FLUCTUATION THEORY OF CONTINUOUS-TIME SKIP-FREE DOWNWARD MARKOV CHAINS WITH APPLICATIONS TO BRANCHING PROCESSES WITH IMMIGRATION

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Abstract. We develop a comprehensive methodology for the fluctuation theory of continuous-time skip-free Markov chains, extending and improving the recent work of Choi and Patie [5] for discrete-time skip-free Markov chains. As a significant application, we use it to derive a full set of fluctuation identities regarding exiting a finite or infinite interval for Markov branching processes with immigration, thereby uncovering many new results for this classical family of continuous-time Markov chains. The theory also allows us to recover in a simple manner fluctuation identities for skip-free downward compound Poisson processes.

1. Introduction

First passage times are ubiquitous in probability theory and, more broadly, in many fields of sciences, such as epidemiology, physics, neurology, insurance and financial mathematics. In spite of the wide applicability of first-passage phenomena, finer characterizations of the law of these random variables beyond as solutions of a boundary value problem associated with a linear operator, have been obtained only for very few and specific classes of Markov processes, including most notably one-dimensional diffusion processes and their discrete-state space analogue birth-death chains on the one hand and processes with stationary and independent increments on the other hand, both for which tailor-made approaches have been developed. Unfortunately, attempts to carry over techniques from these cases have had limited success, providing some evidence towards the need to find a novel way to tackle this issue. In this vein, in the recent paper [5], Choi and Patie develop an original and comprehensive approach based on a combination of techniques such as the theory of Martin boundary and potential theory to provide an expression for the resolvent of a discrete-time skip-free Markov chain in terms of the so-called fundamental excessive functions of three different chains, namely the given skip-free downward (upward) Markov chain, its dual and the original one killed when going above (below) a given level. This allowed them to derive various fluctuation identities involving the first hitting time of a given point and the first exit times out of a given finite or semi-infinite interval.

In this paper we will follow the approach of Choi and Patie [5] to extend their results to the case of continuous-time skip-free Markov chains, while also improving on their methodology as explained in the beginning of Section 3. This is our first main contribution. Besides our time homogeneous chain being skip-free (downward) we further assume that all its states are stable (i.e. no instantaneous states), and allow for state-dependent killing and for explosion with the precise setting being laid out in Section 2. Our second main contribution is to use the general theory presented in Section 3 to derive a collection of fluctuation identities for Markov branching processes with immigration (MBIs), a well-studied class of continuous-time Markov chains. Most of these identities are new as explained at the end of Section 5, which shows the power of our methodology. As a quick illustration of the general theory and in order to contrast it with the case of MBIs we will also derive the same type of

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fluctuation identities for continuous-time skip-free Markov chains with stationary and independent increments, i.e. skip-free compound Poisson processes (CPPs), though for this class the resulting identities are (essentially) known.

Of course for deriving certain probabilities involving first passage times there is the analytical approach. To illustrate this, let $p(x)$ be the probability that our skip-free downward continuous-time Markov chain with state space $E \subset \mathbb{Z}$ and starting at $x \in E$ exits a given interval $(a, b) \subset E$ by hitting a. Then, with some of the terminology explained later on in Section 2, $p(x), x \in [a-1, b+1]$, is harmonic for the Markov chain killed at exiting (a, b) and as a consequence, with Q the Q-matrix of the chain, $p(x)$, $x \geq a$, solves the boundary valued problem

$$\begin{aligned} \sum_{y \in E} Q(x,y)p(y) &= p(x) \quad x \in [a+1, b-1] \\ p(a) &= 1 \text{ and } p(x) = 0 \text{ for } x \geq b \end{aligned}$$

Consequently, $p(x)$ can be computed by solving a linear system of $b - a - 1$ equations. Note that this approach becomes more difficult when E is unbounded and $a = -\infty$ or $b = \infty$ as one then has to deal with an infinite system of equations. Our main theorems in Section 3 provide in particular a structure for the solution of these systems. Besides yielding more insight into the solution, even in the case of a finite system, our results for MBIs and CPPs allow one to compute probabilities like $p(x)$ for these classes in a more efficient way than by numerically solving a system of linear equations, especially when one wants to compute $p(x)$ for various values of a and b.

The rest of the paper is organized as follows. In Section 2 we introduce various notations and concepts that will be needed later on and in particular define the class of Markov chains that we will be working with. Section 3 contains the main results where we, following Choi and Patie [5], express, for a general Markov chain, the resolvent and derive various fluctuation identities in terms of the three fundamental excessive functions, whereas in Sections 4 and 5 we apply these results to the class of skip-free CPPs and MBIs respectively.

2. Preliminaries

Let $E = [\mathfrak{l}, \mathfrak{h}] \cap \mathbb{Z}$ where $\mathfrak{l}, \mathfrak{h} \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $\mathfrak{l} < \mathfrak{h}$. We set $E_\partial = E \cup \{\partial\}$ where ∂ denotes the cemetery state. We always work in a canonical setup where

- Ω is the space of functions $\omega : [0, \infty) \to E_\partial$ with the property that there exists $\zeta(\omega) \in [0, \infty]$ such that ω is E-valued and right-continuous on $[0, \zeta(\omega))$ and $\omega(t) = \partial$ for $t \geq \zeta(\omega)$
- $X = (X_t)_{t \geq 0}$ is the process defined by $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$
- $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of X
- $\mathcal{F}$ is the smallest sigma-algebra containing $\cup_{t \geq 0} \mathcal{F}_t$

The random variable $\zeta = \inf\{t \geq 0;\ X_t = \partial\}$ is the lifetime of X. With $J_0 = 0$ and $J_n = \inf\{t \geq J_{n-1};\ X_t \neq X_{J_{n-1}}\}$ for $n \geq 1$, the random variables $J_1, J_2, \ldots$ are the consecutive jump times of X. We denote by $J_\infty = \lim_{n \to \infty} J_n$ the explosion time.

Let $Q = (Q(x,y))_{x,y \in E}$ be a given Q-matrix on E, i.e. a matrix satisfying

$$0 \leq Q(x,y) < \infty \text{ for } x \neq y \text{ and } \sum_{y \neq x} Q(x,y) \leq Q(x) < \infty$$

where $Q(x) := -Q(x,x)$. We set $Q(x, \partial) = Q(x) - \sum_{y \neq x} Q(x,y) \geq 0$ for $x \in E$. Note that Q is not assumed to be conservative, i.e. $Q(x, \partial)$ is not necessarily equal to 0. Given Q we next specify a continuous-time, time homogeneous Markov chain on $(\Omega, \mathcal{F})$ via the usual jump chain/holding time definition. Namely, we assume $\mathbb{P} = (\mathbb{P}_x)_{x \in E_\partial}$ is a family of probability measures on $(\Omega, \mathcal{F})$ such that

- $\mathbb{P}_x(X_0 = x) = 1$ for $x \in E_\partial$

- For all $n \geq 1$ and $x \in E$, given $J_{n-1} < \infty$ and $X_{J_{n-1}} = x$ the n-th holding time $H_n := J_n - J_{n-1}$ is exponentially distributed with parameter $Q(x)$ if $Q(x) > 0$ and equal to ∞ otherwise and is further conditionally independent from $\mathcal{F}_{J_{n-1}}$
- For all $n \geq 1$, $x \in E$ and $y \in E_\partial$, given $J_{n-1} < \infty$ and $X_{J_{n-1}} = x$, $X_{J_n} = y$ with probability $\frac{Q(x,y)}{Q(x)}$ if $Q(x) > 0$ and X_{J_n} is conditionally independent from $\mathcal{F}_{J_{n-1}}$ and H_n
- X is sent to the cemetery state if explosion occurs, i.e. $X_{J_\infty} = \partial$ if $J_\infty < \infty$.

We call $(X, \mathbb{P})$ the Markov chain generated by Q with state space E. Note that the Markov chain can end up in the cemetery state either due to explosion or due to a transition to ∂ by the jump chain $(X_{J_n})_{n\geq 0}$.

We write $P_t(x,y) = \mathbb{P}_x(X_t = y)$, $x, y \in E$, $t \geq 0$, for the transition kernel of $(X, \mathbb{P})$. From the theory of continuous-time Markov chains, see e.g. Section 2.8 in [12][1], the transition kernel $P_t(x,y)$, $x, y \in E$, $t \geq 0$, is the minimal nonnegative solution of both the forward equation

$$\frac{d}{dt}P_t(x,y) = \sum_{z\in E} P_t(x,z)Q(z,y), \quad P_0(x,y) = \delta_{xy} \tag{2.1}$$

with δ_{xy} the Kronecker delta, as well as the backward equation

$$\frac{d}{dt}P_t(x,y) = \sum_{z\in E} Q(x,z)P_t(z,y), \quad P_0(x,y) = \delta_{xy}$$

For a function $f : E \to \mathbb{R}$, we will use the notation

$$\begin{aligned} P_t f(x) &:= \sum_{y\in E} P_t(x,y)f(y) = \mathbb{E}_x[f(X_t)\mathbb{1}_{\{t<\zeta\}}] \\ Qf(x) &:= \sum_{y\in E} Q(x,y)f(y) \end{aligned}$$

where $\mathbb{E}_x$ is the expectation operator associated with $\mathbb{P}_x$. Throughout the paper we assume the Q-matrix Q is such that, for $x, y \in E$,

$$Q(x, x-1) > 0 \quad \text{if } x > \mathfrak{l} \quad \text{and} \quad Q(x,y) = 0 \quad \text{if } y \leq x-2. \tag{2.2}$$

This assumption means that $(X, \mathbb{P})$ is skip-free downwards, i.e. from any state, $(X, \mathbb{P})$ can reach a lower state with a positive probability but only by visiting all states in between. Next, for any $q > 0$, one defines the q-resolvent as follows

$$G_q(x,y) = \int_0^\infty e^{-qt}P_t(x,y)dt$$

For all $x, y \in E$, $G_q(x,y) < \infty$, and, by the skip-free downward assumption in combination with the right-continuity of the sample paths $G_q(x,y) > 0$ for all $x, y \in E$ such that $y \leq x$. In the case where the Markov chain is transient (i.e. all states in E are transient), the resolvent of $(X, \mathbb{P})$ is defined by $G(x,y) = \lim_{q\to 0} G_q(x,y)$, i.e.

$$G(x,y) = \int_0^\infty P_t(x,y)dt$$

The chain being transient is equivalent to $G(x,y) < \infty$ for all $x, y \in E$ (see e.g. Theorem 3.4.2 in [12]). Note that, $\int_0^\infty \mathbb{P}_x(X_t = y, t < \mathbf{e}_q \wedge \zeta)dt = G_q(x,y)$ where $\mathbf{e}_q$ is an exponentially distributed random variable of parameter $q > 0$ and taken independent of $\mathcal{F}$. Therefore G_q is the (0-)resolvent of the skip-free downward Markov chain generated by the Q-matrix $Q - qI$ with I the identity matrix.

[1]In this reference it is assumed that Q is conservative but the arguments go through if Q is not conservative.

For any $q \geq 0$, we say that a function $h : E \to [0,\infty)$ is q-excessive (resp. q-harmonic) for $(X,\mathbb{P})$, if, for all $x \in E$ and $t \geq 0$,

$$e^{-qt}P_th(x) \leq h(x) \quad \text{(resp. } e^{-qt}P_th(x) = h(x))$$

We simply say excessive (resp. harmonic) when $q = 0$. Note that, if h is q-excessive, then $\lim_{t\downarrow 0} e^{-qt}P_th(x) = h(x)$, which follows by an application of Fatou's Lemma combined with the right-continuity of P_t at $t = 0$. Similarly, we call a measure $\pi : E \to [0,\infty)$ excessive (resp. invariant) if

$$\sum_{x\in E} \pi(x)P_t(x,y) \leq \pi(y), \quad \text{(resp. } \sum_{x\in E} \pi(x)P_t(x,y) = \pi(y)), \text{ for all } y \in E \text{ and } t \geq 0$$

We assume throughout that there exists an excessive measure π for $(X,\mathbb{P})$ that is positive, i.e. $\pi(y) > 0$ for all $y \in E$. Note that in the transient case, a candidate is given by $\pi(y) = \sum_{x\in E} 2^{-\sigma(x)}G(x,y)$ for some injective map $\sigma : E \to \{1,2,\ldots\}$, see also [9, Section 5.2]. We now fix such a positive excessive measure π as our reference measure and we set

$$G_q(x,y) = \mathsf{g}_q(x,y)\pi(y) \tag{2.3}$$

g_q is called the q-resolvent density, and we simply write $\mathsf{g}(x,y) = \mathsf{g}_0(x,y)$ when $(X,\mathbb{P})$ is transient.

