Asymptotic analysis of thin linear elastic layers constrained by two rigid plates

A.B. Movchan[a], N.V. Movchan[a], and G.J. Rodin* [b,c]

[a]Department of Mathematical Sciences, University of Liverpool, L69 7ZL, UK

[b]Oden Institute for Computational Engineering and Sciences

[c]Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712 USA

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*For correspondence contact at gjr@oden.utexas.edu

Abstract

Two asymptotic solutions are presented for linear elastic thin, not necessarily circular, cylindrical layers fully constrained by two rigid plates. Other than being small, the plate displacements and rotations are not restricted, and therefore, in general, a constrained layer is subjected to combined stretching, bending, shearing, and twisting. The first solution is restricted to layers formed by compressible solids, whose Poisson's ratio is not too close to one half. This solution is a superposition of a polynomial field, valid in the bulk of the layer, and a corrective field, which decays exponentially fast away from the cylindrical surface and becomes negligible at distances comparable to the layer thickness. The second solution is not restricted in terms Poisson's ratio but it is correct only to a leading order. This solution unifies leading-order solutions for layers formed by *compressible, nearly incompressible, and incompressible solids*. The unification involves a parameter characterizing the competition between compressibility and thinness.

1 Introduction

This paper is concerned with asymptotic analysis of linear elastic thin cylindrical, not necessarily circular, layers fully constrained by two rigid plates as shown in Figure 1. Other than being small, the plate displacements and rotations are not restricted, and therefore, in general, the induced fields are truly three-dimensional, as the layers deform by combined stretching, bending, shearing, and twisting.

Our first objective is to construct a complete asymptotic solution for layers formed by materials whose Poisson's ratio ν is not too close to one half; we refer to such materials as compressible solids. This solution will be constructed as a superposition of polynomial fields, valid in the bulk of the layer, and corrective fields, valid near the cylindrical surface. Further, it will be established that the corrective fields decay exponentially fast away from the cylindrical surface and become negligible at distances comparable to the layer thickness.

By restricting the asymptotic solution to compressible solids one significantly simplifies the problem. Indeed, if incompressible solids are included, one must address differences in solutions of boundary-value problems governed by Navier's versus Stokes' equations, and for constrained thin layers those differences are significant. To illustrate this point, let us consider stretching of two circular layers each of radius a and thickness $2h$. The first layer is formed by a compressible solid characterized by Lamé constants λ and μ. For this layer, the apparent Young's modulus $\bar{E}$ is given by the expression

[1]:

$$\bar{E} = \lambda + 2\mu\,. \tag{1}$$

The second layer is formed by an incompressible solid. In this case, (1) cannot be used as $\lambda \to \infty$. Rather the correct solution is [2]

$$\hat{E} = \frac{3}{8}\mu\left(\frac{a}{h}\right)^2\,. \tag{2}$$

Thus to reconcile (1) and (2) one needs an alternative expression valid for any λ. This motivates our second objective, which is to construct a unifying asymptotic solution valid for all solids. This objective will be pursued by extending the approach in [3] from axisymmetric to three-dimensional problems.

It is peculiar that the classical papers of Filon [1] and Stefan [2] has had a minimal impact on the pertinent solid mechanics literature. Perhaps Filon's robust approach, not restricted to thin cylinders, is unnecessary complicated for analyzing thin layers, and Stefan's work was overshadowed by Reynold's seminal work [4] published twelve years later.

The recent solid mechanics literature concerned with thin constrained layers dates back to Gent and Lindley [5, 6], who were the first to recognize the significance of testing thin rubber layers bonded to stiff (metal or glass) plates, and taking small rubber compressibility into account. Their analysis of axisymmetric stretching is based on the assumption that the radial velocity profile is parabolic. This and other assumptions are central to subsequent analyses of stretching [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and bending

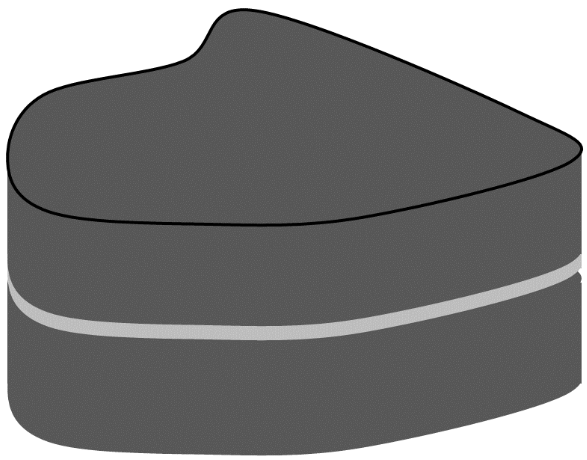

Figure 1: A thin cylindrical layer constrained by two plates.

[8, 10, 17] of thin circular cylindrical layers. Other configurations involve thin cylinders with non-circular platforms [8] and thin layers constrained by curved surfaces [17, 18]. The majority of approximate axisymmetric solutions compare favorably with results of detailed finite element analyses for layers formed by compressible and nearly incompressible solids [3].

Our approach is based on the method of compound asymptotic expansions [19, 20], and it does not involve any assumptions other than the layer is thin. Like other asymptotic approaches, ours is most naturally formulated for problems involving one small parameter. In this work, those problems are for layers formed by either compressible or incompressible solids, where the small parameter is associated with the layer thinness. In contrast, if the solid is nearly incompressible, there are two small parameters. Typically, in this case, one needs to construct multiple solutions, each corresponding to a power-law relationship between the two small parameters. Here we follow [3] and construct a single solution for the two-parameter problem, which includes the solutions of the single-parameter problems as particular cases.

Throughout the paper, we refer to the solid rather than layer compressibility. The former is an intrinsic property, whereas the latter also depends on the layer thinness. In particular, in describing moderately thin rubber layers, it is essential to treat rubber as a nearly incompressible solid. In contrast, for very thin layers, rubber can be treated as a compressible solid. This point has been emphasized in [3] and it will be revisited in Section 5.

Besides [3], this work is most closely related to [8] in terms of the scope of problems and to [17, 18] in terms of reliance on asymptotic analysis. In contrast to [8], we are interested in formal asymptotic analysis of the general problem rather than approximate solutions for rubber layers with circular and non-circular platforms. The emphasis in [17, 18] is on both thin and moderately thick rubber layers constrained by flat and curved surfaces. In contrast, we are concerned with more detailed asymptotic analysis of thin cylindrical layers formed by either compressible or nearly incompressible or incompressible solids.

The remainder of the paper is structured as follows. In Section 2, we formulate the problem and introduce the notation. In Sections 3, we develop a complete asymptotic solution for layers formed by compressible solids. In Section 4 we develop the leading order solution for layers formed by incompressible solids. In Section 5, we develop a unifying leading order solution.

We conclude the paper with a discussion of results and future work in Section 6.

We use both index and direct notations. Boldface Latin letters are reserved for vectors (lowercase) and operators (uppercase). Latin subscripts range from 1 to 3, whereas Greek subscripts range from 1 to 2, and repeated indices imply summation.

2 Problem statement

Consider a thin cylindrical layer Ω of thickness $2h$ with the mid-plane platform ω. The layer is described in Cartesian coordinates chosen so that the origin is at the centroid of ω,

$$(x_1, x_2) \in \omega \, , \quad \text{and} \quad -h < x_3 < h \, . \tag{3}$$

Further, x_1 and x_2 are the principal axes of ω. We denote the top and bottom surfaces of Ω by $\partial\Omega^{\pm}$, and the cylindrical surface by $\partial\Omega^0$ (Fig. 2.) The partial differential equations governing the linear elastic response of Ω are derived by combining the basic equilibrium equations,

$$\sigma_{ij,j} = 0 \, , \tag{4}$$

with a Hooke's law. For compressible and nearly incompressible solids, it is expressed as

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu u_{i,j} + \mu u_{j,i} \, . \tag{5}$$

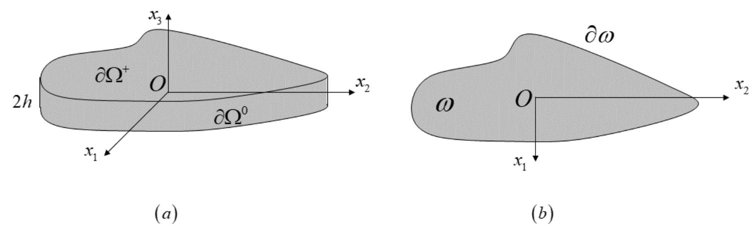

Figure 2: A thin layer Ω: (a) a three-dimensional view with the bottom flat surface $\partial\Omega^-$ not shown, (b) a view of the platform ω in the plane $x_3 = 0$.

