant of Azuma's inequality.

▶ **Lemma 12** ([7, Theorem 6.5])**.** *Let $(Z_i)_{i\geq 0}$ be a discrete-time martingale associated with a filter $\mathcal{F}$ satisfying*

1. $\mathbf{Var}\left[\, Z_i \,\middle|\, \mathcal{F}_{i-1} \,\right] \leq \sigma_i^2$ *for all* $1 \leq i \leq n$*;*
2. $Z_{i-1} - Z_i \leq M$ *for* $1 \leq i \leq n$*.*

Then for any $h \geq 0$,

$$\mathbf{P}\left[\, Z_n - \mathbf{E}\left[\, Z_n \,\right] \leq -h \,\right] \leq \exp\left(-\frac{h^2}{2\cdot\left(\sum_{i=1}^{n}\sigma_i^2 + Mh/3\right)}\right).$$

To apply this the following simple lemma will be useful.

▶ **Lemma 13.** *Let Z be a non-negative random variable. Then,* $\mathbf{Var}\left[\,\log(1+Z)\,\right] \leq \mathbf{Var}\left[\, Z \,\right]$.

Lastly, before beginning the proof of Theorem 9, we first state the following helper claim.

▷ Claim 14. For τ_2 and $\xi := 10^{-30}$ as in Theorem 9 and $1 \leq A \leq B \leq n/2$, we have,

$$\sum_{t=t_1}^{\tau_2-1} \delta_t \leq \frac{4}{\xi^2}\cdot\left(\log\left(\frac{B}{A}\right) + \log(1+C_{\text{grow}}) + 1\right).$$

We can now prove our lower bound on the informed set during the growing phase.

Proof of Theorem 9. Recall that,

$$X_t := \log\left(1 + \frac{|\Delta_{t+1}|}{|I_t|}\right).$$

and if $|I_t| \leq n/2$

$$\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] := \delta_t.$$

Moreover, let us define

$$Y_t := \sum_{s=t_1}^{t-1} (X_s - \mathbf{E}_s[\,X_s\,]).$$

By construction, $(Y_t)_{t=t_1}^{\tau_2-1}$ is a zero-mean martingale with respect to $I_{t_1}, I_{t_1+1}, \dots, I_{\tau_2-1}$. To apply concentration inequalities, we need to provide a bound (M) on $Y_t - Y_{t+1}$ when $|I_t| \leq n/2$. In this case,

$$Y_t - Y_{t+1} = \sum_{s=t_1}^{t-1} (X_s - \mathbf{E}_s[\,X_s\,]) - \sum_{s=t_1}^{t} (X_s - \mathbf{E}_s[\,X_s\,]) = -(X_t - \mathbf{E}_t[\,X_t\,]).$$

Now, using in (a) that $X_t \geq 0$ deterministically, Jensen's inequality in (b), and in (c) the fact that $\mathcal{P}$ is a C_{grow}-growing process, we obtain

$$Y_t - Y_{t+1} \stackrel{(a)}{\leq} \mathbf{E}_t\left[\log\left(1 + \frac{|\Delta_{t+1}|}{|I_t|}\right)\right] \stackrel{(b)}{\leq} \log\left(1 + \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]\right) \stackrel{(c)}{\leq} \log(1 + C_{\text{grow}}) := M. \tag{3}$$

We seek concentration for Y_{τ_2}, however τ_2 may be very large (even unbounded). Thus, we cannot use a standard version of Azuma's inequality, and we need to additionally consider the conditional variances, $\mathbf{Var}_t[\,X_t\,]$. To this end, we bound the variance for any round t with $|I_t| \leq n/2$, by using Lemma 13 in (a),

$$\mathbf{Var}_t[\,X_t\,] = \mathbf{Var}_t\left[\log\left(1 + \frac{|\Delta_{t+1}|}{|I_t|}\right)\right] \stackrel{(a)}{\leq} \mathbf{Var}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] = \frac{1}{|I_t|^2} \cdot \mathbf{Var}_t[\,|\Delta_{t+1}|\,].$$

Using Lemma 2 and by recalling the definition $\delta_t = \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]$, assuming $|I_t| \leq n/2$, we get

$$\mathbf{Var}_t[\,X_t\,] \leq \frac{1}{|I_t|} \cdot \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] = \frac{1}{|I_t|} \cdot \delta_t. \tag{4}$$

Note that by Claim 14, and using that $|I_t| \geq A$ for all $t \geq t_1$,

$$\sum_{t=t_1}^{\tau_2-1} \frac{1}{|I_t|} \cdot \delta_t \leq \frac{1}{A} \cdot \frac{4}{\xi^2}\left(\log\left(\frac{B}{A}\right) + \log(1 + C_{\text{grow}}) + 1\right).$$

We are almost in a position to apply Lemma 12 to Y_{τ_2}. The only slight tweak is that we will work with a martingale also stopped by $\tau := \min\{t \geq t_1 : |I_t| \geq n/2\}$, namely

$$\widehat{Y}_t := Y_{t \wedge \tau_2 \wedge \tau},$$

which is also a zero-mean martingale that satisfies Equations (3) and (4). The reason for this is that the bound (4) assumes the inequality $|I_t| \leq n/2$ holds; we loose nothing doing this because $|B| \leq n/2$.

