Continued Fraction approach to Gauss Reduction Theory

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Abstract. Jordan Normal Forms serve as excellent representatives of conjugacy classes of matrices over algebraically closed fields. Once we know normal forms, we can compute functions of matrices, their main invariants, etc. The situation is more complicated if we search for normal forms for conjugacy classes over fields that are not closed and especially over rings.
In this paper we study $\mathrm{PGL}(2,\mathbb{Z})$-conjugacy classes of $\mathrm{GL}(2,\mathbb{Z})$ matrices. For the ring of integers the Jordan approach has various limitations and in fact it is not effective. The normal forms of conjugacy classes of $\mathrm{GL}(2,\mathbb{Z})$ matrices are provided by an alternative theory, which is known as Gauss Reduction Theory. We introduce new techniques to compute reduced forms in Gauss Reduction Theory in terms of the elements of certain continued fractions. The current approach is based on recent progress in the field of the geometry of numbers. The proposed technique provides an explicit computation of periods of continued fractions for the slopes of eigenvectors.

Keywords: Integer matrices · Gauss Reduction Theory · continued fractions · geometry of numbers.

Introduction

In this paper we study the structure of the conjugacy classes of $\mathrm{GL}(2,\mathbb{Z})$. Recall that $\mathrm{GL}(2,\mathbb{Z})$ is the group of all invertible matrices with integer coefficients. As a consequence the determinants of such matrices are ±1. We say that the matrices A and B from $\mathrm{GL}(2,\mathbb{Z})$ are $\mathrm{PGL}(2,\mathbb{Z})$-*conjugate* if there exists a $\mathrm{GL}(2,\mathbb{Z})$ matrix C such that $B = \pm CAC^{-1}$. In the integer case projectivity simply means that all matrices are considered up to the multiplication by ±1.

Recall that for algebraically closed fields every matrix is conjugate to its Jordan Normal Form. The situation with $\mathrm{GL}(\mathrm{n},\mathbb{Z})$ is not so simple as the set of integer numbers does not have a field structure. A description of $\mathrm{PGL}(2,\mathbb{Z})$-conjugacy classes in the two-dimensional case is the subject of Gauss Reduction Theory. The conjugacy classes are classified by periods of certain periodic continued fractions (for additional information we refer to [17, 13, 18]; for the algorithms of the conjugacy test in $\mathrm{GL}(2,\mathbb{Z})$ see [4, 6]). The first geometric invariants of $\mathrm{GL}(2,\mathbb{Z})$ matrices in the spirit of continued fractions were studied in [9]. The question of the classification of conjugacy classes is closely related to the study of homogeneous forms (see e.g. in [2]) and the theory of Markov and Lagrange spectra (see e.g. in [5]).

Our aim is to study a natural class of reduced matrices that represent every conjugacy class, which are good candidates for normal forms in integer settings. Note that the number of reduced matrices in any $\mathrm{PGL}(2,\mathbb{Z})$-conjugacy class of matrices is finite (see, e.g., in Chapter 7 of [11]). In this paper we approach the following problem.

Problem 1. Find explicit expressions for normal forms $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to a given matrix.

We solve this problem by introducing a new surprising explicit formula to generate all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to a given one via certain long continued fractions that are built using the elements of the matrices. We show how write all the reduced matrices in Section 3. The formula is justified by Theorem 3 which is supplemented by technical statements of Theorem 2, Theorem 4 and Proposition 2. The new method is based on lattice trigonometry introduced in [7, 8] (see also in [11]).

We expect that the computational complexity of the new method is comparable to the algorithm of Chapter 7 in [11]. One of the advantages of the proposed new approach is that it constructs all reduced matrices while the classical algorithms result in a single reduced matrix. In addition all the reduced operators of the proposed approach are explicitly described via geometric invariants, which is potentially useful for the multidimensional case. Recall that the studies of the conjugacy classes of $\mathrm{GL}(\mathrm{n},\mathbb{Z})$ for $n > 2$ were motivated by V. Arnold (see, e.g., in [1]) who revived the notion of multidimensional continued fractions in the sense of Klein ([15, 16]). The first results in higher dimensional cases were obtained in [10] (see also [11], Chapter 21), however the theory is far from its final form even for the case of $n = 3$. We hope that the approach of the current paper will give some hints for numerous open problems in the multidimensional case.

This paper is organized as follows. In Section 1 we start with necessary notions and definitions of geometry of numbers. In particular we introduce the notion of the semigroup of reduced matrices. We discuss three different cases of $\mathrm{GL}(2,\mathbb{Z})$ matrices in general in Section 2. In Section 3 we bring together all the stages of finding all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to a given one. Finally in Section 4 we discuss some technical details used in the construction of reduced matrices.

1 Background

In this section we briefly discuss basic notions used in the computation of reduced matrices. We start in Subsection 1.1 with elementary notions and definitions of lattice geometry. In Subsection 1.2 we define sails of integer angles; and introduce

LLS sequences for broken lines. Further we define LLS sequences for integer angles. Sails and LLS sequences are important invariants related to conjugacy classes of $\mathrm{GL}(2,\mathbb{Z})$ matrices. We continue in Subsection 1.3 with the notion of periods of LLS sequences related to matrices. In Subsection 1.4 we introduce reduced matrices and give a continuant representation for them. We conclude this section with a general definition of difference of sequences in Subsection 1.5.

