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Three loop anomalous dimension of non-singlet quark currents in the RI′ scheme

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Abstract. We renormalize QCD at three loops in the modified regularization invariant, RI′, scheme in arbitrary covariant gauge and deduce that the four loop β-function is equivalent to the $\overline{\text{MS}}$ result. The anomalous dimensions of the scalar, vector and tensor currents are then determined in the RI′ scheme at three loops by considering the insertion of the operator in a quark two-point function. The expression for the scalar current agrees with the quark mass anomalous dimension and we deduce an expression for the four loop RI′ mass anomalous dimension in arbitrary covariant gauge and for any Lie group.

1 Introduction.

The most widely used renormalization prescription in perturbative quantum field theory is the minimal subtraction scheme where only the infinities with respect to the regularization are subtracted from the divergent part of the Green's function to determine the renormalization group functions, [1]. In practice, though, it is more appropriate to use the modified minimal subtraction scheme, $\overline{\mathrm{MS}}$, since the convergence properties of perturbative series in this scheme are improved by additionally absorbing a *finite* part, $\ln(4\pi e^{-\gamma})$, into the renormalization constants, [2], where γ is the Euler-Mascheroni constant. The advantage of using the $\overline{\mathrm{MS}}$ scheme, which is a mass independent scheme, rests in some elegant properties. For example, the $\overline{\mathrm{MS}}$ β-function and anomalous dimension of the quark mass in QCD, are both independent of the covariant gauge fixing parameter, [1, 3]. Moreover, performing computations using $\overline{\mathrm{MS}}$ and dimensional regularization, where the spacetime dimension becomes $d = 4 - 2\epsilon$ and ϵ is the regularizing parameter, one can carry out multiloop calculations to very high order. Indeed in QCD various four loop renormalization group functions are available, [4, 5, 6], which represent the current state of computation. Whilst the $\overline{\mathrm{MS}}$ scheme enjoys these elegant features and has become the standard reference scheme, it has the limitation that it is not a physical renormalization scheme. Examples of schemes which are founded in a more physical origin include, for example, the MOM and $\overline{\mathrm{MOM}}$ schemes, [7, 8]. Though one disadvantage of using physical schemes is that fewer results currently exist to the same multiloop precision as in $\overline{\mathrm{MS}}$. However, it is well known that physical quantities in one scheme can be simply related to the same quantity in other schemes by a conversion function, [7]. Whilst one ordinarily uses dimensionally regularized perturbation theory to compute physical quantities one can also determine such information by using a lattice regularization. The advantage of this approach is that one in principle includes all non-perturbative contributions in a calculation which need to be converted from the lattice scheme to $\overline{\mathrm{MS}}$. For a recent review and applications in determining matrix elements in deep inelastic scattering see, for example, [9] where lattice results were matched to $\overline{\mathrm{MS}}$ results. The scheme used is similar to a modified version of the regularization invariant, RI, scheme known as the RI$'$ scheme, [10]. Therefore, in order to improve lattice estimates one requires the conversion of various renormalization group functions from $\overline{\mathrm{MS}}$ to RI$'$. Recently, this problem has been addressed in the context of quark masses where the conversion functions were produced for all covariant gauges for the quark mass anomalous dimension at four loops, [11, 12]. Indeed the field anomalous dimensions were also deduced to the same order for the $SU(N_c)$ colour group, [12]. Though in practice for the lattice application one only considers one particular gauge which is the Landau gauge. This work of [12] extended the three loop calculation of [11]. One practical feature of these computations was that one only needed to consider ordinary perturbation theory in the massless limit which effectively meant that the conversion functions could be deduced using standard multiloop perturbative tools for massless field theories.

Whilst these papers dealt with the problem of quark masses deduced from the lattice there are other problems where the conversion functions are required. For instance, there is interest in deducing low moments of the structure functions measured in deep inelastic scattering from the lattice, [9, 13]. To improve estimates the conversion factors from the $\overline{\mathrm{MS}}$ scheme to the RI$'$ scheme are required. Therefore, the purpose of this article is twofold. First, given that the RI$'$ scheme is important for relating Landau gauge lattice results to the $\overline{\mathrm{MS}}$ scheme we will renormalize QCD at three loops in the RI$'$ scheme though in a general covariant gauge. Whilst we are ultimately interested in quark currents it is not inconceivable that the anomalous dimensions of operators with gluonic fields will at some time be measured on the lattice and therefore the anomalous dimensions of the gluon (and ghost) fields will need to be determined at the same level as the quarks. Equipped with the fully renormalized QCD Lagrangian in the RI$'$

scheme we will then extend the approach of [12] to deduce RI$'$ information but for the anomalous dimension of a particular quark composite operator which corresponds to the tensor current in QCD. It is of interest since it represents the lowest moment of the transversity operator in deep inelastic scattering, [14]. Given the recent resurgence of experimental and theoretical interest in transversity, (see, for example, [15, 16, 17, 18, 19]), the longer term aim is to provide a more accurate numerical estimate for the associated matrix element prior to experimental data being accumulated at RHIC. One motivation in this approach is to develop the calculational formalism to determine the conversion function for an operator which is a simple extension of [12]. In [12] the quark mass conversion functions were determined by considering the corrections to the massive quark two-point function which is an avenue not immediately available for a composite operator. Nevertheless we will bridge this gap by reconstructing the result of [12] at three loops by first considering the problem of the mass renormalization as the renormalization of the associated composite operator, $\bar{\psi}\psi$, as inserted in a two-point Green's function prior to replacing it by the operator of main interest which is the tensor current. Moreover, given this way of computing we need only use the *massless* version of QCD.

The paper is organised as follows. In section two we discuss the three loop renormalization of QCD in the RI$'$ scheme and provide the renormalization group functions. These results are used in section three to extract the anomalous dimension of the quark mass operator in QCD in RI$'$ which agrees with the earlier Landau gauge result of [12]. Having provided the formalism for treating an operator, we extend that calculation to the tensor current case in section four. Finally, our conclusions are given in section five.

2 RI$'$ scheme at three loops.

We begin by explicitly renormalizing QCD in the RI$'$ scheme at three loops using dimensional regularization. The bare QCD Lagrangian, with the gauge fixed covariantly, is

$$L ~=~ - ~ \frac{1}{4} G^a_{\mu\nu} G^{a \, \mu\nu} ~-~ \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 ~-~ \bar{c}^a \partial^\mu D_\mu c^a ~+~ i \bar{\psi}^{iI} \not{\!\!D} \psi^{iI} \tag{2.1}$$

where A^a_μ is the gluon field, $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$, c^a and $\bar{c}^a$ are respectively the ghost and antighost fields and α is the covariant gauge fixing parameter. The indices range over $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ where N_F and N_A are the respective dimensions of the fundamental and adjoint representations of the colour group whose generators are T^a and structure functions are f^{abc} whilst N_f is the number of quark flavours. The covariant derivatives are defined by

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi ~+~ i g A^a_\mu T^a \psi ~~~,~~~ D_\mu A^a_\nu ~=~ \partial_\mu A^a_\nu ~-~ g f^{abc} A^b_\mu A^c_\nu \\ D_\mu c^a &= \partial_\mu c^a ~-~ g f^{abc} A^b_\mu c^c ~. \end{aligned} \tag{2.2}$$

In renormalizing the full Lagrangian in the RI$'$ scheme, which we will define more precisely later, we must ensure that the scheme is consistent. We can achieve this, for example, by demonstrating that the renormalization constants for the gluon, ghost and quark fields correctly produce the *same* gauge independent coupling constant renormalization at three loops from different Green's functions, thereby ensuring that the Slavnov-Taylor identities are respected. If we regard all the quantities in the QCD Lagrangian, (2.1), as bare and denote them with the subscript $_{\mathrm{o}}$, we introduce the corresponding renormalized quantities by the usual definitions

$$A^{a\,\mu}_{\mathrm{o}} ~=~ \sqrt{Z_A} \, A^{a\,\mu} ~~,~~ c^a_{\mathrm{o}} ~=~ \sqrt{Z_c} \, c^a ~~,~~ \psi_{\mathrm{o}} ~=~ \sqrt{Z_\psi} \psi ~~,~~ g_{\mathrm{o}} ~=~ \mu^\epsilon Z_g \, g ~~,~~ \alpha_{\mathrm{o}} ~=~ Z^{-1}_\alpha Z_A \, \alpha \tag{2.3}$$

where μ is the mass scale introduced to ensure the coupling constant is dimensionless in d-dimensions and $d = 4 - 2\epsilon$ with ϵ the regularizing parameter.

To determine the RI$'$ scheme values of the renormalization constants we first consider the gluon, quark and ghost two-point functions. In [10, 11, 12] the RI$'$ scheme definition of the quark wave function renormalization is given by the Minkowski space condition

$$\lim_{\epsilon \rightarrow 0} \left[Z^{\mathrm{RI}'}_\psi \Sigma_\psi(p) \right] \Bigg|_{p^2 = \mu^2} = \not{p} \tag{2.4}$$

where $\Sigma_\psi(p)$ is the bare (massless) quark two-point function and p is the external quark momentum. As we are considering massless quarks $\Sigma_\psi(p)$ will be proportional to $\not{p}$ and involve poles in ϵ at each order in the strong coupling constant. To contrast with the $\overline{\mathrm{MS}}$ scheme, the RI$'$ scheme definition of the quark wave function is such that one absorbs the complete finite part of the Green's function with respect to ϵ into the renormalization constant. In other words only the $O(1)$ piece is removed and the $O(\epsilon)$ part is ignored. In the $\overline{\mathrm{MS}}$ scheme only the poles in ϵ are removed as well as the finite parts involving powers of $\ln(4\pi e^{-\gamma})$. For completeness we note that in the RI scheme one absorbs the full finite part in the same way as in the RI$'$ scheme, (2.4), but for a different part of the Green's function which is,

$$\lim_{\epsilon \rightarrow 0} \left[\frac{1}{4d} \mathrm{tr} \left(Z^{\mathrm{RI}}_\psi \gamma^\mu \frac{\partial}{\partial p^\mu} \Sigma_\psi(p) \right) \right] \Bigg|_{p^2 = \mu^2} = 1 ~. \tag{2.5}$$

Due to the presence of the derivative this scheme is much more difficult to implement on the lattice compared with (2.4) which is why we are concentrating on RI$'$. For the remaining wave function renormalization constants we define their RI$'$ values in a similar way to (2.4). For the ghost fields $Z^{\mathrm{RI}'}_c$ is determined by

$$\lim_{\epsilon \rightarrow 0} \left[Z^{\mathrm{RI}'}_c \frac{\Sigma_c(p)}{p^2} \right] \Bigg|_{p^2 = \mu^2} = 1 ~. \tag{2.6}$$

