LTH–578

The anomalous dimension of the gluon-ghost mass operator in Yang-Mills theory

D. Dudal[*], H. Verschelde[†]
Ghent University
Department of Mathematical Physics and Astronomy, Krijgslaan 281-S9, B-9000 Gent, Belgium

V.E.R. Lemes, M.S. Sarandy, R. Sobreiro, S.P. Sorella[‡]
UERJ - Universidade do Estado do Rio de Janeiro,
Rua São Francisco Xavier 524, 20550-013 Maracanã, Rio de Janeiro, Brazil.

M. Picariello[§]
Universitá degli Studi di Milano, via Celoria 16, I-20133, Milano, Italy and INFN Milano, Italy

J.A. Gracey[¶]
Theoretical Physics Division
Department of Mathematical Sciences
University of Liverpool
P.O. Box 147, Liverpool, L69 3BX, United Kingdom

Abstract

The local composite gluon-ghost operator $\left(\frac{1}{2}A^{a\mu}A^{a}_{\mu}+\alpha\overline{c}^{a}c^{a}\right)$ is analysed in the framework of the algebraic renormalization in $SU(N)$ Yang-Mills theories in the Landau, Curci-Ferrari and maximal abelian gauges. We show, to all orders of perturbation theory, that this operator is multiplicatively renormalizable. Furthermore, its anomalous dimension is not an independent parameter of the theory, being given by a general expression valid in all these gauges. We also verify the relations we obtain for the operator anomalous dimensions by explicit 3-loop calculations in the $\overline{\text{MS}}$ scheme for the Curci-Ferrari gauge.

[*]Research Assistant of The Fund For Scientific Research-Flanders, Belgium.

[†]david.dudal@ugent.be, henri.verschelde@ugent.be

[‡]vitor@dft.if.uerj.br, sarandy@dft.if.uerj.br, sobreiro@dft.if.uerj.br, sorella@uerj.br

[§]marco.picariello@mi.infn.it

[¶]jag@amtp.liv.ac.uk

1 Introduction

Vacuum condensates are believed to play an important role in the understanding of the non-perturbative dynamics of Yang-Mills theories. In particular, much effort has been devoted to the study of condensates of dimension two built up with gluons and ghosts. For instance, the relevance of the pure gluon condensate $\left\langle A^{a\mu}A_{\mu}^{a}\right\rangle$ in the Landau gauge has been discussed from lattice simulations [1] as well as from a phenomenological point of view [2]. That the operator A_{μ}^{2} has a special meaning in the Landau gauge follows by observing that, due to the transversality condition $\partial_{\mu}A^{a\mu}=0$, the integrated operator $(VT)^{-1}\int d^{4}x\, A_{\mu}^{a}A^{a\mu}$ is gauge invariant, with VT denoting the space-time volume. An effective potential for $\left\langle A^{a\mu}A_{\mu}^{a}\right\rangle$ has been constructed in [3], showing that the vacuum of Yang-Mills favors a nonvanishing value for this condensate, which gives rise to a dynamical mass generation for the gluons.

The operator A_{μ}^{2} in the Landau gauge can be generalized to other gauges such as the Curci-Ferrari gauge and maximal abelian gauge, (MAG). Indeed, as was shown in [4], the mixed gluon-ghost operator[1] $\mathcal{O}=\left(\frac{1}{2}A^{a\mu}A_{\mu}^{a}+\alpha\overline{c}^{a}c^{a}\right)$ turns out to be BRST invariant on-shell, where α is the gauge parameter. Also, the Curci-Ferrari gauge has the Landau gauge, $\alpha=0$, as a special case. Thus, the gluon-ghost condensate $\left(\frac{1}{2}\left\langle A^{a\mu}A_{\mu}^{a}\right\rangle+\alpha\left\langle\overline{c}^{a}c^{a}\right\rangle\right)$ might be suitable for the description of dynamical mass generation in these gauges. Recently, the effective potential for this condensate in the Curci-Ferrari gauge has been constructed in [5] by combining the algebraic renormalization [6] with the local composite operators technique [3, 7], resulting in a dynamical mass generation. In this formalism, an essential step is the renormalizability of the local composite operator related to the condensate, which is fundamental to obtaining its anomalous dimension. It is worth mentioning that the anomalous dimension of the gluon-ghost operator $\mathcal{O}=\left(\frac{1}{2}A^{a\mu}A_{\mu}^{a}+\alpha\overline{c}^{a}c^{a}\right)$ in the Curci-Ferrari gauge, and thus of the gluon operator A_{μ}^{2} in the Landau gauge, has been computed to three loops in the $\overline{\mathrm{MS}}$ renormalization scheme, [8]. In addition, it has been proven [9] by using BRST Ward identities that the anomalous dimension $\gamma_{A_{\mu}^{2}}(a)$ of the operator A_{μ}^{2} in the Landau gauge is not an independent parameter, being expressed as a combination of the gauge beta function, $\beta(a)$, and of the anomalous dimension, $\gamma_{A}(a)$, of the gauge field, according to the relation

$$\gamma_{A_{\mu}^{2}}(a) \;=\; -\left(\frac{\beta(a)}{a}\;+\;\gamma_{A}(a)\right), \qquad a=\frac{g^{2}}{16\pi^{2}}\;. \tag{1.1}$$

