Optimal Impulse Control of a SIR Epidemic

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Abstract

Based on our recent results on the optimal impulse control, we solve explicitly an optimal isolation problem for a specific SIR (susceptible-infective-removed) epidemic model describing, e.g., the spread of AIDS.

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1 Introduction

Mathematical epidemiology goes back to the 1920s [20]. An important motivation behind mathematical modelling of infectious disease spread is the evaluation of alternative control strategies. One approach to this is via optimal control theory. A brief review of applications of control theory to infectious disease models appears in [8]. Control can mean vaccination of succeptibles [5, 6, 23] or isolation/treatment of infectives [6, 9, 25]. In the cited papers, the control was gradual, meaning that it affects the dynamics of the system; the trajectories remain continuous. The optimal control strategy was obtained using the dynamic programming [9, 23] and the maximum principle [5, 6, 25].

Impulse control means that at any time it is possible, for a certain price, to reduce instantly the number of susceptibles (case of vaccination) or infectives (case of isolation). Such models, the case of isolation, were investigated in [1, 21, 24]. Roughly speaking, the optimal strategy prescribes to isolate all the infectives at once, as soon as the number of infectives is below a certain threshold.

Impulse control of various dynamical systems attracts the attention of many researchers, e.g., those in [1, 2, 3, 7, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 26] to mention the most relevant and the most recent works. The underlying system can be described in terms of ordinary [1, 2, 3, 7, 17, 19, 21, 22, 26] or stochastic [13, 18] differential equations. In [10, 11, 12, 15], along with the given deterministic drift, there are spontaneous (or natural) Markov jumps of the state. Such models are called piecewise deterministic Markov processes (PDMPs); the drift is usually described by a fixed flow. On the other hand, if there is no drift and the trajectories are piecewise constant, the model is called a continuous-time Markov decision process (CTMDP) [14]. The *impulse* control means the following: at particular discrete time moments, the decision maker decides to intervene by instantaneously moving the process to some new point in the state space; that new point may be also random in the cases of CTMDP and PDMP [14, 15]. Then, restarting at this new point, the process runs until the next intervention and so on. Sometimes, such control is called "singular control" [18]. The goal is to minimize the total (expected) accumulated cost which may be discounted [3, 10, 11, 12, 14, 15, 18] or not [1, 3, 7, 13, 17, 19]. The popular method of attack to such problems is dynamic programming [3, 10, 11, 12, 14, 15, 18, 22]. In [2, 17, 19, 26], versions of the Pontryagin maximum principle are used.

Apart from the epidemiology, impulse control theory is widely applied to other real-life problems: Internet congestion control [3], reliability [12], economics and finance [18], medicine [17], etc.

In the current paper, we study the modification of the standard Kermack-McKendrick model [20], suggested in [16]. Note that in [1, 21], the authors considered the standard SIR model [20]. Moreover, the solution was based on partially intuitive and special reasoning, rather than on the general impulse control theory. On the opposite, our solution is based on the recently obtained results on the optimal impulse control of general deterministic dynamical systems [24]. Compared with the simple illustrative example presented in [24], the distinguishing feature of the current work is the presence of the positive cost $k > 0$ for initiating the isolation process. If $k > 0$ then, within a wide range of the parameters, it is not optimal to intervene if the current number of infectives, y is small. As a result, the critical area $\mathcal{L}$, where the total instantaneous isolation of all infectives is needed, has the form $0 < \sigma^1(x) \le y \le \sigma^2(x)$. Here x is the current number of susceptibles. Such an optimal control strategy with two thresholds never appeared before.

In Section 2, we provide the detailed statement of the problem, and in Section 3 we solve the stated problem, considering all the qualitatively different cases depending on the parameters. The conclusion is presented in Section 4. The proofs of auxiliary statements are postponed to the appendix.

2 Description of the Mathematical Model

At time $t \ge 0$, denote by $x(t)$, $y(t)$, $z(t)$ the numbers of susceptible, infective and removed individuals present, respectively. 'Removed' individuals may have died, or been isolated from the entire population. We suppose that, in the absence of intervention, the population evolves according to

$$\begin{cases} \dot{x}(t) = -\beta \dfrac{x(t)y(t)}{x(t)+y(t)}; \\ \dot{y}(t) = \beta \dfrac{x(t)y(t)}{x(t)+y(t)} - \gamma y(t); \\ \dot{z}(t) = \gamma y(t) \end{cases} \tag{1}$$

for constant parameters $\beta, \gamma > 0$.

Note that the population is closed, so that if the fixed initial state is $(x(0), y(0), z(0)) = (x_0, y_0, 0)$ then $x(t) + y(t) + z(t) = x_0 + y_0$ for all $t \ge 0$. That is why we consider the two-dimensional dynamical system with the state space $\mathbf{X} = \{(x, y) : \ x, y \ge 0, x + y < N\}$, with a fixed constant $N > x_0 + y_0 > 0$. The third differential equation in (1) will be omitted: the component $z(t)$ can be calculated from $x(t)$, $y(t)$ if needed.

Our model is similar to [4, 9, 16, 23, 24]; it can be justified by the following reasoning. Per time unit, a susceptible comes into contact with l individuals out of existing $(x + y)$: remember, the removed individuals don't participate in the process at all. Therefore, the chance that any one contact is with an infective equals $\frac{y}{x+y}$. Suppose the probability for a susceptible to become an infective due to the contact with any one infective is p. After the time unit has elapsed, the expected number of susceptibles therefore is approximately equal to

$$x(t + \Delta t) = x(t) - lp \frac{y(t)}{x(t) + y(t)} x(t) \Delta t.$$

This yields the first differential equation in (1) with $\beta = lp$. If $\gamma\Delta t$ is the probability of one infective to escape from the population (to become 'removed') during a small time interval Δt, then the dynamics of infectives can be described by the second differential equation in (1). We don't take into account the natural births and deaths of individuals and ignore immigration. Such a mathematical model was suggested in [16]; as claimed in [4], it describes the spread of AIDS (acquired immune deficiency syndrome) well enough. Equations (1) describe also such fatal infectious diseases as smallpox or plague.

Differential equations (1) can be analytically solved [16, 23]:

$$\left.\begin{array}{l} \text{if } \beta \neq \gamma, \quad \text{then} \\ \qquad x(t) = x_0 \dfrac{\left(1+\frac{y_0}{x_0}\right)^{\frac{\beta}{\beta-\gamma}}}{\left(1+\frac{y_0}{x_0}e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}}; \\ \qquad y(t) = y_0 \dfrac{\left(1+\frac{y_0}{x_0}\right)^{\frac{\beta}{\beta-\gamma}} e^{(\beta-\gamma)t}}{\left(1+\frac{y_0}{x_0}e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}}; \\ \text{if } \beta = \gamma, \quad \text{then} \\ \qquad x(t) = x_0 e^{-\frac{\beta y_0 t}{x_0+y_0}}; \quad y(t) = y_0 e^{-\frac{\beta y_0 t}{x_0+y_0}}. \end{array}\right\} \tag{2}$$

From these expressions, it is clear that

$$\frac{y(t)}{x(t)} = \frac{y_0}{x_0} e^{(\beta-\gamma)t} \tag{3}$$

and, for all $t \geq 0$, $x(t), y(t) > 0$, if $x_0, y_0 > 0$. Besides,

$$\lim_{t\to\infty} x(t) = \begin{cases} 0, & \text{if } \gamma \leq \beta; \\ x_0 \left(\frac{x_0}{x_0+y_0}\right)^{\beta/(\gamma-\beta)}, & \text{if } \gamma > \beta; \end{cases} \qquad \lim_{t\to\infty} y(t) = 0. \tag{4}$$

At any time moment t, when the state is $(x(t), y(t))$, it is possible to isolate or remove any number $0 < a \leq y(t)$ of infectives for the price of

$$C^I(x(t), y(t), a) \stackrel{\triangle}{=} k + ca, \tag{5}$$

where $k > 0$ is the cost of the initialisation of the isolation process and $c > 0$ is the cost of the isolation of one unit of infectives. Such an instantaneous isolation is called 'impulse' or 'intervention' and a is its value or size. After the impulse of size a is applied in state $(x(t), y(t))$, the state of the system instantaneously changes to

$$l(x(t), y(t), a) \stackrel{\triangle}{=} (x(t+), y(t+)) = (x(t), y(t) - a).$$

It is natural to put $C^I(x, y, 0) = 0$ for all $x, y \geq 0$. We accept that the action space is $\mathbf{A} = [0, N]$ and modify the introduced expressions for C^I and l:

$$C^I(x(t), y(t), a) = \begin{cases} k + c\min\{a, y(t)\}, & \text{if } a > 0; \\ 0, & \text{if } a = 0; \end{cases} \qquad l(x, y, a) = (x, y - \min\{a, y\}).$$

Clearly, the function C^I is lower semicontinuous and the function l is continuous.

The impulse control strategy is a sequence

$$\pi = (\theta_1, a_1, \theta_2, a_2, \ldots).$$

Here $\theta_1 \geq 0$ is the interval up to the very first impulse of the size a_1; $\theta_i \geq 0$ (for $i > 1$) is the time interval between the impulses $i-1$ and i; a_i is the size of the i-th impulse. We also accept that $\theta_0 = 0$ and $a_0 = 0$. In case an impulse is needed at time moment zero, one can put $\theta_1 = 0$.