The definition of the gluon field requires more care due to it having transverse and longitudinal components. Rendering the former finite determines Z_A whilst the finiteness of the latter component fixes the gauge parameter renormalization constant. In the spirit of the quark and ghost RI$'$ scheme renormalization constants we now define $Z^{\mathrm{RI}'}_A$ and $Z^{\mathrm{RI}'}_\alpha$ through the following conditions. Writing the gluon polarization tensor, $\Pi_{\mu\nu}(p)$, as

$$\Pi_{\mu\nu}(p) = \frac{\Pi_T(p)}{p^2} \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + \Pi_L(p) \frac{p_\mu p_\nu}{(p^2)^2} \tag{2.7}$$

we define the associated RI$'$ scheme renormalization constants as

$$\lim_{\epsilon \rightarrow 0} \left[Z^{\mathrm{RI}'}_A \Pi_T(p) \right] \Bigg|_{p^2 = \mu^2} = 1 \tag{2.8}$$

and

$$\lim_{\epsilon \rightarrow 0} \left[Z^{\mathrm{RI}'}_\alpha \Pi_L(p) \right] \Bigg|_{p^2 = \mu^2} = 1 ~. \tag{2.9}$$

Having established the scheme definition of the field renormalization constants we have computed them explicitly for arbitrary α for (2.1). We have used the MINCER package, [20], written in the symbolic manipulation language FORM, [21, 22], where the Feynman diagrams are generated with QGRAF, [23]. The basic nature of the renormalization conditions are straightforward to implement in FORM using the approach of [24]. Briefly, one computes the appropriate Green's functions in terms of bare parameters and couplings and then introduces the counterterms by rescaling these variables into the renormalized ones using the definition of renormalization constants, (2.3). Also, for Green's functions which ordinarily involve a bare parameter in the tree part, one divides the Green's function by the bare parameter first before rescaling, [24]. Therefore, we quote the results of our calculations. We find

$$
\begin{aligned}
Z_A^{\mathrm{RI}'} &= 1 + \left[\left(\left(\frac{13}{6} - \frac{\alpha}{2} \right) C_A - \frac{4}{3} T_F N_f \right) \frac{1}{\epsilon} + \left(\left(\frac{97}{36} + \frac{\alpha}{2} + \frac{\alpha^2}{4} \right) C_A - \frac{20}{9} T_F N_f \right) \right] a \\
&+ \left[\left(\left(\frac{\alpha^2}{4} - \frac{17\alpha}{24} - \frac{13}{8} \right) C_A^2 + C_A T_F N_f \left(\frac{2}{3}\alpha + 1 \right) \right) \frac{1}{\epsilon^2} \right. \\
&- \left(\left(\frac{\alpha^3}{4} + \frac{\alpha^2}{12} + \frac{331\alpha}{144} - \frac{4115}{432} \right) C_A^2 + 2 C_F T_F N_f - \frac{80}{27} T_F^2 N_f^2 \right. \\
&+ \left. \left(\frac{589}{54} - \frac{14\alpha}{9} + \frac{\alpha^2}{3} \right) C_A T_F N_f \right) \frac{1}{\epsilon} + \left(\left(16\zeta(3) - \frac{55}{3} \right) T_F N_f C_F \right. \\
&+ \frac{400}{81} T_F^2 N_f^2 - \left(\frac{8659}{324} + \frac{20\alpha}{9} + \frac{10\alpha^2}{9} + 8\zeta(3) \right) T_F N_f C_A \\
&+ \left. \left. \left(\frac{83105}{2592} + \frac{701\alpha}{288} + \frac{365\alpha^2}{144} + \frac{11\alpha^3}{16} + \frac{\alpha^4}{8} - 3\zeta(3) + 2\alpha\zeta(3) \right) C_A^2 \right) \right] a^2 \\
&+ \left[\left(\left(\frac{403}{144} + \frac{47\alpha}{48} + \frac{\alpha^2}{6} - \frac{\alpha^3}{8} \right) C_A^3 - \left(\frac{22}{9} + \frac{5\alpha}{6} + \frac{\alpha^2}{3} \right) C_A^2 T_F N_f + \frac{4}{9} C_A T_F^2 N_f^2 \right) \frac{1}{\epsilon^3} \right. \\
&+ \left(\left(\frac{3\alpha^4}{16} + \frac{\alpha^3}{6} + \frac{139\alpha^2}{96} - \frac{5287\alpha}{864} - \frac{2935}{216} \right) C_A^3 \right. \\
&+ \left(\frac{1643}{108} + \frac{953\alpha}{108} - \frac{7\alpha^2}{12} + \frac{\alpha^3}{3} \right) C_A^2 T_F N_f - \left(\frac{110}{27} + \frac{80\alpha}{27} \right) C_A T_F^2 N_f^2 \\
&- \left. \frac{8}{9} C_F T_F^2 N_f^2 + \left(\frac{31}{9} + \alpha \right) C_A C_F T_F N_f \right) \frac{1}{\epsilon^2} \\
&+ \left(\left(\frac{88391}{972} - \frac{19595\alpha}{576} + \frac{1021\alpha^2}{1728} - \frac{235\alpha^3}{96} - \frac{61\alpha^4}{96} - \frac{5\alpha^5}{32} \right. \right. \\
&- \left. \left. \left(\frac{33\alpha^2}{16} - \frac{85\alpha}{12} + \frac{107}{16} \right) \zeta(3) \right) C_A^3 \right. \\
&- \left(\frac{85831}{648} - \frac{217\alpha}{8} + \frac{485\alpha^2}{216} - \frac{3\alpha^3}{4} + \frac{\alpha^4}{6} + \frac{22}{3}\zeta(3) - \frac{16\alpha}{3}\zeta(3) \right) C_A^2 T_F N_f \\
&- \left(\frac{2441}{54} - \frac{52\alpha}{3} + \frac{\alpha^2}{2} - \frac{80}{3}\zeta(3) + 16\alpha\zeta(3) \right) C_A C_F T_F N_f \\
&+ \left(\frac{1477}{27} - \frac{40\alpha}{9} + \frac{40\alpha^2}{27} + \frac{32}{3}\zeta(3) \right) C_A T_F^2 N_f^2 \\
&+ \left. \left(\frac{824}{27} - \frac{64}{3}\zeta(3) \right) C_F T_F^2 N_f^2 + \frac{2}{3} C_F^2 T_F N_f - \frac{1600}{243} T_F^3 N_f^3 \right) \frac{1}{\epsilon} \\
&+ \left(\left(\left(252 + 16\alpha + 8\alpha^2 \right) \zeta(3) - 12\zeta(4) + 80\zeta(5) - \frac{128819}{324} \right. \right. \\
&- \left. \frac{55\alpha}{3} - \frac{55\alpha^2}{6} \right) C_A C_F T_F N_f + \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5) \right) C_F^2 T_F N_f
\end{aligned}
$$

$$
\begin{aligned}
& - \Bigg(\frac{2080363}{3888} + \frac{6115\alpha}{216} + \frac{10993\alpha^2}{432} + \frac{55\alpha^3}{12} + \frac{5\alpha^4}{6} \\
& \quad + \Bigg(\frac{469}{6} + \frac{430\alpha}{9} + \frac{23\alpha^2}{6} \Bigg) \zeta(3) - 9\zeta(4) - \frac{160}{3}\zeta(5) \Bigg) C_A^2 T_F N_f \\
& + \Bigg(\frac{14002}{81} - \frac{416}{3}\zeta(3) \Bigg) C_F T_F^2 N_f^2 - \frac{8000}{729} T_F^3 N_f^3 \\
& + \Bigg(\frac{12043}{81} + \frac{152\alpha}{27} + \frac{100\alpha^2}{27} + 64\zeta(3) + \frac{64\alpha}{9}\zeta(3) \Bigg) C_A T_F^2 N_f^2 \\
& + \Bigg(\frac{44961125}{93312} + \frac{14939\alpha}{432} + \frac{125759\alpha^2}{3456} + \frac{497\alpha^3}{48} + \frac{233\alpha^4}{64} + \frac{45\alpha^5}{64} + \frac{5\alpha^6}{64} \\
& \quad - \Bigg(\frac{1937}{24} - \frac{15431\alpha}{288} - \frac{257\alpha^2}{96} - \frac{91\alpha^3}{96} + \frac{13\alpha^4}{96} \Bigg) \zeta(3) \\
& \quad - \Bigg(\frac{7025}{192} + \frac{115\alpha}{8} + \frac{385\alpha^2}{96} - \frac{5\alpha^3}{24} - \frac{35\alpha^4}{192} \Bigg) \zeta(5) \\
& \quad - \Bigg(\frac{9}{32} + \frac{3\alpha}{8} + \frac{3\alpha^2}{32} \Bigg) \zeta(4) \Bigg) C_A^3 \Bigg) \Bigg] a^3 \; + \; O(a^4) \qquad (2.10)
\end{aligned}
$$

$$
Z_\alpha^{\mathrm{RI}'} \; = \; 1 \; + \; O(a^4) \qquad (2.11)
$$

$$
\begin{aligned}
Z_c^{\mathrm{RI}'} \; = \; & 1 \; + \; \Bigg(\Bigg(\frac{3}{4} - \frac{\alpha}{4} \Bigg) \frac{1}{\epsilon} + 1 \Bigg) C_A a \; + \; \Bigg[\Bigg(\Bigg(\Bigg(\frac{3\alpha^2}{32} - \frac{35}{32} \Bigg) C_A^2 + \frac{1}{2} C_A T_F N_f \Bigg) \frac{1}{\epsilon^2} \\
& - \Bigg(\Bigg(\frac{\alpha^3}{16} + \frac{\alpha^2}{8} + \frac{257\alpha}{288} - \frac{167}{96} \Bigg) C_A^2 + \Bigg(\frac{5}{12} - \frac{5\alpha}{9} \Bigg) C_A T_F N_f \Bigg) \frac{1}{\epsilon} \\
& + \Bigg(\Bigg(\frac{1943}{192} - \frac{7\alpha}{64} + \frac{3\alpha^2}{8} - \Bigg(\frac{15}{16} - \frac{3\alpha}{8} + \frac{3\alpha^2}{16} \Bigg) \zeta(3) \Bigg) C_A^2 - \frac{95}{24} C_A T_F N_f \Bigg) \Bigg] a^2 \\
+ & \Bigg[\Bigg(\Bigg(\frac{2765}{1152} + \frac{35\alpha}{384} - \frac{9\alpha^2}{128} - \frac{5\alpha^3}{128} \Bigg) C_A^3 - C_A^2 T_F N_f \Bigg(\frac{149}{72} + \frac{\alpha}{24} \Bigg) + \frac{4}{9} C_A T_F^2 N_f^2 \Bigg) \frac{1}{\epsilon^3} \\
& + \Bigg(\Bigg(\frac{3\alpha^4}{64} + \frac{11\alpha^3}{96} + \frac{269\alpha^2}{384} + \frac{5\alpha}{96} - \frac{19367}{3456} \Bigg) C_A^3 + C_A C_F T_F N_f \\
& \quad + \Bigg(\frac{1621}{432} - \frac{\alpha}{48} - \frac{5\alpha^2}{12} \Bigg) C_A^2 T_F N_f - \frac{10}{27} C_A T_F^2 N_f^2 \Bigg) \frac{1}{\epsilon^2} \\
& + \Bigg(\Bigg(\frac{241171}{20736} - \frac{117809\alpha}{10368} - \frac{1015\alpha^2}{2304} - \frac{919\alpha^3}{1152} - \frac{11\alpha^4}{64} - \frac{\alpha^5}{32} \\
& \quad + \Bigg(\frac{3\alpha^3}{64} - \frac{45\alpha^2}{64} + \frac{89\alpha}{64} - \frac{39}{64} \Bigg) \zeta(3) \Bigg) C_A^3 \\
& \quad - \Bigg(\frac{9551}{2592} - \frac{21899\alpha}{2592} - \frac{5\alpha^2}{9} - \frac{5\alpha^3}{18} + 3\zeta(3) - 2\alpha\zeta(3) \Bigg) C_A^2 T_F N_f \\
& \quad - \Bigg(\frac{15}{4} - \frac{55\alpha}{12} - 4\zeta(3) + 4\alpha\zeta(3) \Bigg) C_A C_F T_F N_f \\
& \quad - \Bigg(\frac{35}{81} + \frac{100\alpha}{81} \Bigg) C_A T_F^2 N_f^2 \Bigg) \frac{1}{\epsilon} \\
& + \Bigg(\Bigg(\frac{1082353}{7776} - \frac{313\alpha}{768} + \frac{253\alpha^2}{48} + \frac{989\alpha^3}{768} + \frac{3\alpha^4}{16}
\end{aligned}
$$