The aim of this paper is to extend the analysis of [9] to the Curci-Ferrari and maximal abelian gauges. We shall prove that the operator $\left(\frac{1}{2}A^{a\mu}A_{\mu}^{a}+\alpha\overline{c}^{a}c^{a}\right)$ is multiplicatively renormalizable to all orders of perturbation theory. Furthermore, as in the case of the Landau gauge, its anomalous dimension $\gamma_{\mathcal{O}}(a)$ is not an independent parameter of the theory, being given in fact by a general relationship valid in the Landau, Curci-Ferrari and maximal abelian gauges, which is

$$\gamma_{\mathcal{O}}(a) \;=\; -\,2\left(\gamma_{c}(a)\;+\;\gamma_{gc^{2}}(a)\right)\;, \tag{1.2}$$

where $\gamma_{c}(a)$, $\gamma_{gc^{2}}(a)$ are the anomalous dimensions of the Faddeev-Popov ghost c^{a} and of the composite operator $\frac{1}{2}gf^{abc}c^{b}c^{c}$ corresponding to the BRST variation of c^{a}. In other words $sc^{a}=\frac{1}{2}gf^{abc}c^{b}c^{c}$ where s is the BRST operator.

The paper is organized as follows. In section 2, the renormalization of the dimension two operator $\left(\frac{1}{2}A^{a\mu}A_{\mu}^{a}+\alpha\overline{c}^{a}c^{a}\right)$ is considered in detail, by taking the Curci-Ferrari gauge as an

[1] In the case of the maximal abelian gauge, the color index a runs only over the $N(N-1)$ off-diagonal components.

example and the relationship (1.2) is derived. In section 3, we verify the relation we obtain between the anomalous dimensions explicitly at three loops in the Curci-Ferrari gauge in the $\overline{\text{MS}}$ scheme. Section 4 is devoted to the Landau gauge, showing that the expression (1.2) reduces to that of (1.1). In section 5 we shall analyse the maximal abelian gauge, where the results established in [10] for the case of $SU(2)$ will be recovered. Finally, in section 6 we present our conclusions.

2 The gluon-ghost operator in Yang-Mills theories in the Curci-Ferrari gauge

2.1 The Curci-Ferrari action

We begin by reviewing the quantization of pure $SU(N)$ Yang-Mills in the Curci-Ferrari gauge. The pure Yang-Mills action is given by

$$S_{\text{YM}} \;=\; -\frac{1}{4}\int d^4x\, F^{a\,\mu\nu}F^a_{\mu\nu} \;, \tag{2.3}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$. The so called Curci-Ferrari gauge, [11, 12], is defined by the following gauge fixing term

$$\begin{aligned} S_{\text{gf}} \;&=\; \int d^4x \left(b^a\partial_\mu A^{\mu a} + \frac{\alpha}{2}b^a b^a + \overline{c}^a\partial^\mu \left(D_\mu c\right)^a \right. \\ &\qquad \left. -\; \frac{\alpha}{2}gf^{abc}b^a\overline{c}^b c^c - \frac{\alpha}{8}g^2 f^{abc}\overline{c}^a\overline{c}^b f^{cmn}c^m c^n \right) \;, \end{aligned} \tag{2.4}$$

with

$$\left(D_\mu c\right)^a \;=\; \partial_\mu c^a \;+\; gf^{abc}A^b_\mu c^c \;. \tag{2.5}$$

In order to analyse the renormalization of the operator $\left(\frac{1}{2}A^{a\mu}A^a_\mu + \alpha\overline{c}^a c^a\right)$, we have to introduce it in the action by means of a set of external sources. Following [9], it turns out that three external sources J, η^μ and τ^μ are required, so that

$$\begin{aligned} S_{\text{J}} \;&=\; \int d^4x \left(J\left(\frac{1}{2}A^{a\mu}A^a_\mu + \alpha\overline{c}^a c^a\right) + \frac{\xi}{2}J^2 - \eta^\mu A^a_\mu c^a - \tau^\mu\left(\partial_\mu c^a\right)c^a \right. \\ &\qquad \left. -\; \frac{g}{2}\tau^\mu f^{abc}A^a_\mu c^b c^c \right) \end{aligned} \tag{2.6}$$

where ξ is a dimensionless parameter, accounting for the divergences present in the vacuum Green function $\left\langle \left(\frac{1}{2}A^{a\mu}A^a_\mu + \alpha\overline{c}^a c^a\right)_x \left(\frac{1}{2}A^{a\mu}A^a_\mu + \alpha\overline{c}^a c^a\right)_y \right\rangle$, which are proportional to J^2 [5]. The action $(S_{\text{YM}} + S_{\text{gf}} + S_{\text{J}})$ is invariant under the BRST transformations, which read

$$\begin{aligned} sA^a_\mu \;&=\; -\left(D_\mu c\right)^a \;, & sc^a \;&=\; \frac{g}{2}f^{abc}c^b c^c \;, \\ s\overline{c}^a \;&=\; b^a \;, & sb^a \;&=\; -Jc^a \;, \\ s\tau^\mu \;&=\; -\eta^\mu \;, & s\eta^\mu \;&=\; \partial^\mu J \;, \\ sJ \;&=\; 0 \;. \end{aligned} \tag{2.7}$$

Also, due to the introduction of the external sources J, η^μ and τ^μ it follows that the operator s is not strictly nilpotent, namely

$$s^2\Phi \;=\; 0 \;, \quad \left(\Phi = A^{a\mu},\, c^a,\, J,\, \eta^\mu\right) \;,$$

$$
\begin{aligned}
s^2\overline{c}^a &= -\ Jc^a\,, \\
s^2 b^a &= -\ J\frac{g}{2}f^{abc}c^b c^c\,, \\
s^2\tau^\mu &= -\ \partial^\mu J\,.
\end{aligned}
\tag{2.8}
$$

