Efficient stochastic modal decomposition methods for structural stochastic static and dynamic analyses

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Abstract

This paper presents unified and efficient stochastic modal decomposition methods to solve stochastic structural static and dynamic problems. We extend the idea of deterministic modal decomposition method for structural dynamic analysis to stochastic cases. Standard/generalized stochastic eigenvalue equations are adopted to calculate the stochastic subspaces for stochastic static/dynamic problems and they are solved by an efficient reduced-order method. The stochastic solutions of both static and dynamic equations are approximated by stochastic bases of the stochastic subspaces. Original stochastic static/dynamic equations are then transformed into a set of single-degree-of-freedom (SDoF) stochastic static/dynamic equations, which are efficiently solved by the proposed non-intrusive methods. Specifically, a non-intrusive stochastic Newmark method is developed for the solution of SDoF stochastic dynamic equations, and the element-wise division of sample vectors is used to solve the SDoF stochastic static equations. All of these methods have low computational effort and are weakly sensitive to the stochastic dimension, thus the proposed methods avoid the curse of dimensionality successfully. Two numerical examples, including two- and three-dimensional spatial problems with low and high stochastic dimensions, are given to show the efficiency and accuracy of the proposed methods.

Keywords: Stochastic static and dynamic analyses; Stochastic eigenvalue problems; Stochastic reduced-order equations; Stochastic Newmark method; Curse of dimensionality;

1. Introduction

Predicting uncertainty propagation on the physical models has become an essential part in the analysis and design of practical engineering problems. The considerable influence of uncertainties on system behavior has led to the development of numerical methods for uncertainty analysis. In this paper, we focus on developing efficient and unified numerical algorithms for linear stochastic static and dynamic analyses.

Many effective numerical methods have been developed for solving stochastic dynamic and static equations. The Monte Carlo simulation (MCS) and its extensions [1, 2, 3] have been widely used for the analyses. By repeatedly solving the deterministic problem for each sample realization, they are executed in non-intrusive ways and can be readily applied to high-dimensional stochastic problems. However, they are usually computationally expensive since a large number of deterministic problems are solved to achieve high-accuracy stochastic solutions. Several more efficient non-intrusive methods, e.g. adaptive sampling methods and response surface methods [4, 5, 6], are also proposed for the purpose. As a kind of intrusive method, the Galerkin spectral stochastic finite element method and its extensions [7, 8, 9] have received a lot of attention for solving stochastic problems. In this method, the stochastic solution is decomposed into the summation of a series of products of polynomial chaos (PC) bases and deterministic time-independent/time-dependent functions. An augmented deterministic static/dynamic equation is then obtained by using the stochastic Galerkin projection. The size of the augmented equation is huge when dealing with large-scale and high-dimensional stochastic problems, thus this approach suffers from the curse of dimensionality. Several improvements are presented to reduce the computational effort of PC-based methods, e.g. the Krylov-based iteration and the sparse PC approach [10, 11]. Other methods are also explored to efficiently and accurately perform stochastic structural static and dynamic analyses, e.g. the stochastic collocation method [12], the PGD-based method [13, 14], the ANOVA method [15], the direct probability integral method [16, 17], etc.

We mention another kind of method, known as the modal decomposition-type method. Let us simply recall the idea of the modal decomposition method for solving deterministic structural

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dynamic equations [18, 19]. In this method, a set of modes of the target system are calculated using a generalized eigenvalue equation and the time-dependent displacement response is solved by the linear superposition of each mode. It is natural and straightforward to extend the above idea of deterministic modal decomposition to the stochastic case, i.e., stochastic eigenvalue equations are used to calculate a set of stochastic modes of the target stochastic system and the stochastic displacement is solved by the linear superposition of each stochastic mode. However, it is usually not a simple matter since efficient and accurate numerical algorithms are required to solve stochastic eigenvalue problems. Although a lot of methods can be used to solve stochastic eigenvalue equations, e.g. the MCS [20, 21], the perturbation method [22, 23, 24], the PC-based method [25], the subspace iteration approach [26, 27, 28, 29], the stochastic collocation method [30], the polynomial/spline dimensional decomposition methods [31, 32], the homotopy approach[33], the low-rank approximation method [34, 35], etc., only a few effort has been made to apply stochastic eigenvalue algorithms to modal decomposition-based stochastic dynamic analyses.

In the references[36, 37], the PC expansion is adopted to approximate stochastic eigenvalues, stochastic eigenvectors and stochastic responses, and stochastic Galerkin projection is used for their solutions. Similar to classical PC-based methods, this method still suffers from the curse of dimensionality, which is alleviated by a hybrid PC and MCS strategy in the reference[38]. Also, another improvement is developed in the reference[26] by extending deterministic deflated and subspace inverse power methods to PC-based stochastic approaches, which can be applied to well solve the problems with repeated or closely spaced stochastic eigenvalues. The methods that combine PC-based stochastic finite element analyses with deterministic and stochastic reduced-order methods are developed in the reference[39], where deterministic/stochastic reduced bases are obtained by using deterministic/stochastic Krylov subspace techniques. In the reference[40], a high-order perturbation technique coupled with reduced modal subspaces is developed to solve stochastic dynamic systems, which exhibits better performance than the classical perturbation methods. The spline chaos expansion (SCE) and the spline dimensional decomposition (SDD) methods are proposed in the reference[41] for stochastic modal analyses. Similar to the PC expansion, both the two methods can be considered as Fourier-like expansions and the unknown expanded coefficients need to be calculated. An advantage of these two methods is that they can

well capture nonlinear and nonsmooth stochastic solutions. Furthermore, SDD alleviates the curse of dimensionality encountered by SCE and classical PC-based methods.

In this paper, we develop efficient and unified stochastic modal decomposition-based numerical schemes for solving stochastic static and dynamic problems. As discussed above, we construct stochastic subspaces for stochastic static/dynamic equations by solving standard/generalized stochastic eigenvalue equations. An efficient reduced-order method proposed in our prior work [42] is adopted for this purpose. In this method, stochastic eigenvectors are approximated by a set of products of random variables and deterministic vectors, where the deterministic vectors are calculated via a few number of deterministic eigenequations. A reduced-order stochastic eigenvalue problem constructed using the obtained deterministic vectors is then used to solve stochastic eigenvalues of the original problem and random variable coefficients corresponding to the deterministic vectors. Original stochastic static/dynamic equations are transformed into a set of SDoF stochastic static/dynamic equations by using the stochastic subspaces obtained by standard/generalized stochastic eigenequations. Following that, we develop a non-intrusive stochastic Newmark method to solve the SDoF stochastic dynamic equations and an element-wise division of sample vectors to solve the SDoF stochastic static equations. The proposed methods overcome the curse of dimensionality to a great extent since both the stochastic eigenvalue algorithm and the approaches for solving SDoF stochastic static/dynamic equations are not sensitive to the stochastic dimension. Further, the final stochastic solutions of both stochastic static and dynamic problems can be represented as a sum of deterministic vectors with random variable coefficients, where the random variable coefficients are described by random samples. It is a kind of semi-explicit representation and provides a convenient pathway to perform uncertainty quantification.

The paper is organized as follows: We extend the classical modal decomposition method to stochastic dynamic and static problems in Section 2, where an efficient reduced-order method is given to solve stochastic eigenvalue equations and generate stochastic subspaces for the stochastic modal decomposition method in Section 2.3. According to the obtained stochastic subspaces, stochastic static and dynamic equations are solved in Section 3. Algorithm implementations of the proposed methods are then elaborated in Section 4. Following that, two numerical examples are given to demonstrate the efficiency and accuracy of the proposed methods in Section 5, and

conclusions follow in Section 6.

2. Stochastic modal decomposition for stochastic static and dynamic analysis

2.1. Stochastic modal decomposition for stochastic dynamic analysis

Let $(\Theta, \Xi, \mathcal{P})$ be a suitable probability space , where Θ denotes the space of elementary events, Ξ is a σ-algebra defined on Θ and $\mathcal{P}$ is a probability measure. In this paper, we consider a linear stochastic structural dynamic equation

$$\mathbf{M}(\theta)\,\ddot{\mathbf{u}}(t,\theta) + \mathbf{C}(\theta)\,\dot{\mathbf{u}}(t,\theta) + \mathbf{K}(\theta)\,\mathbf{u}(t,\theta) = \mathbf{F}(t,\theta) \tag{1}$$

defined on a deterministic domain $\mathcal{D} \subset \mathbb{R}^d$ with the physical dimension $d = 1, 2, 3$ and the boundary $\partial\mathcal{D}$, where $\mathbf{u}(t,\theta) \in \mathbb{R}^{n\times n_t}$ represents the unknown stochastic solution, n_t is the number of time steps, $\mathbf{u}(t_i,\theta) \in \mathbb{R}^n$ is the stochastic solution at the time step t_i, the symmetric and positive definite matrices $\mathbf{M}(\theta), \mathbf{C}(\theta), \mathbf{K}(\theta) \in \mathbb{R}^{n\times n}$, $\mathbf{F}(t,\theta) \in \mathbb{R}^{n\times n_t}$ are the stochastic mass matrix, the stochastic damping matrix, the stochastic stiffness matrix and the stochastic force vector, respectively, which are obtained by the classical finite element discretization, and the initial values are given by $\mathbf{u}(0,\theta) = \mathbf{u}_0(\theta) \in \mathbb{R}^n$ and $\dot{\mathbf{u}}(0,\theta) = \mathbf{u}_1(\theta) \in \mathbb{R}^n$. In this paper, we only consider the Rayleigh damping, i.e., $\mathbf{C}(\theta) = \varsigma_{\mathrm{M}}(\theta)\,\mathbf{M}(\theta) + \varsigma_{\mathrm{K}}(\theta)\,\mathbf{K}(\theta)$, where $\varsigma_{\mathrm{M}}(\theta)$ and $\varsigma_{\mathrm{K}}(\theta)$ are given (random) parameters. Hence, uncertainties of Eq. (1) may be from stochastic material properties, stochastic forces, stochastic initial values and stochastic parameters in Rayleigh damping.