$$
\begin{aligned}
& - \left(\frac{13483}{576} - \frac{589\alpha}{64} + \frac{509\alpha^2}{192} + \frac{29\alpha^3}{64} + \frac{3\alpha^4}{32} \right) \zeta(3) \\
& + \left(\frac{9}{64} + \frac{3\alpha}{16} + \frac{3\alpha^2}{64} \right) \zeta(4) - \left(\frac{65}{32} + \frac{65\alpha}{32} - \frac{35\alpha^2}{32} + \frac{5\alpha^3}{32} \right) \zeta(5) \Bigg) C_A^3 \\
& + \left(22\zeta(3) + 6\zeta(4) - \frac{899}{24} \right) C_A C_F T_F N_f + \left(\frac{5161}{486} + \frac{8}{9}\zeta(3) \right) C_A T_F^2 N_f^2 \\
& - \left(\frac{165637}{1944} - \frac{13\alpha}{8} + \frac{5\alpha^2}{3} + \left(\frac{29}{9} + 3\alpha - \frac{5\alpha^2}{6} \right) \zeta(3) \right. \\
& \quad \left. + \frac{9}{2}\zeta(4) \right) C_A^2 T_F N_f \Bigg) \Bigg] a^3 + O(a^4) \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
Z_\psi^{\mathrm{RI}'} &= 1 - \left(\frac{\alpha C_F}{\epsilon} + \alpha C_F \right) a + \left[\left(C_F C_A \left(\frac{\alpha^2}{4} + \frac{3\alpha}{4} \right) + \frac{\alpha^2}{2} C_F^2 \right) \frac{1}{\epsilon^2} \right. \\
& - \left(\left(\frac{\alpha^4}{4} + \frac{5\alpha^3}{8} + \frac{\alpha^2}{8} + \frac{133\alpha}{36} + \frac{25}{8} \right) C_F C_A \right. \\
& \quad \left. - \left(1 + \frac{20\alpha}{9} \right) C_F T_F N_f - \left(\frac{3}{4} + \alpha^2 \right) C_F^2 \right) \frac{1}{\epsilon} \\
& \quad + \left(\left(3\zeta(3) + 3\alpha\zeta(3) - \frac{41}{4} - \frac{331\alpha}{36} - \frac{13\alpha^2}{8} - \frac{\alpha^4}{4} \right) C_F C_A \right. \\
& \quad \left. \left. + \left(\frac{7}{2} + \frac{20\alpha}{9} \right) C_F T_F N_f + \left(\frac{5}{8} + \alpha^2 \right) C_F^2 \right) \right] a^2 \\
& + \left[\left(\frac{\alpha}{3} C_A C_F T_F N_f - \left(\frac{3\alpha^2}{4} + \frac{\alpha^3}{4} \right) C_A C_F^2 - \left(\frac{31\alpha}{24} + \frac{3\alpha^2}{8} + \frac{\alpha^3}{12} \right) C_A^2 C_F - \frac{\alpha}{6} C_F^3 \right) \frac{1}{\epsilon^3} \right. \\
& + \left(\frac{8}{9} C_F T_F^2 N_f^2 - \left(\frac{3\alpha}{4} + \frac{\alpha^3}{2} \right) C_F^3 + \left(\frac{2}{3} - \alpha - \frac{20\alpha^2}{9} \right) C_F^2 T_F N_f \right. \\
& \quad - \left(\frac{47}{9} + \frac{8\alpha}{3} + \frac{10\alpha^2}{9} \right) C_A C_F T_F N_f \\
& \quad + \left(\frac{25\alpha}{8} + \frac{53\alpha^2}{18} + \frac{3\alpha^3}{8} + \frac{\alpha^4}{4} - \frac{11}{6} \right) C_A C_F^2 \\
& \quad \left. + \left(\frac{275}{36} + \frac{81\alpha}{16} + \frac{89\alpha^2}{36} + \frac{9\alpha^3}{16} + \frac{\alpha^4}{8} \right) C_A^2 C_F \right) \frac{1}{\epsilon^2} \\
& + \left(\left(\frac{287}{27} + \frac{4919\alpha}{162} + \frac{25\alpha^2}{9} + \frac{10\alpha^3}{9} + 8\alpha\zeta(3) \right) C_A C_F T_F N_f \right. \\
& \quad - \left(\frac{20}{27} + \frac{400\alpha}{81} \right) C_F T_F^2 N_f^2 - \left(\frac{1}{2} + \frac{11\alpha}{8} + \alpha^3 \right) C_F^3 \\
& \quad - \left(1 - \frac{83\alpha}{6} + \frac{40\alpha^2}{9} + 16\alpha\zeta(3) \right) C_F^2 T_F N_f \\
& \quad - \left(\frac{\alpha^5}{8} + \frac{3\alpha^4}{4} + \frac{217\alpha^3}{72} + \frac{289\alpha^2}{72} + \frac{48595\alpha}{1296} + \frac{9155}{432} \right. \\
& \quad \quad \left. - \left(\frac{23}{8} + \frac{11\alpha}{4} - \frac{17\alpha^2}{8} \right) \zeta(3) \right) C_A^2 C_F \\
& \quad \left. + \left(\frac{143}{12} + \frac{107\alpha}{8} + \frac{116\alpha^2}{9} + \frac{9\alpha^3}{4} + \frac{\alpha^4}{2} - \left(4 + 3\alpha + 3\alpha^2 \right) \zeta(3) \right) C_A C_F^2 \right) \frac{1}{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\left(\frac{11887}{81} + \frac{42185\alpha}{648} + \frac{65\alpha^2}{9} + \frac{10\alpha^3}{9} - \frac{52}{3}\zeta(3) - \frac{20\alpha}{3}\zeta(3) \right) C_F C_A T_F N_f \right. \\
&\quad + \left(\frac{79}{6} + \frac{77\alpha}{6} - \frac{40\alpha^2}{9} - 16\zeta(3) - 16\alpha\zeta(3) \right) C_F^2 T_F N_f \\
&\quad - \left(\frac{1570}{81} + \frac{400\alpha}{81} \right) C_F T_F^2 N_f^2 \\
&\quad + \left(\left(\frac{3139}{24} + \frac{553\alpha}{12} + \frac{35\alpha^2}{8} + \frac{13\alpha^3}{12} \right) \zeta(3) + \left(\frac{69}{16} - \frac{3\alpha}{8} - \frac{3\alpha^2}{16} \right) \zeta(4) \right. \\
&\qquad - \left(\frac{165}{4} + \frac{5\alpha}{2} + \frac{5\alpha^2}{4} \right) \zeta(5) - \frac{159257}{648} - \frac{615193\alpha}{5184} \\
&\qquad \left. - \frac{13849\alpha^2}{576} - \frac{1091\alpha^3}{144} - \frac{5\alpha^4}{4} - \frac{\alpha^5}{8} \right) C_F C_A^2 \\
&\quad + \left(\frac{997}{24} + \frac{33\alpha}{2} + \frac{152\alpha^2}{9} + \frac{27\alpha^3}{8} + \frac{\alpha^4}{2} - \left(44 - 11\alpha + 6\alpha^2 + \alpha^3 \right) \zeta(3) \right. \\
&\qquad \left. - 6\zeta(4) + 20(1-\alpha)\zeta(5) \right) C_F^2 C_A \\
&\quad \left. + \left(\frac{73}{12} - \frac{17\alpha}{8} - \alpha^3 + \frac{2\alpha^3}{3}\zeta(3) \right) C_F^3 \right) \Bigg] a^3 \; + \; O(a^4)
\end{aligned}
\tag{2.13}
$$

where $\mathrm{Tr}\left(T^a T^b\right) = T_F \delta^{ab}$, $T^a T^a = C_F I$, $f^{acd} f^{bcd} = C_A \delta^{ab}$, $\zeta(n)$ is the Riemann zeta function, $a = g^2/(16\pi^2)$ and g is the coupling constant appearing in the covariant derivative. At this point it is worth commenting on the status of the variables a and α. As we have computed the renormalization constants for the RI$'$ scheme they correspond to the RI$'$ scheme coupling constant and covariant gauge parameter. When it is necessary we will include a label on the variables to distinguish which scheme they are defined in. As with other schemes they can be related to the corresponding $\overline{\mathrm{MS}}$ variables which we will discuss later. Throughout this article when we quote any renormalization constant the scheme will be denoted on the renormalization constant itself and it will be understood that the variables will be in that scheme as well. For completeness we note,

