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Two loop $\overline{\text{MS}}$ gluon pole mass from the LCO formalism

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Abstract. We compute the pole mass of the gluon in QCD from the local composite operator formalism at two loops in the $\overline{\text{MS}}$ renormalization scheme. For Yang-Mills theory an estimate of the mass at two loops is $2.13\Lambda_{\overline{\text{MS}}}$.

1 Introduction.

There has been recent interest in understanding the role the dimension two gauge invariant gluon mass operator plays in the vacuum structure of Yang-Mills theory and QCD. For instance, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. As was pointed out in [14, 15, 16, 17] the perturbative vacuum one ordinarily uses for high energy computations is not stable and it has been suggested that in the true vacuum, various operators condense developing non-zero vacuum expectation values. Whilst the operator product expansion and QCD sum rules are usually centred on the gauge invariant operators $\left(G^a_{\mu\nu}\right)^2$ and $\bar{\psi}\psi$, where $G^a_{\mu\nu}$ is the gluon field strength and ψ is the quark field, recent work considered the lower dimension operator $\frac{1}{2}A^{a\,2}_\mu$ in the Landau gauge, [3, 8, 9, 10, 11, 12, 13], and various generalizations of it. For instance, there one can construct a non-local gauge invariant dimension two operator which truncates to $\frac{1}{2}A^{a\,2}_\mu$ in the Landau gauge. An example of the role such an operator can play in the infrared structure has been examined in [18] where a massive gauge invariant QCD Lagrangian was studied in connection with vortex solutions. As $\frac{1}{2}A^{a\,2}_\mu$ has the dimensions of a mass operator, it has been the subject of investigating the issue of whether the gluon can develop a mass dynamically. Indeed in the early work of Curci and Ferrari, [19], an extension of this operator, which was on-shell BRST invariant, was included in the usual QCD Lagrangian. More recently another approach, known as the local composite operator (LCO) method, has been developed which avoids the ad hoc inclusion of a gluon mass term [8, 9, 10]. Instead the QCD Lagrangian is modified to introduce an extra scalar field, σ, coupled to $\frac{1}{2}A^{a\,2}_\mu$ and with this one can compute the effective potential of the scalar field. It transpires that due to the development of a non-zero vacuum expectation value for σ the gluon gains a non-zero mass in a vacuum which does not correspond to the (unstable) perturbative one. By contrast, applying the same formalism to QED, [10], the perturbative vaccum is stable and whilst there is another extremum where σ has a non-zero vacuum expectation value, it corresponds to an unstable point.

Within this formalism one can estimate the size of an effective gluon mass at one and two loops. In Yang-Mills theory it is of the order of $2\Lambda_{\overline{\mathrm{MS}}}$ and is stable to the higher order corrections. Whilst this is roughly consistent with other estimates of a gluon mass from a wide range of methods (which are succinctly summarized in Table 15 of Field's article, [20]) the LCO estimates suffer several shortcomings. One of these is that the effective gluon mass used in [8] was the tree object and whilst the effective potential does have the quantum corrections no account of the dressings of the tree quantity were included. Further, whilst all the gluon estimates are of a similar range, it is not clear to what extent the same mass quantity is being measured. For instance, in the quark sector of QCD the quark masses are all measured and compared to the same benchmark, which is the running mass at the scale 2GeV. This is irrespective of whether the pole mass of the quark was determined or, say, the running mass at another scale prior to using the (four loop) quark mass anomalous dimension to run the mass to the standard reference scale. For the same problem for a gluon mass, the anomalous dimension of the $\frac{1}{2}A^{a\,2}_\mu$ gluon mass operator in the Landau gauge is now available at four loops, [21], extending the two, [22], and three loop, [23], results which are all in the $\overline{\mathrm{MS}}$ scheme. Remarkably the operator anomalous dimension is the sum of the gluon and ghost anomalous dimensions in the Landau gauge, [23, 24]. To complete the analysis for any future gluon mass computations one requires, for instance, the relation between the running and pole mass of the gluon. This was initially addressed for the LCO formalism in [25] where the one loop relation between these quantities was given in the $\overline{\mathrm{MS}}$ scheme where the computation extended the one loop calculation of [26] for the Curci-Ferrari Lagrangian (with $N_f = 0$) itself rather than the LCO one. Moreover, an estimate of the pole mass was provided, [25], by converting the effective potential of the classical gluon mass of the LCO Lagrangian into a potential for the gluon pole mass. Remarkably upon extremization and solution of the resultant equation, a mass estimate emerged in QCD which was *independent* of the renormalization scale. For Yang-Mills theory this corresponded to $2.10\Lambda_{\overline{\mathrm{MS}}}$, [25], though the values for $N_f = 2$ and 3 were significantly lower. Given the interest in gluon masses and the absence of relations between the various mass quantities, it is the purpose of this article to extend [25] to two loops by computing the gluon pole mass in the LCO formalism in the $\overline{\mathrm{MS}}$ scheme in the Landau gauge. As a by-product we will be to deduce the same quantity in the Curci-Ferrari Lagrangian to extend the Yang-Mills result of [26]. Another motivation for such an analysis, aside from determining whether the two loop corrections significantly alter the one loop estimates, is to ascertain whether the two loop result using the minimization criterion of [25] retains the one loop renormalization scale independence. Given the fact that by analogy, except for the β-function, the $\overline{\mathrm{MS}}$ anomalous dimensions are renormalization scheme independent at only *one* loop, it would be surprising if a two loop pole mass estimate remained renormalization scale independent.

The article is organised as follows. In section 2, we review the background to the problem including the necessary points of the LCO formalism before discussing the construction of the two loop gluon pole mass in section 3. Equipped with this we perform the analysis to produce a mass estimate in section 4 before concluding with various estimates in section 5.