For incompressible solids, one introduces the pressure p and the incompressibility constraint,

$$u_{i,i} = 0\,. \tag{6}$$

Then Hooke's law is expressed as

$$\sigma_{ij} = -p\delta_{ij} + \mu u_{i,j} + \mu u_{j,i}\,. \tag{7}$$

The boundary conditions are stated by adopting two assumptions. First, $\partial\Omega^0$ is neither loaded nor constrained, so that

$$\sigma_{ij} n_j = 0 \quad \text{on } \partial\Omega^0\,. \tag{8}$$

Second, the layer is perfectly bonded to two rigid plates, one is above $\partial\Omega^+$ and the other is below $\partial\Omega^-$. Accordingly,

$$u_i(x_1, x_2, \pm h) = U_i^{\pm}(x_1, x_2)\,, \tag{9}$$

where $U_i^{\pm}$ are displacements of the plates.

For simplicity of derivations, but without any loss of generality, we assume that the plates are displaced so that rigid body translations and rotations of $\partial\Omega^-$ are opposite to their counterparts of $\partial\Omega^+$. By symmetry, this choice results in $u_i(x_1, x_2, 0) = 0$. Accordingly, basic deformation modes are associated with the following prescriptions of the rigid body motion parameters:

Stretching: The only non-zero rigid body motion parameter is a translation δ_3 along the x_3 axis:

$$U_3^{\pm}(x_1, x_2) = \pm\delta_3 \ , \ \ U_1^{\pm}(x_1, x_2) = U_2^{\pm}(x_1, x_2) = 0 \, . \tag{10}$$

Bending: The only non-zero rigid body motion parameters are small counter-clock-wise rotations ϕ_1 and ϕ_2 about the x_1 and x_2 axes, respectively:

$$U_3^{\pm}(x_1, x_2) = \pm\left(\phi_1 x_2 - \phi_2 x_1\right) \ , \ \ U_1^{\pm}(x_1, x_2) = U_2^{\pm}(x_1, x_2) = 0 \, . \tag{11}$$

Shearing: The only non-zero rigid body motion parameters are translations δ_1 and δ_2 along the x_1 and x_2 axes, respectively:

$$U_1^{\pm}(x_1, x_2) = \pm\delta_1 \ , \ \ U_2^{\pm}(x_1, x_2) = \pm\delta_2 \ , \ \ U_3^{\pm}(x_1, x_2) = 0 \, . \tag{12}$$

Twisting: The only non-zero rigid body motion parameter is a small counter-clock-wise rotation about the x_3 axes:

$$U_1^{\pm}(x_1, x_2) = \mp\phi_3 x_2 \ , \ \ U_2^{\pm}(x_1, x_2) = \pm\phi_3 x_1 \ , \ \ U_3^{\pm}(x_1, x_2) = 0 \, . \tag{13}$$

These four modes can be combined, and the result written in the form

$$\mathbf{U}^{\pm}(x_1, x_2) = \pm \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \pm \begin{bmatrix} -\phi_3 x_2 \\ \phi_3 x_1 \\ \phi_1 x_2 - \phi_2 x_1 \end{bmatrix} . \tag{14}$$

3 Layers formed by compressible solids

In this section, we consider layers formed by compressible solids, which represent most materials.

3.1 Asymptotic solution for the displacements

To facilitate asymptotic analysis, let us introduce a characteristic length of ω and denote this length by a. A precise definition of a is not necessary, and ultimately it will not be included in the solution. Nevertheless, it allows us to associate the layer thinness with the small dimensionless parameter

$$\xi := \frac{h}{a} . \tag{15}$$

Further, a allows us to introduce scaled coordinates

$$X_1 := \frac{x_1}{a}, \quad X_2 := \frac{x_2}{a}, \quad X_3 := \frac{x_3}{h} = \xi^{-1} \frac{x_3}{a} . \tag{16}$$

For compressible solids, (4) and (5) imply Navier's equations,

$$L_{ij} u_j := (\lambda + \mu)\, u_{j,ji} + \mu u_{i,jj} = 0 . \tag{17}$$

If the matrix operator L_{ij} is expressed in scaled coordinates, it becomes a rational function of ξ,

$$\mathbf{L} = \xi^{-2}\mathbf{L}_0 + \xi^{-1}\mathbf{L}_1 + \mathbf{L}_2 \, , \tag{18}$$

where

$$\mathbf{L}_0 = \frac{1}{a^2}\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{bmatrix} \frac{\partial^2}{\partial X_3^2} \, , \tag{19}$$

$$\mathbf{L}_1 = \frac{\lambda+\mu}{a^2}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial X_1} \\ 0 & 0 & \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & 0 \end{bmatrix} \frac{\partial}{\partial X_3} \, , \tag{20}$$

and

$$\mathbf{L}_2 = \frac{\mu}{a^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right) + \frac{\lambda+\mu}{a^2}\begin{bmatrix} \frac{\partial^2}{\partial X_1^2} & \frac{\partial^2}{\partial X_1 \partial X_2} & 0 \\ \frac{\partial^2}{\partial X_1 \partial X_2} & \frac{\partial^2}{\partial X_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{21}$$

Further, in the scaled coordinates, the boundary conditions on $\Omega^{\pm}$, stated in (9) and (14), are rewritten as

$$\mathbf{u}(X_1, X_2, \pm 1) = \pm \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \pm a \begin{bmatrix} -\phi_3 X_2 \\ \phi_3 X_1 \\ \phi_1 X_2 - \phi_2 X_1 \end{bmatrix} . \tag{22}$$

The natural ansatz for the displacement vector is

$$\mathbf{u} = \mathbf{u}^{(0)} + \xi \mathbf{u}^{(1)} + \xi^2 \mathbf{u}^{(2)} + \ldots \, , \tag{23}$$

where it is implied that $\mathbf{u}^{(0)}$, $\mathbf{u}^{(1)}$, and $\mathbf{u}^{(2)}$ are of the same magnitude and independent of ξ. Once (18) through (21) and (23) are substituted in (17), the left-hand side becomes a rational function of ξ whose leading order term is $\mathcal{O}\left(\xi^{-2}\right)$. By equating to zero the coefficients in front of ξ^{-2} we obtain the system of ordinary differential equations:

$$\mathbf{L}_0 \mathbf{u}^{(0)} = 0 \, , \tag{24}$$

which implies that $\mathbf{u}^{(0)}$ is a linear polynomial in X_3. Then (22) implies

$$\mathbf{u}^{(0)} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} X_3 + a \begin{bmatrix} -\phi_3 X_2 \\ \phi_3 X_1 \\ \phi_1 X_2 - \phi_2 X_1 \end{bmatrix} X_3 \, . \tag{25}$$

The next order equation derived from the product $\mathbf{Lu}$ is

$$\mathbf{L}_0 \mathbf{u}^{(1)} = -\mathbf{L}_1 \mathbf{u}^{(0)} = \frac{\lambda + \mu}{a} \begin{bmatrix} \phi_2 \\ -\phi_1 \\ 0 \end{bmatrix} \, . \tag{26}$$

Thus $\mathbf{u}^{(1)}$ is a quadratic polynomial in X_3. Since (22) and (25) imply homogeneous boundary conditions for $\mathbf{u}^{(1)}$, we obtain

$$\mathbf{u}^{(1)} = \frac{\lambda+\mu}{2\mu} a \begin{bmatrix} -\phi_2 \\ \phi_1 \\ 0 \end{bmatrix} \left(1 - X_3^2\right) \,. \tag{27}$$

It is remarkable that $\mathbf{u}^{(1)}$ is associated with bending only. Finally, let us consider the equation for $\mathbf{u}^{(2)}$:

$$\mathbf{L}_0 \mathbf{u}^{(2)} = -\mathbf{L}_1 \mathbf{u}^{(1)} - \mathbf{L}_2 \mathbf{u}^{(0)} \,. \tag{28}$$

It is straightforward to establish from (20), (21), (25), (27) that the right-hand side of this equation is equal to zero. Further, (22) implies that $\mathbf{u}^{(2)}\left(X_1, X_2, \pm 1\right) = 0$, and therefore $\mathbf{u}^{(2)} = 0$. The same logic leads to the conclusion that $\mathbf{u}^{(k)} = 0$ for any $k > 2$. Thus, we regard

$$\mathbf{u} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} X_3 + a \begin{bmatrix} -\phi_3 X_2 \\ \phi_3 X_1 \\ \phi_1 X_2 - \phi_2 X_1 \end{bmatrix} X_3 + \frac{\lambda+\mu}{2\mu} a\xi \begin{bmatrix} -\phi_2 \\ \phi_1 \\ 0 \end{bmatrix} \left(1 - X_3^2\right) \,, \tag{29}$$

as an exact asymptotic approximation. We conclude this subsection by rewriting $\mathbf{u}$ in the physical coordinates,

$$\mathbf{u} = \begin{bmatrix} \delta_1 - \phi_3 x_2 \\ \delta_2 + \phi_3 x_1 \\ \delta_3 + \phi_1 x_2 - \phi_2 x_1 \end{bmatrix} \frac{x_3}{h} + \frac{\lambda+\mu}{2\mu h} \begin{bmatrix} -\phi_2 \\ \phi_1 \\ 0 \end{bmatrix} \left(h^2 - x_3^2\right) \,. \tag{30}$$

As expected, this expression is independent of a. Furthermore, it is valid for any ω, outside a neighbourhood of the lateral boundary.

