The Pseudoscalar Decay Constant

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We discuss insights that may be drawn from our recent 2 flavour $\mathcal{O}(a)$–improved Wilson quark simulations. We discuss the evidence of the onset of chiral logarithms in the pion decay constant. An overview is given of current extrapolation methods and a modification of chiral perturbation theory is presented as an approach for sensibly extrapolating to the physical quark masses.

1. Introduction

In a recent work [1] we presented results from the lightest UKQCD Wilson quark simulations for the mass of the singlet pseudoscalar meson and the pseudoscalar (pion) decay constant. In this work we shall expand upon the pion decay constant extrapolation details and suggest a simplistic first attempt at an improved extrapolation method.

The motivation for this recent calculation of the pseudoscalar decay constant, $f_{\rm PS}$, is the expectation of observing behaviour predicted by Chiral Perturbation Theory (χPT) as the pseudoscalar mass is lowered. In particular evidence of the onset of chiral logs in observables. The lowest order chiral Lagrangian has non-analyticity resulting from loop corrections introduced in the parameters. This non-analytic behaviour provides a good check that the lattice calculation is in the regime where chiral perturbation theory is valid.

Computational constraints have forced us to work at finite lattice spacing, however the choice was made to compare our results to the continuum predictions of χPT. Whilst Chiral perturbation theory has been formulated for finite lattice spacing [2,3], it is at the cost of additional parameters.

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2. Calculating $f_{\rm PS}$

Our results for the pseudoscalar decay constant were extracted from UKQCD simulations at three different quark masses: $\kappa = 0.1358$, 0.1355 and 0.1350. All lattices had $16^3 \times 32$ volumes with $\beta = 5.2$ and a non-perturbatively improved clover action. Using quark propagators with sources on the time planes $t = 0$, 7, 15 and 23 allowed an improvement in the statistics of the simulation. As the value of $m_{\rm PS}L \approx 4$ in each of the cases we would expect some finite volume effects in our calculations. These are taken into consideration [4] in the subsequent analysis.

Table 1
The raw lattice value of $af_{\rm PS}$ is given by using the order a improved expression $(1 + b_A m_{\rm PS})(af_A + c_A af_P)$ and we tabulate these two contributions. [1]

κ	af_A	af_P
0.1358	0.0829(26)	0.1457(78)
0.1355	0.1055(14)	0.1835(44)
0.1350	0.1336(11)	0.2468(33)

Our results for $f_{\rm PS}$ were presented in [1] and are reproduced here in Table 1 and Fig. 1. The lattice value of $af_{\rm PS}$ used in Fig. 1 is found by using the order a improved expression $(1 + b_A m_{\rm PS})(af_A +$

$c_A a f_P$), where $f_{A,P}$ are the axial-vector and pseudoscalar decay constants and their contributions are listed in Table 1.

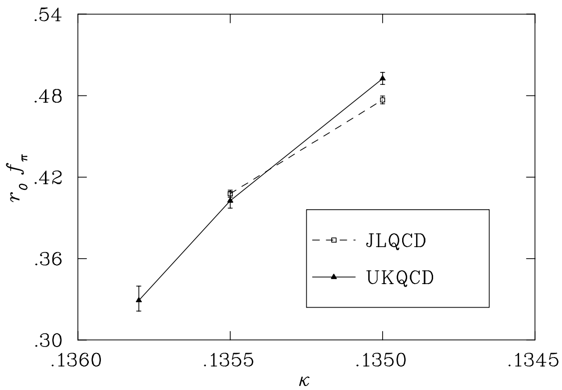

Figure 1. The pseudoscalar decay constant in units of r_0 from UKQCD and JLQCD versus κ.[1]

To aid comparison with the only available comparable data, that of JLQCD [5], we have used the same perturbative formulation of the corrections and also the same prescription for Z_A, the renormalisation factor. Finally as there is a discrepancy in the prescription for evaluating r_0 we apply our determination of r_0 to their data. The comparison of results is shown in Fig. 1. The agreement at the lightest common κ values is pleasing. The exciting feature of our new results is the indication of curvature versus m_q. It is this result that has motivated the current investigation.