Inspired by the classical modal decomposition method for deterministic structural dynamic analysis [18, 43], we approximate the stochastic solution $\mathbf{u}(t,\theta)$ of Eq. (1) as

$$\mathbf{u}(t,\theta) = \sum_{i=1}^{k} \boldsymbol{\varphi}_i(\theta)\, q_i(t,\theta) = \boldsymbol{\Phi}(\theta)\,\mathbf{q}(t,\theta), \tag{2}$$

where $\boldsymbol{\varphi}_i(\theta) \in \mathbb{R}^n$, $i = 1, \cdots, k$ are a set of stochastic vectors that have determined in some way, $\boldsymbol{\Phi}(\theta) = [\boldsymbol{\varphi}_1(\theta), \cdots, \boldsymbol{\varphi}_k(\theta)] \in \mathbb{R}^{n\times k}$ is a stochastic matrix (similar to the deterministic case, we name it as the stochastic subspace), $q_i(t,\theta) \in \mathbb{R}^{1\times n_t}$ are unknown time-dependent coefficients that need to be solved, $\mathbf{q}(t,\theta) = \left[q_1^{\mathrm{T}}(t,\theta), \cdots, q_k^{\mathrm{T}}(t,\theta)\right]^{\mathrm{T}} \in \mathbb{R}^{k\times n_t}$ is the coefficient matrix, k is the

dimension of the stochastic subspace $\mathbf{\Phi}(\theta)$. Substituting Eq. (2) into Eq. (1) we reformulate the stochastic dynamic equation as

$$\mathbf{M}(\theta)\,\mathbf{\Phi}(\theta)\,\ddot{\mathbf{q}}(t,\theta)+\mathbf{C}(\theta)\,\mathbf{\Phi}(\theta)\,\dot{\mathbf{q}}(t,\theta)+\mathbf{K}(\theta)\,\mathbf{\Phi}(\theta)\,\mathbf{q}(t,\theta)=\mathbf{F}(t,\theta), \tag{3}$$

which only has k unknown time-dependent variables $\{q_i(t,\theta)\}_{i=1}^{k}$ if the stochastic subspace $\mathbf{\Phi}(\theta)$ has been known, therefore we can solve it with less computational effort. To construct the stochastic subspace $\mathbf{\Phi}(\theta)$, the stochastic modes $\{\boldsymbol{\varphi}_i(\theta)\}_{i=1}^{k}$ are obtained by solving the following generalized stochastic eigenvalue equation

$$\mathbf{K}(\theta)\,\boldsymbol{\varphi}(\theta)=\lambda(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}(\theta), \tag{4}$$

whose solution algorithm will be discussed in detail in Section 2.3. In this way, the stochastic matrix $\mathbf{\Phi}(\theta)$ meets $\mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{\Phi}(\theta)=\mathbf{I}_k\in\mathbb{R}^{k\times k}$ almost everywhere (a.e.), where $\mathbf{I}_k$ represents the identity matrix. Taking advantage of this and recalling Eq. (2) we have $q_i(t,\theta)=\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{u}(t,\theta)$, which results in $q_i(t,\theta)$ still depending on the random input θ. However, as presented later, $q_i(t,\theta)$ can be solved cheaply via a one-dimensional stochastic problem benefiting from the orthogonality of $\mathbf{\Phi}(\theta)$. Furthermore, the stochastic modes $\boldsymbol{\varphi}_i(\theta)$ and $\boldsymbol{\varphi}_j(\theta)$ corresponding to different stochastic eigenvalues $\lambda_i(\theta)$ and $\lambda_j(\theta)$ (i.e., $i\neq j$) meet the orthogonal conditions

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta)=0,\quad \boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_j(\theta)=0\qquad \text{a.e.}, \tag{5}$$

whose proof is found in Appendix A1. In other words, although Eq. (5) is dependent of the random input, the above orthogonal conditions hold over the spatial domain for each realization of $\theta\in\Theta$, which thus allows the construction of decoupled stochastic problems to solve the coefficients $\{q_i(t,\theta)\}_{i=1}^{k}$.

To solve unknown stochastic coefficients $\{q_i(t,\theta)\}_{i=1}^{k}$ in Eq. (3), multiplying Eq. (3) by $\mathbf{\Phi}^{\mathrm{T}}(\theta)$ from left and taking advantage of the orthogonality in Eq. (5) we have

$$\mathbf{m}(\theta)\,\ddot{\mathbf{q}}(t,\theta)+\mathbf{c}(\theta)\,\dot{\mathbf{q}}(t,\theta)+\mathbf{k}(\theta)\,\mathbf{q}(t,\theta)=\mathbf{f}(t,\theta) \tag{6}$$

with the initial values $\mathbf{q}_0(\theta)=\mathbf{q}(0,\theta)=\mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{u}_0(\theta)\in\mathbb{R}^k$ and $\mathbf{q}_1(\theta)=\dot{\mathbf{q}}(0,\theta)=\mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{u}_1(\theta)\in\mathbb{R}^k$, where the diagonal stochastic matrices $\mathbf{m}(\theta)$, $\mathbf{k}(\theta)$, $\mathbf{c}(\theta)\in\mathbb{R}^{k\times k}$ and the stochastic vector

$\mathbf{f}(t,\theta) \in \mathbb{R}^{k\times n_t}$ are given by

$$\begin{aligned}
\mathbf{m}(\theta) &= \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\mathbf{\Phi}(\theta) = \begin{bmatrix} m_1(\theta) & & 0 \\ & \ddots & \\ 0 & & m_k(\theta) \end{bmatrix}, \\
\mathbf{k}(\theta) &= \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\mathbf{\Phi}(\theta) = \begin{bmatrix} k_1(\theta) & & 0 \\ & \ddots & \\ 0 & & k_k(\theta) \end{bmatrix}, \\
\mathbf{c}(\theta) &= \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{C}(\theta)\,\mathbf{\Phi}(\theta) = \varsigma_{\mathrm{M}}(\theta)\,\mathbf{m}(\theta) + \varsigma_{\mathrm{K}}(\theta)\,\mathbf{k}(\theta), \\
\mathbf{f}(t,\theta) &= \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{F}(t,\theta) = \left[f_1(t,\theta), \cdots, f_k(t,\theta)\right]^{\mathrm{T}}.
\end{aligned} \tag{7}$$

Since the matrices $\mathbf{m}(\theta)$, $\mathbf{k}(\theta)$ and $\mathbf{c}(\theta)$ are diagonal, Eq. (6) can be decoupled into k SDoF stochastic dynamic equations

$$m_i(\theta)\,\ddot{q}_i(t,\theta) + c_i(\theta)\,\dot{q}_i(t,\theta) + k_i(\theta)\,q_i(t,\theta) = f_i(t,\theta) \tag{8}$$

for $i = 1, \cdots, k$, which is further simplified as

$$\ddot{q}_i(t,\theta) + \varpi_i(t,\theta)\,\dot{q}_i(t,\theta) + \lambda_i(\theta)\,q_i(t,\theta) = h_i(t,\theta) \tag{9}$$

by a normalizing procedure

$$\begin{aligned}
\frac{k_i(\theta)}{m_i(\theta)} &= \frac{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_i(\theta)}{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_i(\theta)} = \lambda_i(\theta), \\
\frac{c_i(\theta)}{m_i(\theta)} &= \frac{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{C}(\theta)\,\boldsymbol{\varphi}_i(\theta)}{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_i(\theta)} = \varsigma_{\mathrm{M}}(\theta) + \varsigma_{\mathrm{K}}(\theta)\,\lambda_i(\theta) := \varpi_i(\theta), \\
\frac{f_i(t,\theta)}{m_i(\theta)} &= \frac{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{F}(t,\theta)}{\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_i(\theta)} := h_i(t,\theta).
\end{aligned} \tag{10}$$

In this way, we transform the original stochastic dynamic equation into k SDoF stochastic dynamic equations (9) (i.e., one-dimensional second order stochastic ordinary differential equations), which can be solved by existing ODE solvers [44]. In this paper, we will efficiently solve Eq. (9) by using a stochastic Newmark method in Section 3.2. Further, Eq. (1) degenerates into a deterministic dynamical equation for each realization of $\theta \in \Theta$. In this way, Eq. (2) to Eq. (10) are the same

as the procedure of the classical modal decomposition method. Therefore, the existence and well-posedness of the above stochastic solution can be inherited from the classical modal decomposition method in a probabilistic sense.

2.2. Stochastic modal decomposition for stochastic static analysis

The above procedure for solving stochastic dynamic equations can be readily extended to stochastic static problems. Typically, it is unnecessary to solve deterministic static problems by modal decomposition-type methods due to the well established deterministic solvers. However, efficient stochastic solvers for stochastic static problems are still worth studying. The stochastic modal decomposition method in this paper provides a new approach for this purpose. Specifically, we consider the following linear stochastic finite element equation

$$\mathbf{K}(\theta)\,\mathbf{u}(\theta) = \mathbf{F}(\theta) \tag{11}$$

defined on a deterministic domain $\mathcal{D} \subset \mathbb{R}^d$ with the boundary $\partial\mathcal{D}$, where $\mathbf{u}(\theta) \in \mathbb{R}^n$ is the unknown stochastic solution, the symmetric and positive definite stochastic stiffness matrix $\mathbf{K}(\theta) \in \mathbb{R}^{n\times n}$ is the same as that given in Eq. (1) and $\mathbf{F}(\theta) \in \mathbb{R}^n$ is a time-independent stochastic force vector. Similar to Eq. (2), we approximate the stochastic solution $\mathbf{u}(\theta)$ of Eq. (11) by

$$\mathbf{u}(\theta) = \sum_{i=1}^{k} \boldsymbol{\varphi}_i(\theta)\, q_i(\theta) = \boldsymbol{\Phi}(\theta)\,\mathbf{q}(\theta), \tag{12}$$

where the stochastic vectors $\boldsymbol{\varphi}_i(\theta) \in \mathbb{R}^n$, $i = 1, \cdots, k$ and the stochastic subspace $\boldsymbol{\Phi}(\theta) \in \mathbb{R}^{n\times k}$ has the same form given in Eq. (2) and $\mathbf{q}(\theta) = [q_1(\theta), \cdots, q_k(\theta)]^{\mathrm{T}} \in \mathbb{R}^k$ are unknown time-independent stochastic coefficients that need to be solved. To calculate the stochastic subspace $\boldsymbol{\Phi}(\theta)$, only a standard stochastic eigenvalue equation is required to solve in this case

$$\mathbf{K}(\theta)\,\boldsymbol{\varphi}(\theta) = \lambda(\theta)\,\boldsymbol{\varphi}(\theta), \tag{13}$$

whose solution algorithm is the same as that for Eq. (4) and will be discussed in Section 2.3. Substituting Eq. (12) into Eq. (11) and multiplying it by $\boldsymbol{\Phi}^{\mathrm{T}}(\theta)$ from left we have

$$\mathbf{k}(\theta)\,\mathbf{q}(\theta) = \mathbf{f}(\theta), \tag{14}$$

where the diagonal stochastic matrix $\mathbf{k}(\theta) \in \mathbb{R}^{k\times k}$ and the stochastic vector $\mathbf{f}(\theta) \in \mathbb{R}^{k}$ are given in similar ways as in Eq. (7)

$$\mathbf{k}(\theta) = \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\mathbf{\Phi}(\theta) = \begin{bmatrix} \lambda_1(\theta) & & 0 \\ & \ddots & \\ 0 & & \lambda_k(\theta) \end{bmatrix}, \quad \mathbf{f}(\theta) = \mathbf{\Phi}^{\mathrm{T}}(\theta)\,\mathbf{F}(\theta) = \begin{bmatrix} f_1(\theta) \\ \vdots \\ f_k(\theta) \end{bmatrix}. \tag{15}$$

Thus, Eq. (14) are also decoupled into k SDoF stochastic static equations

$$\lambda_i(\theta)\, q_i(\theta) = f_i(\theta) \tag{16}$$

for $i = 1, \cdots, k$, which are one-dimensional stochastic algebraic equations and will be solved by a non-intrusive method in Section 3.3.