3.2 Stress analysis

Equation (30) presents a remarkably simple displacement field. However, this field was constructed without satisfying the traction-free boundary conditions on the cylindrical surface $\partial\Omega^0$. To address this issue, let us use (5) to calculate the stress field corresponding to (30):

$$\begin{aligned}
\sigma_{11} &= \sigma_{22} = \frac{\lambda\left(\delta_3 + \phi_1 x_2 - \phi_2 x_1\right)}{h} \\
\sigma_{33} &= \frac{(\lambda + 2\mu)\left(\delta_3 + \phi_1 x_2 - \phi_2 x_1\right)}{h} \\
\sigma_{12} &= 0 \\
\sigma_{13} &= \frac{\mu\left(\delta_1 - \phi_3 x_2\right)}{h} + \lambda\phi_2\frac{x_3}{h} \\
\sigma_{23} &= \frac{\mu\left(\delta_2 + \phi_3 x_1\right)}{h} - \lambda\phi_1\frac{x_3}{h}\,.
\end{aligned} \tag{31}$$

To calculate the traction vector, we identify the unit normal $\mathbf{n}$ at a point $P \in \partial\Omega^0$, and parameterize the normal as

$$\mathbf{n} = \begin{bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{bmatrix}, \tag{32}$$

where ψ is the angle between $\mathbf{n}$ and the x_1-axis (Fig. 3). The traction vector calculated from (31) and (32) is confined to the plane spanned by $\mathbf{n}$ and the

x_3-axis. The non-zero components of this vector are

$$\sigma_{nn} = \frac{\lambda\,(\delta_3 + \phi_1 x_2 - \phi_2 x_1)}{h} \tag{33}$$

and

$$\begin{aligned} \sigma_{n3} &= \frac{\mu}{h}\left[(\delta_1 - \phi_3 x_2 h)\cos\psi + (\delta_2 + \phi_3 x_1)\sin\psi\right] \\ &- \lambda\,(\phi_1 \sin\psi - \phi_2 \cos\psi)\,\frac{x_3}{h}\,. \end{aligned} \tag{34}$$

In these equations, it is implied that x_i are the coordinates of P. Note that σ_{nn} is dictated solely by stretching and bending. Further, upon comparison with (31), we conclude that σ_{nn} is of the same magnitude as the normal stresses in the bulk of the layer. The stress σ_{n3} consists of two terms. The first term is dictated solely by shearing and twisting, and it is of the same magnitude as the shear stresses in the bulk of the layer. The second term is dictated solely by bending, and it an order of magnitude smaller than the normal stresses induced by bending in the bulk of the layer. Furthermore, the resultant of the second term is equal to zero. Therefore, effectively, as far as the induced fields are concerned, the second term of σ_{n3} is by two orders of magnitude smaller than the first term.

Equations (33) and (34) demonstrate that the displacement field given in (30) does not satisfy traction-free boundary conditions on $\partial\Omega^0$. To remedy this situation, we need to construct a corrective solution, which will be done in the next subsection.

We conclude this subsection by calculating the internal forces and moments

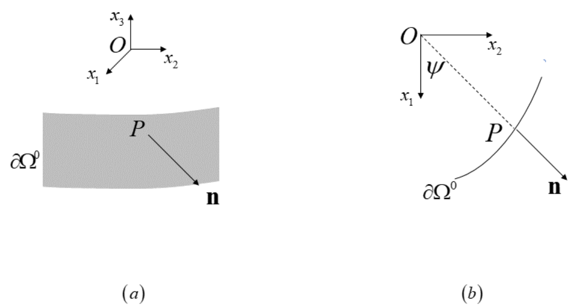

Figure 3: The normal vector at $P \in \partial\Omega^0$: (a) a three-dimensional view, (b) a view in a x_3-plane containing P.

transmitted through ω. This is done by integrating the stress components in (31) evaluated at $x_3 = 0$. The integration is very simple because the coordinate origin is at the centroid of ω, and x_1 and x_2 are the principal axes:

$$
\begin{aligned}
&\text{Shear forces :}\\
&F_1 = \int_\omega \sigma_{13}\mathrm{d}A = \mu A\frac{\delta_1}{h}, \quad F_2 = \int_\omega \sigma_{23}\mathrm{d}A = \mu A\frac{\delta_2}{h}\\
&\text{Normal force :}\\
&F_3 = \int_\omega \sigma_{33}\mathrm{d}A = \bar{E}A\frac{\delta_3}{h}\\
&\text{Bending moments :}\\
&M_1 = \int_\omega x_2\sigma_{33}\mathrm{d}A = \bar{E}I_{11}\frac{\phi_1}{h}, \quad M_2 = -\int_\omega x_1\sigma_{33}\mathrm{d}A = \bar{E}I_{22}\frac{\phi_2}{h}\\
&\text{Twisting moment :}\\
&M_3 = \int_\omega \left(x_1\sigma_{23} - x_2\sigma_{13}\right)\mathrm{d}A = \mu I_p\frac{\phi_3}{h}\ .
\end{aligned}
\tag{35}
$$

Here $\bar{E}$ is the apparent modulus given in (1). The area A and moments of inertia, I_{11}, I_{22}, and I_p are defined with respect to ω. Of course these equations are almost identical to those of elementary beam theory. The only difference is that $\bar{E}$ replaces the true Young's modulus E measured in uni-axial tension.

3.3 Corrective solution

In the previous subsection we established that the displacement field $\mathbf{u}$ in (30) results in the stress field, which does not satisfy (8) on the lateral surface of the thin solid. Accordingly, in this subsection, we construct a corrective displacement field $\mathbf{v}$, so that the stress field corresponding to $\mathbf{u} + \mathbf{v}$ satisfies (8) in an asymptotic sense. Furthermore, we establish that $\mathbf{v}$ decays exponentially fast away from $\partial\Omega^0$ and therefore becomes negligible in the bulk of the layer.

Let us focus on constructing the corrective displacement field $\mathbf{v}$ in a small neighborhood of a point $P \in \partial\Omega^0$. To this end, we first identify O' so that $O' \in \partial\omega$ and $\overrightarrow{O'P}$ is parallel to the x_3-axis. Next, we introduce a local Cartesian coordinate system with the origin at O', the x_1'-axis along the unit normal vector at O', the x_2'-axis along the global x_3-axis, and the x_3'-axis tangential to $\partial\omega$, so that the coordinate system is right-handed (Fig. 4a). In these coordinates, the boundary-value problem for $\mathbf{v}$ is defined as

$$L_{i'j'} v_{j'} = 0 \quad \text{in } \partial\Omega \,, \tag{36}$$

$$v_{i'} = 0 \quad \text{on } \partial\Omega^{\pm} \,, \tag{37}$$

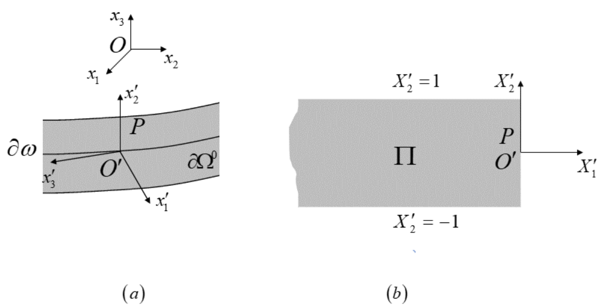

Figure 4: Setting for analysis of corrective fields: (a) local physical coordinates, (b) the semi-infinite strip Π in the plane $X_3' = 0$ shown in scaled coordinates.

and

$$\tau_{1'1'} = -\sigma_{nn} \, , \quad \tau_{1'2'} = -\sigma_{n3} \, , \quad \tau_{1'3'} = 0 \quad \text{on } \partial\Omega^0 \, . \tag{38}$$

In the last equation, $\tau_{i'j'}$ are the components of the stress tensor corresponding to $v_{i'}$, and σ_{nn} and σ_{n3} are from (33) and (34), respectively (Fig. 4a).