Therefore, setting

$$
s^2 \equiv \delta\,, \tag{2.9}
$$

we have $\delta\left(S_{\rm YM}+S_{\rm gf}+S_{\rm J}\right)=0$. The operator δ is related to a global $SL(2,R)$ symmetry [13], which is known to be present in the Landau, Curci-Ferrari and maximal abelian gauges, [14]. Finally, in order to express the BRST and δ invariances in a functional way, we introduce the external action

$$
\begin{aligned}
S_{\rm ext} &= \int d^4x \left(\Omega^{a\mu}\, sA^a_\mu \,+\, L^a\, sc^a\right) \\
&= \int d^4x \left(-\ \Omega^{a\mu}\,(D_\mu c)^a + L^a\,\frac{g}{2}f^{abc}c^b c^c\right),
\end{aligned}
\tag{2.10}
$$

where $\Omega^{a\mu}$ and L^a are external sources invariant under both BRST and δ transformations, coupled to the nonlinear variations of the fields A^a_μ and c^a. It is worth noting that the source L^a couples to the composite operator $\frac{g}{2}f^{abc}c^b c^c$, thus defining its renormalization properties. It is easy to check that the complete classical action

$$
\Sigma \,=\, S_{\rm YM} \,+\, S_{\rm gf} \,+\, S_{\rm J} \,+\, S_{\rm ext} \tag{2.11}
$$

is invariant under BRST and δ transformations

$$
s\Sigma \,=\, 0\ ,\quad \delta\Sigma \,=\, 0\,. \tag{2.12}
$$

When translated into functional form, the BRST and the δ invariances give rise to the following Ward identities for the complete action Σ, namely

- the Slavnov-Taylor identity

$$
\mathcal{S}(\Sigma) \,=\, 0\,, \tag{2.13}
$$

- with

$$
\mathcal{S}(\Sigma)=\int d^4x\left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}}\frac{\delta\Sigma}{\delta A^a_\mu}+\frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a}+b^a\frac{\delta\Sigma}{\delta\overline{c}^a}-Jc^a\frac{\delta\Sigma}{\delta b^a}+\partial^\mu J\frac{\delta\Sigma}{\delta\eta^\mu}-\eta^\mu\frac{\delta\Sigma}{\delta\tau^\mu}\right) \tag{2.14}
$$

- The δ Ward identity

$$
\mathcal{W}(\Sigma) \,=\, 0\,, \tag{2.15}
$$

with

$$
\mathcal{W}(\Sigma)=\int d^4x\left(-Jc^a\frac{\delta\Sigma}{\delta\overline{c}^a}-J\frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta b^a}-\partial^\mu J\frac{\delta\Sigma}{\delta\tau^\mu}\right)\,. \tag{2.16}
$$

2.2 The invariant counterterm and the renormalization constants

We are now ready to analyse the structure of the most general local counterterm compatible with the identities (2.13) and (2.15). Let us begin by displaying the quantum numbers of all fields and sources

	A^a_μ	c^a	$\overline{c}^a$	b^a	L^a	Ω^a_μ	J	η^μ	τ^μ
Gh. number	0	1	-1	0	-2	-1	0	-1	-2
Dimension	1	0	2	2	4	3	2	3	3

(2.17)

In order to characterize the most general invariant counterterm which can be freely added to all orders of perturbation theory, we perturb the classical action Σ by adding an arbitrary integrated local polynomial $\Sigma^{\rm count}$ in the fields and external sources of dimension bounded by four and with zero ghost number, and we require that the perturbed action $(\Sigma + \varepsilon\Sigma^{\rm count})$ satisfies the same Ward identities and constraints as Σ to first order in the perturbation parameter ε, which are

$$
\begin{aligned}
\mathcal{S}(\Sigma + \varepsilon\Sigma^{\rm count}) &= 0 + O(\varepsilon^2) \,, \\
\mathcal{W}\left(\Sigma + \varepsilon\Sigma^{\rm count}\right) &= 0 + O(\varepsilon^2) \,,
\end{aligned}
\tag{2.18}
$$

This amounts to imposing the following conditions on $\Sigma^{\rm count}$

$$
\mathcal{B}_\Sigma \Sigma^{\rm count} = 0 \,,
\tag{2.19}
$$

with

$$
\begin{aligned}
\mathcal{B}_\Sigma = \int d^4x \bigg(& \frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta\Omega^{a\mu}} + \frac{\delta\Sigma}{\delta\Omega^{a\mu}}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a}\frac{\delta}{\delta L^a} \\
& + b^a\frac{\delta}{\delta\overline{c}^a} + \partial_\mu J\frac{\delta}{\delta\eta_\mu} + \eta^\mu\frac{\delta}{\delta\tau^\mu} - Jc^a\frac{\delta}{\delta b^a}\bigg) \,,
\end{aligned}
\tag{2.20}
$$

and

$$
\int d^4x \left(-Jc^a\frac{\delta\Sigma^{\rm count}}{\delta\overline{c}^a} - J\frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma^{\rm count}}{\delta b^a} - J\frac{\delta\Sigma}{\delta b^a}\frac{\delta\Sigma^{\rm count}}{\delta L^a} - \partial^\mu J\frac{\delta\Sigma^{\rm count}}{\delta\tau^\mu}\right) = 0 \,.
\tag{2.21}
$$