2 LCO formalism.

We begin by reviewing the key ingredients of the LCO formalism, [8], we will require. First, we define the QCD Lagrangian in an arbitrary covariant gauge as

$$L^{\mathrm{QCD}} ~=~ -~ \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 ~-~ \bar{c}^a \partial^\mu D_\mu c^a ~+~ i \bar{\psi}^{iI} \not{D} \psi^{iI} \tag{2.1}$$

where α is the gauge fixing parameter, A^a_μ is the gluon field, c^a and $\bar{c}^a$ are the ghost and anti-ghost fields, ψ^{iI} is the quark field and the various indices range over $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ where N_F and N_A are the dimensions of the fundamental and adjoint representations respectively and N_f is the number of quarks. To construct the LCO Lagrangian from L^{QCD} we introduce the path integral $W[J]$ defined by, [8],

$$e^{-W[J]} ~=~ \int \mathcal{D}A^\mu_{\mathrm{o}} \mathcal{D}\psi_{\mathrm{o}} \mathcal{D}\bar{\psi}_{\mathrm{o}} \mathcal{D}c_{\mathrm{o}} \mathcal{D}\bar{c}_{\mathrm{o}} \exp \left[\int d^d x \left(L_{\mathrm{o}} ~-~ \tfrac{1}{2} J_{\mathrm{o}} A^{a\,2}_{\mathrm{o}\,\mu} ~+~ \tfrac{1}{2} \xi_{\mathrm{o}} J^2_{\mathrm{o}} \right) \right] \tag{2.2}$$

where J is the source coupled to the local composite operator $\frac{1}{2} A^{a\,2}_\mu$ in the Landau gauge and the subscript $_{\mathrm{o}}$ denotes bare quantities. To retain renormalizability of the action including the source as well as a homogeneous renormalization group equation an additional term has been introduced. For instance, the term quadratic in J is necessary since the vacuum energy is divergent as can easily be seen by power counting. This term is coupled in via a parameter ξ and its associated counterterm $\delta\xi$ is included when the action is converted to renormalized parameters giving

$$e^{-W[J]} ~=~ \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}c \mathcal{D}\bar{c} \exp \left[\int d^d x \left(L ~-~ \tfrac{1}{2} Z_m J A^{a\,2}_\mu ~+~ \tfrac{1}{2} (\xi + \delta\xi) J^2 \right) \right] \tag{2.3}$$

where Z_m is the gluon mass renormalization constant. It transpires, [8], that in the LCO formalism one can compute the explicit form of ξ by ensuring that $W[J]$ does indeed satisfy a homogeneous renormalization group equation. Consequently since the coupling constant, g, runs in such an equation ξ is constrained to satisfy a differential equation dependent on the β-function and anomalous dimension of the gluon mass operator, $\frac{1}{2} A^{a\,2}_\mu$. This equation can be solved in a coupling constant expansion. As we will require the explicit form here we note that in QCD we have, [8, 10],

$$
\begin{aligned}
\frac{1}{g^2\xi(g)} ~=~ & \left[\frac{(13C_A - 8T_F N_f)}{9N_A} \right. \\
& + \left(2685464 C_A^3 T_F N_f - 1391845 C_A^4 - 213408 C_A^2 C_F T_F N_f - 1901760 C_A^2 T_F^2 N_f^2 \right. \\
& \quad + 221184 C_A C_F T_F^2 N_f^2 + 584192 C_A N_f^3 T_F^3 - 55296 C_F T_F^3 N_f^3 \\
& \quad \left. - 65536 T_F^4 N_f^4 \right) \frac{g^2}{5184\pi^2 N_A (35C_A - 16T_F N_f)(19C_A - 8T_F N_f)} \\
& + \left(\left(62228252520 C_A^6 N_f T_F - 8324745975 C_A^7 - 42525100800 C_A^5 C_F N_f T_F \right. \right. \\
& \quad - 123805256256 C_A^5 N_f^2 T_F^2 + 105262940160 C_A^4 C_F N_f^2 T_F^2 \\
& \quad + 112398515712 C_A^4 N_f^3 T_F^3 - 103719518208 C_A^3 C_F N_f^3 T_F^3 \\
& \quad - 52888043520 C_A^3 N_f^4 T_F^4 + 50866421760 C_A^2 C_F N_f^4 T_F^4 \\
& \quad + 12606898176 C_A^2 N_f^5 T_F^5 - 12419334144 C_A C_F N_f^5 T_F^5 \\
& \quad \left. - 1207959552 C_A N_f^6 T_F^6 + 1207959552 C_F N_f^6 T_F^6 \right) \zeta(3) - 13223737800 C_A^7 \\
& \quad + 5886241060 C_A^6 N_f T_F + 52585806000 C_A^5 C_F N_f T_F + 41351916768 C_A^5 N_f^2 T_F^2 \\
& \quad + 522849600 C_A^4 C_F^2 N_f T_F - 130596636288 C_A^4 C_F N_f^2 T_F^2 \\
& \quad - 67857620736 C_A^4 N_f^3 T_F^3 - 1286267904 C_A^3 C_F^2 N_f^2 T_F^2 \\
& \quad + 128750638080 C_A^3 C_F N_f^3 T_F^3 + 46700324864 C_A^3 N_f^4 T_F^4 \\
& \quad + 1180127232 C_A^2 C_F^2 N_f^3 T_F^3 - 63001780224 C_A^2 C_F N_f^4 T_F^4 \\
& \quad - 16782753792 C_A^2 N_f^5 T_F^5 - 475987968 C_A C_F^2 N_f^4 T_F^4 \\
& \quad + 15308685312 C_A C_F N_f^5 T_F^5 + 3106406400 C_A N_f^6 T_F^6 \\
& \quad + 70778880 C_F^2 N_f^5 T_F^5 - 1478492160 C_F N_f^6 T_F^6 \\
& \quad \left. \left. - 234881024 N_f^7 T_F^7 \right) \frac{g^4}{995328\pi^4 N_A (35C_A - 16T_F N_f)^2 (19C_A - 8T_F N_f)^2} \right] \\
& + O(g^6)
\end{aligned}
\tag{2.4}
$$

where $\mathrm{Tr}\left(T^a T^b\right) = T_F \delta^{ab}$, T^a is the group generator, C_A and C_F are the usual colour group Casimirs and $\zeta(z)$ is the Riemann zeta function. Consequently one uses a Hubbard-Stratanovich transformation to rewrite the exponential in the path integral of $W[J]$. This introduces the additional scalar field σ and gives a generating functional where the source J now couples linearly to a field as opposed to a composite operator. Thus, [8],

$$
e^{-W[J]} ~=~ \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\sigma \, \exp \left[\int d^d x \left(L^\sigma ~-~ \frac{\sigma J}{g} \right) \right] \tag{2.5}
$$

where L^σ is the LCO Lagrangian and is given by, [8],

$$
\begin{aligned}
L^\sigma ~=~ & -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 ~-~ \bar{c}^a \partial^\mu D_\mu c^a ~+~ i \bar{\psi}^{iI} \not{D} \psi^{iI} \\
& - \frac{\sigma^2}{2g^2 \xi(g) Z_\xi} ~+~ \frac{Z_m}{2g\xi(g) Z_\xi} \sigma A^a_\mu A^{a\,\mu} ~-~ \frac{Z_m^2}{8\xi(g) Z_\xi} \left(A^a_\mu A^{a\,\mu} \right)^2
\end{aligned}
\tag{2.6}
$$

where the first few terms of $\xi(g)$ are given by (2.4).