The local asymptotic solution is developed under the assumption that $\partial\omega$ is a smooth curve whose radius of curvature is comparable to a rather than h. Therefore the normalization length along x_3' is chosen to be a. In contrast, since we are interested in a solution in a vicinity of $\partial\Omega^0$, the normalization length along x_1' is chosen to be h. Of course, because the layer is thin, x_2' is normalized with h. Accordingly, the local scaled coordinates are

$$X_1' := \frac{x_1'}{h}, \quad X_2' := \frac{x_2'}{h}, \quad X_3' := \frac{x_3'}{a} \, . \tag{39}$$

To the leading order, this normalization results in splitting the original three-dimensional problem into anti-plane and plane strain problems defined on the semi-infinite strip (Fig. 4b),

$$\Pi = \left\{ \left(X'_1, X'_2 \right) : X'_1 < 0, \; -1 < X'_2 < 1 \right\} .$$

The anti-plane problem is governed by the partial differential equation,

$$\frac{\partial^2 v_{3'}}{\partial X'^2_1} + \frac{\partial^2 v_{3'}}{\partial X'^2_2} = 0 \, , \tag{40}$$

with the boundary conditions

$$v_{3'} \left(X'_1, \pm 1 \right) = 0 \tag{41}$$

and

$$\tau_{1'3'} \left(0, X'_2 \right) = 0 \, . \tag{42}$$

The plane strain problem is governed by the system of partial differential equations for $(x'_1, x'_2) \in \Pi$

$$\begin{aligned} \mu \left(\frac{\partial^2 v_{1'}}{\partial X'^2_1} + \frac{\partial^2 v_{1'}}{\partial X'^2_2} \right) + (\lambda + \mu) \frac{\partial}{\partial X'_1} \left(\frac{\partial v_{1'}}{\partial X'_1} + \frac{\partial v_{2'}}{\partial X'_2} \right) = 0 \, , \\ \mu \left(\frac{\partial^2 v_{2'}}{\partial X'^2_1} + \frac{\partial^2 v_{2'}}{\partial X'^2_1} \right) + (\lambda + \mu) \frac{\partial}{\partial X'_2} \left(\frac{\partial v_{1'}}{\partial X'_1} + \frac{\partial v_{2'}}{\partial X'_2} \right) = 0 \, , \end{aligned} \tag{43}$$

with the boundary conditions

$$v_{1'} \left(X'_1, \pm 1 \right) = v_{2'} \left(X'_1, \pm 1 \right) = 0 \tag{44}$$

and

$$\tau_{1'1'}\left(0, X_2'\right) = -\sigma_{nn}\ , \quad \tau_{1'2'}\left(0, X_2'\right) = -\sigma_{n3}\ . \tag{45}$$

It is clear that the solution of the anti-plane problem is trivial: $v_{3'} = 0$. For the plane strain problem, we follow what is now a textbook approach [21], and seek the solution in the form

$$v_{\alpha'}\left(X_1', X_2'\right) = \exp\left(\Lambda X_1'\right)\tilde{v}_{\alpha'}\left(X_2'\right)\ , \tag{46}$$

with the objective of finding Λ which yields non-trivial solutions to (43) and (44). In what follows, we establish that for any pair of λ and μ there is a unique $\Lambda > 0$ of interest. The condition $\Lambda > 0$ results in an exponentially decaying $\mathbf{v}$ since Π is characterized by $X_1' < 0$.

The equation for Λ is derived in four steps:

1. Equation (46) is substituted in (43), so that the system of partial differential equations for $v_{\alpha'}\left(X_1', X_2'\right)$ becomes a system of ordinary differential equations for $\tilde{v}_{\alpha'}\left(X_2'\right)$.

2. The ordinary differential equations are solved for $\tilde{v}_{\alpha'}\left(X_2'\right)$.

3. The solutions are substituted in (44) to form a homogeneous system of linear algebraic equations for the integration constants.

4. A non-trivial solution for the integration constants is obtained by setting the determinant of the system matrix equal to zero.

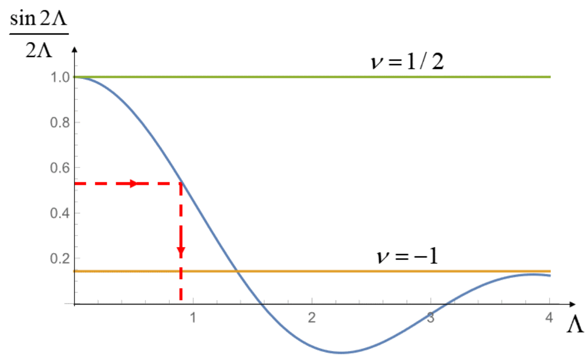

Figure 5: In the domain, where $\frac{1}{7} < \frac{\sin 2\Lambda}{2\Lambda} < 1$, equation (47) has one and only one positive solution.

As a result, one obtains the characteristic equation

$$\frac{\sin 2\Lambda}{2\Lambda} = \frac{\lambda + \mu}{\lambda + 3\mu} = \frac{1}{3 - 4\nu} \, . \tag{47}$$

The restriction on Poisson's ratio, $-1 < \nu < 1/2$, implies that the right-hand side of this equation is in the interval $(1/7, 1)$. The plot shown in Figure 5 contains the curve $\sin 2\Lambda/(2\Lambda)$ and two straight lines. The upper line corresponds to the right-hand side evaluated at $\nu = 1/2$ and the lower one for the right-hand side evaluated at $\nu = -1$. This plot clearly shows that for the segment of the curve bounded by the lines there is one and only one positive solution. For a typical $\nu = 0.3$ the solution is $\Lambda \approx 0.883$, and it is shown by red dashed lines. Thus, in this case, the exponential decay is characterized by the length approximately equal to $h/\Lambda \approx 1.133h$.

4 Layers formed by incompressible solids

In this section, we consider layers formed by incompressible solids. This problem dates back to [2] and its solution can be found in many papers including [8, 10, 17]. We revisit the problem because its solution structure is similar to that of the unifying solution to be developed in the next section. Since both shearing and twisting modes are insensitive to the degree of compressibility, in this section we restrict our attention to the stretching and bending modes.

4.1 Leading order asymptotic solution for the displacements

For an incompressible solid, (4), (6), and (7) yield Stokes' partial differential equations:

$$-p_{,i} + \mu u_{i,jj} = 0 \quad \text{and} \;\; u_{i,i} = 0\,. \tag{48}$$

The boundary conditions on $\partial\Omega^0$ are specified by combining (7) and (8):

$$\sigma_{ij} n_j = -p n_i + \mu u_{i,j} n_j + \mu u_{j,i} n_j = 0 \quad \text{on} \;\; \partial\Omega^0\,. \tag{49}$$

The boundary conditions on $\partial\Omega^\pm$ follow from (9) and (14), with the provision that the kinematic degrees of freedom associated with shearing and twisting are set equal to zero:

$$u_1 = u_2 = 0, \quad u_3 = \pm\left(\delta_3 + \phi_1 x_2 - \phi_2 x_1\right) \quad \text{on} \;\; \partial\Omega^\pm\,. \tag{50}$$

Following, Stefan's solution [2], where the radial displacement is by an order of magnitude larger than the axial one, we adopt the displacement vector in the form

$$\mathbf{u} = \begin{bmatrix} \xi^{-1} u_1^{(-1)} \\ \xi^{-1} u_2^{(-1)} \\ u_3^{(0)} \end{bmatrix} . \tag{51}$$

With this ansatz, in scaled coordinates, the leading order approximation of (48) can be written as the following system of equations:

$$-\frac{\partial p}{\partial X_\alpha} + \frac{\mu}{a\xi^3} \frac{\partial^2 u_\alpha^{(-1)}}{\partial X_3^2} = 0 \, , \tag{52}$$

$$-\frac{\partial p}{\partial X_3} + \frac{\mu}{a\xi} \frac{\partial^2 u_3^{(0)}}{\partial X_3^2} = 0 \, , \tag{53}$$

$$\Theta := \sum_{\alpha=1}^{2} \frac{\partial u_\alpha^{(-1)}}{\partial X_\alpha} + \frac{\partial u_3^{(0)}}{\partial X_3} = 0 \, . \tag{54}$$

Since $u_\alpha^{(-1)}$ and $u_3^{(0)}$ are of the same order of magnitude, (52) and (53) imply that to the leading order

$$\frac{\partial p}{\partial X_3} = 0 \, . \tag{55}$$

This makes integration of (52) with respect to X_3 elementary. With $u_\alpha^{(-1)} = 0$ at $X_3 = \pm 1$, this integration yields

$$u_\alpha^{(-1)} = -\frac{a\xi^3}{2\mu} \frac{\partial p}{\partial X_\alpha} \left(1 - X_3^2\right) \, . \tag{56}$$

Once this equation is substituted in (54) and the resulting equation is integrated with respect to X_3, (50) implies

$$\begin{aligned} u_\alpha &= -\frac{a\xi^2}{2\mu}\frac{\partial p}{\partial X_\alpha}\left(1 - X_3^2\right)\ ,\ \alpha = 1,2, \\ u_3 &= \frac{3}{2}\left(\delta_3 + a\phi_1 X_2 - a\phi_2 X_1\right)\left(X_3 - \frac{1}{3}X_3^3\right)\ , \end{aligned} \tag{57}$$

and

$$\frac{a\xi^3}{3\mu}\left(\frac{\partial^2 p}{\partial X_1^2} + \frac{\partial^2 p}{\partial X_2^2}\right) = \delta_3 + a\phi_1 X_2 - a\phi_2 X_1 \quad \text{in } \omega\ . \tag{58}$$

In the next subsection, we will establish the boundary conditions corresponding to this partial differential equation.