Next, we introduce some auxiliary Markov chains associated with $(X,\mathbb{P})$. First, writing $\widehat{\mathbb{P}} = (\widehat{\mathbb{P}}_y)_{y\in E_\partial}$, the dual $(X,\widehat{\mathbb{P}})$ of $(X,\mathbb{P})$ with respect to π is the Markov chain with state space E generated by the Q-matrix $\widehat{Q}$ given, for any $y,x \in E$, by

$$\widehat{Q}(y,x)\pi(y) = \pi(x)Q(x,y)$$

Note that $\widehat{Q}$ is a well-defined Q-matrix since π is a positive measure, and, for each $y \in E$,

$$\begin{aligned}\pi(y)\sum_{x\in E}\widehat{Q}(y,x) = \sum_{x\in E}\pi(x)Q(x,y) &\leq \liminf_{t\downarrow 0}\sum_{x\neq y}\pi(x)\frac{P_t(x,y)}{t} + \pi(y)\lim_{t\downarrow 0}\frac{P_t(y,y)-1}{t}\\ &= \liminf_{t\downarrow 0}\frac{\sum_{x\in E}\pi(x)P_t(x,y)-\pi(y)}{t}\\ &\leq 0,\end{aligned}$$

where the first inequality is due to Fatou's lemma, and, from the backward equation, $\frac{d}{dt}P_t(x,y)_{|t=0} = Q(x,y)$ and the last inequality is because π is an excessive measure for $(X,\mathbb{P})$. Note that $(X,\widehat{\mathbb{P}})$ is skip-free upwards, i.e. $(-X,\widehat{\mathbb{P}})$ is skip-free downwards. Denoting by $P_t(y,x) := \widehat{\mathbb{P}}_y(X_t = x)$ the transition kernel of $(X,\widehat{\mathbb{P}})$, we easily see that $\frac{\pi(x)P_t(x,y)}{\pi(y)}$, $y,x \in E$, $t \geq 0$, is the minimal nonnegative solution of the backward (or forward) equations associated with $\widehat{Q}$ and therefore

$$\widehat{P}_t(y,x)\pi(y) = \pi(x)P_t(x,y)$$

Denoting by $\widehat{G}_q$ and $\widehat{\mathsf{g}}_q$ respectively the q-resolvent and the q-resolvent density (with respect to π) of the dual, we immediately observe that, for any $y,x \in E$,

$$\widehat{\mathsf{g}}_q(y,x) = \mathsf{g}_q(x,y)$$

Now, denote by T_A the first hitting time of the set $A \subseteq \mathbb{R}$, that is

$$T_A = \inf\{t \geq 0;\ X_t \in A\}$$

with the usual convention that $\inf\{\emptyset\} = \infty$. We will write $T_a := T_{\{a\}}$ for $a \in E$. For $A \subseteq \mathbb{R}$ and $B \subseteq E_\partial$ we write $B^A := B\backslash A$. We will also use the notation $[b$ for the set $[b,\infty)$.

For $A \subseteq \mathbb{R}$ and writing $\mathbb{P}^A = \left(\mathbb{P}_x^A\right)_{x\in E_\partial^A}$, the second auxiliary Markov chain $(X, \mathbb{P}^A)$ is the one with state space E^A and generated by the Q-matrix Q^A given, for any $x, y \in E^A$, by

$$Q^A(x,y) = Q(x,y)$$

Let $P_t^A(x,y) := \mathbb{P}_x^A(X_t = y)$ be the transition kernel of $(X, \mathbb{P}^A)$. By mimicking Steps 1 and 2 of the proof of [12, Theorems 2.8.3 and 2.8.4] one obtains that $\mathbb{P}_x(X_t = y, t < T^A)$, $x, y \in E^A$, $t \geq 0$, is the minimal nonnegative solution of the backward equations associated with Q^A and thus $P_t^A(x,y) = \mathbb{P}_x(X_t = y, t < T_A)$. Therefore, we call $(X, \mathbb{P}^A)$ the Markov chain obtained by killing $(X, \mathbb{P})$ at the stopping time T_A. The semi-group property entails that, for all $t \geq 0$, $x \in E$ and $\Gamma \in \mathcal{F}_t$, $\mathbb{P}_x^A(\Gamma) = \mathbb{P}_x(\Gamma \cap \{t < T_A\})$. Then, for any stopping time T, one can show by first considering stopping times with a discrete range and then approximating a general stopping time by a non-increasing sequence of stopping times with a discrete range, that, for any $\Gamma \in \mathcal{F}_T$ and $x \in E$,

$$\mathbb{P}_x^A(\Gamma \cap \{T < \infty\}) = \mathbb{P}_x(\Gamma \cap \{T < T_A\}) \tag{2.4}$$

The q-resolvent of $(X, \mathbb{P}^A)$ is denoted by $G_q^A(x,y) = \int_0^\infty e^{-qt} P_t^A(x,y)dt$, $x, y \in E^A$, and we extend it to the whole of $E \times E$ by setting $G_q^A(x,y) = 0$ for $(x,y) \notin E^A \times E^A$. The corresponding q-resolvent density is denoted, for any $x, y \in E$, by

$$G_q^A(x,y) = \mathsf{g}_q^A(x,y)\pi(y)$$

We can consider the dual of $(X, \mathbb{P}^A)$ or kill the dual at T_A. Clearly by looking at the Q-matrices involved, the operations of taking the dual (with respect to π) and killing at T_A commute and the following switching identity holds:

$$\widehat{\mathsf{g}}_q^A(y,x) = \mathsf{g}_q^A(x,y), \quad x, y \in E, \tag{2.5}$$

where $\widehat{G}_q^A(y,x)$ is the q-resolvent and $\widehat{\mathsf{g}}_q^A(y,x)$ its density of the dual process killed at T_A. Very closely related to $(X, \mathbb{P}^A)$, is the third auxiliary Markov chain that we introduce. For $A \subseteq \mathbb{R}$ and writing $\overline{\mathbb{P}}^A = \left(\overline{\mathbb{P}}_x^A\right)_{x\in E_\partial}$, let $(X, \overline{\mathbb{P}}^A)$ be the Markov chain with state space E and generated by the Q-matrix $\overline{Q}^A$ given by

$$\overline{Q}^A(x,y) = \begin{cases} Q(x,y) & \text{if } x \in E^A \text{ and } y \in E \\ 0 & \text{if } x \in A \text{ and } y \in E \end{cases}$$

Similarly as for the previous auxiliary Markov chain, one can show that the transition kernel $\overline{P}_t^A(x,y) := \overline{\mathbb{P}}_x^A(X_t = y)$ of $(X, \overline{\mathbb{P}}^A)$ admits the expression $\overline{P}_t^A(x,y) = \mathbb{P}_x(X_{t\wedge T_A} = y)$, $x, y \in E$, $t \geq 0$. Therefore, we call $(X, \overline{\mathbb{P}}^A)$ the Markov chain obtained by stopping $(X, \mathbb{P})$ at T_A. Finally, for $a \in E$, with slight abuse of notation, we write $E_\partial^a := E_\partial \cap [a, \infty)$ and $\overline{\mathbb{P}}^a = \left(\overline{\mathbb{P}}_x^a\right)_{x\in E_\partial^a}$ and denote by $(X, \overline{\mathbb{P}}^a)$ the Markov chain obtained by stopping $(X, \mathbb{P})$ at T_a. Note that the state space of $(X, \overline{\mathbb{P}}^a)$ is chosen to be $E^a := E \cap [a, \infty)$ and not E, which we can do because $(X, \mathbb{P})$ is skip-free downward. Its transition kernel is denoted by $\overline{P}_t^a$. For the convenience of the reader, we gather in the table below the object associated to the different Markov chains we have introduced.

	transition kernel	q-resolvent
$(X, \mathbb{P})$	$P_t(x,y)$	$G_q(x,y) = \mathsf{g}_q(x,y)\pi(y)$
$(X, \widehat{\mathbb{P}})$	$\widehat{P}_t(y,x)\pi(x) = P_t(x,y)\pi(y)$	$\widehat{\mathsf{g}}_q(y,x) = \mathsf{g}_q(x,y)$
$(X, \mathbb{P}^A)$	$P_t^A(x,y) = P_x(X_t = y, t < T_A)$	$G_q^A(x,y) = \mathsf{g}_q^A(x,y)\pi(y)$
$(X, \widehat{\mathbb{P}}^A)$	NA	$\widehat{\mathsf{g}}_q^A(y,x) = \mathsf{g}_q^A(x,y)$
$(X, \overline{\mathbb{P}}^A)$	$\overline{P}_t^A(x,y) = \mathbb{P}_x(X_{t\wedge T_A} = y)$	NA

3. Potential theory and fluctuation identities of continuous-time skip-free Markov Process

The following two theorems are, to some extent, the analogue of the results in Section 3 and the first part of Section 4 in Choi and Patie [5] who deal with discrete-time Markov chains that are skip-free upwards. However, there are some differences with the most notable ones being the following. First, translating their approach to our setting of continuous-time skip-free downward Markov chains, Choi and Patie [5] distinguish between the cases (i) $\mathfrak{l} = -\infty$ or $\mathfrak{l} > -\infty$ with $\mathbb{P}_{\mathfrak{l}}(X_{J_1} \in E\backslash\{\mathfrak{l}\}) = 0$ on the one hand and (ii) $\mathfrak{l} > -\infty$ with $\mathbb{P}_{\mathfrak{l}}(X_{J_1} \in E\backslash\{\mathfrak{l}\}) > 0$ on the other hand, with their results being less comprehensive in the latter case, due to technical difficulties arising in the use of the theory of Martin boundary. To overcome this difficulty, we propose another definition of the fundamental excessive function associated to the killed process than the one in [5, Theorem 3.1(2)] (though both definitions are equivalent in case (i)). Moreover, our viewpoint enables to unify the two cases and at the same time give a more elementary proof that avoids the Martin boundary theory employed in [5]. Second, we allow for state-dependent killing whereas Choi and Patie [5] worked with a constant killing rate. We recall that the notation and terminologies used in the following two theorems can be found in Section 2.

Theorem 3.1. *Let $(X, \mathbb{P})$ be a Markov chain with state space E generated by the Q-matrix Q satisfying (2.2), $q > 0$ (resp. $q \geq 0$ if X is transient) and fix a reference point $\mathfrak{o} \in E$.*

(1) Define, for any $x \in E$,

$$H_q(x) = \lim_{y \downarrow \mathfrak{l}} \frac{G_q(x,y)}{G_q(\mathfrak{o},y)} := \begin{cases} \lim_{y\downarrow -\infty} \frac{G_q(x,y)}{G_q(\mathfrak{o},y)} & \text{if } \mathfrak{l} = -\infty \\ \frac{G_q(x,\mathfrak{l})}{G_q(\mathfrak{o},\mathfrak{l})} & \text{if } \mathfrak{l} > -\infty \end{cases}$$

Then, for any $x, a \in E$ such that $x \geq a$,

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<\zeta\}}\right] = \frac{H_q(x)}{H_q(a)} \tag{3.1}$$

(2) The function H_q is q-excessive for $(X, \mathbb{P})$ and it is the unique positive-valued function on E satisfying the following four properties: (i) non-increasing, (ii) $H_q(\mathfrak{o}) = 1$, (iii) for any $a \in E$, H_q restricted to E^a is q-harmonic for $(X, \overline{\mathbb{P}}^a)$ in the sense that for any $t \geq 0, x \in E^a$, $\mathbb{E}_x[e^{-q(t\wedge T_a)}H_q(X_{t\wedge T_a})\mathbb{1}_{\{t\wedge T_a<\zeta\}}] = H_q(x)$, (iv) if $\mathfrak{h} = \infty$, $q = 0$ and if there exists $x > a$ such that $\mathbb{P}_x(T_a = \zeta = \infty) > 0$, then $\lim_{x\to\infty} H_q(x) = 0$.