Using (2.6) the effective potential $V(\sigma)$ was constructed by standard methods at two loops in the $\overline{\mbox{MS}}$ scheme, [8, 9, 10]. The divergences in the vacuum energy are removed by straightforward renormalization since the Lagrangian, L^σ, retains the renormalizability property. As the two loop effective potential will be required later we recall its explicit form is

$$
\begin{aligned}
V(\sigma) ~=~ & \frac{9N_A}{2} \lambda_1 \sigma'^2 \\
& + \left[\frac{3}{64} \ln \left(\frac{g\sigma'}{\mu^2} \right) + C_A \left(- \frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{249}{128} \lambda_2 + \frac{27}{64} \lambda_3 \right) \right. \\
& \left. ~+~ C_A^2 \left(- \frac{81}{16} \lambda_1 \lambda_2 + \frac{81}{32} \lambda_1 \lambda_3 \right) + \left(- \frac{13}{128} - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right) \right] \frac{g^2 N_A \sigma'^2}{\pi^2} \\
& + \left[C_A \left(- \frac{593}{16384} - \frac{255}{16} C_F \lambda_2 + \frac{36649}{4096} C_F \lambda_3 - \frac{1053}{64} C_F^2 \lambda_1 \lambda_2 + \frac{1053}{128} C_F^2 \lambda_1 \lambda_3 \right. \right. \\
& \quad - \frac{5409}{1024} C_F^2 \lambda_2^2 + \frac{1053}{1024} C_F^2 \lambda_3^2 + \frac{891}{8192} s_2 - \frac{1}{4096} \zeta(2) - \frac{3}{64} \zeta(3) \\
& \quad \left. + \frac{585}{16} \zeta(3) C_F \lambda_2 - \frac{4881}{256} \zeta(3) C_F \lambda_3 \right) \\
& ~+~ C_A^2 \left(- \frac{11583}{128} C_F \lambda_1 \lambda_2 + \frac{11583}{256} C_F \lambda_1 \lambda_3 + \frac{72801}{2048} C_F \lambda_2^2 + \frac{11583}{2048} C_F \lambda_3^2 \right. \\
& \quad + \frac{3159}{128} C_F^2 \lambda_1 \lambda_2^2 + \frac{3159}{512} C_F^2 \lambda_1 \lambda_3^2 + \frac{372015}{16384} \lambda_2 - \frac{189295}{16384} \lambda_3 \\
& \quad + \frac{3159}{16} \zeta(3) C_F \lambda_1 \lambda_2 - \frac{3159}{32} \zeta(3) C_F \lambda_1 \lambda_3 - \frac{1053}{16} \zeta(3) C_F \lambda_2^2 \\
& \quad \left. - \frac{3159}{256} \zeta(3) C_F \lambda_3^2 - \frac{6885}{256} \zeta(3) \lambda_2 + \frac{116115}{8192} \zeta(3) \lambda_3 \right) \\
& ~+~ C_A^3 \left(\frac{34749}{256} C_F \lambda_1 \lambda_2^2 + \frac{34749}{1024} C_F \lambda_1 \lambda_3^2 + \frac{64071}{512} \lambda_1 \lambda_2 - \frac{64071}{1024} \lambda_1 \lambda_3 \right. \\
& \quad - \frac{694449}{16384} \lambda_2^2 - \frac{64071}{8192} \lambda_3^2 - \frac{9477}{32} \zeta(3) C_F \lambda_1 \lambda_2^2 - \frac{9477}{128} \zeta(3) C_F \lambda_1 \lambda_3^2 \\
& \quad \left. - \frac{37179}{256} \zeta(3) \lambda_1 \lambda_2 + \frac{37179}{512} \zeta(3) \lambda_1 \lambda_3 + \frac{12393}{256} \zeta(3) \lambda_2^2 + \frac{37179}{4096} \zeta(3) \lambda_3^2 \right) \\
& ~+~ C_A^4 \left(- \frac{192213}{1024} \lambda_1 \lambda_2^2 - \frac{192213}{4096} \lambda_1 \lambda_3^2 + \frac{111537}{512} \zeta(3) \lambda_1 \lambda_2^2 + \frac{111537}{2048} \zeta(3) \lambda_1 \lambda_3^2 \right) \\
& ~+ \left(- \frac{247}{4096} C_F + \frac{1185}{1024} C_F^2 \lambda_2 - \frac{615}{1024} C_F^2 \lambda_3 + \frac{1}{128} \zeta(2) N_f T_F + \frac{3}{64} \zeta(3) C_F \right) \\
& ~+~ \left[C_A \left(+ \frac{75}{4096} - \frac{315}{1024} C_F \lambda_2 \right) + C_A^2 \left(+ \frac{315}{4096} \lambda_2 \right) + \frac{9}{1024} C_F \right] \ln \left(\frac{g\sigma'}{\mu^2} \right) \\
& \left. ~-~ \frac{9}{4096} C_A \left(\ln \left(\frac{g\sigma'}{\mu^2} \right) \right)^2 \right] \frac{g^4 N_A \sigma'^2}{\pi^4} ~+~ O(g^6)
\end{aligned}
\tag{2.7}
$$

where

$$
\lambda_1 ~=~ [13C_A - 8T_F N_f]^{-1} ~~,~~ \lambda_2 ~=~ [35C_A - 16T_F N_f]^{-1} ~~,~~ \lambda_3 ~=~ [19C_A - 8T_F N_f]^{-1}
\tag{2.8}
$$

