

# Invariance principle for fragmentation processes derived from conditioned stable Galton-Watson trees

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## Abstract

Aldous, Evans and Pitman (1998) studied the behavior of the fragmentation process derived from deleting the edges of a uniform random tree on  $n$  labelled vertices. In particular, they showed that, after proper rescaling, the above fragmentation process converges as  $n \rightarrow \infty$  to the fragmentation process of the Brownian CRT obtained by cutting-down the Brownian CRT along its skeleton in a Poisson manner.

In this work, we continue the above investigation and study the fragmentation process obtained by deleting randomly chosen edges from a critical Galton-Watson tree  $\mathbf{t}_n$  conditioned on having  $n$  vertices, whose offspring distribution belongs to the domain of attraction of a stable law of index  $\alpha \in (1, 2]$ . Our main results establish that, after rescaling, the fragmentation process of  $\mathbf{t}_n$  converges as  $n \rightarrow \infty$  to the fragmentation process obtained by cutting-down proportional to the length on the skeleton of an  $\alpha$ -stable Lévy tree of index  $\alpha \in (1, 2]$ . We further show that the latter can be constructed by considering the partitions of the unit interval induced by the normalized  $\alpha$ -stable Lévy excursion with a deterministic drift studied by Miermont (2001). This extends the result of Bertoin (2000) on the fragmentation process of the Brownian CRT.

KEY WORDS AND PHRASES: Additive coalescent, fragmentation, Galton-Watson trees, spectrally positive stable Lévy processes, stable Lévy tree, Prim's algorithm.

SUBJECT CLASSES: 60J25, 60J90, 60F05, 60G52, 60C05.

## 1 Introduction and main results

Aldous, Evans and Pitman [5, 22, 38] (see also [15, 30]) considered a fragmentation process of a uniform random tree  $\mathbf{t}_n$  on  $n \in \mathbb{N}$  labelled vertices (or Cayley tree with  $n$  vertices) by deleting the edges of  $\mathbf{t}_n$  one by one in uniform random order. More precisely, as time passes, the deletion of edges creates more and more subtrees of  $\mathbf{t}_n$  (connected components) such that the evolution of the ranked vector of sizes (number of vertices) of these subtrees (in decreasing order) evolves as a fragmentation process.

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It turns out that the asymptotic behavior of this fragmentation process, in reverse time, is related to the so-called *standard additive coalescent* [5, 22]. Moreover, this leads to a continuous representation of the standard additive coalescent in terms of the time-reversal of an analogue fragmentation process of the Brownian continuum random tree (Brownian CRT); see [5]. Evans and Pitman [22, Theorem 2] showed that an additive coalescent is a Feller Markov process with values in the infinite ordered set

$$\mathbb{S} := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0 \text{ and } \sum_{i=1}^{\infty} x_i < \infty \right\}, \quad (1)$$

endowed with the  $\ell^1$ -norm,  $\|\mathbf{x}\|_1 = \sum_{i=1}^{\infty} |x_i|$  for  $\mathbf{x} \in \mathbb{S}$ , whose evolution is described formally by: given that the current state is  $\mathbf{x}$ , two terms  $x_i$  and  $x_j$ ,  $i < j$ , of  $\mathbf{x}$  are chosen and merged into a single term  $x_i + x_j$  (which implies some reordering of the resulting sequence) at rate equal to  $x_i + x_j$ . A version of this process defined for times describing the whole real axis is called *eternal*. This model is also closely related to the so-called Marcus-Lushnikov process [31, 29], and in particular, the version studied in [5] is referred to as the standard additive coalescent.

In this work, we shall extend the investigation, that was begun in [5, 22, 38], to the more general situation where one wants to cut-down critical Galton–Watson trees conditioned on having a fixed number of vertices, but whose offspring distribution belongs to the domain of attraction of a stable law. More precisely, consider a critical offspring distribution  $\mu = (\mu(k), k \geq 0)$ , i.e., a probability distribution on the nonnegative integers satisfying  $\sum_{k=0}^{\infty} k\mu(k) = 1$ . In addition, we always implicitly assume that  $\mu(0) > 0$  and  $\mu(0) + \mu(1) < 1$  to avoid degenerate cases, and that  $\mu$  is aperiodic. We say that  $\mu$  belongs to domains of attraction of a stable law of index  $\alpha \in (1, 2]$  if either the variance of  $\mu$  is finite, or if  $\mu([k, \infty)) = k^{-\alpha}L(k)$  as  $k \rightarrow \infty$ , where  $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a function such that  $L(x) > 0$  for  $x \in \mathbb{R}_+$  large enough and  $\lim_{x \rightarrow \infty} L(tx)/L(x) = 1$  for all  $t > 0$  (such function is called slowly varying function). In other terms, if  $(Y_i)_{i \geq 1}$  is a sequence of i.i.d. random variables with distribution  $\mu$ , then there exists a sequence of positive real numbers  $(B_n)_{n \geq 1}$  such that

$$B_n \rightarrow \infty \quad \text{and} \quad \frac{Y_1 + Y_2 + \dots + Y_n - n}{B_n} \xrightarrow{d} Y_\alpha, \quad \text{in distribution as } n \rightarrow \infty \quad (2)$$

to a random variable  $Y_\alpha$  with Laplace exponent given by  $\mathbb{E}[\exp(-\lambda Y_\alpha)] = \exp(-\lambda^\alpha)$  whenever  $\alpha \in (1, 2)$ , and  $\mathbb{E}[\exp(-\lambda Y_2)] = \exp(-\lambda^2/2)$  if  $\alpha = 2$ , for every  $\lambda > 0$  ([23, Section XVII.5] guarantees its existence). In particular, for  $\alpha = 2$ , we have that  $Y_2$  is distributed as a standard Gaussian random variable. The factor  $B_n$  is of order  $n^{1/\alpha}$  (more precisely,  $B_n/n^{1/\alpha}$  is a slowly varying function), and one may take  $B_n = \sigma n^{1/2}$  when  $\mu$  has finite variance  $\sigma^2$ .

We henceforth let  $\mathbf{t}_n$  denote a critical Galton-Watson tree whose offspring distribution  $\mu$  belongs to the domain of attraction of a stable law of index  $\alpha \in (1, 2]$  and refer to it as an  $\alpha$ -stable GW-tree, for simplicity. Following Aldous, Evans and Pitman [5, 22], we are interested in the evolution of the ranked vector of sizes (in decreasing order) of the subtrees created by deleting randomly chosen edges from  $\mathbf{t}_n$ . Indeed, we will consider a continuous-time version of this cutting-down process. Let  $\mathbf{edge}(\mathbf{t}_n)$  be the set of edges of  $\mathbf{t}_n$  and equip each of the edges of  $\mathbf{t}_n$  with i.i.d. uniform random variables (or weights)

$\mathbf{w} = (w_e : e \in \mathbf{edge}(\mathbf{t}_n))$  on  $[0, 1]$  and independently of the tree  $\mathbf{t}_n$ . For  $u \in [0, 1]$ , we then keep the edges of  $\mathbf{t}_n$  with weight smaller than  $u$  and discard the others. Therefore, one obtains a (fragmentation) forest  $\mathbf{f}_n(u)$  conformed by the connected components (or subtrees of  $\mathbf{t}_n$ ) created by the above procedure; see Figure 3. In particular, the forest  $\mathbf{f}_n(u)$  has the same set of vertices as  $\mathbf{t}_n$  but clearly it has a different set of edges given by  $\mathbf{edge}(\mathbf{f}_n(u)) = \{e \in \mathbf{edge}(\mathbf{t}_n) : w_e \leq u\}$ . Let  $\mathbf{F}_n = (\mathbf{F}_n(u), u \in [0, 1])$  be the process given by

$$\mathbf{F}_n(u) = (F_{n,1}(1-u), F_{n,2}(1-u), \dots), \quad \text{for } u \in [0, 1],$$

the sequence of sizes (number of vertices) of the connected components of the forest  $\mathbf{f}_n(1-u)$ , ranked in decreasing order. We have strategically viewed the sequence of sizes of the components of  $\mathbf{f}_n(1-u)$  as an infinite sequence, by completing with an infinite number of zero terms. Plainly as time passes more and more subtrees are created, and thus, the process  $\mathbf{F}_n$  evolves as a fragmentation process. Note also that  $\mathbf{F}_n(0) = (n, 0, 0, \dots)$  and that  $\mathbf{F}_n(1) = (1, 1, \dots, 1, 0, 0, \dots)$  are infinite sequences where the first  $n$  terms are ones in  $\mathbf{F}_n(1)$ . Since we are interested in studying the asymptotic behaviour of  $\mathbf{F}_n$ , we consider the (rescaled in time and space) fragmentation process  $\mathbf{F}_n^{(\alpha)} = (\mathbf{F}_n^{(\alpha)}(t), t \geq 0)$  given by

$$\mathbf{F}_n^{(\alpha)}(t) = \frac{1}{n} \mathbf{F}_n \left( \frac{B_n t}{n} \right), \quad \text{for } 0 \leq t \leq n/B_n, \quad \text{and } \mathbf{F}_n^{(\alpha)}(t) = \frac{1}{n} \mathbf{F}_n(1) \quad \text{for } t > n/B_n, \quad (3)$$

where  $(B_n)_{n \geq 1}$  is a sequence satisfying (2). The process  $\mathbf{F}_n^{(\alpha)}$  takes values on the set  $\mathbb{S}$ . The aim of this paper is to establish a convergence limit result for the fragmentation process  $\mathbf{F}_n^{(\alpha)}$ . To state the precise statement (Theorem 1), it will be convenient to introduce first the limiting object.

Bertoin [8] showed that the fragmentation process of the Brownian CRT in [5] can be constructed by considering the partitions of the unit interval induced by a standard Brownian excursion with drift. This latter is sometimes called the *Brownian fragmentation*. In a similar vein, Miermont [32] built other fragmentation processes from Lévy processes with no positive jumps (or equivalently, negative of spectrally positive Lévy processes). Specifically, let  $X_\alpha^{\text{exc}} = (X_\alpha^{\text{exc}}(s), s \in [0, 1])$  be the normalized excursion (with unit length) of an  $\alpha$ -stable spectrally positive Lévy process of index  $\alpha \in (1, 2]$ ; see Section 3 for a formal definition. In particular,  $X_2^{\text{exc}}$  is the normalized standard Brownian excursion. For every  $t \geq 0$ , define the processes  $Y_\alpha^{(t)} = (Y_\alpha^{(t)}(s), s \in [0, 1])$  and  $I_\alpha^{(t)} = (I_\alpha^{(t)}(s), s \in [0, 1])$  by letting

$$Y_\alpha^{(t)}(s) = X_\alpha^{\text{exc}}(s) - ts \quad \text{and} \quad I_\alpha^{(t)}(s) = \inf_{u \in [0, s]} Y_\alpha^{(t)}(u), \quad \text{for } s \in [0, 1]. \quad (4)$$

For  $t \geq 0$ , we introduce

$$\mathbf{F}^{(\alpha)}(t) = (F_1^{(\alpha)}(t), F_2^{(\alpha)}(t), \dots) \quad (5)$$

as the random element of  $\mathbb{S}$  defined by the ranked sequence (in decreasing order) of the lengths of the intervals components of the complement of the support of the Stieltjes measure  $d(-I_\alpha^{(t)})$ ; note that  $s \mapsto -I_\alpha^{(t)}(s) = \sup_{u \in [0, s]} -Y_\alpha^{(t)}(u)$  is an increasing process. More precisely, the support of  $d(-I_\alpha^{(t)})$  is

defined as the set of times when the process  $Y_\alpha^{(t)}$  reaches a new infimum. On the other hand, it can be shown that the support of  $d(-I_\alpha^{(t)})$  coincides with the so-called ladder time set of  $-Y_\alpha^{(t)}$  which is given by the closure of the set of times when  $Y_\alpha^{(t)}$  is equal to its infimum, i.e.,

$$\mathcal{L}^\alpha(t) := \overline{\{s \in [0, 1] : Y_\alpha^{(t)}(s) = I_\alpha^{(t)}(s)\}};$$

see for example [7, Proposition 1, Chapter VI] and the discussion after that. Then  $\mathbf{F}^{(\alpha)}(t)$  is the lengths of the open intervals in the canonical decomposition of  $[0, 1] \setminus \mathcal{L}^\alpha(t)$  arranged in the decreasing order. The intervals components of the complement of the support of the measure  $d(-I_\alpha^{(t)})$  are also called constancy intervals of  $-I_\alpha^{(t)}$ , and in fact, those intervals corresponds to excursion intervals of  $Y_\alpha^{(t)}$  above its infimum (or equivalently, excursion intervals of the reflected process  $Y_\alpha^{(t)} - I_\alpha^{(t)}$  above 0). It is well-known that  $\mathcal{L}^\alpha(t)$  is a.s. a random closed set with zero Lebesgue measure which implies that  $\mathbf{F}^{(\alpha)}(t) \in \mathbb{S}_1$  a.s., where  $\mathbb{S}_1 \subset \mathbb{S}$  is the space of the elements of  $\mathbb{S}$  with sum 1; see [7, Corollary 5, Chapter VII]. Observe that for every fixed  $0 \leq t < t'$ , the process  $s \rightarrow Y_\alpha^{(t)}(s) - Y_\alpha^{(t')}(s) = (t' - t)s$  is monotone increasing which entails that  $\mathcal{L}^\alpha(t) \subseteq \mathcal{L}^\alpha(t')$ . Then the partition of  $[0, 1]$  induced by  $\mathcal{L}^\alpha(t')$  is finer than that induced by  $\mathcal{L}^\alpha(t)$ . As a consequence, it has been shown by Miermont [32, Proposition 2] (see also [8, Theorem 1] for  $\alpha = 2$ ) that  $\mathbf{F}^{(\alpha)} = (\mathbf{F}^{(\alpha)}(t), t \geq 0)$  is a fragmentation process issued from  $\mathbf{F}^{(\alpha)}(0) = (1, 0, 0, \dots)$ . A precise description of its transition kernel is given in [32, Definition 4]; see Corollary 1 below for some insights. From now on, we will refer to  $\mathbf{F}^{(\alpha)}$  as the  $\alpha$ -stable fragmentation of index  $\alpha \in (1, 2]$ .

We are now able to state our first main result. Let  $\mathbb{D}(I, \mathbb{M})$  be the space of càdlàg functions from an interval  $I \subseteq \mathbb{R}$  to the separable, complete metric space  $(\mathbb{M}, d)$  equipped with the Skorohod topology; (see e.g. [14, Chapter 3] or [25, Chapter VI] for details on this space). We write  $\xrightarrow{d}$  to denote convergence in distribution.

**Theorem 1.** *Let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree of index  $\alpha \in (1, 2]$ . Then, we have that*

$$(\mathbf{F}_n^{(\alpha)}(t), t \geq 0) \xrightarrow{d} (\mathbf{F}^{(\alpha)}(t), t \geq 0), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{D}(\mathbb{R}_+, \mathbb{S}).$$

As mentioned earlier,  $\mathbf{F}^{(2)}$  is exactly the Brownian fragmentation studied by Bertoin [8], that is to say, it corresponds to the fragmentation process derived from the Brownian CRT of Aldous and Pitman [5]; see also [2]. In view of this, the second goal of this paper is to show that indeed  $\mathbf{F}^{(\alpha)}$  is the fragmentation process obtained by cutting-down the “edges” of the  $\alpha$ -stable Lévy tree.