(3) Define

$$\widehat{H}_q(y) = \lim_{x\uparrow\mathfrak{h}} \frac{\widehat{G}_q(y,x)}{\widehat{G}_q(\mathfrak{o},x)},\ y \in E \quad \text{and} \quad \mathbf{H}_q^{[b}(x) = \lim_{y\downarrow\mathfrak{l}} \frac{G_q^{[b}(x,y)}{G_q(\mathfrak{o},y)},\ x \in E$$

Then, writing $\mathsf{c}_q = \mathsf{g}_q(\mathfrak{o},\mathfrak{o})$, we have, for any $x, y \in E$,

$$\mathsf{g}_q(x,y) = \mathsf{c}_q\widehat{H}_q(y)\left(H_q(x) - \mathbf{H}_q^{[y}(x)\right) \tag{3.2}$$

Remark 3.2. (1) The function H_q is not necessarily q-harmonic for $(X, \mathbb{P})$ if $\mathfrak{l} = -\infty$ whereas, in the discrete-time case, H_q is q-harmonic when $\mathfrak{l} = -\infty$, see [5, Theorem 3.1]. For instance when $(X, \mathbb{P})$ is the negative of a birth process which explodes, then $H_0 \equiv 1$ but H_0 is not harmonic since the lifetime is a.s. finite.

(2) If $h_q : E \to (0, \infty)$ is q-harmonic for $(X, \mathbb{P})$ then h_q satisfies property (iii) in Theorem 3.1(2). Indeed, if h_q is q-harmonic, then by the Markov property for any $s \leq t$ and $x \in E$,

$$\mathbb{E}_x[e^{-qt}h_q(X_t)\mathbb{1}_{\{t<\zeta\}}|\mathcal{F}_s] = \mathbb{1}_{\{s<\zeta\}}e^{-qs}\mathbb{E}_{X_s}[e^{-q(t-s)}h_q(X_{t-s})\mathbb{1}_{\{t-s<\zeta\}}] = e^{-qs}h_q(X_s)\mathbb{1}_{\{s<\zeta\}}$$

So $(e^{-qt}h_q(X_t)\mathbb{1}_{\{t<\zeta\}})_{t\geq 0}$ is a martingale and therefore is $(e^{-q(t\wedge T_a)}h_q(X_{t\wedge T_a})\mathbb{1}_{\{t\wedge T_a<\zeta\}})_{t\geq 0}$ for any $a \in E$. This yields (iii).

(3) Property (iv) in Theorem 3.1(2) is needed to characterise H_q when $q = 0$. Indeed, if $\mathfrak{h} = \infty$ and $(X, \mathbb{P})$ is such that $\lim_{t\to\infty} X_t = \infty$ $\mathbb{P}_x$-a.s., then $h \equiv 1$ satisfies properties (i)-(iii) but $H_0 \neq h$.

Theorem 3.3. *Under the assumptions of Theorem 3.1, using its notation, and, for any $q > 0$ (resp. $q \geq 0$ if X is transient), we have the following.*

(a) For any $x, y \in E$,

$$\mathbb{E}_x\left[e^{-qT_y}\mathbb{I}_{\{T_y<\zeta\}}\right] = \frac{H_q(x) - \mathbf{H}_q^{[y}(x)}{H_q(y)} \tag{3.3}$$

In particular, for any $x, a \in E$ such that $x \geq a$,

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<\zeta\}}\right] = \frac{H_q(x)}{H_q(a)} \tag{3.4}$$

(b) For any $x, a, b \in E$ such that $a \leq x < b$,

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<T_{[b}\wedge\zeta\}}\right] = \frac{\mathbf{H}_q^{[b}(x)}{\mathbf{H}_q^{[b}(a)} \tag{3.5}$$

(c) If $\mathfrak{l} = -\infty$, let $\mathbf{H}_q^{[b}(-\infty) := \lim_{x\to-\infty}\mathbf{H}_q^{[b}(x) \in (0, \infty]$. Then, for any $x, b \in E$,

$$\mathbb{E}_x\left[e^{-qT_{[b}}\mathbb{I}_{\{T_{[b}<\zeta\}}\right] = \begin{cases} 1 - \sum_{y\in E^{[b}}(Q(y,\partial)+q)\mathsf{g}_q^{[b}(x,y)\pi(y) - \frac{\mathbf{H}_q^{[b}(x)}{\mathbf{H}_q^{[b}(-\infty)} & \text{if } \mathfrak{l} = -\infty \\ 1 - \sum_{y\in E^{[b}}(Q(y,\partial)+q)\mathsf{g}_q^{[b}(x,y)\pi(y) & \text{if } \mathfrak{l} > -\infty \end{cases} \tag{3.6}$$

where, for any $x, y \in E^{[b}$,

$$\mathsf{g}_q^{[b}(x,y) = \mathsf{c}_q\widehat{H}_q(y)\left(\mathbf{H}_q^{[b}(x) - \mathbf{H}_q^{[y}(x)\right) \tag{3.7}$$

(d) Let $a, b \in E$ such that $a < b$. Assume $f : E \to \mathbb{R}$ is such that $Q|f|(x) < \infty$ for all $x \in E^{(a,b)^{\mathsf{c}}}$. Then, for any $x \in E$,

$$\mathbb{E}_x\left[e^{-qT_{(a,b)^{\mathsf{c}}}}f\left(X_{T_{(a,b)^{\mathsf{c}}}}\right)\mathbb{1}_{\{T_{(a,b)^{\mathsf{c}}}<\zeta\}}\right] = f(x) + \sum_{y\in E^{(a,b)^{\mathsf{c}}}}(Q-qI)f(y)\mathsf{g}_q^{(a,b)^{\mathsf{c}}}(x,y)\pi(y) \tag{3.8}$$

where, for any $x, y \in E^{(a,b)^{\mathsf{c}}}$,

$$\mathsf{g}_q^{(a,b)^{\mathsf{c}}}(x,y) = \mathsf{c}_q\widehat{H}_q(y)\left(\frac{\mathbf{H}_q^{[b}(x)}{\mathbf{H}_q^{[b}(a)}\mathbf{H}_q^{[y}(a) - \mathbf{H}_q^{[y}(x)\right) \tag{3.9}$$

In particular,

$$\mathbb{E}_x\left[e^{-qT_{(a,b)^{\mathsf{c}}}}\mathbb{1}_{\{T_{(a,b)^{\mathsf{c}}}<\zeta\}}\right] = 1 - \sum_{y\in E^{(a,b)^{\mathsf{c}}}}(Q(y,\partial)+q)\,\mathsf{g}_q^{(a,b)^{\mathsf{c}}}(x,y)\pi(y) \tag{3.10}$$

3.1. **Proof of Theorems 3.1 and 3.3.**

Proof of Theorem 3.1. Let $x, y, z, a, b \in E$, and, without loss of generality, we set $q = 0$ and assume that X is transient as the proof follows the same line of reasoning for $q > 0$. For the item (1), we start by proving the following claim:

$$\frac{G(x,y)}{G(\mathfrak{o},y)} = \frac{G(x,x\wedge\mathfrak{o})}{G(o,x\wedge\mathfrak{o})}, \quad y \leq x\wedge\mathfrak{o} \tag{3.11}$$

By the strong Markov property, $G(z,a) = \mathbb{P}_z(T_a < \zeta)G(a,a)$ for any $z, a \in E$. Hence,

$$\frac{G(b,y)}{G(y,y)} = \mathbb{P}_b(T_y < \zeta) = \mathbb{P}_b(T_z < \zeta)\mathbb{P}_z(T_y < \zeta) = \frac{G(b,z)}{G(z,z)}\frac{G(z,y)}{G(y,y)}, \quad y \le z \le b,$$

where the second equality follows by the strong Markov property and the skip-free downward property of $(X,\mathbb{P})$. When $x \le \mathfrak{o}$, take $z = x$ and $b = \mathfrak{o}$, to get for $y \le x \le \mathfrak{o}$, $G(\mathfrak{o},y) = \frac{G(\mathfrak{o},x)}{G(x,x)}G(x,y)$ which shows (3.11) in the case where $x \le \mathfrak{o}$. Similarly, if $\mathfrak{o} \le x$, take $z = \mathfrak{o}$ and $b = x$ instead to get for $y \le \mathfrak{o} \le x$, $G(x,y) = \frac{G(x,\mathfrak{o})}{G(\mathfrak{o},\mathfrak{o})}G(\mathfrak{o},y)$, which shows (3.11) in the case where $x \ge \mathfrak{o}$. From (3.11) it follows that $H(x) = \lim_{y\downarrow \mathfrak{l}} \frac{G(x,y)}{G(\mathfrak{o},y)}$ is well-defined. By the strong Markov property and the skip-free downward property, $G(x,y) = \mathbb{P}_x(T_a < \zeta)G(a,y)$ for $y \le a \le x$. Hence

$$\mathbb{P}_x(T_a < \zeta) = \frac{G(x,y)}{G(a,y)} = \frac{G(x,y)/G(\mathfrak{o},y)}{G(a,y)/G(\mathfrak{o},y)}, \quad y \le a \le x.$$

Now taking limits as $y \downarrow \mathfrak{l}$ yields (3.1). For the item (2), by the semi-group property of P_t, for $y \le x$,

$$\sum_{z\in E} P_t(x,z)G(z,y) = \sum_{z\in E} P_t(x,z)\int_0^\infty P_s(z,y)ds = \int_0^\infty P_{t+s}(x,y)ds = \int_t^\infty P_u(x,y)du < G(x,y)$$

Thus, if $\mathfrak{l} > -\infty$, $P_tH(x) < H(x)$ for all $x \in E$ and hence H is excessive (but not harmonic) for $(X,\mathbb{P})$. On the other hand if $\mathfrak{l} = -\infty$, then H is the limit as $n \to \infty$ of the sequence of excessive functions $H^{(n)}(x) := \frac{G(x,-n)}{G(\mathfrak{o},-n)}$, $-n \in E$, and thus is excessive since, by Fatou's lemma,

$$P_tH = P_t \lim H^{(n)} \le \liminf P_tH^{(n)} \le \liminf H^{(n)} = H$$

By means of (3.1) and the fact that $(X,\mathbb{P})$ is skip-free downward, we deduce that the function H is non-increasing. Obviously, $H(\mathfrak{o}) = 1$ by definition. Fix now $a \in E$ and let $h^a(x) := \mathbb{P}_x(T_a < \zeta)$, $x \in E^a$. Then, for any $x \in E^a$ and $t \ge 0$,

$$\begin{aligned}
\overline{P}^a_t h^a(x) &= \mathbb{E}_x\left[h^a(X_{t\wedge T_a})\mathbb{1}_{\{t\wedge T_a<\zeta\}}\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{t\ge T_a, T_a<\zeta\}}\mathbb{P}_{X_{T_a}}(T_a<\zeta)\right] + \mathbb{E}_x\left[\mathbb{1}_{\{t<T_a\wedge\zeta\}}\mathbb{P}_x(T_a<\zeta|\mathcal{F}_t)\right] \\
&= \mathbb{P}_x(t \ge T_a, T_a < \zeta) + \mathbb{P}_x(t < T_a\wedge\zeta, T_a < \zeta) \\
&= h^a(x)
\end{aligned}$$

where we have used the Markov property in the second equality. Thus, h^a is harmonic for $(X,\overline{\mathbb{P}}^a)$. Then, by (3.1) H restricted to E^a is harmonic for $(X,\overline{\mathbb{P}}^a)$. Let $h : E \to (0,\infty)$ satisfy (i) and (iii). Then for $x \in E^a$,

$$\begin{aligned}
h(x) &= \lim_{t\to\infty} \overline{P}^a_t h(x) = \mathbb{E}_x\left[\lim_{t\to\infty} h(X_{t\wedge T_a})\mathbb{1}_{\{t\wedge T_a<\zeta\}}\right] \\
&= h(a)\mathbb{P}_x(T_a<\zeta) + \lim_{y\to\infty} h(y)\mathbb{P}_x(T_a = \zeta = \infty) \qquad (3.12)
\end{aligned}$$

where the second equality is due to the dominated convergence theorem which is applicable because h is non-increasing and positive, $x \ge a$ and $(X,\mathbb{P})$ is skip-free downward and for the last equality we used that by transience $X_t \to \infty$ as $t \to \infty$ if $T_a = \zeta = \infty$ (which can only happen if $\mathfrak{h} = \infty$). So if h satisfies (i)-(iv), then we conclude $h(x) = h(a)\mathbb{P}_x(T_a < \zeta)$ for all $x \ge a$, which yields that h is proportional to H on E^a by (3.1). Since $a \in E$ was chosen arbitrarily, h is positive-valued and $h(\mathfrak{o}) = H(\mathfrak{o})$, it follows that $h = H$ on E if h satisfies (i)-(iv). Since we have showed that H satisfy (i) and (iii), (3.12) holds for H and thus H satisfies property (iv) by (3.1). We now turn to the proof of (3). By the strong Markov property and the skip-free upward property of the dual, for