Let us conclude this subsection by rewriting the key results in the physical coordinates:

$$\begin{aligned} u_\alpha &= -\frac{h^2}{2\mu}\frac{\partial p}{\partial x_\alpha}\left[1 - \left(\frac{x_3}{h}\right)^2\right]\ , \alpha = 1,2, \\ u_3 &= \frac{3}{2}\left(\delta_3 + \phi_1 x_2 - \phi_2 x_1\right)\left[\left(\frac{x_3}{h}\right) - \frac{1}{3}\left(\frac{x_3}{h}\right)^3\right]\ . \end{aligned} \tag{59}$$

Note that the first equation justifies the assumption that, in the axisymmetric setting, the radial velocity profile is parabolic. The second equation implies that the problem is kinematically determinate for u_3. The governing partial differential equation (58) is rewritten as

$$\frac{h^3}{3\mu}\left(\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2}\right) = \delta_3 + \phi_1 x_2 - \phi_2 x_1 \quad \text{in } \omega\ . \tag{60}$$

As expected, neither (59) nor (60) include a.

4.2 Stress analysis

The stresses corresponding to (59) are calculated from (7):

$$
\begin{aligned}
\sigma_{11} &= -p - \left(h^2 - x_3^2\right) \frac{\partial^2 p}{\partial x_1^2} , \\
\sigma_{22} &= -p - \left(h^2 - x_3^2\right) \frac{\partial^2 p}{\partial x_2^2} , \\
\sigma_{33} &= -p + \frac{3\mu \left(h^2 - x_3^2\right)}{h^3} \left(\delta_3 - \phi_2 x_1 + \phi_1 x_2\right) , \\
\sigma_{12} &= - \left(h^2 - x_3^2\right) \frac{\partial^2 p}{\partial x_1 \partial x_2} \\
\sigma_{13} &= x_3 \left[\frac{\partial p}{\partial x_1} - \frac{\mu \left(3h^2 - x_3^2\right)}{2h^3} \phi_2 \right] , \\
\sigma_{23} &= x_3 \left[\frac{\partial p}{\partial x_2} + \frac{\mu \left(3h^2 - x_3^2\right)}{2h^3} \phi_1 \right] .
\end{aligned}
\tag{61}
$$

To assess the relative significance of the components in (61), let us introduce a reference strain,

$$
\epsilon := \frac{|\delta_3|}{h} + \frac{|\phi_1|\, a}{h} + \frac{|\phi_2|\, a}{h} = \frac{|\delta_3|}{h} + \xi^{-1} \left(|\phi_1| + |\phi_2|\right) . \tag{62}
$$

Then using (60) it is straightforward to establish that $p = \mu \mathcal{O}\left(\xi^{-2}\epsilon\right)$, σ_{13} and σ_{23} are $\mu \mathcal{O}\left(\xi^{-1}\epsilon\right)$, and the remaining deviatoric components are $\mu \mathcal{O}(\epsilon)$. The scaling for p is in agreement with Stefan's solution [2] and in sharp contrast to (31), where all stresses induced by stretching and bending are $\mu \mathcal{O}(\epsilon)$.

The traction vector on $\partial\Omega^0$ corresponding to (61) is calculated in the local coordinates shown in Figure 4:

$$
\begin{aligned}
\sigma_{1'1'} &= -p \\
&- \left(h^2 - x_3^2\right)\left(\frac{\partial^2 p}{\partial x_1^2}\cos^2\psi + \frac{\partial^2 p}{\partial x_1 \partial x_2}\sin 2\psi + \frac{\partial^2 p}{\partial x_2^2}\sin^2\psi\right) , \quad (63) \\
\sigma_{1'2'} &= x_3\left(\frac{\partial p}{\partial x_1}\cos\psi + \frac{\partial p}{\partial x_2}\sin\psi\right) \\
&+ \frac{\mu x_3\left(3h^2 - x_3^2\right)}{2h^3}\left(\phi_1 \sin\psi - \phi_2\cos\psi\right) , \quad (64) \\
\sigma_{1'3'} &= \frac{1}{2}\left(h^2 - x_3^2\right)\left[\left(-\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2}\right)\sin 2\psi + 2\frac{\partial^2 p}{\partial x_1 \partial x_2}\cos 2\psi\right] . (65)
\end{aligned}
$$

In these expressions, ψ is the angle shown in Figure 3. On the right-hand side of (63), $p = \mu\mathcal{O}\left(\xi^{-2}\epsilon\right)$, whereas the second term is $\mu\mathcal{O}\left(\epsilon\right)$. On the right-hand side of (64), both terms are $\mu\mathcal{O}\left(\xi^{-1}\epsilon\right)$. But because the resultant of $\sigma_{1'2'}$ is equal to zero, the effect of $\sigma_{1'2'}$ as if it were $\mu\mathcal{O}\left(\epsilon\right)$. The right-hand side of (65) is $\mu\mathcal{O}\left(\epsilon\right)$. Thus if we set

$$
p = 0 \quad \text{on } \partial\omega \, , \tag{66}
$$

the entire traction vector becomes effectively $\mu\mathcal{O}\left(\epsilon\right)$. Thus the stresses in (61) are $\mu\mathcal{O}\left(\xi^{-2}\epsilon\right)$ whereas the stress fields associated with non-zero tractions on $\partial\Omega^0$ are effectively $\mu\mathcal{O}\left(\epsilon\right)$. Therefore a corrective solution similar to that developed in Section 3.3 is unnecessary here.

We conclude this subsection by observing that both the normal force and bending moment resultants are dominated by the pressure, and therefore their explicit expressions require the solution of (60) and (66). Of course the remaining internal forces and moments are equal to zero.

4.3 Circular layers

Boundary-value problems governed by (60) and (66) are straightforward to solve for circular, elliptical, square, and other canonical domains. In particular, if $\partial\omega$ is a circle of radius a and $\phi_1 = \phi_2 = 0$, the solution of (60) and (66) should recover that of the pressure in Stefan's problem. Accordingly, (60) and (66) are specified for p dependent only on the radial distance r:

$$\frac{h^3}{3\mu}\left(\frac{\mathrm{d}^2 p}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = \delta_3 \quad 0 \leq r < a \tag{67}$$

and

$$p = 0 \quad \text{at } r = a\,. \tag{68}$$

With the standard provision that the solution must be bounded, one can solve (67) and (68) to obtain

$$p = -\frac{3\mu\left(a^2 - r^2\right)\delta_3}{4h^3}\,. \tag{69}$$

The radial displacement corresponding to this pressure field follows from (59):

$$u_r = -\frac{h^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}r}\left[1 - \left(\frac{x_3}{h}\right)^2\right] = -\frac{3r\delta_3}{4h}\left[1 - \left(\frac{x_3}{h}\right)^2\right]\,. \tag{70}$$

The internal force corresponding to (69) is

$$F_3 = -2\pi\int_0^a p(r)r\mathrm{d}r = \frac{3\pi\mu a^4\delta_3}{8h^3}\,. \tag{71}$$

To the leading order these three expressions agree with their counterparts of Stefan's solution. Furthermore, the apparent Young's modulus calculated from (71) coincides with that in (2).

For the bending case, p is a function of both polar coordinates, so that (60) implies

$$\frac{h^3}{3\mu}\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right)=r\left(\phi_1\sin\theta-\phi_2\cos\theta\right)\ . \tag{72}$$

The solution of this equation is sought in the form

$$p(r,\theta)=f(r)\left(\phi_1\sin\theta-\phi_2\cos\theta\right)\ . \tag{73}$$

This form reduces (72) to the ordinary equation for the function f:

$$\frac{h^3}{3\mu}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}-\frac{f}{r^2}\right)=r \quad 0\leq r<a\,. \tag{74}$$

The finite solution for $f(r)$ subjected to the boundary condition $f(a)=0$ is

$$f(r)=-\frac{3\mu\left(a^2-r^2\right)r}{8h^3}\,. \tag{75}$$

This function gives rise to the following expressions:

$$p=-\frac{3\mu r\left(a^2-r^2\right)}{8h^3}\left(\phi_1\sin\theta-\phi_2\cos\theta\right)\ , \tag{76}$$

$$u_r=\frac{3\left(a^2-3r^2\right)}{16h}\left[1-\left(\frac{x_3}{h}\right)^2\right]\left(\phi_1\sin\theta-\phi_2\cos\theta\right)\ , \tag{77}$$

$$u_\theta=\frac{3\left(a^2-r^2\right)}{16h}\left[1-\left(\frac{x_3}{h}\right)^2\right]\left(\phi_1\cos\theta+\phi_2\sin\theta\right)\ , \tag{78}$$

and

$$M_\alpha = \frac{\pi a^6 \mu}{32h^3} \phi_\alpha \,. \tag{79}$$

To a leading order all of the solutions presented in this subsection agree with those published in [8, 10, 17].

