DESY 04-021
Edinburgh 2004/03
Leipzig LU-ITP 2004/007
Liverpool LTH 618

One-loop renormalisation of quark bilinears for overlap fermions with improved gauge actions

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Abstract

We compute lattice renormalisation constants of local bilinear quark operators for overlap fermions and improved gauge actions. Among the actions we consider are the Symanzik, Lüscher-Weisz, Iwasaki and DBW2 gauge actions. The results are given for a variety of ρ parameters. We show how to apply mean field (tadpole) improvement to overlap fermions. The question, what is a good gauge action, is discussed from the perturbative point of view. Finally, we show analytically that the gauge dependent part of the self-energy and the amputated Green functions are independent of the lattice fermion representation, using either Wilson or overlap fermions.

1 Introduction

Lattice calculations at small quark masses require an action with good chiral properties. The same is true for calculations of matrix elements of certain operators, which otherwise mix with operators of opposite chirality. Ginsparg-Wilson fermions [1] have an exact chiral symmetry on the lattice [2], and thus are well suited for these tasks. A further advantage is that they are automatically $O(a)$ improved [3]. Overlap fermions [4, 5, 6] provide a four-dimensional realisation of Ginsparg-Wilson fermions. The massless Neuberger-Dirac operator is defined by

$$D_N = \frac{\rho}{a}\left(1 + \frac{X}{\sqrt{X^\dagger X}}\right) \, , \; X = D_W - \frac{\rho}{a} \, , \tag{1}$$

where D_W is the Wilson-Dirac operator, and ρ is a real parameter corresponding to a negative mass term. At tree level ρ must lie in the range $0 < \rho < 2r$, r being the Wilson parameter. Exact chiral symmetry on the lattice has its price however. Numerical implementations of the overlap operator are significantly more expensive than Wilson fermions. In spite of this difficulty, calculations with overlap fermions are progressing rapidly, and we expect to see many more results in the near future.

It is important to use a good gauge field action. The cost of the overlap operator is governed largely by the condition number of $X^\dagger X$. This number is greatly reduced for improved gauge field actions [7]. For example, for the tadpole improved Lüscher-Weisz action we found a reduction factor of $\gtrsim 3$ compared to the Wilson gauge field action [8]. The reason is that the Lüscher-Weisz action, as many other improved actions [9], suppresses unphysical zero modes, sometimes called dislocations. A reduction in the number of small modes of $X^\dagger X$ appears also to result in an improvement of the locality of the overlap operator [7].

To obtain continuum results from lattice calculations of hadron matrix elements, the underlying operators have to be renormalised. A perturbative calculation of the corresponding renormalisation constants is always the first step. Recently, the renormalisation constants of bilinear [10, 11, 12] and four-quark [13] operators have been computed to one-loop order for overlap fermions and Wilson gauge field action. Our aim is to extend the calculation to general, improved gauge field actions with up to six links, including the tree-level Symanzik action, the Lüscher-Weisz action, the Iwasaki action, and the DBW2 action. In this paper we shall consider local bilinear quark operators first. Operators including covariant derivatives will be treated in a separate publication. Preliminary results of the calculation have been reported in [8].

The paper is organised as follows. In Section 2 and Appendix A we give the details of the calculation. The basic results are given in Section 3. In Appendix B we show that the gauge dependent terms in the Green functions are not only independent of the gauge field action, but also of the type of fermion. Tadpole improvement is an important issue in lattice perturbation theory. In Section 4 and Appendix C we tadpole improve our results. Finally, in Section 5 we give our conclusions.

2 Calculational details

We denote the lattice action by

$$S = S_G + S_F \,, \tag{2}$$

where S_G is the gauge field action, and S_F is the fermion action. Let us discuss the gauge field action first. We consider the following class of improved actions:

$$\begin{aligned} S_G = \frac{6}{g^2} \Bigg[& c_0 \sum_{\text{plaquette}} \frac{1}{3} \,\mathrm{Re\,Tr}\,(1 - U_{\text{plaquette}}) \,+\, c_1 \sum_{\text{rectangle}} \frac{1}{3} \,\mathrm{Re\,Tr}\,(1 - U_{\text{rectangle}}) \\ & + \, c_2 \sum_{\text{chair}} \frac{1}{3} \,\mathrm{Re\,Tr}\,(1 - U_{\text{chair}}) \,+\, c_3 \sum_{\text{parallelogram}} \frac{1}{3} \,\mathrm{Re\,Tr}\,(1 - U_{\text{parallelogram}}) \Bigg] \,, \end{aligned} \tag{3}$$

where $U_{\text{plaquette}}$ is the standard plaquette, while the remaining U's cover all possible closed loops of link matrices of length six along the edges of the hypercubes, as indicated in Fig. 1. It is customary to impose the normalisation condition

$$c_0 + 8c_1 + 16c_2 + 8c_3 = 1 \,. \tag{4}$$

In Section 4 we will also consider actions which fulfill this condition only in the limit $g \to 0$, which is sufficient to ensure the correct continuum limit. Calling the coefficient in front of the plaquette part the lattice coupling β, we have the obvious relation

$$\beta = \frac{6}{g^2}\, c_0 \,. \tag{5}$$

If the improvement is performed only on-shell, one of the six-link contributions can be set to zero [14]. In general, the parameters c_i are chosen according to an approximate renormalisation group (RG) analysis or using tadpole improved perturbation theory.

For perturbation theory expansions, we write the $SU(3)$ link matrices as

$$U_{x,\mu} = \exp\left[\mathrm{i} g a A_\mu\Big(x + \frac{a}{2}\hat{\mu}\Big)\right] \,, \tag{6}$$

where $A_\mu = T^a A^a_\mu$, T^a being the generators of the Lie algebra, and perform a weak coupling expansion in g. The calculations are carried out in a general covariant gauge,

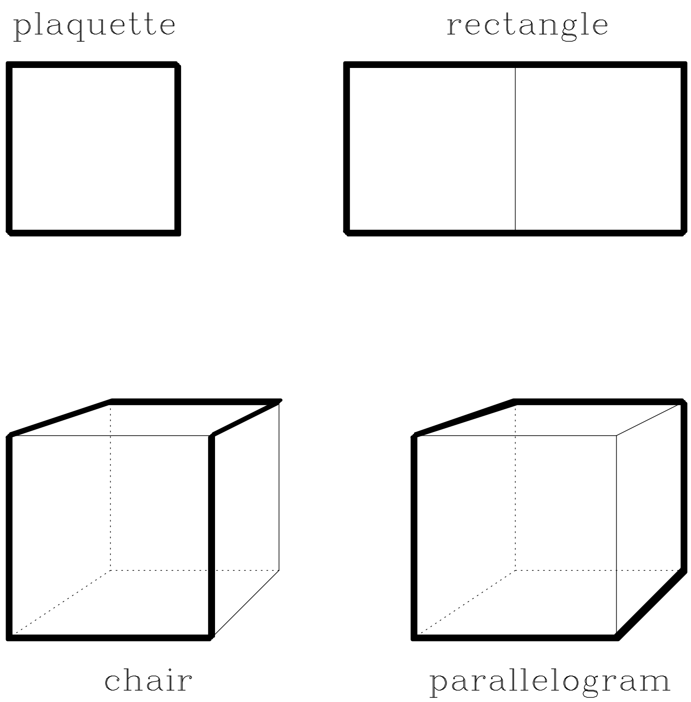

Figure 1: The meaning of the four terms in the gauge field action (3).

distinguished by the gauge parameter $\xi = 1 - \alpha$. Landau gauge corresponds to $\xi = 1$, while the Feynman gauge corresponds to $\xi = 0$. The gluon propagator (in the infinite volume) is obtained from the lowest order expansion of (3):

$$S_{\rm G}^{(0)} = \frac{1}{2}\int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} A_\mu^a(k) \left[G_{\mu\nu}(k) - \frac{\xi}{\xi - 1}\hat{k}_\mu \hat{k}_\nu \right] A_\nu^a(-k)\,, \tag{7}$$

where

$$G_{\mu\nu}(k) = \hat{k}_\mu \hat{k}_\nu + \sum_\rho \left(\hat{k}_\rho^2 \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\rho \delta_{\rho\nu} \right) d_{\mu\rho} \tag{8}$$

and

$$d_{\mu\nu} = (1 - \delta_{\mu\nu}) \left[C_0 - C_1\, a^2 \hat{k}^2 - C_2\, a^2 (\hat{k}_\mu^2 + \hat{k}_\nu^2) \right], \quad \hat{k}_\mu = \frac{2}{a} \sin \frac{a k_\mu}{2}, \quad \hat{k}^2 = \sum_\mu \hat{k}_\mu^2 . \tag{9}$$

The coefficients C_i are related to the coefficients c_i of the improved action by

$$C_0 = c_0 + 8c_1 + 16c_2 + 8c_3\,, \;\; C_1 = c_2 + c_3\,, \;\; C_2 = c_1 - c_2 - c_3\,. \tag{10}$$

(We will usually work with actions in which $C_0 = 1$; see eq. (4).)

In lattice momentum space the gluon propagator $D_{\mu\nu}(k)$ is given by the set of linear equations

$$\sum_\rho \left[G_{\mu\rho}(k) - \frac{\xi}{\xi-1}\hat{k}_\mu \hat{k}_\rho\right] D_{\rho\nu}(k) = \delta_{\mu\nu}\,. \tag{11}$$

For the plaquette action the gluon propagator reads

$$D_{\mu\nu}^{\text{plaquette}}(k) = \frac{1}{\hat{k}^2}\left(\delta_{\mu\nu} - \xi\,\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right)\,. \tag{12}$$

Expressions for the more general action can be found in the literature [15, 16]. We will use dimensional regularisation to regularise the loop integrals. Thus we need to know $D_{\mu\nu}$ for arbitrary dimensions. In general, $D_{\mu\nu}$ can be given in closed form only for integer dimensions. Only for the special case $C_2 = 0$ can we derive explicit expressions for arbitrary dimensions. A way out is to split the gluon propagator into two parts: a singular part, which can easily be extended to arbitrary dimensions, and a finite part, which does not need to be regularised. This is achieved by writing

$$D_{\mu\nu} = C_0^{-1}\, D_{\mu\nu}^{\text{plaquette}} + \Delta D_{\mu\nu}\,. \tag{13}$$

It is readily seen that $D_{\mu\nu}$ and $C_0^{-1}\, D_{\mu\nu}^{\text{plaquette}}$ have the same infrared singularity. The finite part of the gluon propagator $\Delta D_{\mu\nu}$ is obtained by solving (11) for the difference $D_{\mu\nu} - C_0^{-1}\, D_{\mu\nu}^{\text{plaquette}}$ (in four dimensions). It can be written

$$\Delta D_{\mu\nu} = \delta_{\mu\nu} \sum_{n=0}^{4} D_n(k,C)\, \hat{k}_\mu^{2n} + \sum_{n=0}^{4}\sum_{m=0}^{4-n} D_{m,n}(k,C)\, \hat{k}_\mu^{2m+1}\, \hat{k}_\nu^{2n+1}\,. \tag{14}$$

The scalars D_n and $D_{m,n} = D_{n,m}$ are rational functions of $\hat{k}_\mu$ and C_i. They are listed in Appendix A for the case $C_0 = 1$.

The final results cannot be expressed in analytic form (as a function of C_i) anymore. We therefore have to make a choice concerning the parameters. We restrict ourselves to the most popular actions in this class:

<u>Tree-level Symanzik</u> [17]

$$c_1 = -1/12,\; c_2 = c_3 = 0\,. \tag{15}$$

<u>Tadpole improved Lüscher-Weisz (TILW)</u> [18, 14, 19]

$$c_2 = 0\,. \tag{16}$$

Once we have chosen β, the other parameters are fixed [19]:

$$\frac{c_1}{c_0} = -\frac{(1 + 0.4805\,\alpha)}{20\,u_0^2}\,, \quad \frac{c_3}{c_0} = -\frac{0.03325\,\alpha}{u_0^2}\,, \quad \frac{1}{c_0} = 1 + 8\left(\frac{c_1}{c_0} + \frac{c_3}{c_0}\right)\,, \tag{17}$$

where

$$u_0 = \left(\frac{1}{3}\,\mathrm{Tr}\,\langle U_{\mathrm{plaquette}}\rangle\right)^{\frac{1}{4}} \tag{18}$$

and

$$\alpha = -\frac{\log(u_0^4)}{3.06839}\,. \tag{19}$$

We have picked the following values [20]:

β	c_1	c_3
8.60	-0.151791	-0.0128098
8.45	-0.154846	-0.0134070
8.30	-0.159128	-0.0142442
8.20	-0.161827	-0.0147710
8.10	-0.165353	-0.0154645
8.00	-0.169805	-0.0163414

which corresponds to lattice spacings $a = 0.084 - 0.136$ fm. These β values are currently being employed in quenched Monte Carlo simulations, or most probably will be in the near future.