$x, y, b \in E$ such that $y \le b$,

$$\widehat{G}(y,x) = \widehat{G}^{[b}(y,x) + \widehat{\mathbb{P}}_y(T_{[b} < \zeta)\widehat{G}(b,x) = \widehat{G}^{[b}(y,x) + \frac{\widehat{H}(y)}{\widehat{H}(b)}\widehat{G}(b,x)$$

where for the second equality we used the analogue of (3.1) for the dual which is justified because $(X, \widehat{\mathbb{P}})$ is skip-free upward and is transient. Then, by the switching identity (2.5),

$$G^{[b}(x,y) = G(x,y) - G(x,b)\frac{\widehat{H}(y)\pi(y)}{\widehat{H}(b)\pi(b)}, \quad y \le b \tag{3.13}$$

Since by the same arguments $\frac{G(\mathfrak{o},y)}{\pi(y)} = \mathsf{g}(\mathfrak{o},y) = \widehat{\mathsf{g}}(y,\mathfrak{o}) = \widehat{\mathbb{P}}_y(T_{\mathfrak{o}} \le \zeta)\mathsf{g}(\mathfrak{o},\mathfrak{o}) = \mathsf{c}\frac{\widehat{H}(y)}{\widehat{H}(\mathfrak{o})}$ for $y \le \mathfrak{o}$,

$$\frac{G^{[b}(x,y)}{G(\mathfrak{o},y)} = \frac{G(x,y)}{G(\mathfrak{o},y)} - \frac{G(x,b)}{\mathsf{c}\widehat{H}(b)\pi(b)}, \quad y \le b \wedge \mathfrak{o}$$

Taking limits as $y \downarrow \mathfrak{l}$ shows that $\mathbf{H}^{[b}$ is well-defined and $\mathbf{H}^{[b}(x) = H(x) - \frac{G(x,b)}{\mathsf{c}\widehat{H}(b)\pi(b)}$. The expression for the resolvent density of G then follows. □

Proof of Theorem 3.3. As in the proof of Theorem 3.1, and after recalling that the case $q > 0$ corresponds to the case $q = 0$ and changing the Q-matrix to $Q - qI$, without loss of generality, we set $q = 0$ and assume that $(X, \mathbb{P})$ is transient. First, by the strong Markov property $G(x,y) = \mathbb{P}_x(T_y < \zeta)G(y,y)$ for $x, y \in E$. The identity (3.3) then follows from (3.2) since $\mathbf{H}^{[y}(x) = 0$ for $x \ge y$. For the item (b), since $(X, \mathbb{P}^{[b})$ is skip-free downward with all states transient (3.4) is applicable for this Markov chain. Hence, by (2.4) and with $H^{[b}(x) := \lim_{y\downarrow\mathfrak{l}} \frac{G^{[b}(x,y)}{G^{[b}(\mathfrak{o}^{[b},y)}$ where $\mathfrak{o}^{[b} \in E^{[b}$ is a reference point,

$$\mathbb{P}_x(T_a < T_{[b} \wedge \zeta) = \mathbb{P}^{[b}(T_a < \zeta) = \frac{H^{[b}(x)}{H^{[b}(a)}, \quad x \ge a$$

The required identity then follows since $\mathbf{H}^{[b}(x) = \lim_{y\downarrow\mathfrak{l}} \frac{G^{[b}(x,y)}{G^{[b}(\mathfrak{o}^{[b},y)} \frac{G^{[b}(\mathfrak{o}^{[b},y)}{G(\mathfrak{o},y)} = H^{[b}(x)\mathbf{H}^{[b}(o^{[b})$, $x \in E$. Next for the item (d), we first prove (3.9) and (3.7). The latter follows directly from (3.13) and (3.2). Since by the strong Markov property and $(X, \mathbb{P})$ being skip-free downward,

$$\mathsf{g}^{[b}(x,y) = \mathsf{g}^{(a,b)^{\mathsf{c}}}(x,y) + \mathbb{P}_x^{[b}(T_a < \zeta)\mathsf{g}^{[b}(a,y), \quad x \ge a, y \in E$$

(3.9) follows from (3.5) and (3.7). Now we prove (3.8). Note that without loss of generality we can assume that f is bounded because the general case can be proved by approximation. By the forward equation (2.1), $\overline{P}_t^{(a,b)^{\mathsf{c}}}(x,y) = \delta_{xy} + \int_0^t \sum_{z\in E} \overline{P}_s^{(a,b)^{\mathsf{c}}}(x,z)\overline{Q}^{(a,b)^{\mathsf{c}}}(z,y)ds$, $x, y \in E$, $t \ge 0$. Therefore, for $f : E \to \mathbb{R}$ bounded, $x \in E$ and $t \ge 0$, by definition of $\overline{Q}^{(a,b)^{\mathsf{c}}}$,

$$\overline{P}_t^{(a,b)^{\mathsf{c}}} f(x) = f(x) + \sum_{y\in E^{(a,b)^{\mathsf{c}}}} Qf(y) \int_0^t \overline{P}_s^{(a,b)^{\mathsf{c}}}(x,y)ds$$

where the interchanging of the two sums is justified because $E^{(a,b)^{\mathsf{c}}}$ is a finite set. By the transience assumption, $T_{(a,b)^{\mathsf{c}}} \wedge \zeta < \infty$ $\mathbb{P}_x$-a.s. for all $x \in E^{(a,b)^{\mathsf{c}}}$. As f is assumed to be bounded, the dominated convergence theorem yields

$$\begin{aligned}
\mathbb{E}_x\left[f(X_{T_{(a,b)^{\mathbf{c}}}})\mathbf{1}_{\{T_{(a,b)^{\mathbf{c}}}<\zeta\}}\right] &= \lim_{t\to\infty}\mathbb{E}_x\left[f(X_{t\wedge T_{(a,b)^{\mathbf{c}}}})\mathbf{1}_{\{t\wedge T_{(a,b)^{\mathbf{c}}}<\zeta\}}\right] \\
&= \lim_{t\to\infty}\left(\overline{P}_t^{(a,b)^{\mathbf{c}}}f\right)(x) \\
&= f(x)+\sum_{z\in E^{(a,b)^{\mathbf{c}}}} Qf(z)\int_0^\infty \overline{P}_s^{(a,b)^{\mathbf{c}}}(x,z)ds \\
&= f(x)+\sum_{z\in E^{(a,b)^{\mathbf{c}}}} Qf(z)G^{(a,b)^{\mathbf{c}}}(x,z)
\end{aligned}$$

The identity (3.10) then simply follows by taking $f\equiv 1$ and the definition of $Q(x,\partial)$. We now turn to the item (c). When $\mathfrak{l}>-\infty$, we can use the same arguments for (3.10) to prove (3.6) since $E^{[b}$ is a finite set and $T_{[b}\wedge\zeta<\infty$ $\mathbb{P}_x$-a.s. for all $x\in E^{[b}$. Now assume $\mathfrak{l}=-\infty$. By (3.10) and (3.5), for $x,a,b\in E$ such that $a<b,x$,

$$\begin{aligned}
\mathbb{P}_x(T_{[b}<T_a\wedge\zeta) &= \mathbb{P}_x(T_{(a,b)^{\mathbf{c}}}<\zeta)-\mathbb{P}_x(T_a<T_{[b}\wedge\zeta) \\
&= 1-\sum_{z\in E^{(a,b)^{\mathbf{c}}}} Q(z,\partial)G^{(a,b)^{\mathbf{c}}}(x,z)-\frac{\mathbf{H}^{[b}(x)}{\mathbf{H}^{[b}(a)}
\end{aligned}$$

Since $\mathfrak{l}=-\infty$ we have for any $x\in E$, $\mathbb{P}_x$-a.s., $T_a\wedge\zeta\to\zeta$ as $a\to-\infty$ since the Markov chain is sent to the cemetery state when explosion occurs. Hence, (3.6) for $\mathfrak{l}=-\infty$ follows by taking $a\to-\infty$ in the last equation and using the monotone convergence theorem. □

4. Downward skip-free compound Poisson processes

In this section, we apply the methodology described in Section 3 to recover various fluctuation identities for downward skip-free $\mathbb{Z}$-valued compound Poisson processes. We recall that for the continuous-state space analogue of this class of Markov processes, namely spectrally positive Lévy processes, this problem has been well studied and has found an impressive range of applications, such as insurance mathematics, epidemiology, financial mathematics and queuing theory, see [10, 6]. For the discrete-time analogue, i.e. for random walks, Spitzer [14] solved this problem by means of the celebrated Wiener-Hopf factorization and alternative interesting proofs, based either on excursion theory or martingales devices have been proposed, see again [10, 6] and the references therein. However, all of these approaches rely on the stationarity and independent increments property of the random walks/Lévy processes, which makes it difficult to extend them to a wider context. Note that the continuous-time, discrete-state space case considered here has been studied in [15].

In this section we assume that $E=\mathbb{Z}$ and the Q-matrix generating $(X,\mathbb{P})$ is given by, for $x,y\in\mathbb{Z}$,

$$Q(x,y)=\begin{cases}\mu(y-x+1)\alpha & \text{if } y\geq x+1 \text{ or } y=x-1\\ -(\alpha+p) & \text{if } y=x\\ 0 & \text{if } y\leq x-2\end{cases} \tag{4.1}$$

where $\alpha>0$, $\mu=(\mu(j))_{j\geq 0}$ is a probability measure with $\mu(0)>0$ and $\mu(1)=0$ and $p\geq 0$. Note that $Q(x,\partial)=p$ for $x\in\mathbb{Z}$. This means that the holding times are i.i.d. and exponentially distributed and the jump sizes are i.i.d. and independent from the holding times. Thus, $(X,\mathbb{P})$ corresponds to a $\mathbb{Z}$-valued skip-free downward compound Poisson process with intensity α, jump distribution $\mathbb{P}_0(X_{J_1}=j)=\mu(j+1)$, $j\geq -1$, and killed at the constant rate p. For a function $f:\mathbb{Z}\to\mathbb{R}$ we denote its generating function (GF) by

$$f^{\star}(s):=\sum_{y\in\mathbb{Z}} s^y f(y)$$

We have, for $x \in \mathbb{Z}$ and $s \in (0,1]$,

$$Q^\star(x,\cdot)(s) = \psi(s)s^{x-1} \tag{4.2}$$

where, setting $\mu(j) = 0$ for $j \le -1$,

$$\psi(s) = \alpha\left(\mu^\star(s) - s\right) - ps$$

Further, we define

$$\theta = \inf\{s > 0;\ \psi(s) = 0\}$$

Since ψ is convex, $\psi(0) = \alpha\mu_0 > 0$ and $\psi(1) = -p \le 0$, we have $\theta \in (0,1]$ and $\psi(s) > 0$ for $s \in [0,\theta)$. Note that $P_t^\star(x,\cdot)(s)$ is well-defined for $s = e^{iu}$ for $u \in \mathbb{R}$. Using the forward equations, Fubini and (4.2), we get for $t \ge 0$, $x \in \mathbb{Z}$ and $s = e^{iu}$ with $u \in \mathbb{R}$,

$$P_t^\star(x,\cdot)(s) = 1 + \int_0^t \sum_{z\in\mathbb{Z}} P_v(x,z)Q^\star(z,\cdot)(s)dv = 1 + \frac{\psi(s)}{s}\int_0^t P_v(x,\cdot)^\star(s)dv$$

Then, by taking the derivative in t, we get $\frac{d}{dt}P_t^\star(x,\cdot)(s) = \frac{\psi(s)}{s}P_t^\star(x,\cdot)(s)$. Solving this simple ODE with the boundary condition $P_0(x,\cdot)^\star(s) = s^x$ yields, for $t \ge 0$, $x \in \mathbb{Z}$ and $s = e^{iu}$ with $u \in \mathbb{R}$,

$$\mathbb{E}_x\left[s^{X_t}\mathbb{1}_{\{t<\zeta\}}\right] = P_t^\star(x,\cdot)(s) = s^x e^{\frac{\psi(s)}{s}t} \tag{4.3}$$

By analytic extension (4.3) also holds for $s \in (0,1]$. Also (4.3) implies the spatial homogeneity property $P_t(x,y) = P_t(z, y-x+z)$ for $x,y,z \in \mathbb{Z}$. Clearly all states are transient if $p > 0$. Further, as can be deduced from the theory of random walks, if $p = 0$ then all states are transient if and only if $\mathbb{E}_0[X_1] \neq 0$ and by differentiating in s both sides of (4.3) we see that $\mathbb{E}_0[X_1] = \psi'(1) = \alpha(m-1)$ where $m := \sum_{j\ge 2} j\mu(j)$. Hence all states are transient for $(X,\mathbb{P})$ if and only if $p > 0$ or $m \neq 1$. Otherwise all states are recurrent. In order to present the fluctuation identities we introduce the function $W : \mathbb{Z} \to [0,\infty)$ defined by $W(x) = 0$ for $x < 0$, $W(0) = \frac{1}{\alpha\mu(0)}$ and for $x \ge 0$,

$$W(x+1) = \frac{1}{\alpha\mu(0)} + \frac{1}{\mu(0)}\sum_{j=0}^{x} W(j)\left(\overline{\mu}(x-j+1) + \frac{p}{\alpha}\right) \tag{4.4}$$

where $\overline{\mu}(k) = 1 - \sum_{j\le k}\mu(j)$. It is easy to verify that the function W on $\mathbb{Z}\cap[0,\infty)$ is identical to the one defined in Lemma 5.7 below and, hence, from that lemma we get that the GF of W equals $W^\star(s) = \frac{1}{\psi(s)}$ for $s \in [0,\theta)$. We now provide a suite of fluctuation identities for skip-free downward compound Poisson processes (killed at a constant rate). Recall that some of these identities can be found in [15] whereas all of them for the closely related class of spectrally negative Lévy processes can be found in [10, Chapter 8].