5 Unifying solution

In this section, we construct a leading order asymptotic solution which allows us to unify the leading order solution developed in Sections 3 and 4.

5.1 Leading order asymptotic solution for the displacements

Following [3], we parameterize elastic solids using the shear modulus μ and the dimensionless parameter

$$\chi := \sqrt{\frac{3\mu}{\lambda + 2\mu}} \,. \tag{80}$$

This parameter was introduced in [7], and it arises naturally in axisymmetric problems. One can also see its emergence if one seeks a cross-over point and sets $\bar{E} = \hat{E}$ in (1) and (2).

Equation (80) implies that for incompressible solids, $\chi = 0$, for nearly incompressible solids $\chi \ll 1$, and compressible solids $\chi = \mathcal{O}(1)$. Thus a layer formed by a nearly incompressible solid is characterized by two small parameters, χ and ξ, whereas layers formed by either compressible or incompressible solids are characterized by only one small parameter ξ.

Conventionally, problems with two small parameters are solved by identifying power-law relationships between them. For example, if one sets $\chi = \xi$, the operator $\mathbf{L}$ becomes a rational function of ξ whose leading term is proportional to ξ^{-4} rather than ξ^{-2}, as it was the case in (18). Similarly, if one sets $\chi = \sqrt{\xi}$, the operator $\mathbf{L}$ becomes a rational function of ξ whose leading term is proportional to ξ^{-3}. Of course, multiple choices may result in multiple solutions.

Here we follow [3] and introduce the parameter

$$\zeta := \frac{\xi}{\chi}\,, \tag{81}$$

which characterizes the competition between ξ and χ. That is, if the solid is nearly incompressible, so that $\chi \ll \xi$, then $\zeta \gg 1$. On the contrary, if the solid is compressible, so that $\chi = \mathcal{O}(1)$, then $\zeta \gg 1$. In what follows, we regard ζ as neither small nor large, which in effect leaves us with only one small parameter ξ. Accordingly, $\mathbf{L}$ is split as

$$\mathbf{L} = \mathbf{L}' + \mathbf{L}'' = \left(\xi^{-2}\mathbf{L}'_0 + \xi^{-1}\mathbf{L}'_1 + \mathbf{L}'_2\right) + 3\zeta^2\left(\xi^{-4}\mathbf{L}''_0 + \xi^{-3}\mathbf{L}''_1 + \xi^{-2}\mathbf{L}''_2\right)\,, \tag{82}$$

with

$$\mathbf{L}'_0 \;=\; \mu a^{-2}\begin{bmatrix} \frac{\partial^2}{\partial X_3^2} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial X_3^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{83}$$

$$\mathbf{L}_1' = -\mu a^{-2} \begin{bmatrix} 0 & 0 & \frac{\partial^2}{\partial X_1 \partial X_3} \\ 0 & 0 & \frac{\partial^2}{\partial X_2 \partial X_3} \\ \frac{\partial^2}{\partial X_1 \partial X_3} & \frac{\partial^2}{\partial X_2 \partial X_3} & 0 \end{bmatrix}, \tag{84}$$

$$\mathbf{L}_2' = \mu a^{-2} \begin{bmatrix} \frac{\partial^2}{\partial X_2^2} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial X_1^2} & 0 \\ 0 & 0 & \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} \end{bmatrix}, \tag{85}$$

$$\mathbf{L}_0'' = \mu a^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial X_3^2} \end{bmatrix}, \tag{86}$$

$$\mathbf{L}_1'' = \mu a^{-2} \begin{bmatrix} 0 & 0 & \frac{\partial^2}{\partial X_1 \partial X_3} \\ 0 & 0 & \frac{\partial^2}{\partial X_2 \partial X_3} \\ \frac{\partial^2}{\partial X_1 \partial X_3} & \frac{\partial^2}{\partial X_2 \partial X_3} & 0 \end{bmatrix}, \tag{87}$$

$$\mathbf{L}_2'' = \mu a^{-2} \begin{bmatrix} \frac{\partial^2}{\partial X_1^2} & \frac{\partial^2}{\partial X_1 \partial X_2} & 0 \\ \frac{\partial^2}{\partial X_1 \partial X_2} & \frac{\partial^2}{\partial X_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{88}$$

We choose the displacement ansatz for the leading order solution which mimics (51), but rewrite it as

$$\mathbf{u} = \xi^{-1}\mathbf{u}^{(-1)} + \mathbf{u}^{(0)} = \xi^{-1}\begin{bmatrix} u_1^{(-1)} \\ u_2^{(-1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_3^{(0)} \end{bmatrix} . \tag{89}$$

Once (82) and (89) are combined we obtain

$$\mathbf{Lu} = \xi^{-4}\begin{bmatrix} 0 \\ 0 \\ \frac{\partial\Theta}{\partial X_3} \end{bmatrix} + \xi^{-3}\begin{bmatrix} \frac{\partial^2 u_1^{(-1)}}{\partial X_3^2} + 3\zeta^2\frac{\partial\Theta}{\partial X_1} \\ \frac{\partial^2 u_2^{(-1)}}{\partial X_3^2} + 3\zeta^2\frac{\partial\Theta}{\partial X_2} \\ 0 \end{bmatrix} + \dots . \tag{90}$$

These equations are straightforward to integrate with respect to X_3, so that the boundary conditions on $\partial\Omega^{\pm}$,

$$\begin{aligned} u_1^{(-1)}(X_1, X_2, \pm 1) = u_2^{(-1)}(X_1, X_2, \pm 1) = 0 \, , \\ u_3^{(0)}(X_1, X_2, \pm 1) = \pm\left(\delta_3 + a\phi_1 X_2 - a\phi_2 X_1\right) \, , \end{aligned} \tag{91}$$

yield the displacements,

$$\begin{aligned} u_\alpha &= \frac{3}{2}\xi^{-1}\zeta^2\frac{\partial\Theta}{\partial X_\alpha}\left(1 - X_3^2\right) \, , \\ u_3 &= \frac{1}{2}\left(\delta_3 + a\phi_1 X_2 - a\phi_2 X_1\right)\left(3X_3 - X_3^3\right) + \frac{1}{2}\Theta\left(-X_3 + X_3^3\right) \, , \end{aligned} \tag{92}$$

and the partial differential equation for Θ,

$$-\zeta^2\left(\frac{\partial^2\Theta}{\partial X_1^2} + \frac{\partial^2\Theta}{\partial X_2^2}\right) + \Theta = \delta_3 + a\phi_1 X_2 - a\phi_2 X_1 \quad \text{in } \omega \, . \tag{93}$$

The similarities between (92) versus (57) and (93) versus (58) are striking. Thus, the unifying solution is closely related to the solution for layers formed by incompressible solids. Indeed, (92) and (93) are essentially identical to those developed in Section 2.1.3 of [17] for layers formed by nearly incompressible solids.

Following Sections 3 and 4, let us conclude this subsection with rewriting (92) and (93) in the physical coordinates:

$$\begin{aligned} u_\alpha &= \frac{3h}{2\chi^2}\frac{\partial\Theta}{\partial x_\alpha}\left[1-\left(\frac{x_3}{h}\right)^2\right] , \\ u_3 &= \frac{1}{2}\left(\delta_3+\phi_1 x_2-\phi_2 x_1\right)\left[3\left(\frac{x_3}{h}\right)-\left(\frac{x_3}{h}\right)^3\right] \\ &+ \frac{\Theta}{2}\left[-\left(\frac{x_3}{h}\right)+\left(\frac{x_3}{h}\right)^3\right] , \end{aligned} \tag{94}$$

and

$$-\frac{h^2}{\chi^2}\left(\frac{\partial^2\Theta}{\partial x_1^2}+\frac{\partial^2\Theta}{\partial x_2^2}\right)+\Theta=\delta_3+\phi_1 x_2-\phi_2 x_1 \quad \text{in } \omega . \tag{95}$$

As expected, every entry in these equations is unambiguously defined.