Iwasaki [15]

$$c_1 = -0.331,\ c_2 = c_3 = 0\,. \tag{20}$$

DBW2 [21]

$$c_1 = -1.4086,\ c_2 = c_3 = 0\,. \tag{21}$$

In all these cases c_0 is chosen to satisfy eq. (4).

Which action is the best one? They all reduce or eliminate $O(a^2)$ corrections – at the tree level, the one-loop level, or beyond. In Section 4 we compare the actions with regard to their perturbative merits.

The action for massless overlap fermions is given by

$$S_F = \bar{\psi} D_N \psi \,. \tag{22}$$

The Neuberger-Dirac operator D_N was already given in eq. (1). The Wilson-Dirac operator D_W reads

$$D_W = \frac{1}{2}\Big[\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - ar\nabla^\star_\mu\nabla_\mu\Big]\,, \tag{23}$$

where ∇_μ is the lattice forward covariant derivative:

$$\nabla_\mu\,\psi(x) = \frac{1}{a}\Big[U_{x,\mu}\,\psi(x+a\hat{\mu}) - \psi(x)\Big]\,. \tag{24}$$

For $0<\rho<2r$ the correct spectrum of massless fermions without doubling is obtained [6].

The lattice Feynman rules for overlap fermions, being originally derived in [22, 23], are collected in Appendix A. The regularisation of the infrared divergences follows [24], which was adapted from [25]. The calculations are done analytically as far as possible. To do so, we have extended our *Mathematica* programme package, which we developed originally for Wilson [24] and clover fermions [26], to overlap fermions with improved gauge actions.

We performed several tests to check the code. Operators with gamma structures Γ and $\Gamma\gamma_5$ must give the same result. We have performed calculations independently for both sets of gamma matrices. Although the analytic expressions looked very much different, we found complete agreement. Another condition is that all terms $\propto 1/a$ cancel. Furthermore, it is expected [11] that the gauge dependent parts of the one-loop lattice integrals are independent of the particular choice of fermion propagator (be it Wilson or overlap), as a consequence of the Ward identities. This has been verified numerically and analytically. In Appendix B we give an analytic proof.

3 Renormalisation

To obtain finite answers, the lattice operators $\mathcal{O}(a)$ must, in general, be renormalised. Ignoring operator mixing, we define renormalised operators by

$$\mathcal{O}^{\mathcal{S}}(\mu) = Z^{\mathcal{S}}_{\mathcal{O}}(a,\mu)\mathcal{O}(a)\,, \tag{25}$$

where $\mathcal{S}$ denotes the renormalisation scheme. The renormalisation constants $Z_{\mathcal{O}}$ are often defined in the MOM scheme first by computing the gauge fixed quark propagator S_N and the amputated Green function $\Lambda_{\mathcal{O}}$ of the operator $\mathcal{O}$:

$$Z^{MOM}_\psi(a,\mu)\;S_N\,\Big|_{p^2=\mu^2} \;=\; S^{\rm tree}\,, \tag{26}$$

$$\frac{Z^{MOM}_{\mathcal{O}}(a,\mu)}{Z^{MOM}_\psi(a,\mu)}\;\Lambda_{\mathcal{O}}\,\Big|_{p^2=\mu^2} \;=\; \Lambda^{\rm tree}_{\mathcal{O}} + \text{ other Dirac structures}\,. \tag{27}$$

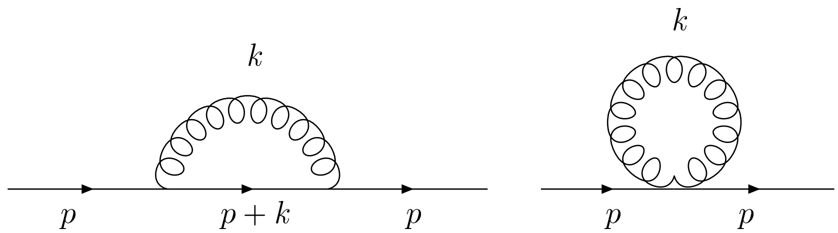

Figure 2: One-loop diagrams contributing to the quark self-energy.

(Note that $Z_\psi = 1/Z_2$.) The renormalisation constants can be converted to the $\overline{MS}$ scheme,

$$Z_{\psi,\mathcal{O}}^{\overline{MS}}(a,\mu) = Z_{\psi,\mathcal{O}}^{\overline{MS},MOM} Z_{\psi,\mathcal{O}}^{MOM}(a,\mu)\,, \tag{28}$$

where $Z_{\psi,\mathcal{O}}^{\overline{MS},MOM}$ is calculable in continuum perturbation theory.

We shall restrict all our numerical calculations to the case $r=1$. The optimal choice for ρ appears to be $\rho \approx 1.4$. We will present results for $\rho = 1.3$, 1.35, 1.4, 1.45 and 1.5, which should cover the most interesting region. Any other value in this region may be obtained by inter- or extrapolation.

Self-energy and wave function renormalisation

Let us consider the massless quark propagator S_N first. The inverse of S_N can be written

$$S_N^{-1} = \mathrm{i}\, \not{p} \left(1 - \frac{g^2\, C_F}{16\pi^2}\, \Sigma_1\right)\,, \tag{29}$$

with $C_F = 4/3$. The diagrams that contribute to Σ_1 to one-loop order are shown in Fig. 2. The integral to be evaluated is [24]

$$\frac{\Sigma_1}{16\pi^2} = \int \frac{d^{\mathrm{D}}k}{(2\pi)^{\mathrm{D}}} \sum_{\mu\nu} \Big[V_{1\,\mu}(p,p+k)\, S_N(p+k)\, V_{1\,\nu}(p+k,p) + V_{2\,\mu\nu}(p,p,k,k)\Big]\, D_{\mu\nu}(k)\,. \tag{30}$$

Explicit expressions for the propagator and vertex functions are given in Appendix A. Putting everything together, we finally obtain

$$\Sigma_1(a,p) = (1-\xi)\,\log(a^2p^2) + 4.79201\,\xi + b_\Sigma\,. \tag{31}$$

The results for b_Σ are given in Table 1 for the gauge field actions and ρ parameters mentioned. The result for the plaquette action ($c_0 = 1$, $c_1 = c_2 = c_3 = 0$) agrees with Refs. [10, 11]. The gauge dependent part of (31) is the same for all actions:

$$(F_0 - \gamma_E + 1)\,\xi = 4.79201\,\xi\,, \tag{32}$$

Action	ρ				
	1.30	1.35	1.40	1.45	1.50
Plaquette	-26.400	-25.031	-23.766	-22.592	-21.501
Symanzik	-21.774	-20.713	-19.732	-18.823	-17.978
TILW, β=8.60	-19.216	-18.329	-17.509	-16.749	-16.044
TILW, β=8.45	-19.124	-18.243	-17.429	-16.675	-15.974
TILW, β=8.30	-18.997	-18.124	-17.319	-16.572	-15.878
TILW, β=8.20	-18.918	-18.051	-17.25	-16.508	-15.819
TILW, β=8.10	-18.817	-17.957	-17.163	-16.426	-15.743
TILW, β=8.00	-18.692	-17.841	-17.054	-16.325	-15.648
Iwasaki	-15.960	-15.295	-14.681	-14.112	-13.584
DBW2	-10.373	-10.096	-9.841	-9.605	-9.386

Table 1: The contribution b_Σ to the self-energy for the various actions and parameters.

with

$$F_0 = 4.3692252338748\,, \quad \gamma_E = 0.57721566\,. \tag{33}$$

We explain the reason for this observation in Appendix B. For the wave function renormalisation constant we then obtain

$$\begin{aligned} Z_\psi^{MOM}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\,\Sigma_1(a,\mu) \\ &= 1 - \frac{g^2 C_F}{16\pi^2}\,\left[2(1-\xi)\log(a\mu) + 4.79201\,\xi + b_\Sigma\right]\,. \end{aligned} \tag{34}$$

In the $\overline{MS}$ scheme this becomes

$$Z_\psi^{\overline{MS}}(a,\mu) = 1 - \frac{g^2 C_F}{16\pi^2}\,\left[2(1-\xi)\log(a\mu) + 3.79201\,\xi + b_\Sigma + 1\right]\,. \tag{35}$$

For Wilson fermions (with $r=1$), on the other hand, we have

$$Z_{\psi,\mathrm{Wilson}}^{\overline{MS}}(a,\mu) = 1 - \frac{g^2 C_F}{16\pi^2}\,\left[2(1-\xi)\log(a\mu) + 3.79201\,\xi + 12.8524\right]\,. \tag{36}$$

We observe that the constant, gauge independent terms in eqs. (35) and (36) have opposite sign. So for $\mu = 1/a$ the overlap Z_ψ is greater than one, while $Z_{\psi,\mathrm{Wilson}}$ is less than one.

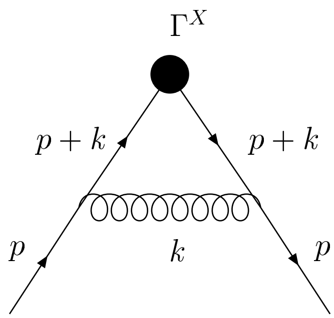

Figure 3: One-loop vertex diagram contributing to the amputated Green functions.

Quark bilinears

Let us consider local operators of the form

$$\mathcal{O}_X = \bar{\psi}(x)\Gamma^X\psi(x) \tag{37}$$

with

$$\Gamma^S = 1\!\!1\,,\ \Gamma^P = \gamma_5\,,\ \Gamma^V = \gamma_\mu\,,\ \Gamma^A = \gamma_\mu\gamma_5\,,\ \Gamma^T = \sigma_{\mu\nu}\gamma_5\,, \tag{38}$$

i.e. $X = S$, P, V, A and T. As the operators are local, operator tadpole and cockscomb diagrams do not contribute [24]. This leaves us to compute the vertex diagram shown in Fig. 3. We denote the amputated Green function of the operator $\mathcal{O}_X$ by Λ^X. Thus we have to evaluate the integral

$$\begin{aligned}\Lambda^X = \Gamma^X + g^2C_F\int\frac{d^{\mathrm{D}}k}{(2\pi)^{\mathrm{D}}}\sum_{\mu\nu}V_{1\,\mu}(p,p+k)\,S_N(p+k)\\ \times\ \Gamma^X\,S_N(p+k)\,V_{1\,\nu}(p+k,k)\,D_{\mu\nu}(k)\,.\end{aligned} \tag{39}$$

The final result is

$$\Lambda^{S,P} = \{1\!\!1,\gamma_5\} + \frac{g^2C_F}{16\pi^2}\left[(\xi-4)\,\log(a^2p^2) - 5.79201\xi + b_{S,P}\right]\,\{1\!\!1,\gamma_5\}\,, \tag{40}$$

$$\begin{aligned}\Lambda^{V,A}_\mu &= \{\gamma_\mu,\gamma_\mu\gamma_5\} + \frac{g^2C_F}{16\pi^2}\Bigg\{\gamma_\mu\left[(\xi-1)\,\log(a^2p^2) - 4.79201\xi + b_{V,A}\right]\\ &\qquad - 2(1-\xi)\,\frac{p_\mu\,\not{p}}{p^2}\Bigg\}\,\{1\!\!1,\gamma_5\}\,,\end{aligned} \tag{41}$$

$$\Lambda^T_{\mu\nu} = \sigma_{\mu\nu}\gamma_5 + \frac{g^2C_F}{16\pi^2}\left[\xi\log(a^2p^2) - 3.79201\xi + b_T\right]\,\sigma_{\mu\nu}\gamma_5\,. \tag{42}$$

Action	ρ				
	1.30	1.35	1.40	1.45	1.50
Plaquette	11.083	11.214	11.343	11.472	11.600
Symanzik	10.621	10.739	10.856	10.972	11.086
TILW, β=8.60	10.275	10.385	10.493	10.599	10.705
TILW, β=8.45	10.261	10.370	10.478	10.584	10.689
TILW, β=8.30	10.241	10.350	10.457	10.563	10.667
TILW, β=8.20	10.229	10.337	10.444	10.549	10.653
TILW, β=8.10	10.213	10.321	10.427	10.532	10.636
TILW, β=8.00	10.192	10.300	10.406	10.510	10.613
Iwasaki	9.641	9.736	9.830	9.922	10.012
DBW2	7.457	7.515	7.571	7.626	7.679

Table 2: The contribution $b_{S,P}$ to $\Lambda^{S,P}$ for various actions and parameters.