The value $\theta_i = +\infty$ is also allowed, meaning that no interventions are planned for the future. Remember that in such case $\lim_{t\to\infty} y(t) = 0$. We assume that $\theta_{i+1} = +\infty$, if $\theta_i = +\infty$ for some $i \geq 1$; the values a_i, a_{i+1} are of no importance; one can put $a_i = a_{i+1} = 0$.

For a fixed control strategy π, the dynamics of the epidemic model is well defined. (x_0, y_0) are the given initial values. On each one interval

$$\left[\sum_{j=0}^{i-1}\theta_j, \sum_{j=0}^{i}\theta_j\right), \qquad i \geq 1,$$

the dynamics is described by the differential equations (1) with initial condition

$$x\left(\sum_{j=0}^{i-1}\theta_j\right) = x\left(\sum_{j=0}^{i-1}\theta_j-\right); \qquad y\left(\sum_{j=0}^{i-1}\theta_j\right) = y\left(\sum_{j=0}^{i-1}\theta_j-\right) - a_{i-1}.$$

The values at the right end are denoted as

$$x_i \triangleq x\left(\sum_{j=0}^{i}\theta_j-\right); \qquad y_i \triangleq y\left(\sum_{j=0}^{i}\theta_j-\right).$$

Now the dynamics of the system, under a fixed control strategy π, can be represented by the following sequence:

$$(x_0, y_0), \theta_1, a_1, (x_1, y_1), \theta_2, a_2, (x_2, y_2), \theta_3, a_3, \ldots.$$

If the values (x_{i-1}, y_{i-1}) and θ_i, a_i are known, then the values (x_i, y_i) can be calculated as described above. We underline that a sequence of simultaneous impulses is allowed; in this case $\theta_i = \theta_{i+1} = \ldots = 0$. The controlled process terminates as soon as $y(t) = 0$.

Below, it is convenient to introduce the continuous two-dimensional flow ϕ coming from the first two differential equations (1):

$$(x(u+t), y(u+t)) = \phi(x(u), y(u), t).$$

The explicit formula for ϕ is given in (2): one should substitute $x(u)$ and $y(u)$ for x_0 and y_0 respectively.

The goal of the impulse control is to minimize the total number of new infectives, up to the end of the epidemic, also taking into account the total price of interventions. Therefore, the running/gradual cost rate to be integrated equals the rate of the spread of the disease:

$$C^g(x, y) = \beta\frac{xy}{x+y}.$$

Altogether, the objective in the described impulse control problem is

$$\mathcal{V}(x_0, y_0, \pi) = \sum_{i=1}^{\infty}\left\{\int_{(0,\theta_i]} C^g(\phi(x_{i-1}, y_{i-1}, t))dt + I\{\theta_i < \infty\}\, C^I(\phi(x_{i-1}, y_{i-1}, \theta_i), a_i)\right\} \to \min_{\pi}. \tag{6}$$

The much simpler case, when $k = 0$ and only one impulse resulting in isolation of all infectives is allowed, was studied in [24]. In the current work, we consider the more interesting and practical situation when $k > 0$ and several successive impulses are allowed.

3 Solution to the Stated Control Problem

We will apply the general optimal impulse control theory, developed in [24], to the formulated control problem. All the conditions introduced in [24] are satisfied. Below, we enlist them for the reader's convenience.

Conditions 1.

- *The functions C^g and C^I are non-negative and lower semicontinuous; moreover, $C^I \geq k > 0$.*
- *The space $\mathbf{A}$ is compact.*
- *The flow ϕ exhibits the semigroup property.*
- *The mappings $(x,y,a) \in \mathbf{X} \times \mathbf{A} \to l(x,y,a)$ and $(x,y,\theta) \in \mathbf{X} \times \mathbb{R}^0_+ \to \phi(x,y,\theta)$ are continuous.*
- *The integral $\int_{(0,+\infty)} C^g(\phi(x,y,u)du$ is a uniformly bounded function of $(x,y) \in \mathbf{X}$.*

Here and all over the text, we use notations $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}^0_+ = [0,+\infty)$.

Note that one can naturally extend the domain of the flow to negative values of time by assuming, for $u > 0$, that $\phi(x,y,-u)$ is equal to $(\tilde{x},\tilde{y})$ if $\phi(\tilde{x},\tilde{y},u) = (x,y)$. With this convention, $\phi(x,y,-u)$ is either uniquely defined or does not exist. In fact, for all $(x,y) \in \mathbf{X}$ there exists $\varepsilon(x,y) > 0$ such that for $0 < u < \varepsilon(x,y)$, $\phi(x,y,-u)$ does exist because the boundary $x+y=N$ is not in $\mathbf{X}$.

Below we formulate the sufficient condition of optimality in the form of the differential Bellman equation.

Proposition 1. *Suppose Conditions 1 are satisfied and function $V : \mathbf{X} \to \mathbb{R}^0_+$ exhibits the following properties:*

i) The integral $\int_{(0,+\infty)} V(\phi(x_0,y_0,t))dt$ is finite for all $(x_0,y_0) \in \mathbf{X}$.

ii) – *Either the limit*

$$\mathcal{F}^V_+(x,y) \triangleq \lim_{t\to 0^+} \Big[\frac{V(\phi(x,y,t)) - V(x,y)}{t} + \frac{1}{t}\int_{(0,t]} C^g(\phi(x,y,u))\,du\Big],$$

exists and equals zero and

$$\inf_{a\in\mathbf{A}} \big[C^I(x,y,a) + V(l(x,y,a)) - V(x,y)\big] > 0, \tag{7}$$

– *or*

$$\underline{\mathcal{F}}^V_-(x,y) \triangleq \underline{\lim}_{t\to 0^-} \Big[\frac{V(\phi(x,y,t)) - V(x,y)}{t} + \frac{1}{t}\int_{[t,0)} C^g(\phi(x,y,u))\,du\Big] \geq 0$$

and

$$\inf_{a\in\mathbf{A}} \big[C^I(x,y,a) + V(l(x,y,a)) - V(x,y)\big] = 0.$$

iii) The function $V(x,y)$ is right lower semicontinuous and left upper semicontinuous along the flow. That is, for all $(x,y) \in \mathbf{X}$ we have

$$\underline{\lim}_{t\to 0^+} V(\phi(x,y,t)) \geq V(x,y) \quad \text{and} \quad \overline{\lim}_{t\to 0^-} V(\phi(x,y,t)) \leq V(x,y).$$

iv) If, for some $(x,y) \in \mathbf{X}$ and $s > 0$ and for all $0 \leq t < s$ the states $\phi(x,y,t)$ are not in the critical area

$$\mathcal{L} \triangleq \{(x,y) \in \mathbf{X} : \inf_{a\in\mathbf{A}} \big[C^I(x,y,a) + V(l(x,y,a)) - V(x,y)\big] = 0\}, \tag{8}$$

then $\lim_{t\to s^-} V(\phi(x,y,t)) = V(\phi(x,y,s))$.

v) For all $(x,y) \in \mathbf{X}$ the set $\{t \in \mathbb{R}^0_+ : \phi(x,y,t) \in \mathcal{L}\}$ is either empty, or contains its infimum.

Then the optimal impulse control strategy π^* is as follows:

(a) If $(x_0,y_0) \in \mathcal{L}$ then $\theta^*_1 = 0$ and $a^*_1 = y_0$: the biggest possible impulse must be applied immediately.

(b) If $(x_0, y_0) \notin \mathcal{L}$ then $\theta_1^* = \inf\{t: \ \phi(x_0, y_0, t) \in \mathcal{L}\}$, $a_1^* = \phi_y((x_0, y_0, \theta_1^*)$: one has to wait when the trajectory $(x(t), y(t))$ reaches the critical area $\mathcal{L}$ and apply the biggest possible impulse. The case $\theta_1^* = +\infty$ and $a_1^* = 0$ is not excluded here.

The proof follows from theorems 1 and 2 in [24].

The Bellman function V and the critical area $\mathcal{L}$ have different form depending on the parameters of the model. Recall that, if $y(t) = 0$ then the epidemic is over.

3.1 Solution in the Case $\beta \geq \gamma$

In this subsection, we show that the function

$$V(x,y) = \begin{cases} cy + k, & \text{if } 0 < y \leq \frac{x-k}{c}; \\ x, & \text{if } y > \max\{\frac{x-k}{c}, 0\}; \\ 0 & \text{if } y = 0 \end{cases}$$

satisfies all the requirements of Proposition 1. The line $k + cy = x$ appears after we equate the total loss of the instantaneous isolation of all infectives, $k + cy$, and the total loss associated with no interventions at all, x. Recall, if $y_0 > 0$, then in the uncontrolled epidemic all the susceptibles will be eventually infected if $\beta \geq \gamma$.