2.3. An iterative algorithm for solving stochastic eigenvalue equations

As discussed above, the key issues of both the stochastic static and dynamic problems are to solve standard/generalized stochastic eigenvalue equations. In this section, we give an effective solution algorithm for this purpose. Here we only consider the generalized stochastic eigenvalue problem Eq. (4), which degenerates to the standard stochastic eigenvalue problem Eq. (13) by simply letting $\mathbf{M}(\theta)$ be the identity matrix $\mathbf{I}_n$. The authors have developed an efficient reduced-order method to solve stochastic eigenvalue problems in the reference[42]. In this section, we recall the idea and apply it to solve the stochastic static and dynamic equations. To solve Eq. (4), we approximate the i-th stochastic eigenvector $\boldsymbol{\varphi}_i(\theta)$ by

$$\boldsymbol{\varphi}_i(\theta) = \sum_{j=1}^{r} \eta_{ij}(\theta)\,\mathbf{d}_j = \mathbf{D}\boldsymbol{\eta}_i(\theta) \tag{17}$$

for $i = 1, 2, \cdots$, where $\mathbf{d}_j \in \mathbb{R}^n$ are deterministic vectors, $\mathbf{D} = [\mathbf{d}_1, \cdots, \mathbf{d}_r] \in \mathbb{R}^{n\times r}$ is a deterministic matrix, $\eta_{ij}(\theta) \in \mathbb{R}$ are scalar random variables, $\boldsymbol{\eta}_i(\theta) = [\eta_{i1}(\theta), \cdots, \eta_{ir}(\theta)]^{\mathrm{T}} \in \mathbb{R}^r$ represents the random variable vector, r is the number of approximate terms, which requires to meet $r \geq k$ in order to provide a high-accuracy approximation. In practical implementations, we simply let the number $r = k$.

Both the random variables $\left\{\eta_{ij}(\theta)\right\}_{j=1}^{r}$ and the deterministic vectors $\left\{\mathbf{d}_j\right\}_{j=1}^{r}$ are not known a priori. According to the reference[42], the vectors $\left\{\mathbf{d}_j\right\}_{j=1}^{r}$ are solved by the following alternating

iteration

$$\mathbb{E}\left\{\lambda_j(\theta)\,\mathbf{K}(\theta)\right\}\mathbf{d}_j = \mathbb{E}\left\{\lambda_j^2(\theta)\,\mathbf{M}(\theta)\right\}\mathbf{d}_j, \tag{18a}$$

$$\left[\mathbf{d}_j^{\mathrm{T}}\mathbf{M}(\theta)\,\mathbf{d}_j\right]\lambda_j(\theta) = \mathbf{d}_j^{\mathrm{T}}\mathbf{K}(\theta)\,\mathbf{d}_j, \tag{18b}$$

where $\mathbb{E}\{\cdot\}$ is the expectation operator. Eq. (18a) is obtained taking advantage of the stochastic Galerkin method [7, 8] for a known random variable $\lambda_j(\theta)$ (or given an initial value). The vector $\mathbf{d}_j$ can be solved efficiently by use of existing deterministic eigenvalue solvers [45]. For the modal decomposition analysis, we only need to solve the first few stochastic eigenvectors and the inverse power iteration is adopted in this paper. In numerical implementations, we make the vector $\mathbf{d}_j$ orthogonal to the obtained vectors $\{\mathbf{d}_i\}_{i=1}^{j-1}$, which achieves by Gram-Schmidt orthogonalization. With the known $\mathbf{d}_j$, Eq. (18b) is obtained by the classical Galerkin procedure. Note that it is a one-dimensional stochastic algebraic equation that can be solved efficiently in a non-intrusive sampling way using n_s random sample realizations. Similarly, the deterministic matrix $\mathbb{E}\left\{\lambda_j(\theta)\,\mathbf{K}(\theta)\right\} = 1/n_s\sum_{i=1}^{n_s}\lambda_j\left(\theta^{(i)}\right)\mathbf{K}\left(\theta^{(i)}\right) \in \mathbb{R}^{n\times n}$ can be estimated cheaply using n_s random sample realizations. We can obtain the couple $\left\{\lambda_j(\theta), \mathbf{d}_j\right\}$ by repeatedly solving Eq. (18a) and Eq. (18b) until convergence. The random variable vector $\boldsymbol{\eta}_i(\theta)$ is then solved based on the known matrix $\mathbf{D} = [\mathbf{d}_1, \cdots, \mathbf{d}_r]$. Substituting Eq. (17) into Eq. (4) and multiplying Eq. (4) by $\mathbf{D}^{\mathrm{T}}$ from left we have

$$\left[\mathbf{D}^{\mathrm{T}}\mathbf{K}(\theta)\,\mathbf{D}\right]\boldsymbol{\eta}_i(\theta) = \lambda_i(\theta)\left[\mathbf{D}^{\mathrm{T}}\mathbf{M}(\theta)\,\mathbf{D}\right]\boldsymbol{\eta}_i(\theta), \tag{19}$$

which is a reduced-order stochastic eigenvalue equation of size r. We adopt the sampling method to solve Eq. (19), which has low computational effort thanking to its small size. In practice, we only need to solve the first few stochastic eigenvectors $\{\boldsymbol{\eta}_i(\theta)\}_{i=1}^{r}$ of the reduced-order stochastic eigenproblem (19) to approximate the stochastic eigenvectors $\{\boldsymbol{\varphi}_i(\theta)\}_{i=1}^{k}$ of the original stochastic eigenproblem (4). It is noted that Eq. (18a), Eq. (18b) and Eq. (19) are weakly dependent on the stochastic dimension, thus the proposed method can be applied to very high-dimensional stochastic problems. Details of numerical implementations of the above iterative process can be found in the reference[42].

3. Solution algorithms based on stochastic modal decomposition

3.1. Stochastic modal decomposition-based stochastic static and dynamic analysis

Recalling the SDoF stochastic dynamic and static problems Eq. (9) and Eq. (16), we can calculate the stochastic eigenvalues $\{\lambda_i(\theta)\}_{i=1}^{k}$ via solving the reduced-order stochastic eigenproblem (19) and no additional computational effort is required. According to Eq. (17), the right-side terms of Eq. (9) and Eq. (16) are calculated by

$$h_i(t,\theta) = \frac{\boldsymbol{\eta}_i^{\mathrm{T}}(\theta)\,\mathbf{D}^{\mathrm{T}}\mathbf{F}(t,\theta)}{\boldsymbol{\eta}_i^{\mathrm{T}}(\theta)\,\mathbf{D}^{\mathrm{T}}\mathbf{M}(\theta)\,\mathbf{D}\boldsymbol{\eta}_i(\theta)}, \tag{20a}$$

$$f_i(\theta) = \boldsymbol{\eta}_i^{\mathrm{T}}(\theta)\,\mathbf{D}^{\mathrm{T}}\mathbf{F}(\theta). \tag{20b}$$

In this way, we reformulate the SDoF stochastic equations (9) and (16) based on the proposed stochastic eigenvalue algorithm. There is no reformulation to the left sides and only slight modifications to the right-side terms are required.

3.2. Solution algorithm for SDoF stochastic dynamic equations

In this section, we extend the deterministic Newmark method to solve the SDoF stochastic dynamic equation (9). According to the classical Newmark method [18, 46], the stochastic solution $q_i(t+\Delta t,\theta)$ at the time $t+\Delta t$ is solved based on the stochastic solution $q_i(t,\theta)$ at the time t and the time increment Δt, which corresponds to

$$s_i(\theta)\,q_i(t+\Delta t,\theta) = z_i(t+\Delta t,\theta), \tag{21}$$

where the random variables $s_i(\theta) = \alpha_1 + \alpha_2\varpi_i(\theta) + \lambda_i(\theta)$ and $z_i(t+\Delta t,\theta) = h_i(t+\Delta t,\theta) + \alpha_{9,k}(\theta)\,q_i(t,\theta) + \alpha_{10,k}(\theta)\,\dot{q}_i(t,\theta) + \alpha_{11,k}(\theta)\,\ddot{q}_i(t,\theta)$, the parameters $\alpha_1 = \frac{1}{\beta\Delta t^2}$, $\alpha_2 = \frac{\gamma}{\beta\Delta t}$, $\alpha_3 = \frac{1}{\beta\Delta t}$, $\alpha_4 = \frac{1}{2\beta} - 1$, $\alpha_5 = \frac{\gamma}{\beta} - 1$, $\alpha_6 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right)$, $\alpha_7 = \Delta t(1-\gamma)$, $\alpha_8 = \gamma\Delta t$ are inherited from the classical Newmark method. They are fixed for the given time discretization and the chosen parameters γ, β. Also, the parameters $\alpha_{9,i}(\theta)$, $\alpha_{10,i}(\theta)$ and $\alpha_{11,i}(\theta)$ are random variables given by $\alpha_{9,i}(\theta) = \alpha_1 + \alpha_2\varpi_i(\theta)$, $\alpha_{10,i}(\theta) = \alpha_3 + \alpha_5\varpi_i(\theta)$ and $\alpha_{11,i}(\theta) = \alpha_4 + \alpha_6\varpi_i(\theta)$. We solve Eq. (21) using a non-intrusive approach

$$\widehat{q_i}\left(t+\Delta t,\widehat{\boldsymbol{\theta}}\right) = \widehat{z_i}\left(t+\Delta t,\widehat{\boldsymbol{\theta}}\right) \oslash \widehat{s_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}, \tag{22}$$

where $\oslash$ is the Hadamard division operator representing the element-wise division of two vectors. It is numerically stable since $s_i(\theta) > 0$ holds for all $\theta \in \Theta$. The random variable $s_i(\theta)$ is fixed for different time t, but the random variable $z_i(t + \Delta t, \theta)$ needs to be updated for each time step. To solve the stochastic solution of the next time step, the first and second derivatives $\ddot{q}_i(t + \Delta t, \theta)$ and $\dot{q}_i(t + \Delta t, \theta)$ are calculated based on $\widehat{q_i}\left(t + \Delta t, \widehat{\boldsymbol{\theta}}\right)$ in a similar way to the classical Newmark method.

3.3. Solution algorithm for SDoF stochastic static equations

In this section, we focus on solving the SDoF stochastic static equation (16). Similar to Eq. (22), it is easily solved by using the proposed non-intrusive approach

$$\widehat{q_i}\left(\widehat{\boldsymbol{\theta}}\right) = \widehat{f_i}\left(\widehat{\boldsymbol{\theta}}\right) \oslash \widehat{\lambda_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}, \tag{23}$$

where the the random sample vectors are given by $\widehat{f_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ and $\widehat{\lambda_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$. We can thus recover the stochastic solution $\mathbf{u}(\theta)$ in Eq. (12) as

$$\mathbf{u}(\theta) = \mathbf{D}\boldsymbol{\Pi}(\theta)\,\mathbf{q}(\theta) = \mathbf{D}\boldsymbol{\Psi}(\theta), \tag{24}$$

where $\boldsymbol{\Pi}(\theta) = [\boldsymbol{\eta}_1(\theta), \cdots, \boldsymbol{\eta}_k(\theta)] \in \mathbb{R}^{r\times k}$ is a random variable matrix and a new random variable vector is given by $\boldsymbol{\Psi}(\theta) = \boldsymbol{\Pi}(\theta)\,\mathbf{q}(\theta) \in \mathbb{R}^r$. In this way, we decouple the stochastic solution $\mathbf{u}(\theta)$ into stochastic and deterministic spaces and all the randomness is embedded into the random variable vector $\boldsymbol{\Psi}(\theta)$, which is more convenient for uncertainty quantification. Also, we can use a similar way to recover the stochastic solution of the stochastic dynamic equation

$$\mathbf{u}(t, \theta) = \mathbf{D}\boldsymbol{\Pi}(\theta)\,\mathbf{q}(t, \theta) = \mathbf{D}\boldsymbol{\Psi}(t, \theta), \tag{25}$$

where $\boldsymbol{\Psi}(t, \theta) = \boldsymbol{\Pi}(\theta)\,\mathbf{q}(t, \theta)$ is a time-dependent random variable vector.