1.1 Basics of integer geometry in the plane

We say that a point is *integer* if its coordinates are integers. A segment is integer if its endpoints are integer. An angle is called *integer* if its vertex is an integer point. We also say that an integer angle is *rational* if its edges contain integer points distinct to the vertex.

An affine transformation is said to be integer if it is a one-to-one mapping of the lattice $\mathbb{Z}^2$ to itself. Note that the set of integer transformations is a semidirect product of the group of translations by an integer vector and the group $\mathrm{GL}(2,\mathbb{Z})$. Two sets are *integer congruent* if there exists an integer affine transformation providing a bijection between these two sets.

Definition 1. *The integer length of an integer segment AB is the number of integer points inside its interior plus one. Denote it by* $\mathrm{l}\ell(\mathrm{AB})$.

The integer sine of a rational angle $\angle ABC$ is the following integer: $\frac{|\det(AB,BC)|}{\mathrm{l}\ell(\mathrm{AB})\cdot\mathrm{l}\ell(\mathrm{BC})}$, *where $|\det(AB,BC)|$ is the absolute value of the determinant of the matrix of the pair of vectors (AB,BC). Denote it by* $\mathrm{lsin}\angle\mathrm{ABC}$.

The integer lengths and sines are invariants of integer affine transformations.

1.2 Sail and LLS sequences

Let us now study an important invariant of angles and broken lines. It will be employed in the proofs, however from computational perspectives one can use the statement of Theorem 4 as the explicit definition of LLS sequences for angles (without appealing to integer geometry).

Let $\angle ABC$ be an integer angle. The boundary of the convex hull of all integer points in the convex closure of $\angle ABC$ except B is called the *sail* of $\angle ABC$.

Note that the sail of a rational angle is a finite broken line, while the sail of an integer angle that is not rational is a broken line infinite on one or both sides.

Definition 2. *Let $A_1,\ldots,A_n$ be a broken line (here we can consider finite or infinite broken lines) such that A_i, A_{i+1}, and O are not in one line for all admissible parameters of i. Define*

$$a_{2k}=\det(OA_k,OA_{k+1}) \quad \textit{and} \quad a_{2k-1}=\frac{\det(A_kA_{k-1},A_kA_{k+1})}{a_{2k-2}a_k}.$$

for all admissible k. The sequence $(a_0,\ldots,a_{2n})$ (or an infinite one respectively) is called the LLS sequence of the broken line $A_0\ldots A_n$.

Definition 3. *Consider an integer angle $\angle ABC$. Let $\ldots A_{i-1}, A_i A_{i+1}, \ldots$ be the sail of $\angle ABC$. Here we consider the broken line directed from the edge AB to the edge BC. Let the LLS sequence for the broken line $\ldots A_{i-1}, A_i A_{i+1}, \ldots$ be $(\ldots a_{2k-1}, a_{2k}, a_{2k+1}, \ldots)$ (finite or infinite). Then the sequence*

$$(\ldots |a_{2k-1}|, |a_{2k}|, |a_{2k+1}|, \ldots)$$

is called the Lattice Length Sine sequence (or simply LLS sequence, for short) of the angle $\angle ABC$ and is denoted by $LLS(\angle ABC)$.

Remark 1. Note that the LLS sequence can be defined for any lattice (not necessarily for the integer lattice).

Remark 2. Consider a rational angle $\angle ABC$ with a positive $\det(AO, BC)$. Then its LLS sequence $(a_0, \ldots, a_{2n})$ consists of an odd number of elements and

$$a_{2k} = \mathrm{l\ell A_k A_{k+1}} \quad \text{and} \quad \mathrm{a_{2k-1} = lsin\angle A_{k-1} A_k A_{k+1}} \quad \text{for all admissible } k.$$

Now let us recall the definition of a continuant.

Definition 4. *Let n be a positive integer. A continuant K_n is a polynomial with integer coefficients defined recursively by*

$$K_{-1}() = 0; \quad K_0() = 1; \quad K_1(a_1) = a_1;$$
$$K_n(a_1, a_2, \ldots, a_n) = a_n K_{n-1}(a_1, a_2, \ldots, a_{n-1}) + K_{n-2}(a_1, a_2, \ldots, a_{n-2}).$$

Remark 3. Note that we have the following general expression relating continued fractions and continuants. For any real numbers $a_1, \ldots, a_n$ it holds that

$$[a_1; a_2 : \cdots : a_n] = \frac{K_n(a_1, a_2, \ldots, a_n)}{K_{n-1}(a_2, \ldots, a_n)}.$$

We use the following important geometric property of LLS sequences.

Theorem 1. ([7] 2008) *Consider a finite broken line $A_1, \ldots, A_n$ with LLS sequence $(a_0, \ldots, a_{2n})$. Let also $A_0 = (1, 0)$ and $A_1 = (1, a_0)$. Then*

$$A_n = \big(K_{2n+1}(a_0, \ldots, a_{2n}), K_{2n}(a_1, \ldots, a_{2n})\big).$$

For further additional information on the geometry of continued fractions see [11].

1.3 LLS periods of $\mathrm{GL}(2, \mathbb{Z})$ matrices

Let M be a (2×2)-matrix with two distinct real eigenvalues. In this case M has two eigenlines. The complement to these eigenlines is a union of four cones. We say that the sails of these cones are the *sails associated to M*.