Now, applying Lemma 12 to $\widehat{Y}_t$ yields that for any $h > 0$ and $t \geq t_1$

$$\mathbf{P}_t\left[\widehat{Y}_t < -h\right] < \exp\left(-\frac{h^2}{2\cdot\left(\frac{1}{A}\cdot\frac{4}{\xi^2}\left(\log\left(\frac{B}{A}\right)+\log\left(1+C_{\text{grow}}\right)+1\right)+\frac{h}{3}\log\left(1+C_{\text{grow}}\right)\right)}\right).$$

Let us set,

$$h := \left(\log\left(\frac{B}{A}\right)+\log(1+C_{\text{grow}})+1\right)^{2/3} \geq 1. \tag{5}$$

Thus, for this h and any round $t \geq t_1$,

$$\begin{aligned}\mathbf{P}_{t_1}\left[\widehat{Y}_t < -h\right] &= \exp\left(-\frac{h^2}{2\left(\frac{4h^{3/2}}{A\cdot\xi^2}+\frac{\log(1+C_{\text{grow}})h}{3}\right)}\right)\\ &\leq \exp\left(-\frac{h^2}{2\left(\frac{4h^{3/2}}{A\cdot\xi^2}+\frac{\log(1+C_{\text{grow}})h^{3/2}}{3}\right)}\right)\\ &= \exp\left(-C_2\cdot h^{1/2}\right),\end{aligned} \tag{6}$$

where C_2 is given by $\left(\frac{8}{A\cdot\xi^2}+\frac{2}{3}\cdot\log\left(1+C_{\text{grow}}\right)\right)^{-1} > 0$. Observe that the right-hand side of (6) is independent of t, this will be important later. However, at this point we must make the following claim:

$$\text{Conditional on } |I_{t_1}| \geq A,\ \{Y_{\tau_2\wedge\tau} \geq -h\}\cap\{\tau_2\wedge\tau < \infty\} \subseteq \{|I_{\tau_2\wedge\tau}| \geq B\}\cap\{\tau_2\wedge\tau<\infty\}. \tag{7}$$

We prove this later, first we show how this, and earlier estimates, will establish the theorem.

Returning to the proof, by (6), we have that for any integer $t \geq 0$,

$$\mathbf{P}_{t_1}\left[Y_{\tau_2\wedge\tau} < -h,\ \tau_2\wedge\tau \leq t\right] \leq \exp\left(-C_2\cdot h^{1/2}\right).$$

Since the above bound holds for any integer $t \geq 0$, it follows that

$$\mathbf{P}_{t_1}\left[Y_{\tau_2\wedge\tau} < -h,\ \tau_2\wedge\tau < \infty\right] \leq \exp\left(-C_2\cdot h^{1/2}\right). \tag{8}$$

Observe that $|I_{\tau_2\wedge\tau}| \leq |I_{\tau_2}|$ by monotonicity ($\mathcal{P}_2$). Using this fact, then (7), and finally (8),

$$\begin{aligned}&\mathbf{P}_{t_1}\left[|I_{\tau_2}| < B \,\middle|\, |I_{t_1}| \geq A\right] \qquad (9)\\ &\leq \mathbf{P}_{t_1}\left[|I_{\tau_2\wedge\tau}| < B \,\middle|\, |I_{t_1}| \geq A\right]\\ &= \mathbf{P}_{t_1}\left[|I_{\tau_2\wedge\tau}| < B, \tau_2\wedge\tau<\infty \,\middle|\, |I_{t_1}| \geq A\right] + \mathbf{P}_{t_1}\left[|I_{\tau_2\wedge\tau}| < B, \tau_2\wedge\tau=\infty \,\middle|\, |I_{t_1}| \geq A\right]\\ &\leq \mathbf{P}_{t_1}\left[Y_{\tau_2\wedge\tau} < -h, \tau_2\wedge\tau<\infty \,\middle|\, |I_{t_1}| \geq A\right] + \mathbf{P}_{t_1}\left[\tau_2\wedge\tau=\infty \,\middle|\, |I_{t_1}| \geq A\right]\\ &\leq \exp\left(-C_2\cdot h^{1/2}\right) + \mathbf{P}_{t_1}\left[\tau_2=\infty \,\middle|\, |I_{t_1}| \geq A\right],\end{aligned}$$

which, recalling the definition (5) of h, gives the bound in the statement.