Remark 1. *Inspired by the decoupled representation* Eq. (24) *and* Eq. (25)*, we can also transform the stochastic dynamic equation (1) into a reduced-order stochastic dynamic equation*

$$\left[\mathbf{D}^{\mathrm{T}}\mathbf{M}(\theta)\,\mathbf{D}\right]\ddot{\mathbf{q}}(t, \theta) + \left[\mathbf{D}^{\mathrm{T}}\mathbf{C}(\theta)\,\mathbf{D}\right]\dot{\mathbf{q}}(t, \theta) + \left[\mathbf{D}^{\mathrm{T}}\mathbf{K}(\theta)\,\mathbf{D}\right]\mathbf{q}(t, \theta) = \mathbf{D}^{\mathrm{T}}\mathbf{F}(t, \theta) \tag{26}$$

by introducing $\mathbf{u}(t,\theta) = \mathbf{D}\mathbf{q}(t,\theta)$*, and transform the stochastic static equation (11) into a reduced-order stochastic static equation*

$$\left[\mathbf{D}^{\mathrm{T}}\mathbf{K}(\theta)\mathbf{D}\right]\mathbf{q}(\theta) = \mathbf{D}^{\mathrm{T}}\mathbf{F}(\theta) \tag{27}$$

by introducing $\mathbf{u}(\theta) = \mathbf{D}\mathbf{q}(\theta)$*, where the matrix* $\mathbf{D}$ *is given in* Eq. (17)*,* $\mathbf{q}(\theta)$ *and* $\mathbf{q}(t,\theta)$ *are the unknown stochastic coefficients that need to be solved. We can solve both* Eq. (26) *and* Eq. (27) *by sample-based methods, i.e., solving them for each sample realization* $\theta^{(i)}$*,* $i = 1, \cdots, n_s$*, which is still cheap since the sizes of the reduced-order equations are small in most cases. We remark that similar reduced-order equations have been studied in the references[47, 48], but in which the deterministic matrix* $\mathbf{D}$ *is obtained by other iterative algorithms and they are considered as a kind of deterministic reduced-order method. As a comparison, the proposed methods in this paper are kinds of stochastic reduced-order methods. In this paper, by taking advantage of the orthogonality, only SDoF stochastic dynamic/static equations require to be solved, while one has to solve k degrees-of-freedom systems of stochastic linear dynamic/static equations for* Eq. (26) *and* Eq. (27)*, and the size k has a significant influence on the computational effort.*

4. Algorithm implementation

In this section, we give the details of algorithm implementations of the proposed methods. The method for solving stochastic dynamic equations is summarized in Algorithm 1. To generate the stochastic subspace, stochastic eigenvalues and stochastic eigenvectors are solved in step 1 via the stochastic eigenvalue Algorithm 3. After that, the SDoF stochastic dynamic equation (9) is solved by two loops, where the outer loop that is from step 2 to step 8 is used to solve the stochastic solution $q_i(t,\theta)$ corresponding to the i-th stochastic mode, and the inner loop that is from step 3 to step 8 is used to solve the stochastic solution $q_i\left(t_j,\theta\right)$ of the time step t_j. For the inner loop, the random sample vectors $\widehat{s_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ is precomputed once, but $\widehat{z_i}\left(t_j,\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ is calculated for each time step based on the results of previous time step. Also, the first and second derivatives are required to be calculated in step 8. After the two loops, the stochastic solution $\mathbf{u}(t,\theta)$ is reconstructed in step 9.

Algorithm 1 Stochastic modal decomposition for solving stochastic dynamic equations

1: Solve stochastic eigenvalues and eigenvectors via the stochastic eigenvalue Algorithm 3
2: **for** $i = 1, \cdots, k$ **do**
3: Calculate $\widehat{s_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$
4: **for** $j = 1, \cdots, n_t$ **do**
5: Calculate the i-th right-side term $\widehat{h_i}\left(t_j, \widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ by Eq. (20a)
6: Calculate $\widehat{z_i}\left(t_j, \widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$
7: Solve the stochastic solution $\widehat{q_i}\left(t_j, \widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ at the time step t_j by Eq. (22)
8: Calculate the second and first derivatives $\widehat{\ddot{q}_i}\left(t_j, \widehat{\boldsymbol{\theta}}\right), \widehat{\dot{q}_i}\left(t_j, \widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ at the time step t_j
9: Recover the stochastic solution $\mathbf{u}(t, \theta)$ using Eq. (25)

Further, the proposed method for solving stochastic static equations is listed in Algorithm 2. The stochastic eigenvalue Algorithm 3 is still used in step 1 to generate the stochastic subspace. Only one loop that is from step 2 to step 4 is involved to solve the stochastic solution $q_i(\theta)$ corresponding to the i-th stochastic mode. Compared to the stochastic dynamic case, both the random sample vector $\widehat{\lambda_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ and $\widehat{f_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ are calculated once. Thus the computational effort for solving the stochastic solution $\widehat{q_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ in step 4 is very low. Finally the stochastic solution $\mathbf{u}(t, \theta)$ is reconstructed in step 5. Furthermore, we highlight both Algorithms 1 and 2 have good parallelizability since Eq. (9) and Eq. (16) can be solved in parallel for each stochastic mode $i = 1, \cdots, k$. Their solution processes are completely independent.

Algorithm 2 Stochastic modal decomposition for solving stochastic static equations

1: Solve stochastic eigenvalues and eigenvectors via the stochastic eigenvalue Algorithm 3
2: **for** $i = 1, \cdots, k$ **do**
3: Calculate the i-th right-side term $\widehat{f_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ by Eq. (20b)
4: Solve the stochastic solution $\widehat{q_i}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ by Eq. (23)
5: Recover the stochastic solution $\mathbf{u}(\theta)$ using Eq. (24)

Both Algorithms 1 and 2 require generating the stochastic subspace using the stochastic eigenvalue algorithm, which has been studied in detail in the reference[42] and only a simplified version

of which is adopted in this paper. To clearly implement the proposed methods, we review its implementation in Algorithm 3. The stochastic stiffness matrices are assembled in step 1. It is noted that only a standard stochastic eigenvalue equation is solved for stochastic static problems and it is achieved by letting the stochastic mass matrix $\mathbf{M}(\theta) = \mathbf{I}_n$. There are two loops involved, where the outer loop that is from step 2 to step 9 is used to solve all couples $\{\lambda_i(\theta), \mathbf{d}_i\}_{i=1}^{j}$ and the inner loop that is from step 4 to step 7 is used to calculate each couple $\{\lambda_i(\theta), \mathbf{d}_i\}$. The stopping criteria of the inner and outer loops are given by $\epsilon_{\mathbf{d},m} = \left\|\mathbf{d}_j^{(m)} - \mathbf{d}_j^{(m-1)}\right\|$ and $\epsilon_{\mathrm{ev},j} = \mathbb{E}\left\{\lambda_1^2(\theta)\right\} \Big/ \sum_{i=1}^{j} \mathbb{E}\left\{\lambda_i^2(\theta)\right\}$, respectively.

Algorithm 3 Iterative algorithm for solving stochastic eigenvalue equations

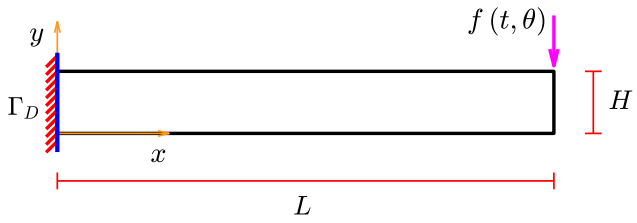
1: Assemble stochastic matrices $\mathbf{K}(\theta)$ and $\mathbf{M}(\theta)$ (= $\mathbf{I}_n$ if *Stochastic static problems*)
2: **while** $\epsilon_{\mathrm{ev},j} > \epsilon_{\mathrm{ev}}$ **do**
3: Initialize the random sample vector $\widehat{\lambda}_j^{(0)}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$
4: **while** $\epsilon_{\mathbf{d},m} > \epsilon_{\mathbf{d}}$ **do**
5: Solve the deterministic vector $\mathbf{d}_j^{(m)}$ by the inverse power iteration
6: Solve the random variable $\widehat{\lambda}_j^{(m)}\left(\widehat{\boldsymbol{\theta}}\right) \in \mathbb{R}^{n_s}$ by Eq. (18b)
7: Calculate the iterative error $\epsilon_{\mathbf{d},m}$, $m \leftarrow m + 1$
8: Update the matrix $\mathbf{D} = \left[\mathbf{D}, \mathbf{d}_j\right] \in \mathbb{R}^{n \times j}$
9: Calculate the iterative error $\epsilon_{\mathrm{ev},j}$, $j \leftarrow j + 1$
10: Calculate reduced-order eigenpairs $\{\lambda_i(\theta), \boldsymbol{\eta}_i(\theta)\}_{i=1}^{k}$ by Eq. (19)
11: Calculate the i-th original stochastic eigenvector $\boldsymbol{\varphi}_i(\theta) = \mathbf{D}\boldsymbol{\eta}_i(\theta)$ via Eq. (17)

5. Numerical examples

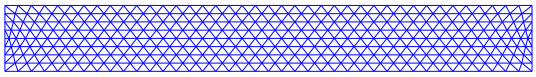
In this section, we test the proposed stochastic modal decomposition (SMD) methods with the aid of two numerical examples. For both examples, the stopping criterion $\epsilon_{\mathbf{d},m}$ of the inner loop of Algorithm 3 is set as 1×10^{-3}, and $\epsilon_{\mathrm{ev},j}$ of the outer loop of Algorithm 3 is set as 1×10^{-6}. The parameters in Rayleigh damping are simply set as $\varsigma_{\mathrm{M}}(\theta) = 10$ rad/s and $\varsigma_{\mathrm{K}}(\theta) = 0$ s/rad, which is just a simple setting and the values can be changed to fit more realistic situations.

Parameters of the stochastic Newmark method are $\beta = 0.25$ and $\gamma = 0.5$. Reference solutions are obtained by directly solving Eq. (1) in the time domain using the standard MCS-based stochastic Newmark method. It has been verified that 1×10^4 MCS can reach converged reference solutions. Furthermore, to eliminate the influence caused by sampling processes, the same 1×10^4 random sample realizations are used to the proposed SMD methods too. All examples are performed on a desktop computer (sixteen cores, Intel Core i7, 2.50GHz), but only one core is used for the numerical implementation.

5.1. Example 1: a two-dimensional beam



(a) Beam model.



(b) Finite element mesh of the beam.

Figure 1: Geometric model of the beam and its finite element mesh.