Definition 5. *We say that a sequence of positive integers is an LLS sequence of M if this sequence is the LLS sequence of one of the sails associated to M.*

Remark 4. It turns out that in the case of $\mathrm{GL}(2,\mathbb{Z})$ matrices with real irrational eigenvalues the LLS sequences of all associated sails coincide up to a possible index shift and reversal (see Section 7 of [11]). So the LLS sequence is uniquely defined by the matrix in this case.

We conclude this subsection with the following fundamental definition.

Definition 6. *Let M be a $\mathrm{GL}(2,\mathbb{Z})$ matrix with real irrational eigenvalues. Then its LLS sequence is periodic. In addition M^2 acts as a periodic shift on every one of the sails. Assume that M^2 shifts the sail by n vertices. Then any period of length n is called an LLS period of M. (Here we write the elements of the period in order from a vertex v on the sail to the vertex $M^2(v)$ on the sail.)*

Remark 5. Note that matrices inverse to each other have reversed periods.

1.4 Reduced matrices and continuants

In this section we introduce reduced matrices. Their elements have a nice representation in terms of continuants. Let us fix the following notation.

Definition 7. *Let a be a real number, denote $M_a = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$.*
Now let $(a_1, \ldots, a_n)$ be any sequence of real numbers, we set

$$M_{a_1,\ldots,a_n} = \prod_{k=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix}.$$

Definition 8. *Consider a sequence of positive integers $(a_1, \ldots, a_n)$. Then the matrix $M_{a_1,\ldots,a_n}$ is said to be reduced.*

There are two main benefits for the proposed choice of reduced matrices. Firstly, they form a semigroup with respect to matrix multiplication. Secondly, there is the following explicit description of such matrices.

Proposition 1. *Let $n \geq 0$ and let $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Then we have*

$$M_{a_1,\ldots,a_n} = \begin{pmatrix} K_{n-2}(a_2, \ldots, a_{n-1}) & K_{n-1}(a_2, \ldots, a_n) \\ K_{n-1}(a_1, a_2, \ldots, a_{n-1}) & K_n(a_1, a_2, \ldots, a_n) \end{pmatrix}.$$

In addition, we have $\det M_{a_1,\ldots,a_n} = (-1)^n$.

Example 1. Consider

$$M_{3,-3,-2,5} = M_3 \cdot M_{-3} \cdot M_{-2} \cdot M_5.$$

Hence M is represented by the following sequence: $(3, -3, -2, 5)$. By Proposition 1 we immediately have

$$M_{3,-3,-2,5} = \begin{pmatrix} K_2(-3,-2) & K_3(-3,-2,5) \\ K_3(3,-3,-2) & K_4(3,-3,-2,5) \end{pmatrix}.$$

Therefore,

$$M_{3,-3,-2,5} = \begin{pmatrix} 7 & 32 \\ 19 & 87 \end{pmatrix}.$$

Here we actually have

$$\frac{19}{7} = [3 : -3 : -2] \quad \text{and} \quad \frac{87}{19} = [3 : -3 : -2 : 5].$$

Note also that

$$\det M = (-1)^4 = 1.$$

Proof of Proposition 1. The proof is done by induction in n.

Base of induction. For $n = 1, 2$ we have respectively

$$M_{a_1} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} = \begin{pmatrix} K_{-1}() & K_0() \\ K_0() & K_1(a_1) \end{pmatrix}.$$
$$M_{a_1,a_2} = M_{a_1}M_{a_2} = \begin{pmatrix} 1 & a_2 \\ a_1 & 1 + a_1a_2 \end{pmatrix} = \begin{pmatrix} K_0() & K_1(a_2) \\ K_1(a_1) & K_2(a_1, a_2) \end{pmatrix}.$$

Step of induction. We have

$$\begin{aligned} M_{a_1,\ldots,a_{n+1}} &= M_{a_1,\ldots,a_n} \cdot M_{a_{n+1}} = \\ &\begin{pmatrix} K_{n-2}(a_2, \ldots, a_{n-1}) & K_{n-1}(a_2, \ldots, a_n) \\ K_{n-1}(a_1, \ldots, a_{n-1}) & K_n(a_1, \ldots, a_n) \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix} = \\ &\begin{pmatrix} K_{n-1}(a_2, \ldots, a_n) & K_{n-2}(a_2, \ldots, a_{n-1}) + a_{n+1}K_{n-1}(a_2, \ldots, a_n) \\ K_n(a_1, \ldots, a_n) & K_{n-1}(a_1, \ldots, a_{n-1}) + a_{n+1}K_n(a_1, \ldots, a_n) \end{pmatrix} = \\ &\begin{pmatrix} K_{n-1}(a_2, \ldots, a_n) & K_n(a_2, \ldots, a_{n+1}) \\ K_n(a_1, \ldots, a_n) & K_{n+1}(a_1, \ldots, a_{n+1}) \end{pmatrix}. \end{aligned}$$

The last equality is a classical relation for the numerators and denominators of continued fractions (see, e.g., in [14] or in [11]). This concludes the proof for the induction step.

Finally, since $\det M_a = -1$ we have $\det M = (-1)^n$. □

1.5 Difference of sequences

Finally let us give the following general combinatorial definition.