It remains to prove the claimed containment in (7). For that we analyze the behavior of $|I_{\tau_2\wedge\tau}|$ when the event $\{Y_{\tau_2\wedge\tau} \geq -h\}\cap\{\tau_2\wedge\tau<\infty\}$ holds. We will split into two cases.

In the first case $\{Y_{\tau_2\wedge\tau} \geq -h\} \cap \{\tau < \infty, \tau \leq \tau_2\}$. Hence, $|I_{\tau_2\wedge\tau}| = |I_\tau| \geq n/2 \geq B$.

In the second case $\{Y_{\tau_2\wedge\tau} \geq -h\} \cap \{\tau_2 < \infty, \tau_2 < \tau\}$. Thus, $Y_{\tau_2\wedge\tau} = Y_{\tau_2}$, and deterministically we have,

$$Y_{\tau_2} = \sum_{t=t_1}^{\tau_2-1} (X_t - \mathbf{E}_t[X_t]) \geq -h.$$

Rearranging this, we get that,

$$\begin{aligned}\sum_{t=t_1}^{\tau_2-1} X_t \geq \sum_{t=t_1}^{\tau_2-1} \mathbf{E}_t[X_t] - h &= \sum_{t=t_1}^{\tau_2-1} \mathbf{E}_t\left[\log\left(1 + \frac{|\Delta_{t+1}|}{|I_t|}\right)\right] - h \\ &\geq \gamma \cdot \sum_{t=t_1}^{\tau_2-1} \log\left(1 + \mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right]\right) - h,\end{aligned}$$

where the last inequality follows from Lemma 11, and $\gamma := \left(1 - (1-\xi)\cdot|I_t|^{-\xi}\right)^2$ for $\xi := 10^{-30}$. Since $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] = \delta_t$ for rounds t with $|I_t| \leq n/2$, we conclude that

$$\sum_{t=t_1}^{\tau_2-1} X_t \geq \gamma \cdot \sum_{t=t_1}^{\tau_2-1} \log(1+\delta_t) - h = \gamma \cdot \sum_{t=t_1}^{\tau_2-1} \log(1+\delta_t) - \left(\log\left(\tfrac{B}{A}\right) + \log(1+C_{\text{grow}}) + 1\right)^{\frac{2}{3}}.$$

Finally, by using that

$$\sum_{t=t_1}^{\tau_2-1} \log(1+\delta_t) \geq \frac{\log(\frac{B}{A}) + \left(\log\left(\frac{B}{A}\right) + \log(1+C_{\text{grow}}) + 1\right)^{2/3}}{\gamma},$$

we conclude that $\sum_{t=t_1}^{\tau_2-1} X_t \geq \log(\frac{B}{A})$, i.e. that $|I_{\tau_2}| - |I_{t_1}| \geq B - A$, and thus $|I_{\tau_2}| \geq B$. ◀

3.2 Shrinking phase: $|U_t| \in [C, D]$

In this section we consider the shrinking of the number of informed vertices. We prove an upper bound on the number of uninformed vertices after a stopping time $\tau_3 \geq t_2$, which now aggregates over the expected shrinking factors between round t_2 and $\tau_3 - 1$.

▶ **Theorem 15.** *Let $(G_t)_{t\geq 0}$ be any sequence of d_t-regular n-vertex graphs and consider a C_{shrink}-shrinking process $\mathcal{P}$ with expected shrinking factors δ_t. Let C, D be thresholds satisfying $n/2 \geq C \geq D \geq \frac{3}{4}$ and $t_2 \geq 0$ be a round such that $|U_{t_2}| \leq C$. We define a stopping time $\tau_3 \in \mathbb{N} \cup \{\infty\}$ as*

$$\tau_3 := \min\left\{s \geq t_2 \colon \sum_{t=t_2}^{\tau_3-1} \log(1-\delta_t) \leq -\frac{1}{\gamma}\left(\log\left(\frac{C}{D}\right) + \left(\log\left(\frac{C}{D}\right) - \log(1-C_{\text{shrink}}) + 1\right)^{\frac{2}{3}}\right)\right\},$$

where

$$\gamma := \left(1 - \min\left(\frac{1}{2(1-C_{\text{shrink}})\cdot D}, \frac{1}{2}\right)\right).$$

Then there is a constant $C_2 > 0$ such that

$$\mathbf{P}_{t_2}\left[|U_{\tau_3}| > D \;\middle|\; |U_{t_2}| \leq C\right] \leq \exp\left(-C_2 \cdot \left(\log\left(\frac{C}{D}\right)\right)^{1/3}\right) + \mathbf{P}_{t_2}\left[\tau_3 = \infty \;\middle|\; |U_{t_2}| \leq C\right].$$

The proof of Theorem 15 follows a similar flow to the proof of Theorem 9.