In this example, we consider a two-dimensional beam shown in Fig. 1a, which is subjected to a stochastic force $f(t,\theta)$, $t \in [0,T]$. Its finite element mesh is depicted in Fig. 1b and includes $n_p = 385$ nodes and $n_e = 672$ linear triangle elements. The stochastic solution is $u(x,y,\theta) = 0$ on the boundary Γ_D. Geometric and material parameters are given by length $L = 8$m, width $H = 1$m, duration $T = 2$s, mass density $2.75 \times 10^3 \mathrm{kg} \cdot \mathrm{m}^{-3}$, Poisson's ratio 0.3. The Young's modulus is modeled as a Gaussian random field with the mean value $E_0(x,y) = 60$GPa and the modified exponential-type covariance function [49, 50]

$$C_{EE}(x_1,y_1;x_2,y_2) = \sigma_E^2 \exp\left(-\frac{|x_1-x_2|}{l_x} - \frac{|y_1-y_2|}{l_y}\right)\left(1+\frac{|x_1-x_2|}{l_x}\right)\left(1+\frac{|y_1-y_2|}{l_y}\right), \tag{28}$$

where the standard deviation $\sigma_E = 0.1E_0(x, y)$ and the correlation lengths $l_x = 8\text{m}$ and $l_y = 1\text{m}$. By use of the Karhunen–Loève (KL) expansion [7, 51, 52], the random field $E(x, y, \theta)$ is approximated by the following series expansion

$$E(x, y, \theta) = E_0(x, y) + \sum_{i=1}^{r_E} \xi_i(\theta) \sqrt{\kappa_i} E_i(x, y), \tag{29}$$

where r_E is the number of truncated terms, $\{\xi_i(\theta)\}_{i=1}^{r_E}$ are independent standard Gaussian random variables and $\{\kappa_i, E_i(x, y)\}$ are the eigenvalues and eigenvectors of the covariance function solved by the following homogeneous Fredholm integral equation of the second kind

$$\int_{\mathcal{D}} C_{EE}(x_1, y_1; x_2, y_2) E_i(x_1, y_1) \mathrm{d}x_1 \mathrm{d}y_1 = \kappa_i E_i(x_2, y_2). \tag{30}$$

In practical implementation, to ensure that the Young's modulus is positive, the random samples $\theta^{(i)}$ such that $\min_{x,y,z\in\mathcal{D}} E\left(x, y, z, \theta^{(i)}\right) \leq 1 \times 10^{-3}\text{GPa}$ are dropped out.

Further, the force $f(t, \theta)$ is also considered as a Gaussian random process with the covariance function $C_{ff} = \sigma_f^2 \exp(|t_1 - t_2|/l_t)$ and its mean function $f_0(t)$ (kN) is

$$f_0(t) = 100 - 100\exp(-3t)(1 - \sin(10\pi t)). \tag{31}$$

The stochastic force is thus approximated by KL expansion

$$f(t, \theta) = f_0(t) + \sum_{i=1}^{r_f} \eta_i(\theta) \sqrt{\chi_i} f_i(t), \tag{32}$$

where r_f is the number of the truncation and the eigenvalues and eigenvectors $\{\chi_i, f_i(t)\}$ are solved by an Eq. (30)-like integral equation.

5.1.1. Case 1: dynamic analysis

This case considers the stochastic dynamic analysis of the given model. The time step $\Delta t = 0.01\text{s}$ is adopted for the time discretization and the number of total time steps is 201. For both this case and the static analysis in the next section, the truncated numbers r_E in Eq. (29) and r_f in Eq. (32) are set as 10 and 20 stochastic dimensions are thus involved in total. The iterative errors $\epsilon_{\text{ev},j}$ in Algorithm 3 corresponding to different retained terms $\left\{\mathbf{d}_j\right\}$ are shown in Fig. 2. Only 6 terms are retained to achieve the specified precision, which indicates the good convergence rate of

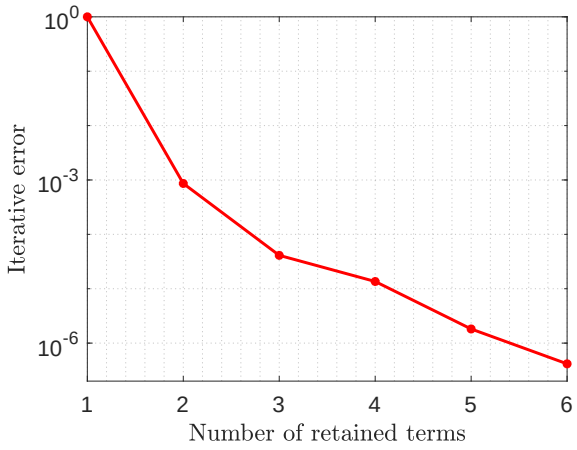


Figure 2: Iterative errors of different numbers of retained terms.

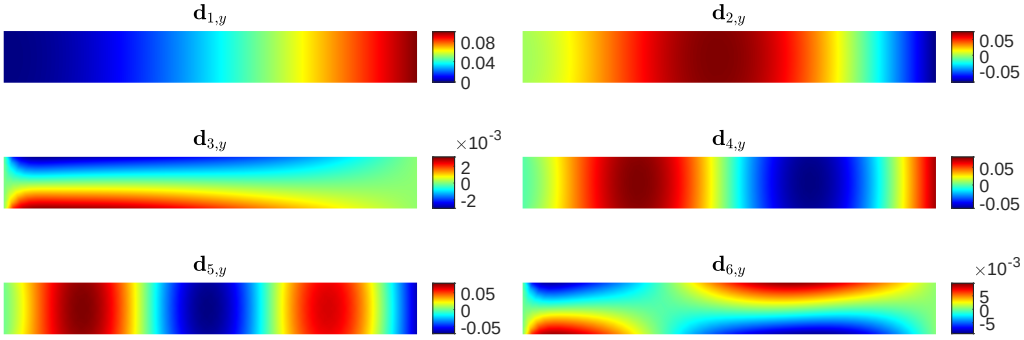


Figure 3: Deterministic vectors $\left\{\mathbf{d}_{j,y}\right\}_{j=1}^{6}$ in the y direction.

the proposed method. It is noted that since the stochastic eigenvalues $\left\{\lambda_j(\theta)\right\}_{j=1}^{6}$ are recomputed by Eq. (19) and those used to calculate the iterative error $\epsilon_{\text{ev},j}$ are not the final solutions, $\epsilon_{\text{ev},j}$ is only considered as an indicator error. In our experience, it is a good checker for a high-accuracy

Table 1: Computational times for solving the components in Eq. (25) and Eq. (26).

Time for	$\mathbf{D}$	$\mathbf{\Pi}(\theta)$	$\mathbf{q}(t,\theta)$	Total time (s)
SMD	1.47	1.24	5.73	8.44
ROM	1.47	–	7.12	8.59
MCS	–	–	–	1.03×10^4

approximation of the stochastic solution. Deterministic vectors $\left\{\mathbf{d}_{j,y}\right\}_{j=1}^{6}$ in the y direction are seen from Fig. 3. Similar to the deterministic case, more high-order modes are captured as the number of retained terms increases, which provides high-accuracy deterministic bases to approximate the stochastic solution. To verify the computational accuracy of Algorithm 3, we compare the probability density functions (PDFs) of the stochastic eigenvalues solved by Eq. (19) and MCS in Fig. 4, which demonstrates that all stochastic eigenvalues obtained by Algorithm 3 are well matched with that by MCS.

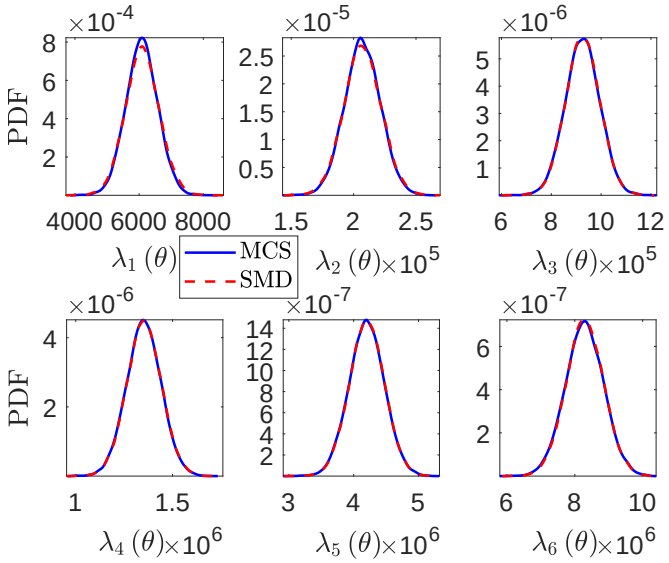


Figure 4: PDFs of the stochastic eigenvalues $\{\lambda_i(\theta)\}_{i=1}^{6}$.

Based on the stochastic eigenvalues and the stochastic eigenvectors obtained by Algorithm 3, we solve the original stochastic dynamic equation using Algorithm 1. To test the computational accuracy of the proposed method, PDFs of the stochastic displacements of the point $(x, y) = (8, 0.5)$ at the time $t = 1$s in the y direction obtained by the proposed SMD and MCS are compared in Fig. 5, which shows that the proposed SMD has a good agreement with MCS. In Fig. 5, we also provide the PDF obtained by the reduced-order method (ROM) given in Remark 1, which also achieves a good agreement with MCS. Hence, combining Algorithm 3 and the ROM in Remark 1 is also suggested to be a good choice to solve stochastic dynamic problems. To show the computational efficiency, computational times of all three methods are listed in Table 1. The times for calculating the deterministic matrix $\mathbf{D} \in \mathbb{R}^{770\times6}$, the stochastic matrix $\mathbf{\Pi}(\theta) \in \mathbb{R}^{6\times6}$ and the

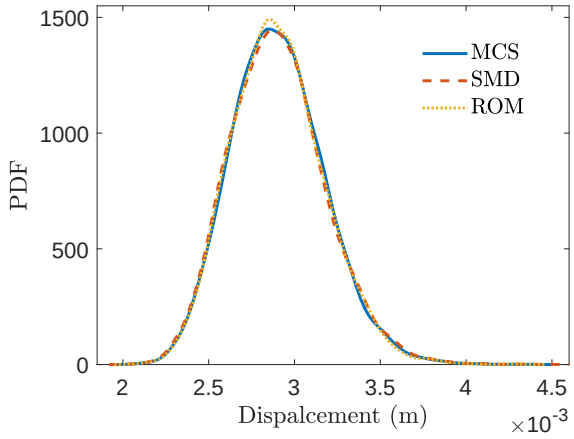


Figure 5: PDFs of the stochastic displacements $u(x, y, t, \theta)$ of the point $(x, y) = (8, 0.5)$ at the time $t = 1$s in the y direction obtained by MCS, the proposed SMD and the ROM given in Remark 1, respectively.

time-dependent stochastic vector $\mathbf{q}(t, \theta) \in \mathbb{R}^{6\times 201}$ represent the computational costs of step 2 to step 9 of Algorithm 3, step 10 of Algorithm 3 and step 2 to step 8 of Algorithm 1, respectively. It is seen that both SMD and ROM are much more efficient than MCS and a thousandfold speedup is achieved. The ROM does not involve the calculation of the stochastic matrix $\mathbf{\Pi}(\theta)$ since the deterministic matrix $\mathbf{D}$ is used to construct the reduced-order dynamic equation directly. SMD is a bit faster than ROM in solving the time-dependent stochastic vector $\mathbf{q}(t, \theta)$ since only SDoF stochastic dynamic equations are solved in SMD, while small systems of linear stochastic dynamic equations require to be solved in ROM. As the number of retained terms $\{\mathbf{d}_i\}$ and/or the number of total time steps increase, the computational effort for solving $\mathbf{q}(t, \theta)$ by ROM is slightly higher than that of SMD.

We highlight that the proposed method has very high accuracy even for each sample realization $\theta^{(i)}$. To show this point, five sample realizations of the stochastic displacements $u(x, y, t, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction obtained by the proposed SMD and MCS are compared in Fig. 6. For each sample realization and each time step, the solution obtained by SMD well matches that by MCS. In this sense, the proposed SMD also provides a high-accuracy reduced-order model for the original stochastic dynamic equation. As discussed in Remark 1, the ROM can also achieve this purpose. We do not provide further numerical illustrations for this point here.