Firstly, let us show that the integral $\int_{(0,+\infty)} V(\phi(x_0, y_0, t))dt$ is finite for all $(x_0, y_0) \in \mathbf{X}$. The cases $x_0 = 0$ and $y_0 = 0$ are trivial. Assume that $x_0, y_0 > 0$. According to (3) and keeping in mind that component x decreases with time, we conclude that, for all $t > 0$,

$$(x(t), y(t)) \in \left\{(x,y) > 0: \ \frac{y}{x} \geq \frac{y_0}{x_0}, \ x < x_0\right\}.$$

If $y_0 \geq \frac{x_0 - k}{c}$, then

$$y(t) \geq x(t)\,\frac{y_0}{x_0} \geq x(t)\Big(\frac{1}{c} - \frac{k}{cx_0}\Big) > x(t)\Big(\frac{1}{c} - \frac{k}{cx(t)}\Big),$$

and hence,

$$y(t) > \frac{x(t) - k}{c} \text{ for all } t > 0.$$

Therefore, for such initial values (x_0, y_0),

$$\int_{(0,+\infty)} V(\phi(x_0, y_0, t))dt = \int_{(0,+\infty)} x(t)dt.$$

According to (3),

$$x(t) = y(t)\frac{x_0}{y_0}e^{-(\beta-\gamma)t}.$$

Since function $y(t) < N$ is uniformly bounded, in case $\beta > \gamma$, the integral $\int_{(0,+\infty)} V(\phi(x_0, y_0, t))dt$ is finite. If $\beta = \gamma$, its finiteness follows directly from (2).

In case $y_0 \leq \frac{x_0 - k}{c}$ and $\beta > \gamma$, again using (3), we see that, at some finite value of $t = t^*$, $y(t^*) = \frac{x(t^*) - k}{c}$: $t^* < \frac{\ln(x_0) - \ln(y_0 c)}{\beta - \gamma}$. (On the right, there is the time moment when the slope $\frac{y(t)}{x(t)}$ equals $\frac{1}{c}$, bigger than $\frac{y}{x}$ for any point lying on the line $y = \frac{x-k}{c}$.) If $y_0 \leq \frac{x_0 - k}{c}$ and $\beta = \gamma$, then $y(t^*) = \frac{x(t^*) - k}{c}$ at the time moment

$$t^* = -\frac{x_0 + y_0}{\beta y_0}\ln\left(\frac{k}{x_0 - cy_0}\right) < \infty.$$

In each case,

$$\int_{(0,+\infty)} V(\phi(x_0, y_0, t))dt = \int_{(0,t^*)} (cy(t) + k)dt + \int_{[t^*,+\infty)} x(t)dt < \infty,$$

and the property i) in Proposition 1 is satisfied.

Since function V is continuous on $\mathbf{X} \cap (\mathbb{R} \times \mathbb{R}_+)$ and on $\mathbf{X} \cap (\mathbb{R} \times \{0\})$ and each trajectory $\{(x(t), y(t)), t \geq 0\}$ is entirely contained either in $\mathbf{X} \cap (\mathbb{R} \times \mathbb{R}_+)$, or in $\mathbf{X} \cap (\mathbb{R} \times \{0\})$, properties iii) and iv) are fulfilled.

The set $\mathcal{L}$ defined in (8) has the form

$$\mathcal{L} = \{(x, y) \in \mathbf{X} \cap (\mathbb{R}_+)^2 : \ y \leq \frac{x-k}{c}\}.$$

Indeed, if $x = 0$ or $y = 0$, then $\inf_{a\in\mathbf{A}}\{C^I(x, y, a) + V(l(x, y, a)) - V(x, y)\} = k > 0$ and $(x, y) \notin \mathcal{L}$. Other cases are studied below.

If $0 < y \leq \frac{x-k}{c}$ then

$$\begin{aligned} C^I(x, y, a) + V(l(x, y, a)) - V(x, y) &= k + c\ \min\{a, y\} + V(x, y - \min\{a, y\}) - (cy + k) \\ &= \begin{cases} k + ca + c(y - a) + k - (cy + k) = k > 0, & \text{if } a < y; \\ k + cy + 0 - (cy + k) = 0, & \text{if } a \geq y, \end{cases} \end{aligned}$$

so that $(x, y) \in \mathcal{L}$.

If $y > \max\{\frac{x-k}{c}, 0\}$ then

$$\begin{aligned} C^I(x, y, a) + V(l(x, y, a)) - V(x, y) = k + c\ \min\{a, y\} + V(x, y - \min\{a, y\}) - x \\ = \begin{cases} k + cy + 0 - x > 0, & \text{if } a \geq y; \\ k + ca + c(y - a) + k - x = 2k + cy - x > 0, & \text{if } y - \frac{x-k}{c} < a < y; \\ k + ca + x - x > 0, & \text{if } a \leq y - \frac{x-k}{c}. \end{cases} \end{aligned}$$

Hence $(x, y) \notin \mathcal{L}$.

As for the requirement v), it has already been shown at the beginning of this subsection that, if $(x, y) \notin \mathcal{L}$, then $\phi(x, y, t) \notin \mathcal{L}$ for all $t \geq 0$. On the other hand, for $(x, y) \in \mathcal{L}$, obviously, $\inf\{t \in \mathbb{R}^0_+ : \ \phi(x, y, t) \in \mathcal{L}\} = 0$. Requirement v) is fulfilled.

Now show that the Bellman equation ii) in Proposition 1 is satisfied.

If $(x, y) \in \mathcal{L}$ then

$$\underline{\mathcal{F}}^V_-(x, y) \quad = \quad \beta\frac{xy}{x+y} + \frac{\partial V}{\partial y}\left[\beta\frac{xy}{x+y} - \gamma y\right] = \frac{y}{x+y}\left[\beta(1+c)x - \gamma c(x+y)\right].$$

On the boundary $y = \frac{x-k}{c}$, the expression $\frac{\partial V}{\partial y}$ means the left derivative. Since $y \leq \frac{x-k}{c}$, we conclude that

$$\underline{\mathcal{F}}^V_-(x, y) \geq \frac{xy}{x+y}[\beta(1+c) - \gamma(c+1)] = \frac{xy}{x+y}(\beta - \gamma)(c+1) \geq 0,$$

so that the second case of Item ii) is realised.

The cases $x = 0$ and $y = 0$ are trivial: $\inf_{a\in\mathbf{A}}\{C^I(x, y, a) + V(l(x, y, a)) - V(x, y)\} = k > 0$ and $\underline{\mathcal{F}}^V_-(x, y) = 0$.

If $(x, y) \notin \mathcal{L}$ and $(x, y) \in (\mathbb{R}_+)^2$ then

$$\inf_{a\in\mathbf{A}}[C^I(x, y, a) + V(l(x, y, a)) - V(x, y)] > 0$$

and

$$\mathcal{F}^V_+(x, y) = \beta\frac{xy}{x+y} + \frac{\partial V}{\partial x}\left[-\beta\frac{xy}{x+y}\right] = 0,$$

so that the first case of Item ii) is realised.

According to Proposition 1, the optimal strategy prescribes to isolate immediately all the infectives in case $y_0 > 0$ and $x_0 \geq k + cy_0$. Otherwise, don't isolate at all, as the trajectory never reaches the critical area $\mathcal{L}$ if $(x_0, y_0) \notin \mathcal{L}$. The straight line $x = k + cy$ is a dispersal line: the optimal trajectories go away from it. Remember, the goal of the control is to save susceptibles from being infected, and isolation is reasonable only when there are many susceptibles to be saved:

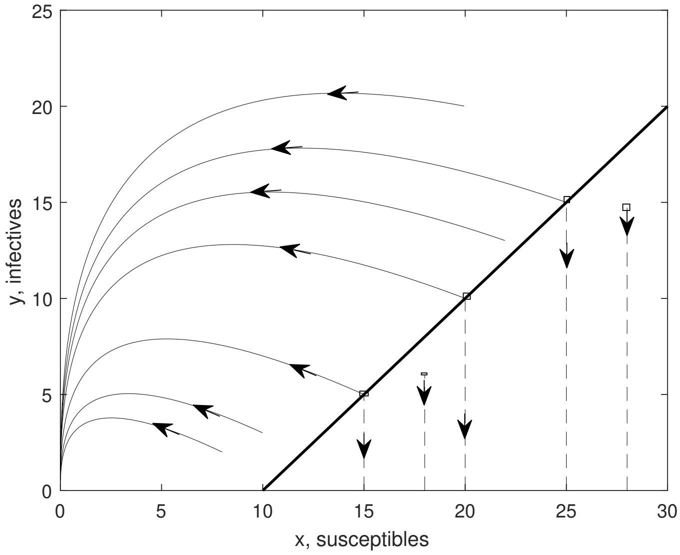

Figure 1: Susceptible–Infective dynamics under optimal control with $c = 1$, $k = 10$, $\beta = 0.05$ and $\gamma = 0.02$.