and μ is the renormalization scale which incorporates the usual factor of $4\pi e^{-\gamma}$ into the MS renormalization scale where γ is the Euler-Mascheroni constant. Minimizing $V(\sigma)$ with respect to the quantity σ one discovers that the classical perturbative vacuum at $\langle \sigma \rangle = 0$ is unstable and that there is a stable vacuum for a value of $\langle \sigma \rangle \neq 0$. Estimates for the value of $\langle \sigma \rangle$ were given in [8] by assuming that

$$
\frac{dV(\sigma)}{d\sigma} ~=~ 0
\tag{2.9}
$$

and then choosing the renormalization scale μ to be such that there were no logarithms in the equation relating the coupling constant to the value of $\langle \sigma \rangle$. Using the renormalization group properties relating the coupling constant to the scale μ and hence the fundamental scale Λ_{QCD} in the $\overline{\text{MS}}$ scheme, the estimate for $\langle \sigma \rangle$ was obtained which was relatively stable to two loop corrections. In [8] a subsequent estimate was deduced for an effective gluon mass. This will be illustrated in more detail in a later section.

However, as indicated earlier this was essentially the classical or bare gluon mass in the region of the stable minimum of the effective potential for σ. A more appropriate quantity to examine would be a gluon mass where some account of the quantum corrections were included. In this article we will use the pole mass of the gluon as constructed from the LCO Lagrangian at two loops building on the previous one loop analysis of [25].

3 Two loop gluon pole mass.

In this section we construct the relation between the running gluon mass of the LCO Lagrangian and the pole mass which is defined to be the pole of the one particle irreducible gluon polarization tensor. In [27] the simpler two dimensional Gross-Neveu model was studied and the relation between the analogous quantities was determined. The final mass estimates compared favourably with the known exact mass gap. Whilst we will use [27] as a basis for the QCD computation there are significant differences aside from the space-time dimensionality. The first is that L^σ has more interactions and basic fields as well as the gauge property. Second, and partly as a consequence of the previous point, it is not possible to fully construct the gluon 2-point function for all momenta and then deduce the pole of the propagator. This is also due to the fact that not all relevant basic 2-point two loop Feynman diagrams can be written in terms of closed known analytic functions for all values of the momenta. To circumvent these difficulties we have followed the strategy and algorithm of a similar model in the context of the weak sector of the full standard model. Our approach is based on the series of articles [28, 29, 30, 31] which applies the ON-SHELL algorithm to the relation of the vector gauge boson poles masses in $\overline{\text{MS}}$ to their bare values. This package, [28, 29], is designed to determine the value of two loop Feynman diagrams with massless propagators in addition to a propagator with a mass which is the on-shell value whose pole mass one is interested in. It uses dimensional regularization in $d = 4 - 2\epsilon$ dimensions. One can extend the approach of [28, 29] to integrals with more than one scale by expanding in an appropriate ratio of masses which is assumed to be small. In our case this complication does not occur.

The ON-SHELL package, [28, 29], is written in the symbolic manipulation language FORM, [32], and for L^σ we have generated the relevant one and two loop one particle irreducible Feynman diagrams using the QGRAF package, [33]. This is converted into a FORM readable format before applying the ON-SHELL procedure to determine the value of each individual diagram when the external momentum is set to its on-shell value. For the LCO Lagrangian we are interested in there are 5 one loop diagrams and 39 two loop diagrams to evaluate.

The remaining issue is to construct the pole mass itself from the integral contributing to the gluon 2-point polarization. If we define the transverse part of the correction to the polarization tensor by

$$\Pi_{\mu\nu}(p) \;=\; \Pi(p^2, m^2) \left[\eta_{\mu\nu} \;-\; \frac{p_\mu p_\nu}{p^2} \right] \tag{3.1}$$

where p is the external momentum then the pole mass is defined to be that value of p^2 which is the solution to, [30],

$$p^2 \ - \ m^2 \ - \ \Pi(p^2,m^2) \ = \ 0 \ . \tag{3.2}$$

If we write the perturbative expansion of the transverse part of the polarization tensor as

$$\Pi(p^2,m^2) \ = \ \sum_{n=1}^{\infty} \Pi_n(p^2,m^2) g^{2n} \tag{3.3}$$

then to two loops one can solve the pole mass condition iteratively to obtain the pole mass, s_p, as, [30],

$$s_p \ = \ m^2 \ + \ \Pi_1(m^2,m^2) g^2 \ + \ \left(\Pi_2(m^2,m^2) \ + \ \Pi_1(m^2,m^2) \Pi_1'(m^2,m^2) \right) g^4 \ + \ O(g^6) \tag{3.4}$$

where

$$\Pi_1'(m^2,m^2) \ = \ \left. \frac{\partial}{\partial p^2} \Pi_1(p^2,m^2) \right|_{p^2=m^2} \tag{3.5}$$

and here $m^2 = m^2(\mu)$ is the running mass. The actual values of $\Pi_i(m^2,m^2)$ are obtained from the ON-SHELL package, [28, 29]. In determining the two loop part of (3.4) from the one loop diagrams, we have expanded the bare coupling constant and bare mass in terms of the renormalized variables before applying the one and two loop ON-SHELL routines. As a check that the final expression we obtain for the pole mass is correct we note that first the full 2-point function $\Pi(m^2,m^2)$ itself has to be finite at two loops after renormalization with the usual Landau gauge renormalization constants, [34, 35, 36, 37], and, [8, 9, 10, 22],

$$\begin{aligned} Z_\xi^{-1} \ &= \ 1 \ + \ \left(\frac{13}{6} C_A - \frac{4}{3} T_F N_f \right) \frac{g^2}{16\pi^2 \epsilon} \\ &+ \ \left[\left(1464 C_A^2 T_F N_f - 1365 C_A^3 - 384 C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^2} \right. \\ &\quad + \ \left(5915 C_A^3 - 6032 C_A^2 T_F N_f - 1248 C_A C_F T_F N_f + 1472 C_A T_F^2 N_f^2 \right. \\ &\quad\quad \left. \left. + \ 768 C_F T_F^2 N_f^2 \right) \frac{1}{\epsilon} \right] \frac{g^4}{6144\pi^4 (35 C_A - 16 T_F N_f)} \ + \ O(g^6) \end{aligned} \tag{3.6}$$