3. Extrapolation Comparisons

The limitation of all current lattice calculation of the pseudoscalar decay constant to large quark masses necessitates some form of extrapolation to the physical quark (or equivalently pion) mass. Over the last few years it has become accepted within lattice groups that some form of *chirally motivated* extrapolation is appropriate [6,7,8,9], by using insights from chiral perturbation theory, when extrapolating physical observables. The situation is no different in the case of $f_{\rm PS}$, however opinions of what is *appropriate* differ between groups.

The chiral perturbation prediction for the pion mass dependence of $f_{\rm PS}$ is [4,10]:

$$\frac{f_{\rm PS}}{f_\pi(0)} = 1 - 2\left(\frac{m_{\rm PS}}{4\pi f_\pi(0)}\right)^2 \log\left(\frac{m_{\rm PS}^2}{\Lambda_4^2}\right) + \cdots \tag{1}$$

where $f_\pi(0)$ is the pseudoscalar (pion) decay constant in the chiral limit.

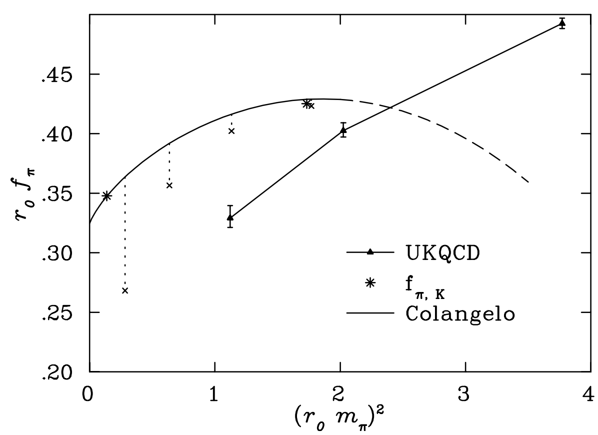

Figure 2. The pseudoscalar decay constant in units of r_0 from UKQCD versus the squared pseudoscalar meson mass. Also shown is an expression including chiral perturbation theory terms to order $m_{\rm PS}^4$ which has been fitted (see ref. [4] where we use $\mu = 0.75$ GeV and $\tilde{r}_F(\mu) = -2$) to agree with the experimental values of f_π and f_K which are shown (*). An estimate [4] of the finite size effect expected from chiral perturbation theory (to order $m_{\rm PS}^2$) is shown by the vertical lines.

Recent work [4] has provided a way of estimating the expected finite size effects. The quark loops generate logarithmic corrections, but moreover they are the source of the finite size effects. Using $L = 1.5$ fm and $m_\pi = 400$ MeV, which are close to our values a suggestion of the size of the corrections may be determined. The solid curve shown is an expression including chiral perturbation theory terms to order $m_{\rm PS}^4$ which has been fitted (see Ref. [4] where we use $\mu = 0.75$ GeV and $\tilde{r}_F(\mu) = -2$) to agree with the experimental values of f_π and f_K which are shown (*). An estimate [4] of the finite size effect expected from chiral perturbation theory (to order $m_{\rm PS}^2$) is shown by the vertical lines. As an experiment, we also continued the χPT prediction for $f_{\rm PS}$ to almost twice the kaon mass (dashed curve). It is clear the lattice simulation behaviour in this regime is fundamentally different from the naïve predictions of χPT. This discrepancy indicates χPT (as formulated) has a limited range of overlap with current simulations.