Proceeding as in [9], it turns out that the most general local invariant counterterm compatible with the Ward identities (2.13) and (2.15) contains six independent parameters denoted by σ, a_1, a_2, a_3, a_4 and a_5, and is given by

$$
\begin{aligned}
\Sigma^{\rm count} = \int d^4x \bigg(& \frac{\sigma}{4}F^{a\mu\nu}F^a_{\mu\nu} + (a_3 - a_4)(D_\mu F^{\mu\nu})^a A^a_\nu + \frac{a_1}{2}b^ab^a + a_2 b^a\partial^\mu A^a_\mu \\
& + (a_2 - a_3)\overline{c}^a\partial^2 c^a + (a_4 - a_2)gf^{abc}\overline{c}^a\partial^\mu\left(c^bA^c_\mu\right) \\
& + \frac{(\alpha a_4 - a_1)}{2}gf^{abc}b^a\overline{c}^bc^c + (\alpha a_4 - \frac{a_1}{2})\frac{g^2}{4}f^{abc}\overline{c}^a\overline{c}^bf^{cmn}c^mc^n \\
& + a_3\Omega^{a\mu}\partial_\mu c^a + a_4 gf^{abc}\Omega^{a\mu}A^b_\mu c^c - \frac{a_4}{2}L^agf^{abc}c^bc^c \\
& + \frac{(a_2 + a_3)}{2}JA^{a\mu}A^a_\mu + a_1 J\overline{c}^ac^a + a_5\frac{\xi}{2}J^2 - a_2\eta^\mu A^a_\mu c^a \\
& + (a_2 - a_3)\tau^\mu c^a\partial_\mu c^a + (a_4 - a_2)\frac{g}{2}\tau^\mu f^{abc}A^a_\mu c^bc^c\bigg) \,.
\end{aligned}
\tag{2.22}
$$

It therefore remains to discuss the stability of the classical action. In other words to check that $\Sigma^{\rm count}$ can be reabsorbed in the classical action Σ by means of a multiplicative renormalization of the coupling constant g, the parameters α and ξ, the fields $\{\phi = A, c, \overline{c}, b\}$ and the sources $\{\Phi = J, \eta, \tau, L, \Omega\}$, namely

$$\Sigma(g,\xi,\alpha,\phi,\Phi) + \varepsilon\Sigma^{\mathrm{count}} = \Sigma(g_0,\xi_0,\alpha_0,\phi_0,\Phi_0) + O(\varepsilon^2) \ , \tag{2.23}$$

with the bare fields and parameters defined as

$$\begin{array}{rclcrclcrcl}
A^a_{0\mu} &=& Z_A^{1/2} A^a_\mu &,& \Omega^a_{0\mu} &=& Z_\Omega \Omega^a_\mu &,& \tau_{0\mu} &=& Z_\tau \tau_\mu \\
c^a_0 &=& Z_c^{1/2} c^a &,& L^a_0 &=& Z_L L^a &,& g_0 &=& Z_g g \ , \\
\overline{c}^a_0 &=& Z_{\overline{c}}^{1/2} \overline{c}^a &,& J_0 &=& Z_J J &,& \alpha_0 &=& Z_\alpha \alpha \ , \\
b^a_0 &=& Z_b^{1/2} b^a &,& \eta_{0\mu} &=& Z_\eta \eta_\mu &,& \xi_0 &=& Z_\xi \xi \ .
\end{array} \tag{2.24}$$

The parameters σ, a_1, a_2, a_3, a_4 and a_5 turn out to be related to the renormalization of the gauge coupling constant g, of the gauge parameter α, and of L^a, c^a, A^a_μ, and ξ respectively, according to

$$\begin{array}{rcl}
Z_g &=& 1 + \varepsilon \dfrac{\sigma}{2} \ , \\
Z_\alpha &=& 1 + \varepsilon \left(\dfrac{a_1}{\alpha} - \sigma - 2a_2 + 2a_3 - 2a_4 \right) \ , \\
Z_L &=& 1 + \varepsilon \left(-a_2 - \dfrac{\sigma}{2} + a_3 - a_4 \right) \ , \\
Z_c^{1/2} &=& 1 + \varepsilon \left(\dfrac{-a_3 + a_2}{2} \right) \ , \\
Z_A^{1/2} &=& 1 + \varepsilon \left(-\dfrac{\sigma}{2} + a_3 - a_4 \right) , \\
Z_\xi &=& 1 + \varepsilon \left(a_5 - 2\sigma - 2a_2 + 2a_3 - 4a_4 \right) \ .
\end{array} \tag{2.25}$$

Concerning the other fields and the sources Ω^a_μ, η_μ, and τ_μ, it can be verified that they are renormalized as

$$\begin{array}{rclcrclcrcl}
Z_{\overline{c}} &=& Z_c \ , && Z_b^{1/2} &=& Z_L^{-1} &,& Z_\Omega &=& Z_L Z_A^{-1/2} Z_c^{1/2} \\
Z_\eta &=& Z_L^{-1} Z_c^{-1/2} &,& Z_\tau &=& 1 \ . &&&&
\end{array} \tag{2.26}$$

Finally, for the source J, one has

$$Z_J = Z_L^{-2} Z_c^{-1} , \tag{2.27}$$

from which it follows that

$$\gamma_{\mathcal{O}}(a) = -2\Big(\gamma_c(a) + \gamma_{gc^2}(a) \Big) \ , \tag{2.28}$$

where $\gamma_c(a)$ and $\gamma_{gc^2}(a)$ are the anomalous dimensions of the Faddeev-Popov ghost c^a and of the composite operator $\frac{g}{2} f^{abc} c^b c^c$, defined as

$$\begin{array}{rclcrclcrcl}
\gamma_c(a) &=& \mu\partial_\mu \ln Z_c^{1/2} && \gamma_{gc^2}(a) &=& \mu\partial_\mu \ln Z_L && \gamma_{\mathcal{O}}(a) &=& \mu\partial_\mu \ln Z_J \\
\dfrac{\beta(a)}{a} &=& \mu\partial_\mu \ln Z_g^{-1} && \gamma_\alpha(a) &=& \mu\partial_\mu \ln Z_\alpha^{-1} &&&&
\end{array} \tag{2.29}$$

where μ is the renormalization scale.