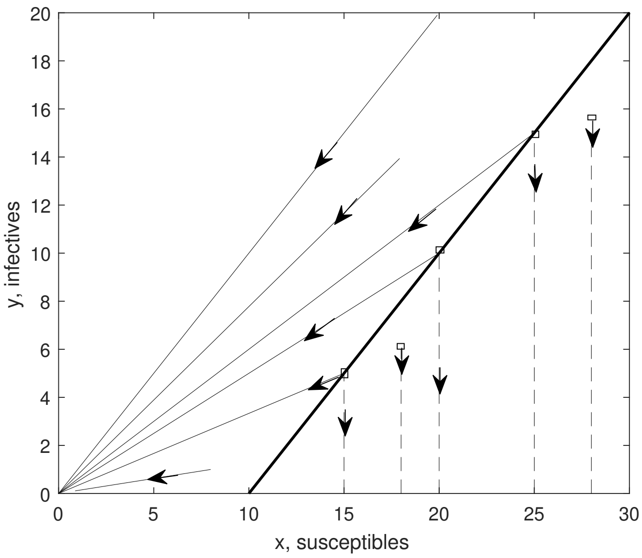

Figure 2: Susceptible–Infective dynamics under optimal control with $c = 1$, $k = 10$, $\beta = 0.1$ and $\gamma = 0.1$.

$x_0 \geq k + cy_0$ because otherwise the cost of isolation, $k + cy_0$, is bigger than the profit for saving susceptibles (i.e., x_0). It is also obvious that isolating a part of infectives makes not much sense as, if there remain infectives, then all the susceptibles will become infected because $\lim_{t\to\infty} x(t) = 0$ (see (4)). Finally, isolation in two or more steps is also not optimal because the cost of the initialisation of the isolation process $k > 0$ will be paid twice or more.

Figures 1 and 2 illustrate the dynamics under the optimal impulse control. The dashed arrows correspond to the impulses (interventions).

3.2 Solution in the Case $\beta < \gamma$

Here, very much depends on the value of c. If c is big then the cost of isolation is too big and it is optimal not to apply impulses at all. The case when c is small is most challenging.

3.2.1 Case $c \geq \frac{\beta}{\gamma-\beta}$

Here, it is optimal not to intervene at all, as the cost c is too high. Consequently,

$$V(x,y) = x\left[1 - \left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right].$$

(See (4).)

Below we show that all the requirements of Proposition 1 are satisfied.

i) The case $x_0 = 0$ is trivial. For $x_0 > 0$, according to (3),

$$V(\phi(x_0,y_0,t)) \quad = \quad x(t)\left[1 - \left(1 + \frac{y_0}{x_0}e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}\right].$$

For big values of t, the square bracket here is of the order $e^{-(\gamma-\beta)t}$ because

$$\lim_{t\to\infty} \frac{1 - \left(1 + \frac{y_0}{x_0}e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}}{e^{-(\gamma-\beta)t}} = \frac{\beta\frac{y_0}{x_0}}{\gamma-\beta}.$$

Therefore, for some constant $0 < K < \infty$,

$$\left[1 - \left(1 + \frac{y_0}{x_0}e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}\right] \leq Ke^{-(\gamma-\beta)t}.$$

Since $x(t) < N$ we conclude that

$$\int_{(0,+\infty)} V(\phi(x_0,y_0,t))dt \leq \frac{NK}{\gamma-\beta} < \infty.$$

ii) At all $(x,y) \in \mathbf{X}$, function V is continuously differentiable. Hence

$$\begin{aligned}
\mathcal{F}_+^V(x,y) \quad &= \quad \frac{\partial V}{\partial x}\left[-\frac{\beta xy}{x+y}\right] + \frac{\partial V}{\partial y}\left[\frac{\beta xy}{x+y} - \gamma y\right] + \frac{\beta xy}{x+y} \\
&= \quad \left\{\left[1 - \left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right] - \frac{\beta y}{\gamma-\beta}\cdot\frac{x^{\frac{\beta}{\gamma-\beta}}}{(x+y)^{\frac{\gamma}{\gamma-\beta}}}\right\}\cdot\left[-\frac{\beta xy}{x+y}\right] \\
&\quad + \left\{\frac{\beta x}{\gamma-\beta}\cdot\frac{x^{\frac{\beta}{\gamma-\beta}}}{(x+y)^{\frac{\gamma}{\gamma-\beta}}}\right\}\left[\frac{\beta xy}{x+y} - \gamma y\right] + \frac{\beta xy}{x+y} = 0.
\end{aligned}$$

For

$$\begin{aligned}
g_1(x,y,a) \quad &\triangleq \quad C^I(x,y,a) + V(l(x,y,a)) - V(x,y) = k + ca + V(x,y-a) - V(x,y) \qquad (9) \\
&= \quad k + ca + x\left[1 - \left(\frac{x}{x+y-a}\right)^{\frac{\beta}{\gamma-\beta}}\right] - x\left[1 - \left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right],
\end{aligned}$$

where $0 \leq a \leq y$, note that the derivative, with respect to a, equals

$$\frac{\partial g_1}{\partial a} = \frac{\partial}{\partial a}\left\{k + ca - x\left(\frac{x}{x+y-a}\right)^{\frac{\beta}{\gamma-\beta}} + x\left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right\} = c - \frac{\beta}{\gamma-\beta}\cdot\left(\frac{x}{x+y-a}\right)^{\frac{\gamma}{\gamma-\beta}},$$

and the latter function of a decreases with $a \in [0, y]$, meaning that the function g_1 is concave with respect to a. Moreover, the derivative $\frac{\partial g_1}{\partial a}$ has the minimal value $c - \frac{\beta}{\gamma-\beta} \geq 0$, attained at $a = y$. Therefore,

$$\inf_{a \in \mathbf{A}} [C^I(x, y, a) + V(l(x, y, a)) - V(x, y)] = C^I(x, y, 0) + V(l(x, y, 0)) - V(x, y) = k > 0.$$

Requirement ii) is fulfilled. Item iii) is valid because the function V is continuous. The critical area $\mathcal{L} = \emptyset$ is empty, and Items iv) and v) follow, too.

The dynamics is illustrated on Figure 3.

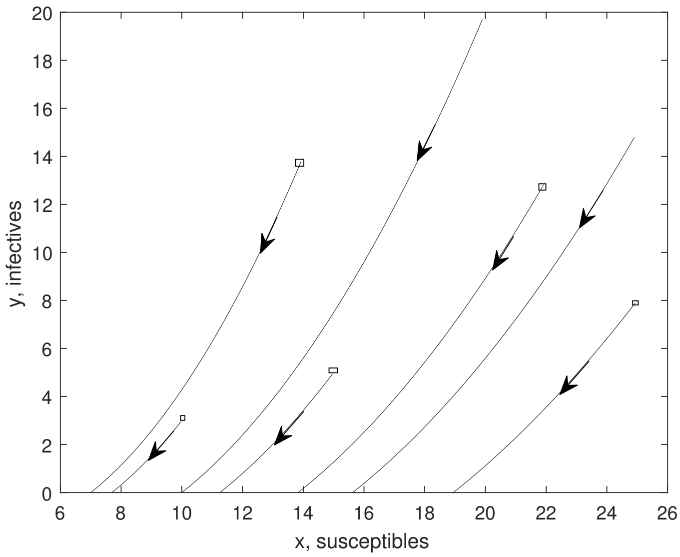

Figure 3: Susceptible–Infective dynamics under optimal control with $c = 2$, $k = 0.5$, $\beta = 0.1$ and $\gamma = 0.2$: no impulses are applied.

3.2.2 Case $c < \frac{\beta}{\gamma-\beta}$

First of all, we construct the critical area $\mathcal{L}$. After that, we present function V and check that all the requirements in Proposition 1 are satisfied.

Similarly to Section 3.1, we firstly equate the total loss of the instantaneous isolation of all infectives, $k + cy$, and the total loss associated with no interventions at all, if $x > 0$:

$$k + cy = x \left[1 - \left(1 + \frac{y}{x} \right)^{-\frac{\beta}{\gamma-\beta}} \right] \tag{10}$$

(see (4)). Denoting $w \triangleq \frac{y}{x}$, we obtain the following equations

$$\left. \begin{aligned} x &= \frac{k}{1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}}; \\ y &= \frac{kw}{1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}}. \end{aligned} \right\} \tag{11}$$

When w approaches zero, $y \to \frac{k(\gamma-\beta)}{\beta - c(\gamma-\beta)} + 0$ and $x \to \infty$. Equation

$$1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}} = 0 \tag{12}$$

has a unique positive solution $\tilde{w}^*$ because the left-hand side is a concave function of w, equals zero at $w = 0$, increases at zero, and goes to minus infinity as $w \to \infty$. In particular, the derivative of the function in the left-hand side of (12) at $w = \tilde{w}^*$ is negative, that is,

$$-c + \frac{\beta}{\gamma - \beta}(1 + \tilde{w}^*)^{-\frac{\gamma}{\gamma-\beta}} < 0. \tag{13}$$

As a result, $x \to \infty$, $y \to \infty$ when $w \to \tilde{w}^* - 0$ and $x < 0$, $y < 0$ when $w > \tilde{w}^*$. Therefore, solution to (10) in the area $x, y > 0$ is given in the parametric form by (11), where $0 < w < \tilde{w}^*$. Note also that equation (12) has no positive solutions if $c \geq \frac{\beta}{\gamma-\beta}$.