JLQCD

Recent progress in the extension of χPT has been attempted by the JLQCD collaboration [5]. Their approach used χPT as motivation for their functional form which, in the dynamical case ($\kappa_{\rm sea} = \kappa_{\rm val}$) may be written as:

$$f_{\rm PS} = A + Bm_{\rm PS}^2 + Cm_{\rm PS}^4 \qquad (2)$$

The analytic (even powers of $m_{\rm PS}$) terms of Eq. (2) are consistent with both χPT and the JLQCD data set, as seen in Fig. 3. The success of such a fit may be seen to vindicate "leaving the problem of the chiral logarithm [...] for future publication." [5]

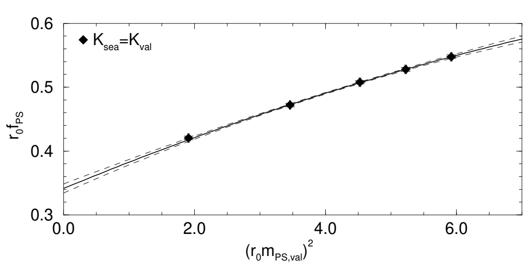

Figure 3. Chiral extrapolation of the pseudoscalar meson decay constant. The fit is made using Eq. (2). [5]

The problem with this approach is the selective nature of what parts of χPT are used. The chiral logarithm term that occurs with a power of the pseudoscalar mass less than $m_{\rm PS}^4$ is ignored. Yet this is the greatest contribution to the curvature in the small mass regime. The form is also inconsistent with the large mass behaviour of $f_{\rm PS}$ which experience has shown to be proportional to $m_{\rm PS}^2$ up to pseudoscalar masses of the order of 1.4 GeV^2. Thus, any perceived successes of this form of extrapolation must be tempered by the clear inconsistencies with the behaviour of QCD as established by χPT.

Dyson–Schwinger Equation

An interesting alternative insight is that promoted by the Dyson–Schwinger Equation (DSE) community. The Dyson–Schwinger equations, whilst maintaining a Poincaré covariant framework, provide a setting for a non-perturbative chiral symmetry preserving truncation that leads to an efficient one–parameter rainbow–ladder model [11]. In this DSE approach, the pion is well–understood and is essentially model–independent; it is both the Goldstone boson and a bound state of strongly dressed quarks [12,13]. This reduction of QCD to an understandable case benefits us as the circumstances of the lattice (that is equal mass quarks as discussed later) may be investigated.

The DSE approach of Tandy [14] to the pseudoscalar decay constant is presented in Fig. 4 by the solid curve. For the dashed curve, the same calculated decay constant is plotted against the pseudoscalar mass obtained from m_q via the Gell-Mann–Oakes–Renner relation, as would be used by χPT. This alternative approach differs from χPT and the lattice calculations as quark loops are explicitly excluded from the calculation. It would thus be expected that the exclusion will, in the light quark mass region, result in a behaviour similar to that expected by the JLQCD collaboration in Eq. (2) — no logarithms, and yet be suggestive of the lattice calculations in the region where quark loops become suppressed. Tandy parameterised the DSE results with the following phenomenologically motivated form [11]:

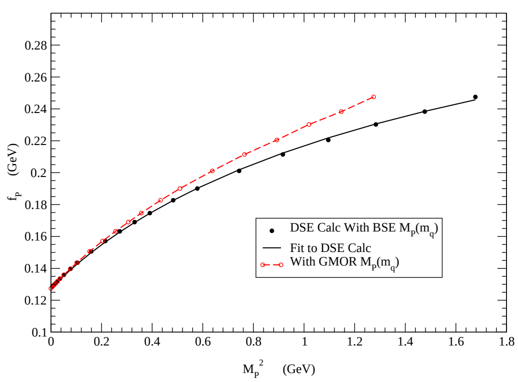

Figure 4. The pseudoscalar decay constant versus the square of the pseudoscalar mass both calculated within the Dyson–Schwinger framework. The solid line is the exact pseudoscalar mass, whilst the dashed curve is the mass obtained by using the Gell-Mann–Oakes–Renner relationship. [15]

$$f_{\rm PS} = \sqrt{\frac{f_\pi^2(0) + A_0 m_{\rm PS}}{1.0 + A_1 m_{\rm PS} + A_2 m_{\rm PS}^2}} \tag{3}$$

Whilst reproducing the correct QCD behaviour at large $m_{\rm PS}$ it is incorrect at small quark masses. As was discussed above this should not be considered a failure, as the model that motivated this form lacks the chiral behaviour that induces the logarithm in Eq. (1).