$$
\begin{aligned}
Z_\psi^{\overline{\mathrm{MS}}} &= 1 \; - \; \alpha C_F \frac{a}{\epsilon} \; + \; \left[\left(C_F C_A \left(\frac{\alpha^2}{4} + \frac{3\alpha}{4} \right) + \frac{\alpha^2}{2} C_F^2 \right) \frac{1}{\epsilon^2} \right. \\
&\qquad\qquad \left. - \left(C_F C_A \left(\frac{\alpha^2}{8} + \alpha + \frac{25}{8} \right) \; - \; C_F T_F N_f \; - \; \frac{3}{4} C_F^2 \right) \frac{1}{\epsilon} \right] a^2 \\
&\quad + \left[\left(\frac{\alpha}{3} C_A C_F T_F N_f - \left(\frac{3\alpha^2}{4} + \frac{\alpha^3}{4} \right) C_A C_F^2 \right. \right. \\
&\qquad \left. - \left(\frac{31\alpha}{24} + \frac{3\alpha^2}{8} + \frac{\alpha^3}{12} \right) C_A^2 C_F - \frac{\alpha^3}{6} C_F^3 \right) \frac{1}{\epsilon^3} \\
&\qquad + \left(\frac{8}{9} C_F T_F^2 N_f^2 - \frac{3\alpha}{4} C_F^3 + \left(\frac{2}{3} - \alpha \right) C_F^2 T_F N_f - \left(\frac{47}{9} + \alpha \right) C_A C_F T_F N_f \right. \\
&\qquad\quad \left. + \left(\frac{\alpha^3}{8} + \alpha^2 + \frac{25\alpha}{8} - \frac{11}{6} \right) C_A C_F^2 + \left(\frac{275}{36} + \frac{73\alpha}{24} + \frac{3\alpha^2}{4} + \frac{\alpha^3}{8} \right) C_A^2 C_F \right) \frac{1}{\epsilon^2} \\
&\qquad + \left(- \frac{20}{27} C_F T_F^2 N_f^2 - \frac{1}{2} C_F^3 - C_F^2 T_F N_f + \left(\frac{287}{27} + \frac{17\alpha}{12} \right) C_A C_F T_F N_f \right.
\end{aligned}
$$

$$- \left(\frac{5\alpha^3}{48} + \frac{13\alpha^2}{32} + \frac{263\alpha}{96} + \frac{9155}{432} - \left(\frac{23}{8} - \frac{\alpha}{4} - \frac{\alpha^2}{8} \right) \zeta(3) \right) C_A^2 C_F$$
$$+ \left. \left. \left(\frac{143}{12} - 4\zeta(3) \right) C_A C_F^2 \right) \frac{1}{\epsilon} \right] a^3 + O(a^4) \qquad (2.14)$$

where the variables a and α correspond to those of the $\overline{\mathrm{MS}}$ scheme. There are several checks on these results. First, we have verified that the correct three loop $\overline{\mathrm{MS}}$ renormalization constants, [25, 26, 27, 28, 29], emerge with the programmes we have written prior to extracting the results for the RI$'$ scheme. Second, the gauge parameter, α, does not get renormalized as in the $\overline{\mathrm{MS}}$ scheme so that gauge invariance is not destroyed. Next, our three loop result for $Z_\psi^{\mathrm{RI}'}$ agrees with the four loop result of [12] in the *Landau* gauge. They do not agree for non-zero α. The reason for this is that in the RI$'$ scheme used in [12] the values taken for Z_A and Z_α were different from those given above. More specifically the $\overline{\mathrm{MS}}$ expressions were used which only differ in the case $\alpha \neq 0$. However, using the scheme adopted in [12] we have reproduced the expressions given there for all α which again confirms the correctness of our programming. However, as we believe the full RI$'$ scheme we have introduced here is more *natural* we will always present subsequent results with reference to the above renormalization constants. Indeed it is not inconceivable that at some point one would require the renormalization of an operator with gluon content in RI$'$ and therefore one would require our $Z_A^{\mathrm{RI}'}$. Nevertheless all results should be consistent with [12] in the Landau gauge, $\alpha = 0$.

Whilst the results for the field renormalizations are deduced from the two-point functions we have yet to establish their consistency. This can be verified by renormalizing several three-point functions to obtain the same coupling constant renormalizations. We have examined both the quark gluon and ghost gluon vertices. However, with the wave function renormalization containing a finite part at one loop the method of renormalizing the vertices in the RI$'$ scheme cannot be the same as for the two-point functions. In other words one cannot absorb a finite part from the three-point functions into the definition of the coupling constant renormalization. For instance, at one loop the quark gluon vertex would give an N_f dependent finite part to that coupling constant renormalization constant. However, this N_f dependence cannot be matched in the gluon ghost vertex which is N_f independent at one loop. Although it is possible to follow this route and accommodate the problem of the finite parts in the three-point vertices it will lead to an MOM or $\overline{\mathrm{MOM}}$ renormalization scheme which we are not considering here, [7, 8]. Therefore, to define a three loop renormalization constant for Z_g in the RI$'$ scheme which is consistent with the Slavnov-Taylor identities we define the Z_g renormalization for each vertex in an $\overline{\mathrm{MS}}$ way. In other words we define Z_g by only absorbing the infinities without removing any finite parts, aside from powers of $\ln(4\pi e^{-\gamma})$. With this we can consistently deduce the same coupling constant renormalization from both three-point functions. More concretely we have

$$\lim_{\epsilon \rightarrow 0} \left[Z_\psi^{\mathrm{RI}'} \left(Z_A^{\mathrm{RI}'} \right)^{\frac{1}{2}} Z_g^{\mathrm{RI}'} G^{\mu\, a}_{A\bar{\psi}\psi}(p) \right] \Bigg|_{p^2 = \mu^2} = G^{\mu\, a \;\mathrm{finite}}_{A\bar{\psi}\psi} \qquad (2.15)$$

and

$$\lim_{\epsilon \rightarrow 0} \left[Z_c^{\mathrm{RI}'} \left(Z_A^{\mathrm{RI}'} \right)^{\frac{1}{2}} Z_g^{\mathrm{RI}'} G_{A\bar{c}c}(p) \right] \Bigg|_{p^2 = \mu^2} = G^{\mathrm{finite}}_{A\bar{c}c} \qquad (2.16)$$

where G_i^{finite} are treated as finite with respect to ϵ and are *not* unity and we have omitted the overall structure function from the ghost gluon Green's function. Though the tree part of each G_i^{finite} is unity since we have first divided $G_i(p)$ by the bare coupling constant before rescaling as in the method of [24]. To evaluate $Z_g^{\mathrm{RI}'}$ explicitly we have again used Mincer, [20]. However, to do this correctly the external momentum of one leg must be nullified as Mincer can only be applied to two-point functions. To avoid potential spurious infrared infinities arising in this case we have chosen to nullify an external quark or ghost leg respectively leaving p as the overall external momentum. Consequently we find from both three-point functions that

$$\begin{aligned}
Z_g^{\mathrm{RI}'} &= 1 + \left(\frac{2}{3} T_F N_f - \frac{11}{6} C_A \right) \frac{a}{\epsilon} + \left[\left(\frac{121}{24} C_A^2 + \frac{2}{3} T_F^2 N_f^2 - \frac{11}{3} C_A T_F N_f \right) \frac{1}{\epsilon^2} \right. \\
&+ \left. \left(C_F T_F N_f + \frac{5}{3} C_A T_F N_f - \frac{17}{6} C_A^2 \right) \frac{1}{\epsilon} \right] a^2 \\
&+ \left[\left(\frac{605}{36} C_A^2 T_F N_f - \frac{55}{9} C_A T_F^2 N_f^2 + \frac{20}{27} T_F^3 N_f^3 - \frac{6655}{432} C_A^3 \right) \frac{1}{\epsilon^3} \right. \\
&\quad + \left(\frac{22}{9} C_F T_F^2 N_f^2 - \frac{121}{18} C_A C_F T_F N_f - \frac{979}{54} C_A^2 T_F N_f + \frac{110}{27} C_A T_F^2 N_f^2 + \frac{2057}{108} C_A^3 \right) \frac{1}{\epsilon^2} \\
&\quad + \left(\frac{205}{54} C_A C_F T_F N_f - \frac{22}{27} C_F T_F^2 N_f^2 + \frac{1415}{162} C_A^2 T_F N_f \right. \\
&\qquad - \left. \left. \frac{79}{81} C_A T_F^2 N_f^2 - \frac{1}{3} C_F^2 T_F N_f - \frac{2857}{324} C_A^3 \right) \frac{1}{\epsilon} \right] a^3 + O(a^4)
\end{aligned} \tag{2.17}$$

which is the same as in the $\overline{\mathrm{MS}}$ scheme, [29, 28], though it could only have differed in the three loop term which is scheme dependent. Thus to this order the RI′ and $\overline{\mathrm{MS}}$ β-functions coincide. In our conventions we have, for all α,

$$\begin{aligned}
\beta^{\mathrm{RI}'}(a_{\mathrm{RI}'}) &= - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a_{\mathrm{RI}'}^2 - \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a_{\mathrm{RI}'}^3 \\
&+ \left[2830 C_A^2 T_F N_f - 2857 C_A^3 + 1230 C_A C_F T_F N_f - 316 C_A T_F^2 N_f^2 \right. \\
&\quad \left. - 108 C_F^2 T_F N_f - 264 C_F T_F^2 N_f^2 \right] \frac{a_{\mathrm{RI}'}^4}{54} + O(a_{\mathrm{RI}'}^5)
\end{aligned} \tag{2.18}$$

where we have explicitly indicated the scheme of the coupling constant as a subscript. Given that the three loop term of the β-function can be different we have constructed the relation between the coupling constants of the RI′ and $\overline{\mathrm{MS}}$ schemes explicitly. To do this we parallel the approach of [30] where the same procedure was followed to establish the relation between various $\overline{\mathrm{MOM}}$ coupling constants and the $\overline{\mathrm{MS}}$ coupling. Since we have renormalized two different three-point functions we have to check that both give equivalent results and are consistent with the Slavnov-Taylor identities. We define

$$a_{\mathrm{RI}'}(\mu) = a_{\overline{\mathrm{MS}}}(\mu) \left[\frac{1}{\Pi_T^{\overline{\mathrm{MS}} \text{ finite}}(p) \left(\Sigma_c^{\overline{\mathrm{MS}} \text{ finite}}(p) \right)^2} \left[\frac{G_{A\bar{c}c}^{\overline{\mathrm{MS}} \text{ finite}}}{G_{A\bar{c}c}^{\mathrm{RI}' \text{ finite}}} \right]^2 \right] \Bigg|_{p^2 = \mu^2} \tag{2.19}$$

and

$$a_{\mathrm{RI}'}(\mu) = a_{\overline{\mathrm{MS}}}(\mu) \left[\frac{1}{\Pi_T^{\overline{\mathrm{MS}} \text{ finite}}(p) \left(\Sigma_\psi^{\overline{\mathrm{MS}} \text{ finite}}(p) \right)^2} \left[\frac{G_{A\bar{\psi}\psi}^{(1)\, \overline{\mathrm{MS}} \text{ finite}}}{G_{A\bar{\psi}\psi}^{(1)\, \mathrm{RI}' \text{ finite}}} \right]^2 \right] \Bigg|_{p^2 = \mu^2} \tag{2.20}$$

where the Green's functions on the right hand side are the finite expressions. Moreover, since we are concerned with the finite part of the Green's function after renormalization $G_{A\bar{\psi}\psi}^{(1)}$ corresponds to a particular Lorentz projection of the full Green's function $G_{A\bar{\psi}\psi}^{\mu\, a}$, [30]. In particular, for the momentum routing we are considering, we have defined, [30],

$$G_{A\bar{\psi}\psi}^{\mu\, a}(p) = T^a \left[G_{A\bar{\psi}\psi}^{(1)}(p) \gamma^\mu + G_{A\bar{\psi}\psi}^{(2)}(p) \left(\gamma^\mu - \frac{\not{p} p^\mu}{p^2} \right) \right] \tag{2.21}$$

where

$$T^a G^{(1)}_{A\bar{\psi}\psi}(p) = \frac{1}{4p^2} \mathrm{tr} \left[p_\mu \not{p} G^{\mu\, a}_{A\bar{\psi}\psi}(p) \right]$$

$$T^a G^{(2)}_{A\bar{\psi}\psi}(p) = \frac{1}{4(d-1)} \left[\mathrm{tr} \left(\gamma_\mu G^{\mu\, a}_{A\bar{\psi}\psi}(p) \right) - d\, \mathrm{tr} \left(\frac{\not{p} p_\mu}{p^2} G^{\mu\, a}_{A\bar{\psi}\psi}(p) \right) \right] . \quad (2.22)$$