The  $\alpha$ -stable Lévy tree of index  $\alpha \in (1, 2]$  is the continuum random tree analogue of (discrete)  $\alpha$ -stable GW-trees. They were introduced by Duquesne and Le Gall [20], and in particular, they also appear as scaling limits of  $\alpha$ -stable GW-trees. In brief, the  $\alpha$ -stable Lévy tree  $\mathcal{T}_\alpha = (\mathcal{T}_\alpha, d_\alpha, \rho_\alpha)$  is a random compact metric space  $(\mathcal{T}_\alpha, d_\alpha)$  with one distinguished element  $\rho \in \mathcal{T}_\alpha$  called the root such that  $(\mathcal{T}_\alpha, d_\alpha)$  is a tree-like space in that for  $v, w \in \mathcal{T}_\alpha$ , there is a unique non-self-crossing path  $[v, w]$  from  $v$  to  $w$  in  $\mathcal{T}_\alpha$ , whose length equals  $d_\alpha(v, w)$ . The leaves  $\text{Lf}(\mathcal{T}_\alpha)$  of  $\mathcal{T}_\alpha$  are those points that do not belong to the interior of any path leading from one point to another, and the skeleton of the tree is the set  $\text{Sk}(\mathcal{T}_\alpha) = \mathcal{T}_\alpha \setminus \text{Lf}(\mathcal{T}_\alpha)$  of non-leaf points. The  $\alpha$ -stable Lévy tree  $\mathcal{T}_\alpha$  is naturally endowed with a uniform

probability measure  $\mu_\alpha$  (*the mass measure*) that is supported on  $\text{Lf}(\mathcal{T}_\alpha)$ , and a unique  $\sigma$ -finite measure  $\lambda_\alpha$  (*the length measure*) carried by  $\text{Sk}(\mathcal{T}_\alpha)$  that assigns measure  $d(v, w)$  to the geodesic path between  $v$  and  $w$  in  $\mathcal{T}_\alpha$ .

Following Aldous-Pitman's fragmentation [5] of the Brownian CRT, the analogue of deleting randomly chosen edges in  $\mathbf{t}_n$  is to cut the skeleton of  $\mathcal{T}_\alpha$  by a Poisson point process of cuts with intensity  $dt \otimes \lambda_\alpha(dv)$  on  $[0, \infty) \times \mathcal{T}_\alpha$ . For all  $t \geq 0$ , define an equivalence relation  $\sim_t$  on  $\mathcal{T}_\alpha$  by saying that  $v \sim_t w$ , for  $v, w \in \mathcal{T}_\alpha$ , if and only if, no atom of the Poisson process that has appeared before time  $t$  belongs to the path  $[v, w]$ . These cuts split the  $\alpha$ -stable Lévy tree into a (continuum) forest, that is a countably infinite set of smaller subtrees (connected components) of  $\mathcal{T}_\alpha$ . Let  $\mathcal{T}_{\alpha,1}^{(t)}, \mathcal{T}_{\alpha,2}^{(t)}, \dots$  be the distinct equivalence classes for  $\sim_t$  (connected components of  $\mathcal{T}_\alpha$ ), ranked according to the decreasing order of their  $\mu_\alpha$ -masses. The subtrees  $(\mathcal{T}_{\alpha,i}^{(t)}, i \geq 1)$  are nested as  $t$  varies, that is, for every  $0 \leq t < t'$  and  $i \geq 1$ , there exists  $j \geq 1$  such that  $\mathcal{T}_{\alpha,i}^{(t')} \subset \mathcal{T}_{\alpha,j}^{(t)}$ . Let  $\mathbf{F}_{\mathcal{T}_\alpha} = (\mathbf{F}_{\mathcal{T}_\alpha}(t), t \geq 0)$  be the process given by

$$\mathbf{F}_{\mathcal{T}_\alpha}(t) = (\mu_\alpha(\mathcal{T}_{\alpha,1}^{(t)}), \mu_\alpha(\mathcal{T}_{\alpha,2}^{(t)}), \dots), \quad t \geq 0,$$

where  $\mathbf{F}_{\mathcal{T}_\alpha}(0) = (1, 0, 0, \dots)$ . Indeed,  $\mathbf{F}_{\mathcal{T}_\alpha}$  is a fragmentation process in the sense that  $\mathbf{F}_{\mathcal{T}_\alpha}(t')$  is obtained by splitting at random the elements of  $\mathbf{F}_{\mathcal{T}_\alpha}(t)$ , for  $0 \leq t < t'$ . We call  $\mathbf{F}_{\mathcal{T}_\alpha}$  the fragmentation process of the  $\alpha$ -stable Lévy tree. In particular,  $\mathbf{F}_{\mathcal{T}_2}$  is the fragmentation process of the Brownian CRT introduced in [5, Section 2.2]. Note that  $\mathbf{F}_{\mathcal{T}_\alpha}$  takes values in  $\mathbb{S}$ , and that Lemma 7 below shows that  $\mathbf{F}_{\mathcal{T}_\alpha}(t) \in \mathbb{S}_1$  a.s., for every  $t \geq 0$ . We can now state our second main result.

**Proposition 1.** *We have that*

$$(\mathbf{F}^{(\alpha)}(t), t \geq 0) \stackrel{d}{=} (\mathbf{F}_{\mathcal{T}_\alpha}(t), t \geq 0),$$

where  $\stackrel{d}{=}$  means equal in distribution (in the sense of finite-dimensional distributions).

Theorem 3 in [5] shows that the time-reversed fragmentation process of the Brownian CRT, i.e.  $(\mathbf{F}_{\mathcal{T}_2}(e^{-t}), t \in \mathbb{R})$ , is a version of the standard additive coalescent providing an explicit construction of this last process. In general, Miermont [32, Section 6] has shown that the time-reversed  $\alpha$ -stable fragmentation process, i.e.  $(\mathbf{F}^{(\alpha)}(e^{-t}), t \in \mathbb{R})$ , is an eternal additive coalescent as described by Evans and Pitman [22]. More precisely, it is a mixing of so-called *extremal coalescents* of Aldous and Pitman [6] (see also [9]) which exact law is given in [32, Proposition 3]. Thus, Proposition 1 implies that this eternal additive coalescent can also be constructed from the  $\alpha$ -stable Lévy tree by Poisson splitting along its skeleton. On the other hand, Theorem 1 and Proposition 1 clearly generalize Bertoin's work [8] and moreover, complete Miermont's [32] one by identifying the distribution of the  $\alpha$ -stable fragmentation with that of the fragmentation process of the  $\alpha$ -stable Lévy tree. In particular, Bertoin [10] proved that  $\mathbf{F}^{(2)}$  (or equivalently,  $\mathbf{F}_{\mathcal{T}_2}$ ) is a so-called *self-similar fragmentation process* of index  $1/2$ . However, Miermont [33] has already pointed out that  $\mathbf{F}^{(\alpha)}$  (and therefore  $\mathbf{F}_{\mathcal{T}_\alpha}$ ), for  $\alpha \in (1, 2)$ , is not a self-similar fragmentation due to the existence of points in  $\mathcal{T}_\alpha$  with infinite degree.

The proof of Theorem 1 uses some of the ideas developed in [15] where only the case of Cayley tree was treated. However, in our more general framework, there are technical challenges that do not appear in [15], mostly due to the lack of some properties that only the Cayley tree satisfies. To prove Theorem 1, we use the so-called *Prim's algorithm* [40] to obtain a consistent ordering on the vertices of the forest created by deleting randomly chosen edges from  $\mathbf{t}_n$  that we refer to as the *Prim order*. Informally, given  $\mathbf{t}_n$  whose edges are equipped with non-negative and distinct weights, and a starting vertex, say  $v$  of  $\mathbf{t}_n$ , the Prim's algorithm explores a connected component from  $v$ , each time visiting a neighbouring vertex which connecting edge possesses the smallest weight; see Section 4. Then every time an edge is removed and a new connected component is created, the Prim order of the vertices in the new forest always remains the same. This will allow us to precisely encode this forest (and in particular, the sizes of connected components) using a discrete analogue of the process  $Y_\alpha^{(t)}$  defined in (4) that we refer to as the *Prim path*. We then show that this (properly rescaled) Prim path indeed converges to its continuous version. Finally, we use the results in [9] to develop a general approach for the convergence of fragmentation processes encoded by functions in  $\mathbb{D}([0, 1], \mathbb{R})$  to conclude our proof.

There are some of the key differences with the proof for Cayley trees in [15]. For example, the convergence of the encoding processes in [15] uses a bound (see in (10) in [15]) that is only known to hold for Cayley trees (or Galton-Watson trees where  $\mu$  has some exponential finite moment). In [15], the authors mostly work with convergence of continuous processes. This is no longer possible in our framework, since our encoding processes are discontinuous due to the nature of the  $\alpha$ -stable GW-trees. The above makes an important difference at the technical level.

The proof of Proposition 1 follows along the lines of that of Theorem 3 in [5] for the Brownian CRT (see also the proof of Proposition 13 in [6]). Informally, we use the convergence of rescaled  $\alpha$ -stable GW-trees toward the  $\alpha$ -stable Lévy tree  $\mathcal{T}_\alpha$  in order to approximate the fragmentation process of  $\mathcal{T}_\alpha$ .

The rest of the paper is organized as follows. In Section 2, we discuss some connections with some combinatorial and probabilistic models: additive coalescents, parking schemes, laminations and Bernoulli bond-percolation. In Section 3, we recall some facts about stable Lévy processes, bridges and excursions that will be important for our proofs. Section 4 is devoted to the introduction of Galton-Watson trees as well as the formal definition of the exploration process (the Prim path) associated with the fragmentation forest. The asymptotic behavior of the Prim path is studied in Section 5. Finally, the proofs of Theorem 1 and Proposition 1 are given in Section 6 and Section 7, respectively.

## 2 Further remarks

In this section, we comment on our main results and highlight some connections with previous works.

**Additive coalescents.** A Cayley tree of size  $n$  can be viewed as a Galton-Watson tree with Poissonian offspring distribution of parameter 1 and conditioned to have  $n$  vertices, where the labels are assigned to the vertices uniformly at random. In particular, Aldous, Evans and Pitman fragmentation process [5, 22, 38], say  $\mathbf{F}_n^+ = (\mathbf{F}_n^+(t), t \geq 0)$ , corresponds precisely to  $\mathbf{F}_n^{(\alpha)}$  in (3), with  $\alpha = 2$  and  $B_n = n^{1/2}$ . The fragmentation process  $\mathbf{F}_n^+$  leads to a representation of an additive coalescent by an appropriate

time reversal, that is, the exponential time-change  $t \rightarrow e^{-t}$ . Specifically,  $(\mathbf{F}_n^+(e^{-t}), t \geq -(1/2) \ln n)$  is an additive coalescent starting at time  $-(1/2) \ln n$  from the state  $(1/n, 1/n, \dots, 1/n, 0, 0, \dots) \in \mathbb{S}$  (or equivalently, from the component sizes in Marcus-Lushnikov model with  $n$  initial masses  $1/n$ ). Evans and Pitman [22] (see also [5, Proposition 2]) showed that this time-reversed version of  $\mathbf{F}_n^+$  converges in distribution to the standard additive coalescent, i.e.,  $(\mathbf{F}_{\mathcal{T}_2}(e^{-t}), t \in \mathbb{R})$ .

Aldous and Pitman [6] (see also [22, Construction 5]) also studied the fragmentation process derived by cutting-down *birthday trees*. They are a family of trees that generalizes the Cayley tree in allowing “weights” on the vertices. Aldous and Pitman showed that this fragmentation process, suitable rescaled, converges to the fragmentation process associated of the continuum counterpart of birthday trees, the *inhomogeneous continuum random trees* (ICRT). Moreover, the time-reversed version of the fragmentation process of the ICRT can be viewed as version of an eternal additive coalescent. On the other hand, Bertoin [9] has proved that the fragmentation process of the ICRT can also be constructed by considering the partitions of the unit interval induced by certain bridges with exchangeable increments.

**Parking schemes.** Chassaing and Louchard [16] have provided yet another representation of the standard additive coalescent as parking schemes related to the *Knuth’s parking problem*; see also [17, 30]. Bertoin and Miermont [13] extended the work [16] and relate the Knuth’s parking problem for caravans to different versions of eternal additive coalescent. On the other hand, the Knuth’s parking problem bear some similarities with the dynamics of an aggregating server studied by Bertoin [9] that also relate to the additive coalescent.

**Lamination process.** In a recent work, Thévenin [42] has provided a geometric representation of the fragmentation process  $\mathbf{F}_{\mathcal{T}_\alpha}$  by a new lamination-valued process. In particular, Theorem 1.1 in [42] combined with Proposition 1 allows to deduce the exact distribution of the ranked sequence (in decreasing order) of the masses of the faces of this lamination-valued process.

**Bernoulli bond-percolation.** Bernoulli bond-percolation on finite connected graphs is perhaps the simplest example of a percolation model. In this model, each edge in the connected graph is removed with probability  $1 - p \in (0, 1)$ , and it is kept with probability  $p$ , independently of the other edges. This induces a partition of the set of vertices of the graph into connected components usually referred to as clusters. It should be intuitively clear that there is a link between Bernoulli bond-percolation on  $\alpha$ -stable GW-trees and their associated fragmentation processes. More precisely, let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree. For  $u \in [0, 1]$ , recall that continuous-time cutting-down procedure of  $\mathbf{t}_n$  described in the introduction results in a random forest of connected components. Indeed, the probability that a given edge of  $\mathbf{t}_n$  has not yet been removed at time  $u$  is exactly  $u$ . Thus, the configuration of the connected components at time  $u$  is precisely that resulting from Bernoulli bond-percolation on  $\mathbf{t}_n$  with parameter  $u$ . A natural problem in this setting is then to investigate the asymptotic behavior of the sizes (number of vertices) of the largest clusters for appropriate percolation regimes. In this direction, let  $(B_n)_{n \geq 1}$  be a sequence of positive real numbers satisfying (2). An application of Theorem 1 shows that for the percolation parameter  $1 - (B_n/n)t$  with a fixed  $t \geq 0$ , the sequence of sizes of the clusters ranked in

decreasing order and renormalized by a factor of  $1/n$  (i.e.  $\mathbf{F}_n^{(\alpha)}(t)$ ) converges in distribution, as  $n \rightarrow \infty$ , to  $\mathbf{F}^{(\alpha)}(t)$ . In particular, Theorem 2 in [32] allows us to describe explicitly the distribution of  $\mathbf{F}^{(\alpha)}$  at fixed times. Let  $(p_s(z), z \in \mathbb{R}, s \geq 0)$  be the family of densities of the distribution of a strictly stable spectrally positive Lévy process with index  $\alpha \in (1, 2]$ ; see Section 3.

**Corollary 1.** *For  $t > 0$ , let  $a_1^{(\alpha)}(t) > a_2^{(\alpha)}(t) > \dots$  be the atoms of a Poisson measure on  $(0, \infty)$  with intensity  $\Lambda_\alpha^{(t)}(dz) := z^{-1}p_z(-tz)\mathbb{1}_{\{z>0\}}dz$ , ranked in decreasing order. Then*

$$\mathbf{F}^{(\alpha)}(t) \stackrel{d}{=} \left( (a_1^{(\alpha)}(t), a_2^{(\alpha)}(t), \dots) \mid \sum_{i=1}^{\infty} a_i^{(\alpha)}(t) = 1 \right).$$

Following Bertoin's [12] work about Bernoulli bond-percolation on random trees. The percolation regime  $1 - (B_n/n)t$  on  $\mathbf{t}_n$  corresponds to the so-called supercritical regime. Indeed, the result in Corollary 1 has already been proved by Pitman [38] for Cayley trees. Furthermore, it has been shown in [5, 8] that the distribution of  $\mathbf{F}^{(2)}(t)$  is equal to that of the ranked jump sizes (in decreasing order) of a stable subordinator of index  $1/2$  over the interval  $[0, t]$ , conditionally on being 1 at time  $t$ . In general, for  $t > 0$ ,  $\Lambda_\alpha^{(t)}(dz) := z^{-1}p_z(-tz)\mathbb{1}_{\{z>0\}}dz$  is the Lévy measure of a not killed pure jump subordinator and  $a_1^{(\alpha)}(t) > a_2^{(\alpha)}(t) > \dots$  is the ranked jump sizes of this subordinator before time  $t$ ; see [32]. We refer to [37] and [39, Section 8.1] for more information about the distribution of the jumps of a subordinator.

### 3 Stable Lévy processes, bridges and excursions

In this section, we recall several results about stable Lévy processes without negative jumps and refer the interesting reader to [7, Chapter VIII] or the work of Chaumont [18] for further details.