in the $\overline{\mathrm{MS}}$ scheme. This is useful since it checks that the expansion of the one loop diagrams has been performed correctly when the coupling constant and mass are replaced by their renormalized variables. The quantity $\Pi_1'(m^2,m^2)$ is itself clearly finite by simple power counting. Finally, we arrive at our expression for the two loop $\overline{\mathrm{MS}}$ pole mass using the LCO Lagrangian which, with $s_p = m^2_{\mathrm{LCO}}$, is

$$\begin{aligned} m^2_{\mathrm{LCO}} \ &= \ \left[1 \ + \ \left(\left(\frac{287}{576} \ - \ \frac{3}{64} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \ - \ \frac{11\pi\sqrt{3}}{128} \right) C_A - \frac{1}{9} T_F N_f \right) \frac{g^2}{\pi^2} \right. \\ &+ \ \left(\left(\frac{1}{8} \zeta(3) - \frac{39}{256} + \frac{1}{128} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) T_F N_f C_F \right. \\ &\quad + \ \left(- \frac{2801}{55296} + \frac{99}{1024} S_2 - \frac{647}{6912} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) + \frac{3}{512} \left[\ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right]^2 \right. \\ &\quad\quad \left. + \ \frac{379}{4608} \zeta(2) - \frac{1}{8} \zeta(3) \right) T_F N_f C_A \\ &\quad + \ \left(\frac{3}{32} S_2 - \frac{11}{3456} + \frac{11}{1536} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \sqrt{3} T_F N_f C_A \end{aligned}$$

$$
\begin{aligned}
&+ \left(-\frac{7}{432} + \frac{1}{54} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) - \frac{1}{72}\zeta(2) + \frac{\pi^2}{144} \right) T_F^2 N_f^2 \\
&+ \left(\frac{3}{2048} - \frac{9}{2048} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) C_F C_A \\
&+ \left(-\frac{105}{2048} + \frac{315}{2048} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \frac{C_F C_A^2}{[35C_A - 16T_F N_f]} \\
&+ \left(\frac{9737}{24576} - \frac{3069}{8192} S_2 + \frac{11461}{221184} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right. \\
&\qquad \left. - \frac{51}{8192} \left[\ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right]^2 - \frac{59}{2304}\zeta(2) + \frac{231}{8192}\zeta(3) \right) C_A^2 \\
&+ \left(\frac{105}{8192} - \frac{315}{8192} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \frac{C_A^3}{[35C_A - 16T_F N_f]} \\
&+ \left(-\frac{1413}{32768} S_2 - \frac{12503}{221184} + \frac{77}{24576} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \sqrt{3}\pi C_A^2 \\
&+ \left. \left. \frac{17\pi^2}{2304} C_A^2 \right) \frac{g^4}{\pi^4} + O(g^6) \right] m^2(\mu)
\end{aligned}
\tag{3.7}
$$

where $S_2 = (4\sqrt{3}/3) Cl_2(\pi/3)$ and $Cl_2(x)$ is the Clausen function.

Another check on the symbolic manipulation routines we have written was to consider the Curci-Ferrari Lagrangian, [19], in the Landau gauge which is effectively QCD with a gluon mass included by hand. The corresponding Lagrangian is

$$
L^{\mathrm{mQCD}} = L^{\mathrm{QCD}} + \tfrac{1}{2} m^2 A^a_\mu A^{a\,\mu} - \alpha m^2 \bar{c}^a c^a
\tag{3.8}
$$

where we have included the ghost mass term for completeness. As the one loop pole mass for this theory was given in [26, 25], it does not require much more effort to produce the two loop $\overline{\mathrm{MS}}$ correction using the same symbolic manipulation programmes. In this case the same QGRAF output is used but with a null σ vertex and the usual quartic gluon interaction. The same check that the 2-point function is finite was satisfied. By contrast to (3.7) we find that the pole mass for (3.8) is, with $s_p = m^2_{\mathrm{CF}}$,

$$
\begin{aligned}
m^2_{\mathrm{CF}} = \Bigg[1 &+ \left(\left(\frac{313}{576} - \frac{35}{192} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) - \frac{11\pi\sqrt{3}}{128} \right) C_A \right. \\
&\qquad \left. + \left(\frac{1}{12} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) - \frac{5}{36} \right) T_F N_f \right) \frac{g^2}{\pi^2} \\
&+ \left(\left(\frac{1}{8}\zeta(3) - \frac{119}{768} + \frac{1}{64} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) T_F N_f C_F \right. \\
&\qquad + \left(-\frac{20335}{165888} + \frac{297}{1024} S_2 + \frac{13}{13824} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) + \frac{95}{4608} \left[\ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right]^2 \right. \\
&\qquad\qquad \left. + \frac{91}{1536}\zeta(2) - \frac{1}{8}\zeta(3) \right) T_F N_f C_A \\
&\qquad + \left(\frac{3}{32} S_2 + \frac{11}{13824} - \frac{11}{2304} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \sqrt{3} T_F N_f C_A
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(-\frac{5}{648} + \frac{7}{432} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) - \frac{1}{144} \left[\ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right]^2 + \frac{\pi^2}{144} \right) T_F^2 N_f^2 \\
&+ \left(\frac{163265}{331776} - \frac{5643}{8192} S_2 - \frac{11057}{110592} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right. \\
&\quad \left. - \frac{875}{73728} \left[\ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right]^2 - \frac{51}{2048} \zeta(2) + \frac{231}{8192} \zeta(3) \right) C_A^2 \\
&+ \left(-\frac{1413}{32768} S_2 - \frac{13933}{221184} + \frac{1661}{73728} \ln \left(\frac{m^2(\mu)}{\mu^2} \right) \right) \sqrt{3} \pi C_A^2 \\
&+ \left. \left. \frac{17\pi^2}{2304} C_A^2 \right) \frac{g^4}{\pi^4} + O(g^6) \right] m^2(\mu) \ . \qquad (3.9)
\end{aligned}
$$

Although the Lagrangian for QCD in the non-linear Curci-Ferrari gauge formally differs from that for the Landau gauge in relation to the ghost gluon interaction term for $\alpha = 0$, we have checked that the same pole mass emerges as (3.9) for the usual covariant Landau gauge fixing.