Theorem 4.1. *We have the following.*

(1) If $p > 0$ or $m \neq 1$, then the resolvent of $(X,\mathbb{P})$ is given by, for $x,y \in \mathbb{Z}$,

$$G(x,y) = -\frac{\theta^{x-y}}{\psi'(\theta)} - W(y-x-1)$$

(2) In any case, for $x,y \in \mathbb{Z}$,

$$\mathbb{P}_x(T_y < \zeta) = \theta^{x-y} + \psi'(\theta)W(y-x-1)$$

(3) In any case, for $a \le x \le b-1$,

$$\mathbb{P}_x(T_a < T_{[b} \wedge \zeta) = \frac{W(b-x-1)}{W(b-a-1)}$$

$$\mathbb{P}_x(T_{(a,b)^{\mathsf{c}}} < \zeta) = 1 + p\sum_{z=0}^{b-x-2} W(z) - \frac{W(b-x-1)}{W(b-a-1)} p \sum_{z=0}^{b-a-2} W(z)$$

$$\mathbb{P}_x(T_{[b} < \zeta) = \begin{cases} 1 + p\sum_{z=0}^{b-x-2} W(z) - \frac{p\theta}{1-\theta} W(b-x-1) & \textit{if } p > 0 \textit{ or } m > 1 \\ 1 + \psi'(1) W(b-x-1) & \textit{if } p = 0 \textit{ and } m \le 1 \end{cases}$$

Proof. We first assume $p > 0$ which entails that all states are transient. We choose for the reference point $\mathfrak{o} = 0$, and for the reference measure $\pi \equiv 1$. Note that by spatial homogeneity $\sum_{y\in\mathbb{Z}} P_t(x,y) = \sum_{y\in\mathbb{Z}} P_t(0, y-x) \le 1$ for any $x \in \mathbb{Z}$ and so $\pi \equiv 1$ is excessive for $(X, \mathbb{P})$. We have by (4.3), for any $x \in \mathbb{Z}$,

$$\sum_{y\in\mathbb{Z}} P_t(x,y)\theta^y = \theta^x \exp\left(\frac{\psi(\theta)}{\theta} t\right) = \theta^x$$

So $x \mapsto \theta^x$ is harmonic for $(X, \mathbb{P})$ and so, via Remark (3.2) and since $\theta \in (0,1)$ as $p > 0$, satisfies properties (i)-(iv) in Theorem 3.1(2). We now simply write $H = H_0, \widehat{H} = \widehat{H}_0, \mathbf{H}^{[y} = \mathbf{H}_0^{[y}$. Hence, $H(x) = \theta^x$. Furthermore, as $\widehat{G}(y,x) = G(x,y) = G(-y,-x)$ for all $y \in \mathbb{Z}$, we have by the definitions of H and $\widehat{H}$ in Theorem 3.1, for all $y \in \mathbb{Z}$,

$$\widehat{H}(y) = \lim_{x\uparrow\infty} \frac{\widehat{G}(y,x)}{\widehat{G}(0,x)} = \lim_{x\uparrow\infty} \frac{G(-y,-x)}{G(0,-x)} = \lim_{z\downarrow-\infty} \frac{G(-y,z)}{G(0,z)} = H(-y) = \theta^{-y}$$

Since $\psi(s) < 0$ for $s \in (\theta, 1]$ we have by (4.3), for $s \in (\theta, 1]$,

$$G^\star(0,\cdot)(s) = \int_0^\infty e^{\frac{\psi(s)}{s} t} dt = -\frac{s}{\psi(s)}$$

By (3.2) and recalling $\mathbf{H}^{[y}(0) = 0$ for $y \le 0$, we have for $s \in (\theta, 1]$,

$$\begin{aligned} \sum_{y\ge1} s^y G(0,y) &= G^\star(0,\cdot)(s) - \sum_{y\le0} \mathsf{c}(s/\theta)^y = -\frac{s}{\psi(s)} - \frac{\mathsf{c}}{1-\theta/s} \\ &= -\frac{(s-\theta)s + \mathsf{c}\psi(s)s}{\psi(s)(s-\theta)} \end{aligned} \tag{4.5}$$

Since $G(0,y) \le 1/p$ for all $y \in \mathbb{Z}$, we must have $\sum_{y\ge1} s^y G(0,y) < \infty$ for all $s \in [0,1]$. Hence by continuity of a power series

$$-\lim_{s\downarrow\theta} \frac{(s-\theta)s + \mathsf{c}\psi(s)s}{\psi(s)(s-\theta)} = \lim_{s\downarrow\theta} \sum_{y\ge1} s^y G(0,y) \in [0,\infty)$$

But by L'Hôpital, if $\mathsf{c} \ne -\frac{1}{\psi'(\theta)}$, then

$$\left|\lim_{s\downarrow\theta} \frac{(s-\theta)s + \mathsf{c}\psi(s)s}{\psi(s)(s-\theta)}\right| = \left|\lim_{s\downarrow\theta} \frac{s-\theta+s(1+\mathsf{c}\psi'(s)) + \mathsf{c}\psi(s)}{\psi'(s)(s-\theta)+\psi(s)}\right| = \infty$$

So we must have $\mathsf{c} = -\frac{1}{\psi'(\theta)}$. Then by analytic continuation, (4.5) holds for all $s \in [0,1]$. Therefore, since $G(0,y) = \mathsf{c}\theta^{-y}(1 - \mathbf{H}^{[y}(0))$ for $y \in \mathbb{Z}$ by (3.2), we have for $s \in [0,\theta)$,

$$\sum_{y\ge1} (s/\theta)^y \mathbf{H}^{[y}(0) = -\frac{1}{\mathsf{c}} \sum_{y\ge1} s^y G(0,y) + \frac{s/\theta}{1-s/\theta} = -\psi'(\theta)\frac{s}{\psi(s)}$$

Hence, $\mathbf{H}^{[y}(0) = -\psi'(\theta)\theta^y W(y-1)$. Since $W^\star(s) = \frac{1}{\psi(s)}$ for $s \in [0,\theta)$, we have $G(0,y) = \mathsf{c}\theta^{-y}(1 - \mathbf{H}^{[y}(0)) = -\frac{\theta^{-y}}{\psi'(\theta)} - W(y-1)$. By spatial homogeneity $G(x,y) = G(0,y-x)$ which proves item (1) for the case $p > 0$ and further implies by (3.2), for $x, y \in \mathbb{Z}$,

$$\mathbf{H}^{[y}(x) = -\psi'(\theta)\theta^y W(y-x-1)$$

If $p = 0$, we consider for $q > 0$ the process $(X, \mathbb{P}^{(q)})$ which is defined to be the process $(X, \mathbb{P})$ but killed at rate q, i.e. its Q-matrix is given by (4.1) but with $p = q$. Then clearly from (4.3), the transition kernel of $(X, \mathbb{P}^{(q)})$ is given by $P_t^{(q)}(x,y) = e^{-qt}P_t(x,y)$. By the same reasoning as for (2.4), we further have for any stopping time T,

$$\mathbb{P}_x^{(q)}(\Gamma \cap \{T < \infty\}) = \mathbb{E}_x\left[e^{-qT}\mathbb{1}_{\Gamma \cap \{T<\infty\}}\right], \quad \Gamma \in \mathcal{F}_T,\ x \in \mathbb{Z} \tag{4.6}$$

Since all states are transient for $(X, \mathbb{P}^{(q)})$ we have with the obvious notation,

$$G_q(x,y) = \int_0^\infty e^{-qt}P_t(x,y)dy = -\frac{\theta(q)^{x-y}}{\psi'(\theta(q)) - q} - W^{(q)}(y-x-1), \quad x, y \in \mathbb{Z}$$

By the monotone convergence theorem $G_q(x,y) \to G(x,y)$ as $q \downarrow 0$ and it is easy to see by convexity of ψ that $\theta(q) \to \theta$ and thus also $\psi'(\theta(q)) \to \psi'(\theta)$ as $q \downarrow 0$. Further, via (4.4) it is easily seen that $W^{(q)}(x) \to W(x)$ as $q \downarrow 0$ for any $x \in \mathbb{Z}$. Since $\psi'(\theta) \neq 0$ when $p = 0$ and $m \neq 1$, we can conclude that item (1) also holds when $m \neq 1$. Note that the above expressions found for $H(x)$, $\widehat{H}(y)$ (given the same reference measure $\pi \equiv 1$) and $\mathbf{H}^{[y}(x)$ still hold for the case $p = 0$ and $m \neq 1$ (for which we know all states are transient as well). Then the identities in items (2) and (3) in the case $p > 0$ or $m \neq 1$ immediately follow from Theorem 3.3 in combination with, relevant for the last identity in item (3),

$$\lim_{x\to\infty} W(x) = \begin{cases} \infty & \text{if } p > 0 \text{ or } m > 1 \\ -\frac{1}{\psi'(1)} & \text{if } p = 0 \text{ and } m < 1 \end{cases}$$

In order to see this, note that $\lim_{x\to\infty} W(x) = \lim_{s\uparrow 1} \Delta W^\star(s)$ where $\Delta W(x) = W(x) - W(x-1)$, $x \in \mathbb{Z}$, and that (i) $\theta < 1$ if $p > 0$ or $m > 1$ and so $\lim_{s\uparrow 1}\Delta W^\star(s) \geq \lim_{s\uparrow\theta}\Delta W^\star(s) = \lim_{s\uparrow\theta}\frac{1-s}{\psi(s)} = \infty$ and (ii) $\theta = 1$ and $\psi(1) = 0$ if $p = 0$ and $m < 1$ and so $\lim_{s\uparrow 1}\Delta W^\star(s) = \lim_{s\uparrow 1}\frac{1-s}{\psi(s)} = -\frac{1}{\psi'(1)}$. If on the other hand, $p = 0$ and $m = 1$, then $\theta = 1$, $\psi'(1) = 0$ and all states are recurrent. Hence item (2) and the last two identities in item (3) still hold when $p = 0$ and $m = 1$. The first identity in item (3), in the case where $p = 0$ and $m = 1$, can be proved by considering the process $(X, \mathbb{P}^{(q)})$ for $q > 0$ and taking limits as $q \downarrow 0$ while using (4.6). □

5. Markov Branching Processes with Immigration

In this section, we apply the methodology developed in Section 3 to establish the fluctuation theory of Markov branching processes with immigration, which are downward skip-free continuous-time Markov chains. We emphasize that although this class of processes have been intensively studied, only the law of the downward first passage time, that is the one with continuous crossing, has been characterized through its Laplace transform, see [4, 16] and in the continuous state space setting [7]. Further remarks about the literature can be found in Remark 5.5.

We assume that $E = \mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$ is the set of nonnegative integers and the Q-matrix generating $(X, \mathbb{P})$ is given by, for $x, y \in \mathbb{Z}_+$,

$$Q(x,y) = \begin{cases} \alpha\mu(y-x+1)x + \beta\nu(y-x) & \text{if } y \geq x+1 \\ -(\alpha+p)x - \beta - r\mathbb{I}_{\{\beta=0\}} & \text{if } y = x \\ \mu(0)\alpha x & \text{if } y = x-1 \\ 0 & \text{if } y \leq x-2 \end{cases} \tag{5.1}$$

where $\alpha, r > 0$, $\mu = (\mu(j))_{j\geq 0}$ is a probability measure with $\mu(0) > 0$ and $\mu(1) = 0$, $p, \beta \geq 0$, $\nu = (\nu(j))_{j\geq 1}$ is a probability measure. Note that, for $x \in \mathbb{Z}_+$,

$$Q(x, \partial) = px + r\mathbb{I}_{\{\beta=0\}} \tag{5.2}$$

We call $(X, \mathbb{P})$ a Markov branching process with immigration (MBPI) with parameters $(\alpha, \mu, p, \beta, \nu, r)$. If $\beta = 0$, we call the MBPI a Markov branching process (MBP) with parameters (α, μ, p, r).