**Spectrally positive stable Lévy processes.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the underlying probability space. A strictly stable spectrally positive Lévy process with index  $\alpha \in (1, 2]$  is a random process  $X_\alpha = (X_\alpha(s), s \geq 0)$  with paths in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ , which has independent and stationary increments, no negative jumps and such that  $\mathbb{E}[\exp(-\lambda X_\alpha(s))] = \exp(cs\lambda^\alpha)$  for every  $s, \lambda \geq 0$ , and some constant  $c > 0$ . An important feature of  $X_\alpha$  is the so-called scaling property: for every real constant  $k > 0$ , the process  $(k^{-1/\alpha} X_\alpha(ks), s \geq 0)$  has the same distribution as  $X_\alpha$ . Then, in this work, we can and we will take  $c = 1$  if  $\alpha = (1, 2)$ , and  $c = 1/2$  if  $\alpha = 2$ , without loss of generality. In particular, for  $\alpha = 2$ , the process  $X_2$  is the standard Brownian motion on the positive real line.

**Stable bridge and stable normalized excursion.** The stable Lévy bridge  $X_\alpha^{\text{br}} = (X_\alpha^{\text{br}}(s), s \in [0, 1])$  is a random process with paths in  $\mathbb{D}([0, 1], \mathbb{R})$  that can informally be defined as the process  $X_\alpha$  conditioned to be at level 0 at time 1. This conditioning can be made rigorous and we refer to [18] for details. The normalized excursion  $X_\alpha^{\text{exc}} = (X_\alpha^{\text{exc}}(s), s \in [0, 1])$  of a spectrally positive  $\alpha$ -stable Lévy process with unit lifetime (or  $\alpha$ -stable excursion for simplicity) is a random process with paths in  $\mathbb{D}([0, 1], \mathbb{R})$  that can be thought as the process  $X_\alpha^{\text{br}}$  conditioned to stay nonnegative between times 0 and 1. Let us make this more precise and formally define the process  $X_\alpha^{\text{exc}}$ . We consider



the so-called Vervaat transform (or Vervaat excursion) introduced by Takács [41] and used by Vervaat [43] to change a bridge type function in  $\mathbb{D}([0, 1], \mathbb{R})$  into an excursion. More precisely, a bridge is a function  $g \in \mathbb{D}([0, 1], \mathbb{R})$  such that  $g(0) = g(1) = g(1-) = 0$ . For any  $g \in \mathbb{D}([0, 1], \mathbb{R})$ , we set  $\bar{\mu}(g) := \inf\{s \in [0, 1] : g(s-) \wedge g(s) = \inf_{u \in [0, 1]} g(u)\}$ , i.e., the smallest location of the infimum of  $g$ . Then, we define the Vervaat transform  $\mathbf{V}$  of a bridge  $g \in \mathbb{D}([0, 1], \mathbb{R})$  by

$$\mathbf{V}(g)(s) := \begin{cases} g(s + \bar{\mu}(g)) - \inf_{u \in [0, 1]} g(u) & \text{if } s \leq 1 - \bar{\mu}(g), \\ g(s + \bar{\mu}(g) - 1) - \inf_{u \in [0, 1]} g(u) & \text{if } s \geq 1 - \bar{\mu}(g). \end{cases}$$

Clearly,  $\mathbf{V}(g)$  is a path in  $\mathbb{D}([0, 1], \mathbb{R})$  which only takes nonnegative values and  $\mathbf{V}(g)(0) = \mathbf{V}(g)(1) = 0$ . It is well-known that a stable bridge  $X_\alpha^{\text{br}}$  satisfies  $X_\alpha^{\text{br}}(0) = X_\alpha^{\text{br}}(1) = X_\alpha^{\text{br}}(1-) = 0$ . Moreover,  $X_\alpha^{\text{br}}$  reaches its infimum at a unique random time that  $\bar{\mu}_\alpha := \bar{\mu}(X_\alpha^{\text{br}})$ ; see [18]. Thus, we formally define the  $\alpha$ -stable excursion as the Vervaat transform of the stable bridge  $X_\alpha^{\text{br}}$ , i.e.,  $X_\alpha^{\text{exc}} := \mathbf{V}(X_\alpha^{\text{br}})$ . We refer to the work of Chaumont [18] (see also [7, Chapter VIII]) for other constructions of the process  $X_\alpha^{\text{exc}}$  via path transformations, or alternatively, using arguments from excursion theory of Markov processes. A useful property (see [18, Theorem 4]) that one can deduce from the above construction is that

$$\bar{\mu}_\alpha \text{ and } X_\alpha^{\text{exc}} \text{ are independent and } \bar{\mu}_\alpha \text{ is uniformly distributed on } [0, 1]. \quad (6)$$

## 4 The coding of Galton-Watson trees and their fragmentation

In this section, we formally introduce the family of critical Galton-Watson trees and explain how they can be coded by different functions, namely the so-called Łukasiewicz path and a similar path derived by the Prim's algorithm. The latter provides an alternative order on the vertices of the tree, which we refer to as the Prim order. Following [15], we will see how the Prim's order of the vertices can be used to define a consistent exploration process of the fragmentation forest that stores all the information of the sizes of its connected components. Finally, we prove a distributional property for this exploration process that will be a crucial ingredient in the proof of Theorem 1.

**Plane trees.** We follow the formalism of Neveu [35]. Let  $\mathbb{N} = \{1, 2, \dots\}$  be the set of positive integers, set  $\mathbb{N}^0 = \{\emptyset\}$  and consider the set of labels  $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ . For  $u = (u_1, \dots, u_n) \in \mathbb{U}$ , we denote by  $|u| = n$  the length (or generation, or height) of  $u$ ; if  $v = (v_1, \dots, v_m) \in \mathbb{U}$ , we let  $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$  be the concatenation of  $u$  and  $v$ . A plane tree is a nonempty, finite subset  $\tau \subset \mathbb{U}$  such that: (i)  $\emptyset \in \tau$ ; (ii) if  $v \in \tau$  and  $v = uj$  for some  $j \in \mathbb{N}$ , then  $u \in \tau$ ; (iii) if  $u \in \tau$ , then there exists an integer  $c(u) \geq 0$  such that  $ui \in \tau$  if and only if  $1 \leq i \leq c(u)$ . We will view each vertex  $u$  of a tree  $\tau$  as an individual of a population whose  $\tau$  is the genealogical tree. The vertex  $\emptyset$  is called the root of the tree and for every  $u \in \tau$ ,  $c(u)$  is the number of children of  $u$  (if  $c(u) = 0$ , then  $u$  is called a leaf, otherwise,  $u$  is called an internal vertex). The total progeny (or size) of  $\tau$  will be denoted by  $\zeta(\tau) = \text{Card}(\tau)$  (i.e., the number of vertices of  $\tau$ ). We denote by  $\mathbb{T}$  the set of plane trees and for each  $n \in \mathbb{N}$ , by  $\mathbb{T}_n$  the set of plane trees with  $n$  vertices, or equivalently  $n - 1$  edges.

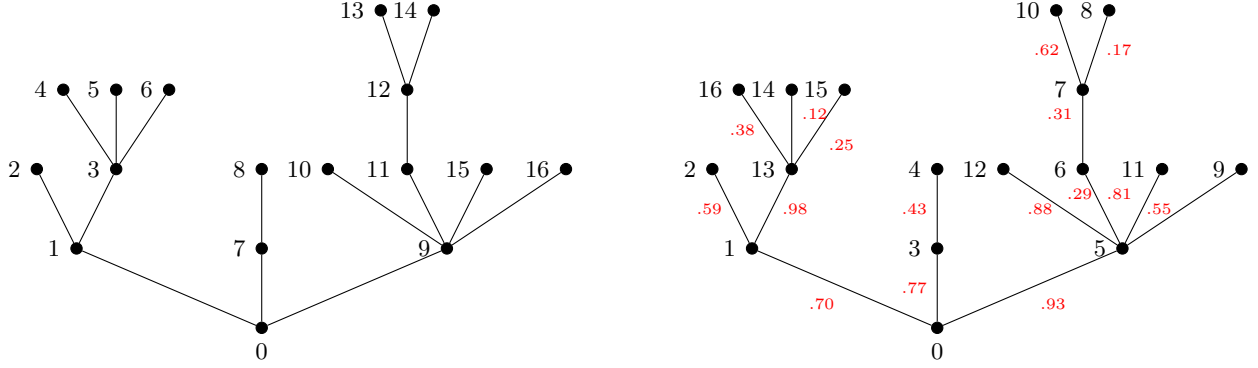
**Galton-Watson trees.** Let  $\mu$  be a probability measure on  $\mathbb{Z}_+$  which satisfies  $\mu(0) > 0$ , expectation  $\sum_{k=0}^{\infty} k\mu(k) = 1$  and such that  $\mu(0) + \mu(1) < 1$ . The law of a critical Galton–Watson tree with offspring distribution  $\mu$  is the unique probability measure  $\mathbb{P}_\mu$  on  $\mathbb{T}$  satisfying: (i)  $\mathbb{P}_\mu(c(\emptyset) = k) = \mu(k)$  for every  $k \geq 0$ ; (ii) For every  $k \geq 1$  such that  $\mu(k) > 0$ , conditioned on the event  $\{c(\emptyset) = k\}$ , the subtrees that stem from the children of the root  $\{u \in \mathbb{U} : 1u \in \tau\}, \dots, \{u \in \mathbb{U} : ku \in \tau\}$  are independent and distributed as  $\mathbb{P}_\mu$ . Otter [36] shows that the law  $\mathbb{P}_\mu$  is given by the explicit formula  $\mathbb{P}_\mu(\tau) = \prod_{u \in \tau} \mu(c(u))$ . A random tree whose distribution is  $\mathbb{P}_\mu$  will be called a Galton–Watson tree with offspring distribution  $\mu$ . We also denote by  $\mathbb{P}_\mu^{(n)}$  the law on  $\mathbb{T}_n$  of a Galton–Watson tree with offspring distribution  $\mu$  conditioned to have  $n$  vertices, providing that this conditioning makes sense.

**Coding planar trees by discrete paths.** In this work, we will use two different orderings of the vertices of a tree  $\tau \in \mathbb{T}$ :

- (i) **Lexicographical ordering.** Given  $v, w \in \tau$ , we write  $v \prec_{\text{lex}} w$  if there exists  $z \in \tau$  such that  $v = z(v_1, \dots, v_n)$ ,  $w = z(w_1, \dots, w_m)$  and  $v_1 < w_1$ .
- (ii) **Prim ordering.** Let  $\mathbf{edge}(\tau)$  be the set of edges of  $\tau$  and consider a sequence of distinct and positive weights  $\mathbf{w} = (w_e : e \in \mathbf{edge}(\tau))$  (i.e., each edge  $e$  of  $\tau$  is marked with a different and positive weight  $w_e$ ). Given two distinct vertices  $u, v \in \tau$ , we write  $\{u, v\}$  for the edge connecting  $u$  and  $v$  in  $\tau$ . Let us describe the Prim order  $\prec_{\text{prim}}$  of the vertices in  $\tau$ , that is,  $\emptyset = u(0) \prec_{\text{prim}} u(1) \prec_{\text{prim}} \dots \prec_{\text{prim}} u(\zeta(\tau) - 1)$ . We will use the notation  $V_i$  for the set  $\{u(0), \dots, u(i - 1)\}$ , for  $0 \leq i \leq \zeta(\tau)$ . First set  $u(0) = \emptyset$  and  $V_0 = \{u(0)\}$ . Suppose that for some  $0 \leq i \leq \zeta(\tau) - 1$ , the vertices  $u(0), \dots, u(i - 1)$  have been defined. Consider the weights  $\{w_{\{u,v\}} : u \in V_i, v \notin V_i\}$  of edges between a vertex of  $V_i$  and another outside of  $V_i$ . Since all the weights are distinct, the minimum weight in  $\{w_{\{u,v\}} : u \in V_i, v \notin V_i\}$  is reached at an edge  $\{\tilde{u}, \tilde{v}\}$  where  $\tilde{u} \in V_i$  and  $\tilde{v} \notin V_i$ . Then set  $u(i) = \tilde{v}$ . This iterative procedure completely determines the Prim order  $\prec_{\text{prim}}$ .

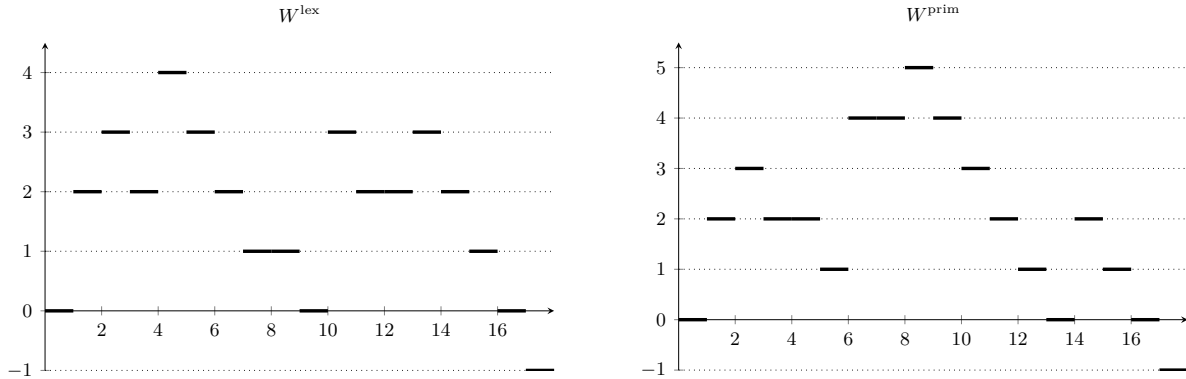
For  $*$   $\in$   $\{\text{lex}, \text{prim}\}$ , we associate to every ordering  $\emptyset = u(0) \prec_* u(1) \prec_* \dots \prec_* u(\zeta(\tau) - 1)$  of the vertices of  $\tau$  a path  $W^* = (W^*(k), 0 \leq k \leq \zeta(\tau))$ , by letting  $W^*(0) = 0$  and for  $0 \leq k \leq \zeta(\tau) - 1$ ,  $W^*(k + 1) = W^*(k) + c(u(k)) - 1$ , where we recall that  $c(u(k))$  denotes the number of children of the vertex  $u(k) \in \tau$ . Observe that  $W^*(k + 1) - W^*(k) = c(u(k)) - 1 \geq -1$  for every  $0 \leq k \leq \zeta(\tau) - 1$ , with equality if and only if  $u(k)$  is a leaf of  $\tau$ . Note also that  $W^*(k) \geq 0$ , for every  $0 \leq k \leq \zeta(\tau) - 1$ , but  $W^*(\zeta(\tau)) = -1$ . We shall think of such a path as the step function on  $[0, \zeta(\tau)]$  given  $s \mapsto W^*([s])$ . The path  $W^{\text{lex}}$  is commonly called Łukasiewicz path of  $\tau$ , and from now on we refer to  $W^{\text{prim}}$  as the Prim path; see Figure 2. See [28] for more details and properties on the Łukasiewicz path.

The procedure just described to obtain the Prim ordering is known as Prim’s algorithm (or Prim–Jarník algorithm); see [40]. This algorithm associates to any properly weighted graph its unique minimum spanning tree. In practice, one could also consider that  $\mathbf{w}$  is a sequence of i.i.d. positive random variables such that they are all distinct a.s. and independent of the tree. See Figure 1 for an illustration of the previous orderings of the vertices in a tree.



**Figure 1:** From left to right, a plane tree with vertices labeled in lexicographical order and a weighted plane tree with vertices labeled in Prim order.

Define the probability measure  $\hat{\mu}$  on  $\{-1, 0, 1, \dots\}$  by  $\hat{\mu}(k) = \mu(k + 1)$  for every  $k \geq -1$ . Let  $X = (X(k), k \geq 0)$  be a random walk which starts at 0 with jump distribution  $\hat{\mu}$  and define also the time  $\zeta_1 = \inf\{k \geq 0 : X(k) = -1\}$ . In the Prim ordering, consider that the weights  $\mathbf{w}$  is a sequence of i.i.d. positive random variables such that they are distinct a.s. and independent of the tree.