4 Analysis.

The next stage of our analysis is to produce an estimate for the pole mass. In [27] another method of estimating the Gross-Neveu mass gap was used. This required knowledge of the full 2-point function as a function of the external momentum. This approach is not available to us for QCD since the technology does not exist to compute the gluon 2-point function *exactly* as a function of momentum. Instead we simply extend the argument of [25] to two loops. In [8] an estimate for an effective gluon mass was obtained by examining the location of the minimum of $V(\sigma)$ using (2.9). The effective gluon mass is essentially the bare mass of L^σ. It does not take account of quantum corrections. In [25] it was argued that a more appropriate quantity to estimate from the effective potential was $m^2_{\rm LCO}$ itself. Specifically the quantity $V^{\rm eff}(m^2_{\rm LCO})$ was constructed by inverting the one loop part of (3.7) to obtain $m^2(\mu)$ as a function of $m^2_{\rm LCO}$ before substituting for $m^2(\mu)$ using the relation with $\langle \sigma \rangle$

$$
m^2(\mu) = \frac{9 N_A \langle \sigma \rangle}{[13 C_A - 8 T_F N_f] g \xi(g)} \ . \qquad (4.1)
$$

Thus we find the potential, truncated to one loop, is

$$
\begin{aligned}
V^{\rm eff}\left(m^2_{\rm LCO}\right) = & \left[\frac{9}{2}\lambda_1 + \left(-\frac{29}{128} + \frac{3}{64} \ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right. \right. \\
& + C_A \left(-\frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{183}{64} \lambda_1 \right. \\
& \left. \quad - \frac{249}{128} \lambda_2 + \frac{27}{64} \lambda_3 + \frac{99}{128} \pi \sqrt{3} \lambda_1 \right) \\
& \left. + C_A^2 \left(-\frac{81}{16} \lambda_1 \lambda_2 + \frac{81}{32} \lambda_1 \lambda_3 \right) + \frac{27}{64} C_A \lambda_1 \ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) \right) \frac{g^2}{\pi^2} \\
& \left. + O(g^4) \right] \frac{(13 C_A - 8 T_F N_f)^2}{81 N_A} g^2 \xi^2(g) m^4_{\rm LCO} \ . \qquad (4.2)
\end{aligned}
$$

Using the minimization criterion, [25],

$$
\frac{d V^{\rm eff}(m^2_{\rm LCO})}{d m^2_{\rm LCO}} = 0 \qquad (4.3)
$$

the following condition

$$
\begin{aligned}
0 \;=\; & \Bigg[\frac{9}{2}\lambda_1 + \left(-\frac{13}{64} + \frac{3}{64} \ln\left(\frac{m^2_{\mathrm{LCO}}}{\mu^2} \right) - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right. \\
& + C_A \left(-\frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{339}{128} \lambda_1 \right. \\
& \left. - \frac{249}{128}\lambda_2 + \frac{27}{64}\lambda_3 + \frac{99}{128}\pi\sqrt{3}\lambda_1 \right) \\
& \left. + C_A^2 \left(-\frac{81}{16}\lambda_1\lambda_2 + \frac{81}{32}\lambda_1\lambda_3 \right) + \frac{27}{64} C_A \lambda_1 \ln\left(\frac{m^2_{\mathrm{LCO}}}{\mu^2} \right) \right) \frac{g^2}{\pi^2} \\
& + O(g^4) \Bigg] \frac{(13C_A - 8T_F N_f)^2}{81 N_A} g^2 \xi^2(g) m^2_{\mathrm{LCO}}
\end{aligned}
\tag{4.4}
$$

emerged where we have not included the explicit expansion of $\xi(g)$. This is because it would introduce an unnecessary truncation error into the estimates for the pole mass. Ignoring the trivial solution of $m^2_{\mathrm{LCO}} = 0$ which corresponds to the unstable vacuum, the non-trivial condition determines the pole mass estimate at one loop. To solve this the renormalization scale μ was parametrically related to m^2_{LCO} by $m^2_{\mathrm{LCO}} = s\mu^2$ which leaves a parametric relation between the running coupling constant and m^2_{LCO}

$$
\begin{aligned}
y \;=\; & 36 C_A (16 T_F N_f - 35 C_A) \Big[\left(3465\pi\sqrt{3} + 4620\ln(s) - 25690 \right) C_A^2 - 864 C_F T_F N_f \\
& + \left(19240 - 1584\pi\sqrt{3} - 3792\ln(s) \right) C_A T_F N_f \\
& + (768\ln(s) - 3328) T_F^2 N_f^2 \Big]^{-1}
\end{aligned}
\tag{4.5}
$$

where $y = C_A g^2/(16\pi^2)$. However, from the one loop β-function the coupling constant can be related to the fundamental scale $\Lambda_{\overline{\mathrm{MS}}}$ using

$$
\frac{g^2(\mu)}{16\pi^2} = \left[\beta_0 \ln\left[\frac{\mu^2}{\Lambda^2_{\overline{\mathrm{MS}}}} \right] \right]^{-1}
\tag{4.6}
$$

where

$$
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F N_f \; .
\tag{4.7}
$$

Hence, the μ independent estimate for the pole mass emerged, [25],

$$
\begin{aligned}
m_{\mathrm{LCO}} \;=\; & \Lambda^{(N_f)}_{\overline{\mathrm{MS}}} \exp\Big[- \Big(\left(3465\pi\sqrt{3} - 25690 \right) C_A^2 - 864 C_F T_F N_f \\
& + \left(19240 - 1584\pi\sqrt{3} \right) C_A T_F N_f - 3328 T_F^2 N_f^2 \Big) \\
& \Big(24 (11C_A - 4T_F N_f)(35 C_A - 16 T_F N_f) \Big)^{-1} \Big] \; .
\end{aligned}
\tag{4.8}
$$