Therefore, we have provided a purely algebraic proof of the multiplicative renormalizability of the gluon-ghost operator to all orders of perturbation theory. In particular, we have been able to show, as explicitly exhibited in (2.28), that the anomalous dimension of $\left(\frac{1}{2} A^{a\mu} A^a_\mu + \alpha \overline{c}^a c^a \right)$

is not an independent parameter of the theory, being given by a combination of the anomalous dimensions $\gamma_c(a)$ and $\gamma_{gc^2}(a)$. It is worth mentioning that it has also been proven, [15], for the Curci-Ferrari gauge that the anomalous dimension of the ghost operators $\frac{g}{2} f^{abc} c^b c^c$, $\frac{g}{2} f^{abc} \overline{c}^b c^c$ and $\frac{g}{2} f^{abc} \overline{c}^b \overline{c}^c$ are the same.

Although we did not consider matter fields in the previous analysis, it can be checked that the renormalizability of $\mathcal{O}$ and the relation (2.28) remain unchanged if matter fields are included.

3 Three loop verification

In this section, we will explicitly verify the relation (2.28) up to three loop order in the Curci-Ferrari gauge in a particular renormalization scheme, $\overline{\text{MS}}$. The values for the β-function and the anomalous dimensions of the gluon, ghost, the operator $\mathcal{O}$ and the gauge parameter α have already been calculated in the presence of matter fields in [8]. For completeness we note that for an arbitrary colour group these are

$$
\begin{aligned}
\beta(a) &= - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 ~-~ \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3 \\
&\quad + \Big[2830 C_A^2 T_F N_f - 2857 C_A^3 + 1230 C_A C_F T_F N_f - 316 C_A T_F^2 N_f^2 \\
&\qquad - 108 C_F^2 T_F N_f - 264 C_F T_F^2 N_f^2 \Big] \frac{a^4}{54} ~+~ O(a^5) ~, \qquad (3.30) \\
\gamma_A(a) &= \left[(3\alpha - 13) C_A + 8 T_F N_f \right] \frac{a}{6} \\
&\quad + \left[\left(\alpha^2 + 11\alpha - 59 \right) C_A^2 + 40 C_A T_F N_f + 32 C_F T_F N_f \right] \frac{a^2}{8} \\
&\quad + \Big[\left(54\alpha^3 + 909\alpha^2 + (6012 + 864\zeta(3))\alpha + 648\zeta(3) - 39860 \right) C_A^3 \\
&\qquad - (2304\alpha + 20736\zeta(3) - 58304) C_A^2 T_F N_f + (27648\zeta(3) + 320) C_A C_F T_F N_f \\
&\qquad - 9728 C_A T_F^2 N_f^2 - 2304 C_F^2 T_F N_f - 5632 C_F T_F^2 N_f^2 \Big] \frac{a^3}{1152} ~+~ O(a^4) \qquad (3.31) \\
\gamma_c(a) &= (\alpha - 3) C_A \frac{a}{4} ~+~ \left[\left(3\alpha^2 - 3\alpha - 95 \right) C_A^2 + 40 C_A T_F N_f \right] \frac{a^2}{48} \\
&\quad + \Big[\left(162\alpha^3 + 1485\alpha^2 + (3672 - 2592\zeta(3))\alpha - (1944\zeta(3) + 63268) \right) C_A^3 \\
&\qquad - (6048\alpha - 62208\zeta(3) - 6208) C_A^2 T_F N_f - (82944\zeta(3) - 77760) C_A C_F T_F N_f \\
&\qquad + 9216 C_A T_F^2 N_f^2 \Big] \frac{a^3}{6912} ~+~ O(a^4) ~, \qquad (3.32) \\
\gamma_{\mathcal{O}}(a) &= - \left[16 T_F N_f + (3\alpha - 35) C_A \right] \frac{a}{6} \\
&\quad - \left[280 C_A T_F N_f + (3\alpha^2 + 33\alpha - 449) C_A^2 + 192 C_F T_F N_f \right] \frac{a^2}{24} \\
&\quad - \Big[\left((2592\alpha + 1944)\zeta(3) + 162\alpha^3 + 2727\alpha^2 + 18036\alpha - 302428 \right) C_A^3 \\
&\qquad - (62208\zeta(3) + 6912\alpha - 356032) C_A^2 T_F N_f + (82944\zeta(3) + 79680) C_A C_F T_F N_f \\
&\qquad - 49408 C_A T_F^2 N_f^2 - 13824 C_F^2 T_F N_f - 33792 C_F T_F^2 N_f^2 \Big] \frac{a^3}{3456} ~+~ O(a^4) ~. \qquad (3.33) \\
\gamma_\alpha(a) &= \alpha \left[\frac{a}{4} C_A ~+~ (\alpha + 5) C_A^2 \frac{a^2}{16} \right.
\end{aligned}
$$

$$+ \; 3C_A^2 \left[\left(\alpha^2 + 13\alpha + 67\alpha \right) C_A - 40 T_F N_f \right] \frac{a^3}{128} \right] \; + \; O(a^4) \tag{3.34}$$