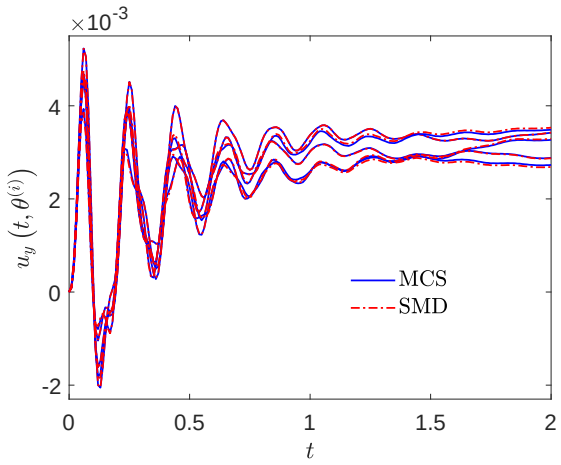


Figure 6: Five sample realizations of the stochastic displacements $u(x, y, t, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction obtained by MCS and the proposed SMD, respectively.

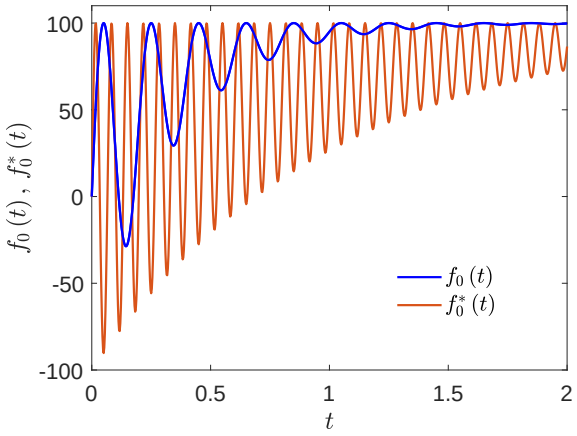


Figure 7: Comparison of the mean functions $f_0(t)$ in Eq. (31) and $f_0^*(t)$ in Eq. (33).

Further, it is seen from Fig. 6 that the stochastic displacements decay quickly. To demonstrate the promising performance of the proposed method, we consider a slowly decaying stochastic excitation by changing the mean function $f_0(t)$ of the stochastic force $f(t, \theta)$ in Eq. (32) to $f_0^*(t)$

$$f_0^*(t) = 100 - 100 \exp(-t)(1 - \sin(30\pi t)), \tag{33}$$

which decays much more slowly than $f_0(t)$, as compared in Fig. 7. It is noted that only the stochastic excitation is changed here, so the above stochastic eigenvectors can be reused. In this way, only the right-side term $h_i(t, \theta)$ in Eq. (9) is recalculated, which is computationally cheap.

Five sample realizations of the stochastic displacements $u(x, y, t, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction under the slowly decaying stochastic excitation obtained by MCS and the proposed SMD are shown in Fig. 8, which indicates that the proposed method is still very accurate for each sample realization $\theta^{(i)}$.

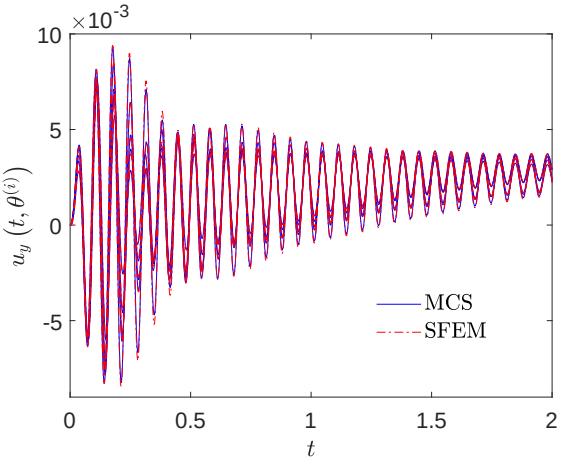


Figure 8: Five sample realizations of the stochastic displacements $u(x, y, t, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction under the slowly decaying stochastic excitation obtained by MCS and the proposed SMD, respectively.

5.1.2. Case 2: static analysis

In this case, we consider the stochastic static analysis of the problem and a stochastic vertical force given by $f(t = T, \theta)$ is applied to the same position. To generate the stochastic subspace, a standard stochastic eigenvalue equation is solved using Algorithm 3 and 6 terms $\left\{\mathbf{d}_j\right\}_{j=1}^{6}$ are retained to meet the specified precision. As shown in Fig. 9, the PDFs of the stochastic displacements $u(x, y, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction obtained by both the proposed SMD and ROM have comparable accuracy with that obtained by MCS, which indicates that the proposed two methods still work well for stochastic static problems. Computational times of the three methods for this case are listed in Table 2, which shows that both SMD and ROM are still much more efficient than MCS. SMD is very cheap due to only the element-wise division operation of sample vectors involved, which weakly depends on the sample size. While the computational cost for solving the reduced-order stochastic finite element equation arising in ROM linearly depends on the sample size. The total cost of ROM is a bit cheaper than that of SMD since there is no cost

spent on the solution of the stochastic matrix $\mathbf{\Pi}(\theta)$.

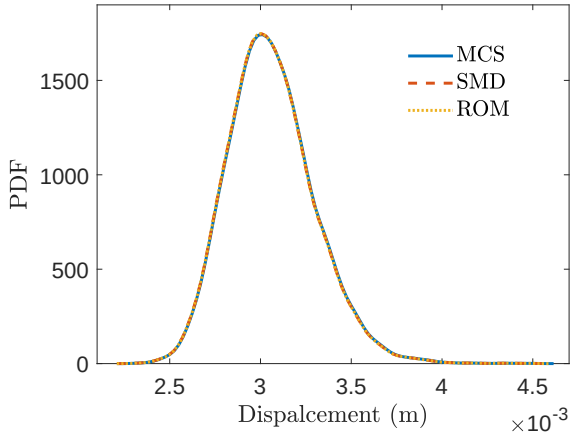


Figure 9: PDFs of the stochastic displacements $u(x, y, \theta)$ of the point $(x, y) = (8, 0.5)$ in the y direction obtained by MCS, the proposed SMD and the ROM given in Remark 1, respectively.

Table 2: Computational times for solving the components in Eq. (24) and Eq. (27).

Time for	$\mathbf{D}$	$\mathbf{\Pi}(\theta)$	$\mathbf{q}(\theta)$	Total time (s)
SMD	1.44	1.28	0.12	2.84
ROM	1.44	–	0.68	2.12
MCS	–	–	–	467.36

5.1.3. Case 3: high-dimensional dynamic analysis

In this case, we perform the dynamic analysis with high stochastic dimensions. The truncated numbers r_E in Eq. (29) and r_f in Eq. (32) are set as 50 and 100 stochastic dimensions are involved in total. Other parameters of the problem are the same as that given in Section 5.1.1. 10 vectors $\left\{\mathbf{d}_j\right\}_{j=1}^{10}$ are retained by a generalized high-dimensional stochastic eigenproblem. Computational times of the proposed SMD and MCS for this case are shown in Table 3. As a comparison, corresponding computational costs for the low-dimensional case studied in Section 5.1.1 are also listed in Table 3. It is seen that SMD is still much cheaper than MCS for high-dimensional stochastic

Table 3: Computational times for solving the components in Eq. (25) and Eq. (26) when the stochastic dimension is 100.

Time for	$\mathbf{D}$	$\mathbf{\Pi}(\theta)$	$\mathbf{q}(t,\theta)$	Total time (s)	MCS
$r = 20$	1.47	1.24	5.73	8.44	1.03×10^4
$r = 100$	1.81	1.39	7.66	10.86	1.14×10^4

cases. Compared to the low-dimensional case, the computational times for solving all three components $\mathbf{D}$, $\mathbf{\Pi}(\theta)$ and $\mathbf{q}(t,\theta)$ has only increased a little bit, which indicates that the proposed SMD is weakly dependent on the stochastic dimension. In this way, the curse of dimensionality arising in high-dimensional stochastic problems is avoided successfully. Further, to show the computational accuracy of high-dimensional stochastic cases, PDFs of the stochastic displacements $u(x,y,t,\theta)$ of the point $(x,y) = (8,0.5)$ at the time $t = 1$s in the y direction obtained by MCS and SMD are compared in Fig. 10, which demonstrates that the proposed SMD still has high accuracy for high-dimensional stochastic cases. It is noted that the PDFs in Fig. 10 are close to that in Fig. 5 since some last truncated terms in Eq. (29) and Eq. (32) have little contributions to the randomness. We can adopt some truncation criteria to choose the truncated numbers. In this paper, we just use high-dimensional truncation to test the proposed method. In the practical implementation,

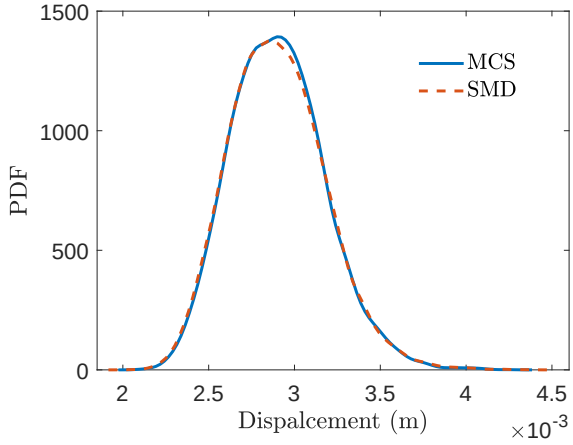
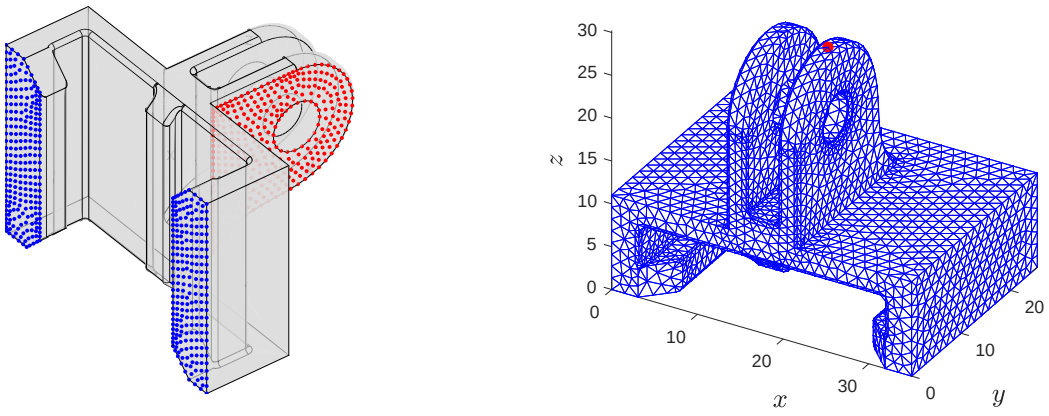


Figure 10: PDFs of the stochastic displacements $u(x,y,t,\theta)$ of the point $(x,y) = (8,0.5)$ at the time $t = 1$s in the y direction obtained by MCS and the proposed SMD when the stochastic dimension is 100.

the proposed SMD performs the same calculation for both low- and high-dimensional cases. If the high-dimensional truncation contributes a lot to the random fields, only more vectors $\mathbf{d}_j$ are retained to approximate the final stochastic solution.

5.2. Example 2: a three-dimensional mechanical part

This example considers a three-dimensional mechanical part shown in Fig. 11a, where the Dirichlet boundary condition $u(x, y, z, \theta) = 0$ holds on the blue surface. An external force P is applied to the red surface along the x direction (see Fig. 11b) and its value is $500\text{kN}/\text{m}^2$ for $t \in [0, t_0]$ and 0 for $t \in (t_0, 1]$, where the duration $t_0 = 0.1\text{s}$. The finite element mesh is depicted in Fig. 11b (unit: cm) and it includes $n_p = 6685$ nodes, $n_e = 26819$ linear tetrahedral elements and 20055 degrees of freedom in total.