5.2 Stress analysis

The stresses are calculated from the displacements by substituting (94) in (5). Upon retaining leading order terms only we obtain:

$$\begin{aligned} \sigma_{11}=\sigma_{22}=\sigma_{33}&=\frac{3\mu\Theta}{\chi^2 h} , \\ \sigma_{12}&=\frac{3h\mu\left(h^2-x_3^2\right)}{\chi^2 h}\frac{\partial^2\Theta}{\partial x_1\partial x_2} , \\ \sigma_{\alpha 3}&=-\frac{3\mu x_3}{\chi^2 h}\frac{\partial\Theta}{\partial x_\alpha} . \end{aligned} \tag{96}$$

The traction vector on $\partial\Omega^0$ corresponding to (96) is calculated in the local coordinates shown in Figures 3 and 4:

$$
\begin{aligned}
\sigma_{1'1'} &= \frac{3\mu}{\chi^2 h}\Theta \\
\sigma_{1'2'} &= -\frac{3\mu x_3}{\chi^2 h}\left(\cos\psi\frac{\partial\Theta}{\partial x_1} + \sin\psi\frac{\partial\Theta}{\partial x_2}\right) , \\
\sigma_{1'3'} &= -\frac{3\mu}{\chi^2 h}\left(h^2 - x_3^2\right)\cos 2\psi\frac{\partial^2\Theta}{\partial x_1\partial x_2} .
\end{aligned}
\tag{97}
$$

Like in (96), these expressions contain only leading order terms. It is clear that $\sigma_{1'1'}$ is one order of magnitude larger than $\sigma_{1'2'}$. In addition, the resultant of $\sigma_{1'2'}$ is equal to zero, and therefore, effectively, the fields induced by $\sigma_{1'1'}$ are two orders of magnitude larger than those induced by $\sigma_{1'2'}$. Since $\sigma_{1'1'}$ is two orders of magnitude larger than $\sigma_{1'3'}$, the traction component $\sigma_{1'1'}$ dominates the other two components by two orders of magnitude. Consequently, the best way to reduce the traction vector on $\partial\Omega^0$ is by setting

$$
\Theta = 0 \quad \text{on } \partial\omega . \tag{98}
$$

With this provision, the stresses associated with non-zero tractions is two orders of magnitude less than those in the bulk of the layer. Therefore, as it was the case in Section 4, a corrective solution is not necessary for the leading order analysis.

The task of calculating the normal forces and bending moments transmitted through ω cannot be completed without specifying Θ. Therefore, this task is not pursued here.

5.3 Circular layers

Boundary-value problems governed by (95) are not as amenable to analytical solutions as those governed by (60). Nevertheless analytical solutions are possible to express in terms of Bessel's function if $\partial\omega$ is a circle [8, 10, 17].

For stretching of a circular layer with radius a, $\phi_1 = \phi_2 = 0$, the problem becomes axisymmetric. Accordingly, Θ becomes a function of r only, and (94), (95), and (98) are rewritten as

$$\begin{aligned} u_r &= \frac{3h}{2\chi^2}\frac{\mathrm{d}\Theta}{\mathrm{d}r}\left[1-\left(\frac{x_3}{h}\right)^2\right] , \\ u_3 &= \frac{1}{2}\delta_3\left[3\left(\frac{x_3}{h}\right)-\left(\frac{x_3}{h}\right)^3\right]+\frac{\Theta}{2}\left[-\left(\frac{x_3}{h}\right)+\left(\frac{x_3}{h}\right)^3\right] , \end{aligned} \tag{99}$$

$$-\frac{h^2}{\chi^2}\left(\frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\Theta}{\mathrm{d}r}\right)+\Theta=\delta_3 , \tag{100}$$

and

$$\Theta = 0 \quad \text{at } r = a . \tag{101}$$

These equations can be readily solved in terms of Bessel's functions:

$$\Theta = \delta_3\left[1-\frac{I_0\left(\frac{r\chi}{h}\right)}{I_0\left(\frac{a\chi}{h}\right)}\right] , \tag{102}$$

$$\begin{aligned} u_r &= -\frac{3}{2\chi}\frac{I_1\left(\frac{r\chi}{h}\right)}{I_0\left(\frac{a\chi}{h}\right)}\left[1-\left(\frac{x_3}{h}\right)^2\right]\delta_3 , \\ u_3 &= \frac{1}{2}\frac{x_3}{h}\left\{2+\left[1-\left(\frac{x_3}{h}\right)^2\right]\frac{I_0\left(\frac{r\chi}{h}\right)}{I_0\left(\frac{a\chi}{h}\right)}\right\}\delta_3 . \end{aligned} \tag{103}$$

The axial force required to realizes the stretch is

$$F_3 = \frac{3\mu}{\chi^2}\pi a^2 \frac{I_2\left(\frac{a\chi}{h}\right)}{I_0\left(\frac{a\chi}{h}\right)}\frac{\delta_3}{h} \, . \tag{104}$$

For bending, governed by prescribed rotations ϕ_1 and ϕ_2, Θ is a function of both polar coordinates, so that (95) is rewritten as

$$-\frac{h^2}{\chi^2}\left(\frac{\partial^2\Theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\right) + \Theta = r\left(\phi_1 \sin\theta - \phi_2 \cos\theta\right) \, . \tag{105}$$

The solution of this equation is sought in the form

$$\Theta(r,\theta) = f(r)\left(\phi_1 \sin\theta - \phi_2 \cos\theta\right) \, . \tag{106}$$

This form reduces (105) to the ordinary differential equation for the function $f(r)$:

$$\left[-\frac{h^2}{\chi^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right) + 1\right] f = r \, . \tag{107}$$

The finite solution for $f(r)$ subjected to the boundary condition $f(a) = 0$ is

$$f(r) = r - \frac{a I_1\left(\frac{r\chi}{h}\right)}{I_1\left(\frac{a\chi}{h}\right)} \, . \tag{108}$$

This function gives rise to the displacements

$$
\begin{aligned}
u_r &= \frac{3\left(h^2 - z^2\right)}{2h\chi^2}\left[1 - \frac{I_0\left(\frac{r\chi}{h}\right) + I_2\left(\frac{r\chi}{h}\right)}{2\zeta I_1\left(\frac{1}{\zeta}\right)}\right]\left(\phi_1 \sin\theta - \phi_2 \cos\theta\right) \ , \\
u_\theta &= \frac{3\left(h^2 - z^2\right)}{2h\chi^2}\left[1 - \frac{aI_1\left(\frac{r\chi}{h}\right)}{rI_1\left(\frac{a\chi}{h}\right)}\right]\left(\phi_1 \cos\theta + \phi_2 \sin\theta\right) \ , \\
u_3 &= \frac{z}{2h^3}\left[a\left(h^2 - z^2\right)\frac{I_1\left(\frac{r\chi}{h}\right)}{I_1\left(\frac{a\chi}{h}\right)} + 2h^2 r\right]\left(\phi_1 \sin\theta - \phi_2 \cos\theta\right) \ .
\end{aligned}
\tag{109}
$$

The required bending moments are

$$
M_\alpha = \frac{3\mu}{\chi^2}\frac{\pi a^4}{4}\frac{I_3\left(\frac{a\chi}{h}\right)}{I_1\left(\frac{a\chi}{h}\right)}\frac{\phi_\alpha}{h} \ . \tag{110}
$$

To a leading order all of the solutions presented in this subsection agree with those published in [8, 10, 17] for layers formed by nearly incompressible solids.

5.4 Unification of solutions

In this subsection, we establish that the leading order solutions developed in Sections 3 and 4 in scaled coordinates can be recovered (92), (93), and (98). Let us begin by showing that both $\bar{E}$, defined in (1), and $\hat{E}$, defined in (2), can be recovered from $\tilde{E}$, the apparent Young's modulus derived for the unifying solution. To this end, we use (104) to obtain

$$
\tilde{E} := \frac{F_3}{\pi a^2} \div \frac{\delta_3}{h} = \frac{3\mu}{\chi^2}\frac{I_2\left(\frac{a\chi}{h}\right)}{I_0\left(\frac{a\chi}{h}\right)} = \frac{3\mu}{\chi^2}\frac{I_2\left(\frac{1}{\zeta}\right)}{I_0\left(\frac{1}{\zeta}\right)} \ . \tag{111}
$$

For layers formed by compressible solids with $\chi = \mathcal{O}(1)$, $\zeta \ll 1$. In this regime, to the leading order,

$$\frac{I_2\left(\frac{1}{\zeta}\right)}{I_0\left(\frac{1}{\zeta}\right)} = 1\,, \tag{112}$$

and therefore $\tilde{E}$ is reduced to

$$\tilde{E} = \frac{3\mu}{\chi^2} = \lambda + 2\mu = \bar{E}\,. \tag{113}$$

For layers formed by incompressible solids with $\chi \to 0$, so that for $\chi \ll \xi$ and $\zeta \to \infty$. In this regime, to the leading order,

$$\frac{I_2\left(\frac{1}{\zeta}\right)}{I_0\left(\frac{1}{\zeta}\right)} = \frac{1}{8\zeta^2} = \frac{\chi^2}{8\xi^2}\,, \tag{114}$$

and therefore $\tilde{E}$ is reduced to

$$\tilde{E} = \frac{3\mu}{8\xi^2} = \hat{E}\,. \tag{115}$$

Now let us proceed with establishing that (92), (93), and (98) reduce to the leading order solution for layers formed by compressible (incompressible) solids for $\zeta \ll 1$ ($\zeta \gg 1$). We restrict our analysis to the stretching and bending modes. Then, for compressible solids, according to (25), (92), (93), and (98) must reduce to

$$
\begin{aligned}
u_\alpha &= 0\,, \\
u_3 &= \left(\delta_3 + a\phi_1 X_2 - a\phi_2 X_1\right) X_3\,. \qquad (116)
\end{aligned}
$$

For incompressible solids, (92), (93), and (98) must reduce to (57), (58), and (66).