For the ghost gluon vertex we do not need to take into account several projections since there is only one Lorentz structure for that vertex with one external momentum nullified. In addition to computing $a_{\mathrm{RI}'}$ as a function of $a_{\overline{\mathrm{MS}}}$ for both vertices we will also need the relation of the covariant gauge parameter in one scheme with that in the other since, for instance, $G^{\mathrm{RI}' \text{ finite}}_{A\bar{c}c}$ depends on $a_{\mathrm{RI}'}$ and $\alpha_{\mathrm{RI}'}$. This is achieved by, [30],

$$\alpha_{\mathrm{RI}'} = \frac{Z^{\mathrm{RI}'}_A}{Z^{\overline{\mathrm{MS}}}_A} \alpha_{\overline{\mathrm{MS}}} . \quad (2.23)$$

By solving (2.19), (2.20) and (2.23) iteratively we have determined the relationships between the coupling constants and covariant gauge parameters in both schemes at three loops. From both the vertices we find

$$a_{\mathrm{RI}'} = a_{\overline{\mathrm{MS}}} + O\left(a^5_{\overline{\mathrm{MS}}} \right) \quad (2.24)$$

for all gauges and for the covariant gauge parameter

$$\begin{aligned}
\alpha_{\mathrm{RI}'} = & \left[1 + \left(\left(-9\alpha^2_{\overline{\mathrm{MS}}} - 18\alpha_{\overline{\mathrm{MS}}} - 97 \right) C_A + 80 T_F N_f \right) \frac{a_{\overline{\mathrm{MS}}}}{36} \right. \\
& + \left(\left(18\alpha^4_{\overline{\mathrm{MS}}} - 18\alpha^3_{\overline{\mathrm{MS}}} + 190\alpha^2_{\overline{\mathrm{MS}}} - 576\zeta(3)\alpha_{\overline{\mathrm{MS}}} + 463\alpha_{\overline{\mathrm{MS}}} + 864\zeta(3) - 7143 \right) C_A^2 \right. \\
& \quad + \left(-320\alpha^2_{\overline{\mathrm{MS}}} - 320\alpha_{\overline{\mathrm{MS}}} + 2304\zeta(3) + 4248 \right) C_A T_F N_f \\
& \quad \left. + (-4608\zeta(3) + 5280) C_F T_F N_f \right) \frac{a^2_{\overline{\mathrm{MS}}}}{288} \\
& + \left(\left(-486\alpha^6_{\overline{\mathrm{MS}}} + 1944\alpha^5_{\overline{\mathrm{MS}}} + 4212\zeta(3)\alpha^4_{\overline{\mathrm{MS}}} - 5670\zeta(5)\alpha^4_{\overline{\mathrm{MS}}} - 18792\alpha^4_{\overline{\mathrm{MS}}} \right. \right. \\
& \quad + 48276\zeta(3)\alpha^3_{\overline{\mathrm{MS}}} - 6480\zeta(5)\alpha^3_{\overline{\mathrm{MS}}} - 75951\alpha^3_{\overline{\mathrm{MS}}} - 52164\zeta(3)\alpha^2_{\overline{\mathrm{MS}}} \\
& \quad + 2916\zeta(4)\alpha^2_{\overline{\mathrm{MS}}} + 124740\zeta(5)\alpha^2_{\overline{\mathrm{MS}}} + 92505\alpha^2_{\overline{\mathrm{MS}}} - 1303668\zeta(3)\alpha_{\overline{\mathrm{MS}}} \\
& \quad + 11664\zeta(4)\alpha_{\overline{\mathrm{MS}}} + 447120\zeta(5)\alpha_{\overline{\mathrm{MS}}} + 354807\alpha_{\overline{\mathrm{MS}}} + 2007504\zeta(3) \\
& \quad + 8748\zeta(4) + 1138050\zeta(5) - 10221367) C_A^3 \\
& \quad + \left(12960\alpha^4_{\overline{\mathrm{MS}}} - 8640\alpha^3_{\overline{\mathrm{MS}}} - 129600\zeta(3)\alpha^2_{\overline{\mathrm{MS}}} - 147288\alpha^2_{\overline{\mathrm{MS}}} + 698112\zeta(3)\alpha_{\overline{\mathrm{MS}}} \right. \\
& \qquad - 312336\alpha_{\overline{\mathrm{MS}}} + 1505088\zeta(3) - 279936\zeta(4) \\
& \qquad - 1658880\zeta(5) + 9236488) C_A^2 T_F N_f \\
& \quad + \left(248832\zeta(3)\alpha^2_{\overline{\mathrm{MS}}} - 285120\alpha^2_{\overline{\mathrm{MS}}} + 248832\zeta(3)\alpha_{\overline{\mathrm{MS}}} - 285120\alpha_{\overline{\mathrm{MS}}} \right. \\
& \qquad - 5156352\zeta(3) + 373248\zeta(4) - 2488320\zeta(5) + 9293664) C_A C_F T_F N_f \\
& \quad + \left(-38400\alpha^2_{\overline{\mathrm{MS}}} - 221184\zeta(3)\alpha_{\overline{\mathrm{MS}}} + 55296\alpha_{\overline{\mathrm{MS}}} \right. \\
& \qquad - 884736\zeta(3) - 1343872) C_A T_F^2 N_f^2 \\
& \quad + (-3068928\zeta(3) + 4976640\zeta(5) - 988416) C_F^2 T_F N_f \\
& \quad \left. \left. + (2101248\zeta(3) - 2842368) C_F T_F^2 N_f^2 \right) \frac{a^3_{\overline{\mathrm{MS}}}}{31104} \right] \alpha_{\overline{\mathrm{MS}}} + O\left(a^4_{\overline{\mathrm{MS}}} \right) . \quad (2.25)
\end{aligned}$$

The former expression is consistent with the three loop RI′ β-function being equivalent to the $\overline{\mathrm{MS}}$ one for arbitrary covariant gauge. Moreover, since the three loop term of the transformation is also absent this implies that the *four* loop RI$'$ β-function is also equivalent to the *four* loop $\overline{\mbox{MS}}$ β-function in all gauges. The non-trivial relation between the gauge parameters will be crucial in carrying out checks on the renormalization group functions.

We have calculated the renormalization group functions for the various wave function renormalizations directly from the renormalization constants themselves. In particular we used

$$\begin{aligned}
\gamma^{\mbox{RI}'}_A(a) &= \beta(a)\frac{\partial \ln Z^{\mbox{RI}'}_A}{\partial a} ~+~ \alpha\gamma^{\mbox{RI}'}_\alpha(a)\frac{\partial \ln Z^{\mbox{RI}'}_A}{\partial \alpha} \\
\gamma^{\mbox{RI}'}_\alpha(a) &= \left[\beta(a)\frac{\partial \ln Z^{\mbox{RI}'}_\alpha}{\partial a} ~-~ \gamma^{\mbox{RI}'}_A(a)\right]\left[1 ~-~ \alpha\frac{\partial \ln Z^{\mbox{RI}'}_\alpha}{\partial \alpha}\right]^{-1} \\
\gamma^{\mbox{RI}'}_\psi(a) &= \beta(a)\frac{\partial \ln Z^{\mbox{RI}'}_\psi}{\partial a} ~+~ \alpha\gamma^{\mbox{RI}'}_\alpha(a)\frac{\partial \ln Z^{\mbox{RI}'}_\psi}{\partial \alpha} \\
\gamma^{\mbox{RI}'}_c(a) &= \beta(a)\frac{\partial \ln Z^{\mbox{RI}'}_c}{\partial a} ~+~ \alpha\gamma^{\mbox{RI}'}_\alpha(a)\frac{\partial \ln Z^{\mbox{RI}'}_c}{\partial \alpha}
\end{aligned} \qquad (2.26)$$

though $Z^{\mbox{RI}'}_\alpha = 1$ at three loops implies that

$$\gamma^{\mbox{RI}'}_A(a) ~=~ -~ \gamma^{\mbox{RI}'}_\alpha(a) \qquad (2.27)$$

to the same order which corresponds to the gluon propagator being transverse. Hence, from the renormalization constants we have computed we find, in four dimensions, that

$$\begin{aligned}
\gamma^{\mbox{RI}'}_A(a) &= [8T_FN_f - (13-3\alpha)C_A]\frac{a}{6} \\
&- \left[\left(27\alpha^3 - 90\alpha^2 - 426\alpha + 3727\right)C_A^2 + \left(72\alpha^2 + 240\alpha - 3616\right)C_AT_FN_f\right. \\
&\quad \left. -~ 864C_FT_FN_f + 640T_F^2N_f^2\right]\frac{a^2}{216} \\
&+ \left[51200T_F^3N_f^3 - 15552C_F^2T_FN_f + (331776\zeta(3) - 487296)\,C_FT_F^2N_f^2\right. \\
&\quad -~ \left(486\alpha^5 + 3078\alpha^4 + 10260\alpha^3 - 1458\zeta(3)\alpha^2 - 25965\alpha^2 + 86184\zeta(3)\alpha\right. \\
&\quad\quad -~ 173406\alpha - 175446\zeta(3) + 2127823)\,C_A^3 ~-~ \left(648\alpha^4 + 216\alpha^3 + 47808\alpha^2\right. \\
&\quad\quad +~ 10368\zeta(3)\alpha + 126480\alpha - 254016\zeta(3) - 2501184)\,C_A^2T_FN_f \\
&\quad -~ \left(7776\alpha^2 - 62208\zeta(3)\alpha + 71280\alpha + 725760\zeta(3) - 1131408\right)C_AC_FT_FN_f \\
&\quad \left. +~ \left(11520\alpha^2 + 19200\alpha - 165888\zeta(3) - 751680\right)C_AT_F^2N_f^2\right]\frac{a^3}{7776} ~+~ O(a^4)
\end{aligned} \qquad (2.28)$$