Action	ρ				
	1.30	1.35	1.40	1.45	1.50
Plaquette	6.343	6.345	6.348	6.351	6.354
Symanzik	6.332	6.333	6.335	6.338	6.340
TILW, β=8.60	6.325	6.327	6.329	6.330	6.332
TILW, β=8.45	6.325	6.327	6.328	6.330	6.332
TILW, β=8.30	6.325	6.326	6.328	6.330	6.332
TILW, β=8.20	6.325	6.326	6.328	6.330	6.331
TILW, β=8.10	6.324	6.326	6.328	6.329	6.331
TILW, β=8.00	6.324	6.326	6.327	6.329	6.331
Iwasaki	6.316	6.318	6.319	6.320	6.321
DBW2	6.302	6.302	6.303	6.303	6.304

Table 3: The contribution $b_{V,A}$ to $\Lambda^{V,A}$ for various actions and parameters.

Action	ρ				
	1.30	1.35	1.40	1.45	1.50
Plaquette	4.096	4.056	4.016	3.977	3.938
Symanzik	3.958	3.914	3.871	3.828	3.785
TILW, β=8.60	3.851	3.804	3.759	3.713	3.668
TILW, β=8.45	3.846	3.800	3.754	3.708	3.664
TILW, β=8.30	3.840	3.793	3.747	3.702	3.657
TILW, β=8.20	3.836	3.789	3.743	3.698	3.652
TILW, β=8.10	3.831	3.784	3.738	3.692	3.647
TILW, β=8.00	3.825	3.778	3.731	3.686	3.640
Iwasaki	3.651	3.601	3.551	3.501	3.452
DBW2	2.943	2.881	2.819	2.759	2.698

Table 4: The contribution b_T to Λ^T for various actions and parameters.

The finite terms $b_{S,P}$, $b_{V,A}$ and b_T are collected in Tables 2, 3 and 4. For the renormalisation constants we then obtain, using (27) and (34),

$$
\begin{aligned}
Z_{S,P}^{MOM}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left[-6\log(a\mu) - \xi + b_{S,P} + b_\Sigma\right] , && (43)\\
Z_{V,A}^{MOM}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left(b_{V,A} + b_\Sigma\right) , && (44)\\
Z_{T}^{MOM}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left[2\log(a\mu) + \xi + b_T + b_\Sigma\right] . && (45)
\end{aligned}
$$

In the $\overline{MS}$ scheme the renormalisation constants read

$$
\begin{aligned}
Z_{S,P}^{\overline{MS}}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left[-6\log(a\mu) - 5 + b_{S,P} + b_\Sigma\right] , && (46)\\
Z_{V,A}^{\overline{MS}}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left(b_{V,A} + b_\Sigma\right) , && (47)\\
Z_{T}^{\overline{MS}}(a,\mu) &= 1 - \frac{g^2 C_F}{16\pi^2}\left[2\log(a\mu) + 1 + b_T + b_\Sigma\right] . && (48)
\end{aligned}
$$

The conversion factors $Z_{\mathcal{O}}^{\overline{MS},MOM}$ are universal, and are the same as for the plaquette action and Wilson fermions [24].

4 Tadpole improvement

The appearance of gluon tadpoles in lattice perturbation theory make the bare coupling constant g into a poor expansion parameter. It was proposed [27] that the perturbative series should be rearranged in order to get rid of these contributions. Tadpole improvement is a technique for summing, to all orders, the numerically large perturbative contributions arising from tadpole diagrams. It is implemented by a mean field renormalisation of the link matrices, $U_{x,\mu} \to U_{x,\mu}/u_0$, where u_0 is the mean value of the link, defined to be the fourth root of the expectation value of the plaquette 'measured' in Monte Carlo simulations, as given in eq. (18). In this Section we apply tadpole improvement, better called mean field improvement, to our results, and to operators involving overlap fermions in general.

Our situation is more complicated than the standard case for two reasons: firstly we are using overlap fermions rather than Wilson or clover fermions, and secondly we are using gauge actions which are more complicated than the basic plaquette action. Both these changes require some thought.

First, let us run briefly through the procedure for the case of Wilson fermions. Tadpole improved renormalisation constants are defined by

$$Z_{\mathcal{O}}^{TI} = Z_{\mathcal{O}}^{MF} \left(\frac{Z_{\mathcal{O}}}{Z_{\mathcal{O}}^{MF}} \right)_{\text{pert}} , \tag{49}$$

where $Z_{\mathcal{O}}^{MF}$ is the mean field approximation of $Z_{\mathcal{O}}$, while the second factor on the r.h.s. is computed in perturbation theory. We find $Z_{\mathcal{O},\text{Wilson}}^{MF}$ by looking at the operator Green function and quark propagator in the mean field approximation. The mean field result for the amputated Green function for an operator with n_D covariant derivatives is

$$\Lambda_{\mathcal{O}}^{MF} = u_0^{n_D} \Lambda_{\mathcal{O}}^{\text{tree}} . \tag{50}$$

This does not depend on the choice of fermion action. In the mean field approximation the inverse quark propagator for massless Wilson fermions is

$$(S_{Wilson}^{MF})^{-1} = \mathrm{i}\, \not{p}\, u_0 + O(a) , \tag{51}$$

giving

$$Z_{\psi,Wilson}^{MF} = u_0 . \tag{52}$$

Combining eqs. (50) and (52) gives the familiar result

$$Z_{\mathcal{O},\text{Wilson}}^{MF} = u_0^{1-n_D} . \tag{53}$$

The result changes when we consider overlap fermions. The mean field result for the amputated Green functions, eq. (50), is unchanged. However, Z_ψ for overlap fermions is no longer given by (52), which will lead to changes in $Z_\mathcal{O}$ as well. To find Z_ψ we need to calculate the overlap operator in the mean field approximation. The mean field result for the Wilson-Dirac operator in momentum space is

$$\begin{aligned} D_W^{MF} &= \frac{1}{a}\left(u_0 \sum_\mu \mathrm{i}\,\gamma_\mu \sin ap_\mu + 4r - u_0\, r \sum_\mu \cos ap_\mu \right) \\ &= \mathrm{i}\,\not{p}\, u_0 + \frac{4r}{a}(1-u_0) + \frac{r}{2} a p^2 u_0 + O(a^2)\,. \end{aligned} \tag{54}$$

Inserting this result into the Neuberger-Dirac operator (1) gives

$$D_N^{MF} = \frac{\rho\, u_0}{\rho - 4r(1-u_0)} \left[\mathrm{i}\,\not{p} + \frac{1}{2}\, a\, p^2\, \frac{u_0}{\rho - 4r(1-u_0)} + O(a^2) \right] . \tag{55}$$

We can now compute Z_ψ^{MF}:

$$Z_\psi^{MF} = \frac{\rho\, u_0}{\rho - 4r(1-u_0)}\,. \tag{56}$$

If we combine this result with eq. (50), we finally obtain

$$Z_\mathcal{O}^{MF} = \frac{\rho\, u_0^{1-n_D}}{\rho - 4r(1-u_0)}\,. \tag{57}$$

It is required that $\rho > 4r(1-u_0)$. The reason for this inequality is easy to understand. Reverting temporarily to writing the fermion matrix in terms of a hopping parameter κ, the ρ definition in eq. (1) is equivalent to $\rho = 4r - 1/(2\kappa)$. To really have a negative mass in X, so that the overlap procedure works, requires $\kappa > \kappa_c$, which implies $\rho > 4r - 1/(2\kappa_c)$. In the mean field approximation $\kappa_c = 1/(8ru_0)$, leading to the inequality $\rho > 4r(1-u_0)$. In other words, the additive renormaliation of mass, which occurs with Wilson fermions, means that in the interacting case $\rho > 0$ is not enough to cause X to have a negative mass. We need ρ to be large enough to overcome this additional mass.

Note that Z_ψ^{MF} will be larger than 1 for reasonable values of ρ, while in the Wilson case $Z_{\psi,Wilson}^{MF} < 1$. Our mean field calculation gives an explanation of the observation in Section 3, that the one-loop b_Σ's have the opposite sign for overlap fermions than for Wilson fermions.

Let us now turn to the calculation of the second, perturbative factor on the r.h.s. of eq. (49). Having removed the large tadpole contributions, we need to re-express the

Action	β	k_u	k_u^{TI}	u_0^4
Plaquette	6.0	π^2	π^2	0.59368
Symanzik		$0.732524\,\pi^2$		
TILW	8.60	$0.590078\,\pi^2$	$0.549643\,\pi^2$	0.66018
TILW	8.45	$0.585039\,\pi^2$	$0.543338\,\pi^2$	0.65176
TILW	8.30	$0.578127\,\pi^2$	$0.534971\,\pi^2$	0.64252
TILW	8.20	$0.573860\,\pi^2$	$0.529705\,\pi^2$	0.63599
TILW	8.10	$0.568378\,\pi^2$	$0.523106\,\pi^2$	0.62894
TILW	8.00	$0.561610\,\pi^2$	$0.515069\,\pi^2$	0.62107
Iwasaki	9.485	$0.420531\,\pi^2$	$0.379397\,\pi^2$	0.67066
Iwasaki	8.026	$0.420531\,\pi^2$	$0.370379\,\pi^2$	0.59561
DBW2	12.745	$0.153264\,\pi^2$	$0.131857\,\pi^2$	0.72759
DBW2	10.662	$0.153264\,\pi^2$	$0.128275\,\pi^2$	0.67332

Table 5: The parameters k_u, k_u^{TI} and the average plaquette u_0^4 for various actions and β values. See eq. (5) for the definition of β. The plaquette values for the TILW, Iwasaki and DBW2 actions are taken from [20], [28] and [29], respectively.

perturbative series in terms of the tadpole improved parameters, which must satisfy

$$\frac{c_0^{TI}}{g_{TI}^2} = u_0^4\,\frac{c_0}{g^2}\,, \tag{58}$$

$$\frac{c_i^{TI}}{g_{TI}^2} = u_0^6\,\frac{c_i}{g^2}\,, \quad i = 1,2,3\,, \tag{59}$$

because the plaquette term in the action is a four-link term, while the other three terms in the action are six-link objects. These conditions are not enough to fully fix the tadpole improved parameters, because we have four equations with five unknowns, leaving us with some freedom of choice. The simplest choice is to define

$$g_{TI}^2 = \frac{g^2}{u_0^4}\,, \tag{60}$$

$$c_0^{TI} = c_0 \tag{61}$$

$$c_i^{TI} = u_0^2\,c_i\,, \quad i = 1,2,3\,. \tag{62}$$

However, it is important to notice that, although this definition makes the formulae for the tadpole improved parameters very simple, with this choice

$$C_0^{TI} \equiv c_0^{TI} + 8c_1^{TI} + 16c_2^{TI} + 8c_3^{TI} \neq 1 \,, \tag{63}$$

which modifies many of the formulae we wrote down in earlier sections. This means, we have to replace every g^2 by g_{TI}^2 and every c_1, c_2 and c_3 by c_1^{TI}, c_2^{TI} and c_3^{TI}, respectively, while keeping c_0 unchanged. The effect of introducing tadpole improved coefficients (62) is that the rescaled gluon propagator remains of the same form as we change u_0, thus ensuring fast convergence.