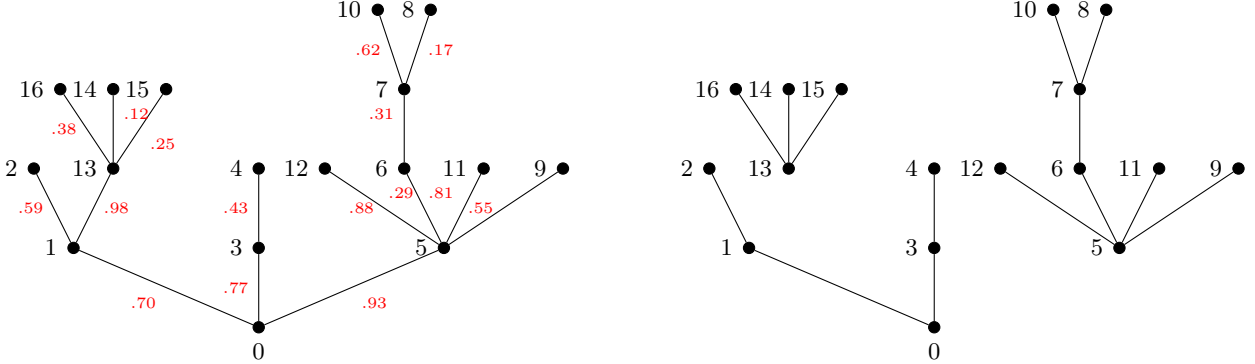
**Figure 2:** In the left, the Łukasiewicz path of the plane tree in Figure 1. In the right, the Prim path of the plane tree in Figure 1

**Proposition 2.** *For every  $* \in \{\text{lex}, \text{prim}\}$ , if we sample a plane tree according to  $\mathbb{P}_\mu$ , then  $W^*$  is distributed as  $(X(0), X(1), \dots, X(\zeta_1))$ . In particular, the total progeny of the sample plane tree has the same distribution as  $\zeta_1$ .*

*Proof.* The proof for the Łukasiewicz path can be found in [28, Proposition 1.5]. For the Prim path the proof follows from a simple adaptation of that of [28, Proposition 1.5]; see also [15, Lemmas 15 and 16] for an alternative approach.  $\square$

**Fragmentation of a plane tree.** Consider  $\tau \in \mathbb{T}$  and let  $\mathbf{edge}(\tau)$  denote its set of edges. Equip the edges of  $\tau$  with i.i.d. uniform random variables (or weights)  $\mathbf{w} = (w_e : e \in \mathbf{edge}(\tau))$  on  $[0, 1]$  and independently of the tree  $\tau$ . In particular, for a vertex  $v \in \tau$  with  $c(v) \geq 1$  children, we write

$(w_{v,k}, 1 \leq k \leq c(v))$  for the weights of the edges connecting  $v$  with its children. For  $t \in [0, 1]$ , we then keep the edges of  $\tau$  with weight smaller than  $t$  and discard the others. This gives rise to a forest  $\mathbf{f}_\tau(t)$  with the same set of vertices as  $\tau$  but with set of edges given by  $\mathbf{edge}(\mathbf{f}_\tau(t)) = \{e \in \mathbf{edge}(\tau) : w_e \leq t\}$ . Furthermore, each vertex  $v \in \mathbf{f}_\tau(t)$  has  $c_t(v) = \sum_{k=1}^{c(v)} \mathbb{1}_{\{w_{v,k} \leq t\}}$  children if  $c(v) \geq 1$ ; otherwise,  $c_t(v) = 0$  whenever  $c(v) = 0$ . In what follows, we refer to the forest  $\mathbf{f}_\tau(t)$  associated to a plane tree  $\tau$  and uniform weights  $\mathbf{w}$  as *the fragmented forest* at time  $t \in [0, 1]$ , or simply, *fragmentation forest*; see Figure 3. In this work we restrict ourselves to the case uniform i.i.d. weights, but certainly some of the forthcoming results can be extended for more general sequences of weights.



**Figure 3:** A plane tree with uniform random weights in the left side. In the right side, the forest created by keeping the edges with weight at most  $t = .92$ . The vertices are labelled according to the Prim ordering.

**Prim exploration of the fragmentation forest.** For a plane tree  $\tau \in \mathbb{T}$  and sequence of i.i.d. uniform random weights  $\mathbf{w}$  on  $[0, 1]$ , let  $\mathbf{f}_\tau(t)$  be the fragmentation forest of  $\tau$  at time  $t \in [0, 1]$ . Let us now explain how to explore the subtree components of the forest  $\mathbf{f}_\tau(t)$  by using the approach outlined in [15, page 532] (see also [4]). For  $t \in [0, 1]$ , denote by  $\mathbf{Neigh}_t(v) := \{u \in \mathbf{f}_\tau(t) : \{u, v\} \in \mathbf{edge}(\mathbf{f}_\tau(t))\}$  the set of neighbours of  $v \in \mathbf{f}_\tau(t)$ . For a set of vertices  $V$  of  $\mathbf{f}_\tau(t)$ , let also  $\mathbf{Neigh}_t(V) := (\bigcup_{v \in V} \mathbf{Neigh}_t(v)) \setminus V$ , the set of neighbours of vertices in  $V$  but not in  $V$ . We associate to the prim ordering  $\emptyset = u(0) \prec_{\text{prim}} u(1) \prec_{\text{prim}} \dots \prec_{\text{prim}} u(\zeta(\tau) - 1)$  of the vertices of  $\tau$  the following exploration process of  $\mathbf{f}_\tau(t)$  (recall that  $\mathbf{f}_\tau(t)$  and  $\tau$  have the same set of vertices). The first visited vertex is  $v_t(0) = u(0)$ . Suppose that we have explored the vertices  $V_k = \{v_t(0), \dots, v_t(k-1)\}$  at some time  $1 \leq k \leq \zeta(\tau)$ . If  $k = \zeta(\tau)$ , we have finished the exploration, and otherwise, one has two possibilities:

- (i) if  $\mathbf{Neigh}_t(V_k) \neq \emptyset$ , then  $v_t(k)$  is the next vertex according to the order  $\prec_{\text{prim}}$  that belongs to  $\mathbf{Neigh}_t(V_k)$ , or
- (ii) if  $\mathbf{Neigh}_t(V_k) = \emptyset$ , then  $v_t(k)$  is the next vertex according to the order  $\prec_{\text{prim}}$  that belongs to  $\tau \setminus V_k$ .

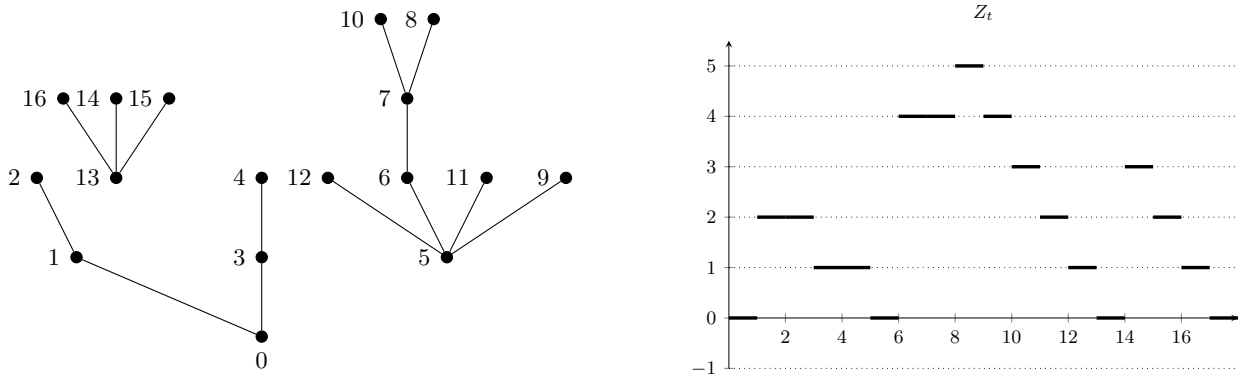
This exploration process results in an order for the vertices of  $\mathbf{f}_\tau(t)$  (equivalently, to the vertices of  $\tau$ ) that we denote by  $\prec_{\text{prim}}$  (i.e.  $\emptyset = v_t(0) \prec_{\text{prim}} v_t(1) \prec_{\text{prim}} \dots \prec_{\text{prim}} v_t(\zeta(\tau) - 1)$ ) and call *Prim*

*exploration.* An important feature of the Prim exploration of  $\mathbf{f}_\tau(t)$  is that the Prim ordering  $<_{\text{prim}}$  of its vertices is preserved for all values of  $t \in [0, 1]$ . More precisely, for  $t_1, t_2 \in [0, 1]$ ,  $v_{t_1}(k) = v_{t_2}(k)$ , for all  $0 \leq k \leq \zeta(\tau) - 1$ ; see Figure 3 for an example when  $t_1 = 1$  and  $t_2 = .92$ . This is a consequence of the algorithm to obtain the Prim ordering of the vertices in  $\tau$  which associates to any properly weighted graph its unique minimum spanning tree. We henceforth write  $\prec_{\text{prim}}$  instead of  $<_{\text{prim}}$  and remove the subindex  $t$  from our notation, i.e., we write  $\emptyset = v(0) \prec_{\text{prim}} v(1) \prec_{\text{prim}} \cdots \prec_{\text{prim}} v(\zeta(\tau) - 1)$  for the vertices of  $\mathbf{f}_\tau(t)$  in Prim order, which is the same as the Prim ordering of the vertices of the tree  $\tau$ ,  $\emptyset = u(0) \prec_{\text{prim}} u(1) \prec_{\text{prim}} \cdots \prec_{\text{prim}} u(\zeta(\tau) - 1)$  presented earlier.

Following the presentation of [15, pages 532-533], one can associate to the Prim ordering of the vertices of  $\mathbf{f}_\tau(t)$ , an *exploration path*  $Z_t = (Z_t(k), 0 \leq k \leq \zeta(\tau) + 1)$  by letting  $Z_t(0) = Z_t(\zeta(\tau) + 1) = 0$ , and for  $1 \leq k \leq \zeta(\tau)$ ,  $Z_t(k) = \text{Card}(\mathbf{Neigh}_t(V_{k-1}))$ . Furthermore, let  $\mathbf{CC}(\mathbf{f}_\tau(t))$  be the set of connected components of  $\mathbf{f}_\tau(t)$ . Then [15, Lemma 14] shows that

$$\text{Card}(\{k \in \{1, \dots, \zeta(\tau)\} : Z_t(k) = 0\}) = \text{Card}(\mathbf{CC}(\mathbf{f}_\tau(t))),$$

and that the successive sizes of the connected components ordered by the exploration coincide with the distances between successive 0's in the sequence  $Z_t = (Z_t(k), 0 \leq k \leq \zeta(\tau) + 1)$ ; see Figure 4.



**Figure 4:** In the left side, the forest of Figure 3. In the right side, its exploration path  $Z_t$ . The vertices are labelled according to the Prim ordering.

In this work, and in analogy with the coding paths of  $\tau$  introduced earlier, we will consider a slight modification of the exploration path  $Z_t$ . More precisely, define the Prim path  $W_t^{\text{prim}} = (W_t^{\text{prim}}(k), 0 \leq k \leq \zeta(\tau))$  by letting  $W_t^{\text{prim}}(0) = 0$ , and for  $0 \leq k \leq \zeta(\tau) - 1$ ,  $W_t^{\text{prim}}(k + 1) = W_t^{\text{prim}}(k) + c_t(v_t(k)) - 1$ , where  $c_t(v)$  denotes the number of children of  $v \in \mathbf{f}_\tau(t)$ . We shall also think of such a path as the step function on  $[0, \zeta(\tau)]$  given by  $s \mapsto W_t^{\text{prim}}(\lfloor s \rfloor)$ .

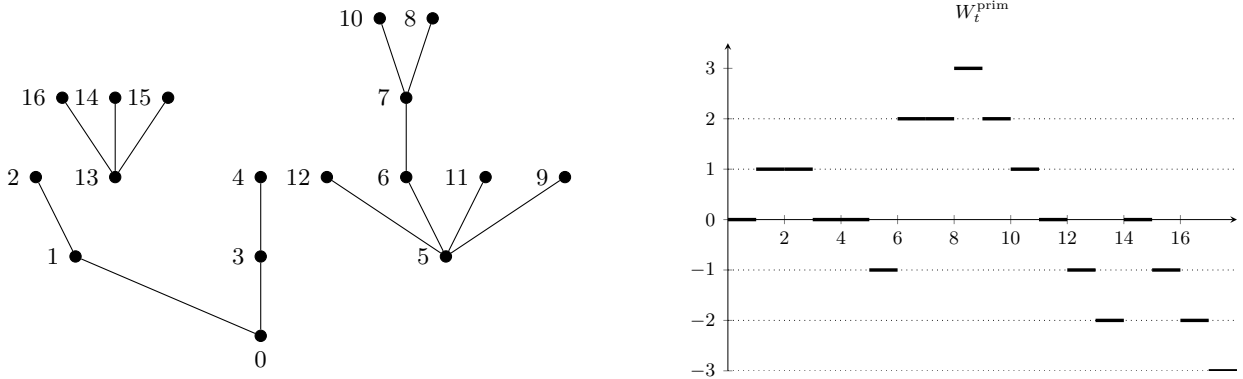
**Lemma 1.** *Let  $\tau \in \mathbb{T}$  and  $\mathbf{w}$  be a sequence of i.i.d. uniform random weights on  $[0, 1]$  which is also independent of  $\tau$ . For any time  $t \in [0, 1]$ ,*

$$\text{Card} \left( \left\{ k \in \{1, \dots, \zeta(\tau)\} : W_t^{\text{prim}}(k) = \min_{0 \leq m \leq k} W_t^{\text{prim}}(m) \right\} \right) = \text{Card}(\mathbf{CC}(\mathbf{f}_\tau(t))),$$

Moreover, the successive sizes of the connected components of  $\mathbf{f}_\tau(t)$  ordered by the exploration process coincide with the distances between successive new minimums in the sequence  $(W_t^{\text{prim}}(k), 0 \leq k \leq \zeta(\tau))$ .

*Proof.* The result is an immediate consequence of the previous discussion.  $\square$

Indeed, the sizes of the connected components of  $\mathbf{f}_\tau(t)$  coincides with the length of the excursions of the walk  $W_t^{\text{prim}}$  above its minimum; see Figure 5.



**Figure 5:** In the left side, the forest of Figure 3 with vertices labelled according to the Prime ordering. In the right side, its Prim path  $W_t^{\text{prim}}$ .

Following Proposition 2, the Prim path of the fragmentation forest associated to a critical Galton-Watson tree with offspring distribution  $\mu$  can also be related to a random walk. Recall that  $X = (X(k), k \geq 0)$  denotes a random walk that starts at 0 and has jump distribution  $\hat{\mu}$  on  $\{-1, 0, 1, \dots\}$ . Recall also that we write  $\zeta_1 = \inf\{k \geq 0 : X(k) = -1\}$ . Denote by  $\xi = (\xi(k), k \geq 1)$  the increments of  $X$ , i.e.  $\xi(k) = X(k) - X(k-1)$ , for  $k \geq 1$ . Let  $(U_k(j))_{k,j \geq 1}$  be a sequence of i.i.d. uniform random variables on  $[0, 1]$ . For  $t \in [0, 1]$ , define  $\xi_t = (\xi_t(k), k \geq 1)$  by letting

$$\xi_t(k) = \sum_{j=1}^{\xi(k)+1} \mathbb{1}_{\{U_k(j) \leq t\}}, \quad \text{for } t \in [0, 1], \quad k \geq 1,$$

with the convention  $\sum_{j=1}^0 \mathbb{1}_{\{U_k(j) \leq t\}} = 0$ . Hence,  $\xi_0(k) = 0$ ,  $\xi_1(k) = \xi(k) + 1$  and for any  $k \geq 1$ , the mapping  $t \mapsto \xi_t(k)$  is non-decreasing. Let  $X_t = (X_t(k), k \geq 0)$  be the process defined by

$$X_t(0) = 0 \quad \text{and} \quad X_t(k) = \sum_{i=1}^k (\xi_t(i) - 1), \quad \text{for } t \in [0, 1], \quad k \geq 1. \quad (7)$$

**Proposition 3.** *Sample a plane tree  $\mathbf{t}$  according to  $\mathbb{P}_\mu$ , i.e., consider a critical Galton-Watson tree  $\mathbf{t}$  with offspring  $\mu$ . Let  $\mathbf{w} = (w_e : e \in \text{edge}(\mathbf{t}))$  be a sequence of i.i.d. uniform random weights on  $[0, 1]$  which is also independent of  $\mathbf{t}$ . Then, the Prim path  $W_t^{\text{prim}}$  satisfies*

$$(W_t^{\text{prim}}(0), W_t^{\text{prim}}(1), \dots, W_t^{\text{prim}}(\zeta(\mathbf{t})))_{t \in [0, 1]} \stackrel{d}{=} (X_t(0), X_t(1), \dots, X_t(\zeta_1))_{t \in [0, 1]},$$

where  $\stackrel{d}{=}$  means equal in distribution (in the sense of finite-dimensional distributions).