where the anomalous dimension of $\mathcal{O}$ in our conventions is given by (-4) times the result quoted in [8]. The group Casimirs are tr $\left(T^a T^b\right) = T_F \delta^{ab}$, $T^a T^a = C_F I$, $f^{acd} f^{bcd} = \delta^{ab} C_A$, N_f is the number of quark flavours and $\zeta(n)$ is the Riemann zeta function. Our definition here of $\gamma_\alpha(a)$, which denotes the running of α, differs from that of [8] due to a different definition of Z_α. For computational reasons, it turns out to be more convenient to consider the renormalization of the ghost operators $f^{abc} c^b c^c$, $f^{abc} \overline{c}^b c^c$ and $f^{abc} \overline{c}^b \overline{c}^c$ instead of $g f^{abc} c^b c^c$, $g f^{abc} \overline{c}^b c^c$ and $g f^{abc} \overline{c}^b \overline{c}^c$ respectively. We note that we have first verified that to three loops the anomalous dimension of each of the three operators are in fact equal, in agreement with [15]. Accordingly, we find

$$\begin{aligned}
\gamma_{c^2}(a) \;&=\; \frac{3}{2} C_A a \;+\; \left[(18\alpha + 95)\, C_A^2 - 40 C_A T_F N_f \right] \frac{a^2}{24} \\
&+ \left[\left(621\alpha^2 + (7182 + 2592\zeta(3))\alpha + (1944\zeta(3) + 63268) \right) C_A^3 \right. \\
&\quad - \; (432\alpha + 62208\zeta(3) + 6208)\, C_A^2 T_F N_f + (82944\zeta(3) - 77760)\, C_A C_F T_F N_f \\
&\quad \left. - \; 8960 C_A T_F^2 N_f^2 \right] \frac{a^3}{3456} \;+\; O(a^4) \; .
\end{aligned} \tag{3.35}$$

We have deduced this result using the MINCER package, [16], written in FORM, [17], where the Feynman diagrams are generated in FORM input format by QGRAF, [18]. For instance, for $f^{abc} \overline{c}^b c^c$ there are 529 diagrams to determine at three loops and 376 for the operator $f^{abc} c^b c^c$ where each is inserted in the appropriate ghost two-point function. The same FORM converter functions of [8] were used here. Since the operator $f^{abc} \overline{c}^b c^c$ has the same ghost structure as the operator $\bar{c}^a c^a$, we were able merely to replace the Feynman rule for the operator insertion of $\bar{c}^a c^a$ in the ghost two-point function with the new operator and use the same routine which determined $\gamma_{\mathcal{O}}(a)$ in [8]. However, as $f^{abc} c^b c^c$ has a different structure we had to generate a new QGRAF set of diagrams to renormalize this operator. That the anomalous dimensions of both operators emerged as equivalent at three loops for all α provides a strong check on our programming as well as justifying the general result of section 2. Now, taking into account the extra factor g, the anomalous dimension $\gamma_{gc^2}(a)$ is found to be

$$\begin{aligned}
\gamma_{gc^2}(a) \;&=\; \left[8 T_F N_f - 13 C_A \right] \frac{a}{6} \;+\; \left[(6\alpha - 59)\, C_A^2 + 40 C_A T_F N_f + 32 C_F T_F N_f \right] \frac{a^2}{8} \\
&+ \left[\left(207\alpha^2 + (2394 + 864\zeta(3))\alpha + (648\zeta(3) - 39860) \right) C_A^3 \right. \\
&\quad - \; (144\alpha + 20736\zeta(3) - 58304)\, C_A^2 T_F N_f + (27648\zeta(3) + 320)\, C_A C_F T_F N_f \\
&\quad \left. - \; 9728 C_A T_F^2 N_f^2 - 2304 C_F^2 T_F N_f - 5632 C_F T_F^2 N_f^2 \right] \frac{a^3}{1152} \;+\; O(a^4) \; .
\end{aligned} \tag{3.36}$$

It is then easily checked from the expressions (3.33), (3.34) and (3.36) that, up to three loop order,

$$\gamma_{\mathcal{O}}(a) \;=\; - \; 2 \left(\gamma_c(a) \;+\; \gamma_{gc^2}(a) \right) \; . \tag{3.37}$$

It is worth mentioning that the renormalizability of the operator $\mathcal{O}$ was already discussed in [19] from the viewpoint of the massive Curci-Ferrari model. Whilst the relation (2.28) was not explicitly given in [19], it is possible to obtain the relation from that analysis. Although the relation (2.28) has been established in the case of the Curci-Ferrari gauge, it expresses a general property of the gluon-ghost operator which remains valid also in the Landau and maximal abelian gauges, as will be shown in the following sections.

4 The Landau gauge

The Landau gauge is a particular case of the Curci-Ferrari gauge, corresponding to $\alpha = 0$. The Landau gauge is known to possess further additional Ward identities [6, 20], implying that the renormalization constants Z_L and Z_c can be expressed in terms of Z_g and Z_A, according to [9]

$$Z_L = Z_A^{1/2} \quad , \quad Z_c \;=\; Z_g^{-1} Z_A^{-1/2} . \tag{4.38}$$

Therefore, it follows that (2.27) reduces to

$$Z_J \;=\; Z_g Z_A^{-1/2} \tag{4.39}$$

from which the expression (1.1) is recovered, providing a nontrivial check of the validity of the general relationship (1.2).