3.1. Modified χPT

The final approach we will mention is a minimalistic modification of χPT as motivated by our previous work [6,7,8,16]. In these investigations we found that a change in the form of the regularisation of χPT admits an extended radius of convergence whilst maintaining an exact agreement with the results of the dimensional regularised solution. Recent work of the Adelaide group [17] has show that this exact agreement is indeed obtainable between different formulations of the regulariser, and it is indeed a requirement of a regularisation scheme.

The changes we make to the dimensional regularised result of χPT to form our Modified χPT are two:

- Terms of $\mathcal{O}(m_{\rm PS}^4)$ and higher in the full expression of Eq. (1) are neglected,
- An additional parameter $\mathcal{B}m_{\rm PS}^2$ is introduced to absorb the physics ignored by the truncation.

The form used for extrapolating thus becomes:

$$\frac{f_{\rm PS}}{f_\pi(0)} = 1 - 2\left(\frac{m_{\rm PS}}{4\pi f_\pi(0)}\right)^2 \times \log\left(\frac{m_{\rm PS}^2}{\mathcal{B}m_{\rm PS}^2 + \Lambda^2}\right) \tag{4}$$

where the new term $\mathcal{B}m_{\rm PS}^2$ suppresses the chiral logarithms above a certain energy scale and provides the missing analytic behaviour near the chiral limit. A similar approach has been attempted by JLQCD [18].

The advantages of this form for extrapolating results from lattice QCD calculations are that it reproduces χPT near the chiral limit, but it also has the intermediate mass behaviour of QCD that the lattice has shown. This simple form allows an extension of the radius of convergence of χPT. It must be noted that this form is indeed naïve in the way the two mass limits are enforced but it does suggests hope for a more rigorous FRR solution that has been achieved elsewhere [17]. Such an extension is part of an ongoing investigation.

This modified χPT approach is more suited to extrapolating lattice QCD calculations, at the current juncture, than the dimensionally regularised approach of Eq. (1) for another reason too: it has fewer fit parameters. As shown in Fig. 1 near the chiral limit we currently have, at best, 5 data points, and fitting purely to our UKQCD data we would have only three data points. The $\mathcal{O}(p^6)$ low–energy constants expansion of Eq. (1) (see Ref. [4] for the full expression) has seven parameters: $f_\pi(0)$, $\Lambda_1, \ldots, \Lambda_4$, $\tilde{r}^F(\mu)$ and μ; each of which must be fit to some data. This has been undertaken in [19,20], but in the comparison of lattice QCD to experimental results, these pa-