For the perturbative calculation of $Z_{\mathcal{O}}^{MF}$ we need to know the perturbative expansion of u_0 [27, 19]:

$$u_0 = 1 - \frac{g_{TI}^2 C_F}{16\pi^2}\, k_u^{TI} \,, \tag{64}$$

where k_u^{TI} is given by the integral

$$k_u^{TI} = 4\pi^2 a^4 \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \left[\hat{k}_4^2\, D_{11}(k, C^{TI}) - \hat{k}_1 \hat{k}_4\, D_{14}(k, C^{TI})\right] . \tag{65}$$

Here $D_{11}(k, C^{TI})$ and $D_{14}(k, C^{TI})$ are components of the gluon propagator (13), with $C_0 = 1$, $C_1 = c_2 + c_3$ and $C_2 = c_1 - c_2 - c_3$ being replaced by $C_0^{TI} = c_0 + 8c_1^{TI} + 16c_2^{TI} + 8c_3^{TI}$, $C_1^{TI} = u_0^2\, C_1$ and $C_2^{TI} = u_0^2\, C_2$, respectively. The numerical values of k_u^{TI} are given in Table 5 for the various actions. For comparison, we also give the non-tadpole improved result (i.e. with coefficients C_1 and C_2), which we call k_u. The 'measured' values of u_0^4 are also collected in Table 5. We then obtain

$$Z_{\mathcal{O}\,\mathrm{pert}}^{MF} = 1 - \frac{g_{TI}^2 C_F}{16\pi^2} \left(1 - n_D - \frac{4r}{\rho}\right) k_u^{TI} \,. \tag{66}$$

Let us now introduce constants $B_{\mathcal{O}}(\rho, C)$ by writing the original one-loop renormalisation constants of Section 3 as

$$Z_{\mathcal{O}} = 1 - \frac{C_F\, g^2}{16\pi^2} \left[\gamma_{\mathcal{O}} \log(a\mu) + B_{\mathcal{O}}(\rho, C)\right] \,, \tag{67}$$

where $\gamma_{\mathcal{O}}$ is the anomalous dimension of $\mathcal{O}$. We then obtain tadpole improved renormalisation constants for $r = 1$:

$$Z_{\mathcal{O}}^{TI} = \frac{\rho\, u_0^{1-n_D}}{\rho - 4(1 - u_0)} \left\{1 - \frac{g_{TI}^2 C_F}{16\pi^2} \left[\frac{\gamma_{\mathcal{O}}}{C_0^{TI}} \log(a\mu) + B_{\mathcal{O}}(\rho, C^{TI}) - \left(1 - n_D - \frac{4}{\rho}\right) k_u^{TI}\right]\right\} \,, \tag{68}$$

where $B_{\mathcal{O}}(\rho, C^{TI})$ is the analogue of $B_{\mathcal{O}}(\rho, C)$, with C_0, C_1 and C_2 being replaced by C_0^{TI}, C_1^{TI} and C_2^{TI}, respectively. In the following we shall use the abbreviation

$$B_{\mathcal{O}}^{TI} = B_{\mathcal{O}}(\rho, C^{TI}) - \left(1 - n_D - \frac{4}{\rho}\right) k_u^{TI} \,, \tag{69}$$

giving

$$Z_{\mathcal{O}}^{TI} = \frac{\rho\, u_0^{1-n_D}}{\rho - 4(1-u_0)} \left\{ 1 - \frac{g_{TI}^2 C_F}{16\pi^2} \left[\frac{\gamma_{\mathcal{O}}}{C_0^{TI}} \log(a\mu) + B_{\mathcal{O}}^{TI} \right] \right\} . \tag{70}$$

So far we have only tadpole improved the gluon propagator in our calculation of the perturbative factor on the r.h.s. of eq. (49), but not the fermion propagator. There is no need to do so for Wilson fermions. However, already for clover fermions we have seen [26] that the fermion propagator ought to be improved as well. If we want the rescaled fermion propagator (see eq. (55))

$$\left[\mathrm{i}\, \not{p} + \frac{1}{2}\, a\, p^2\, \frac{u_0}{\rho - 4r(1-u_0)} \right]^{-1} \tag{71}$$

to have the same form as we change u_0, we must replace ρ by

$$\rho^{TI} = \frac{\rho - 4(1-u_0)}{u_0} \tag{72}$$

for $r = 1$. An alternative derivation of ρ^{TI}, without expanding as a power series in a, is given in Appendix C. This defines 'fully tadpole improved' renormalisation constants

$$Z_{\mathcal{O}}^{FTI} = \frac{\rho\, u_0^{1-n_D}}{\rho - 4(1-u_0)} \left\{ 1 - \frac{g_{TI}^2 C_F}{16\pi^2} \left[\frac{\gamma_{\mathcal{O}}}{C_0^{TI}} \log(a\mu) + B_{\mathcal{O}}^{FTI} \right] \right\} , \tag{73}$$

with

$$B_{\mathcal{O}}^{FTI} = B_{\mathcal{O}}(\rho^{TI}, C^{TI}) - \left(1 - n_D - \frac{4}{\rho^{TI}} \right) k_u^{TI} . \tag{74}$$

Before we present numbers for $Z_{\mathcal{O}}^{TI}$ and $Z_{\mathcal{O}}^{FTI}$, let us make a few general remarks concerning the choice of gauge action. Only if g_{TI} is a 'good' expansion parameter, can we expect the perturbative series to converge fast. It is generally accepted that $g_{\overline{MS}}(\mu)$ is a good expansion parameter for appropriate choices of μ. To one-loop order we have

$$\frac{1}{g_{\overline{MS}}^2(\mu)} - \frac{1}{g^2(a)} = 2b_0 \left(\log \frac{\mu}{\Lambda_{\overline{MS}}} - \log \frac{1}{a\Lambda_{\mathrm{lat}}} \right) = 2b_0 \log(a\mu) + d_g + N_f\, d_f , \tag{75}$$

where $b_0 = (11 - 2/3 N_f)/(4\pi)^2$, and N_f is the number of flavours. (It is appropriate to consider the case of general N_f here.) The ratio of Λ parameters is thus given by

$$\frac{\Lambda_{\mathrm{lat}}}{\Lambda_{\overline{MS}}} = \exp\left(\frac{d_g + d_f}{2b_0} \right) . \tag{76}$$

Upon inserting (60) and (64), we obtain

$$\frac{1}{g_{\overline{MS}}^2(\mu)} - \frac{1}{g_{TI}^2(a)} = 2b_0 \left(\log \frac{\mu}{\Lambda_{\overline{MS}}} - \log \frac{1}{a\Lambda_{\mathrm{lat}}^{TI}} \right) = 2b_0 \log(a\mu) + d_g + N_f\, d_f + \frac{k_u^{TI}}{3\pi^2} , \tag{77}$$

Action	$\Lambda_{\rm lat}/\Lambda_{\overline{MS}}$		$\Lambda^{TI}_{\rm lat}/\Lambda_{\overline{MS}}$	
	$N_f=0$	$N_f=2$	$N_f=0$	$N_f=2$
Plaquette	0.0347	0.0172	0.380	0.262
Symanzik	0.184	0.115	1.06	0.843
Iwasaki	2.13	1.86	5.82	5.86
DBW2	45.4	60.7	65.6	92.2

Table 6: Ratios of Λ parameters.

giving

$$\frac{\Lambda^{TI}_{\rm lat}}{\Lambda_{\overline{MS}}} = \exp\left(\frac{d_g + N_f\, d_f + k_u^{TI}/6\pi^2}{2b_0}\right) . \tag{78}$$

The coefficient d_g is known for some of our actions [30]:

$$\begin{array}{c|c} \text{Action} & d_g \\ \hline \text{Plaquette} & -0.4682 \\ \text{Symanzik} & -0.2361 \\ \text{Iwasaki} & 0.1053 \\ \text{DBW2} & 0.5317 \end{array} \tag{79}$$

The coefficient d_f is known for $\rho = 1.4$ [31]:

$$d_f = -0.01449\,. \tag{80}$$

Unfortunately, d_g is not known for the Lüscher-Weisz action, nor is it known for tadpole improved coefficients (61) and (62). We expect the numbers for the Lüscher-Weisz action to be close to the result for the Symanzik action though. In Table 6 we have computed the ratios of Λ parameters $\Lambda_{\rm lat}/\Lambda_{\overline{MS}}$ and $\Lambda^{TI}_{\rm lat}/\Lambda_{\overline{MS}}$ for coefficients (79) and (80) and, to be consistent, with k_u^{TI} replaced by k_u. We see that tadpole improvement drives $\Lambda_{\rm lat}$ towards $\Lambda_{\overline{MS}}$ for the Symanzik action, giving $g_{TI} \approx g_{\overline{MS}}$, so that g_{TI} appears to be a good expansion parameter. This is not the case for the Iwasaki action, and even less so for the DBW2 action. We thus may conclude that the Symanzik action, and possibly the Lüscher-Weisz action as well, is a suitable lattice gauge field action not only from the non-perturbative perspective, but also from the perturbative point of view, if supplemented with tadpole improvement.

Action	β	$B_{S,P}$	$Z_{S,P}^{\overline{MS}}$	$B_{S,P}^{TI}$	$Z_{S,P}^{TI,\overline{MS}}$	$B_{S,P}^{FTI}$	$Z_{S,P}^{FTI,\overline{MS}}$
Symanzik	8.45	−13.876	1.139				
TILW	8.60	−12.016	1.164	−1.554	1.295	−3.329	1.341
TILW	8.45	−11.951	1.168	−1.593	1.309	−3.421	1.359
TILW3	8.30	−11.862	1.173	−1.644	1.325	−3.527	1.380
TILW	8.20	−11.806	1.176	−1.677	1.337	−3.601	1.395
TILW	8.10	−11.736	1.180	−1.717	1.350	−3.683	1.412
TILW	8.00	−11.648	1.184	−1.766	1.366	−3.778	1.432
Iwsasaki	9.485	−9.851	1.192	−2.527	1.334	−3.851	1.381
Iwsasaki	8.026	−9.851	1.227	−2.610	1.481	−4.379	1.573
DBW2	12.745	−7.270	1.354	−4.095	1.506	−4.543	1.541
DBW2	10.662	−7.270	1.423	−4.122	1.681	−4.666	1.739

Table 7: The constants $B_{S,P}$ and $Z_{S,P}^{\overline{MS}}(a\mu = 1)$ for various levels of improvement.

Action	β	$B_{V,A}$	$Z_{V,A}^{\overline{MS}}$	$B_{V,A}^{TI}$	$Z_{V,A}^{TI,\overline{MS}}$	$B_{V,A}^{FTI}$	$Z_{V,A}^{FTI,\overline{MS}}$
Symanzik	8.45	−13.397	1.134				
TILW	8.60	−11.180	1.153	−0.099	1.258	−1.342	1.290
TILW	8.45	−11.100	1.156	−0.101	1.268	−1.384	1.303
TILW3	8.30	−10.990	1.160	−0.105	1.280	−1.431	1.319
TILW	8.20	−10.922	1.163	−0.106	1.289	−1.464	1.330
TILW	8.10	−10.835	1.166	−0.109	1.299	−1.410	1.343
TILW	8.00	−10.727	1.169	−0.112	1.311	−1.539	1.358
Iwsasaki	9.485	−8.362	1.163	−0.271	1.252	−1.196	1.285
Iwsasaki	8.026	−8.362	1.193	−0.204	1.356	−1.469	1.422
DBW2	12.745	−3.538	1.172	−0.277	1.204	−0.582	1.228
DBW2	10.662	−3.538	1.206	−0.230	1.264	−0.607	1.304

Table 8: The same as Table 7, but for $B_{V,A}$ and $Z_{V,A}^{\overline{MS}}(a\mu = 1)$.

Action	β	B_T	$Z_T^{\overline{MS}}$	B_T^{TI}	$Z_T^{TI,\overline{MS}}$	B_T^{FTI}	$Z_T^{FTI,\overline{MS}}$
Symanzik	8.45	-14.861	1.148				
TILW	8.60	-12.750	1.174	-1.100	1.283	-2.128	1.310
TILW	8.45	-12.675	1.178	-1.082	1.295	-2.146	1.324
TILW3	8.30	-12.571	1.183	-1.062	1.308	-2.163	1.340
TILW	8.20	-12.507	1.186	-1.048	1.318	-2.177	1.352
TILW	8.10	-12.424	1.190	-1.032	1.328	-2.189	1.365
TILW	8.00	-12.322	1.194	-1.012	1.341	-2.202	1.380
Iwsasaki	9.485	-10.130	1.197	-1.029	1.279	-1.773	1.306
Iwsasaki	8.026	-10.130	1.233	-0.806	1.388	-1.851	1.442
DBW2	12.745	-6.021	1.293	-0.364	1.210	-0.581	1.228
DBW2	10.662	-6.021	1.351	-0.164	1.257	-0.446	1.287

Table 9: The same as Table 7, but for B_T and $Z_T^{\overline{MS}}(a\mu = 1)$.

In Tables 7, 8 and 9 we compare tadpole improved and unimproved renormalisation constants for $\rho = 1.4$ and some selected values of β, which are widely being used in Monte Carlo simulations. The DBW2 couplings $\beta = 10.662$ and 12.745 correspond to $\beta = 0.87$ and 1.04, respectively, in the notation of [32]. The 'measured' and perturbative plaquette values are taken from Table 5. The tadpole improved renormalisation constants given earlier in [8] referred to unimproved coefficients c_i and unimproved parameter ρ.