As another internal check of our computations, we note that we should also find in the Landau gauge that $\gamma_{gc^2}(a) = \gamma_A(a)$, as is obvious from (4.38). It can indeed be checked from (3.31) and (3.36) that

$$\gamma_{gc^2}(a)\Big|_{\alpha=0} \;=\; \gamma_A(a)|_{\alpha=0} \; . \tag{4.40}$$

5 The maximal abelian gauge

As is well known, the maximal abelian gauge is a nonlinear partial gauge fixing allowing for a residual $U(1)^{N-1}$ local invariance [10, 21, 22, 23]. In the following, a Landau type gauge fixing will be assumed for this local residual invariance. The Slavnov-Taylor and the δ Ward identities (2.13) and (2.15) can be straightforwardly generalized to this case. It is useful to recall that the gauge field is now decomposed into its off-diagonal and diagonal components

$$A^a_\mu T^a = A^i_\mu T^i + A^\alpha_\mu T^\alpha \, , \tag{5.41}$$

where the index i labels the $N-1$ generators T^i of the Cartan subalgebra of $SU(N)$. The remaining $N(N-1)$ off-diagonal generators T^α will be labelled by the index α. Accordingly, for the Faddeev-Popov ghost c^a we have

$$c^a T^a = c^i T^i + c^\alpha T^\alpha \, , \tag{5.42}$$

with

$$\begin{aligned} sc^\alpha &= g f^{\,\alpha\beta i} c^\beta c^i + \frac{g}{2} f^{\,\alpha\beta\gamma} c^\beta c^\gamma , \\ sc^i &= \frac{g}{2} f^{\,i\alpha\beta} c^\alpha c^\beta . \end{aligned} \tag{5.43}$$

Also, the group index of the gluon-ghost operator runs only over the off-diagonal components, namely

$$\mathcal{O}_{\rm MAG} \;=\; \left(\frac{1}{2} A^{\alpha\mu} A^\alpha_\mu + \alpha \overline{c}^\alpha c^\alpha \right) \, . \tag{5.44}$$

Denoting respectively by $\widetilde{Z}$ and Z the renormalization factors of the off-diagonal and diagonal components of the fields, it follows that, according to the relationship (2.28), the output of the Slavnov-Taylor and δ Ward identities gives

$$\gamma_{\mathcal{O}_{\rm MAG}}(a) \;=\; -\; 2 \left(\widetilde{\gamma}_{c^\alpha}(a) + \widetilde{\gamma}_{gc^2}(a) \right) \, , \tag{5.45}$$

where $\widetilde{\gamma}_{c^\alpha}(a)$ and $\widetilde{\gamma}_{gc^2}(a)$ are the anomalous dimensions of the off-diagonal ghost c^α and of the composite operator $\left(gf^{\,\alpha\beta i}c^\beta c^i + \frac{g}{2}f^{\,\alpha\beta\gamma}c^\beta c^\gamma\right)$ which corresponds to the BRST variation of c^α. Moreover, as shown in [22], the use of the Landau gauge for the local residual $U(1)^{N-1}$ invariance allows for a further Ward identity. This identity, called the diagonal ghost Ward identity in [22], implies that the anomalous dimension $\widetilde{\gamma}_{gc^2}(a)$ can be expressed as

$$\widetilde{\gamma}_{gc^2}(a) \;=\; \frac{\beta(a)}{a} \;-\; \widetilde{\gamma}_{c^\alpha}(a) \;-\; \gamma_{c^i}(a)\;, \tag{5.46}$$

where $\gamma_{c^i}(a)$ is the anomalous dimension of the diagonal ghost c^i. Therefore, for the expression of $\gamma_{\mathcal{O}_{\rm MAG}}(a)$ we obtain

$$\gamma_{\mathcal{O}_{\rm MAG}}(a) \;=\; -\;2\left(\frac{\beta(a)}{a} \;-\; \gamma_{c^i}(a)\right)\;, \tag{5.47}$$

a result which is in complete agreement with that already found in [10] for the case of $SU(2)$. Finally, it is worth mentioning that the anomalous dimensions of the diagonal and off-diagonal components of the fields have been computed at one-loop order in [10, 23], so that (2.28) gives explicit knowledge of the one-loop anomalous dimension of the gluon-ghost operator in the maximal abelian gauge.

6 Conclusion

We have shown that the mass dimension two gluon-ghost operator $\mathcal{O} = \frac{1}{2}A^a_\mu A^{\mu a} + \alpha\overline{c}^a c^a$ is multiplicatively renormalizable in the Landau, Curci-Ferrari and maximal abelian gauges. Further, we were able to establish a general relation between the anomalous dimension of $\mathcal{O}$, the Faddeev-Popov ghost c^a and the dimension two ghost operator $gf^{abc}c^bc^c$, as expressed by the eq.(1.2). This relation has been derived within the framework of the algebraic renormalization [6], following from the Slavnov-Taylor identity (2.13). As such, it extends to all orders of perturbation theory and is renormalization scheme independent, for any scheme preserving the Slavnov-Taylor identity. It has been explicitly verified up to three loops in the $\overline{\rm MS}$ scheme in the Curci-Ferrari gauge.

Furthermore, due to additional Ward identities that exist in the Landau gauge [6] and in the MAG [22], we were able to rewrite the relation (1.2) for the anomalous dimension for $\mathcal{O}$ in terms of the β-function and the anomalous dimension of the gluon and/or ghost fields. In particular, concerning the maximal abelian gauge, it is worth underlining that the multiplicative renormalizability of the gluon-ghost operator, eq.(5.47), is a necessary ingredient towards the construction of a renormalizable effective potential for studying the possible condensation of the gluon-ghost operator and the ensuing dynamical mass generation, as done in the Landau [3] and Curci-Ferrari [5] gauges.