MBPI's were introduced by Sevast'yanov [13] and are continuous-time skip-free downward Markov chains with transition rates that are affine in the state variable and with $Q(x, x-1) = \alpha\mu(0)x > 0$. For more background information on MBPI's/MBP's we refer to Li et al. [11, Section 1], Harris [8, Chapter V], Athreya and Ney [3, Chapter III] and the references therein. Note that when $\beta = 0$, in the limit case $r = 0$, $(X, \mathbb{P})$ is still an MBP in the sense of these references. However, 0 is an absorbing state in this case which prevents the use of our theory developed in Section 3 due to the lack of the existence of a positive excessive reference measure. Nevertheless, as we shall see below, this case can be dealt with by a limiting procedure.

We recall that for $f : \mathbb{Z}_+ \to \mathbb{R}$, $f^\star(s) := \sum_{y\in\mathbb{Z}_+} s^y f(y)$ denotes its generating function (GF), and, for $f, g : \mathbb{Z}_+ \to \mathbb{R}$ we denote by $f * g : \mathbb{Z}_+ \to \mathbb{R}$ the convolution of f and g, i.e.

$$f * g(x) = \sum_{y=0}^{x} f(x-y)g(y)$$

The branching and immigration mechanisms ψ and ϕ of a MBPI $(X, \mathbb{P})$, are defined, for any $s \in [0, 1]$, by

$$\begin{aligned} \psi(s) &= \alpha\left(\mu^\star(s) - s\right) - ps \\ \phi(s) &= \beta\left(1 - \nu^\star(s)\right) + r\mathbb{I}_{\{\beta=0\}} \end{aligned} \tag{5.3}$$

where we have set $\nu(0) = 0$. Moreover, we have $\psi(0) = \alpha\mu(0) > 0$, $\psi(1) = -p \leq 0$, and, since ψ is convex, $\psi(s) > 0$ for $s \in [0, \theta)$ where

$$\theta = \inf\{s > 0;\ \psi(s) = 0\} \in (0, 1]$$

From (5.1), we get that, for any $x \geq 0$ and $s \in [0, 1]$,

$$\sum_{y\in\mathbb{Z}_+} s^y Q(x, y) = \psi(s)xs^{x-1} - \phi(s)s^x \tag{5.4}$$

where we understand that $xs^{x-1} = 0$ when $x = 0$ and $s = 0$. Before we derive the fluctuation identities for the MBPI using the theory from Section 3, we need to introduce further notation. For a measure $\mu = (\mu(k))_{k\geq 0}$, we denote by $\overline{\mu} = (\overline{\mu}(k))_{k\geq 0}$ its tail, i.e. $\overline{\mu}(k) = 1 - \sum_{0\leq j\leq k} \mu(j), k \geq 0$. Then, $\overline{\mu} = (\overline{\mu}(k))_{k\geq 0}$ and $\overline{\nu} = (\overline{\nu}(k))_{k\geq 0}$ are the tail measures of μ and ν where we set $\nu(0) := 0$. We introduce the following functions.

(1) Let $W, \Delta_W, \kappa_q : \mathbb{Z}_+ \to [0, \infty)$ be defined as follows. For any $k \geq 0$,

$$W(k) = \sum_{j=0}^{k} \Delta_W(j)$$

where $\Delta_W(0) = \frac{1}{\alpha\mu(0)}$ and recursively

$$\Delta_W(k+1) = \frac{1}{\mu(0)} \sum_{j=0}^{k} \Delta_W(j) \left(\overline{\mu}(k-j+1) + \frac{p}{\alpha}\right)$$

and, for any $q \geq 0$,

$$\kappa_q(k) = \beta\, \Delta_W * \overline{\nu}(k) + (q + r\mathbb{I}_{\{\beta=0\}})W(k)$$

which is easily seen to be positive for any $k \in \mathbb{Z}_+$.

(2) For any $q \geq 0$, the positive measure $\pi_q = (\pi_q(k))_{k\geq 0}$ and the signed measure $\upsilon_q = (\upsilon_q(k))_{k\geq 0}$ are defined recursively by $\pi_q(0) = \upsilon_q(0) = 1$, and, for $k \geq 0$,

$$\pi_q(k+1) = \frac{\pi_q * \kappa_q(k)}{k+1} \quad \text{and} \quad \upsilon_q(k+1) = -\frac{\upsilon_q * \kappa_q(k)}{k+1} \tag{5.5}$$

As above, we simply write $\pi = \pi_0$ and $\upsilon = \upsilon_0$.

We proceed with the following main results which first characterize the q-resolvent density of MBPI along with the transience-recurrence dichotomy, and, in a subsequent claim, the Laplace transform of first exit times of an interval. The proof of this claim is given in Section 5.1.

Theorem 5.1. *Assume $(X, \mathbb{P})$ is a MBPI with parameters $(\alpha, \mu, p, \beta, \nu, r)$. Then the positive measure $\pi = \pi_0$ defined in (5.5) is an invariant reference measure for $(X, \mathbb{P})$.*

(1) The q-resolvent density of $(X, \mathbb{P})$ is given, for any $x, y \in \mathbb{Z}_+$ and $q > 0$, by

$$\mathsf{g}_q(x,y) = \mathsf{c}_q \widehat{H}_q(y)\left(H_q(x) - \mathbf{H}_q^{[y}(x)\right) \tag{5.6}$$

where $\mathsf{c}_q = \int_0^\theta \frac{1}{\psi(v)} e^{-\int_0^v \frac{\phi(u)+q}{\psi(u)}du} dv$,

$$H_q(x) = \frac{1}{\mathsf{c}_q}\int_0^\theta \frac{v^x}{\psi(v)} e^{-\int_0^v \frac{\phi(u)+q}{\psi(u)}du} dv \tag{5.7}$$

$$\widehat{H}_q(y) = \frac{\pi_q(y)}{\pi(y)} \tag{5.8}$$

and

$$\mathbf{H}_q^{[y}(x) = \frac{1}{\mathsf{c}_q \pi_q(y)} \sum_{l=0}^{y-x-1} \frac{\pi_q(y-x-1-l)}{x+l+1} W * \upsilon_q(l)\mathbb{I}_{\{y\geq x+1\}} \tag{5.9}$$

or, equivalently, for any $y \geq x+1$ and $s \in [0, \theta)$,

$$\sum_{y\in\mathbb{Z}_+} \mathbf{H}_q^{[y}(x)\pi_q(y)s^y = \frac{1}{\mathsf{c}_q}\int_0^s \frac{v^x}{\psi(v)} e^{\int_v^s \frac{\phi(u)+q}{\psi(u)}du} dv \tag{5.10}$$

(2) Moreover, for any $q > 0$, the functions $x \mapsto H_q(x)$ and $x \mapsto \mathbf{H}_q^{[y}(x)$ (resp. $y \mapsto \widehat{H}_q(y)$) are non-increasing (resp. non-decreasing) on $\mathbb{Z}_+$.

(3) $(X, \mathbb{P})$ is transient, i.e. for all $x, y \in \mathbb{Z}_+$, $\mathsf{g}(x,y) = \lim_{q\to 0} \mathsf{g}_q(x,y) < \infty$, if and only if, for some $0 < \epsilon < \theta$,

$$\int_\epsilon^\theta \frac{1}{\psi(v)} e^{\int_0^v \frac{\phi(u)}{\psi(u)}du} dv < \infty \tag{5.11}$$

As a by-product of these previous results and Theorem 3.1, we deduce the following.

Corollary 5.2. *(1) For any $x \geq a \geq 0$ and $q > 0$,*

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<\zeta\}}\right] = \frac{H_q(x)}{H_q(a)} \tag{5.12}$$

(2) For any $0 \leq a \leq x \leq b-1$ and $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<T_{[b}\wedge\zeta\}}\right] = \frac{\mathbf{H}_q^{[y}(x)}{\mathbf{H}_q^{[y}(a)}$$

(3) For any $x, y \geq 0$ and $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_y}\mathbb{I}_{\{T_y<\zeta\}}\right] = \frac{H_q(x) - \mathbf{H}_q^{[y}(x)}{H_q(y)}$$

(4) For any $0 \leq x \leq b$ and $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_{b]}}\mathbb{I}_{\{T_{b]}<\zeta\}}\right] = 1 - \mathsf{c}_q \sum_{y=0}^{b-1}(py+q)\left(\mathbf{H}_q^{[b}(x) - \mathbf{H}_q^{[y}(x)\right)\pi_q(y)$$

(5) For any $a \leq x \leq b$ and $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_{(a,b)^{\mathsf{c}}}}\right] = 1 - \mathsf{c}_q \sum_{y=a+1}^{b-1}(py+q)\left(\frac{\mathbf{H}_q^{[b}(x)}{\mathbf{H}_q^{[b}(a)}\mathbf{H}_q^{[y}(a) - \mathbf{H}_q^{[y}(x)\right)\pi_q(y)$$

We first point out that the fluctuation identities above still hold by taking the limit $q \to 0$. Moreover, the limiting case $r \to 0$ when $\beta = 0$, that is $(X, \mathbb{P})$ is an MBP with 0 as an absorbing state, cannot be directly handled with our approach as there does not exist a positive excessive measure. However, due to the structure of MBP's, the fluctuation identities in the case $r = 0$ and $q > 0$ are easily derived from the previous Corollary by replacing in the expressions both r and q by $q/2$. Finally, the case $r = q = 0$ is given in the following.

Corollary 5.3. *Let $(X, \mathbb{P})$ be an MBP with $r = 0$, then, for any $x \geq 0$ and $y \geq 1$,*

$$G(x,y) = \frac{1}{y}\left(\theta^x W(y-1) - W(y-x-1)\right) \tag{5.13}$$

where we have set $W(y) = 0$ for $y \leq -1$. Moreover, for $x, y \in \mathbb{Z}_+$,

$$\mathbb{P}_x(T_y < \zeta) = e^{(x-y)\ln\theta}\left(1 - \frac{W(y-x-1)}{W(y-1)}\mathbb{I}_{\{y \geq x+1\}}\right)$$

and, for any $0 \leq x \leq b$,

$$\mathbb{P}_x(T_{b]} < \zeta) = 1 + p\sum_{y=0}^{b-x-2} W(y) - \frac{W(b-x-1)}{W(b-1)}\left(1 + p\sum_{y=0}^{b-2} W(y)\right)$$

Finally, for any $0 \leq a \leq x \leq b$,

$$\mathbb{P}_x(T_a < T_{[b} \wedge \zeta) = \frac{W(b-x-1)}{W(b-a-1)}$$

and

$$\mathbb{P}_x(T_{(a,b)^{\mathsf{c}}} < \zeta) = 1 + p\sum_{y=0}^{b-x-2} W(y) - \frac{W(b-x-1)}{W(b-a-1)}p\sum_{y=0}^{b-a-2} W(y)$$

Remark 5.4. The class of MBPI's is invariant by killing according to a linear multiplicative functional. More specifically, if $(X, \mathbb{P})$ is a MBPI with parameters $(\alpha, \mu, p, \beta, \nu, r)$ (including the limiting case $r = 0$), and $(X, \mathbb{P}^{(\gamma)})$ a MBPI with parameters $(\alpha, \mu, p+\gamma, \beta, \nu, r), \gamma \geq 0$, then using (5.2) and a classical result in the theory of Markov processes, one can show for any stopping time T, $q > 0$ and $x, \gamma \geq 0$,

$$\mathbb{E}_x\left[e^{-qT - \gamma\int_0^T X_s ds}\mathbb{1}_{\{T<\infty\}}\right] = \mathbb{E}_x^{(\gamma)}\left[e^{-qT}\mathbb{1}_{\{T<\infty\}}\right]$$

where $\mathbb{E}^{(\gamma)}$ is the expectation operator associated with $\mathbb{P}^{(\gamma)}$. This entails that all the identities in Theorem 5.1 can be rephrased for the couple $(T, \int_0^T X_s ds)$ where $(X, \mathbb{P})$ is a MBPI with parameter $p \geq 0$. As an example, for any $0 \leq a \leq x \leq b-1$, one has

$$\mathbb{E}_x\left[e^{-qT_a-\gamma\int_0^{T_a} X_s ds}\mathbb{1}_{\{T_a<T_{[b}\wedge\zeta\}}\right] = \mathbb{E}_x^{(\gamma)}\left[e^{-qT_a}\mathbb{1}_{\{T_a<T_{[b}\wedge\zeta\}}\right] = \frac{\mathbf{H}^{[b}_{q,\gamma}(x)}{\mathbf{H}^{[b}_{q,\gamma}(a)}$$

where $\mathbf{H}^{[b}_{q,\gamma}$ is the one associated with the MBPI process with parameters $(\alpha, \mu, p+\gamma, \beta, \nu, r)$.