How good is the mean field approximation and tadpole improvement? The ultimate test is to compare with non-perturbative determinations of Z, which at present we can only do in a single case (to be discussed later). However, there is also a powerful internal test, which we can make with our perturbative coefficients. If the mean field Z^{MF} of eqs. (56) and (57) is a good approximation, then we should always find that the one-loop coefficient of Z^{MF} is very close to the one-loop coefficient of Z_ψ or Z_O, i.e. that

$$B_O^{sub}(\rho, C) \equiv B_O(\rho, C) - \left(1 - n_D - \frac{4}{\rho}\right) k_u \tag{81}$$

is small, whatever gauge action or ρ we use. We test this in Fig. 4, where we compare the original values of B_V with the subtracted ones defined in (81). The original coefficients B_V have large negative values, and depend strongly on the choice of gauge action and on the value of ρ. The subtracted coefficients B_V^{sub} are much closer to zero, and depend only weakly on gauge action or ρ. So we see that the mean field approximation of eqs. (56)

and (57) is very good at the one-loop level. If we had naively used the Wilson fermion result (53), this test would have failed completely, not even having the right sign. In Fig. 5 we show the subtracted coefficients in more detail. We can see that they all lie

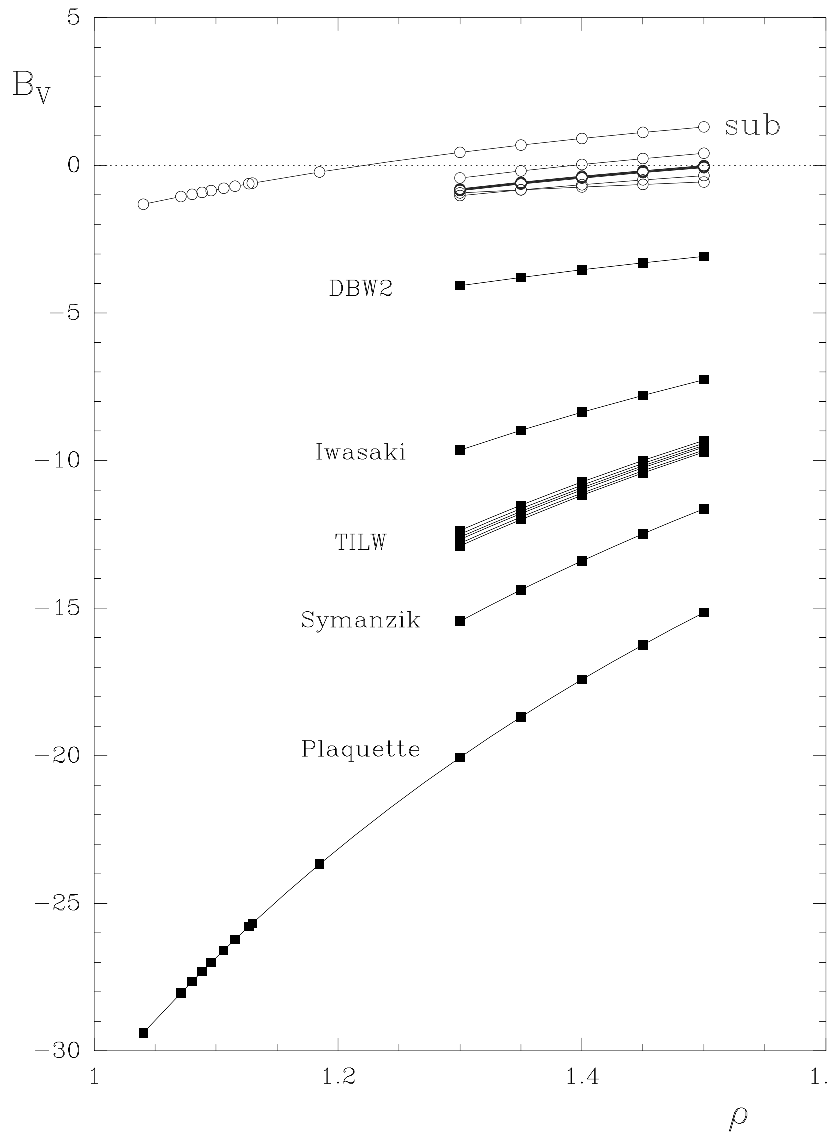

Figure 4: A test of the mean field approximation. We compare the perturbative B_V (filled squares), as defined in (67), with the subtracted B_V^{sub} (open circles) defined in (81). The subtracted coefficients are much smaller, showing that the mean field approximation is very good at the one-loop level.

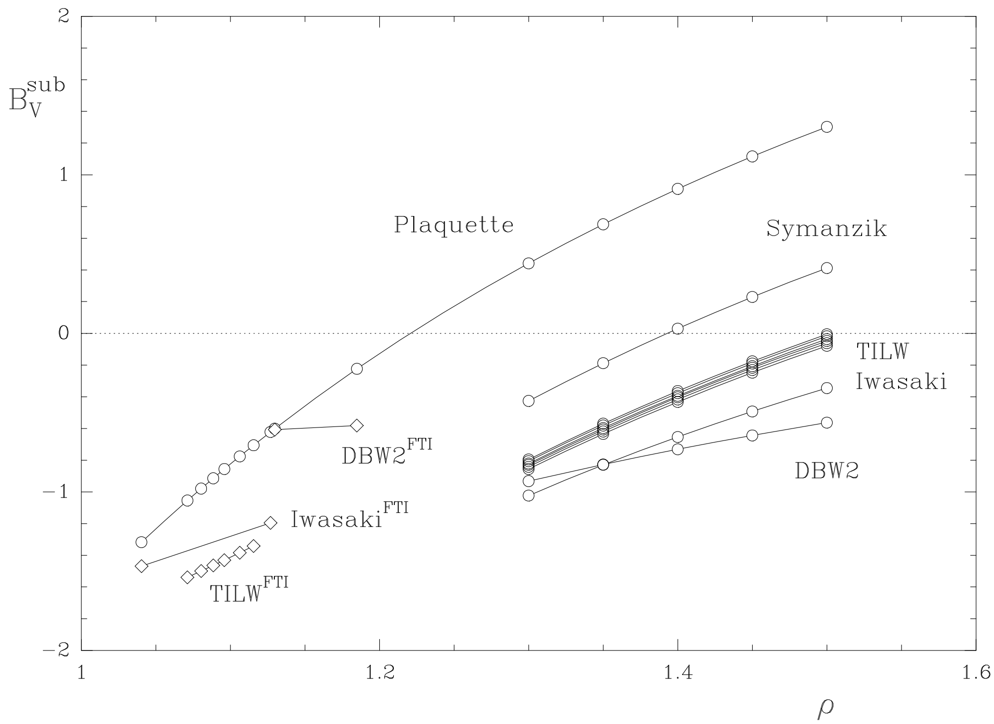

Figure 5: The subtracted points from Fig. 4 shown in more detail. The open circles show results calculated with unimproved gauge actions, the open diamonds with gauge actions modified according to eqs. (61), (62) and (72). All actions considered give very similar results.

in a fairly narrow band, with points calculated from the original actions (solid points) and the tadpole improved actions (open points) in agreement. This shows how well the dependence of Z on gauge action and ρ is described by the mean field approximation, leaving only a small residue to be described by perturbation theory.

As far as we can tell from the single non-perturbative result we have found for the TILW action at $\beta = 8.45$ [8]), corresponding to a lattice spacing of $a \approx 0.1$ fm, tadpole improvement drives the perturbative number closer to the non-perturbative value. But the discrepancy is still of the order of 10%.

An alternative improvement scheme has been proposed in [33], in which $Z_{\mathcal{O}}^{MF}$ is replaced by the non-perturbatively computed renormalisation constant of the local vector current Z_V^{NP}:

$$Z_{\mathcal{O}}^{VI} = Z_V^{NP} \left(\frac{Z_{\mathcal{O}}}{Z_V} \right)_{\rm pert} . \tag{82}$$

$Z_{S,P}^{VI,\overline{MS}}$	$Z_{V,A}^{NP}$	$Z_{T}^{VI,\overline{MS}}$
1.478	1.416	1.439

Table 10: The renormalisation constants $Z_{S,P}^{VI,\overline{MS}}(a\mu = 1)$ and $Z_{T}^{VI,\overline{MS}}(a\mu = 1)$ for the TILW action at $\beta = 8.45$. The value of $Z_{V,A}^{NP}$ was used as input.

This gives

$$\begin{aligned} Z_{\mathcal{O}}^{VI} &= Z_V^{NP}\left\{1 - \frac{C_F g_{TI}^2}{16\pi^2}\left[\frac{\gamma_{\mathcal{O}}}{C_0^{TI}}\log(a\mu) + B_{\mathcal{O}}^{FTI} - B_V^{FTI}\right]\right\} \\ &\equiv Z_V^{NP}\left\{1 - \frac{C_F g_{TI}^2}{16\pi^2}\left[\frac{\gamma_{\mathcal{O}}}{C_0^{TI}}\log(a\mu) + B_{\mathcal{O}}^{VI}\right]\right\} . \end{aligned} \tag{83}$$

For the TILW action at $\beta = 8.45$ and $\rho = 1.4$ we found $Z_V^{NP} (= Z_A^{NP}) = 1.416$ [8]. We used this number to compute $Z_{\mathcal{O}}^{VI}$ in Table 10. This method is not applicable to operators with $n_D > 0$ covariant derivatives. In that case one would have to replace the local vector current by an operator with an equal number (i.e. n_D) of covariant derivatives.

5 Summary

We have computed the renormalisation constants of local quark bilinear operators for overlap fermions and improved gauge field actions with up to six links. The computations have been performed in general covariant gauge using the symbolic language *Mathematica*. This gave us complete control over the Lorentz and spin structure, the cancellation of infrared divergences, as well as the cancellation of $1/a$ singularities. The price to pay is high. The extension to improved gauge field actions blew up the calculation tremendously. In some instances we had to deal with $O(10^4)$ terms. Based on generalised lattice Ward identities we were able to show analytically that the gauge dependent part of the self-energy and the amputated Green functions does not depend on the choice of lattice fermions.

In the limit $c_0 = 1$, $c_1 = c_2 = c_3 = 0$ our results agree with previous calculations employing the plaquette action [10, 11]. Comparing overlap with Wilson fermions, we notice that the one-loop corrections have opposite sign. This is mainly caused by changes in the quark self-energy, and can be understood through a mean field calculation.

We have formulated mean field (tadpole) improvement for overlap fermions. For the

Symanzik action the boosted coupling g_{TI} turns out to be close to $g_{\overline{MS}}$, which makes g_{TI} a good expansion parameter. We thus may expect that the perturbative series converges rapidly. For the Iwasaki action, and in particular for the DBW2 action, this appears not to be true. In this case boosted perturbation theory might even worsen the situation.

We have presented results for a variety of parameters and couplings, which cover most of the parameter values used in recent Monte Carlo simulations. We are happy to supply numbers for different choices of parameters on request. Details of our results for operators with covariant derivatives [8] will be given elsewhere.

Acknowledgements

This work is supported by DFG under contract FOR 465 (Forschergruppe Gitter-Hadronen-Phänomenologie), which we gratefully acknowledge. We thank Tom Blum for providing us with the plaquette values for the DBW2 action. The authors would like to thank Haris Panagopoulos for pointing out the misprints in Appendix A.