Definition 9. *Let $m > n$ be two non-negative integers and consider two sequences of real numbers $S_a = (a_1, \ldots, a_m)$ and $S_b = (b_1, \ldots, b_n)$. We say that there exists a difference of S_a and S_b if there exists $k \leq m + 1$ such that the following conditions are fulfilled:* (*i*) *$b_i = a_i$ for $1 \leq i < k$;*
(*ii*) *either $k = m + 1$ or $b_k \neq a_k$;* (*iii*) *$b_{k+i} = a_{k+i+m-n}$ for $0 \leq i \leq n - k$.*
In this case we denote $S_a - S_b = (a_k, a_{k+1}, \ldots, a_{k+n-m-1})$.

Example 2. (*i*) We have $(1, 2, 3, 4, 5, 6, 7, 8) - (1, 2, 3, 6, 7, 8) = (4, 5)$.
(*ii*) The expression $(1, 2, 3, 4, 5, 6, 7, 8) - (1, 5, 8)$ is not defined.

2 Three cases of $\mathrm{GL}(2,\mathbb{Z})$ matrices

It is natural to split the matrices of $\mathrm{GL}(2,\mathbb{Z})$ into three cases with respect to their spectra (set of eigenvalues). We distinguish the cases of complex, rational, and real irrational spectra. The cases of complex and rational cases are rather straightforward, they are not included in Gauss Reduction Theory. The case of real irrational spectra is more complicated, it is central for this paper. Let us now briefly discuss these three cases in this section.

Case of complex spectra: We start with $\mathrm{GL}(2,\mathbb{Z})$ matrices whose characteristic polynomials have a pair of complex conjugate roots. There are exactly three $\mathrm{PGL}(2,\mathbb{Z})$-conjugacy classes of such matrices (these classes are perfectly distinguished by traces of matrices). They are represented by

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

The author does not have a link to the proof of the classification in the complex case, however it is a classical result. The complete proof will be shortly available in the second edition of [11].

Case of rational spectra: It turns out that such matrices have eigenvalues equal to ± 1, any of rational spectra matrices are $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to exactly one of the following matrices

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad m \geq 0, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

(Note that the rational spectra case contains the degenerate case of coinciding roots.) For the proofs in the rational spectra case see [3].

Case of real irrational spectra: This case is the most complicated. It is described by a so-called Gauss Reduction Theory, which is based on Euclidean type algorithms providing a descent to reduced matrices (see Chapter 7 of [11]). It is interesting to note that the number of reduced matrices integer congruent to a given one is finite and equal to the number of elements in the minimal period of the regular continued fraction for the tangent of the slope of any eigenvector of the matrix. In the next section we introduce an alternative algorithm based on explicit expressions for reduced matrices that originated in geometry of numbers.

3 Techniques to find reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to a given one

Let us outline the main stages of the reduced matrices construction. All the statements involved in it are proven in the next section. The construction is based on general Theorem 3 and several supplementary technical statements.

Remark 6. The proposed algorithm provides an answer to Problem 1.

Goal of the algorithm. List all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to M.

Input data. We are given a $\mathrm{GL}(2,\mathbb{Z})$ matrix. Namely we have $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$.

Step 1. Starting with any point P_0 we set $P_1 = M^4(P_0)$ and $P_2 = M^6(P_0)$ and compute $LLS(\angle P_0OP_1)$ and $LLS(\angle P_0OP_2)$ using Theorem 4.

Step 2. By Proposition 2 one of the periods of the LLS sequence for M is a half of $LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1)$. We take the first half of this sequence, so let the period be $(a_1, \ldots, a_n)$ and let the lengths of the minimal possible periods be m.

Step 3. Now we can write down the reduced matrices in accordance with Theorem 2 and Proposition 1.

Output. All the reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to M will be of the form

$$\begin{pmatrix} K_{n-2}(a_{k+2}, \ldots, a_{k+n-1}) & K_{n-1}(a_{k+2}, \ldots, a_{k+n}) \\ K_{n-1}(a_{k+1}, a_{k+2}, \ldots, a_{k+n-1}) & K_n(a_{k+1}, a_{k+2}, \ldots, a_{k+n}) \end{pmatrix}, k = 0, \ldots, m-1.$$

Example 3. **Input:** Find all reduced matrices for the matrix $M = \begin{pmatrix} 7 & -30 \\ -10 & 43 \end{pmatrix}$.

Step 1. Starting with any point $P_0 = (1,1)$ set

$$\begin{aligned} P_1 &= M^4(P_0) = (-2875199, 4119201) \quad \text{and} \\ P_2 &= M^6(P_0) = (-7182245951, 10289762449). \end{aligned}$$

Let us first compute $LLS(\angle P_0OP_1)$. First of all note that

$$\varepsilon = -\mathrm{sign}\frac{1}{1} = -1, \ \delta = \mathrm{sign}\frac{-2875199}{4119201} = -1 \quad \text{and} \quad \det(\mathrm{OP}_1, \mathrm{OP}_2)\cdot(-1) > 0.$$

Hence we take the following odd regular continued fractions: $\frac{1}{1} = [1]$ and

$$\left|\tfrac{-2875199}{4119201}\right| = \tfrac{2875199}{4119201} = [0; 1:2:3:4:1:2:3:4:1:2:3:4:1:2:3:3].$$

Now we combine these two continued fractions in accordance with Theorem 4:

$$\begin{aligned} &[-1; 0:0:-1:-2:-3:-4:-1:-2:-3:-4:-1:-2:-3:-4: \\ &-1:-2:-3:-3] = \tfrac{-6994400}{4119201}. \end{aligned}$$

We have

$$\left|\frac{-6994400}{4119201}\right| = \frac{6994400}{4119201} = [1; 1:2:3:4:1:2:3:4:1:2:3:4:1:2:3:3].$$

Therefore, $LLS(\angle P_0OP_1) = (\underline{1,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3}, \overline{3})$.