Remark 5.5. Most of the identities presented in Theorem 5.1 and Corollary 5.2 are new in the sense that also no analogues have appeared in the literature for the discrete-time analogue of MBPI's, i.e. Galton-Watson processes, or the continuous-state space analogue of MBPI's, i.e. continuous-state branching processes with immigration (CBI's). For the special case of MBP's some of these identities are covered in the literature because the Lamperti transform entails that a MBP is a time-changed skip-free downward compound Poisson process stopped at hitting 0 and killed at a constant rate, see [3, Theorem III.11.3] for the Lamperti transform for nondecreasing MBP's. This implies that, in the MBP case, the last two identities in Corollary 5.3 for $y \leq x$ are identical to those for skip-free downward compound Poisson processes, which is confirmed by comparing these results with Theorem 4.1, see also [10, Theorem 12.8 and Corollary 12.9] for the continuous-state space case. With immigration the only identity that, to our knowledge, has appeared before in the literature is the one for the downwards hitting probability, which from (5.12) in Corollary 5.2 reads, for any $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<\zeta\}}\right] = \frac{\int_0^\theta \frac{v^x}{\psi(v)}\exp\left(-\int_0^v \frac{\phi(u)+q}{\psi(u)}du\right)dv}{\int_0^\theta \frac{v^a}{\psi(v)}\exp\left(-\int_0^v \frac{\phi(u)+q}{\psi(u)}du\right)dv}, \quad 0 \leq a \leq x \tag{5.14}$$

Namely, the analogue of this identity for CBI's has been derived from Duhalde et al. [7] and the MBPI case can be found in [16], see also [4] for the case $\beta = 0$ where the methodology developed therein is used.

In Corollary 5.3, we see that the expressions simplify for the special case of MBP's for $r = q = 0$. This is also the case for another subclass of MBPI's.

Example 5.6. Assume the parameters $(\alpha, \mu, p, \beta, \nu, r)$ of the MBPI are such that $0 < m < 1 + \frac{p}{\alpha}$, $\beta = \alpha m > 0$, $\nu(j) = \frac{(j+1)\mu(j+1)}{m}$ for $j \geq 1$ and $q = p - \alpha(m-1)$, where $m := \sum_{j\geq 2} j\mu(j)$. Then $\phi(s) + q = -\psi'(s)$ for all $s \geq 0$ and consequently, for $s \in [0,\theta)$ and $y \geq 0$, via Lemma 5.7 and Theorem 5.1, we have $\pi_q^\star(s) = \frac{\psi(0)}{\psi(s)}$,

$$\begin{aligned}
\int_0^\theta \frac{v^x}{\psi(v)}\exp\left(-\int_0^v \frac{\phi(u)+q}{\psi(u)}du\right)dv &= \frac{1}{\psi(0)}\frac{\theta^{x+1}}{x+1} \\
\sum_{y\in\mathbb{Z}_+} \mathbf{H}_q^{[y}(x)\mathsf{c}_q\pi_q(y)s^y &= \frac{1}{\psi(s)}\frac{s^{x+1}}{x+1}
\end{aligned}$$

This yields, for $x, y \geq 0$,

$$H_q(x) = \frac{\theta^x}{x+1},\ \pi_q(y) = \psi(0)W(y) \text{ and } \mathbf{H}_q^{[y}(x)\pi_q(y) = \frac{\psi(0)}{\theta}\frac{W(y-x-1)}{x+1}$$

5.1. **Proof of Theorem 5.1.** We start with the following lemma.

Lemma 5.7. *The GF's of Δ_W, W, κ_q, π_q and υ_q are given, for any $q \geq 0$ and $s \in [0, \theta)$, by*

$$\begin{aligned} \Delta_W^\star(s) &= \frac{1-s}{\psi(s)}, \qquad W^\star(s) = \frac{1}{\psi(s)}, \qquad \kappa_q^\star(s) = \frac{\phi(s)+q}{\psi(s)} \\ \pi_q^\star(s) &= e^{\int_0^s \frac{\phi(u)+q}{\psi(u)}du} \quad \text{ and } \quad \upsilon_q^\star(s) = e^{-\int_0^s \frac{\phi(u)+q}{\psi(u)}du} \end{aligned}$$

Proof. Let $f : \mathbb{Z}_+ \to [0, \infty)$ with $f(0) = 0$ be given and let f^{*n} denote the n-fold convolution of f with $f^{*0}(x) = \delta_{0x}$. It is well-known that, given $u_0 \geq 0$, $u(x) = u_0 \sum_{n \geq 0} f^{*n}(x)$ is well-defined and satisfies the renewal equation

$$u(x+1) = \sum_{j=0}^{x} u(j) f(x+1-j), \quad x \geq 0, \quad u(0) = u_0 \tag{5.15}$$

see e.g. [2, Theorem V.2.4]. Hence, if $f^\star(s) < 1$ for some $s > 0$, then $u^\star(s) = \frac{u_0}{1-f^\star(s)} < \infty$. Let $\eta(x) = \frac{1}{\mu(0)}(\overline{\mu}(x) + \frac{p}{\alpha})$ for $x \geq 1$ and set $\eta(0) = 0$. Since, for $s \in [0, 1)$, $\overline{\mu}^\star(s) = \frac{1-\mu^\star(s)}{1-s}$, we obtain that

$$\eta^\star(s) = \frac{1}{\mu(0)} \left(\frac{1-\mu^\star(s)}{1-s} - (1-\mu(0)) + \frac{p}{\alpha(1-s)} \right) = 1 - \frac{\psi(s)}{\alpha\mu(0)(1-s)}$$

That is $\eta^\star(s) < 1$ for $s \in [0, \theta)$, and we conclude that, for $s \in [0, \theta)$,

$$\Delta_W^\star(s) = \frac{\Delta_W(0)}{1-\eta^\star(s)} = \frac{1-s}{\psi(s)}$$

Consequently, for any $s \in [0, \theta)$,

$$W^\star(s) = \frac{\Delta_W^\star(s)}{1-s} = \frac{1}{\psi(s)}$$

Next, since $\overline{\nu}^\star(s) = \frac{1-\nu^\star(s)}{1-s}$, we have, for any $s \in [0, \theta)$,

$$\kappa_q^\star(s) = \beta \left(\frac{1-s}{\psi(s)} \frac{1-\nu^\star(s)}{1-s} \right) + \frac{q}{\psi(s)} = \frac{\phi(s)+q}{\psi(s)}$$

From the definitions of π_q and υ_q it follows that $|\upsilon_q(x)| \leq \pi_q(x) \leq u(x)$ where u is given by the renewal sequence (5.15) with $f(j) = \kappa_q(j-1)$, $j \geq 1$. Since $f^\star(s) = \kappa_q^\star(s)s$ there exists $\epsilon > 0$ such that $f^\star(s) < 1$ and consequently $u^\star(s) < \infty$ for any $s \in [0, \epsilon]$. Hence, the mappings $s \mapsto \pi_q^\star(s)$ and $s \mapsto \upsilon_q^\star(s)$ are well-defined for $s \in [0, \epsilon]$. By the recursive definitions of π_q and υ_q, we deduce that

$$\begin{aligned} \frac{d}{ds}\pi_q^\star(s) &= \pi_q^\star(s)\frac{\phi(s)+q}{\psi(s)} \quad \text{with } \pi_q^\star(0) = 1 \\ \frac{d}{ds}\upsilon_q^\star(s) &= -\upsilon_q^\star(s)\frac{\phi(s)+q}{\psi(s)} \quad \text{with } \upsilon_q^\star(0) = 1 \end{aligned}$$

By solving these ODE's, we get $\pi_q^\star(s) = e^{\int_0^s \frac{\phi(u)+q}{\psi(u)}du} = \frac{1}{\upsilon_q^\star(s)}$ for $s \in [0, \epsilon]$. Since the middle part of the last equation is well-defined for all $s \in [0, \theta)$ it follows by properties of power series that this last equation holds for all $s \in [0, \theta)$. ∎

The next two lemmas are more of less covered in Li et al. [11, Section 2], though we provide a proof since [11] has a different starting definition of MBPI's and uses different notation.

Lemma 5.8. *Define $I_\psi : [0, \theta) \to [0, \infty)$ by $I_\psi(s) = \int_0^s \frac{1}{\psi(u)}\mathrm{d}u$. The function I_ψ is bijective and its inverse $I_\psi^{-1} : [0, \infty) \to [0, \theta)$ is well-defined. Then, the GF of the MBPI $(X, \mathbb{P})$ is given, for any $x \in \mathbb{Z}_+$, $t \geq 0$ and $s \in [0, \theta)$, by*

$$\mathbb{E}_x \left[s^{X_t} \mathbb{1}_{\{t<\zeta\}} \right] = \Psi_t^x(s)\Phi_t(s) \tag{5.16}$$

where, recalling that $\pi^\star$ (resp. $\upsilon^\star$) is the GF of $\pi = \pi_0$ (resp. $\upsilon = \upsilon_0$), we have set

$$\Psi_t(s) = I_\psi^{-1}(t + I_\psi(s)) \tag{5.17}$$

$$\Phi_t(s) = \pi^\star(s)\upsilon^\star(\Psi_t(s)) \tag{5.18}$$

Proof. Note that I_ψ is bijective since it is increasing, continuous, $I_\psi(0) = 0$, and, as ψ is a power series with $\psi(\theta) = 0$, $\lim_{s\uparrow\theta} I_\psi(s) = \infty$. The first part of the proof of [11, Theorem 2.2] in combination with [1, p. 83] shows that, for $x, y \in \mathbb{Z}_+$ and $t \geq 0$, $P_t(x, y)$ is the unique nonnegative solution to the forward equation satisfying $\sum_{y\in\mathbb{Z}_+} P_t(x, y) \leq 1$. Let $\widetilde{P}_t(x, y) = P_t(0, \cdot) * \breve{P}_t(x, \cdot)(y)$ where $\breve{P}_t(x, y)$ is the transition kernel associated with the MBP with parameters $(\alpha, \mu, p, 0)$ whose Q-matrix we denote by $\breve{Q}$. Clearly, $\widetilde{P}_t(x, y) \geq 0$ and $\sum_{y\in\mathbb{Z}_+} \widetilde{P}_t(x, y) \leq 1$. Since P_t, respectively $\breve{P}_t$, satisfies the forward equation associated with Q, respectively $\breve{Q}$, see e.g. [12, Section 2.8], we have

$$\begin{aligned}
\frac{d}{dt}\widetilde{P}_t(x, y) &= \sum_{l=0}^{y+1}\sum_{k=0}^{y+1} P_t(0, l)\breve{P}_t(x, k)\left(Q(l, y-k) + \breve{Q}(k, y-l)\right) \\
&= \sum_{l=0}^{y+1}\sum_{k=0}^{y+1} P_t(0, l)\breve{P}_t(x, k)Q(l+k, y) \\
&= \sum_{m=0}^{y+1}\sum_{l=0}^{m} P_t(0, l)\breve{P}_t(x, m-l)Q(m, y) \\
&= \sum_{m\in\mathbb{Z}_+} \widetilde{P}_t(x, m)Q(m, y)
\end{aligned}$$

where the second equality is due to (5.1) and where we have set $Q(x, -1) = \breve{Q}(x, -1) = 0$. So $\widetilde{P}_t$ is a solution to the forward equations and therefore $P_t = \widetilde{P}_t$. From [1, Theorem 3.1(2)], see also [8, Theorem V.4.1][2], we get that $\breve{P}_t$ satisfies the branching property $\breve{P}_t(x, y) = \breve{P}_t(1, \cdot)^{*x}(y)$ where $*x$ denotes the x-fold convolution. Combining this with $P_t(x, y) = P_t(0, \cdot) * \breve{P}_t(x, \cdot)(y)$, we see that (5.16) holds with $\Psi_t(s) = \breve{P}_t^\star(1, \cdot)(s)$ and $\Phi_t(s) = P_t^\star(0, \cdot)(s)$. Since P_t, respectively $\breve{P}_t$, also satisfies the backward equations associated with Q, respectively $\breve{Q}$, we have with the help of Fubini, for $s \in [0, 1]$ and $t \geq 0$,

$$\begin{aligned}
\breve{P}_t^\star(1, \cdot)(s) &= s + \int_0^t \sum_{y\in\mathbb{Z}_+} \breve{Q}(1, y)\breve{P}_u^\star(y, \cdot)(s)du \\
P_t^\star(0, \cdot)(s) &= 1 + \int_0^t \sum_{y\in\mathbb{Z}_+} Q(0, y)P_u^\star(y, \cdot)(s)du
\end{aligned}$$

Then, by taking derivatives in t and using (5.4), we get, for $s \in [0, 1]$ and $t \geq 0$,

$$\frac{d}{dt}\Psi_t(s) = \psi(\Psi_t(s)) \text{ with } \Psi_0(s) = s \tag{5.19}$$

$$\frac{d}{dt}\Phi_t(s) = -\phi(\Psi_t(s))\Phi_t(s) \text{ with } \Phi_0(s) = 1 \tag{5.20}$$

Since ψ is locally Lipschitz on $(-1, 1)$, and, for any $s \in [0, \theta)$, the right-hand side of (5.17) solves, for all $t \geq 0$, the ODE (5.19), it follows by a classical argument that (5.17) holds. Furthermore, the linear ODE (5.20) has a unique solution, and, using (5.19) and Lemma 5.7, the right-hand side of (5.18) solves, for all $t \geq 0$, (5.20) for any given $s \in [0, \theta)$, which implies that (5.18) holds. ∎

[2] In these references it is assumed that $p = 0$ but the arguments go through for $p > 0$.