(a) Model of the mechanical part.

(b) Finite element mesh of the mechanical part.

Figure 11: Geometric model of the mechanical part and its finite element mesh.

The Young's modulus is modeled as a Gaussian random field with the mean value $E_0(x, y) = 209\text{GPa}$ and the covariance function

$$C_{EE}(x_1, y_1, z_1; x_2, y_2, z_2) = \sigma_E^2 \exp\left(-\frac{|x_1 - x_2|}{l_x} - \frac{|y_1 - y_2|}{l_y} - \frac{|z_1 - z_2|}{l_z}\right), \tag{34}$$

where the standard derivation $\sigma_E = 0.1E_0(x, y)$, the correlation lengths are $l_{x_i} = \max x_i - \min x_i$, $x_i = x, y, z$. In the numerical investigation, we adopt Eq. (29)-like 10-term truncated KL expansion to approximate the Young's modulus and the random samples $\theta^{(i)}$ such that $\min_{x,y,z\in\mathcal{D}} E\left(x, y, z, \theta^{(i)}\right) \leq$

1×10^{-3}GPa are dropped out to ensure that all realizations are positive. Other material properties are given by mass density $7.80 \times 10^{3}\text{kg} \cdot \text{m}^{-3}$ and Poisson's ratio 0.3.

5.2.1. Case 1: dynamic analysis

In this case, the time step $\Delta t = 0.01$s is adopted for the time discretization and 100 time steps are thus obtained. The iterative errors $\epsilon_{\text{ev},j}$ of different retained terms $\left\{\mathbf{d}_j\right\}_j$ are shown in Fig. 12 and it requires 22 terms to meet the specified precision. Compared to the case in Example 5.1.1, more terms are retained due to a more complex geometry involved. The corresponding first six deterministic vectors $\left\{\mathbf{d}_{j,x}\right\}_{j=1}^{6}$ in the x direction are depicted in Fig. 13. It is seen that several local modes of the upper part are well captured.

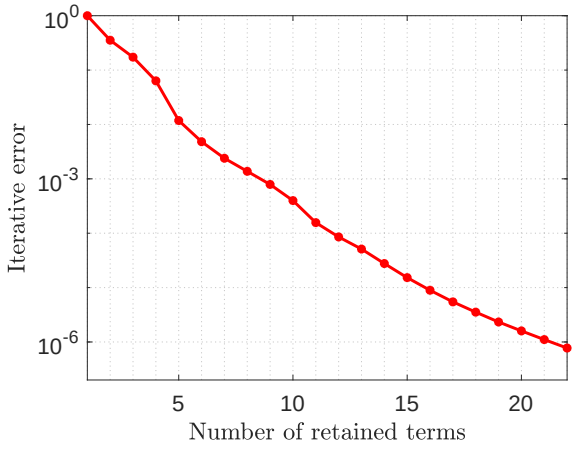


Figure 12: Iterative errors of different numbers of retained terms.

PDFs of the stochastic displacements of the red point (shown in Fig. 11b) at the time $t = 0.5$s in the x direction obtained by the proposed SMD and MCS are compared in Fig. 14, which again verifies the high accuracy of the proposed method. Since the external force disappears at the time $t = 0.1$s, for the time $t > 0.1$s, the problem can be considered as a free stochastic vibration with the stochastic initial displacement and velocity given at the time $t = 0.1$s. The stochastic displacement at the time $t = 0.5$s induced by the free vibration is related to the stochastic initial values and the stochastic stiffness matrix, and their values may be positive or negative. Also, five sample realizations of the stochastic displacement $u(x, y, z, t, \theta)$ of the red point in the x direction obtained by the proposed SMD and MCS are compared in Fig. 15. For each sample realization, the results

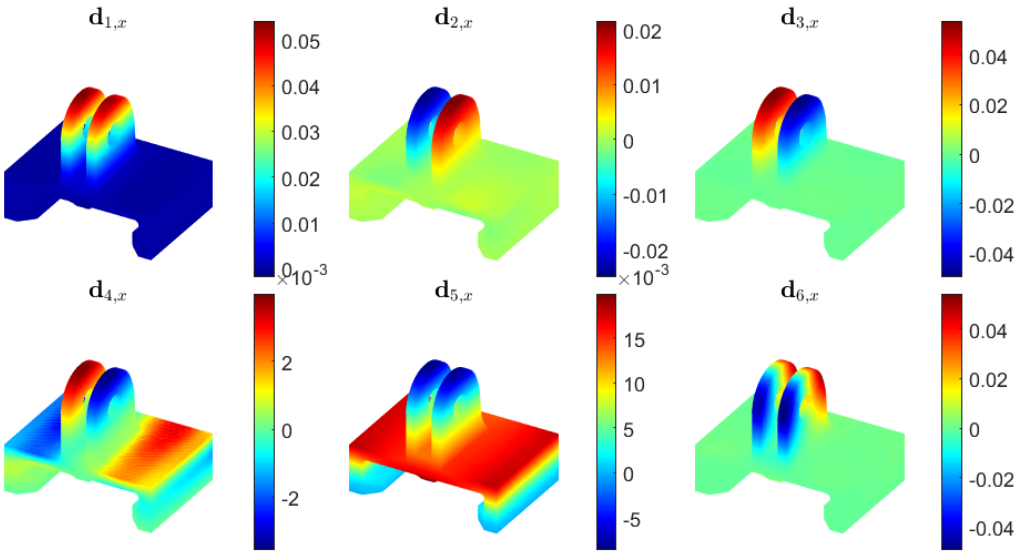


Figure 13: The first six deterministic vectors $\{\mathbf{d}_{i,x}\}_{i=1}^{6}$ in the x direction.

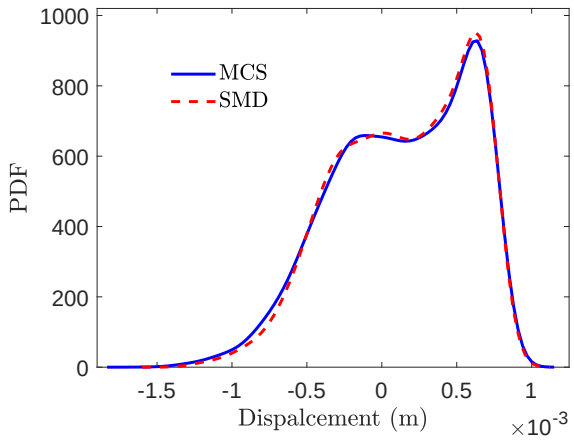


Figure 14: PDFs of the stochastic displacements $u(x, y, z, t, \theta)$ of the red point (shown in Fig. 11b) at the time $t = 0.5$s in the x direction obtained by MCS, the proposed SMD, respectively.

obtained by the two methods are in good accordance, thus using the proposed SMD as a stochastic reduced-order method is still accurate enough. Computational times for this case are listed in the first line of Table 4, which indicates that the total computational time of the proposed method is much less than that of MCS. Most computational effort is used for the deterministic matrix $\mathbf{D} \in \mathbb{R}^{20055\times 22}$ since solving large-scale deterministic eigenvalue equations is time-consuming. The computational time for solving the stochastic matrix $\mathbf{\Pi}(\theta) \in \mathbb{R}^{22\times 22}$ only depends on the

number of retained terms, thus it is cheap to be solved in both this example and the Example 5.1. The computational cost for the time-dependent stochastic vector $\mathbf{q}(t,\theta) \in \mathbb{R}^{22\times 101}$ is a bit more expensive in this case due to a larger number of the retained terms, but it is still cheap enough compared to the total computational time.

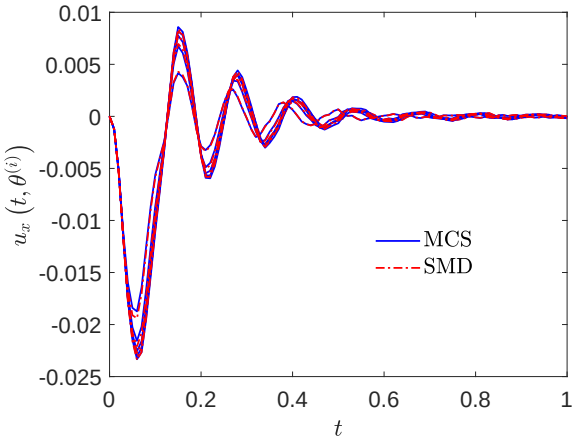


Figure 15: Five sample realizations obtained by MCS and the proposed SMD, respectively.

5.2.2. Case 2: static analysis

The stochastic static analysis of the problem is performed in this case. The external force P is applied to the model in the same way but its value is fixed as $500\text{kN}/\text{m}^2$. 20 terms $\left\{\mathbf{d}_j\right\}_{j=1}^{20}$ are obtained by solving the standard stochastic eigenvalue equation. PDFs of the stochastic displacement $u(x,y,z,\theta)$ of the red point (shown in Fig. 11b) in the x direction obtained by the proposed SMD and MCS are depicted in Fig. 16, which indicates that the proposed method still provides comparable accuracy to MCS. If a higher-accuracy stochastic solution is required in some cases, it can be achieved by increasing the number of retained terms $\left\{\mathbf{d}_j\right\}_j$. Computational times for this case are listed in the second line of Table 4. Similar to the stochastic dynamic case, the most computational cost is used for the deterministic matrix $\mathbf{D} \in \mathbb{R}^{20055\times 20}$. In this case, the computational cost for each component in Table 4 is less than that for the dynamic case, especially for solving the time-independent stochastic vector $\mathbf{q}(\theta) \in \mathbb{R}^{20}$, since only 20 time-independent stochastic algebraic equations need to be solved.

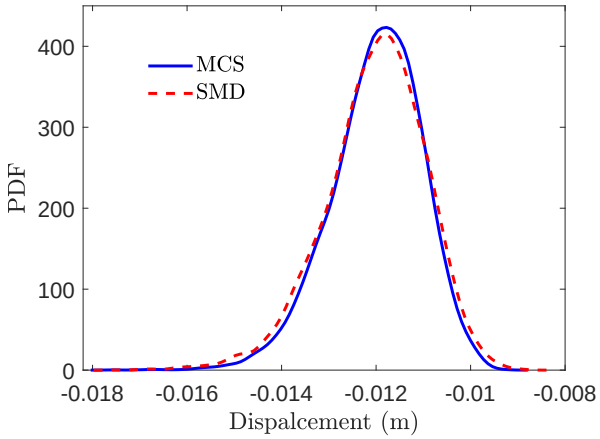


Figure 16: PDFs of the stochastic displacements $u(x, y, z, \theta)$ of the red point (shown in Fig. 11b) in the x direction obtained by MCS and the proposed SMD, respectively.

Table 4: Computational times for solving the components in Eq. (25) and Eq. (26).