$$\begin{aligned}
\gamma^{\mbox{RI}'}_\psi(a) &= \alpha C_Fa ~+~ \left[\left(9\alpha^3 + 45\alpha^2 + 223\alpha + 225\right)C_A - 54C_F - (80\alpha+72)\,T_FN_f\right]\frac{C_Fa^2}{36} \\
&+ \left[\left(162\alpha^5 + 1377\alpha^4 + 7578\alpha^3 + 1134\zeta(3)\alpha^2 + 22608\alpha^2\right.\right. \\
&\quad -~ 23004\zeta(3)\alpha + 113080\alpha - 39690\zeta(3) + 179811)\,C_A^2 \\
&\quad -~ \left(648\alpha^3 + 1944\alpha^2 - 15552\zeta(3) + 52272\right)C_AC_F \\
&\quad -~ \left(1440\alpha^3 + 7200\alpha^2 + 5184\zeta(3)\alpha + 63616\alpha - 10368\zeta(3) + 110016\right)C_AT_FN_f \\
&\quad +~ (20736\zeta(3)\alpha - 23760\alpha + 6048)\,C_FT_FN_f \\
&\quad \left. +~ (6400\alpha + 14976)\,T_F^2N_f^2 + 1944C_F^2\right]\frac{C_Fa^3}{1296} ~+~ O(a^4)
\end{aligned} \qquad (2.29)$$

and

$$
\begin{aligned}
\gamma_c^{\mathrm{RI}'}(a) &= [\alpha-3]\frac{C_A a}{4} + \left[\left(9\alpha^3+18\alpha^2+88\alpha-813\right)C_A-(80\alpha-312)T_FN_f\right]\frac{C_Aa^2}{144} \\
&+ \left[\left(162\alpha^5+891\alpha^4+972\zeta(3)\alpha^3+1503\alpha^3+4050\zeta(3)\alpha^2-2070\alpha^2\right.\right. \\
&\quad - 15876\zeta(3)\alpha+46363\alpha+34182\zeta(3)-471909)C_A^2 \\
&\quad - \left(1440\alpha^3+2880\alpha^2+7776\zeta(3)\alpha+39208\alpha-33696\zeta(3)-322680\right)C_AT_FN_f \\
&\quad + (20736\zeta(3)\alpha-23760\alpha-62208\zeta(3)+79056)C_FT_FN_f \\
&\quad \left. + (6400\alpha-48000)T_F^2N_f^2\right]\frac{C_Aa^3}{5184} + O(a^4)
\end{aligned}
\tag{2.30}
$$

where we use the same convention for the renormalization group functions as for the renormalization constants in that the variables a and α are in the scheme indicated on the renormalization group function itself. The expression for $\gamma_\psi^{\mathrm{RI}'}(a)$ agrees with the three loop expression for the colour group $SU(N_c)$ in the Landau gauge given in [12]. Moreover, a final check on our calculation resides in the fact that in constructing these RI$'$ scheme renormalization group functions the correct double and triple poles in ϵ in the renormalization constants have been determined in the computation. If they were not correct then *finite* renormalization group functions would *not* have emerged.

In [12] the quark anomalous dimension was computed explicitly by first determining the appropriate function which converts the $\overline{\mathrm{MS}}$ result to the RI$'$ expression following a standard procedure which is discussed in, for example, [31]. As a final check on our wave function renormalization group functions in the RI$'$ scheme we have also computed them from the conversion functions which are defined as

$$
C_A(a,\alpha) = \frac{Z_A^{\mathrm{RI}'}}{Z_A^{\overline{\mathrm{MS}}}} ~~,~~ C_c(a,\alpha) = \frac{Z_c^{\mathrm{RI}'}}{Z_c^{\overline{\mathrm{MS}}}} ~~,~~ C_\psi(a,\alpha) = \frac{Z_\psi^{\mathrm{RI}'}}{Z_\psi^{\overline{\mathrm{MS}}}} ~.
\tag{2.31}
$$

It is important to appreciate how these functions are explicitly constructed. They are functions of the two parameters a and α in the same scheme. However, the RI$'$ scheme renormalization constants depend on $a_{\mathrm{RI}'}$ and $\alpha_{\mathrm{RI}'}$ which therefore must be converted to their $\overline{\mathrm{MS}}$ counterparts. In the following expressions for the conversion functions, and those we give later, we have chosen to express them in terms of the $\overline{\mathrm{MS}}$ variables and omitted the corresponding subscript. We found

$$
\begin{aligned}
C_A(a,\alpha) &= 1 + \left[\left(9\alpha^2+18\alpha+97\right)C_A-80T_FN_f\right]\frac{a}{36} \\
&+ \left[\left(810\alpha^3+2430\alpha^2+5184\zeta(3)\alpha+2817\alpha-7776\zeta(3)+83105\right)C_A^2\right. \\
&\quad - (2880\alpha+20736\zeta(3)+69272)C_AT_FN_f+(41472\zeta(3)-47520)C_FT_FN_f \\
&\quad \left. + 12800T_F^2N_f^2\right]\frac{a^2}{2592} \\
&+ \left[\left(17010\zeta(5)\alpha^4-12636\zeta(3)\alpha^4+64638\alpha^4-51516\zeta(3)\alpha^3+19440\zeta(5)\alpha^3\right.\right. \\
&\quad + 322947\alpha^3+203148\zeta(3)\alpha^2-8748\zeta(4)\alpha^2-374220\zeta(5)\alpha^2+1094553\alpha^2 \\
&\quad + 4636764\zeta(3)\alpha-34992\zeta(4)\alpha-1341360\zeta(5)\alpha+1457685\alpha \\
&\quad - 7531056\zeta(3)-26244\zeta(4)-3414150\zeta(5)+44961125)C_A^3 \\
&\quad + \left(15552\zeta(3)\alpha^2-303912\alpha^2-3670272\zeta(3)\alpha-890064\alpha-7293888\zeta(3)\right. \\
&\qquad + 839808\zeta(4)+4976640\zeta(5)-49928712)C_A^2T_FN_f \\
&\quad + (746496\zeta(3)\alpha-855360\alpha+23514624\zeta(3)-1119744\zeta(4)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 7464960\zeta(5) - 37099872) C_A C_F T_F N_f \\
&+ (663552\zeta(3)\alpha + 64512\alpha + 5971968\zeta(3) + 13873536) C_A T_F^2 N_f^2 \\
&+ (9206784\zeta(3) - 14929920\zeta(5) + 2965248) C_F^2 T_F N_f \\
&+ (16130304 - 12939264\zeta(3)) C_F T_F^2 N_f^2 \\
&- 1024000 T_F^3 N_f^3 \Big] \frac{a^3}{93312} + O(a^4)
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
C_c(a,\alpha) &= 1 + C_A a \\
&+ \left[\left(72\alpha^2 - 36\zeta(3)\alpha^2 + 72\zeta(3)\alpha - 21\alpha - 180\zeta(3) + 1943 \right) C_A - 760 T_F N_f \right] \frac{C_A a^2}{192} \\
&+ \Big[\Big(29241\alpha^3 - 11178\zeta(3)\alpha^3 - 4860\zeta(5)\alpha^3 - 56862\zeta(3)\alpha^2 + 1458\zeta(4)\alpha^2 \\
&\quad + 34020\zeta(5)\alpha + 102789\alpha^2 + 254826\zeta(3)\alpha + 5832\zeta(4)\alpha - 63180\zeta(5)\alpha \\
&\quad - 3510\alpha - 728082\zeta(3) + 4374\zeta(4) - 63180\zeta(5) + 4329412) C_A^2 \\
&\quad + (42984\alpha - 67392\zeta(3)\alpha - 100224\zeta(3) - 139968\zeta(4) - 2650192) C_A T_F N_f \\
&\quad + (684288\zeta(3) + 186624\zeta(4) - 1165104) C_F T_F N_f \\
&\quad + (27648\zeta(3) + 330304) T_F^2 N_f^2 \Big] \frac{C_A a^3}{31104} + O(a^4)
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
C_\psi(a,\alpha) &= 1 - \alpha C_F a \\
&+ \left[\left(8\alpha^2 + 5 \right) C_F - \left(9\alpha^2 - 24\zeta(3)\alpha + 52\alpha - 24\zeta(3) + 82 \right) C_A + 28 T_F N_f \right] \frac{C_F a^2}{8} \\
&+ \Big[\Big(1728\zeta(3)\alpha^3 - 11880\alpha^3 + 25272\zeta(3)\alpha^2 - 972\zeta(4)\alpha^2 - 6480\zeta(5)\alpha^2 \\
&\quad - 63747\alpha^2 + 181440\zeta(3)\alpha - 1944\zeta(4)\alpha - 12960\zeta(5)\alpha - 358191\alpha \\
&\quad + 678024\zeta(3) + 22356\zeta(4) - 213840\zeta(5) - 1274056) C_A^2 \\
&\quad + \Big(12312\alpha^2 - 5184\zeta(3)\alpha^3 - 31104\zeta(3)\alpha^2 + 59616\alpha^2 + 57024\zeta(3)\alpha \\
&\qquad - 103680\zeta(5)\alpha + 85536\alpha - 228096\zeta(3) - 31104\zeta(4) \\
&\qquad + 103680\zeta(5) + 215352) C_A C_F \\
&\quad + \left(3456\zeta(3)\alpha^3 - 5184\alpha^3 - 11016\alpha + 31536 \right) C_F^2 \\
&\quad + (124056\alpha - 41472\zeta(3)\alpha - 89856\zeta(3) + 760768) C_A T_F N_f \\
&\quad + (68256 - 82944\zeta(3) - 28512\alpha) C_F T_F N_f \\
&\quad - 100480 T_F^2 N_f^2 \Big] \frac{C_F a^3}{5184} + O(a^4) \, .
\end{aligned}
\tag{2.34}
$$

With these the RI′ scheme renormalization group functions can be determined from

$$
\begin{aligned}
\gamma_i^{\mathrm{RI}'}\left(a_{\mathrm{RI}'}\right) &= \gamma_i^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right) + \beta\left(a_{\overline{\mathrm{MS}}}\right) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C_i\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right) \\
&\quad + \alpha_{\overline{\mathrm{MS}}} \gamma_\alpha^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C_i\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right)
\end{aligned}
\tag{2.35}
$$

where $i = A$, c or ψ and we have included the scheme dependence of the variables explicitly though to the order we are working to the β-function is the same in both schemes. For each of the three cases we have computed the right hand side of (2.35) in terms of the $\overline{\mathrm{MS}}$ variables and then converted to their RI′ counterparts before verifying that the same previous expressions correctly emerge in terms of the RI′ scheme variables. This completes the full three loop renormalization of the QCD Lagrangian in the RI′ scheme.