Similarly we get

$$LLS(\angle P_0OP_2) = (\underline{1,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3}, \boxed{4,\ 1,\ 2,\ 3,\ 4,\ 1,\ 2,\ 3}, \overline{3}).$$

(Here we show the difference of the sequences in the box.)

Step 2. By Proposition 2 one of the periods of the LLS sequence for M is a half of the sequence $LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1) = (4,1,2,3,4,1,2,3)$, which is $(4,1,2,3)$. The minimal possible period is of length 4 (so $m = 4$).

Step 3. We can write down the reduced matrices in accordance with Theorem 2 and Proposition 1 for all distinct periods of length 4, i.e. for

$$(4,1,2,3), \quad (1,2,3,4), \quad (2,3,4,1), \quad \text{and} \quad (3,4,1,2).$$

Output. Finally applying Proposition 1 to these four sequences we have the list of all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to M:

$$\begin{pmatrix} K_2(1,2) & K_3(1,2,3) \\ K_3(4,1,2) & K_4(4,1,2,3) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 14 & 47 \end{pmatrix}, \quad \begin{pmatrix} 7 & 30 \\ 10 & 43 \end{pmatrix}, \quad \begin{pmatrix} 13 & 16 \\ 30 & 37 \end{pmatrix}, \quad \begin{pmatrix} 5 & 14 \\ 16 & 45 \end{pmatrix}.$$

(We show continuants only for the first matrix and omit them for the others.)

4 Technical aspects of reduced matrices computation

In this section we show some technical statements involved in the justification of the above algorithm. We start in Subsection 4.1 with writing periods of LLS sequences for reduced matrices. In Subsection 4.2 we explain how to list all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to the given one (the reduced matrices are given in terms of LLS periods of original matrices). Then we show in general how to compute LLS sequences of angles in Subsection 4.3. Finally in Subsection 4.4 we state how to compute the periods of LLS sequences.

4.1 Continued fraction enumeration of reduced matrices

Let us find a period of the LLS sequence for matrices $M_{a_1,a_2,\ldots,a_n}$.

Theorem 2. *Let n, $a_1,\ldots,a_n$ be positive integers. Then one of the periods of the LLS sequence for $M_{a_1,a_2,\ldots,a_n}$ is $(a_1,a_2,\ldots,a_n)$.*

Proof. Consider the sequence of integer points $(x_k,y_k) = M^k_{a_1,a_2,\ldots,a_n}(1,0)$ for positive integer values of k. By Proposition 1 and Definition 7 for every k we know the coordinates x_k and y_k via continuants. So from the general theory of continued fractions they are relatively prime and further by Remark 3 they satisfy

$$\frac{y_k}{x_k} = [(a_1;a_2:\cdots:a_n)^k].$$

Therefore, all the points (x_k,y_k) are vertices of the sail of the periodic continued fraction $\alpha = [(a_1;a_2:\cdots:a_n)]$. (This is a classical statement of geometry of numbers (Theorem 3.1 of [11]).) This immediately implies that the direction of the vector $(1,\alpha)$ is the limiting direction for the sequence of directions for the vectors (x_k, y_k), and in particular that $\lim\limits_{k\to\infty}(y_k/x_k) = \alpha$. Hence $(1, \alpha)$ is one of the eigenvectors corresponding to the maximal eigenvalue (and thus the eigenvalues are both real and distinct). By construction the LLS sequence for α is periodic with period $(a_1, a_2, \ldots, a_n)$.

Finally the sail for α from some element coincides with the sail for M. Since the sail for M is periodic, the period is the same as for α, i.e. $(a_1, a_2, \ldots, a_n)$. □

4.2 Matrices $\mathrm{PGL}(2, \mathbb{Z})$-conjugate to a given one

The following theorem produces the list of all reduced matrices $\mathrm{PGL}(2, \mathbb{Z})$-conjugate to a given one.

Theorem 3. *Let M be a $\mathrm{GL}(2, \mathbb{Z})$ matrix and let $(a_1, \ldots, a_n)$ be a period of the LLS sequence corresponding to M. Finally let m be the minimal length of the period of the LLS sequence. Then the list of all reduced matrices $\mathrm{PGL}(2, \mathbb{Z})$-conjugate to M consists of the following m matrices:*

$$M_{a_{1+k}, \ldots, a_{n+k}}, \qquad k = 1, \ldots, m.$$

Let us first prove the following lemma.

Lemma 1. *Two operators have the same LLS sequences if and only if their unions of eigenlines are integer congruent to each other.*

Proof. The LLS sequence is an invariant of arrangements of two lines with respect to integer congruences, hence they are the same if the unions of eigenlines for the operators are integer congruent to each other.

Now let the unions of eigenlines have the same LLS sequences. Pick any of the four angles for the first unions of the eigenlines. Now we pick another angle for the second union of the eigenlines in such a way that their subsequences of integer lengths and integer sines coincide respectively. This is always possible as the adjacent angles have the same LLS sequence with subsequence of integer angles equal to the subsequence of integer sines and vice versa. This is a consequence of a classical duality of sails for adjacent angles ([11], Proposition 8.5).