Remark 1. *The left-hand side of (12) is positive for all $w \in (0, \tilde{w}^*)$.*

The parametric equation (11) exhibits the following properties. When w increases, y also increases, and x initially decreases from infinity and goes to infinity thereafter:

$$\frac{dx}{dw} = \frac{k\left(c - \frac{\beta}{\gamma-\beta}(1+w)^{-\frac{\gamma}{\gamma-\beta}}\right)}{\left[1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}\right]^2}. \tag{14}$$

The minimal value of x equals

$$\hat{x} = \frac{k}{1 - \left(\frac{c(\gamma-\beta)}{\beta}\right)^{\frac{\beta}{\gamma}} - c\left[\left(\frac{\beta}{c(\gamma-\beta)}\right)^{\frac{\gamma-\beta}{\gamma}} - 1\right]}$$

and corresponds to

$$\hat{w} = \left(\frac{\beta}{c(\gamma - \beta)}\right)^{\frac{\gamma-\beta}{\gamma}} - 1.$$

$\hat{y}$ comes from (11) at $w = \hat{w}$. One can easily show that $\hat{w} < \tilde{w}^*$: the left-hand side of (12) is positive at $\hat{w}$. When w approaches $\tilde{w}^* - 0$, taking into account (13) we obtain that the difference $\Delta(w) \triangleq x(w)\tilde{w}^* - y(w)$ approaches (from above)

$$\Delta \triangleq \lim_{w \to \tilde{w}^* - 0} \frac{k(\tilde{w}^* - w)}{1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}} = \frac{k}{c - \frac{\beta}{\gamma-\beta}(1+\tilde{w}^*)^{-\frac{\gamma}{\gamma-\beta}}} > 0.$$

Now it is clear that function $x = G(y)$, the solution to (10), has domain $\left(\frac{k(\gamma-\beta)}{\beta - c(\gamma-\beta)}, \infty\right)$ and two asymptotes:

$$\begin{aligned} \lim_{y \to \frac{k(\gamma-\beta)}{\beta - c(\gamma-\beta)} + 0} G(y) &= +\infty; \\ \lim_{y \to \infty} \left(G(y) - \frac{y + \Delta}{\tilde{w}^*}\right) &= 0. \end{aligned}$$

Functions (11) and $x = G(y)$ for $0 < w < \tilde{w}^*$ are presented on Figure 4.

Remark 2. *Since, under an arbitrarily fixed $y > 0$, the right-hand side of (10) approaches zero as $x \to 0$, we conclude that*

$$\begin{aligned} k + cy > x\left[1 - \left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}\right] &\iff G(y) > x; \\ k + cy < x\left[1 - \left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}\right] &\iff G(y) < x. \end{aligned}$$

Quite formally, we put $G(y) \triangleq \infty$ for $y \leq \frac{k(\gamma-\beta)}{\beta - c(\gamma-\beta)}$.

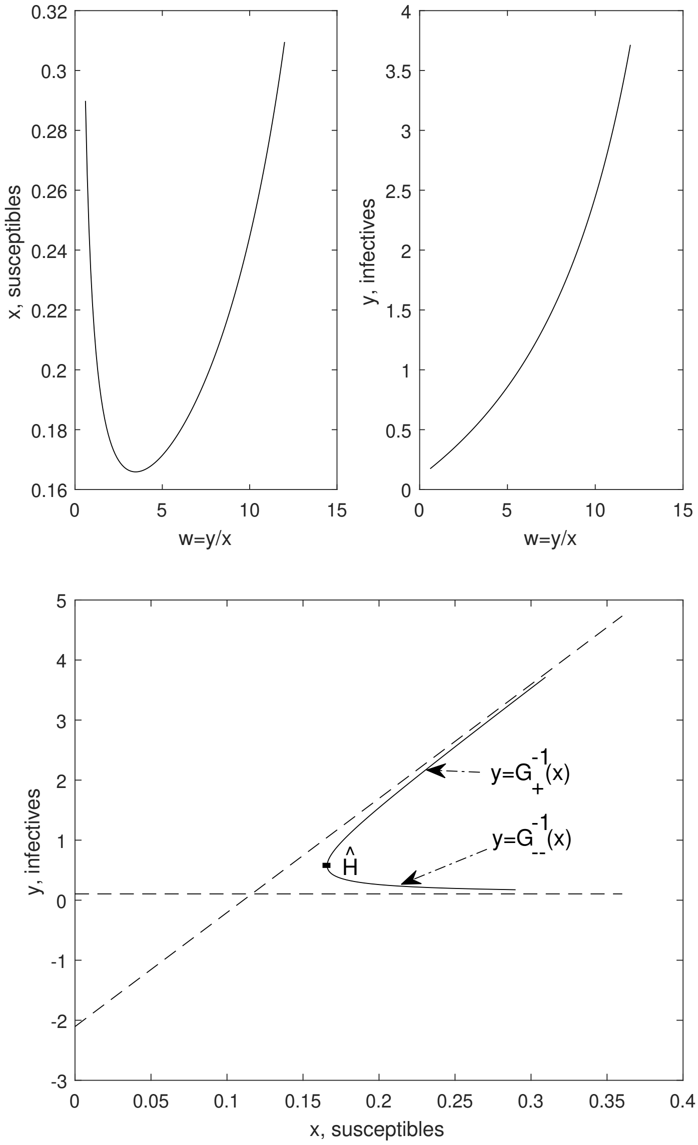

Figure 4: Functions $x(w)$, $y(w)$ and $x = G(y)$ at $c = 0.05$, $k = 0.1$, $\beta = 0.05$ and $\gamma = 0.1$. $\hat{y} \triangleq y(\hat{w})$; $\hat{H}$ is the point with coordinates $(\hat{x}, \hat{y})$. Asymptotes of $G(x)$: $y = \frac{k(\gamma-\beta)}{\beta-c(\gamma-\beta)}$ and $y = xw^* - \Delta$ are shown with the dashed lines.

We expect that the impulses of the size $a = y$ are needed in the area $x \geq G(y)$ leading to expression $V(x, y) = k + cy$, but one must be also sure that

$$\frac{\partial V}{\partial x}\left(-\beta\frac{xy}{x+y}\right) + \frac{\partial V}{\partial y}\left(\beta\frac{xy}{x+y} - \gamma y\right) + \beta\frac{xy}{x+y} \geq 0,$$

i.e.,

$$c\left(\beta\frac{x}{x+y} - \gamma\right) + \beta\frac{x}{x+y} \geq 0 \Longleftrightarrow \frac{y}{x} \leq w^* \triangleq \frac{\beta + c\beta - c\gamma}{c\gamma},$$

and the critical area decreases:

Proposition 2. *The strict inequality*

$$w^* < \tilde{w}^*$$

is valid.

The proofs of this and further propositions are postponed to the appendix.

Now, the critical area is as follows:

$$\mathcal{L} \triangleq \{(x,y) \in \mathbf{X} : x \geq \max\{G(y),\ \frac{y}{w^*}\}\}, \tag{15}$$

see Figure 5.

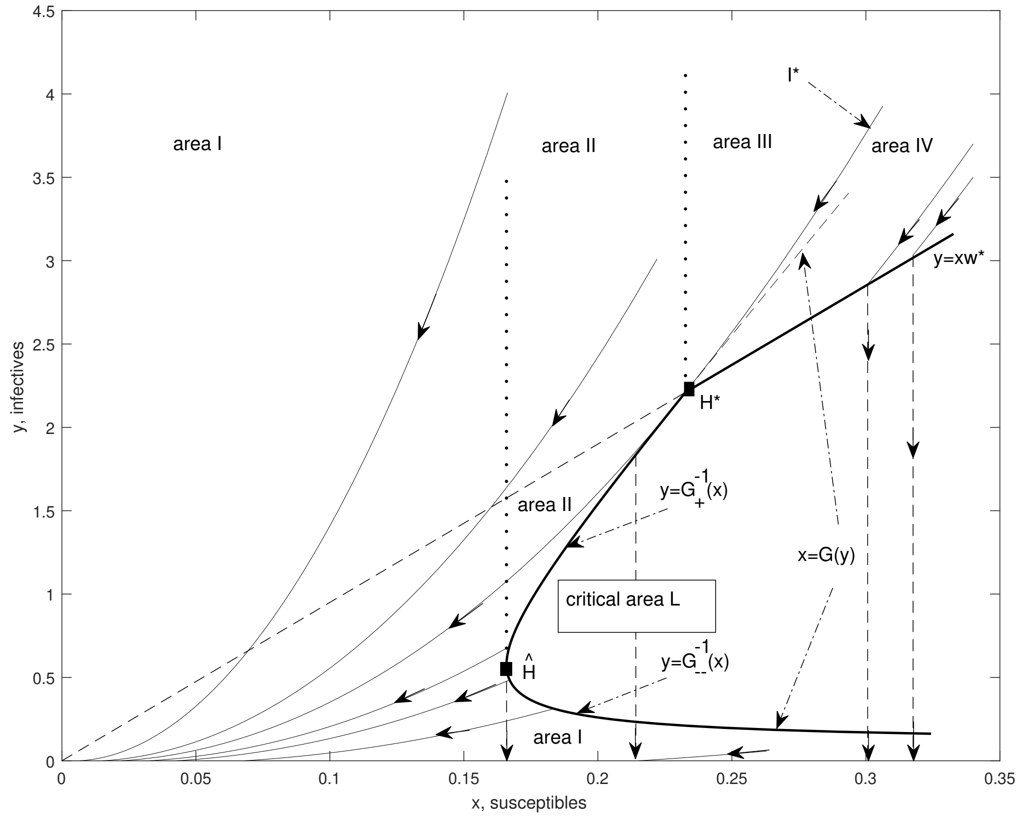

Figure 5: Susceptible–Infective dynamics under optimal control with $c = 0.05$, $k = 0.1$, $\beta = 0.05$ and $\gamma = 0.1$. Dotted lines separate the areas I,II, and III; vertical dashed lines indicate the impulses.