Appendix A

We set $a = 1$ throughout this Appendix. Let us first consider the gluon propagator. We introduce the abbreviation

$$\hat{k}^n = \sum_{\mu=1}^{4} \hat{k}_\mu^n \,, \quad n = 2, 4, \cdots . \tag{84}$$

The coefficients D_n and $D_{m,n}$ in eq. (14) can be written in the form

$$D_n = \frac{d_n}{D\,(\hat{k}^2)^2}, \quad D_{m,n} = \frac{d_{m,n}}{D\,(\hat{k}^2)^2}\,, \tag{85}$$

where $d_{m,n} = d_{n,m}$. For D, in the case $C_0 = 1$, we find

$$\begin{aligned} D = {} & (1 - C_1\,\hat{k}^2)^3\,(\hat{k}^2)^2 - C_2\,(1 - C_1\,\hat{k}^2)^2\,\hat{k}^2\left[(\hat{k}^2)^2 + 2\,\hat{k}^4\right] \\ & + \frac{1}{2}\,C_2^2\,(1 - C_1\,\hat{k}^2)\left[(\hat{k}^2)^4 + (\hat{k}^2)^2\,\hat{k}^4 + 2\,(\hat{k}^4)^2 + 2\,\hat{k}^2\,\hat{k}^6\right] \\ & + \frac{1}{6}\,C_2^3\left[24\,\hat{k}^{10} - 7\,(\hat{k}^2)^3\,\hat{k}^4 + 15\,\hat{k}^2\,(\hat{k}^4)^2 + 12\,(\hat{k}^2)^2\,\hat{k}^6\right. \\ & \left. - 26\,\hat{k}^4\,\hat{k}^6 - 24\,\hat{k}^2\,\hat{k}^8\right], \end{aligned} \tag{86}$$

and for d_n and $d_{m,n}$ we obtain

$$
\begin{aligned}
d_0 \;&= -D\,\hat{k}^2 + (1 - C_1\,\hat{k}^2)^2\,(\hat{k}^2)^3 - C_2\,(1 - C_1\,\hat{k}^2)\,\Big[7\,(\hat{k}^2)^2\hat{k}^4 \\
&\quad - 3\,(\hat{k}^4)^2 - 8\,\hat{k}^2\,\hat{k}^6 + 6\,\hat{k}^8\Big] + \frac{1}{2}\,C_2^2\,\Big[30\,\hat{k}^{10} + 9\,(\hat{k}^2)^2\,\hat{k}^6 \\
&\quad - 25\,\hat{k}^4\,\hat{k}^6 + 12\,\hat{k}^2\,((\hat{k}^4)^2 - 2\,\hat{k}^8)\Big]\,, \\
d_1 \;&= C_2\,(1 - C_1\,\hat{k}^2)\,\Big[-2\,(\hat{k}^2)^3 + 9\,\hat{k}^2\,\hat{k}^4 - 6\,\hat{k}^6\Big] \\
&\quad + \;C_2^2\,\Big[4\,(\hat{k}^2)^2\hat{k}^4 - 7\,(k^4)^2 - 12\,\hat{k}^2\,\hat{k}^6 + 14\,\hat{k}^8\Big]\,, \\
d_2 \;&= 6\,C_2\,(1 - C_1\,\hat{k}^2)\,\Big[(\hat{k}^2)^2 - \hat{k}^4\Big] + \frac{1}{2}C_2^2\,\Big[3\,(\hat{k}^2)^3 - 29\,\hat{k}^2\,\hat{k}^4 \\
&\quad + 28\,\hat{k}^6\Big]\,, \\
d_3 \;&= -6\,C_2\,(1 - C_1\,\hat{k}^2)\,\hat{k}^2 - 2\,C_2^2\,\Big[4(\hat{k}^2)^2 - 7\,\hat{k}^4\Big]\,, \\
d_4 \;&= 15\,C_2^2\,\hat{k}^2\,, \\
d_{0,0} &= D - (1 - C_1\,\hat{k}^2)^2(\hat{k}^2)^2 + C_2\,(1 - C_1\,\hat{k}^2)\,\Big[5\,\hat{k}^2\,\hat{k}^4 - 2\,\hat{k}^6\Big] \\
&\quad + C_2^2\,\Big[-3(\hat{k}^4)^2 - 3\,\hat{k}^2\,\hat{k}^6 + 4\,\hat{k}^8\Big]\,, \\
d_{1,1} &= -6\,C_2\,(1 - C_1\,\hat{k}^2)\,\hat{k}^2 - C_2^2\,\Big[(\hat{k}^2)^2 - 7\,\hat{k}^4\Big]\,, \\
d_{2,2} &= d_{1,3} \;\; = -14\,C_2^2\,, \\
d_{0,1} &= C_2\,(1 - C_1\,\hat{k}^2)\,\Big[2\,(\hat{k}^2)^2 - 3\,\hat{k}^4\Big] - C_2^2\,\Big[3\,\hat{k}^2\,\hat{k}^4 - 5\,\hat{k}^6\Big]\,, \\
d_{0,2} &= -6\,C_2\,(1 - C_1\,\hat{k}^2)\,\hat{k}^2 - \frac{3}{2}\,C_2^2\,\Big[(\hat{k}^2)^2 - 5\,\hat{k}^4\Big]\,, \\
d_{0,3} &= 6\,C_2\,(1 - C_1\,\hat{k}^2)\; + 8\,C_2^2\,\hat{k}^2\,, \\
d_{0,4} &= -15\,C_2^2\,, \\
d_{1,2} &= 6\,C_2\,(1 - C_1\,\hat{k}^2)\; + 7\,C_2^2\,\hat{k}^2\,.
\end{aligned}
\tag{87}
$$

(The coefficient $d_{m,n}$ should not be confused with $d_{\mu\nu}$ of eq. (9)). Note that D_n and $D_{m,n}$ do not depend on the choice of (covariant) gauge, i.e. on ξ. Both D_n and $D_{m,n}$ vanish in the limit $C_1 = C_2 = 0$.

Let us now turn to the lattice Feynman rules. We omit the colour factors and the gauge coupling here. We introduce the following abbreviations:

$$\omega(p) \;=\; \omega(-p) \;=\; \sqrt{\sum_\mu \sin^2 p_\mu + b(p)^2}\,, \quad b(p) = b(-p) = \frac{r}{2}\,\hat{p}^2 - \rho\,, \tag{88}$$

$$\omega(p_2,p_1) \;=\; \sqrt{\sum_\mu \sin p_{2\,\mu} \sin p_{1\,\mu} + b(p_2)b(p_1)}\,, \tag{89}$$

$$X_0(p) \;=\; \mathrm{i}\sum_\mu \gamma_\mu \sin p_\mu + b(p)\,, \quad X_0^\dagger(p)X_0(p) = \omega(p)^2\,. \tag{90}$$

The Wilson quark propagator and its inverse are of the form

$$S_W(p) = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin p_\mu + \frac{r}{2}\,\hat{p}^2}{\sum_\mu \sin^2 p_\mu + \left(\frac{r}{2}\,\hat{p}^2\right)^2}\,, \quad S_W^{-1}(p) = \mathrm{i}\sum_\mu \gamma_\mu \sin p_\mu + \frac{r}{2}\,\hat{p}^2 = X_0(p) + \rho\,. \tag{91}$$

The Wilson quark-quark-gluon (qqg) and quark-quark-gluon-gluon (qqgg) vertices are given by

$$V^W_{1\,\mu}(p_2,p_1) \;=\; -\mathrm{i}\,\gamma_\mu \cos\frac{(p_2+p_1)_\mu}{2} - r\sin\frac{(p_2+p_1)_\mu}{2}\,, \tag{92}$$

$$V^W_{2\,\mu\nu}(p_2,p_1) \;=\; -\frac{1}{2}\delta_{\mu\nu}\left[-\mathrm{i}\,\gamma_\mu \sin\frac{(p_2+p_1)_\mu}{2} + r\cos\frac{(p_2+p_1)_\mu}{2}\right]\,, \tag{93}$$

with incoming and outgoing quark momenta being denoted by p_1 and p_2, respectively. Note that

$$X_0(p) + \rho = S_W^{-1}(p) = -\sum_\mu \hat{p}_\mu V^W_{1\,\mu}(p,0) \equiv -\hat{p}\,V_1^W(p,0)\,. \tag{94}$$

The overlap quark propagator and its inverse are of the form

$$S_N(p) = \frac{X_0^\dagger(p) + \omega(p)}{2\rho\,[\omega(p)+b(p)]}\,, \quad S_N^{-1}(p) = \rho\left[\frac{X_0(p)}{\omega(p)} + 1\right]\,. \tag{95}$$

For the overlap qqg vertex we obtain

$$V^N_{1\,\mu}(p_2,p_1) = \frac{\rho}{\omega(p_2)+\omega(p_1)}\left[V^W_{1\,\mu}(p_2,p_1) - \frac{X_0(p_2)}{\omega(p_2)} V^{W\,\dagger}_{1\,\mu}(p_2,p_1)\frac{X_0(p_1)}{\omega(p_1)}\right]\,, \tag{96}$$

and for the qqgg vertex we derive

$$V^N_{2\,\mu\nu}(p_2,p_1,k_1,k_2) = V^N_{21\,\mu\nu}(p_2,p_1) + V^N_{22\,\mu\nu}(p_2,p_1,k_1,k_2)\,, \tag{97}$$

with

$$V^N_{21\,\mu\nu}(p_2,p_1) \;=\; \frac{\rho}{\omega(p_2)+\omega(p_1)}\left[V^W_{2\,\mu\nu}(p_2,p_1) - \frac{X_0(p_2)}{\omega(p_2)} V^{W\,\dagger}_{2\,\mu\nu}(p_2,p_1)\frac{X_0(p_1)}{\omega(p_1)}\right], \quad (98)$$

$$V^N_{22\,\mu\nu}(p_2,p_1,k_1,k_2) \;=\; \frac{\rho}{2(\omega(p_2)+\omega(p_1))}\Big[W_{\mu\nu}(p_2,p_1,k_1) + W_{\nu\mu}(p_2,p_1,k_2)\Big], \quad (99)$$

where k_1 and k_2 are the gluon momenta with $p_1+k_1+k_2=p_2$, and the tensor $W_{\mu\nu}$ is defined by

$$\begin{aligned} W_{\mu\nu}(p_2,p_1,k) = {} & \frac{1}{\omega(p_2)+\omega(p_1+k)}\,\frac{1}{\omega(p_1+k)+\omega(p_1)}\,\Big[X_0(p_2)\,V^{W\,\dagger}_{1\,\mu}(p_2,p_1+k) \\ & \times V^W_{1\,\nu}(p_1+k,p_1) + V^W_{1\,\mu}(p_2,p_1+k)\,X_0^\dagger(p_1+k)\,V^W_{1\,\nu}(p_1+k,p_1) \\ & + V^W_{1\,\mu}(p_2,p_1+k)\,V^{W\,\dagger}_{1\,\nu}(p_1+k,p_1)\,X_0(p_1) \\ & - \frac{\omega(p_2)+\omega(p_1+k)+\omega(p_1)}{\omega(p_2)\,\omega(p_1+k)\,\omega(p_1)} X_0(p_2)\,V^{W\,\dagger}_{1\,\mu}(p_2,p_1+k)\,X_0(p_1+k) \\ & \times V^{W\,\dagger}_{1\,\nu}(p_1+k,p_1)\,X_0(p_1)\Big]. \end{aligned} \quad (100)$$

In Appendix B we need explicit expressions for the vertices contracted with the gluon momentum $k=p_2-p_1$ and $k=k_1=-k_2$, respectively:

$$\begin{aligned} \hat{k}\,V^{W,N}_1(p_2,p_1) &= \sum_\mu \hat{k}_\mu V^{W,N}_{1\,\mu}(p_2,p_1)\,, \\ \hat{k}\,V^{W,N}_2(p,p,k,-k)\,\hat{k} &= \sum_{\mu\nu} \hat{k}_\mu V^{W,N}_{2\,\mu\nu}(p,p,k,-k)\,\hat{k}_\nu\,. \end{aligned} \quad (101)$$

Using the short-hand notation

$$\omega=\omega(k)\,,\quad b=b(k)\,,\quad X_0=X_0(k)\,,\quad \hat{k}\,V^{W,N}_1 = \hat{k}\,V^{W,N}_1(k,0) = \hat{k}\,V^{W,N}_1(0,k)\,, \quad (102)$$

we obtain

$$\hat{k}\,V^{W\,\dagger}_1\,\hat{k}\,V^W_1 = \omega^2+2\rho\,b+\rho^2\,,\quad \hat{k}\,V^W_1\,X_0^\dagger\,\hat{k}\,V^W_1 = 2\omega^2\rho+\omega^2 X_0+\rho^2 X_0^\dagger \quad (103)$$

and

$$\hat{k}\,V^N_1 = \frac{\rho}{\omega}\left(\hat{k}\,V^W_1+\rho-\omega\right). \quad (104)$$

Differentiating $X_0^\dagger X_0=\omega^2$ with respect to k_μ, and making use of $V^W_{1\,\mu}(k,k)=-\partial X_0/\partial k_\mu$ we find

$$V^N_{1\,\mu}(k,k) = \frac{\rho}{\omega^3}\Big[\omega^2 V^W_{1\,\mu}(k,k) + (\cos k_\mu + rb)\,\sin k_\mu\,X_0\Big]. \quad (105)$$

From

$$
\begin{aligned}
&\hat{k}\, V_1^W(p+k,p) = X_0(p) - X_0(p+k)\,, \\
&X_0(p)\,\hat{k}\, V_1^{W\,\dagger}(p,p+k) + \hat{k}\, V_1^W(p,p+k)\, X_0(p+k) = \omega(p)^2 - \omega(p+k)^2\,, \\
&X_0(p)X_0^\dagger(p+k) + X_0(p+k)X_0^\dagger(p) = 2\,\omega(p,p+k)^2
\end{aligned}
\tag{106}
$$

we finally obtain

$$
\begin{aligned}
\hat{k}\, V_{22}^N(p,p,k,-k)\,\hat{k} = \frac{\rho}{\omega(p)}\Bigg[&\frac{\omega(p)^2 - \omega(p,p+k)^2}{\omega(p)^2} X_0(p) \\
&+ \frac{\omega(p+k) - \omega(p)}{\omega(p+k)} X_0(p+k)\Bigg]\,.
\end{aligned}
\tag{107}
$$

Appendix B

Gauge dependence in position space

General argument

We have seen in the main text that the gauge dependent terms in the Green functions are independent of the gauge action and of the fermion action: they do not depend on the coefficients c_1, c_2, c_3 and on ρ, and are the same for overlap fermions and clover fermions [26]. The numbers are, however, different on the lattice and in the continuum, so this is not simply due to an ultraviolet convergent integral, where the lattice regularisation plays no role. The independence from the gauge action is fairly easy to explain, but the independence of the fermion action is a remarkable observation, and is something which we ought to understand.