Such angles are integer congruent. This follows from the fact that the sail is uniquely reconstructed by the LLS sequence, one of its vertices, and the direction of one of the adjacent edges to this vertex. Once LLS sequence is reconstructed, the integer angle is reconstructed itself (here we assume that we consider the angles with vertex at the origin). For further details we refer to [11], Theorem 4.11. Since the angles are integer congruent, the unions of eigenlines are integer congruent as well. □

Proof of Theorem 3. By Lemma 1 we know that two operators have the same LLS sequences if and only if their unions of eigenlines are integer congruent to each other. In Theorem 2 we showed that the LLS sequence of $M_{b_1, b_2, \ldots, b_r}$ has a period

$$(b_1, b_2, \ldots, b_r).$$

Therefore, by Lemma 1, M could be congruent only to reduced matrices whose units of eigenlines are integer congruent to the units of eigenlines

$$\pm M_{a_{1+k},\ldots,a_{n+k}}$$

for $k=1,\ldots,m$ (these are the only matrices with LLS sequences of length n that have such LLS sequences). By the structure of the sails of reduced operators (as the first segment of the sail in the positive octant containing $(1,0)$ is orthogonal to the x-axis) reduced matrices with congruent units of eigenlines have coinciding eigenlines. By a general statement in geometry of numbers, the operators with coinciding eigenlines are the elements of the same Dirichlet group and in the two-dimensional case they are some rational powers of each other (see Section 8.1 of [11]).

The LLS sequence of M^2 shifts the LLS sequence by n, and hence the reduced matrices integer conjugate to $\pm M$ should be defined by sequences of length n (or, equivalently, that they have periods of length n). Note that the powers of matrices $M_{a_{1+k},\ldots,a_{n+k}}$ are defined by a sequence of length n if and only if the exponents are either 1 or -1.

In the case of the exponent equal to 1 we have matrices $M_{a_{1+k},\ldots,a_{n+k}}$ for $k=1,\ldots,m$ themselves. In case of the exponent equal to -1 the LLS sequences are reversed, so this case is possible only for palindromic sequences, and hence we arrive to the same matrices $M_{a_{1+k},\ldots,a_{n+k}}$ for $k=1,\ldots,m$. Therefore, the list of all reduced matrices $\mathrm{PGL}(2,\mathbb{Z})$-conjugate to M consists of m matrices of the form $M_{a_{1+k},\ldots,a_{n+k}}$ for $k=1,\ldots,m$. This concludes the proof. □

4.3 Computation of LLS sequences for rational angles

In this subsection we formulate a theorem that provides an explicit formula for the LLS sequence of a given matrix. This formula is very much in the spirit of generalized Perron Identity introduced in our recent paper [12].

Theorem 4. *Consider two linearly independent integer vectors $A=(p,q)$ and $B=(r,s)$. We assume that none of them are proportional either to $(1,0)$ or to $(0,1)$. Let two sequences of integers $(a_0,a_1,\ldots,a_{2m})$ and $(b_0,b_1,\ldots,b_{2n})$ be defined as the sequences of elements of the odd regular continued fractions of*

- *$|q/p|$ and $|s/r|$ in case of $\det(OA,OB)\cdot\mathrm{sign}\frac{\mathrm{p}}{\mathrm{q}}<0$;*
- *$|p/q|$ and $|r/s|$ in case of $\det(OA,OB)\cdot\mathrm{sign}\frac{\mathrm{p}}{\mathrm{q}}>0$.*

Further we set $\varepsilon=-\mathrm{sign}\frac{\mathrm{p}}{\mathrm{q}}$ and $\delta=\mathrm{sign}\frac{\mathrm{r}}{\mathrm{s}}$. Denote also

$$\alpha=[\varepsilon a_{2m}:\varepsilon a_{2m-1}:\cdots:\varepsilon a_1:\varepsilon a_0:0:\delta b_0:\delta b_1:\cdots:\delta b_{2n}].$$

Let $|\alpha|=[c_0;c_1:\cdots:c_{2k}]$ be the regular odd continued fraction for $|\alpha|$. Set

- *$S=(c_0,c_1,\ldots,c_{2k})$ in the case $c_0\neq 0$;*
- *$S=(c_2,\ldots,c_{2k})$ in the case $c_0=0$.*

Then S is the LLS sequence for the angle $\angle AOB$.

Remark 7. In fact it is possible to simplify the computation of the continued fraction for α. Namely we take

$$\alpha = [\varepsilon a_{2m} : \varepsilon a_{2m-1} : \cdots : \varepsilon a_1 : \varepsilon a_0 : 0 : \delta w];$$

where $w = s/r$ if $\det(OA, OB) \cdot \mathrm{sign}\frac{\mathrm{p}}{\mathrm{q}} < 0$ and $w = r/s$ otherwise.

We continue with the following remark.

Remark 8. Recall one technical statement for angles represented by slopes with tangents less than 1: the angles represented by the continued fractions

$$[0; a_1 : a_2 : \cdots : a_{2n}] \quad \text{and} \quad [a_2; \cdots : a_{2n}]$$

are integer congruent. In particular, they have the same LLS sequences.

Proof of Theorem 4. First we set $E = (1, 0)$. Consider the broken line that is a concatenation of the sail of the angle $\angle AOE$ (in case the last edge of this sail is not vertical we add the infinitesimal edge EE of zero integer length with vertical direction and 0 integer length) and the sail for the angle $\angle EOB$ (again we add another infinitesimal edge EE in case the first edge of the sail of the angle is not vertical).