Equipped with the two loop pole mass of (3.7), we now repeat the above analysis by first inverting m^2_{LCO} to obtain $m^2(\mu)$ at two loops as a function of m^2_{LCO}. Substituting this into (2.7) and truncating at two loops, we find

$$
\begin{aligned}
V^{\mathrm{eff}}\left(m^2_{\mathrm{LCO}} \right) \;=\; & \Bigg[\frac{9}{2}\lambda_1 + \left(-\frac{29}{128} - \frac{207}{32} C_F \lambda_2 + \frac{117}{32} C_F \lambda_3 \right. \\
& + C_A \left(-\frac{351}{8} C_F \lambda_1 \lambda_2 + \frac{351}{16} C_F \lambda_1 \lambda_3 - \frac{183}{64}\lambda_1 \right.
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{249}{128}\lambda_2 + \frac{27}{64}\lambda_3 + \frac{99}{128}\pi\sqrt{3}\lambda_1\Big) \\
& + C_A^2\left(-\frac{81}{16}\lambda_1\lambda_2 + \frac{81}{32}\lambda_1\lambda_3\right) + \frac{3}{64}\ln\left(\frac{m_{\rm LCO}^2}{\mu^2}\right) \\
& + \frac{27}{64}C_A\lambda_1\ln\left(\frac{m_{\rm LCO}^2}{\mu^2}\right)\Bigg)\frac{g^2}{\pi^2} \\
+ & \Bigg(N_fT_F\left(-\frac{71}{1152} - \frac{1}{128}\zeta(2) + \frac{1}{128}\pi^2\right) \\
& + C_FC_A\left(+\frac{567}{256}\lambda_1 - \frac{23067}{2048}\lambda_2 + \frac{27133}{4096}\lambda_3 - \frac{117}{64}\zeta(3)\lambda_1\right. \\
& \qquad \left. + \frac{585}{16}\zeta(3)\lambda_2 - \frac{4881}{256}\zeta(3)\lambda_3\right) \\
& + C_FC_A\pi\left(-\frac{2277}{2048}\sqrt{3}\lambda_2 + \frac{1287}{2048}\sqrt{3}\lambda_3\right) \\
& + C_FC_A^2\left(-\frac{127287}{2048}\lambda_1\lambda_2 + \frac{16029}{512}\lambda_1\lambda_3 + \frac{72801}{2048}\lambda_2^2 + \frac{11583}{2048}\lambda_3^2\right. \\
& \qquad + \frac{3159}{16}\zeta(3)\lambda_1\lambda_2 - \frac{3159}{32}\zeta(3)\lambda_1\lambda_3 \\
& \qquad \left. - \frac{1053}{16}\zeta(3)\lambda_2^2 - \frac{3159}{256}\zeta(3)\lambda_3^2\right) \\
& + C_FC_A^2\pi\left(-\frac{3861}{512}\sqrt{3}\lambda_1\lambda_2 + \frac{3861}{1024}\sqrt{3}\lambda_1\lambda_3\right) \\
& + C_FC_A^3\left(+\frac{34749}{256}\lambda_1\lambda_2^2 + \frac{34749}{1024}\lambda_1\lambda_3^2\right. \\
& \qquad \left. - \frac{9477}{32}\zeta(3)\lambda_1\lambda_2^2 - \frac{9477}{128}\zeta(3)\lambda_1\lambda_3^2\right) \\
& + C_F\left(-\frac{1015}{4096} + \frac{3}{16}\zeta(3)\right) \\
& + C_F^2C_A\left(-\frac{1053}{64}\lambda_1\lambda_2 + \frac{1053}{128}\lambda_1\lambda_3 - \frac{5409}{1024}\lambda_2^2 + \frac{1053}{1024}\lambda_3^2\right) \\
& + C_F^2C_A^2\left(+\frac{3159}{128}\lambda_1\lambda_2^2 + \frac{3159}{512}\lambda_1\lambda_3^2\right) \\
& + C_F^2\left(+\frac{1185}{1024}\lambda_2 - \frac{615}{1024}\lambda_3\right) \\
& + C_A\left(+\frac{9709}{73728} + \frac{891}{4096}S_2 + \frac{137}{2048}\zeta(2) - \frac{3}{16}\zeta(3)\right) \\
& + C_A\pi\left(-\frac{605}{12288}\sqrt{3} + \frac{27}{256}\sqrt{3}S_2\right) \ + \ C_A\pi^2\left(+\frac{13}{1024}\right) \\
& + C_A^2\left(+\frac{8019}{4096}S_2\lambda_1 - \frac{19845}{16384}\lambda_1 + \frac{389957}{16384}\lambda_2 - \frac{193687}{16384}\lambda_3\right. \\
& \qquad - \frac{2631}{4096}\zeta(2)\lambda_1 + \frac{12897}{8192}\zeta(3)\lambda_1 \\
& \qquad \left. - \frac{6885}{256}\zeta(3)\lambda_2 + \frac{116115}{8192}\zeta(3)\lambda_3\right) \\
& + C_A^2\pi\left(-\frac{32211}{32768}\sqrt{3}S_2\lambda_1 - \frac{1193}{8192}\sqrt{3}\lambda_1 - \frac{2739}{8192}\sqrt{3}\lambda_2 + \frac{297}{4096}\sqrt{3}\lambda_3\right) \\
& + C_A^2\pi^2\left(+\frac{2217}{32768}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \ C_A^3 \left(+ \frac{1050543}{8192}\lambda_1\lambda_2 - \frac{32859}{512}\lambda_1\lambda_3 - \frac{694449}{16384}\lambda_2^2 - \frac{64071}{8192}\lambda_3^2 \right. \\
&\qquad - \frac{37179}{256}\zeta(3)\lambda_1\lambda_2 + \frac{37179}{512}\zeta(3)\lambda_1\lambda_3 \\
&\qquad \left. + \frac{12393}{256}\zeta(3)\lambda_2^2 + \frac{37179}{4096}\zeta(3)\lambda_3^2 \right) \\
&+ \ C_A^3\pi \left(- \frac{891}{1024}\sqrt{3}\lambda_1\lambda_2 + \frac{891}{2048}\sqrt{3}\lambda_1\lambda_3 \right) \\
&+ \ C_A^4 \left(- \frac{192213}{1024}\lambda_1\lambda_2^2 - \frac{192213}{4096}\lambda_1\lambda_3^2 \right. \\
&\qquad \left. + \frac{111537}{512}\zeta(3)\lambda_1\lambda_2^2 + \frac{111537}{2048}\zeta(3)\lambda_1\lambda_3^2 \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) N_f T_F \left(+ \frac{1}{32} \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_F C_A \left(- \frac{153}{2048}\lambda_1 - \frac{117}{128}\lambda_2 + \frac{351}{1024}\lambda_3 \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_F C_A^2 \left(- \frac{11259}{2048}\lambda_1\lambda_2 + \frac{1053}{512}\lambda_1\lambda_3 \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_F \left(+ \frac{9}{512} \right) \ + \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_A \left(- \frac{511}{4096} \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_A\pi \left(+ \frac{33}{2048}\sqrt{3} \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_A^2 \left(+ \frac{657}{8192}\lambda_1 - \frac{27}{256}\lambda_2 + \frac{81}{2048}\lambda_3 \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_A^2\pi \left(- \frac{99}{4096}\sqrt{3}\lambda_1 \right) \\
&+ \ \ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) C_A^3 \left(- \frac{1053}{8192}\lambda_1\lambda_2 + \frac{243}{1024}\lambda_1\lambda_3 \right) \\
&+ \ \left. \left(\ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right)\right)^2 C_A \left(+ \frac{9}{1024} \right) \right) \frac{g^4}{\pi^4} \\
&+ \ O(g^6) \Bigg] \frac{(13C_A - 8T_F N_f)^2}{81 N_A} g^2 \xi^2(g) m_{\mathrm{LCO}}^4
\end{aligned}
\tag{4.9}
$$