For $\zeta \ll 1$, we recognize that (93) becomes an algebraic equation whose solution is

$$
\Theta = \delta_3 + a\phi_1 X_2 - a\phi_2 X_1\,. \qquad (117)
$$

Once this solution is substituted in (92), we obtain u_3 exactly as in (116), and $u_\alpha = \mathcal{O}(\xi)$. Note that Θ in (117) does not satisfy the boundary conditions on $\partial\omega$. To address this issue, let us introduce the function

$$
\hat{\Theta} := \Theta - \left(\delta_3 + a\phi_1 X_2 - a\phi_2 X_1\right)\,, \qquad (118)
$$

which according to (93) and (98) satisfies the partial differential equation

$$
-\zeta^2 \left(\frac{\partial^2 \hat{\Theta}}{\partial X_1^2} + \frac{\partial^2 \hat{\Theta}}{\partial X_2^2} \right) + \hat{\Theta} = 0 \quad \text{in } \omega \qquad (119)
$$

and the boundary conditions

$$
\hat{\Theta} = \delta_3 + a\phi_1 X_2 - a\phi_2 X_1 \quad \text{on } \partial\omega\,. \qquad (120)
$$

The partial differential equation is characterized by an exponentially decaying fundamental solution. This property implies that an integral operator mapping the Dirichlet data on $\partial\omega$ onto $\hat{\Theta}$ in ω exhibits the same exponential decay [22]. The characteristic length scale of this decay is the square root of the coefficient in (95). Thus, for compressible solids, that length is comparable to h. Consequently, the effect of improper boundary conditions associated with the solution given by (117) is negligible, and similar to that for the corrective solution developed in Section 3.

For $\zeta \gg 1$, (93) reduces to Poisson's equation

$$-\zeta^2 \left(\frac{\partial^2 \Theta}{\partial X_1^2} + \frac{\partial^2 \Theta}{\partial X_2^2} \right) = \delta_3 + a\phi_1 X_2 - a\phi_2 X_1 \,. \tag{121}$$

This equation becomes equivalent to (58) if

$$\Theta = -\frac{h\chi^2}{3\mu} p \,. \tag{122}$$

Once this expression is substituted in (92), the expression for u_α becomes identical to that in (57). The equivalence of solutions for u_3 in (57) and (92) is established once we impose the inequality

$$|\Theta| \ll |\delta_3| + a\,|\phi_1 X_2| + a\,|\phi_2 X_1| \ , \tag{123}$$

which follows from (93). Finally, by substituting (124) in (96) we obtain that

$$p = -\frac{1}{3}\sigma_{ii} \,, \tag{124}$$

which establishes the equivalence.

6 Discussion

In this paper, we developed a complete asymptotic solution for a thin, not necessarily circular, linear elastic layer fully constrained by two rigid plates, which subject the layer to combined stretching, bending, shearing, and twisting (Fig. 1). From a practical perspective, the most useful solution is for layers formed by compressible solids characterized by $\chi = \mathcal{O}(1)$, as it covers most materials except rubber. This solution is presented in Section 3, and it consists of two components. The first component $\mathbf{u}$, given in (29), satisfies the partial differential equations and the boundary conditions at the plates, but not on the cylindrical surface $\partial\Omega^0$. Although $\mathbf{u}$ appears as a two-term asymptotic solution, actually, it is exact, as all higher order terms are equal to zero. The second component $\mathbf{v}$ is the corrective solution introduced as a remedy for the failure of $\mathbf{u}$ to satisfy traction-free boundary conditions $\partial\Omega^0$. We constructed $\boldsymbol{v}$ as a local asymptotic solution and established that it decays exponentially away from $\partial\Omega^0$, and the characteristic length of the decay is $\mathcal{O}(h)$. Thus, the corrective solution is negligible in the bulk of the layer.

The unifying solution developed in Section 5 and stated in (94), (95), and (98) also can be regarded as the leading order solution for layers formed by nearly incompressible solids. This becomes evident once (94), (95), and (98) are compared with those in Section 2.1.3 of [17]. Further, to the leading order, the solutions for circular layers in Section 5.3 coincide with those in [8, 10, 17].

The unifying solution was motivated by our earlier work on stretching of circular layers. There we relied on the Love-Galerkin formalism [23, 24] which gives rise to the governing equations valid for all $\chi \geq 0$. That solution is different from that given in (103) and (104). The difference can be explained by recalling that Love-Galerkin's potential Φ must be a bi-harmonic function. That is, in scaled cylindrical coordinates, with

$$R := \frac{r}{a}\,, \tag{125}$$

Φ must satisfy

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \xi^{-2}\frac{\partial^2}{\partial X_3^2}\right)^2 \Phi = 0\,. \tag{126}$$

For the asymptotic approximation

$$\Phi = \Phi^{(0)} + \xi\Phi^{(1)} + \xi^2\Phi^{(2)} + \ldots\,, \tag{127}$$

(126) yields

$$\xi^{-4}\frac{\partial^4\Phi^{(0)}}{\partial X_3^4} + \xi^{-3}\frac{\partial^4\Phi^{(1)}}{\partial X_3^4} + \xi^{-2}\left[\frac{\partial^4\Phi^{(2)}}{\partial X_3^4} + 2\frac{\partial^2}{\partial X_3^2}\left(\frac{\partial^2\Phi^{(0)}}{\partial R^2} + \frac{1}{R}\frac{\partial\Phi^{(0)}}{\partial R}\right)\right] + \ldots = 0. \tag{128}$$

The equation associated with ξ^{-4} states that $\Phi^{(0)}$ is a cubic polynomial in X_3, and its coefficients are functions of R determined from non-homogeneous boundary conditions on $\partial\Omega^{\pm}$. In contrast, the equation associated with ξ^{-3} involves homogeneous boundary conditions for $\Phi^{(1)}$, and therefore $\Phi^{(1)} = 0$. Finally, for $\Phi^{(2)}$ the boundary conditions are homogeneous, but the equation is not, and therefore $\Phi^{(2)}$ is not equal to zero. Thus, the solution based on $\Phi^{(0)}$ is accurate to the first order. Indeed, this assertion is confirmed upon comparison of the displacements and (103) and [3]. Similarly, one can attribute differences between (103) and existing approximate axisymmetric solutions to higher-order terms.

According to (35), for layers formed by compressible solids, $\bar{E}$ measured under stretching conditions equals to the apparent modulus measured under bending conditions. In contrast, according to (71) and (79), for layers formed by incompressible solids, $\hat{E}$ measured under stretching conditions is three times larger than the apparent modulus measured under bending conditions. This difference is not that surprising as stretching is a more effective mechanism for generating high tri-axial tension, which dominates the response of layers formed by incompressible solids.

Another unifying attribute of the problem is the characteristic equation (47) for the corrective field $\mathbf{v}$. Let us rewrite it as

$$\frac{\sin 2\Lambda}{2\Lambda} = \frac{3-\chi^2}{3+\chi^2}\,. \tag{129}$$

For $\chi \ll 1$, the root of this equation is $\Lambda \approx \chi$. Therefore, in this regime, the solid is nearly incompressible and the characteristic length of the decay is

$$l := \frac{h}{\chi}\,. \tag{130}$$

This length is ubiquitous to existing approximate solutions for layers formed by nearly incompressible solids. In particular, it is used in [17] for normalization purposes. As a result, (95) is rewritten so that coefficient in front of the Laplacian is equal to one. In this paper, l appears in (95), which governs the unifying solution, and therefore should be relevant to the response of all layers, not just those formed by nearly incompressible solids.

Results of numerical experiments reported in [3] suggest that the effect of the corrective solution is essentially independent of χ. To this end, we observe that for small χ the exponential decay is weak but the traction vector on $\partial\Omega^0$ is small. In contrast, for large χ, the exponential decay is strong but the traction vector on $\partial\Omega^0$ is large. This interplay between the exponential decay and the traction vector on $\partial\Omega^0$ may explain why the effect of the corrective solution is essentially independent of χ.

Let us conclude by observing that the simplicity of solution for layers formed by compressible solids allows one to construct simple solutions for multi-layers including interphases [25]. This opens interesting avenues for assessing mechanical properties of interphases experimentally.

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