Note that this broken line L has the following properties:
— it starts at the ray OA and ends at the ray OB;
— the direction of the first edge is towards the interior of the angle $\angle AOB$.

Then the angle is integer congruent to the angle $\angle EOC$ with $C = (1, \alpha)$ where $|\alpha|$ is defined by the LLS sequence of the above broken line as

$$\alpha = [\varepsilon a_{2m} : \varepsilon a_{2m-1} : \cdots : \varepsilon a_1 : \varepsilon a_0 : 0 : \delta b_0 : \delta b_1 : \cdots : \delta b_{2n}].$$

The proof for this formula is given by the study of numerous straightforward cases of various signs for p, q, r, s and $\det(OA, OB)$.

Let us study the case $p, q, r, s > 0, \det(OA, OB) < 0$. In this case, the first part of the broken line L will be the sail of $\angle AOE$ passed clockwise. Hence the elements of the LLS sequence will be reversed and negative to the values of the LLS sequence for $\angle AOE$. Note that in the case of $q/p < 1$ we end up with an infinitesimal (zero integer length) vertical vector which additionally brings two elements: the element $\lfloor p/q \rfloor$ for the angle with the vertical line passing through E, and the element 0 indicating that we stay at E. Then we switch to the second sail. Both sails are starting vertically (or asymptotically vertical in the case of a_1 or b_1 are zeroes), hence the angle between the edges corresponding to a_0 and b_0 is zero. So we add a zero element to the LLS sequence for L here. Finally we continue back following the sail of the angle $\angle EOB$, which is described by the continued fraction $[b_0 : b_1 : \cdots : b_{2n}]$ (here again we have $b_0 = 0$ and $b_1 = \lfloor s/r \rfloor$ for the case of $r/s < 1$). Hence the LLS sequence of the broken line L is

$$(-a_{2m}, -a_{2m-1}, \ldots, -a_1, -a_0, 0, b_0, b_1, \ldots, b_{2n}).$$

Finally we get $\alpha = [-a_{2m} : -a_{2m-1} : \cdots : -a_1 : -a_0 : 0 : b_0 : b_1 : \cdots : b_{2n}]$.

The cases for the rest choices of signs for p, q, r, s and $\det(OA, OB)$ are considered similarly, so we omit them here.

Now let $|\alpha| = [c_0; c_1 : \cdots : c_{2k}]$. Therefore (c.f. Remark 8) the LLS sequence for $\angle EOC$ is either $(c_0, c_1, c_2, \ldots, c_{2k})$ if $c_0 \neq 0$, or $(c_2, \ldots, c_{2k})$ otherwise. □

4.4 Periods of the LLS sequences corresponding to matrices

In this subsection we show how to extract periods of the LLS sequence for a given matrix.

Proposition 2. *Let a* $\mathrm{GL}(2,\mathbb{Z})$ *matrix* M *have distinct irrational eigenvalues (not necessarily positive). Let also* P_0 *be any non-zero integer point. Denote* $P_1 = M^4(P_0)$ *and* $P_2 = M^6(P_0)$. *Then there exists a difference* $LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1)$, *which is a period of the LLS sequence for* M *repeated twice.*

Remark 9. The obtained period of the LLS sequence is not necessarily minimal.

We start the proof with the following lemma.

Lemma 2. *Let a* $\mathrm{GL}(2,\mathbb{Z})$ *matrix* M *have distinct irrational positive eigenvalues. Let also* P_0 *be any non-zero integer point. Denote* $P_1 = M^2(P_0)$ *and* $P_2 = M^3(P_0)$. *Then there exists a difference* $LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1)$, *which is a period of the LLS sequence for* M.

Remark 10. It is not enough to consider the difference of the LLS sequences for the angles $\angle P_0OP_1$ and $\angle P_0OQ$ (where $Q = M(P_0)$), as it is not possible to determine the last integer sine of the period then. Let us illustrate this with the following example.

Consider a matrix $M = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and the point $P = (4, -1)$. Then

$$Q = M(P_0) = (2,1), \quad P_1 = M^2(P_0) = (4,5), \quad \text{and} \quad P_2 = M^3(P_0) = (14,19).$$

The LLS sequences for the angles $\angle P_0OQ$, $\angle P_0OP_1$ and $\angle P_0OP_2$ are respectively

$$(1,4,1); \quad (1,3,1,3,1); \quad \text{and} \quad (1,3,1,2,1,3,1).$$

We have

$$(1,3,1,2,1,3,1) - (1,3,1,3,1) = (2,1)$$

which is the correct period for the LLS sequence of M, while the difference $(1,3,1,3,1) - (1,4,1)$ is not even defined.

Proof of Lemma 2. Set $Q = M(P_0)$. First of all note that $\angle P_0OQ$ is a fundamental domain of one of the angles C whose edges are eigenvectors of M up to the action of the group of (integer) powers of M. Hence it contains at least one vertex of the sail. Denote this vertex by v. Then the angle $\angle P_0OP_2$ contains vertices $v_0 = v$, $v_1 = M(v)$, and $v_2 = M^2(v)$. Thus by convexity reasons, the sail for the angle $\angle P_0OP_2$ contains the part of the sail of C between v_0 and v_2. Namely there will be four parts of the sail:

- S_1: a part of the sail contained in P_0Ov_0;
- S_2: a part of the sail contained in v_0Ov_1;
- S_3: a part of the sail contained in v_1Ov_2;
- S_4: a part of the sail contained in v_2OP_2.