The important properties of the critical area and the integral curves looks as follows.

Proposition 3. *i) The point $H^*(x^*, y^*)$ coming from (11) under $w = w^*$ is the only common point of the integral curve (trajectory) of the original equation (1) through H^*, denoted as I^*, and the curve $x = G(y)$. For $x > x^*$, the trajectory I^* is above the curve $x = G(y)$, and for $x < x^*$, the trajectory I^* is to the left from the curve $x = G(y)$. Point H^* is the unique point of tangency of the curve $x = G(y)$ and trajectory I^*.*

ii) The trajectories of the system (1), to the right from I^* (and above H^*), enter the area $\{(x,y) :\ x \geq G(y)\}$ and enter the critical area $\mathcal{L}$; the trajectories, starting from the points on the curve $x = G(y)$ below H^*, exit the critical area $\mathcal{L}$.

It follows from Proposition 3-i) that $w^* > \hat{w}$: the point H^* is on the upper (concave) branch of the curve $x = G(y)$ because, on the integral curves, $\frac{dy}{dx} > 0$. Below, it is convenient to denote the upper (lower) branch of $x = G(y)$ as $y = G_+^{-1}(x)$ ($y = G_-^{-1}(x)$). The point $\hat{H}$ belongs to the both branches $\{(x, y) : \ y = G_-^{-1}(x), x \geq \hat{x}\}$ and $\{(x, y) : \ y = G_+^{-1}(x), x \geq \hat{x}\}$.

Below, we show that all the requirements of Proposition 1 are satisfied for the following function V:

- In the area outside $\mathcal{L}$ and excluding zone IV between the trajectory I^* and the straight line $\frac{y}{x} = w^*$, no impulses are applied and, similarly to Subsubsection 3.2.1,

$$V(x, y) = V_1(x, y) = x \left[1 - \left(\frac{x}{x + y} \right)^{\frac{\beta}{\gamma - \beta}} \right].$$

  This area is split in three parts I,II and III, as is shown on Figure 5. The accuracy of this split, as well as the shape of area IV follow from Proposition 3-iii); the formal description of the areas I,II,III, and IV is given below.

- In the critical area $\mathcal{L}$, the maximal impulse $a = y$ is needed and

$$V(x, y) = V_2(x, y) = k + cy.$$

- In the gulf IV between the trajectory I^* and the straight line $\frac{y}{x} = w^*$, the maximal impulse is needed after the trajectory enters the critical area $\mathcal{L}$, so that

$$V(x, y) = V_3(x, y) = x + (cw^* - 1)x \frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma - \beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma - \beta}}} + k.$$

  Here, $x - x\frac{\left(1+\frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}}$ is the total number of the new infectives over the time interval before the intervention; $w^* x\frac{\left(1+\frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}}$ is the number of infectives to be isolated at the moment of intervention.

One can easily check that function V is continuous in $\mathbf{X}$, so that the requirements iii) and iv) of Proposition 1 are obviously satisfied.

It was shown in Subsubsection 3.2.1 that, starting from any point from the areas I,II or III, $\int_{(0,\infty)} V(\phi(x_0, y_0, t))dt < \infty$. Since, starting from any point $(x_0, y_0) \in \mathbf{X}$, the trajectory $\phi(x_0, y_0, t)$ reaches the area I in the finite time, we conclude that the requirement i) of Proposition 1 is also satisfied.

The critical area $\mathcal{L}$ is closed; hence the requirement v) is satisfied, too.

Below, we show that condition ii) of Proposition 1 is valid as well.

Consider the critical area $\mathcal{L}$. It was built, based on the condition that

$$\frac{\partial V_2}{\partial x} \left(-\beta \frac{xy}{x + y} \right) + \frac{\partial V_2}{\partial y} \left(\beta \frac{xy}{x + y} - \gamma y \right) + \beta \frac{xy}{x + y} \geq 0.$$

On the upper boundary $y = xw^*$, at $x \geq x^*$, one has to take the right derivatives

$$\frac{\partial V_3}{\partial x} \left(-\beta \frac{xy}{x + y} \right) + \frac{\partial V_3}{\partial y} \left(\beta \frac{xy}{x + y} - \gamma y \right) + \beta \frac{xy}{x + y} = 0$$

because in fact V_3 is the solution to this partial differential equation with the boundary condition $k + cy$ on the straight line $\frac{y}{x} = w^*$. One can certainly check the required equality straightforwardly.

For the impulses $a \leq y - G_-^{-1}(x)$, we have

$$C^I(x, y, a) + V(l(x, y, a)) - V(x, y) = k + ca + k + c(y - a) - (k + cy) = k > 0.$$

For the impulses $y - G_-^{-1}(x) < a \leq y$, we have

$$C^I(x,y,a) + V(l(x,y,a)) - V(x,y) = k + ca + x\left[1 - \left(\frac{x}{x+y-a}\right)^{\frac{\beta}{\gamma-\beta}}\right] - (k+cy) \geq 0$$

with the minimal value zero at $a = y$. Indeed, for $y - G_-^{-1}(x) < a < y$, since $(y-a) < G_-^{-1}(x) < \tilde{w}^* x$, we have

$$\left[1 - \left(\frac{x}{x+y-a}\right)^{\frac{\beta}{\gamma-\beta}}\right] > \frac{c(y-a)}{x}$$

according to Remark 1. Condition ii) of Proposition 1 (second part of it) is satisfied in the critical area $\mathcal{L}$.

In the areas I,II,III and IV, equation

$$\begin{aligned}
\mathcal{F}_+^V(x,y) &= \lim_{t\to 0^+}\Big[\frac{V(\phi(x,y,t)) - V(x,y)}{t} + \frac{1}{t}\int_{(0,t]} C^g(\phi(x,y,u))\,du\Big] \\
&= \frac{\partial V}{\partial x}\left(-\beta\frac{xy}{x+y}\right) + \frac{\partial V}{\partial y}\left(\beta\frac{xy}{x+y} - \gamma y\right) + \beta\frac{xy}{x+y} = 0
\end{aligned}$$

is satisfied. Similarly to what was said about function V_3 above, function V_1 is the solution to the required partial differential equation with the zero boundary condition on the axis $y = 0$. The straightforward calculation was presented in Subsubsection 3.2.1. Now, it remains to check the condition (7) in the areas I,II,III and IV, which are defined as follows:

Area I in $\mathbf{X}$: $\{(x,y):\ x < \hat{x}\} \cup \{(x,y):\ x \geq \hat{x};\ y < G_-^{-1}(x)\}$
Area II in $\mathbf{X}$: $\{(x,y):\ \hat{x} \leq x \leq x^*,\quad y > G_+^{-1}(x)\}$
Area III in $\mathbf{X}$: $\{(x,y):\ x > x^*,\ y > I^*(x)\}$
Area IV in $\mathbf{X}$: $\{(x,y):\ x > x^*,\ xw^* < y \leq I^*(x)\}$

Here and below, $I^*(x)$ is the value of y on the trajectory I^*, corresponding to x.

Proposition 4. *In all the areas I,II,III, and IV, inequality (7) is satisfied.*

If $\beta < \gamma$ and $c < \frac{\beta}{\gamma-\beta}$ then the critical area $\mathcal{L}$, where the total impulses $a = y$ should be applied, is given by equation (15). The whole areas I,II, and III are free from interventions/impulses: if y, the number of infectives, is big, it decreases by itself quickly enough; if y is small, the epidemic dies out by itself. Interventions/impulses are needed only if the value of y is moderate and the number of susceptibles, x (which can be saved from infection by isolating infectives) is big enough. In the area IV, one has to wait until the number of infectives decreases and reaches the critical area $\mathcal{L}$; at that moment, the total impulse $a = y$ should be applied.

4 Conclusion

We provided the explicit optimal solution to the stated impulse control problem for the SIR epidemic. Similarly to [1, 21, 24], if the intervention is desirable, then it is optimal to isolate all the infectives instantly in one go. (Note, in [24], only the total isolations were allowed.) On the other hand, as Figure 5 shows, for a fixed x, the number of susceptibles, the critical area, where intervention is needed, can have the shape $0 < y_{min}(x) \leq y \leq y_{max}(x)$. To the best of our knowledge, such optimal strategies did not appear before. Of course, this is a consequence of positive $k > 0$, the cost for initiating the isolation process.

It is interesting to look what happens in the limiting situation, when $k \to 0$.