4 Applications

In this section we will apply our general results to more concrete credibility functions, protocols and graph classes. We do not give an exhaustive list of all results that could be derived from our analysis framework, but instead choose to analyze some natural models with decaying credibility, and show that despite the flexible and abstract nature of the framework, we can recover some known results. Roughly speaking, in this section we will first present results that are very general but not necessarily tight, followed by more specific results that are asymptotically tight up to lower order terms.

We will now outline the general approach followed in this section. To control the growth of $|I_t|$ we break the process into j phases defined by time steps $[t_i, t_{i+1})$ for $1 \le i \le j$. With each phase i we associate two values A_i and B_i, where $A_i < B_i$, such that at the beginning of the i-th phase the informed set has size at least A_i and w.h.p. when the phase ends the informed set has size at least B_i. We use the size of the informed set at the end of the previous phase as a lower bound on the size of the informed set throughout the current phase (i.e. $B_{i-1} = A_i$). The w.h.p. guarantees on the length and growth of phases are provided by Corollary 18 and Corollary 19 (which are direct consequences of Theorem 9 and Theorem 15 respectively). These results also give us expressions for the time to finish the phase i.e. $t_{i+1} - t_i$.

▶ **Definition 16.** *For a round $t \ge 0$ and any subset $I \subseteq V$ with $1 \le |I| \le n-1$, let*

$$\delta_t(I) := \mathbf{E}_t\left[\,\delta_t \mid I_t = I\,\right] = \frac{1}{\min(|I_t|, |U_t|)} \cdot \mathbf{E}_t\left[\,|\Delta_{t+1}| \mid I_t = I\,\right],$$

be the expected growth factor, conditional on $I_t = I$ (this is in fact, a deterministic quantity). Further, for a fixed range of $[A, B]$, we define a worst-case lower bound on the expected growth factor (which only depends on t) by

$$\delta_t^{[A,B]} := \min_{I \subseteq V\colon A \le |I| \le B} \delta_t(I).$$

Note that $\delta_t(I)$ depends on the structure of the set I (e.g., the conductance), as well as on $q(t)$. However, for the more coarse quantity $\delta_t^{[A,B]}$, we only need $A \le |I| \le B$. In order to separate these two factors, we also define the following deterministic quantities,

$$\Phi(t) := \min_{\substack{I \subseteq V\colon\\ 1 \le |I| \le n-1}} \frac{\delta_t(I)}{q(t)} \qquad \text{and} \qquad \Psi(t) := \max_{\substack{I \subseteq V\colon\\ 1 \le |I| \le n-1}} \frac{\delta_t(I)}{q(t)}. \tag{10}$$

Moreover, we define $\Phi := \min_{t \ge 0} \Phi(t)$ and $\Psi := \max_{t \ge 0} \Psi(t)$.

▶ **Definition 17.** *For any subset $I \subseteq V$ with $1 \le |I| \le k \le n-1$,*

$$\phi_k := \min_{1 \le |I| \le k} \varphi(I).$$

The following corollary is a direct consequence of Theorem 9.

▶ **Corollary 18.** *Let $(G_t)_{t \ge 0}$ be any sequence of regular n-vertex graphs and consider a C_{grow}-growing process $\mathcal{P}$. Let A, B be thresholds satisfying $1 \le A \le B \le n/2$. Moreover, let $\nu_t^{[A,B]}$ be deterministic quantities such that $\nu_t^{[A,B]} \le \delta_t^{[A,B]}$ for all $t \ge 0$. Let $t' \ge 0$ be any round such that $|I_{t'}| \ge A$, and define $t^* \in \mathbb{N} \cup \{\infty\}$ as*

$$t^* := \min\left\{ s \ge t'\colon \sum_{t=t_1}^{s-1} \log\left(1 + \nu_t^{[A,B]}\right) \ge \frac{\log\left(\frac{B}{A}\right) + \left(\log\left(\frac{B}{A}\right) + \log(1 + C_{\text{grow}}) + 1\right)^{2/3}}{\left(1 - (1-\xi) \cdot A^{-\xi}\right)^2} \right\},$$

(11)

where, $\xi := 10^{-30}$*. Assume that* $t^* < \infty$*, then there is a constant* $C_2 > 0$ *such that*

$$\mathbf{P}_{t'}\left[|I_{t^*}| < B \;\middle|\; |I_{t'}| \geq A \right] \leq \exp\left(-C_2 \cdot \left(\log\left(\frac{B}{A}\right)\right)^{1/3}\right).$$

The following corollary is a direct consequence of Theorem 15.