*Proof.* For  $t \in [0, 1]$ , we write  $V(0) = \emptyset, V(1), \dots, V(\zeta(\mathbf{t}) - 1)$  for the vertices of  $\mathbf{f}_t(t)$  listed in Prim order. To simplify the notation, for  $t \in [0, 1]$  and  $k = 0, \dots, \zeta(\tau) - 1$ , we will write  $c_t(V(k)) = c_t(k)$  for the number of children of the vertex  $V(k)$  in  $\mathbf{f}_t(t)$ . Recall that  $\mathbf{f}_t(1) = \mathbf{t}$  and  $c_1(V(k)) = c(V(k))$ . In particular, we will write  $c(k) = c(V(k))$ . To prove our claim, it is enough to check that

$$(c_t(0), c_t(1), \dots, c_t(\zeta(\tau) - 1))_{t \in [0, 1]} \stackrel{d}{=} (\xi_t(1), \dots, \xi_t(\zeta_1))_{t \in [0, 1]},$$

in the sense of finite-dimensional distributions.

Consider the infinite tree  $\mathbb{U}$  and denote by  $\mathbf{edge}(\mathbb{U})$  its set of edges. Denote by  $\mathbf{Neigh}(v_0) := \{u \in \mathbb{U} : \{u, v_0\} \in \mathbf{edge}(\mathbb{U})\}$  the set of neighbours of  $v_0 \in \mathbb{U}$ . For  $r \in \mathbb{N}$  and a set of vertices  $S_r := \{v_0, v_1, \dots, v_{r-1}\}$  of  $\mathbb{U}$ , we also write  $\mathbf{Neigh}(S_r) := (\bigcup_{v \in S_r} \mathbf{Neigh}(v)) \setminus S_r$  for the set of neighbours of vertices in  $S_r$  but not in  $S_r$ . For  $v_0 = \emptyset, v_1 \in \mathbf{Neigh}(v_0), \dots, v_r \in \mathbf{Neigh}(\{v_0, v_1, \dots, v_{r-1}\})$ , define the event

$$\mathbf{N}(v_0, v_1, \dots, v_{r-1}) := \{V(0) = v_0, V(1) = v_1, \dots, V(\zeta(\mathbf{t}) - 1) = v_r\} \cap \{\zeta(\mathbf{t}) = r\}.$$

For  $r \in \mathbb{N}$  and  $k_0, k_1, \dots, k_{r-1} \in \mathbb{N} \cup \{0\}$ , we also define the event

$$\mathbf{C}(k_0, k_1, \dots, k_{r-1}) := \{c(0) = k_0, c(1) = k_1, \dots, c(\zeta(\mathbf{t}) - 1) = k_r\} \cap \{\zeta(\mathbf{t}) = r\}.$$

For simplicity, given a measurable set  $A$ , we write  $\mathbb{E}[\cdot; A] = \mathbb{E}[\cdot \mathbf{1}_A]$ , and given a finite collection of measurable sets  $A_1, \dots, A_i$ , we shall write  $\mathbb{E}[\cdot; A_1, \dots, A_i] = \mathbb{E}[\cdot \mathbf{1}_{A_1 \cap \dots \cap A_i}]$ , for  $i \in \mathbb{N}$ . For fixed  $n \in \mathbb{N}$ , we set  $0 \leq t_1 \leq \dots \leq t_n \leq 1$ , and for  $i = 1, 2, \dots, n$  and  $r \in \mathbb{N}$ , consider  $g_0^i, g_1^i, \dots, g_{r-1}^i$  nonnegative functions on  $\{0, 1, \dots\}$ . Hence,

$$\begin{aligned} & \mathbb{E} \left[ \prod_{i=1}^n g_0^i(c_{t_i}(0)) g_1^i(c_{t_i}(1)) \cdots g_{r-1}^i(c_{t_i}(\zeta(\mathbf{t}) - 1)); \mathbf{N}(v_0, v_1, \dots, v_{r-1}), \mathbf{C}(k_0, k_1, \dots, k_{r-1}), \zeta(\mathbf{t}) = r \right] \\ &= \mathbb{E} \left[ \prod_{i=1}^n g_0^i(c_{t_i}(v_0)) g_1^i(c_{t_i}(v_1)) \cdots g_{r-1}^i(c_{t_i}(v_{r-1})) ; \mathbf{N}(v_0, v_1, \dots, v_{r-1}), \mathbf{C}(k_0, k_1, \dots, k_{r-1}), \zeta(\mathbf{t}) = r \right]. \end{aligned}$$

For  $t \in [0, 1]$  and  $p = 0, \dots, \zeta(\tau) - 1$ , recall that if  $c(p) \geq 1$ , then  $c_t(p) = \sum_{i=1}^{c(p)} \mathbf{1}_{\{w_{V(p), i} \leq t\}}$ . Otherwise,  $c_t(p) = 0$  whenever  $c(p) = 0$ . Then, in the event  $\mathbf{N}(v_0, v_1, \dots, v_{r-1}) \cap \mathbf{C}(k_0, k_1, \dots, k_{r-1}) \cap \{\zeta(\mathbf{t}) = r\}$ , we have that  $c_t(v_p) = \sum_{i=1}^{k_p} \mathbf{1}_{\{w_{v_p, i} \leq t\}}$ ; with the convention that the sum is equal to zero if it is empty. Define the random variables,  $\kappa_t(p) = \sum_{j=1}^{k_p} \mathbf{1}_{\{U_{p+1}(j) \leq t\}}$ ; with the convention that the sum is equal to zero whenever it is empty. Since the weights  $\mathbf{w}$  are independent of the tree, we see that

$$\begin{aligned} & \mathbb{E} \left[ \prod_{i=1}^n g_0^i(c_{t_i}(0)) g_1^i(c_{t_i}(1)) \cdots g_{r-1}^i(c_{t_i}(\zeta(\mathbf{t}) - 1)); \mathbf{N}(v_0, v_1, \dots, v_{r-1}), \mathbf{C}(k_0, k_1, \dots, k_{r-1}), \zeta(\mathbf{t}) = r \right] \\ &= \prod_{p=0}^{r-1} \mathbb{E} \left[ g_p^1(\kappa_{t_1}(p)) \cdots g_p^n(\kappa_{t_n}(p)) \right] \mathbb{P}(\mathbf{N}(v_0, v_1, \dots, v_{r-1}) \cap \mathbf{C}(k_0, k_1, \dots, k_{r-1}) \cap \zeta(\mathbf{t}) = r). \end{aligned}$$

Therefore, by summing over all possible,  $k_p$ 's and  $v_p$ 's, Proposition 2 implies that

$$\mathbb{E} \left[ \prod_{i=1}^n g_0^i(c_{t_i}(0)) g_1^i(c_{t_i}(1)) \cdots g_{r-1}^i(c_{t_i}(r-1)); \zeta(\mathbf{t}) = r \right] = \prod_{k=0}^{r-1} \mathbb{E} \left[ g_k^1(\xi_{t_1}(k+1)) \cdots g_k^n(\xi_{t_n}(k+1)); \zeta_1 = r \right],$$

which concludes our proof.  $\square$

## 5 Convergence of the exploration processes

Recall that  $\mathbb{P}_\mu^{(n)}$  denotes the law of a critical Galton-Watson tree with offspring distribution  $\mu$  conditioned to have  $n \in \mathbb{N}$  vertices. For every  $n \in \mathbb{N}$ , for which  $\mathbb{P}_\mu^{(n)}$  is well-defined, sample a plane tree on  $\mathbb{T}_n$ , say  $\mathbf{t}_n$ , according to  $\mathbb{P}_\mu^{(n)}$ , i.e.,  $\mathbf{t}_n$  is a critical Galton-Watson tree conditioned to have  $n$  vertices. Through this section we assume that  $\mu$  belongs to the domain of attraction of a stable law of index  $\alpha \in (1, 2]$ , and refer to  $\mathbf{t}_n$  as an  $\alpha$ -stable GW-tree. We will always let  $\mathbf{w} = (w_e : e \in \text{edge}(\mathbf{t}_n))$  be a sequence of i.i.d. uniform random weights on  $[0, 1]$  which is also independent of  $\mathbf{t}_n$ . We write  $W_n^{\text{lex}} = (W_n^{\text{lex}}(\lfloor nu \rfloor), u \in [0, 1])$  for the associated time-scaled Łukasiewicz path of  $\mathbf{t}_n$ . We also write  $W_n^{\text{prim}} = (W_n^{\text{prim}}(\lfloor nu \rfloor), u \in [0, 1])$  for the time-scaled Prim path of  $\mathbf{t}_n$  with respect to the weights  $\mathbf{w}$ .

The asymptotic behavior of large  $\alpha$ -stable GW-trees is well understood, in particular through scaling limits of their associated Łukasiewicz paths; see, e.g., [19]. In this section, we first show that the Prim path of  $\mathbf{t}_n$  has the same asymptotic behavior as its Łukasiewicz path. Then, we use this as a stepping stone to study the Prim path of the fragmentation forest of  $\mathbf{t}_n$  associated to the weights  $\mathbf{w}$ . Recall that  $X_\alpha^{\text{exc}} = (X_\alpha^{\text{exc}}(u), u \in [0, 1])$  denotes the  $\alpha$ -stable excursion of index  $\alpha$ ; see Section 3.

**Theorem 2.** *Let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree, and let  $(B_n)_{n \geq 1}$  be a sequence of positive real numbers satisfying (2). For  $*$   $\in$   $\{\text{lex}, \text{prim}\}$ , we have that*

$$\left( \frac{1}{B_n} W_n^*(\lfloor nu \rfloor), u \in [0, 1] \right) \xrightarrow{d} (X_\alpha^{\text{exc}}(u), u \in [0, 1]), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{D}([0, 1], \mathbb{R}).$$

*Proof.* The proof for the Łukasiewicz path can be found in [19, Theorem 3.1]. For the Prim path the result follows from that of the Łukasiewicz path and Proposition 2.  $\square$

For  $s \in [0, 1]$ , let  $\mathbf{f}_n(s)$  be the fragmentation forest of  $\mathbf{t}_n$  at time  $s$ . Denote by  $W_{n,s}^{\text{prim}} = (W_{n,s}^{\text{prim}}(\lfloor nu \rfloor), u \in [0, 1])$  the time-scaled Prim path of  $\mathbf{f}_n(s)$ . In particular,  $W_{n,1}^{\text{prim}}$  is exactly  $W_n^{\text{prim}}$ . For fixed  $t \geq 0$ , consider the sequence  $(s_n(t))_{n \geq 1}$  of positive times given by

$$s_n(t) = 1 - \frac{B_n}{n} t,$$

where  $(B_n)_{n \geq 1}$  a sequence of positive real numbers satisfying (2). Define the process  $W_n^{(t)} = (W_n^{(t)}(u), u \in [0, 1])$  by letting

$$W_n^{(t)}(u) = \frac{1}{B_n} W_{n,s_n(t)}^{\text{prim}}(\lfloor nu \rfloor), \quad \text{for } u \in [0, 1]. \quad (8)$$



Later, in the proof of Theorem 1, we will refer to the process  $W_n^{(t)}$  as the (normalized and time-scaled) Prim path of the fragmentation forest at time  $s_n(t)$ , i.e.,  $\mathbf{f}(s_n(t))$ . We then set  $W_n = (W_n^{(t)}, t \geq 0)$ . From the previous section, the mapping  $t \mapsto W_n^{(t)}$  is non-increasing in  $t$  which implies that the process  $W_n$  has càdlàg paths. Thus, we will view  $(t, u) \mapsto W_n^{(t)}(u)$  as a random variable taking values in the space  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$  of  $\mathbb{D}([0, 1], \mathbb{R})$ -valued càdlàg functions on  $\mathbb{R}$  equipped with the Skorokhod topology. In other words, for fixed  $t \geq 0$ ,  $W_n^{(t)}$  is a random variable in  $\mathbb{D}([0, 1], \mathbb{R})$ .

We introduce the continuous counterpart of the process  $W_n$ . For every  $t \geq 0$ , let  $Y_\alpha^{(t)} = (Y_\alpha^{(t)}(u), u \in [0, 1])$  be defined by  $Y_\alpha^{(t)}(u) = X_\alpha^{\text{exc}}(u) - tu$ , for  $u \in [0, 1]$ . In particular, for  $t = 0$ ,  $Y_\alpha^{(0)} = X_\alpha^{\text{exc}}$  and we sometimes write  $X_\alpha^{\text{exc}}$  instead of  $Y_\alpha^{(0)}$ , for simplicity. Then, define the process  $Y_\alpha = (Y_\alpha^{(t)}, t \geq 0)$ .

The following theorem is the main result of this section.

**Theorem 3.** *We have the convergence*

$$(W_n^{(t)}, t \geq 0) \xrightarrow{d} (Y_\alpha^{(t)}, t \geq 0), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R})).$$

Theorem 3 generalizes [15, Theorem 10]. Specifically, in [15], the authors only consider the case when  $\mathbf{t}_n$  is a GW-tree with  $\mu$  being the law of a Poisson random variable of parameter 1 (i.e.,  $\mathbf{t}_n$  is a Cayley tree) while our setting is clearly more general. As in most proofs for convergence of stochastic processes, the proof of Theorem 3 consists in two steps: convergence of the finite-dimensional distributions and tightness of the sequence of processes  $(W_n)_{n \geq 1}$ . To accomplish the above, recall the random walk connected to the Prim path of the fragmentation forest of the  $\alpha$ -stable GW-tree  $\mathbf{t}_n$  (Proposition 3). More precisely, for  $s \in [0, 1]$ , let  $X_s = (X_s(k), k \geq 0)$  be the stochastic process defined in (7). For  $n \in \mathbb{N}$  and  $t \geq 0$ , define the process  $Y_n^{(t)} = (Y_n^{(t)}(u), u \in [0, 1])$  by letting

$$Y_n^{(t)}(u) = \frac{1}{B_n} Y_{s_n(t)}(\lfloor nu \rfloor), \quad \text{for } u \in [0, 1],$$

and set  $Y_n = (Y_n^{(t)}, t \geq 0)$ . From Proposition 3, we see that  $W_n$  has the same finite-dimensional distribution as  $Y_n$  under the conditional probability distribution  $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot | \zeta_1 = n)$ . In the following, we will always work with the process  $Y_n$  (or  $Y_n^{(t)}$ ) under the conditional probability distribution  $\mathbb{P}_n$ , and to keep the notation simple, we will continue to write  $Y_n$  (and  $Y_n^{(t)}$ ) also for the conditional version.

**Finite-dimensional distributions.** We start with two observations that will be used quite often. Proposition 2 and Theorem 2 imply that

$$\left( Y_n^{(0)}(u), u \in [0, 1] \right) \xrightarrow{d} \left( X_\alpha^{\text{exc}}(u), u \in [0, 1] \right), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{D}([0, 1], \mathbb{R}). \quad (9)$$

For  $g \in \mathbb{D}([0, 1], \mathbb{R})$ , we write  $\|g\|_\infty := \sup_{u \in [0, 1]} |g(u)|$ . Since the supremum is a continuous functional on  $\mathbb{D}([0, 1], \mathbb{R})$  (see e.g. [25, Proposition 2.4 in Chapter VI]), (9) implies that

$$\|W_n^{(0)}\|_\infty \xrightarrow{d} \|X_\alpha^{\text{exc}}\|_\infty, \quad \text{as } n \rightarrow \infty, \quad \text{in distribution and } \|X_\alpha^{\text{exc}}\|_\infty < \infty \text{ a.s.} \quad (10)$$

We continue with the convergence of the finite-dimensional distributions.