3 Quark mass anomalous dimension in the RI$'$ scheme.

We now turn to the problem of deducing similar renormalization constants for the composite quark currents of the form $\mathcal{O}_{\mathcal{A}} = \bar{\psi} \mathcal{A} \psi$ where $\mathcal{A} = 1$, γ^μ or $\sigma^{\mu\nu}$ with $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ where the latter corresponds to the tensor current. As the lattice data is available for the insertion of the operator at zero momentum to deduce the conversion functions we will insert each of $\mathcal{O}_{\mathcal{A}}$ at zero momentum into a quark two-point function and examine the divergence structure of

$$G_{\mathcal{O}_{\mathcal{A}}}(p) ~=~ \langle \psi(p) ~ [\bar{\psi} \mathcal{A} \psi](0) ~ \bar{\psi}(-p) \rangle ~=~ \langle \psi(p) ~ \mathcal{O}_{\mathcal{A}}(0) ~ \bar{\psi}(-p) \rangle ~. \tag{3.1}$$

In order to demonstrate the validity of this approach we must first reconstruct the anomalous dimension for the mass, [12], which corresponds to the operator $\mathcal{A} = 1$. Therefore, defining the renormalization constant $Z_{\bar{\psi}\psi}$ of the operator in the usual way by,

$$\mathcal{O}_{1\,\mathrm{o}} ~=~ Z_{\bar{\psi}\psi} \mathcal{O}_1 \tag{3.2}$$

the RI$'$ scheme value is given by the condition

$$\lim_{\epsilon \rightarrow 0} \left[Z^{\mathrm{RI}'}_{\bar{\psi}\psi} Z^{\mathrm{RI}'}_\psi \langle \psi(p) \mathcal{O}_{\bar{\psi}\psi}(0) \bar{\psi}(-p) \rangle \right] \Bigg|_{p^2 \, = \, \mu^2} ~=~ 1 \tag{3.3}$$

where the wave function renormalization constants arise from the external fields. In the $\overline{\mathrm{MS}}$ scheme this results in a gauge independent renormalization constant to all orders, [1, 3]. However, in RI$'$ this is not the case since

$$\begin{aligned}
Z^{\mathrm{RI}'}_{\bar{\psi}\psi} ~=~ & 1 ~-~ \left(\frac{3}{\epsilon} + 4 + \alpha \right) C_F a ~+~ \Bigg[\left(\frac{9}{2} C_F^2 + \frac{11}{2} C_F C_A - 2 T_F N_f C_F \right) \frac{1}{\epsilon^2} \\
& +~ \left(\frac{5}{3} T_F N_f C_F - \frac{97}{12} C_F C_A + \left(\frac{45}{4} + 3\alpha \right) C_F^2 \right) \frac{1}{\epsilon} \\
& +~ \Bigg(\left(\frac{83}{6} + \frac{20\alpha}{9} \right) T_F N_f C_F + \left(18\zeta(3) - \frac{1285}{24} - \frac{223\alpha}{36} - \frac{5\alpha^2}{4} - \frac{\alpha^3}{4} \right) C_F C_A \\
& \quad +~ \left(\frac{19}{8} - 12\zeta(3) + 4\alpha + \alpha^2 \right) C_F^2 \Bigg) \Bigg] a^2 \\
& +~ \Bigg[\left(\frac{88}{9} T_F N_f C_F C_A + 6 T_F N_f C_F^2 - \frac{16}{9} T_F^2 N_f^2 C_F - \frac{121}{9} C_F C_A^2 - \frac{33}{2} C_F^2 C_A - \frac{9}{2} C_F^3 \right) \frac{1}{\epsilon^3} \\
& \quad +~ \Bigg(\frac{40}{27} T_F^2 N_f^2 C_F - \frac{484}{27} T_F C_F C_A + \left(2\alpha - \frac{5}{3} \right) T_F N_f C_F^2 + \frac{1679}{54} C_F C_A^2 \\
& \quad +~ \left(\frac{49}{12} - \frac{11\alpha}{2} \right) C_F^2 C_A - \left(\frac{63}{4} + \frac{9\alpha}{2} \right) C_F^3 \Bigg) \frac{1}{\epsilon^2} \\
& +~ \Bigg(\left(\frac{556}{81} + 16\zeta(3) \right) T_F N_f C_F C_A - \left(\frac{197}{6} + 16\zeta(3) + \frac{25\alpha}{3} \right) T_F N_f C_F^2 + \frac{140}{81} T_F^2 N_f^2 C_F \\
& \quad -~ \frac{11413}{324} C_F C_A^2 + \left(\frac{4889}{24} - 54\zeta(3) + \frac{80\alpha}{3} + \frac{15\alpha^2}{4} + \frac{3\alpha^3}{4} \right) C_F^2 C_A \\
& \quad +~ \left(36\zeta(3) - \frac{205}{8} - \frac{45\alpha}{4} - 3\alpha^2 \right) C_F^3 \Bigg) \frac{1}{\epsilon} \\
& -~ \Bigg(\left(\frac{616}{9} \zeta(3) - 4\alpha\zeta(3) - \frac{95387}{243} - 24\zeta(4) - \frac{3976\alpha}{81} - \frac{50\alpha^2}{9} - \frac{10\alpha^3}{9} \right) T_F N_f C_F C_A \\
& \quad +~ \left(\frac{128}{3} \zeta(3) + 8\alpha\zeta(3) + 24\zeta(4) - \frac{1109}{9} + \frac{115\alpha}{18} + \frac{40\alpha^2}{9} \right) T_F N_f C_F^2
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{7514}{243} + \frac{400\alpha}{81} + \frac{32}{9}\zeta(3)\right) T_F^2 N_f^2 C_F + \left(\frac{3360023}{3888} + \frac{28351\alpha}{324} + \frac{157\alpha^2}{9}\right. \\
&+ \left. \frac{421\alpha^3}{72} + \frac{17\alpha^4}{16} + \frac{\alpha^5}{8} - \frac{31193}{72}\zeta(3) - \frac{65\alpha}{4}\zeta(3) + \frac{7\alpha^2}{8}\zeta(3) + 60\zeta(5)\right) C_F C_A^2 \\
&+ \left(\left(\frac{962}{3} + 43\alpha + 3\alpha^2\right)\zeta(3) - 20\zeta(5) - \frac{18781}{72} - \frac{6089\alpha}{72}\right. \\
&- \left.\frac{170\alpha^2}{9} - 4\alpha^3 - \frac{\alpha^4}{2}\right) C_F^2 C_A + \left(\frac{3227}{12} + \frac{31\alpha}{8} + 4\alpha^2 + \alpha^3\right. \\
&- \left.\left.\left(58 - 6\alpha + 6\alpha^2\right)\zeta(3) - 120\zeta(5)\right) C_F^3 \right)\Bigg] a^3 \; + \; O(a^4) \qquad (3.4)
\end{aligned}
$$

which agrees with the quark mass renormalization constant $Z_m^{\mathrm{RI}'}$ deduced at four loops in [12] in the Landau gauge in our conventions. Thus we have demonstrated the equivalence of our operator method with the massive propagator approach of [12]. Further, we have checked that the correct three loop $\overline{\mathrm{MS}}$ quark mass anomalous dimension, [32, 33], emerges from our programmes. Therefore, from $Z_m^{\mathrm{RI}'}$ we can deduce the corresponding renormalization group function. With

$$
\gamma_{\bar\psi\psi}^{\mathrm{RI}'}(a) \;=\; -\;\beta(a)\frac{\partial \ln Z_{\bar\psi\psi}^{\mathrm{RI}'}}{\partial a} \;-\; \alpha\gamma_\alpha^{\mathrm{RI}'}(a)\frac{\partial \ln Z_{\bar\psi\psi}^{\mathrm{RI}'}}{\partial \alpha} \qquad (3.5)
$$

we find

$$
\begin{aligned}
\gamma_{\bar\psi\psi}^{\mathrm{RI}'}(a) \;=\;& -\;3C_F a \;-\; [(185 + 9\alpha + 3\alpha^2)C_A \;+\; 9C_F \;-\; 52T_F N_f]\frac{C_F a^2}{6} \\
&+ \Big[\left(108\alpha^3 + 324\alpha^2 - 1944 - 19008\zeta(3)\right) C_A C_F \\
&\quad - \left(117428 + 5634\alpha + 1905\alpha^2 + 405\alpha^3 + 54\alpha^4 - 28512\zeta(3)\right) C_A^2 \\
&\quad + \left(480\alpha^2 + 2088\alpha + 62960\right) C_A T_F N_f \;-\; 13932 C_F^2 \\
&\quad + (16632 - 3456\zeta(3))\, C_F T_F N_f \;-\; 6848 T_F^2 N_f^2)\Big] \frac{C_F a^3}{216} \;+\; O(a^4) \qquad (3.6)
\end{aligned}
$$

which agrees with [12] in the Landau gauge, aside from an overall factor stemming from our conventions which are the same as [34]. We have also derived the same expression by constructing the conversion function $C_{\mathcal{O}_{\mathcal{A}}}(a,\alpha)$ with $\mathcal{A} = 1$ where

$$
C_{\mathcal{O}_{\mathcal{A}}}(a,\alpha) \;=\; \frac{Z_{\mathcal{O}_{\mathcal{A}}}^{\mathrm{RI}'}}{Z_{\mathcal{O}_{\mathcal{A}}}^{\overline{\mathrm{MS}}}} \;. \qquad (3.7)
$$

Thus, using

$$
\begin{aligned}
\gamma_{\mathcal{O}_{\mathcal{A}}}^{\mathrm{RI}'}(a_{\mathrm{RI}'}) \;=\;& \gamma_{\mathcal{O}_{\mathcal{A}}}^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right) \;-\; \beta\left(a_{\overline{\mathrm{MS}}}\right)\frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C_{\mathcal{O}_{\mathcal{A}}}\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right) \\
&-\; \alpha_{\overline{\mathrm{MS}}}\gamma_\alpha^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right)\frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C_{\mathcal{O}_{\mathcal{A}}}\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right) \qquad (3.8)
\end{aligned}
$$

with the explicit expression

$$
\begin{aligned}
C_{\bar\psi\psi}(a,\alpha) \;=\;& 1 \;-\; (\alpha+4)C_F a \;+\; \Big[\left(24\alpha^2 + 96\alpha - 288\zeta(3) + 57\right) C_F \\
&+\; 332 T_F N_f - \left(18\alpha^2 + 84\alpha - 432\zeta(3) + 1285\right) C_A\Big] \frac{C_F a^2}{24}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big[\Big(15552\alpha^3 + 89424\alpha^2 - 23328\alpha^2\zeta(3) + 573804\alpha \\
&\quad - 334368\alpha\zeta(3) - 2493504\zeta(3) + 155520\zeta(5) + 2028348\big) C_A C_F \\
&\quad - \Big(13122\alpha^3 - 8748\alpha^2\zeta(3) + 71685\alpha^2 - 103032\alpha\zeta(3) \\
&\qquad + 357777\alpha - 3368844\zeta(3) + 466560\zeta(5) + 6720046\big) C_A^2 \\
&\quad + (113400\alpha - 31104\zeta(3)\alpha \\
&\qquad - 532224\zeta(3) + 186624\zeta(4) + 3052384) C_A T_F N_f \\
&\quad + (62208\alpha\zeta(3) - 123120\alpha - 331776\zeta(3) \\
&\qquad - 186624\zeta(4) + 958176) C_F T_F N_f \\
&\quad - \Big(7776\alpha^3 - 46656\alpha^2\zeta(3) + 31104\alpha^2 + 46656\alpha\zeta(3) + 30132\alpha \\
&\qquad - 451008\zeta(3) - 933120\zeta(5) + 2091096\big) C_F^2 \\
&\quad - (27648\zeta(3) + 240448) T_F^2 N_f^2\Big] \frac{C_F a^3}{7776} ~+~ O(a^4)
\end{aligned}
\tag{3.9}
$$