▶ **Corollary 19.** *Let* $(G_t)_{t\geq 0}$ *be any sequence of regular* n*-vertex graphs and consider a* C_{shrink}*-shrinking process* $\mathcal{P}$*. Let* C, D *be thresholds that satisfy* $n/2 \geq C \geq D \geq \frac{3}{4}$*. Moreover, let* $\nu_t^{[C,D]}$ *be deterministic quantities such that* $\nu_t^{[C,D]} \leq \delta_t^{[C,D]}$ *for all* $t \geq 0$*. Let* $t' \geq 0$ *be a round such that* $|U_{t'}| \leq C$*. We define* $\widehat{t} \in \mathbb{N} \cup \{\infty\}$ *as*

$$\widehat{t} := \min\left\{ s \geq t' : \sum_{t=t_2}^{s-1} \log\left(1 - \nu_t^{[C,D]}\right) \leq -\frac{1}{\widetilde{\gamma}}\left(\log\left(\tfrac{C}{D}\right) + \left(\log\left(\tfrac{C}{D}\right) - \log\left(1 - C_{\text{shrink}}\right) + 1\right)^{\frac{2}{3}}\right)\right\},$$

where

$$\widetilde{\gamma} := \left(1 - \min\left(\frac{1}{2(1 - C_{\text{shrink}}) \cdot D}, \frac{1}{2}\right)\right).$$

Assume that $\widehat{t} < \infty$*, then there is a constant* $C_2 > 0$ *such that*

$$\mathbf{P}_{t'}\left[|U_{\widehat{t}}| > D \;\middle|\; |U_{t'}| \leq C \right] \leq \exp\left(-C_2 \cdot \left(\log\left(\frac{C}{D}\right)\right)^{1/3}\right).$$

4.1 Arbitrary Credibility

The following upper bound is relatively straightforward to prove.

▶ **Theorem 20.** *Let* $(G_t)_{t\geq 0}$ *be any sequence of regular* n*-vertex graphs, and* $q(t)$ *be an arbitrary credibility function. Let* $T \geq 1$ *be a deterministic number of rounds such that for some small* $\rho \in (0,1)$ *(not necessarily constant) it holds that,*

$$\sum_{t=0}^{T-1} \log\Big(1 + \Psi(t) \cdot q(t)\Big) \leq \log n + \log \rho.$$

Then, $\mathbf{E}[\,|I_T|\,] \leq \rho \cdot n$*, and hence by Markov's inequality, for any* $\eta > 0$ *(not necessarily constant),*

$$\mathbf{P}\left[\,|I_T| \leq \rho \cdot n^{1+\eta}\,\right] \leq n^{-\eta}.$$

Next, we state two central results lower bounding the number of informed vertices, which both hold for arbitrary credibility functions. The first one is simple to prove.

▶ **Theorem 21.** *Let* $(G_t)_{t\geq 0}$ *be any sequence of regular* n*-vertex graphs and* $\kappa > 0$ *be any constant. Consider a process* $\mathcal{P}$ *which is both a* C_{grow}*-growing process and a* C_{shrink}*-shrinking process, where* $C_{\text{shrink}} \leq 1 - n^{-\kappa}$*, with an arbitrary credibility function* $q(t)$*. If* T *is a number of rounds satisfying,*

$$\sum_{t=0}^{T-1} \log\left(1 + \delta_t^{[1,n-1]}\right) \geq (2/\xi + \kappa) \cdot \log n,$$

where $\xi := 10^{-30}$ *then, we have*

$$\mathbf{P}\left[\,|I_T| = n\,\right] \geq 1 - o(1).$$

The next result applies to PUSH and PULL.