As a final remark, we point out that, from the 3-loop expressions given in section 3, it is easily checked that the following relations holds in the Curci-Ferrari gauge:

$$\gamma_{\mathcal{O}}(a) \;=\; -\;\left(\frac{\beta(a)}{a} \;+\; \gamma_A(a)\right), \tag{6.48}$$

$$\gamma_{gc^2}(a) \;=\; \gamma_A(a) \;-\; 2\gamma_\alpha(a)\;. \tag{6.49}$$

Up to now, we do not know if these relations are valid to all orders. They do not follow from the Slavnov-Taylor identity (2.13). Nevertheless, although eqs.(6.48), (6.49) have been obtained in a particular renormalization scheme, i.e. the $\overline{\rm MS}$ scheme, it could be interesting to search for additional Ward identities in the Curci-Ferrari gauge which, as in the case of the Landau gauge [9], could allow for a purely algebraic proof of eqs.(6.48), (6.49). Notice in fact that, when $\alpha = 0$, eq.(6.48) yields the anomalous dimension of the composite operator A^2_μ in the Landau gauge. Also, eq.(6.49) reduces to the relation (4.40) of the Landau gauge, since $\gamma_\alpha(a) \equiv 0$ if $\alpha = 0$.

Acknowledgments. M.P. is grateful to R. Ferrari for useful discussions. We acknowledge the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil), the Fundação de Amparo a Pesquisa do Estado do Rio de Janeiro (Faperj), the SR2-UERJ, the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) and the MIUR-Italy for financial support. D.D. would like to thank the kind hospitality of the Theoretical Physics Department of the UERJ where part of this work was completed.

References

[1] P. Boucaud, G. Burgio, F. Di Renzo, J. P. Leroy, J. Micheli, C. Parrinello, O. Pène, C. Pittori, J. Rodriguez-Quintero, C. Roiesnel and K. Sharkey, *JHEP* **0004** (2000) 006;

P. Boucaud, A. Le Yaouanc, J. P. Leroy, J. Micheli, O. Pène and J. Rodriguez-Quintero, *Phys. Rev.* **D63** (2001) 114003;

P. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pène, F. De Soto, A. Donini, H. Moutarde and J. Rodriguez-Quintero, *Phys. Rev.* **D66** (2002) 034504;

Ph. Boucaud, F. De Soto, A. Le Yaouanc, J.P. Leroy, J. Micheli, H. Moutarde, O. Pène and J. Rodriguez-Quintero, *Phys. Rev.* **D67** (2003) 074027.

[2] F. V. Gubarev, L. Stodolsky and V. I. Zakharov, *Phys. Rev. Lett.* **86** (2001) 2220;

F. V. Gubarev and V. I. Zakharov, *Phys. Lett.* **B501** (2001) 28.

[3] H. Verschelde, K. Knecht, K. Van Acoleyen and M. Vanderkelen, *Phys. Lett.* **B516** (2001) 307, Erratum-ibid., to appear.

[4] K.-I. Kondo, *Phys. Lett.* **B514** (2001) 335.

K.-I. Kondo, T. Murakami, T. Shinohara & T. Imai, *Phys. Rev.* **D65** (2002) 085034.

[5] D. Dudal, H. Verschelde, V. E. Lemes, M. S. Sarandy, S. P. Sorella and M. Picariello, *Gluon-ghost condensate of mass dimension 2 in the Curci-Ferrari gauge*, *Ann. Phys.*, to appear, hep-th/0302168.

[6] O. Piguet and S. P. Sorella, *Algebraic Renormalization*, Monograph series **m28**, Springer Verlag, 1995

[7] K. Knecht and H. Verschelde, *Phys. Rev.* **D64** (2001) 085006.

[8] J. A. Gracey, *Phys. Lett.* **B552** (2003) 101.

[9] D. Dudal, H. Verschelde, and S.P. Sorella, *Phys. Lett.* **B555** (2003) 126.

[10] U. Ellwanger and N. Wschebor, *Int. J. Mod. Phys.* **A18** (2003) 1595

[11] I. Ojima, *Z. Phys.* **C13** (1982) 173;
R. Delbourgo and P.D. Jarvis, *J. Phys.* **A15** (1982) 611;
L. Baulieu and J. Thierry-Mieg, *Nucl. Phys.* **B197** (1982) 477.

[12] G. Curci and R. Ferrari, *Nuovo Cim.* **A32** (1976) 151;
G. Curci and R. Ferrari, *Phys. Lett.* **B63** (1976) 91.

[13] F. Delduc and S.P. Sorella, *Phys. Lett.* **B231** (1989) 408.

[14] D. Dudal, V.E.R. Lemes, M. Picariello, M.S. Sarandy, S.P. Sorella, and H. Verschelde, *JHEP* **0212** (2002) 008.

[15] D. Dudal, H. Verschelde, V. E. Lemes, M. S. Sarandy, S. P. Sorella, M. Picariello, A. Vicini, J. A. Gracey, *JHEP* **0306** (2003) 003.

[16] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *Nucl. Phys.* **B174** (1980) 345.

[17] J.A.M. Vermaseren, math-ph/0010025.

[18] P. Nogueira, *J. Comput. Phys.* **105** (1993) 279.

[19] J. de Boer, K. Skenderis, P. van Nieuwenhuizen and A. Waldron, *Phys. Lett.* **B367** (1996) 175

[20] A. Blasi, O. Piguet, S.P. Sorella, *Nucl. Phys.* **B356** (1991) 154.

[21] H. Min, T. Lee and P.Y. Pac, *Phys. Rev.* **D32** (1985) 440.

[22] A.R. Fazio, V.E.R. Lemes, M.S. Sarandy and S.P. Sorella, *Phys. Rev.* **D64** (2001) 085003.

[23] T. Shinohara, T. Imai, K.-I. Kondo, hep-th/0105268.