Lemma 5.9. *For any $q \geq 0$, the positive measure π_q defined in* (5.5) *is a q-invariant measure for $(X, \mathbb{P})$. In particular, $\pi = \pi_0$ is a positive invariant reference measure for $(X, \mathbb{P})$.*

Proof. We need to prove that, for $q \geq 0$, $e^{-qt} \sum_{x \in \mathbb{Z}_+} \pi_q(x) P_t(x, y) = \pi_q(y)$, for all $y, t \geq 0$. To this end, let $t, y \geq 0$ and $s \in [0, \theta)$. Then, by using successively Tonelli theorem, (5.16), (5.17) which entails that $\Psi_t(s) \in [0, \theta)$ and (5.18), we have that

$$\begin{aligned}
\sum_{y \in \mathbb{Z}_+} \sum_{x \in \mathbb{Z}_+} \pi_q(x) P_t(x, y) s^y &= \sum_{x \in \mathbb{Z}_+} \pi_q(x) \sum_{y \in \mathbb{Z}_+} P_t(x, y) s^y = \sum_{x \in \mathbb{Z}_+} \pi_q(x) \Psi_t(s)^x \Phi_t(s) \\
&= \pi_q^\star(\Psi_t(s)) \Phi_t(s) \\
&= \pi^\star(\Psi_t(s)) e^{q I_\psi(\Psi_t(s))} \pi^\star(s) \upsilon^\star(\Psi_t(s)) \\
&= e^{qt} e^{q I_\psi(s)} \pi^\star(s) \\
&= e^{-qt} \pi_q^\star(s)
\end{aligned}$$

where we used, for the two last identities, the fact that $\pi^\star(s) = 1/\upsilon^\star(s)$, (5.17) and $\pi_q^\star(s) = e^{q I_\psi(s)} \pi^\star(s)$. By injectivity of the GF, this provides the claim. □

We proceed by providing the expression of the GF of the q-resolvent, from which the integral test (5.11) for the transience-recurrence dichotomy is derived.

Lemma 5.10. *For any $x \geq 0, q > 0$ and $s \in [0, \theta)$,*

$$G_q^\star(x, .)(s) = \pi_q^\star(s) \int_0^\theta \frac{\upsilon^x}{\psi(\upsilon)} e^{-\int_0^\upsilon \frac{\phi(u)+q}{\psi(u)} du} d\upsilon - \int_0^s \frac{\upsilon^x}{\psi(\upsilon)} e^{\int_\upsilon^s \frac{\phi(u)+q}{\psi(u)} du} d\upsilon \tag{5.21}$$

Moreover, if (5.11) *holds then, for any $x \geq 0$ and $s \in [0, \theta)$,*

$$G^\star(x, .)(s) = \pi^\star(s) \int_0^\theta \frac{\upsilon^x}{\psi(\upsilon)} e^{-\int_0^\upsilon \frac{\phi(u)}{\psi(u)} du} d\upsilon - \int_0^s \frac{\upsilon^x}{\psi(\upsilon)} e^{\int_\upsilon^s \frac{\phi(u)}{\psi(u)} du} d\upsilon \tag{5.22}$$

Thus, $(X, \mathbb{P})$ is transient if and only if (5.11) *holds.*

Proof. By Lemma 5.8 and by performing the change of variables $t = I_\psi(\upsilon) - I_\psi(s)$, we have, for any $x \geq 0$ and $s \in [0, \theta)$,

$$\begin{aligned}
G_q^\star(x, .)(s) &= \sum_{y \in \mathbb{Z}_+} G_q(x, y) s^y = \int_0^\infty e^{-qt} \sum_{y \in \mathbb{Z}_+} P_t(x, y) s^y dt \\
&= \int_0^\infty e^{-qt} \left(I_\psi^{-1}(t + I_\psi(s)) \right)^x \pi^\star(s) \upsilon^\star \left(I_\psi^{-1}(t + I_\psi(s)) \right) dt \\
&= e^{q I_\psi(s)} \pi^\star(s) \int_s^\theta e^{-q I_\psi(\upsilon)} \frac{\upsilon^x \upsilon^\star(\upsilon)}{\psi(\upsilon)} \mathrm{d}\upsilon \\
&= \pi_q^\star(s) \int_0^\theta \frac{\upsilon^x \upsilon_q^\star(\upsilon)}{\psi(\upsilon)} d\upsilon - \pi_q^\star(s) \int_0^s \frac{\upsilon^x \upsilon_q^\star(\upsilon)}{\psi(\upsilon)} d\upsilon \\
&= \pi_q^\star(s) \int_0^\theta \frac{\upsilon^x}{\psi(\upsilon)} e^{-\int_0^\upsilon \frac{\phi(u)+q}{\psi(u)} du} d\upsilon - \int_0^s \frac{\upsilon^x}{\psi(\upsilon)} e^{\int_\upsilon^s \frac{\phi(u)+q}{\psi(u)} du} d\upsilon
\end{aligned}$$

where, for the last identity, we used Lemma 5.7. We complete the proof by letting $q \downarrow 0$ and using the transience-recurrence dichotomy property of continuous-time Markov chains. □

5.1.1. *End of the proof of Theorem 5.1.* We first recall that the characterization of the invariant measure is given in Lemma 5.9. Next, from the definition of H_q in Theorem 3.1(1), i.e. $H_q(x) = \frac{G_q(x,0)}{G_q(0,0)}, x \geq 0$, where we used $\mathfrak{o} = 0$ as reference point. Since $G_q(x,0) = G_q^\star(x,.)(0), x \geq 0$, by invoking Lemma 5.7 again, we deduce from the identity (5.21) that $G_q(x,0) = \pi_q^\star(0)\int_0^\theta \frac{v^x v_q^\star(v)}{\psi(v)}dv = \int_0^\theta \frac{v^x}{\psi(v)}e^{-\int_0^v \frac{\phi(u)+q}{\psi(u)}du}dv$, which provides the expression of $H_q(x)$ and of $\mathsf{c}_q = \mathsf{g}_q(0,0) = \frac{G_q^\star(0,.)(0)}{\pi(0)} = \int_0^\theta \frac{1}{\psi(v)}e^{-\int_0^v \frac{\phi(u)+q}{\psi(u)}du}dv$, since $\pi(0) = 1$. Next, using the definitions (5.8) and (5.9), one gets, for any $x \geq 0, q > 0$ and $s \in [0,\theta)$,

$$\begin{aligned}
\sum_{y\in\mathbb{Z}_+} \mathsf{c}_q \widehat{H}_q(y)\mathbf{H}_q^{[y}(x)\pi(y)s^y &= \mathsf{c}_q \sum_{y\in\mathbb{Z}_+} \pi_q(y)\mathbf{H}_q^{[y}(x)s^y \\
&= \sum_{y\in\mathbb{Z}_+} s^y \sum_{l=0}^{y-x-1} \frac{\pi_q(y-(x+l+1))}{x+l+1} W * v_q(l)\mathbb{I}_{\{y\geq x+1\}} \\
&= \pi_q^\star(s) \sum_{l\in\mathbb{Z}_+} \frac{W * v_q(l)}{l+x+1}s^{x+1+l} \\
&= \pi_q^\star(s) \int_0^s \sum_{l\in\mathbb{Z}_+} W * v_q(l)v^{x+l}dv \\
&= \pi_q^\star(s) \int_0^s v^x W^\star(v)v_q^\star(v)dv \\
&= \int_0^s \frac{v^x}{\psi(v)}e^{\int_v^s \frac{\phi(u)+q}{\psi(u)}du}dv \qquad (5.23)
\end{aligned}$$

where, for the fourth equality, we have used Fubini Theorem which is applicable because clearly $|v_q(l)| \leq \pi_q(l)$ for all $l \geq 0$ by their definitions, and the last one follows from Lemma 5.7. Putting pieces together and using (3.2) and (5.8), we get, for any $x \geq 0, q > 0$ and $s \in [0,\theta)$,

$$\begin{aligned}
\sum_{y\in\mathbb{Z}_+} \mathsf{g}_q(x,y)\pi(y)s^y &= \sum_{y\in\mathbb{Z}_+} \mathsf{c}_q \widehat{H}_q(y)\left(H_q(x) - \mathbf{H}_q^{[y}(x)\right)\pi(y)s^y \\
&= \mathsf{c}_q H_q(x) \sum_{y\in\mathbb{Z}_+} \pi_q(y)s^y - \mathsf{c}_q \sum_{y\geq x+1} \pi_q(y)\mathbf{H}_q^{[y}(x)s^y \\
&= \pi_q^\star(s)\int_0^\theta \frac{v^x}{\psi(v)}e^{-\int_0^v \frac{\phi(u)+q}{\psi(u)}du}dv - \int_0^s \frac{v^x}{\psi(v)}e^{\int_v^s \frac{\phi(u)+q}{\psi(u)}du}dv
\end{aligned}$$

which is the expression (5.21). Using Theorem 3.1, and the injectivity of the GF, we deduce the expression (5.6) of the q-resolvent density. The fluctuation identities, for $q > 0$, are a by-product of Theorem 3.3. The last claims of the Theorem follow readily.

5.2. **Proof of Corollary 5.3.** We now turn to the limiting case $r = 0$, when $(X,\mathbb{P})$ is a MBP, i.e. $\beta = 0$. It is easy to prove by induction that $\lim_{q\downarrow 0}\pi_q(y) = \lim_{q\downarrow 0} v_q(y) = 0$ and $\lim_{q\downarrow 0}\frac{\pi_q(y)}{q} = \frac{W(y-1)}{y}$ for $y \geq 1$ and further by an integration by parts, for $x \geq 0$,

$$\begin{aligned}
\lim_{q\downarrow 0} q\int_0^\theta \frac{v^x}{\psi(v)}e^{-\int_0^v \frac{q}{\psi(u)}du}dv &= \lim_{q\downarrow 0}\left(1 + \int_0^\theta xv^{x-1}e^{-\int_0^v \frac{q}{\psi(u)}du}dv\right) \\
&= \theta^x \qquad (5.24)
\end{aligned}$$

Using Theorem 5.1, replacing in the expression of the q-resolvent both r and q by $q/2$, and invoking, while taking the limit $q \to 0$, the monotone convergence theorem, we deduce, for all $x \geq 0$ and $y \geq 1$, that

$$\begin{aligned} G(x,y) &= \lim_{q \downarrow 0} \frac{\pi_q(y)}{q} q \int_0^\theta \frac{v^x}{\psi(v)} e^{-\int_0^v \frac{q}{\psi(u)} du} dv - \frac{W(y-x-1)}{y} \\ &= \frac{\theta^x W(y-1)}{y} - \frac{W(y-x-1)}{y} \end{aligned}$$

which finishes the proof of (5.13). In the same way, using Corollary 5.2, we get the remaining identities.

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