▶ **Theorem 22.** *Let $(G_t)_{t\geq 0}$ be a sequence of regular n-vertex strong expander graphs, with largest non-trivial eigenvalues $(\lambda_t)_{t\geq 0}$ and let $\lambda := \sup_{t\geq 0}\lambda_t$. Consider the* PUSH *or* PULL *model and let $q(t)$ be an arbitrary credibility function such that for,*

$$\varepsilon := 1 - \max_{t\geq \frac{1}{2\log(2)}\cdot\log(n)} q(t),$$

we have that $\varepsilon \geq \frac{1}{\log n}$. Let $\mathcal{P} \in \{PUSH, PULL\}$, and assume that $T_{\mathcal{P}}$ and $q(t)$ satisfy,

$$\sum_{t=0}^{T_{\mathcal{P}}} \log\left(1+q(t)\right) \geq \frac{1}{\gamma_{\mathcal{P}}} \cdot \frac{\log n + 7\left(\log n\right)^{2/3}}{\left(1-(1-\xi)\cdot(\log n)^{-\xi}\right)^2},$$

where $\xi := 10^{-30}$, $\gamma_{\text{PULL}} := 1-\lambda$, and $\gamma_{\text{PUSH}} := 1 - 7\sqrt{\lambda + 1/\log n}$. Then,

$$\mathbf{P}\left[\, |I_{T_{\mathcal{P}}}| \geq n\cdot\left(1-\exp(-\sqrt{\log n})\right)\right] \geq 1-o(1).$$

Matching previous works [10, 15], for PULL and fixed $q(t) \in (0,1)$ our result implies that in $(1+o(1))\cdot\frac{\log n}{\log(1+q)}$ rounds the majority of the vertices get informed. The same result also holds for PUSH. However, it is important to note that in the results above we do not consider the time to inform *all* n vertices, see Section 4.5 for more results on this model.

4.2 Power-Law Credibility

In this part we consider a natural credibility function with a polynomial decay.

▶ **Definition 23** (Power-law credibility)**.** *Let $\alpha \in (0,\infty)$ be any constant . Then, the power-law credibility function is defined for any round $t \geq 0$ as*

$$q_\alpha(t) := (t+1)^{-\alpha}.$$

In particular, in the first round the credibility function is 1*.*

We first observe that if $\alpha > 1$, we only inform a constant number of vertices in expectation.

▶ **Proposition 24.** *Let $(G_t)_{t\geq 0}$ be any sequence of regular graphs, and consider a Growing Process such that $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] \leq C_{\text{grow}}\cdot q(t)$ for all $t \geq 0$. Then, for any constant $\alpha > 1$, there is a constant $\kappa = \kappa(\alpha) > 0$, such that for any $T \geq 0$,*

$$\mathbf{E}[\,|I_T|\,] \leq \kappa.$$

The condition $\mathbf{E}_t\left[\frac{|\Delta_{t+1}|}{|I_t|}\right] \leq C_{\text{grow}}\cdot q(t)$ is a refinement of $\mathcal{P}_2$ in Definition 1, and is satisfied by the PULL, PUSH, PUSH-PULL processes as shown in Lemma 3 by choosing C_{grow} as 1, 1, and 2, respectively.

The next result considers the regime $\alpha \leq 1$, and proves that after a sufficiently long time, the rumor reaches all n vertices. In particular, when $\alpha = 1$, the spreading time becomes polynomial in n (even if $(G_t)_{t\geq 0}$ was a sequence of expander graphs).

▶ **Theorem 25.** *Let $(G_t)_{t\geq 0}$ be any sequence of regular n-vertex graphs, and consider a process $\mathcal{P}$ that is both a C_{grow}-growing process and a C_{shrink}-shrinking process, where $C_{\text{shrink}} < 1$ is constant, with a power law credibility function. Then, for any constant $\alpha < 1$, there are constants $0 < \kappa_1 := \kappa_1(\alpha) < \kappa_2 := \kappa_2(\alpha)$ such that for any $T_1 \leq \kappa_1\cdot(\frac{1}{\Psi}\cdot\log n)^{1/(1-\alpha)}$, $T_2 \geq \kappa_2\cdot(\frac{1}{\Phi}\cdot\log n)^{1/(1-\alpha)}$ and any $\eta > 0$ we have,*

(*i*) $\mathbf{P}\left[\,|I_{T_1}| < n^{1/2+\eta}\,\right] \geq 1 - n^{-\eta}$,
(*ii*) $\mathbf{P}\left[\,|I_{T_2}| = n\,\right] \geq 1 - o(1)$.

Further, if $\alpha = 1$*, then there are constants* $0 < \kappa_1 < \kappa_2$*, such that for any* $T_1 \leq \left(\frac{1}{\Psi} \cdot n\right)^{\kappa_1}$ *and* $T_2 \geq \left(\frac{1}{\Phi} \cdot n\right)^{\kappa_2}$,

(*iii*) $\mathbf{P}\left[\,|I_{T_1}| < n^{1/2+\eta}\,\right] \geq 1 - n^{-\eta}$,
(*iv*) $\mathbf{P}\left[\,|I_{T_2}| = n\,\right] \geq 1 - o(1)$.