**Lemma 2.** For  $k, m \in \mathbb{N}$ , and for any  $u_1, \dots, u_k \in [0, 1]$  and  $t_1, \dots, t_m \in \mathbb{R}_+$ , we have that

$$\left( Y_n^{(t_i)}(u_r) : 0 \leq r \leq k, 0 \leq i \leq m \right) \xrightarrow{d} \left( Y_\alpha^{(t_i)}(u_r) : 0 \leq r \leq k, 0 \leq i \leq m \right), \text{ as } n \rightarrow \infty.$$

*Proof.* By the Skorokhod representation theorem, we can assume that (9) and (10) hold almost surely. For  $u \in [0, 1]$  and  $t \geq 0$ , we have that

$$W_n^{(t)}(u) = \frac{1}{B_n} \sum_{k=1}^{\lfloor nu \rfloor} (\xi_{s_n(t)}(k) - 1) = \frac{1}{B_n} \sum_{k=1}^{\lfloor nu \rfloor} \left( -1 + \sum_{i=1}^{\xi(k)+1} \mathbb{1}_{\{U_k(i) \leq 1 - tB_n/n\}} \right).$$

Since  $\sum_{k=1}^{\lfloor nu \rfloor} (\xi(k) + 1) = \lfloor nu \rfloor + B_n Y_n^{(0)}(u)$ , we see that

$$Y_n^{(t)}(u) = S_n^{(t)}(u) - \frac{1}{B_n} \lfloor nu \rfloor + \frac{1}{B_n} \left( 1 - \frac{B_n}{n} t \right) (\lfloor nu \rfloor + B_n Y_n^{(0)}(u)), \quad (11)$$

where we set

$$S_n^{(t)}(u) = \frac{1}{B_n} \sum_{k=1}^{\lfloor nu \rfloor} \sum_{i=1}^{\xi(k)+1} \left( \mathbb{1}_{\{U_k(i) \leq 1 - tB_n/n\}} - \left( 1 - \frac{B_n}{n} t \right) \right). \quad (12)$$

For fixed  $t \geq 0$ , the terms in the sum (12) are independent centred random variables whose variance is bounded by  $tB_n/n$ . Moreover, these terms are also independent of  $(\xi(k), 1 \leq k \leq n)$ . Since the number of terms in the sum (12) is bounded by  $n + B_n \|Y_n^{(0)}\|_\infty$ , the Chebyshev's inequality together with (10) implies that  $S_n^{(t)}(u) \rightarrow 0$ , as  $n \rightarrow \infty$ , in probability. For the remaining terms at the right-hand side of (11), we see that (9) implies that

$$-\frac{1}{B_n} \lfloor nu \rfloor + \frac{1}{B_n} \left( 1 - \frac{B_n}{n} t \right) (\lfloor nu \rfloor + B_n Y_n^{(0)}(u)) \rightarrow X_\alpha^{\text{exc}}(u) - tu, \text{ as } n \rightarrow \infty,$$

almost surely. Finally, for any  $u \in [0, 1]$  and  $t \geq 0$ ,  $Y_n^{(t)}(u) \rightarrow Y_\alpha^{(t)}(u)$ , as  $n \rightarrow \infty$ , in probability, which implies our claim.  $\square$

**Tightness.** Since we are going to work with processes with sample paths in the set  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$  equipped with the Skorokhod topology, we start by recalling some aspects of this space of càdlàg functions and refer to [14, Chapter 3] (or [25, Chapter VI]) for details. Fix a separable, complete metric space  $(\mathbb{M}, d)$ , and consider the space  $\mathbb{D}(\mathbb{R}_+, \mathbb{M})$  of càdlàg functions from  $\mathbb{R}_+$  to  $\mathbb{M}$ . For  $a > 0$ ,  $0 < \delta < 1$  and  $k \in \mathbb{N}$ , a sequence  $\Delta_{a,k} = \{0 = t_0 < t_1 < \dots < t_k = a\}$  of subdivisions of  $[0, a]$  is called  $\delta$ -sparse if it satisfies  $\min_{1 \leq i \leq k} (t_i - t_{i-1}) \geq \delta$ . The so-called modified modulus of continuity in  $\mathbb{D}(\mathbb{R}_+, \mathbb{M})$  is given by

$$\tilde{\omega}(\delta, a, d; g) := \inf_{\Delta_{a,k}} \max_{1 \leq i \leq k} \sup_{r, r' \in [t_{i-1}, t_i]} d(g(r), g(r')), \text{ for } g \in \mathbb{D}(\mathbb{R}_+, \mathbb{M}),$$

where the infimum extends over all  $\delta$ -sparse sets  $\Delta_{a,k}$ . Let  $\Theta$  denote the class of strictly increasing, continuous mappings of  $[0, 1]$  onto itself. For  $\theta \in \Theta$ , we put

$$\|\theta\|^\circ := \sup_{0 \leq r < r' \leq 1} \left| \log \frac{\theta(r') - \theta(r)}{r' - r} \right|,$$

and recall that the Skorokhod metric in  $\mathbb{D}([0, 1], \mathbb{M})$  is defined by

$$\text{Sk}_d(g, g') := \inf_{\theta \in \Theta} \left\{ \|\theta\|^\circ \vee \sup_{0 \leq r \leq 1} d(g(r), g'(\theta(r))) \right\}, \quad \text{for } g, g' \in \mathbb{D}([0, 1], \mathbb{M}),$$

where the infimum extends over all  $\theta \in \Theta$  such that  $\|\theta\|^\circ < \infty$  and  $\sup_{0 \leq r \leq 1} d(g(r), g'(\theta(r))) < \infty$ . It is well-known that the metric space  $(\mathbb{D}([0, 1], \mathbb{M}), \text{Sk}_d)$  is complete and separable; see [14, Theorem 12.2, Chapter 3]. In particular, if  $\mathbb{M} = \mathbb{R}$ , we will consider the separable and complete metric space  $(\mathbb{R}, |\cdot|)$ , where  $|\cdot|$  is the Euclidean metric. We then write, for  $g, g' \in \mathbb{D}([0, 1], \mathbb{R})$ ,  $\text{Sk}_{|\cdot|}(g, g')$ .

**Lemma 3.** *For any  $a > 0$  and  $\varepsilon, \varepsilon' > 0$ , there exists  $0 < \delta < 1$  such that*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(\tilde{\omega}(\delta, a, \text{Sk}_{|\cdot|}; Y_n) \geq \varepsilon) \leq \varepsilon'. \quad (13)$$

*In particular, the sequence of stochastic processes  $(Y_n)_{n \geq 1}$  is tight on  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ .*

As a preparation for the proof of Lemma 3, we need a technical result. For  $t \geq 0$  and  $g^{(t)} = (g^{(t)}(u), u \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$ , recall that we write  $\|g^{(t)}\|_\infty = \sup_{u \in [0, 1]} |g^{(t)}(u)|$ . Then for  $g = (g^{(t)}, t \geq 0) \in \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ ,  $a > 0$  and  $0 < \delta < 1$ , define the modulus

$$\omega(\delta, a; g) := \sup\{\|g^{(t)} - g^{(t')}\|_\infty : |t - t'| < \delta, 0 \leq t, t' \leq a\}.$$

For  $t \geq 0$ , let  $S_n^{(t)} = (S_n^{(t)}(u), u \in [0, 1])$  be the process defined in (12) and set  $S_n = (S_n^{(t)}, t \geq 0)$ .

**Lemma 4.** *For any  $a > 0$  and  $\varepsilon, \varepsilon' > 0$ , there exists  $0 < \delta < 1$  such that*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(\omega(\delta, a; S_n) \geq \varepsilon) \leq \varepsilon'.$$

We postpone the proof of Lemma 4 for later and continue with the proof of Lemma 3.

*Proof of Lemma 3.* Suppose that we have proven (13) in Lemma 3. Thanks to the arbitrariness of  $\varepsilon, \varepsilon' > 0$ , one can see that for each  $a > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}[\tilde{\omega}(\delta, a, \text{Sk}_{|\cdot|}; Y_n) \wedge 1] = 0.$$

Then, [26, Theorem 16.10, Chapter 16] and Lemma 2 show that the sequence of processes  $(Y_n)_{n \geq 1}$  is tight on  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ . So, it suffices to prove (13) to finish the proof of Lemma 3.

Fix  $a > 0$ , and observe from (11) that for  $0 \leq t_1 < t_2 \leq a$  and  $u \in [0, 1]$ ,

$$Y_n^{(t_1)}(u) - Y_n^{(t_2)}(u) = S_n^{(t_1)}(u) - S_n^{(t_2)}(u) + \frac{\lfloor nu \rfloor}{n}(t_2 - t_1) + \frac{B_n}{n}(t_2 - t_1)Y_n^{(0)}(u).$$

Since the identity map on  $[0, 1]$  belongs to  $\Theta$ , the triangle inequality implies that

$$\begin{aligned} \text{Sk}_{|\cdot|}(Y_n^{(t_1)}, Y_n^{(t_2)}) &= \inf_{\theta \in \Theta} \left\{ \|\theta\|^\circ \vee \sup_{u \in [0, 1]} |Y_n^{(t_1)}(u) - Y_n^{(t_2)}(\theta(u))| \right\} \\ &\leq \|S_n^{(t_1)} - S_n^{(t_2)}\|_\infty + (t_2 - t_1) + a \frac{B_n}{n} \|Y_n^{(0)}\|_\infty. \end{aligned}$$

For the set  $[0, a)$  and each  $0 < \delta < \min\{a/2, 1/2\}$ , we can have a  $\delta$ -sparse set  $\Delta_{a,k}$  satisfying  $\delta \leq t_i - t_{i-1} \leq 2\delta$ , for  $1 \leq i \leq k$ . Then,

$$\tilde{\omega}(\delta, a, \text{Sk}_{|\cdot|}; Y_n) \leq \omega(2\delta, a; S_n) + 2\delta + 2\delta \frac{B_n}{n} \|Y_n^{(0)}\|_\infty, \quad \text{for } 0 < \delta < \min\{a/2, 1/2\}.$$

Then, (13) follows from the previous inequality, the convergence in (10) and Lemma 4.  $\square$

*Proof of Lemma 4.* For  $n \in \mathbb{N}$  and  $a > 0$ , set  $t_0 = 0$  and  $t_r = ra/\lceil B_n \rceil$ , for  $r = 1, \dots, \lceil B_n \rceil$ . For  $0 \leq r < r' \leq \lceil B_n \rceil$ , define the process  $Z_{r,r',n} = (Z_{r,r',n}(u), u \in [0, 1])$  by letting

$$Z_{r,r',n}(u) := S_n^{(t_r)}(u) - S_n^{(t_{r'})}(u) = \frac{1}{B_n} \sum_{k=1}^{\lfloor nu \rfloor} \sum_{i=1}^{\xi(k)+1} \left( \mathbf{1}_{\{1 - \frac{B_n}{n} t_{r'} < U_k(i) \leq 1 - \frac{B_n}{n} t_r\}} - \frac{B_n}{n} (t_{r'} - t_r) \right).$$

Recall that  $\sum_{k=1}^{\lfloor nu \rfloor} (\xi(k) + 1) = \lfloor nu \rfloor + B_n Y_n^{(0)}(u)$ . For  $r = 0, 1, \dots, \lceil B_n \rceil - 1$  and  $t_r \leq t \leq t_{r+1}$ ,

$$\begin{aligned} |S_n^{(t_r)}(u) - S_n^{(t)}(u)| &\leq |S_n^{(t_r)}(u) - S_n^{(t_{r+1})}(u)| + \frac{a}{\lceil B_n \rceil n} (\lfloor nu \rfloor + B_n Y_n^{(0)}(u)) \\ &\leq |Z_{r,r+1,n}(u)| + \frac{a}{\lceil B_n \rceil} + \frac{1}{n} \|Y_n^{(0)}\|_\infty. \end{aligned}$$

For  $n$  large enough, the triangle inequality together with the previous inequality implies that

$$\begin{aligned} \omega(\delta, a; S_n) &\leq 2 \sup\{\|S_n^{(t_r)} - S_n^{(t)}\|_\infty : 0 \leq r \leq \lceil B_n \rceil - 1, t_r \leq t \leq t_{r+1}\} \\ &\quad + \sup\{\|S_n^{(t_r)} - S_n^{(t_{r'})}\|_\infty : 0 \leq r < r' \leq \lceil B_n \rceil, |t_r - t_{r'}| < \delta\} \\ &\leq 2 \sup\{\|Z_{r,r+1,n}\|_\infty : 0 \leq r \leq \lceil B_n \rceil - 1\} + \frac{2a}{\lceil B_n \rceil} + \frac{2}{n} \|Y_n^{(0)}\|_\infty \\ &\quad + \sup\{\|Z_{r,r',n}\|_\infty : 0 \leq r < r' \leq \lceil B_n \rceil, |t_r - t_{r'}| < \delta\}. \end{aligned} \tag{14}$$

for  $0 < \delta < 1$ . We will prove that for all  $\varepsilon > 0$ , there is a constant  $C_{\varepsilon,p} > 0$  such that for all  $p \geq 2$ ,

$$\mathbb{P}_n(\|Z_{r,r',n}\|_\infty \geq \varepsilon) \leq C_{\varepsilon,p} B_n^{-p/2} (1 - B_n/n)^{p/2} (t_{r'} - t_r)^{p/2}, \quad 0 \leq r < r' \leq \lceil B_n \rceil. \tag{15}$$

Then, Lemma 4 will follow from (10), (14) and the union bound.

Observe that  $\|Z_{r,r',n}\|_\infty = \sup_{1 \leq m \leq n} |Z_{r,r',n}(m/n)|$ . By Etemadi's inequality, we have that

$$\mathbb{P}_n (\|Z_{r,r',n}\|_\infty \geq \varepsilon) \leq 3 \sup_{1 \leq m \leq n} \mathbb{P}_n (|Z_{r,r',n}(m/n)| \geq \varepsilon/3), \quad (16)$$

for all  $\varepsilon > 0$ . On the one hand, the terms in the sum  $Z_{r,r',n}(m/n)$  are independent centred random variables with variance bounded by  $a/n$ . On the other hand, these terms are also independent of the random variables  $(\xi(k), 1 \leq k \leq n)$ . Moreover, the number of terms in the sum  $Z_{r,r',n}(m/n)$  is bounded by  $\sum_{k=1}^n (\xi(k) + 1) = n + B_n Y_n^{(0)}(1) = n - B_n$ , under  $\mathbb{P}_n$ . By the Marcinkiewicz-Zygmund inequality, it is not difficult to see that, for  $p \geq 2$ ,

$$\mathbb{E}[|Z_{r,r',n}(m/n)|^p] \leq C_p B_n^{-p/2} (1 - B_n/n)^{p/2} (t_{r'} - t_r)^{p/2},$$

for some constant  $C_p > 0$ . So, (15) follows from (16) and Chebyshev's inequality.  $\square$

We have now all the ingredients to prove Theorem 3.

*Proof of Theorem 3.* Theorem 3 is a consequence of Proposition 3, Lemma 2 and Lemma 3.  $\square$

## 6 Proof of Theorem 1

In this section, we prove Theorem 1. We start by developing a general approach for the convergence of fragmentation processes encoded by functions in  $\mathbb{D}([0, 1], \mathbb{R})$ . Recall that  $\mathbb{S}$  denotes the space defined in (1) endowed with the  $\ell^1$ -norm. For an increasing function  $h = (h(s), s \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$ , we write

$$\mathbf{F}(h) := (F_1(h), F_2(h), \dots) \in \mathbb{S}$$

for the sequence of the lengths of the intervals components of the complement of the support of the Stieltjes measure  $dh$ , arranged in decreasing order; we tacitly understand  $\mathbf{F}(h)$  as an infinite sequence, by completing with an infinite number of zero terms. Let  $\text{Supp}(dh)$  denote the support of  $dh$  and note that  $(0, 1) \setminus \text{Supp}(dh)$  is the union of all open intervals on each of which the function  $h$  is constant. For any  $g = (g(s), s \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$  such that  $g(0) = 0$ , let  $\hat{g} = (\hat{g}(s), s \in [0, 1])$  be given by

$$\hat{g}(s) := \inf_{u \in [0, s]} g(u), \quad s \in [0, 1].$$

Note that  $-\hat{g}(s) = \sup_{u \in [0, s]} (-g(u))$ , then  $-\hat{g}$  is an increasing function in  $\mathbb{D}([0, 1], \mathbb{R})$ . In particular, the Stieltjes measure  $d(-\hat{g})$  is well-defined and  $\text{Supp}(d(-\hat{g}))$  is given by the set of points where the function  $g$  reaches a new infimum. We call constancy interval of  $-\hat{g}$  any interval component of  $(0, 1) \setminus \text{Supp}(d(-\hat{g}))$ . Indeed, those constancy intervals corresponds to excursion intervals of  $g$  above its infimum (or equivalently, excursion intervals of the function  $g - \hat{g}$  above 0).