where again we have not substituted for $\xi(g)$. As this general expression is rather cumbersome, in order to illustrate the two loop analysis we concentrate for the moment on the case of $SU(3)$ with $N_f = 0$ when we simply have

$$
\begin{aligned}
V^{\mathrm{eff}}\left(m_{\mathrm{LCO}}^2\right)\Big|_{SU(3)}^{N_f=0} \ = \ & \left[\frac{3}{26} + \left(99\sqrt{3}\pi + 132\ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) - 800 \right) \frac{g^2}{1664\pi^2} \right. \\
& + \ \left(1038312\sqrt{3}\pi\ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) + 2174607\sqrt{3}\pi S_2 - 4831320\sqrt{3}\pi \right. \\
& \qquad + \ 640224\left(\ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right)\right)^2 - 8656704\ln\left(\frac{m_{\mathrm{LCO}}^2}{\mu^2}\right) \\
& \qquad - \ 1752192\zeta(3) + 1516143\pi^2 + 26815536 S_2
\end{aligned}
$$

$$+ \ 2936668 \Big) \frac{g^4}{24281088\pi^4} \ + \ O(g^6) \Big] \frac{169}{72} g^2 \xi^2(g) m^4_{\rm LCO} \ . \tag{4.10}$$

Solving (4.3) as before and discarding the trivial solution yields the two loop correction to (4.4) for $SU(3)$ Yang-Mills theory,

$$\begin{aligned} 0 \ &= \ \frac{3}{13} \ + \ \left(99\sqrt{3}\pi + 132 \ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) - 734 \right) \frac{g^2}{832\pi^2} \\ &+ \ \left(1038312\sqrt{3}\pi \ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) + 2174607\sqrt{3}\pi S_2 - 4312164\sqrt{3}\pi \right. \\ &\quad + \ 640224 \left(\ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) \right)^2 - 8016480 \ln \left(\frac{m^2_{\rm LCO}}{\mu^2} \right) - 1752192\zeta(3) \\ &\quad \left. + \ 1516143\pi^2 + 26815536 S_2 - 1391684 \right) \frac{g^4}{12140544\pi^4} \ + \ O(g^6) \ . \end{aligned} \tag{4.11}$$

In analysing this along the lines of the one loop case, it transpires that the resulting two loop correction for the $m^2_{\rm LCO}$ estimate is not μ independent. Therefore, we choose to return to the procedure of [8] and select the scale μ^2 so as to remove the logarithms in (4.11). This fixes $g(\mu)$ to a particular numerical value but using it the mass scale is recovered from the two loop extension of (4.6)

$$\frac{g^2(\mu)}{16\pi^2} \ = \ \left[\beta_0 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}} \right] \right]^{-1} \left[1 \ - \ \beta_1 \left[\beta_0^2 \ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}} \right] \right]^{-1} \ln \left[\ln \left[\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}} \right] \right] \right] \tag{4.12}$$

where

$$\beta_1 \ = \ \frac{34}{3} C_A^2 \ - \ 4 C_F T_F N_f \ - \ \frac{20}{3} C_A T_F N_f \ . \tag{4.13}$$

N_f	1 loop	2 loop
0	2.10	2.13
2	1.74	2.21
3	1.55	2.32

Table 1. One and two loop estimates of $m_{\rm LCO}/\Lambda^{(N_f)}_{\overline{\rm MS}}$ for $SU(3)$.

We have obtained estimates for both groups $SU(2)$ and $SU(3)$ for several quark flavours. These are summarized in Tables 1 and 2. Several features emerge. First, for Yang-Mills interestingly the two loop correction is less than 2% percent of the one loop value which suggests that our approximation is reliable. Unfortunately when quarks are included for both colour groups the situation is different with the two loop estimates being significantly larger than the one loop ones. Though for $SU(3)$ they are of a similar size as the Yang-Mills value. Given the stability of the two loop results for $N_f = 0$ compared with $N_f \neq 0$, it would suggest that the analysis when quarks are present is lacking some stabilising ingredient. One possibility is that for full QCD one actually requires quark masses to be included.

N_f	1 loop	2 loop
0	2.10	2.13
2	1.54	2.29
3	1.24	2.58

Table 2. One and two loop estimates of $m_{\mathrm{LCO}}/\Lambda^{(N_f)}_{\overline{\mathrm{MS}}}$ for $SU(2)$.

5 Discussion.

We have produced estimates for the gluon pole mass in QCD from the local composite operator method which systematically introduces an extra scalar field coupled to the gluon mass operator into the Lagrangian. The one loop renormalization scale independent estimate of [25] is stable to the two loop corrections for Yang-Mills theory but in the presence of quarks the estimates were significantly different. To improve the convergence for this case one could introduce masses for the quarks either by hand or by extending the LCO formalism to include the analogous mass operator vacuum expectation value $\langle\bar{\psi}\psi\rangle$ which is clearly beyond the scope of the present article. Moreover, if one accepts that a gluon mass emerges dynamically in QCD, one would then have to include gluon mass corrections in the estimates of the pole mass of the quarks. Although we have followed one procedure to deduce estimates for the gluon pole mass, other methods are possible. Indeed knowledge of the full momentum dependence of the gluon 2-point function would allow for the possibility of repeating the two loop analysis which was carried out in [27] using Grunberg's method of effective charges, [38, 39].

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