4.3 Additive Credibility

▶ **Definition 26** (Additive credibility). *Let* $\alpha \in (0,1)$ *. Then, the additive credibility function is defined for any round* $t \geq 0$ *as*

$$q_\alpha(t) = q(t) := (1 - t \cdot \alpha)^+,$$

where $z^+ = \max(z, 0)$*. In particular, in the first round* ($t = 0$) *the credibility function is* 1*.*

In comparison to the power-law credibility function, the additive credibility function has a time-independent decrease. As we will see below, the interesting regime (for expanders) is when $\alpha = \Theta(1/\log n)$. That means, unlike the power-law-credibility, the additive credibility function remains close to 1 for a significant number of rounds. However, after $O(\log^2 n)$ steps, the credibility becomes polynomially small; much smaller than any power-law credibility at this point.

Let us consider the additive credibility function in the `PUSH` and `PULL` model for regular graphs. We also observe that if we let $T = 1/\alpha$, I_T is the maximal set of informed vertices in every execution, as $q(t) = 0$ for $t \geq T$. We start by proving an upper bound on I_T for $T = 1/\alpha$, followed by a lower bound. We remark that, due to the specific nature of $q(t)$, we can use Stirling's approximation to determine a rather precise threshold for the parameter α.

▶ **Theorem 27.** *Let* $(G_t)_{t \geq 0}$ *be a sequence of regular* n*-vertex strong expander graphs, and consider the* `PUSH` *or* `PULL` *protocol with an additive credibility function. Let* $\mathcal{P} \in \{PUSH, PULL\}$*.*

(*i*) *Let* $\alpha \geq \frac{\log\left(\frac{4}{e}\right)}{\log n + \log \zeta}$*, where* $\frac{1}{n} < \zeta < \frac{1}{\sqrt{2} \cdot 2}$ *. Then, for any* $T := 1/\alpha$ *and for any* $\eta > 0$ *(not necessarily constant),*

$$\mathbf{P}\left[\,|I_T| \leq \sqrt{2} \cdot \zeta \cdot n^{1+\eta}\,\right] \geq 1 - n^{-\eta}.$$

(*ii*) *Furthermore, let* $\alpha \leq \dfrac{\log\left(\frac{4}{e}\right)}{\log\left(2\sqrt{2} \cdot \exp\left(\frac{1}{\gamma_{\mathcal{P}}} \cdot \frac{\log n + 7(\log n)^{2/3}}{(1-(1-\xi)\cdot(\log n)^{-\xi})^2}\right)\right)}$*, for* $\gamma_{\mathcal{P}}$ *as in Theorem 22. Then, for* $T := 1/\alpha$*,*

$$\mathbf{P}\left[\,|I_T| \geq n \cdot \left(1 - \exp(-\sqrt{\log n})\right)\,\right] \geq 1 - o(1).$$

4.4 Multiplicative Credibility

▶ **Definition 28** (Multiplicative credibility). *Let* $\alpha \in (0,1)$ *. Then, the multiplicative credibility function is defined for any round* $t \geq 0$ *as*

$$q_\alpha(t) := (1 - \alpha)^t.$$

In particular, in the first round the credibility function is 1*.*

The next result is the multiplicative analogue of Theorem 27.

▶ **Theorem 29.** *Let $(G_t)_{t\geq 0}$ be any sequence of regular n-vertex strong expander graphs, and consider the* PUSH *or* PULL *protocol with a multiplicative credibility function . Then, there are constants $\kappa_1 \leq \frac{1}{2}$ and $\kappa_2 \geq \frac{1}{8}$, such that the following holds.*

(i) If $\alpha \geq \frac{\kappa_1}{\log n}$, then for any $T \geq 1$, $\mathbf{E}_t\left[\,|I_T|\,\right] \leq \sqrt{n}$, and hence for any $\eta > 0$ (not necessarily constant),

$$\mathbf{P}\left[\,|I_T| \leq n^{1/2+\eta}\right] \geq 1 - n^{-\eta}.$$

(ii) Further, if $\alpha \leq \frac{\kappa_2}{\log n}$, then, for any $T \geq 4\log n$,

$$\mathbf{P}\left[\,|I_T| \geq n \cdot \left(1 - \exp(-(\log n)^{1/2})\right)\right] \geq 1 - o(1).$$

▶ Remark 30. We believe that with a more refined analysis it would be also possible to show that $\kappa_2 \geq (1 - o(1)) \cdot \kappa_1$, but for the sake of space we only show a weaker dichotomy here.

4.5 Fixed Credibility

Here, we consider $q(t) = q$ to be constant over time (however, $q(t)$ may depend on n). This model was studied in previous works [10, 15] on complete graphs and strong expanders (1), respectively (under the guise of "robustness"). Here we provide upper bounds for the spreading time of the PUSH, PULL and PUSH-PULL model on regular strong expander graphs, using our framework. As the analysis between the protocols are very similar, we will only give details in the case of PUSH here.