In this subsection we give a brief explanation of this phenomenon in configuration space, while in the next subsection we give a more formal argument using the Feynman rules.

At the one-loop level we can split the functional integral over the gauge fields into an integral over the (physical) transverse gauge fields and the (unphysical) longitudinal gauge fields. The integral over the longitudinal fields is a simple Gaussian integral, given by the gauge fixing term added to the action. The contribution of the longitudinal modes is a universal factor multiplying the Landau gauge Green function

$$
G(x,g^2,\xi) = G(x,g^2,\xi=1)\left[1 - (1-\xi)\, g^2 C_F \int \frac{d^4k}{(2\pi)^4}\, \frac{1-\cos kx}{(\hat{k}^2)^2} + \cdots\right], \tag{108}
$$

where G can be any quark Green function, e.g. the quark propagator or any of the three-point functions we consider in this paper. See Appendix A of Ref. [34], where this is proved to all orders for non-compact QED. It is not clear whether this result will hold to all orders for a non-Abelian group, but it is certainly true at the one-loop level.

Equation (108) gives us an exact expression for the $O(g^2)$ gauge dependent term in configuration space

$$G^{\rm gauge}(x) = g^2 C_F\, \xi\, G^{\rm tree}(x) \int \frac{d^4k}{(2\pi)^4} \frac{1-\cos kx}{(\hat{k}^2)^2} \,. \tag{109}$$

When we Fourier transform this expression to find the gauge term in momentum space, the integral is dominated by $x^2 \sim 1/p^2$. Since we are interested in $a^2p^2 \ll 1$, we have to consider what happens to eq. (109) when $x^2 \gg a^2$. Although the tree-level Green functions do depend on the fermion action when $x^2 \sim a^2$, at large distances all fermion actions agree: they simply tend to the continuum value $G^{\rm tree}_{\rm cont}(x)$. Since this is the region that dominates the Fourier transform, we see immediately that the gauge term will be the same for all fermion actions (or to be more precise, that differences in the gauge term due to differing choices of fermion are suppressed by powers of a). This completes our argument for the universality of the gauge dependent terms.

Note that although the gauge dependent terms are independent of both gauge and fermion action, they would change if we used a different lattice discretisation of the gauge fixing term $(\partial_\mu A_\mu)^2$, as advocated in [35].

Calculating the gauge dependent contributions

Equation (109) not only shows that the gauge dependent terms are universal, it also allows us to calculate them. At large distances ($x^2 \gg a^2$) the integral in eq. (109) takes the value

$$\int \frac{d^4k}{(2\pi)^4} \frac{1-\cos kx}{(\hat{k}^2)^2} \quad \longrightarrow \quad \frac{1}{16\pi^2}\Big(F_0 + \gamma_E + \log\frac{x^2}{4a^2}\Big) \,. \tag{110}$$

In position space the massless quark propagator $S^{\rm tree}_{\rm cont}(x)$ and the three-point function $G^{\rm tree}_{\rm cont}(x)$ are

$$\begin{aligned} S^{\rm tree}_{\rm cont}(x) &= \frac{1}{2\pi^2}\frac{\not{x}}{(x^2)^2}\,, \\ G^{\rm tree}_{\rm cont}(x) &= \gamma_\rho \Gamma^F \gamma_\sigma\; \frac{1}{8\pi^2}\left[2\frac{x_\rho x_\sigma}{(x^2)^2} - \frac{\delta_{\rho\sigma}}{x^2}\right] \,. \end{aligned} \tag{111}$$

Let us first calculate the gauge term in the propagator. We have argued that this will be

$$\frac{g^2C_F}{16\pi^2}\xi\; \mathcal{F}\left[S^{\rm tree}_{\rm cont}(x)\Big(F_0+\gamma_E+\log\frac{x^2}{4a^2}\Big)\right] \,, \tag{112}$$

where $\mathcal{F}$ is the Fourier transform. Some details for performing $\mathcal{F}$ are given at the end of this subsection. From

$$\mathcal{F}\left[S^{\rm tree}_{\rm cont}(x)\right] = \frac{1}{{\rm i}\,\not{p}} \tag{113}$$

and

$$\mathcal{F}\left[S^{\rm tree}_{\rm cont}(x)\log\frac{x^2}{4a^2}\right] = \frac{1}{{\rm i}\,\not{p}}[1-2\gamma_E-\log(a^2p^2)] \tag{114}$$

we get

$$S(p,g^2,\xi) = S(p,g^2,\xi=0) + \frac{1}{{\rm i}\,\not{p}}\,\frac{g^2C_F}{16\pi^2}\,\xi\,\left[F_0-\gamma_E+1-\log(a^2p^2)\right], \tag{115}$$

which agrees with eqs. (31), (32) in the body of the paper.

Now we calculate the gauge dependent parts of the *non-amputated* three-point functions:

$$\mathcal{F}\left[G^{\rm tree}_{\rm cont}(x)\left(F_0+\gamma_E+\log\frac{x^2}{4a^2}\right)\right] = \frac{1}{{\rm i}\,\not{p}}\,\Gamma^X\,\frac{1}{{\rm i}\,\not{p}}\left[F_0-\gamma_E+2-\log(a^2p^2)\right]+\frac{1}{2}\,\frac{\gamma_\rho\,\Gamma^X\gamma_\rho}{p^2}\,, \tag{116}$$

which leads to

$$G^X(p,g^2,\xi) = G^X(p,g^2,0)+\frac{g^2C_F}{16\pi^2}\xi\left\{\frac{1}{{\rm i}\,\not{p}}\Gamma^X\frac{1}{{\rm i}\,\not{p}}\left[F_0-\gamma_E+2-\log(a^2p^2)\right]+\frac{\gamma_\rho\Gamma^X\gamma_\rho}{2p^2}\right\}, \tag{117}$$

where Γ^X is defined in (38). When we amputate this using the gauge dependent quark propagator (115) we find

$$\Lambda^X(p,g^2,\xi) = \Lambda^X(p,g^2,0)+\frac{g^2C_F}{16\pi^2}\xi\left\{\Gamma^X\left[-F_0+\gamma_E+\log(a^2p^2)\right]-\frac{1}{2p^2}\,\not{p}\gamma_\rho\Gamma^X\gamma_\rho\,\not{p}\right\}, \tag{118}$$

valid for general Γ^X. The final term simplifies when we put in specific choices for Γ^X, giving

$$\begin{aligned}
\Lambda^{S,P}(p,g^2,\xi) &= \Lambda^{S,P}(p,g^2,0)+\frac{g^2C_F}{16\pi^2}\xi\left[-F_0+\gamma_E-2+\log(a^2p^2)\right]\{1\!\!1,\gamma_5\}\,,\\
\Lambda^{V,A}_\mu(p,g^2,\xi) &= \Lambda^{V,A}_\mu(p,g^2,0)+\frac{g^2C_F}{16\pi^2}\xi\left\{\gamma_\mu\left[-F_0+\gamma_E-1+\log(a^2p^2)\right]+2\frac{p_\mu\,\not{p}}{p^2}\right\}\{1\!\!1,\gamma_5\}\,,\\
\Lambda^{T}_{\mu\nu}(p,g^2,\xi) &= \Lambda^{T}_{\mu\nu}(p,g^2,0)+\frac{g^2C_F}{16\pi^2}\xi\left[-F_0+\gamma_E+\log(a^2p^2)\right]\sigma_{\mu\nu}\gamma_5\,,
\end{aligned} \tag{119}$$

which reproduces all the gauge dependent terms in eq. (42). Equation (109) still holds if a quark mass is included, so it should also reproduce the mass dependence of the gauge terms previously calculated in [26].

The integrals needed to perform the Fourier transform into momentum space are easily found by introducing an additional integration over α, to make the x integrations into Gaussians. As an example, let us transform $S^{\rm tree}_{\rm cont}(x)(x^2/4a^2)^\delta$ into momentum space:

$$\begin{aligned}\frac{1}{2\pi^2}\gamma^\mu \int d^4x\, e^{-{\rm i}px}\frac{x^\mu}{(x^2)^2}\left(\frac{x^2}{4a^2}\right)^\delta &= \frac{1}{2\pi^2}\gamma^\mu \int d^4x\, e^{-{\rm i}px} x^\mu \int_0^\infty \frac{d\alpha\, \alpha^{1-\delta}}{\Gamma(2-\delta)} e^{-\alpha x^2}(4a^2)^{-\delta} \\ &= -\frac{{\rm i}\not{p}}{4}\frac{(4a^2)^{-\delta}}{\Gamma(2-\delta)}\int_0^\infty d\alpha\, \alpha^{-2-\delta} e^{-p^2/4\alpha} \\ &= -\frac{{\rm i}\not{p}}{p^2}\frac{\Gamma(1+\delta)}{\Gamma(2-\delta)}(a^2p^2)^{-\delta}\,.\end{aligned} \tag{120}$$

Expanding both sides to first order in δ, we get the result (114). The integrals needed for the Fourier transform of the three-point functions can be obtained in the same way as outlined above. We obtain

$$\int d^4x\, e^{-{\rm i}px}\frac{1}{x^2}\left(1+\delta\log\frac{x^2}{4a^2}\right) = \frac{4\pi^2}{p^2}\left\{1+\delta\left[-2\gamma_E - \log(a^2p^2)\right]\right\} \tag{121}$$

and

$$\begin{aligned}\int d^4x\, e^{-{\rm i}px}\frac{x_\rho x_\sigma}{(x^2)^2}\left(1+\delta\log\frac{x^2}{4a^2}\right) = 2\pi^2\frac{\delta_{\rho\sigma}}{p^2}&\left\{1+\delta\left[1-2\gamma_E - \log(a^2p^2)\right]\right\} \\ - 4\pi^2\frac{p_\rho p_\sigma}{(p^2)^2}&\left\{1+\delta\left[2-2\gamma_E - \log(a^2p^2)\right]\right\}\,.\end{aligned} \tag{122}$$

Gauge Dependence in Momentum Space

Here we calculate explicitly the gauge dependent one-loop contributions to the quark self-energy and the amputated Green functions using the Feynman rules. We now set $a = 1$ again.