For  $g = (g^{(t)}, t \geq 0) \in \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ , we let  $g^{(t)} = (g^{(t)}(s), s \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$ , for  $t \geq 0$ . Similarly, for  $n \in \mathbb{N}$ , we write  $g_n = (g_n^{(t)}, t \geq 0) \in \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$  such that, for each  $t \geq 0$ ,  $g_n^{(t)} =$

$(g_n^{(t)}(s), s \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$ . If  $g^{(t)}(0) = 0$  (resp.  $g_n^{(t)}(0) = 0$ ), we define  $\hat{g}^{(t)} = (\hat{g}^{(t)}(s), s \in [0, 1])$  (resp.  $\hat{g}_n^{(t)} = (\hat{g}_n^{(t)}(s), s \in [0, 1])$ ) by letting

$$\hat{g}^{(t)}(s) := \inf_{u \in [0, s]} g^{(t)}(u) \quad \left( \text{resp. } \hat{g}_n^{(t)}(s) := \inf_{u \in [0, s]} g_n^{(t)}(u) \right), \quad s \in [0, 1].$$

The following result is the key ingredient in the proof of Theorem 1. Recall that  $\mathbb{S}_1 \subset \mathbb{S}$  denotes the space of the elements of  $\mathbb{S}$  with sum 1.

**Lemma 5.** *On some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $(g_n)_{n \geq 1}$  be a sequence of random elements of  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$  such that  $g_n^{(t)}(0) = 0$ , for  $n \in \mathbb{N}$  and  $t \geq 0$ . Suppose that for any fixed  $0 \leq t_* \leq t^* < \infty$  and any  $\varepsilon > 0$ , there exists  $K, N \in \mathbb{N}$  such that for any  $k \geq K$  and  $n \geq N$*

$$\inf_{t \in [t_*, t^*]} \sum_{i=1}^k F_i(-\hat{g}_n^{(t)}) \geq \sup_{t \in [t_*, t^*]} \lim_{r \rightarrow \infty} \sum_{i=1}^r F_i(-\hat{g}_n^{(t)}) - \varepsilon, \quad \text{almost surely.} \quad (17)$$

Assume further that there exists a random  $g \in \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$  such that  $g^{(t)}(0) = 0$ , for  $t \geq 0$ , and

(i)  $g_n \xrightarrow{d} g$ , as  $n \rightarrow \infty$ , in the space  $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ ,

for every fixed  $t \geq 0$ ,

(ii)  $g^{(t)}(s) \wedge g^{(t)}(s-) > \hat{g}^{(t)}(s)$ , for every  $s \in (a, b)$  whenever  $(a, b) \subset [0, 1]$  is an interval of constancy for the function  $-\hat{g}^{(t)}$ .

(iii)  $\mathbf{F}(-\hat{g}^{(t)}) \in \mathbb{S}_1$ ,

where (ii) and (iii) hold almost surely. Then,

$$(\mathbf{F}(-\hat{g}_n^{(t)}), t \geq 0) \xrightarrow{d} (\mathbf{F}(-\hat{g}^{(t)}), t \geq 0), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{D}(\mathbb{R}_+, \mathbb{S}).$$

*Proof.* By the Skorokhod representation theorem, we can and we will work in a probability space where the convergence in (i) together with (ii) and (iii) holds almost surely. By (i), there exists a dense subset  $D$  of  $\mathbb{R}_+$  such that for any fixed  $k \in \mathbb{N}$  and collection  $0 \leq t_1 < t_2 < \dots < t_k < \infty$  with  $t_1, \dots, t_k \in D$ , we have that a.s.,

$$(g_n^{(t_1)}, \dots, g_n^{(t_k)}) \rightarrow (g^{(t_1)}, \dots, g^{(t_k)}), \quad \text{as } n \rightarrow \infty,$$

in  $\mathbb{D}([0, 1], \mathbb{R})^{\otimes k}$  (i.e., the  $k$ -fold space of  $\mathbb{D}([0, 1], \mathbb{R})$ ). Then [9, Lemma 4] implies that a.s.,

$$(\mathbf{F}(-\hat{g}_n^{(t_1)}), \dots, \mathbf{F}(-\hat{g}_n^{(t_k)})) \rightarrow (\mathbf{F}(-\hat{g}^{(t_1)}), \dots, \mathbf{F}(-\hat{g}^{(t_k)})), \quad \text{as } n \rightarrow \infty,$$

in  $\mathbb{S}^{\otimes k}$  (i.e., the  $k$ -fold space of  $\mathbb{S}$  equipped with the  $\ell^1$ -norm). Note that the conditions in [9, Lemma 4] are satisfied by our assumptions (in fact, one has to apply [9, Lemma 4] to  $-g_n$  and  $-g$ ). This shows the convergence of the finite-dimensional distributions of the sequence of processes  $((\mathbf{F}(-\hat{g}_n^{(t)}), t \geq 0))_{n \geq 1}$  to those of the process  $(\mathbf{F}(-\hat{g}^{(t)}), t \geq 0)$ .

To finish with the proof, we need to show that the sequence of processes  $((\mathbf{F}(-\hat{g}_n^{(t)}), t \geq 0))_{n \geq 1}$  is tight in  $\mathbb{D}(\mathbb{R}_+, \mathbb{S})$ . Indeed, it is enough to see that for each  $0 \leq t_* \leq t^* < \infty$  the sequence of processes  $((\mathbf{F}(-\hat{g}_n^{(t)}), t \in [t_*, t^*]))_{n \geq 1}$  is tight in  $\mathbb{D}([t_*, t^*], \mathbb{S})$ . But this follows by showing that  $((\mathbf{F}(-\hat{g}_n^{(t)}), t \in [t_*, t^*]))_{n \geq 1}$  satisfies the conditions (a), (b) and (c) of [15, Lemma 22] with  $p = 1$ . To see this, for every fixed  $t \in [t_*, t^*]$ , one has that  $\|\mathbf{F}(-\hat{g}_n^{(t)})\|_1 \leq 1$  which implies conditions (a) and (b) in [15, Lemma 22]. Condition (c) in [15, Lemma 22] is (17) in our statement.  $\square$

Finally, we are in position to prove our main result Theorem 1.

*Proof of Theorem 1.* Let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree of index  $\alpha \in (1, 2]$ . Recall that  $(B_n)_{n \geq 1}$  denotes a sequence of positive real numbers satisfying (2). For  $t \geq 0$ , let  $W_n^{(t)}$  be the (normalized and time-scaled) Prim path defined in (8) of the fragmentation forest at time  $s_n(t) = 1 - (B_n/n)t$ , i.e.  $\mathbf{f}(s_n(t))$ , associated to  $\mathbf{t}_n$  and the i.i.d. uniform random weights  $\mathbf{w}$ . Define the process  $I_n^{(t)} = (I_n^{(t)}(u), u \in [0, 1])$  by letting

$$I_n^{(t)}(u) = \inf_{s \in [0, u]} W_n^{(t)}(s), \quad \text{for } s \in [0, 1].$$

Recall that  $\mathbf{F}_n^{(\alpha)} = (\mathbf{F}_n^{(\alpha)}(t), t \geq 0)$  stands for the fragmentation process of  $\mathbf{t}_n$  defined in (3). From Lemma 1 and the preceding discussion, it is clear that  $\mathbf{F}_n^{(\alpha)}(t) = \mathbf{F}(-I_n^{(t)})$ , for  $t \geq 0$ . Let  $Y_\alpha^{(t)}$  and  $I_\alpha^{(t)}$  be the processes defined in (4), and recall that the  $\alpha$ -stable fragmentation process,  $\mathbf{F}^{(\alpha)} = (\mathbf{F}^{(\alpha)}(t), t \geq 0)$ , is given by  $\mathbf{F}^{(\alpha)}(t) = \mathbf{F}(-I_\alpha^{(t)})$ , for  $t \geq 0$ . Note that for all  $t \geq 0$ ,  $W_n^{(t)}(0) = Y_\alpha^{(t)}(0) = 0$ . Then, to prove Theorem 1, one only needs to check that the processes  $W_n = (W_n^{(t)}, t \geq 0)$  and  $Y_\alpha = (Y_\alpha^{(t)}, t \geq 0)$  satisfy the conditions of Lemma 5.

We start by verifying that the process  $Y_\alpha$  fulfills (i), (ii) and (iii) of Lemma 5. Indeed, (i) has been proven in Theorem 3. The process  $X_\alpha^{\text{br}}$  has exchangeable increments due to the stationary and independent increments of the stable Lévy process  $X_\alpha$ ; see e.g., [26, Chapters 11 and 16]. Then, (ii) follows along the lines of the proof of Lemma 7 (i) in [9] thanks to the property in (6). To prove that  $Y_\alpha^{(t)}$  fulfills condition (iii) for every  $t \geq 0$ , recall that the support of the Stieltjes measure  $d(-I_\alpha^{(t)})$  coincides with the ladder time set  $\mathcal{L}^\alpha(t)$  of  $Y_\alpha^{(t)}$ , which is a random closed set with zero Lebesgue measure. The latter follows from [7, Corollary 5, Chapter VII] but alternatively, it can be deduced from (6) by following the same argument as in [9, Proof of Lemma 7]. Since  $\mathbf{F}(-I_\alpha^{(t)})$  is defined as the ranked sequence of the lengths of the open intervals in the canonical decomposition of  $[0, 1]/\mathcal{L}^\alpha(t)$ , condition (iii) follows.

We now check that the sequence  $(W_n)_{n \geq 1}$  fulfills (17). Note that, for every  $t \geq 0$ ,  $\|\mathbf{F}(-I_n^{(t)})\|_1 = 1$ . Fix  $t_*, t^*$  such that  $0 \leq t_* \leq t^* < \infty$ . For every  $t \in [t_*, t^*]$  and  $m \in \mathbb{N}$ ,

$$\|\mathbf{F}(-I_n^{(t)})\|_1 - \sum_{i=1}^m \mathbf{F}_i(-I_n^{(t)}) = \sum_{i>m} \mathbf{F}_i(-I_n^{(t)})$$

reaches its maximum at  $t = t_*$ . Then for (17) to be satisfied, it suffices that for any  $\varepsilon > 0$ , there exists

$m \in \mathbb{N}$  and  $n \in \mathbb{N}$  such that

$$\sum_{i=1}^m \mathbf{F}_i(-I_n^{(t_*)}) \geq \lim_{r \rightarrow \infty} \sum_{i=1}^r \mathbf{F}_i(-I_n^{(t_*)}) - \varepsilon = 1 - \varepsilon. \quad (18)$$

This would imply that for any  $t \in [t_*, t^*]$ , we have that  $\sum_{i=1}^m \mathbf{F}_i(-I_n^{(t)}) \geq 1 - \varepsilon$ , which shows that  $(W_n)_{n \geq 1}$  satisfies (17).

Theorem 3 implies that  $(W_n^{(t)}, t \in [t_*, t^*]) \rightarrow (Y_\alpha^{(t)}, t \in [t_*, t^*])$ , in distribution, as  $n \rightarrow \infty$ , in the space  $\mathbb{D}([t_*, t^*], \mathbb{D}([0, 1], \mathbb{R}))$ . By the Skorokhod representation theorem, we can and we will work on a probability space on which this convergence holds almost surely. Since we have proven that the process  $Y_\alpha^{(t_*)}$  fulfills (iii) of Lemma 5, for any  $\varepsilon > 0$ , there exists an  $m \in \mathbb{N}$  such that  $\sum_{i=1}^m \mathbf{F}_i(-I_\alpha^{(t_*)}) \geq 1 - \varepsilon/2$ . On the other hand, recall that  $Y_\alpha^{(t_*)}$  fulfills (ii) of Lemma 5. Then [9, Lemma 4] implies that a.s.,  $\mathbf{F}(-I_n^{(t_*)}) \rightarrow \mathbf{F}(-I_\alpha^{(t_*)})$ , as  $n \rightarrow \infty$  in the space  $\mathbb{S}$  with the  $\ell^1$ -norm. Hence, a.s. for all  $n$  large enough,  $\sum_{i=1}^m \mathbf{F}_i(-I_n^{(t_*)}) \geq 1 - \varepsilon$ , which proves (18).  $\square$

## 7 Proof of Proposition 1

In this section, we prove Proposition 1. The proof follows along the lines of the proof of Proposition 13 in Aldous and Pitman [6] (see also Theorem 3 in [5]). We provide enough details to convince the reader that everything can be carried out as in [6], but also to make this work self contained.

**The  $\alpha$ -stable Lévy tree.** Recall that an  $\alpha$ -stable Lévy tree  $\mathcal{T}_\alpha = (\mathcal{T}_\alpha, d_\alpha, \rho_\alpha, \mu_\alpha)$  of index  $\alpha \in (1, 2]$  is a random compact rooted measure that arises naturally as the scaling limit of large  $\alpha$ -stable GW-trees. More precisely, let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree, view it as a rooted metric measure tree  $\mathbf{t}_n = (\mathbf{t}_n, d_n^{\text{gr}}, \rho_n, \mu_n^{\text{nod}})$ , where  $\mathbf{t}_n$  is identified as its set of  $n$  vertices  $\{v_1, \dots, v_n\}$ ,  $d_n^{\text{gr}}$  is the graph-distance on  $\mathbf{t}_n$ ,  $\rho_n \in \mathbf{t}_n$  is the root (the initial individual in the population) and  $\mu_n^{\text{nod}} := \frac{1}{n} \sum_{i=1}^n \delta_{v_i}$  is the uniform measure on the set of vertices of  $\mathbf{t}_n$ ; here  $\delta_v$  is the Dirac measure in the point  $v \in \mathbf{t}_n$ . Let  $(B_n)_{n \geq 1}$  be a sequence of positive real numbers satisfying (2) and consider the rescaled  $\alpha$ -stable GW-tree  $(B_n/n) \cdot \mathbf{t}_n = (\mathbf{t}_n, (B_n/n) \cdot d_n^{\text{gr}}, \rho_n, \mu_n^{\text{nod}})$ . Then it is well-known, by results of Aldous [3] and Duquesne [19], that

$$(\mathbf{t}_n, (B_n/n) \cdot d_n^{\text{gr}}, \rho_n, \mu_n^{\text{nod}}) \xrightarrow{d} (\mathcal{T}_\alpha, d_\alpha, \rho_\alpha, \mu_\alpha), \quad n \rightarrow \infty, \quad (19)$$

for the *pointed Gromov-Hausdorff-Prohorov* (pGHP) topology. (see for example [34, Proposition 9], [1, Theorem 2.5] and reference therein for background on the pGHP topology.) We list some useful properties of the  $\alpha$ -stable Lévy tree and the rescaled  $\alpha$ -stable GW-tree.

(T1) The mass measure  $\mu_\alpha$  is non-atomic and it is supported on  $\text{Lf}(\mathcal{T}_\alpha)$ , a.s.; see [21, Theorem 4.6].

For  $k \in \mathbb{N}$ , let  $V_1^n, \dots, V_k^n$  be independent random vertices of  $\mathbf{t}_n$  with common distribution  $\mu_n^{\text{nod}}$ . Let  $\mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n)$  be the reduced subtree of  $\mathbf{t}_n$  by its root and the vertices  $\mathbf{V}_k^n = (V_1^n, \dots, V_k^n)$  (i.e.,  $\mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n)$  is the compact rooted metric space  $(\llbracket \mathbf{v}_k \rrbracket, d_n^{\text{gr}}, \rho)$ , where the distance  $d_n^{\text{gr}}$  in the right-hand



side is tacitly understood to be restricted to the appropriate space). Let also  $(B_n/n) \cdot \mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n)$  be the space  $\mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n)$  with distances multiplied by  $B_n/n$ . Similarly, we let  $V_1, \dots, V_k$  be independent random points (leaves) of  $\mathcal{T}_\alpha$  with common distribution  $\mu_\alpha$ , and write  $\mathcal{R}(\mathcal{T}_\alpha, \mathbf{V}_k)$  for the reduced subtree of  $\mathcal{T}_\alpha$  by its root and the vertices  $\mathbf{V}_k = (V_1, \dots, V_k)$ .