Here S_2 and S_3 are periods of the sail for the angle $\angle P_0OP_2$.

Now by the same reason we have v_0 and v_1 in the sail for angle $\angle P_0OP_1$. We have the following parts:

- S'_1: a part of the sail contained in P_0Ov_0;
- S'_2: a part of the sail contained in v_0Ov_1;
- S'_3: a part of the sail contained in v_1OP_1.

Note that

$$S'_1 = S_1, \quad S'_2 = S_2 \cong S_3, \quad \text{and} \quad S'_3 \cong S_4.$$

Therefore, the difference of the LLS sequences for the angle $\angle P_0OP_2$ and the angle $\angle P_0OP_1$ is precisely the period of the LLS sequence between the points v_1 and v_2. This period corresponds to M as $M(v_1) = v_2$. This concludes the proof. □

Proof of Proposition 2. First of all let us study the LLS sequences of reduced operators. Let $M = M_{a_1,\ldots,a_n}$ be a reduced operator for the sequence of positive integers $(a_1, \ldots, a_n)$. Then from Definition 7 we have

$$M^2 = M^2_{a_1,\ldots,a_n} = M_{a_1,\ldots,a_n,a_1,\ldots,a_n}.$$

Hence the period of the LLS sequence of M^2 is twice the period of M.

For an arbitrary M we know that

$$M^2 \cong M_{a_1,\ldots,a_n,a_1,\ldots,a_n} = M^2_{a_1,\ldots,a_n}.$$

Hence M itself is $\mathrm{PGL}(2,\mathbb{Z})$-congruent to $M_{a_1,\ldots,a_n}$ Therefore, the period of the LLS sequence corresponding to M^2 will be twice the period of the LLS sequence for M. By Lemma 2 the difference $LLS(\angle P_0OP_3) - LLS(\angle P_0OP_2)$ exists and it is a period for M^2. Finally by the above the resulting sequence is a period of the LLS sequence for M repeated twice. □

References

1. Arnold, V. I.: Continued fractions (In Russian). Moscow Center of Continuous Mathematical Education, Moscow (2002).
2. Arnold, V. I.: Arithmetics of binary quadratic forms, symmetry of their continued fractions and geometry of their de Sitter world (dedicated to the 50th anniversary of IMPA), Bull. Braz. Math. Soc. (N.S.), **34**(1), pp. 1–42 (2003).
3. Baake, M., Roberts, J. A. G.: Reversing symmetry group of $\mathrm{GL}(2,\mathbb{Z})$ and $\mathrm{PGL}(2,\mathbb{Z})$ matrices with connections to cat maps and trace maps, J. Phys. A, **30**(5), pp. 1549–1573 (1997).

4. Campbell, J. T., Trouy, E. C.: When are two elements of GL$(2, \mathbb{Z})$ similar? Linear Algebra Appl., **157**, pp. 175-184 (1991).
5. Cusick, T. W., Flahive, M. E.: The Markoff and Lagrange spectra, Mathematical Surveys and Monographs, **30**, American Mathematical Society, Providence, RI, 1989.
6. Henniger, J. P.: Factorization and similarity in GL(2,Z), Linear Algebra Appl. **251**, pp. 223–237, (1997).
7. Karpenkov, O.: Elementary notions of lattice trigonometry, Math. Scand., **102**(2), pp. 161–205 (2008).
8. Karpenkov, O.: On irrational lattice angles, Funct. Anal. Other Math., **2**(2–4), pp. 221–239 (2009).
9. Karpenkov, O.: On determination of periods of geometric continued fractions for twodimensional algebraic hyperbolic operators. Math. Notes, **88**(1–2), pp. 28–38 (2010). Russian version: Mat. Zametki, **88**(1), pp. 30–42 (2010).
10. Karpenkov, O.: Multidimensional Gauss Reduction Theory for conjugacy classes of SL$(n, \mathbb{Z})$, J. Théor. Nombres Bordeaux, **25**(1), pp. 99–109 (2013).
11. Karpenkov, O.: Geometry of continued fractions, Algorithms and Computation in Mathematics, **26**. Springer-Verlag, Berlin, (2013).
12. Karpenkov, O., Van-Son, M.: Perron identity for arbitrary broken lines, J. Théor. Nombres Bordeaux, **31**(1), pp. 131–144 (2019).
13. Katok S.: Continued fractions, hyperbolic geometry and quadratic forms. In MASS selecta, pp. 121–160. Amer. Math. Soc., Providence, RI (2003).
14. Khinchin, A. Ya.: Continued fractions, FISMATGIS, Moscow (1961).
15. Klein, F.: Ueber eine geometrische Auffassung der gewöhnliche Kettenbruchentwicklung, Nachr. Ges. Wiss. Göttingen Math-Phys. Kl., **3** pp 352–357 (1895).
16. Klein, F.: Sur une représentation géométrique de développement en fraction continue ordinaire, Nouv. Ann. Math., **15**(3), pp 327–331 (1896).
17. Lewis, J., Zagier, D.: Period functions and the Selberg zeta function for the modular group. In The mathematical beauty of physics (Saclay, 1996), Adv. Ser. Math. Phys., vol. **24** , pp. 83–97. World Sci. Publ., River Edge, NJ, (1997).
18. Manin, Y. I., Marcolli, M.: Continued fractions, modular symbols, and noncommutative geometry. Selecta Math. (N.S.), **8**(3), pp. 475–521 (2002).