- In case $\beta \geq \gamma$, the dispersal line (see Figures 1 and 2) transforms to $x = cy$.
- In case $\beta < \gamma$ and $c < \frac{\beta}{\gamma-\beta}$, the point $\hat{H}$ approaches $(0,0)$. (See Figure 5.) It means that the branch $y = G_-^{-1}(x)$ transforms to the straight line $y = 0$, and the upper boundary of the critical area $\mathcal{L}$ becomes just the straight line $y = xw^*$. Exactly this critical area was obtained in [24], where the case of $k = 0$ was investigated.

5 Acknowledgement

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6 Appendix

Proof of Proposition 2. It is sufficient to check that the value of the right-hand part of equation (12) is positive, if we substitute $w = w^*$:

$$\frac{(1+c)(\gamma-\beta)}{\gamma} - \left(\frac{\beta+c\beta}{c\gamma}\right)^{-\frac{\beta}{\gamma-\beta}}.$$

After we substitute $\delta \triangleq \frac{\beta}{\gamma} \in (0,1)$, need to check that, for all $c \in \left(0, \frac{\delta}{1-\delta}\right)$,

$$(1+c)(1-\delta) > \left(\frac{c}{\delta(1+c)}\right)^{\frac{\delta}{1-\delta}} \iff \frac{c}{(1+c)^{\frac{1}{\delta}}} < \delta(1-\delta)^{\frac{1-\delta}{\delta}}. \tag{16}$$

The left-hand side increases with c because

$$\frac{d}{dc}\left[\frac{c}{(1+c)^{\frac{1}{\delta}}}\right] = \frac{1-\frac{c}{\delta}(1+c)^{-1}}{(1+c)^{\frac{1}{\delta}}} > 0:$$

$\delta - \frac{c}{1+c} = \frac{\delta - c(1-\delta)}{1+c} \in (0,\delta)$ as $c \in \left(0, \frac{\delta}{1-\delta}\right)$. Therefore, the supremum of the left-hand side of (16) equals

$$\frac{\frac{\delta}{1-\delta}}{\left(1+\frac{\delta}{1-\delta}\right)^{\frac{1}{\delta}}} = \delta(1-\delta)^{\frac{1}{\delta}-1}$$

and is not attained. The proof is completed. □

Proof of Proposition 3. i) Introduce the function

$$\mathcal{Q}(x,y) = x\Big[1 - c\,\frac{y}{x} - \Big(1+\frac{y}{x}\Big)^{-\frac{\beta}{\gamma-\beta}}\Big];$$

then the domain $\{(x,y) : x \geq G(y)\}$ can be given by the inequality $\mathcal{Q}(x,y) \geq k$ and the curve $x = G(y)$ is just the level curve of $\mathcal{Q}$, represented as $\{(x,y) : \mathcal{Q}(x,y) = k\}$. In terms of the auxiliary function $w = y/x$, $\mathcal{Q}$ can be written as

$$\mathcal{Q}(x,y) = x\big[1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}\big].$$

The partial derivatives of $\mathcal{Q}$ are

$$\mathcal{Q}'_x(x,y) = \big[1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}\big] + w\Big[c - \frac{\beta}{\gamma-\beta}(1+w)^{-\frac{\gamma}{\gamma-\beta}}\Big],$$

$$\mathcal{Q}'_y(x,y) = -\Big[c - \frac{\beta}{\gamma-\beta}(1+w)^{-\frac{\gamma}{\gamma-\beta}}\Big].$$

In terms of the function w, the vector field (1) can be written as

$$\dot{x} = -\frac{xw}{1+w}\cdot\beta, \qquad \dot{y} = -\frac{xw}{1+w}\cdot(\gamma(1+w)-\beta)$$

Take a solution $x(t)$, $y(t)$, $t \in \mathbb{R}$ of (1) with positive $x(0)$, $y(0)$. As $t \to -\infty$, $x(t)$ increases and, in view of (3), the corresponding value $w(t) = y(t)/x(t)$ goes to $+\infty$. It follows that $\lim_{t\to-\infty} \mathcal{Q}(x(t), y(t)) = -\infty$. On the other hand, as $t \to +\infty$, $x(t)$ decreases and $w(t) \to 0$; hence $\lim_{t\to+\infty} \mathcal{Q}(x(t), y(t)) = 0$.

Now calculate the derivative of $\mathcal{Q}$ along the vector field (1),

$$\mathcal{Q}'_x \cdot \dot{x} + \mathcal{Q}'_y \cdot \dot{y} = -\frac{xw}{1+w}\Big\{\beta\big[1 - cw - (1+w)^{-\frac{\beta}{\gamma-\beta}}\big] + \beta w\Big[c - \frac{\beta}{\gamma-\beta}(1+w)^{-\frac{\gamma}{\gamma-\beta}}\Big]$$

$$-(\gamma(1+w)-\beta)\Big[c - \frac{\beta}{\gamma-\beta}(1+w)^{-\frac{\gamma}{\gamma-\beta}}\Big]\Big\} = -\frac{xw}{1+w}\, c\gamma\,(w^* - w).$$

Thus, $\mathcal{Q}$ increases along the trajectory when $w > w^*$, reaches its maximal value when $w = w^*$, and decreases when $w < w^*$.

A trajectory of the vector field (1) is uniquely defined by the point of intersection with the line $y = w^* x$.

If the point of intersection lies to the left of H^* (with $x < x^*$) then the maximal value of $\mathcal{Q}(x(t), y(t))$, equal to

$$x\left[1 - cw^* - (1+w^*)^{-\frac{\beta}{\gamma-\beta}}\right] < x^*\left[1 - cw^* - (1+w^*)^{-\frac{\beta}{\gamma-\beta}}\right] = k,$$

is less than k, and therefore, the trajectory does not intersect the domain $\{(x,y) : \mathcal{Q}(x,y) \geq k\} = \{(x,y) : x \geq G(y)\}$.

If the point of intersection coincides with H^* then the maximal value of $\mathcal{Q}(x(t), y(t))$ equals k and is attained at $(x(t^*), y(t^*)) = H^*$. Denote the corresponding trajectory by I^*. The part of I^* corresponding to $t < t^*$ lies outside the domain $\{(x,y) : \mathcal{Q}(x,y) \geq k\}$, and additionally, $x(t) > x^*$ and $y(t) > w^* x(t)$. This implies that this part of the trajectory is above the curve $x = G(y)$. The part of I^* corresponding to $t > t^*$ also lies outside the domain $\{(x,y) : \mathcal{Q}(x,y) \geq k\}$, and additionally, $x(t) < x^*$ and $y(t) < w^* x(t)$; this part of the trajectory lies to the left of the curve $x = G(y)$.

The smooth curves $x = G(y)$ and I^* have a single point, H^*, in common. Further, the derivative of $\mathcal{Q}$ along the trajectory is zero at the corresponding point t^*. It follows that $(\dot{x}(t^*), \dot{y}(t^*))$ is orthogonal to $\nabla\mathcal{Q}(H^*) = (\mathcal{Q}'_x(H^*), \mathcal{Q}'_y(H^*))$ and thereby is parallel to the level curve of $\mathcal{Q}$ $\{(x,y) :\ \mathcal{Q}(x,y) = k\}$. Equivalently, $(\dot{x}(t^*), \dot{y}(t^*))$ is parallel to the tangent to the curve $x = G(y)$ at H^*. The tangency of the curves, and thus, Item i), is proved.

If the point of intersection lies to the right of H^* then the maximal value of $\mathcal{Q}(x(t), y(t))$ is greater than k, and therefore, a certain part $x(t)$, $y(t)$, $\tau_1 \leq t \leq \tau_2$, of the trajectory lies in the domain

$$\{(x,y) : \mathcal{Q}(x,y) \geq k\} = \{(x,y) :\ x > G(y)\}.$$

The trajectory lies to the right of I^*. The value t^* maximizing $\mathcal{Q}(x(t), y(t))$ lies in the open segment (τ_1, τ_2). The point $(x(\tau_1), y(\tau_1))$ where the trajectory enters the domain satisfies the relation $y(\tau_1)/x(\tau_1) > w^*$, and therefore, lies above H^*. The point $(x(\tau_2), y(\tau_2))$ where the trajectory leaves the domain satisfies $y(\tau_2)/x(\tau_2) < w^*$, and therefore, lies below H^*. The trajectory enters $\mathcal{L}$ at some point $(x(t^*), y(t^*))$ and leaves it at $(x(\tau_2), y(\tau_2))$. Thus, Item ii) is proved. □

Proof of Proposition 4.

(a) Consider the area I. In Subsubsection 3.2.1, it was shown that the function g_1 (9) is concave with respect to a for all x and y. Therefore, it is sufficient to check the extreme values $a = 0$ and $a = y$:

$$\begin{aligned} g_1(x,y,0) &= k > 0; \\ g_1(x,y,y) &= k + cy + 0 - x\left[1 - \left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right] > 0 \end{aligned}$$

according to Remark 2: $G(y) > x$ in the areas I,II and III. (In the latter case, Proposition 3-i) is in use.) Condition (7) is satisfied.