Time for	$\mathbf{D}$	$\mathbf{\Pi}(\theta)$	$\mathbf{q}(t, \theta)$	Total time (s)	MCS
Dynamic case	489.29	3.14	13.25	505.68	3.89×10^4
Static case	417.81	2.90	0.21	420.92	7.02×10^3

6. Conclusions

In this paper, we presented novel stochastic modal decomposition-based numerical schemes for solving both stochastic static and dynamic problems. By an efficient reduced-order method, standard/generalized stochastic eigenvalue equations are first solved to calculate stochastic eigenvectors and generate stochastic subspaces. The stochastic solutions of both stochastic static and dynamic problems are then represented as stochastic linear combinations of bases of the stochastic subspaces. Original stochastic static/dynamic equations are thus transformed into a set of SDoF stochastic static/dynamic equations. The SDoF stochastic dynamic equations can be efficiently solved by a non-intrusive stochastic Newmark approach and the SDoF stochastic static equations are solved by an element-wise division operation of random sample vectors. The proposed methods can be applied to high-dimensional stochastic problems without any modification and much extra computational effort, which has been verified via a numerical example of up to a hundred

stochastic dimensions. In these senses, the proposed methods provide effective ways and novel perspectives for structural static and dynamic analysis involving uncertainties.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix

A1. Proof of Eq. (5)

Proof. Multiplying the stochastic eigenequation (4) corresponding to the stochastic mode $\boldsymbol{\varphi}_j(\theta)$ (and $\boldsymbol{\varphi}_i(\theta)$) by $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)$ (and $\boldsymbol{\varphi}_j^{\mathrm{T}}(\theta)$) from left we have

$$\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_j(\theta) = \lambda_j(\theta)\,\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta)\,, \tag{35}$$

$$\boldsymbol{\varphi}_j^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_i(\theta) = \lambda_i(\theta)\,\boldsymbol{\varphi}_j^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_i(\theta)\,. \tag{36}$$

Due to $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_j(\theta) \equiv \boldsymbol{\varphi}_j^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_i(\theta)$ and $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta) \equiv \boldsymbol{\varphi}_j^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_i(\theta)$, subtracting Eq. (36) from Eq. (35) we have

$$\big(\lambda_j(\theta) - \lambda_i(\theta)\big)\,\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta) = 0 \qquad \text{a.e.,} \tag{37}$$

thus $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta) = 0$ holds due to the stochastic eigenvalues $\lambda_j(\theta) \neq \lambda_i(\theta)$. Further, substituting $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{M}(\theta)\,\boldsymbol{\varphi}_j(\theta) = 0$ into Eq. (35) we have $\boldsymbol{\varphi}_i^{\mathrm{T}}(\theta)\,\mathbf{K}(\theta)\,\boldsymbol{\varphi}_j(\theta) = 0$. □

References

[1] M. Papadrakakis, V. Papadopoulos, Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation, Computer Methods in Applied Mechanics Engineering 134 (1996) 325–340.

[2] R. E. Caflisch, Monte carlo and quasi-monte carlo methods, Acta Numerica 7 (1998) 1–49.

[3] M. B. Giles, Multilevel Monte Carlo methods, Acta Numerica 24 (2015) 259–328.

[4] J. N. Fuhg, A. Fau, U. Nackenhorst, State-of-the-art and comparative review of adaptive sampling methods for kriging, Archives of Computational Methods in Engineering (2020) 1–59.

[5] A. I. Khuri, S. Mukhopadhyay, Response surface methodology, Wiley Interdisciplinary Reviews: Computational Statistics 2 (2010) 128–149.

[6] A. Doostan, A. Validi, G. Iaccarino, Non-intrusive low-rank separated approximation of high-dimensional stochastic models, Computer Methods in Applied Mechanics and Engineering 263 (2013) 42–55.

[7] R. G. Ghanem, P. D. Spanos, Stochastic finite elements: a spectral approach, Courier Corporation, 2003.

[8] D. Xiu, G. E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic dierential equations, SIAM Journal on Scientific Computing 24 (2002) 619–644.

[9] H. G. Matthies, A. Keese, Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations, Computer Methods in Applied Mechanics Engineering 194 (2005) 1295–1331.

[10] M. F. Pellissetti, R. G. Ghanem, Iterative solution of systems of linear equations arising in the context of stochastic finite elements, Advances in Engineering Software 31 (2000) 607–616.

[11] G. Blatman, B. Sudret, An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis, Probabilistic Engineering Mechanics 25 (2010) 183–197.

[12] M. Eldred, Recent advances in non-intrusive polynomial chaos and stochastic collocation methods for uncertainty analysis and design, 50th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference (2009) 2274.

[13] G.-Y. Lee, Y.-H. Park, A proper generalized decomposition-based harmonic balance method with arc-length continuation for nonlinear frequency response analysis, Computers & Structures 275 (2023) 106913.

[14] M. Chevreuil, A. Nouy, Model order reduction based on proper generalized decomposition for the propagation of uncertainties in structural dynamics, International Journal for Numerical Methods in Engineering 89 (2012) 241–268.

[15] L. Gao, C. Audouze, P. B. Nair, Anchored analysis of variance Petrov–Galerkin projection schemes for linear stochastic structural dynamics, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 471 (2015) 20150023.

[16] G. Chen, D. Yang, Direct probability integral method for stochastic response analysis of static and dynamic structural systems, Computer Methods in Applied Mechanics and Engineering 357 (2019) 112612.

[17] H. Huo, W. Xu, W. Wang, G. Chen, D. Yang, New non-intrusive stochastic finite element method for plate

structures, Computers & Structures 268 (2022) 106812.

[18] K.-J. Bathe, Finite element procedures, Klaus-Jurgen Bathe, 2006.

[19] T. J. Hughes, The finite element method: linear static and dynamic finite element analysis, Courier Corporation, 2012.

[20] H. Pradlwarter, G. Schuëller, G. Szekely, Random eigenvalue problems for large systems, Computers & Structures 80 (2002) 2415–2424.

[21] A. D. Gilbert, R. Scheichl, Multilevel quasi-Monte Carlo for random elliptic eigenvalue problems ii: Efficient algorithms and numerical results, arXiv preprint arXiv:2103.03407 (2021).

[22] S. Adhikari, M. Friswell, Random matrix eigenvalue problems in structural dynamics, International Journal for Numerical Methods in Engineering 69 (2007) 562–591.

[23] B. Pascual, S. Adhikari, Hybrid perturbation-Polynomial Chaos approaches to the random algebraic eigenvalue problem, Computer Methods in Applied Mechanics and Engineering 217 (2012) 153–167.

[24] P. Hao, H. Tang, Y. Wang, T. Wu, S. Feng, B. Wang, Stochastic isogeometric buckling analysis of composite shell considering multiple uncertainties, Reliability Engineering & System Safety 230 (2023) 108912.

[25] R. Ghanem, D. Ghosh, Efficient characterization of the random eigenvalue problem in a polynomial chaos decomposition, International Journal for Numerical Methods in Engineering 72 (2007) 486–504.

[26] E. Pagnacco, E. S. de Cursi, R. Sampaio, Subspace inverse power method and polynomial chaos representation for the modal frequency responses of random mechanical systems, Computational Mechanics 58 (2016) 129–149.

[27] K. Lee, B. Sousedik, Inexact methods for symmetric stochastic eigenvalue problems, SIAM/ASA Journal on Uncertainty Quantification 6 (2018) 1744–1776.

[28] D. Ghosh, R. Ghanem, An invariant subspace-based approach to the random eigenvalue problem of systems with clustered spectrum, International Journal for Numerical Methods in Engineering 91 (2012) 378–396.

[29] S. Adhikari, S. Chakraborty, Random matrix eigenvalue problems in structural dynamics: An iterative approach, Mechanical Systems and Signal Processing 164 (2022) 108260.

[30] J. Lan, X. Dong, Z. Peng, W. Zhang, G. Meng, Uncertain eigenvalue analysis by the sparse grid stochastic collocation method, Acta Mechanica Sinica 31 (2015) 545–557.

[31] S. Rahman, A solution of the random eigenvalue problem by a dimensional decomposition method, International Journal for Numerical Methods in Engineering 67 (2006) 1318–1340.

[32] S. Rahman, R. Jahanbin, A spline dimensional decomposition for uncertainty quantification in high dimensions, SIAM/ASA Journal on Uncertainty Quantification 10 (2022) 404–438.

[33] B. Huang, H. Zhang, K.-K. Phoon, Homotopy approach for random eigenvalue problem, International Journal for Numerical Methods in Engineering 113 (2018) 450–478.

[34] P. Benner, A. Onwunta, M. Stoll, A low-rank inexact Newton–Krylov method for stochastic eigenvalue prob-

lems, Computational Methods in Applied Mathematics 19 (2019) 5–22.

[35] H. C. Elman, T. Su, Low-rank solution methods for stochastic eigenvalue problems, SIAM Journal on Scientific Computing 41 (2019) A2657–A2680.

[36] K. Sepahvand, S. Marburg, H.-J. Hardtke, Stochastic structural modal analysis involving uncertain parameters using generalized polynomial chaos expansion, International Journal of Applied Mechanics 3 (2011) 587–606.

[37] D. Ghosh, Application of the random eigenvalue problem in forced response analysis of a linear stochastic structure, Archive of Applied Mechanics 83 (2013) 1341–1357.

[38] S. Sarkar, D. Ghosh, A hybrid method for stochastic response analysis of a vibrating structure, Archive of Applied Mechanics 85 (2015) 1607–1626.

[39] J. Yang, B. Faverjon, H. Peters, S. Marburg, N. Kessissoglou, Deterministic and stochastic model order reduction for vibration analyses of structures with uncertainties, Journal of Vibration and Acoustics 139 (2017).

[40] S. Kasinos, A. Palmeri, M. Lombardo, S. Adhikari, A reduced modal subspace approach for damped stochastic dynamic systems, Computers & Structures 257 (2021) 106651.

[41] S. Rahman, R. Jahanbin, Orthogonal spline expansions for uncertainty quantification in linear dynamical systems, Journal of Sound and Vibration 512 (2021) 116366.

[42] Z. Zheng, M. Beer, U. Nackenhorst, An efficient reduced-order method for stochastic eigenvalue analysis, International Journal for Numerical Methods in Engineering 123 (2022) 1–23.

[43] K.-J. Bathe, S. Ramaswamy, An accelerated subspace iteration method, Computer Methods in Applied Mechanics and Engineering 23 (1980) 313–331.

[44] J. C. Butcher, Numerical methods for ordinary differential equations, John Wiley & Sons, 2016.

[45] Y. Saad, Numerical methods for large eigenvalue problems, SIAM, 2011.

[46] M. Paz, Structural dynamics: theory and computation, Springer Science & Business Media, 2012.

[47] Z. Zheng, M. Beer, H. Dai, U. Nackenhorst, A weak-intrusive stochastic finite element method for stochastic structural dynamics analysis, Computer Methods in Applied Mechanics and Engineering 399 (2022) 115360.

[48] Z. Zheng, H. Dai, Structural stochastic responses determination via a sample-based stochastic finite element method, Computer Methods in Applied Mechanics and Engineering 381 (2021) 113824.

[49] P. D. Spanos, M. Beer, J. Red-Horse, Karhunen–Loève expansion of stochastic processes with a modified exponential covariance kernel, Journal of Engineering Mechanics 133 (2007) 773–779.

[50] M. G. Faes, M. Broggi, P. D. Spanos, M. Beer, Elucidating appealing features of differentiable auto-correlation functions: A study on the modified exponential kernel, Probabilistic Engineering Mechanics (2022) 103269.

[51] Z. Zheng, H. Dai, Simulation of multi-dimensional random fields by Karhunen–Loève expansion, Computer Methods in Applied Mechanics and Engineering 324 (2017) 221–247.

[52] Z. Zheng, H. Dai, Y. Wang, W. Wang, A sample-based iterative scheme for simulating non-stationary non-Gaussian stochastic processes, Mechanical Systems and Signal Processing 151 (2021) 107420.