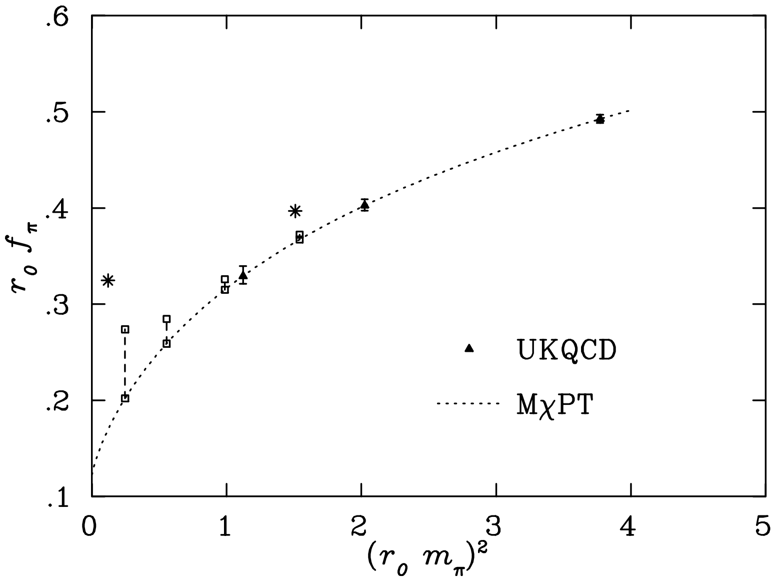

Figure 5. The pseudoscalar decay constant in units of r_0 from UKQCD versus the squared pseudoscalar meson mass. The experimental values of f_π and f_K are shown (*). The dotted curve is the fit of the Modified χPT result of Eq. (4). The vertical lines indicate expected finite size effects at various pseudoscalar meson masses, as in Fig. 2.

rameters must independently be determined by the theory.

Conversely the simplistic approach of Eq. (4) has only three parameters $f_\pi(0)$, Λ and $\mathcal{B}$. In comparing these two approaches the expectation that the non-analytic behaviour should be the same is an important success. The values of Λ (Eq. (4)) and Λ_4 (Eq. (1)) are consistent within the large uncertainties as previously discussed in [7]. The value of Λ_4 from [4] is $1.25\ ^{+0.15}_{-0.13}$ GeV whilst we find $\Lambda = 0.93 \pm 0.42$ GeV. In a similar vein to Ref. [16] some of the physics we have neglected, i.e. higher powers of $m_{\rm PS}$, has been absorbed into the parameter $\mathcal{B}$. We present our fit of Eq. (4) to our UKQCD data in Fig. 5. The vertical dashed lines are the same finite size error estimation calculations as discussed in Sec. 3.

It is clear that even with a finite size correction to the UKQCD calculations there is a discrepancy between the data obtained from lattice simulations and the experimentally measured decay constant of the K meson. This discrepancy highlights some of the limitations of the current results. The finite size corrections that have been applied to this data are indicative only and do not necessarily account for all the finite size effects in the simulation. The benefit is that they reinforce the intuition that finite size effects become less important as the quark masses get larger.

The discussion herein of an extrapolation of the quark mass ignores the other important extrapolation that has not been undertaken in these simulations; that is the continuum extrapolation. In this work we have chosen a value of r_0 as suggested by [21,22], however the uncertainty of this value is 5%. Additionally, the value of Z_A we use to renormalise our result is determined perturbatively. At first order the correction is 25% and the systematic error at second order is expected to be up to 5%. Whilst the value of Z_A remains determined perturbatively we would expect that part of the discrepancy in our results to remain.

Finally there is a fundamental difference between the object that is calculated at non-zero quark masses and the pseudoscalar meson that is experimentally detected. The K meson conceptually is a *strange* quark and a *light* quark in a sea of *light* quarks. The pseudoscalar meson calculated on the lattice with the same mass consists of two quarks with masses half of that of the strange quark in a sea of equivalently heavy quarks. The DSE calculations shown in Fig. 4 have this behaviour, but the experimental points of Figs. 2 and 5 are different objects. There is an expectation that the difference between a $\langle qQ\rangle$ object and $\langle \frac{Q}{2}\frac{Q}{2}\rangle$ meson is not large, but this is yet to be quantified.

4. Conclusion

We have presented a review of our recent publication [1] and in particular the discussion of the extraction of the pseudoscalar decay constant on the lattice. We have shown that this data set contains an indication of the onset of chiral logarithms for the first time in this quantity. Additionally we have discussed how the need to extrapolate to the chiral limit still exists and furthermore that the suggestion of curvature necessitates the use of chiral perturbation theory. Finally we discuss some approaches and present a first attempt at a form that respects chiral perturbation theory whilst being applicable in the region where the lattice calculations are still being performed.

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