(b) Consider the area II and let (x, y) be a fixed point from there. For actions

$$a \in \mathbf{B} \stackrel{\triangle}{=} \mathbf{A} \setminus [y - G_+^{-1}(x),\ y - G_-^{-1}(x)],$$

the new point $l(x, y, a)$ belongs either to the area II, or to the area I. According to Item (a) above,

$$\inf_{a \in \mathbf{B}} [C^I(x, y, a) + V(l(x, y, a)) - V(x, y)] = \inf_{a \in \mathbf{B}} g_1(x, y, a) > 0.$$

It remains to consider $a \in [y - C_+^{-1}(x),\ y \in G_-^{-1}(x)]$, when $l(x, y, a) \in \mathcal{L}$:

$$\begin{aligned} C^I(x, y, a) + V(l(x, y, a)) - V(x, y) &= k + ca + (k + c(y - a)) - x \left[1 - \left(\frac{x}{x + y}\right)^{\frac{\beta}{\gamma - \beta}}\right] \\ &= k + k + cy - x \left[1 - \left(\frac{x}{x + y}\right)^{\frac{\beta}{\gamma - \beta}}\right] > k > 0 \end{aligned}$$

according to Remark 2: $G(y) > x$. Condition (7) is satisfied.

(c) Consider the area III in $\mathbf{X}$ and let (x, y) be a fixed point from there.

For actions

$$a \in \mathbf{B} \stackrel{\triangle}{=} \mathbf{A} \setminus [y - I^*(x), y - G_-^{-1}(x)],$$

the new point $l(x, y, a)$ belongs either to the area III, or to the area I. According to Item (a) above,

$$\inf_{a \in \mathbf{B}} g_1(x, y, a) \geq \min\{g_1(x, y, 0),\ g_1(x, y, y)\} > 0.$$

For actions

$$a \in [y - xw^*,\ y - G_-^{-1}(x)],$$

the new point $l(x, y, a)$ belongs to $\mathcal{L}$ and, similarly to Item (b) above,

$$C^I(x, y, a) + V(l(x, y, a)) - V(x, y) > k > 0$$

according to Remark 2: $G(y) > x$ because of Proposition 3-i).

For actions

$$a \in [y - I^*(x),\ y - xw^*),$$

the new point $l(x, y, a)$ belongs to the area IV and

$$\begin{aligned} g_2(x, y, a) &\stackrel{\triangle}{=} C^I(x, y, a) + V(l(x, y, a)) - V(x, y) \\ &= k + ca + x + (cw^* - 1)x \frac{\left(1 + \frac{y - a}{x}\right)^{-\frac{\beta}{\gamma - \beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma - \beta}}} + k - x \left[1 - \left(\frac{x}{x + y}\right)^{\frac{\beta}{\gamma - \beta}}\right]. \end{aligned}$$

Similarly to Subsubsection 3.2.1, the function g_2 is concave with respect to a: the derivative

$$\frac{\partial g_2}{\partial a} = c + (cw^* - 1) \frac{\beta}{\gamma - \beta} \frac{\left(1 + \frac{y - a}{x}\right)^{-\frac{\gamma}{\gamma - \beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma - \beta}}}$$

decreases with a because $cw^* - 1 = \frac{(c+1)(\beta - \gamma)}{\gamma} < 0$. Therefore, it is sufficient to check the extreme values.

$$\begin{aligned} g_2(x, y, y - xw^*) &= k + c(y - xw^*) + x + (cw^* - 1)x + k - x \left[1 - \left(\frac{x}{x + y}\right)^{\frac{\beta}{\gamma - \beta}}\right] \\ &= k + cy - x \left[1 - \left(\frac{x}{x + y}\right)^{\frac{\beta}{\gamma - \beta}}\right] + k > k > 0 \end{aligned}$$

according to Remark 2: $G(y) > x$ because of Proposition 3-i).

$$g_2(x,y,y-I^*(x)) \;=\; k + c(y - I^*(x)) + x + (cw^* - 1)x\frac{\left(1+\frac{I^*(x)}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}} + k$$
$$-x\left[1-\left(\frac{x}{x+y}\right)^{\frac{\beta}{\gamma-\beta}}\right].$$

On the trajectory I^*, because of continuity of V, we have

$$V_3(x, I^*(x)) = x + (cw^* - 1)x\frac{\left(1+\frac{I^*(x)}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}} + k = V_1(x, I^*(x)) = x\left[1-\left(\frac{x}{x+I^*(x)}\right)^{\frac{\beta}{\gamma-\beta}}\right].$$

For the direct proof, one can substitute

$$x(t) = \frac{k}{1 - cw^* - (1+w^*)^{-\frac{\beta}{\gamma-\beta}}} \cdot \frac{(1+w^*)^{\frac{\beta}{\beta-\gamma}}}{\left(1+w^*e^{(\beta-\gamma)t}\right)^{\frac{\beta}{\beta-\gamma}}}$$

and

$$I^*(x(t)) = y(t) = x(t)w^*e^{(\beta-\gamma)t}$$

for arbitrary $t \in (-\infty, +\infty)$. These expressions come from (2), (3) and (11). Now we are in the framework of Item (a):

$$g_2(x,y,y-I^*(x)) = g_1(x,y,y-I^*(x)) > 0.$$

Condition (7) is satisfied.

(d) Consider the area IV in $\mathbf{X}$ and let (x,y) be a fixed point from there.

For actions

$$a \in [0, y - xw^*),$$

the new point $l(x,y,a)$ again belongs to the area IV and, as in Item (c) above, function

$$\begin{aligned} g_3(x,y,a) \;&\triangleq\; C^I(x,y,a) + V(l(x,y,a)) - V(x,y) \\ &= k + ca + x + (cw^*-1)x\frac{\left(1+\frac{y-a}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}} + k \\ &\quad -\left[x + (cw^*-1)x\frac{\left(1+\frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right]. \end{aligned}$$

is concave with respect to a. Therefore, it is sufficient to check the extreme values:

$$\begin{aligned} g_3(x,y,0) \;&=\; k > 0; \\ g_3(x,y,y-xw^*) \;&=\; k + cy - x - (cw^*-1)x\frac{\left(1+\frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}}. \end{aligned}$$

The latter function is convex with respect to y: its derivative equals

$$c + (cw^*-1)\frac{\beta}{\gamma-\beta}\cdot\frac{\left(1+\frac{y}{x}\right)^{-\frac{\gamma}{\gamma-\beta}}}{(1+w^*)^{-\frac{\beta}{\gamma-\beta}}},$$

increasing (with respect to y) function, because $cw^* - 1 = \frac{(c+1)(\beta-\gamma)}{\gamma} < 0$. The infimum of that derivative corresponds to the infimum of y, equal to xw^*:

$$c + (cw^*-1)\frac{\beta}{\gamma-\beta}\cdot\frac{1}{1+w^*} = 0.$$

Thus, in the area IV,

$$c + (cw^* - 1)\frac{\beta}{\gamma - \beta} \cdot \frac{\left(1 + \frac{y}{x}\right)^{-\frac{\gamma}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} > 0$$

and

$$g_3(x, y, y - xw^*) > g_3(x, xw^*, xw^* - xw^*) = k > 0. \tag{17}$$

For actions

$$a \in [y - xw^*,\ y - G_-^{-1}(x)],$$

the new point $l(x, y, a)$ belongs to $\mathcal{L}$ and

$$\begin{aligned} C^I(x, y, a) + V(l(x, y, a)) - V(x, y) &= k + ca + (k + c(y - a)) \\ &\quad - \left[x + (cw^* - 1)x\frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right] \\ &= g_3(x, y, y - xw^*) > k > 0 \end{aligned}$$

according to the above calculations.

For actions

$$a \in (y - G_-^{-1}(x), y],$$

the new point $l(x, y, a)$ belongs to the area I and

$$\begin{aligned} g_4(x, y, a) &\triangleq C^I(x, y, a) + V(l(x, y, a)) - V(x, y) = k + ca + x\left[1 - \left(\frac{x}{x + y - a}\right)^{\frac{\beta}{\gamma-\beta}}\right] \\ &\quad - \left[x + (cw^* - 1)x\frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right]. \end{aligned}$$

Similarly to Subsubsection 3.2.1, (see (9)) this function is concave with respect to a. Therefore, it is sufficient to check the extreme values.

$$\begin{aligned} \lim_{a \to y - G_-^{-1}(x)} g_4(x, y, a) &= k + c(y - G_-^{-1}(x)) + x\left[1 - \left(\frac{x}{x + G_-^{-1}(x)}\right)^{\frac{\beta}{\gamma-\beta}}\right] \\ &\quad - \left[x + (cw^* - 1)x\frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right] \\ &= k + c(y - G_-^{-1}(x)) + k + cG_-^{-1}(x) \\ &\quad - \left[x + (cw^* - 1)x\frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right] = g_3(x, y, y - xw^*) > k > 0 \end{aligned}$$

according to (17). Recall that, for $y = G_-^{-1}(x) \Longrightarrow x = G(y)$, equality (10) is valid by definition.

$$\begin{aligned} g_4(x, y, y) &= k + cy - \left[x + (cw^* - 1)x\frac{\left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}}{(1 + w^*)^{-\frac{\beta}{\gamma-\beta}}} + k\right] \\ &= g_3(x, y, y - xw^*) - k > g_3(x, xw^*, 0) - k > 0 \end{aligned}$$

again by (17).

Condition (7) is satisfied.

The proof is completed. $\square$

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