(T2) For every fixed  $k \in \mathbb{N}$ ,  $(B_n/n) \cdot \mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n) \xrightarrow{d} \mathcal{R}(\mathcal{T}_\alpha, \mathbf{V}_k)$ , as  $n \rightarrow \infty$ , for the *pointed Gromov-Hausdorff* topology. This follows from (19), [34, Proposition 10] and [24, Lemma 35].

Define the empirical (random) measures

$$\mu_{n,k}^{\text{nod}} := \frac{1}{k} \sum_{i=1}^k \delta_{V_i^n} \quad \text{and} \quad \mu_{\alpha,k} := \frac{1}{k} \sum_{i=1}^k \delta_{V_i}, \quad (20)$$

(T3) The Glivenko-Cantelli Theorem implies that  $\mu_{n,k}^{\text{nod}} \rightarrow \mu_n^{\text{nod}}$  and  $\mu_{\alpha,k} \rightarrow \mu_\alpha$ , almost surely, as  $k \rightarrow \infty$ , in the weakly sense.

(T4) Theorem 3 in [3] shows that the family of reduced subtrees  $(\mathcal{R}(\mathcal{T}_\alpha, \mathbf{V}_k), k \in \mathbb{N})$  satisfies the so-called leaf-tight property, i.e.  $\inf_{2 \leq i < \infty} d_\alpha(V_1, V_i) = 0$ , almost surely.

**Exchangeable random partitions.** Let  $\mathcal{P}_\infty$  be the set of partitions of the set of positive integers  $\mathbb{N} = \{1, 2, \dots\}$ . Lemma 2.6 in [11] shows that  $\mathcal{P}_\infty$  can be endowed with an ultra-metric  $d_{\mathcal{P}_\infty}$  such that  $(\mathcal{P}_\infty, d_{\mathcal{P}_\infty})$  is compact. A partition  $\Pi \in \mathcal{P}_\infty$  is a countable collection  $\Pi = (\Pi(i), i \in \mathbb{N})$  of pairwise disjoint subsets of  $\mathbb{N}$  (also called blocks) such that  $\bigcup_{i \in \mathbb{N}} \Pi(i) = \mathbb{N}$ . For e.g., an equivalence relation  $\sim$  on the set  $\mathbb{N}$  can be identified with a partition of  $\mathbb{N}$  into equivalence classes. In particular, a random equivalence relation on  $\mathbb{N}$  can be identified with a random partition of  $\mathbb{N}$ . An exchangeable random partition  $\Pi$  is a  $\mathcal{P}_\infty$ -valued random variable whose restriction  $\Pi_k = \Pi|_{[k]}$  to the set  $[k] := \{1, \dots, k\}$  has an invariant distribution under the action of permutations of  $[k]$ , for every  $k \in \mathbb{N}$ .

Following Kingman's theory [27], we recall some useful properties of exchangeable random partitions. For  $k \in \mathbb{N}$  and a partition  $\Pi \in \mathcal{P}_\infty$ , let  $\Pi_k = (\Pi_k(i), i \in \mathbb{N})$  be the restriction of  $\Pi$  to  $[k]$ , and let  $\#\Pi_k^\downarrow = (\#\Pi_k^\downarrow(i), i \in \mathbb{N})$  be the decreasing rearrangement of the block sizes (number of elements) of  $\Pi_k$  such that  $\#\Pi_k^\downarrow(i) = 0$  whenever  $\Pi_k$  has fewer than  $i$  blocks. Let  $\mathbb{S}_{\leq 1} \subset \mathbb{S}$  be the space of the elements of  $\mathbb{S}$  with sum less than or equal to 1. Recall also that  $\mathbb{S}_1 \subset \mathbb{S}$  denotes the space of the elements of  $\mathbb{S}$  with sum 1.

(P1) Let  $\Pi$  be an exchangeable random partition. Theorem 2.1 in [11] and the Fatou's lemma show that the asymptotic ranked frequencies (in decreasing order)

$$|\Pi(i)^\downarrow| := \lim_{k \rightarrow \infty} \frac{\#\Pi_k^\downarrow(i)}{k}, \quad \text{for } i \in \mathbb{N}, \text{ exist a.s. and } (|\Pi(i)^\downarrow|, i \in \mathbb{N}) \in \mathbb{S}_{\leq 1}.$$

(P2)  $(|\Pi(i)^\downarrow|, i \in \mathbb{N}) \in \mathbb{S}_1$  a.s. if and only if  $\{1\}$  is not a class (i.e., the singleton  $\{1\}$  is not a block) of  $\Pi$  a.s.; see [11, Proposition 2.8].

(P3) For each  $n \in \mathbb{N} \cup \{\infty\}$ , let  $\Pi^{(n)}$  be an exchangeable random partition, and write  $(|\Pi^{(n)}(i)|^\downarrow, i \in \mathbb{N})$  for the sequence of asymptotic ranked frequencies of its blocks in decreasing order. For  $k \in \mathbb{N}$ , let  $\mathcal{P}_k$  be the set of partitions of  $[k]$  endowed with the discrete topology. Then, Proposition 2.9 in [11] implies that

$$\Pi^{(n)}|_{[k]} \xrightarrow{d} \Pi^{(\infty)}|_{[k]}, \quad \text{as } n \rightarrow \infty, \quad \text{for each } k \in \mathbb{N}, \quad \text{in the space } \mathcal{P}_k$$

if and only if

$$(|\Pi^{(n)}(i)|^\downarrow, i \in \mathbb{N}) \xrightarrow{d} (|\Pi^{(\infty)}(i)|^\downarrow, i \in \mathbb{N}), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{S}_{\leq 1}, \quad (21)$$

where  $\mathbb{S}_{\leq 1}$  is given the topology of pointwise convergence (or equivalently, the uniform distance in [11, Proposition 2.1] which makes  $\mathbb{S}_{\leq 1}$  compact).

**Lemma 6.** *Suppose that (21) holds and that  $(|\Pi^{(\infty)}(i)|^\downarrow, i \in \mathbb{N}) \in \mathbb{S}_1$  almost surely. Then,*

$$(|\Pi^{(n)}(i)|^\downarrow, i \in \mathbb{N}) \xrightarrow{d} (|\Pi^{(\infty)}(i)|^\downarrow, i \in \mathbb{N}), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } (\mathbb{S}_1, \ell_1).$$

*Proof.* The proof follows by a simple application of Fatou's lemma and Scheffé's lemma.  $\square$

**Fragmentation processes.** Following ideas of Aldous and Pitman [6], the framework of exchangeable random partitions provides a different interpretation for the fragmentation processes associated to  $\alpha$ -stable Lévy trees and  $\alpha$ -stable GW-trees.

Consider an  $\alpha$ -stable Lévy tree  $\mathcal{T}_\alpha = (\mathcal{T}_\alpha, d_\alpha, \rho_\alpha, \mu_\alpha)$  together with a Poisson point process of cuts on its skeleton with intensity  $dt \otimes \lambda_\alpha(dv)$  on  $[0, \infty) \times \mathcal{T}_\alpha$ , where  $\lambda_\alpha$  is the length measure associated to  $\mathcal{T}_\alpha$ . Recall that for all  $t \geq 0$  we defined an equivalence relation  $\sim_t$  on  $\mathcal{T}_\alpha$  by saying that  $v \sim_t w$ , for  $v, w \in \mathcal{T}_\alpha$ , if and only if no atom of the Poisson process that has appeared before time  $t$  belongs to the path  $[v, w]$ . We use the above to define a random equivalence relation on  $\mathbb{N}$ . Let  $V_1, V_2, \dots$  be a sequence of independent random points of  $\mathcal{T}_\alpha$  with common distribution  $\mu_\alpha$ . For  $t \geq 0$  and  $i, j \in \mathbb{N}$ , we say  $i \sim_{\alpha, t} j$  if and only if  $V_i \sim_t V_j$ . In particular, we let  $\Pi_\alpha^{(t)} = (|\Pi_\alpha^{(t)}(i)|^\downarrow, i \in \mathbb{N})$  be the random partition of  $\mathbb{N}$  induced by the equivalence classes of the equivalence relation  $\sim_{\alpha, t}$  on  $\mathbb{N}$ .

**Lemma 7.** *For every  $t \geq 0$ ,  $\Pi_\alpha^{(t)}$  is exchangeable. In particular,  $\Pi_\alpha^{(t)}$  is proper a.s., i.e., the asymptotic ranked frequencies  $(|\Pi_\alpha^{(t)}(i)|^\downarrow, i \in \mathbb{N})$  (in decreasing order) of  $\Pi_\alpha^{(t)}$  belongs to  $\mathbb{S}_1$  almost surely.*

*Proof.* The first claim follows from the fact that for every  $k \in \mathbb{N}$  the distribution of the reduced subtree  $\mathcal{R}(\mathcal{T}_\alpha, \mathbf{V}_k)$  of  $\mathcal{T}_\alpha$  is invariant under any permutation of the points (leaves)  $V_1, \dots, V_k$ , i.e.  $\mathcal{R}(\mathcal{T}_\alpha, \mathbf{V}_k)$ . To prove the second part, note that the probability that  $1 \sim_{\alpha, t} j$  is  $\exp(-td_\alpha(V_1, V_j))$ , for  $j \geq 2$ . Then (T4) implies that  $\{1\}$  is not a class a.s., and our claim follows from (P2).  $\square$

**Corollary 2.** *For every  $t \geq 0$ , we have that  $\mathbf{F}_{\mathcal{T}_\alpha}(t) = (|\Pi_\alpha^{(t)}(1)|^\downarrow, |\Pi_\alpha^{(t)}(2)|^\downarrow, \dots)$  almost surely.*

*Proof.* For  $k \in \mathbb{N}$ , let  $\Pi_{\alpha,k}^{(t)} = (\Pi_{\alpha,k}^{(t)}(i), i \in \mathbb{N})$  be the restriction of  $\Pi_{\alpha}^{(t)}$  to  $[k]$ , and let  $\#\Pi_{\alpha,k}^{(t),\downarrow} = (\#\Pi_{\alpha,k}^{(t),\downarrow}(i), i \in \mathbb{N})$  be the decreasing rearrangement of the block sizes of  $\Pi_{\alpha,k}^{(t)}$  such that  $\#\Pi_{\alpha,k}^{(t),\downarrow}(i) = 0$  whenever  $\Pi_{\alpha,k}^{(t)}$  has fewer than  $i$  blocks. Let  $A_{\alpha,1}^{(t)}, A_{\alpha,2}^{(t)}, \dots$  be the distinct equivalence classes for  $\sim_t$ . Then, the vector  $(\#\Pi_{\alpha,k}^{(t),\downarrow}(1), \#\Pi_{\alpha,k}^{(t),\downarrow}(2), \dots)$  is equal to the ranked vector  $(\mu_{\alpha,k}(A_{\alpha,1}^{(t)}), \mu_{\alpha,k}(A_{\alpha,2}^{(t)}), \dots)$  in decreasing order. Thus, our claim follows from **(T3)** and **(P1)**.  $\square$

Consider now the (rescaled)  $\alpha$ -stable GW-tree  $(B_n/n) \cdot \mathbf{t}_n = (\mathbf{t}_n, (B_n/n) \cdot d_n^{\text{gr}}, \rho_n, \mu_n^{\text{nod}})$ , where  $(B_n)_{n \geq 1}$  is a sequence of positive real numbers satisfying (2). For  $t \geq 0$ , recall that the fragmentation forest at time  $s_n(t) = 1 - (B_n/n)t$ , that is  $\mathbf{f}_n(s_n(t))$ , is obtained by keeping those edges in  $\mathbf{t}_n$  with uniform weight smaller than  $s_n(t)$ . As for the fragmentation process of the  $\alpha$ -stable Lévy tree, we can define a random equivalence relation on  $\mathbb{N}$ . Let  $V_1^n, V_2^n, \dots$  be a sequence of independent random vertices of  $\mathbf{t}_n$  with common distribution  $\mu_n^{\text{nod}}$ . For  $t \geq 0$  and  $i, j \in \mathbb{N}$ , we say  $i \sim_{n,t} j$  if and only if there is no cut edge on the path from  $V_i^n$  to  $V_j^n$  before time  $s_n(t)$ . In particular, we let  $\Pi_n^{(t)} = (\Pi_n^{(t)}(i), i \in \mathbb{N})$  be the random partition of  $\mathbb{N}$  induced by the equivalence classes of the equivalence relation  $\sim_{n,t}$  on  $\mathbb{N}$ .

**Lemma 8.** *For every  $t \geq 0$ ,  $\Pi_n^{(t)}$  is exchangeable. In particular,  $\mathbf{F}_n^{(\alpha)}(t) = (|\Pi_n^{(t)}(1)|^{\downarrow}, |\Pi_n^{(t)}(2)|^{\downarrow}, \dots)$  almost surely, where  $(|\Pi_n^{(t)}(i)|^{\downarrow}, i \in \mathbb{N})$  are the asymptotic ranked frequencies of  $\Pi_n^{(t)}$  in decreasing order.*

*Proof.* This follows along the lines of the proofs of Lemma 7 and Corollary 2  $\square$

Now we are able to prove Proposition 1.

*Proof of Proposition 1.* Let  $\mathbf{t}_n$  be an  $\alpha$ -stable GW-tree, and for every fixed  $t \geq 0$ , view the (time-scaled) continuous cutting-down procedure of  $\mathbf{t}_n$  as a (rescaled) Bernoulli process of cuts on its set of edges, that is, every edge of  $\mathbf{t}_n$  is cut at time  $t$  with probability  $(B_n/n)t$ . Then, at time  $t \geq 0$ , the sequence of sizes of the connected components of  $\mathbf{t}_n$  in decreasing order and renormalized by a factor  $1/n$  is given by  $\mathbf{F}_n^{(\alpha)}(t)$ . For every  $k \in \mathbb{N}$  fixed, it should be clear that **(T2)** implies that, as  $n \rightarrow \infty$ , the above (rescaled) Bernoulli process of cuts on  $\mathbf{t}_n$  (viewed as a rooted metric measure tree) up to time  $t$  and restricted to  $\mathcal{R}(\mathbf{t}_n, \mathbf{V}_k^n)$  converges (in distribution) to the Poisson point process of cuts on the skeleton of  $\mathcal{T}_{\alpha}$  with intensity  $ds \otimes \lambda_{\alpha}(dv)$  restricted to  $[0, t] \times \mathcal{R}(\mathcal{T}_{\alpha}, \mathbf{V}_k)$ . In fact, this convergence holds jointly with that in **(T2)**. For every  $t \geq 0$ , it follows that

$$\Pi_n^{(t)}|_{[k]} \xrightarrow{d} \Pi_{\alpha}^{(t)}|_{[k]}, \quad \text{as } n \rightarrow \infty, \quad \text{for each } k \in \mathbb{N}, \quad \text{in the space } \mathcal{P}_k.$$

Property **(P3)**, Lemma 7, Corollary 2 and Lemma 8 imply that

$$\mathbf{F}_n^{(\alpha)}(t) \xrightarrow{d} \mathbf{F}_{\mathcal{T}_{\alpha}}(t), \quad \text{as } n \rightarrow \infty, \quad \text{in the space } \mathbb{S}_{\leq 1},$$

where  $\mathbb{S}_{\leq 1}$  is given the topology of pointwise convergence. Since Lemma 7 also shows that  $\mathbf{F}_{\mathcal{T}_{\alpha}}(t) \in \mathbb{S}_1$  a.s., Lemma 6 entails that the above convergence holds in  $(\mathbb{S}, \ell^1)$ . This shows the convergence of the one-dimensional distribution of  $\mathbf{F}_n^{(\alpha)}$  to  $\mathbf{F}_{\mathcal{T}_{\alpha}}$ . In general, the same argument can be used to obtain the convergence of the finite-dimensional distributions thanks to the convergence of the (rescaled) Bernoulli process of cuts to the Poisson point process of cuts. Finally, Proposition 1 follows from Theorem 1.  $\square$

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