▶ **Theorem 31** (cf. [10]). *Let $(G_t)_{t\geq 0}$ be any sequence of regular n-vertex strong expander graphs. Let the credibility function $q(t) = q$ be constant in $(0, 1-\varepsilon]$ for some constant $\varepsilon > 0$ and define the following times*

- $T_{PUSH} := (1 + o(1)) \cdot \left(\frac{1}{\log(1+q)} + \frac{1}{q}\right) \cdot \log n,$
- $T_{PULL} := (1 + o(1)) \cdot \left(\frac{1}{\log(1+q)} - \frac{1}{\log(1-q)}\right) \cdot \log n,$
- $T_{PUSH\text{-}PULL} := (1 + o(1)) \cdot \left(\frac{1}{\log(1+2q)} + \frac{1}{q - \log(1-q)}\right) \cdot \log n.$

Then for each $\mathcal{P} \in \{PUSH, PULL, PUSH\text{-}PULL\}$ we have

$$\mathbf{P}\left[\,|I_{T_{\mathcal{P}}}| = n\,\right] \geq 1 - o(1).$$

We note that the corresponding result [10, Theorem 1.2] in the original paper is stated only for static graphs, however it is likely that the methods in that paper would also extend to dynamic graphs.

In the proof of Theorem 31 we divide the process into 6 phases, based on the size of informed set. In each phase, we apply either Corollary 18 (if $|I_t| \leq n/2$) or Corollary 19 (if $|I_t| \geq n/2$), using deterministic lower bounds on the growth/shrinking factors. An overview of the running times of these phases, and the size of the informed set when they start/finish, is given in Table 3, also for the PULL and PUSH-PULL processes.

5 Conclusions

In this work, we presented a general framework for analyzing spreading processes with a time-dependent credibility function. The key idea is to link the spreading progress to an aggregate sum of growth (or shrinking) factors over consecutive rounds. In that way, our approach generalizes various previous works that were based on estimating the worst-case

Phase	Start/finish sizes	PUSH	PULL	PUSH-PULL
1	$A = 1, B = \log n$	$\frac{\log\log n}{\log(1+q)}$	$\frac{\log\log n}{\log(1+q)}$	$\frac{\log\log n}{\log(1+2q)}$
2	$A = \log n, B = \frac{n}{\log n}$	$\frac{\log n}{\log(1+q)}$	$\frac{\log n}{\log(1+q)}$	$\frac{\log n}{\log(1+2q)}$
3	$A = \frac{n}{\log n}, B = \frac{n}{2}$	$\frac{\log\log n}{\log(1+q)}$	$\frac{\log\log n}{\log(1+q)}$	$\frac{\log\log n}{\log(1+2q)}$
4	$C = n/2,\ D = \frac{n}{\log n}$	$\frac{1}{q}\log\log n$	$\frac{\log\log n}{-\log(1-q)}$	$\frac{\log\log n}{q-\log(1-q)}$
5	$C = \frac{n}{\log n}, D = \log n$	$\frac{1}{q}\log n$	$\frac{\log n}{-\log(1-q)}$	$\frac{\log n}{q-\log(1-q)}$
6	$C = \log n, D = \frac{3}{4}$	$\frac{1}{q}\log\log n$	$\frac{\log\log n}{-\log(1-q)}$	$\frac{\log\log n}{q-\log(1-q)}$

Table 3 Runtimes for PUSH, PULL and PUSH-PULL for different phases obtained by Corollary 18 (row 1,2,3) and Corollary 19 (row 4,5,6), all bounds hold w.h.p.. The upper bounds contained within cells shaded in yellow hold up to a multiplicative $(1 + o(1))$ factor and it is these bound which contribute to the to total run time, all other bounds hold up to a multiplicative constant and are negligible. Our result for PULL holds only when q is bounded away from 1, the remaining cases where q is equal (or tending to) 1 are covered in [15, 10].

growth across all sets via the conductance of the graph. We also obtained several dichotomy results in terms of the number of vertices that get informed, both for general and more concrete credibility functions (see Section 4).

In terms of open problems, a natural direction is to generalize our main technical results from regular graphs to arbitrary graphs, which we believe to be doable. Another avenue for future research is to allow more complex interactions between the credibility function $q(t)$ and the evolving set of informed vertices I_t, which could more accurately model an external influence on the network (e.g., fact-checkers). Lastly, one could consider more general spreading processes including other epidemic models (e.g., SIR model or independent cascade model), majority dynamics or variants of the voter model, in which informed vertices may also become uninformed in future steps.

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