The generalised lattice Ward identity

As can be easily checked, the generalised lattice Ward identity for Wilson and overlap fermions to lowest order is of the form

$$S^{-1}_{W,N}(p_2) - S^{-1}_{W,N}(p_1) = \sum_\mu (\widehat{p_1 - p_2})_\mu V^{W,N}_{1\,\mu}(p_2,p_1) = (\widehat{p_1-p_2})V^{W,N}_1(p_2,p_1)\,, \tag{123}$$

where inverse propagators and vertices are defined in Appendix A. ¿From this identity it can be seen that

$$\begin{aligned} \mathbb{1} &= -\hat{k}\, V_1^{W,N}(p,p+k)\, S_{W,N}(p+k) + S_{W,N}^{-1}(p) S_{W,N}(p+k) \\ &= -S_{W,N}(p+k)\,\hat{k}\, V_1^{W,N}(p+k,p) + S_{W,N}(p+k) S_{W,N}^{-1}(p)\,, \end{aligned} \tag{124}$$

which leads in the limit $p \to 0$ to

$$\mathbb{1} = -\hat{k}\, V_1^{W,N}(0,k)\, S_{W,N}(k) = -S_{W,N}(k)\,\hat{k}\, V_1^{W,N}(k,0)\,. \tag{125}$$

In addition we have the ordinary lattice Ward identity to lowest order (with zero gauge boson momentum)

$$\frac{\partial}{\partial p_\mu} S_{W,N}^{-1}(p) = -V_{1\,\mu}^{W,N}(p,p)\,. \tag{126}$$

Differentiating (124) with respect to p_μ, and taking the limit $p \to 0$, we get

$$\frac{\partial}{\partial p_\mu} S_{W,N}(p+k)\Big|_{p=0} \hat{k}\, V_1^{W,N}(k,0) = -S_{W,N}(k) \frac{\partial}{\partial p_\mu}\, \hat{k}\, V_1^{W,N}(p+k,p)\Big|_{p=0} + \mathrm{i} S_{W,N}(k)\, \gamma_\mu\,. \tag{127}$$

Using $S_{W,N}^{-1} S_{W,N} = 1$ and (126), (127), another useful form can be derived:

$$\frac{\partial}{\partial p_\mu}\, \hat{k}\, V_1^{W,N}(p+k,p)\Big|_{p=0} = V_{1\,\mu}^{W,N}(k,k) + \mathrm{i}\gamma_\mu\,. \tag{128}$$

Gauge dependent one-loop corrections

We consider the gauge dependent part of the gluon propagator in a general covariant gauge,

$$D_{\mu\nu}^{\mathrm{gauge}}(k) = -\frac{\hat{k}_\mu \hat{k}_\nu}{(\hat{k}^2)^2}\,, \tag{129}$$

and use the short-handed notation for the D-dimensional one-loop integration variable

$$\int_k \equiv \int \frac{d^{\mathrm{D}} k}{(2\pi)^{\mathrm{D}}}\,. \tag{130}$$

For finite integrals we replace the dimension D by four. The basic divergent lattice integral (in dimensional regularisation) is

$$\int_k \frac{1}{(\hat{k}^2)^2} = \frac{1}{16\pi^2} \left[\frac{2}{\mathrm{D}-4} + F_0 - \log 4\pi - \log(a^2\mu^2) \right]. \tag{131}$$

Furthermore, we use the finite lattice integral

$$\int_k \frac{1}{\hat{k}^2} = Z_0\,. \tag{132}$$

First we demonstrate that the gauge dependent contribution of one-loop corrections to local quark bilinears does not depend on the particular representation (Wilson or overlap) of the lattice fermions. Since the local operators do not contain gluon operators, we have to consider only the vertex correction to the amputated Green function:

$$I^X(p) = g^2 C_F \sum_{\mu\nu} \int_k V_{1\,\mu}^{W,N}(p,p+k)\, S_{W,N}(p+k)\, \Gamma^X\, S_{W,N}(p+k)\, V_{1\,\nu}^{W,N}(p+k,k)\, D_{\mu\nu}^{\rm gauge}(k)\,. \tag{133}$$

The corresponding correction is ultraviolet (UV) logarithmically divergent. Using the technique for analytic handling of the divergences, we have to consider

$$I^{X\,{\rm lat}}(p) = I^X(0) + I^{X\,{\rm cont}}(p)\,. \tag{134}$$

With the propagator (129), and using eq. (125), we get at zero momentum the expected result

$$\begin{aligned} I^X(0) &= -g^2 C_F \int_k \frac{1}{(\hat{k}^2)^2} \hat{k}\, V_1^{W,N}(0,k)\, S_{W,N}(k)\, \Gamma^X\, S_{W,N}(k)\, \hat{k}\, V_1^{W,N}(k,0) \\ &= -g^2 C_F\, \Gamma^X \int_k \frac{1}{(\hat{k}^2)^2}\,. \end{aligned} \tag{135}$$

The pole cancels with that of the continuum contribution, $I^{X\,{\rm cont}}(p)$, while the one-loop correction is independent of the form of the lattice propagator as result of the Ward identity.

For operators with derivatives one has to consider the Taylor expansion of the lattice integrals up to the corresponding order of the UV divergence. In that case additionally the $O(g)$ and $O(g^2)$ contributions of the operators have to be considered. It is not difficult to perform a similar proof.

The sunrise diagram (i.e. the left diagram of Fig. 2) for the quark self-energy is of the form

$$I^{\rm sunrise}(p) = g^2 C_F \sum_{\mu\nu} \int_k V_{1\,\mu}^{W,N}(p,p+k) S_{W,N}(p+k) V_{1\,\nu}^{W,N}(p+k,p) D_{\mu\nu}^{\rm gauge}(k)\,. \tag{136}$$

Since that one-loop integral is UV linearly divergent, we have to calculate (using dimensional regularisation)

$$I^{\rm sunrise}(p) = I^{\rm sunrise}(0) + \sum_\alpha p_\alpha \frac{\partial}{\partial p_\alpha} I^{\rm sunrise}(p)\Big|_{p=0} + I^{\rm cont}(p)\,. \tag{137}$$

For the finite tadpole contribution (D=4) we have

$$I^{\rm tadpole}(p) = g^2 C_F \sum_{\mu\nu} \int_k V_{2\,\mu\nu}^{W,N}(p,p,k,-k) D_{\mu\nu}^{\rm gauge}(k)\,. \tag{138}$$

The gauge dependent quark self-energy contribution $I^{\rm lat}(p)$ is then given as sum of (136) and (138). To show the independence on the lattice fermion representation, we transform the difference $I^{\rm lat}(p) - I^{\rm cont}(p)$, using eqs. (125)-(128), to the form

$$I^{\rm lat}(p) - I^{\rm cont}(p) = g^2 C_F \left[{\rm i}\, \not{p} \int_k \frac{1}{(\hat{k}^2)^2} - (J_1^{W,N} + J_2^{W,N} + J_3^{W,N}) \right] \tag{139}$$

where

$$\begin{aligned} J_1^{W,N}(\mathbb{1}) &= -\int_k \frac{1}{(\hat{k}^2)^2} \hat{k}\, V_1^{W,N}(k,0)\,, && (140)\\ J_2^{W,N}(\not{p}) &= -\sum_\mu p_\mu \int_k \frac{1}{(\hat{k}^2)^2} \left[V_{1\,\mu}^{W,N}(k,k) + {\rm i}\gamma_\mu \right]\,, && (141)\\ J_3^{W,N}(\mathbb{1},\not{p}) &= \int_k \frac{1}{(\hat{k}^2)^2} \hat{k}\, V_2^{W,N}(p,p,k,-k)\, \hat{k}\,. && (142) \end{aligned}$$

It remains to show that the sum of integrals $J_1^{W,N} + J_2^{W,N} + J_3^{W,N}$ vanishes. For Wilson fermions we immediately obtain

$$J_1^W(\mathbb{1}) = \frac{r}{2} Z_0\,, \quad J_2^W(\not{p}) = -\frac{1}{8}\,{\rm i}\,\not{p}\, Z_0\,, \quad J_3^W(\mathbb{1},\not{p}) = \frac{1}{8}\,{\rm i}\,\not{p}\, Z_0 - \frac{r}{2} Z_0\,. \tag{143}$$

Therefore the sum is zero.

Integrals for overlap fermions

In the following we use the abbreviations (102). Using (104), and taking into account the symmetry in the integration, we get immediately

$$J_1^N(\mathbb{1}) = \rho \int_k \frac{1}{(\hat{k}^2)^2} \frac{\omega + b}{\omega}\,. \tag{144}$$

To calculate J_2^N we use the symmetric part of $V_{1\,\mu}^N(k,k)$ in k and get

$$J_2^N(\not{p}) = -{\rm i} \sum_\mu p_\mu \gamma_\mu \int_k \frac{1}{(\hat{k}^2)^2} \frac{1}{\omega^3} \left[\omega^2(\omega - \rho\cos k_\mu) + \rho\, \sin^2 k_\mu (\cos k_\mu + rb) \right]\,. \tag{145}$$

To calculate J_3^N, we split it into two pieces, $J_3^N = J_{31}^N + J_{32}^N$, according to eq. (97). After symmetrisation and taking $\sum_\mu V_{21\,\mu\mu}^N(p,p)$ in the zero lattice spacing limit, we find for the k independent part

$$J_{31}^N(\not{p}) \equiv \int_k \frac{1}{(\hat{k}^2)^2}\, \hat{k}\, V_{21}^N(p,p)\, \hat{k} = {\rm i}\, \not{p}\, \frac{1}{8} \left(1 - 4\frac{r}{\rho} \right) Z_0\,. \tag{146}$$

For the cancellation to be shown, we write this result in the form

$$J_{31}^{N}(\not{p}) = \mathrm{i}\sum_{\mu} p_\mu \gamma_\mu \int_k \frac{1}{(\hat{k}^2)^2}\left(1 - \cos k_\mu - \frac{b+\rho}{\rho}\right). \tag{147}$$

The k dependent vertex part arising from (99) contributes both to the unit matrix $\mathbb{1}$ and to $\not{p}$. After integrating over k we have to take the zero spacing limit. This is achieved by performing a Taylor expansion around $p = 0$ up to linear terms in p:

$$\begin{aligned} J_{32}^{N}(\mathbb{1}, \not{p}) &= \int_k \frac{1}{(\hat{k}^2)^2}\left[\hat{k}\, V_{22}^{N}(0,0,k,-k)\,\hat{k} + \sum_\mu p_\mu \frac{\partial}{\partial p_\mu}\hat{k}\, V_{22}^{N}(p,p,k,-k)\,\hat{k}\Big|_{p=0}\right] \\ &= J_{32}^{N}(\mathbb{1}) + J_{32}^{N}(\not{p})\,. \end{aligned} \tag{148}$$

Using the form (107) we get

$$\begin{aligned} J_{32}^{N}(\mathbb{1}) &= -\rho \int_k \frac{1}{(\hat{k}^2)^2}\frac{\omega + b}{\omega}\,, \\ J_{32}^{N}(\not{p}) &= \sum_\mu \mathrm{i}\, p_\mu \gamma_\mu \int_k \frac{1}{(\hat{k}^2)^2}\left[\frac{b+\rho}{\rho} + \frac{(\omega-\rho)\cos k_\mu}{\omega} + \frac{\rho \sin^2 k_\mu\,(\cos k_\mu + rb)}{\omega^3}\right]. \end{aligned} \tag{149}$$

We see that $J_{32}^{N}(\mathbb{1})$ cancels the contribution from the sunrise diagram (144), and

$$J_{2}^{N}(\not{p}) + J_{31}^{N}(\not{p}) + J_{32}^{N}(\not{p}) = 0\,. \tag{150}$$

6 Appendix C

In this appendix we give a more complete derivation of the mean field overlap fermion propagator and tadpole improved ρ, without having to expand in powers of a.

To calculate the mean field overlap fermion propagator, we start from the mean field value of the Wilson-Dirac operator in momentum space,

$$D_W^{\mathrm{MF}}(u_0, r) = \frac{1}{a}\left(u_0 \sum_\mu \mathrm{i}\,\gamma_\mu \sin a p_\mu + 4r - u_0\, r \sum_\mu \cos a p_\mu\right). \tag{151}$$

This can be written in terms of the tree-level Dirac operator

$$D_W^{\mathrm{MF}}(u_0, r) = u_0 D_W^{\mathrm{tree}}(r) + \frac{4r}{a}(1 - u_0)\,. \tag{152}$$

Let us now calculate the overlap operator from $D_W^{\rm MF}$. The first step is to find the X operator

$$
\begin{aligned}
X^{\rm MF}(u_0,r,\rho) &\equiv D_W^{\rm MF}(u_0,r) - \frac{\rho}{a} \\
&= u_0 \left[D_W^{\rm tree}(r) - \frac{\rho - 4r(1-u_0)}{a\,u_0} \right] \\
&= u_0 X^{\rm tree}(r,\rho^{TI})\,,
\end{aligned} \tag{153}
$$

where

$$
\rho^{TI} = \frac{\rho - 4r(1-u_0)}{u_0}\,. \tag{154}
$$

We are now ready to calculate the mean field overlap operator

$$
\begin{aligned}
D_N^{\rm MF}(u_0,r,\rho) &= \frac{\rho}{a}\left[1 + \frac{X^{\rm MF}(u_0,r,\rho)}{\sqrt{X^{\rm MF\,\dagger}(u_0,r,\rho) X^{\rm MF}(u_0,r,\rho)}} \right] \\
&= \frac{\rho}{a}\left[1 + \frac{X^{\rm tree}(r,\rho^{TI})}{\sqrt{X^{\rm tree\,\dagger}(r,\rho^{TI}) X^{\rm tree}(r,\rho^{TI})}} \right] \\
&= \frac{\rho}{\rho^{TI}} D_N^{\rm tree}(r,\rho^{TI}) \equiv Z_\psi^{\rm MF} D_N^{\rm tree}(r,\rho^{TI})\,.
\end{aligned} \tag{155}
$$

In other words, the mean field overlap operator is proportional to the tree-level overlap operator, calculated with the same r but with a tadpole improved ρ. The constant of proportionality is

$$
Z_\psi^{\rm MF} = \frac{\rho}{\rho^{TI}} = \frac{\rho u_0}{\rho - 4r(1-u_0)}\,. \tag{156}
$$

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