we find exact agreement. Moreover, this conversion function agrees with that given in [12] when restricted to the Landau gauge. However, in checking the three loop expression in the RI$'$ scheme only the contribution up to and including the two loop term is required. The three loop part is only relevant for the four loop anomalous dimension. Therefore, since the four loop $\overline{\mathrm{MS}}$ quark mass anomalous dimension is available, [5, 6], for an arbitrary colour group we can deduce that the four loop correction to (3.6) is

$$
\begin{aligned}
\gamma^{\mathrm{RI}'}_{\bar\psi\psi}(a) ~=~& -3C_F a ~-~ [(185 + 9\alpha + 3\alpha^2)C_A ~+~ 9C_F ~-~ 52T_F N_f]\frac{C_F a^2}{6} \\
&+ \Big[\Big(108\alpha^3 + 324\alpha^2 - 1944 - 19008\zeta(3)\Big) C_A C_F \\
&\quad - \Big(117428 + 5634\alpha + 1905\alpha^2 + 405\alpha^3 + 54\alpha^4 - 28512\zeta(3)\Big) C_A^2 \\
&\quad + \Big(480\alpha^2 + 2088\alpha + 62960\Big) C_A T_F N_f ~-~ 13932 C_F^2 \\
&\quad + (16632 - 3456\zeta(3)) C_F T_F N_f ~-~ 6848 T_F^2 N_f^2)\Big] \frac{C_F a^3}{216} \\
&+ \Big[\Big(-1215\alpha^6 - 13608\alpha^5 - 90801\alpha^4 - 8262\zeta(3)\alpha^3 - 368064\alpha^3 + 104004\zeta(3)\alpha^2 \\
&\quad - 1397826\alpha^2 + 940734\zeta(3)\alpha - 4554684\alpha + 39004740\zeta(3) \\
&\quad - 1710720\zeta(5) - 92569118\big) C_A^3 C_F \\
&\quad + \Big(2916\alpha^5 + 28188\alpha^4 - 23328\zeta(3)\alpha^3 + 136404\alpha^3 - 252720\zeta(3)\alpha^2 \\
&\qquad + 377136\alpha^2 - 1717200\zeta(3)\alpha + 429300\alpha - 22203072\zeta(3) \\
&\qquad - 1710720\zeta(5) + 10355148\big) C_A^2 C_F^2 \\
&\quad + \Big(12960\alpha^4 + 103032\alpha^3 + 34992\zeta(3)\alpha^2 \\
&\qquad + 677952\alpha^2 - 316224\zeta(3)\alpha + 3021840\alpha - 14239152\zeta(3) \\
&\qquad - 1244160\zeta(5) + 73217928\big) C_A^2 C_F T_F N_f \\
&\quad + \Big(-3888\alpha^4 + 46656\zeta(3)\alpha^3 - 11664\alpha^3 + 241056\zeta(3)\alpha^2 - 5832\alpha^2 \\
&\qquad - 1236384\zeta(3)\alpha - 103032\alpha - 1601856\zeta(3) \\
&\qquad + 10264320\zeta(5) - 33960384\big) C_A C_F^3 \\
&\quad + \Big(-25920\alpha^3 - 62208\zeta(3)\alpha^2 + 18144\alpha^2 + 694656\zeta(3)\alpha + 230688\alpha \\
&\qquad + 1347840\zeta(3) - 1244160\zeta(5) + 20983248\big) C_A C_F^2 T_F N_f
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(- 57600\alpha^2 + 82944\zeta(3)\alpha - 449280\alpha \\
&\quad + 580608\zeta(3) - 16599552 \Big) C_A C_F T_F^2 N_f^2 \\
&+ \Big(- 62208\alpha^2 + 373248\zeta(3)\alpha + 31104\alpha \\
&\quad - 3856896\zeta(3) + 9745920 \Big) C_F^3 T_F N_f \\
&+ \left(- 165888\zeta(3)\alpha + 41472\alpha + 2571264\zeta(3) - 6653952 \right) C_F^2 T_F^2 N_f^2 \\
&+ \left(2612736\zeta(3) + 1225692 \right) C_F^4 + 1025536 C_F T_F^3 N_f^3 \\
&+ \left(248832 - 1866240\zeta(3) \right) \frac{d_A^{abcd} d_F^{abcd}}{N_F} \\
&+ \left(3732480\zeta(3) - 497664 \right) N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \Bigg] \frac{a^4}{7776} ~+~ O(a^5)
\end{aligned}
\tag{3.10}
$$

where $\frac{d_A^{abcd} d_F^{abcd}}{N_F}$ and $\frac{d_F^{abcd} d_F^{abcd}}{N_F}$ are the quartic Casimirs associated with light-by-light topologies, [4, 6], and the coupling constant and gauge parameter are in the RI$'$ scheme. The four loop expression is in agreement with the Landau gauge expression of [12] for an $SU(N_c)$ colour group.

As an additional check on the renormalization of a composite operator in the RI$'$ scheme we have also considered the case of $\mathcal{A} = \gamma^\mu$. One reason for examining this operator arises from the fact that as it is now a Lorentz vector the Green's function, $G^\mu_{\bar{\psi}\gamma^\mu\psi}(p)$, is not merely proportional to γ^μ. Instead by Lorentz symmetry

$$
G^\mu_{\bar{\psi}\gamma^\mu\psi}(p) ~=~ \langle \psi(p) ~ [\bar{\psi}\gamma^\mu\psi](0) ~ \bar{\psi}(-p) \rangle ~=~ \Sigma^{(1)}_{\bar{\psi}\gamma^\mu\psi}(p)\gamma^\mu ~+~ \Sigma^{(2)}_{\bar{\psi}\gamma^\mu\psi}(p)\frac{p^\mu \not{p}}{p^2}
\tag{3.11}
$$

where the amplitudes $\Sigma^{(i)}_{\bar{\psi}\gamma^\mu\psi}(p)$ depend on the coupling constant. They are determined by the relations

$$
\begin{aligned}
\Sigma^{(1)}_{\bar{\psi}\gamma^\mu\psi}(p) &= \frac{1}{4(d-1)} \left[\mathrm{tr} \left(\gamma_\mu G^\mu_{\bar{\psi}\gamma^\mu\psi}(p) \right) ~-~ \mathrm{tr} \left(\frac{p_\mu \not{p}}{p^2} G^\mu_{\bar{\psi}\gamma^\mu\psi}(p) \right) \right] \\
\Sigma^{(2)}_{\bar{\psi}\gamma^\mu\psi}(p) &= -\frac{1}{4(d-1)} \left[\mathrm{tr} \left(\gamma_\mu G^\mu_{\bar{\psi}\gamma^\mu\psi}(p) \right) ~-~ d \, \mathrm{tr} \left(\frac{p_\mu \not{p}}{p^2} G^\mu_{\bar{\psi}\gamma^\mu\psi}(p) \right) \right] ~.
\end{aligned}
\tag{3.12}
$$

The renormalization constant for the operator is determined from

$$
\mathcal{O}_{\gamma^\mu \, \mathrm{o}} ~=~ Z_{\bar{\psi}\gamma^\mu\psi} \mathcal{O}_{\gamma^\mu} ~.
\tag{3.13}
$$

and renormalizing the Green's function we find

$$
Z^{\overline{\mathrm{MS}}}_{\bar{\psi}\gamma^\mu\psi} ~=~ 1 ~+~ O(a^4) ~.
\tag{3.14}
$$

The non-renormalization of this current rests in the fact that it corresponds to a physical operator and therefore on general grounds its anomalous dimension vanishes. (See, for example, [31].) Furthermore, as the vector current has been inserted at zero momentum the γ^μ component of its Green's function must obey the Slavnov-Taylor identity and be equivalent to the finite part of the quark two-point function after renormalization in the same scheme. In computing the quark wave function anomalous dimension in the previous section we have also determined the finite part in the $\overline{\mathrm{MS}}$ scheme and it is reassuring to note that both it and

$$
\begin{aligned}
\Sigma^{(1) ~ \overline{\mathrm{MS}} ~ \mathrm{finite}}_{\bar{\psi}\gamma^\mu\psi}(p) \Big|_{p^2 = \mu^2} &= 1 ~+~ \alpha C_F a \\
&+ \left[\left(\frac{41}{4} + \frac{13\alpha}{2} + \frac{9\alpha^2}{8} - 3(1+\alpha)\zeta(3) \right) C_F C_A \right.
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{7}{2} T_F N_f C_F - \frac{5}{8} C_F^2 \Big] a^2 \\
+ & \left[\frac{1570}{81} T_F^2 N_f^2 + \left(16\zeta(3) - \frac{79}{6} - \frac{3\alpha}{2} \right) T_F N_f C_F \right. \\
& + \left(\frac{52}{3}\zeta(3) + 8\alpha\zeta(3) - \frac{11887}{81} - \frac{1723\alpha}{72} \right) T_F N_f C_A \\
& + \left(\frac{159257}{648} + \frac{39799\alpha}{576} + \frac{787\alpha^2}{64} + \frac{55\alpha^3}{24} \right. \\
& \quad - \left(\frac{3139}{24} + 35\alpha + \frac{39\alpha^2}{8} + \frac{\alpha^3}{3} \right) \zeta(3) \\
& \quad + \left(\frac{3\alpha^2}{16} + \frac{3\alpha}{8} - \frac{69}{16} \right) \zeta(4) \\
& \quad \left. + \left(\frac{165}{4} + \frac{5\alpha}{2} + \frac{5\alpha^2}{4} \right) \zeta(5) \right) C_A^2 \\
& + \left(20(\alpha - 1)\zeta(5) + 6\zeta(4) + \left(44 - 17\alpha + \alpha^3 \right) \zeta(3) \right. \\
& \quad \left. - \frac{997}{24} + 4\alpha + \frac{3\alpha^2}{2} - \frac{\alpha^3}{8} \right) C_F C_A \\
& \left. - \left(\frac{73}{12} - \frac{7\alpha}{8} + \frac{2\alpha^3}{3}\zeta(3) \right) C_F^2 \right] C_F a^3 ~+~ O(a^4) \qquad (3.15)
\end{aligned}
$$

are in exact agreement which provides an additional check on our programming. Further, it is the finite part of the first term which determines the relation of the $\overline{\mbox{MS}}$ scheme to the RI$'$ scheme. Such a feature of extra contributions will persist in the tensor current case but the vector operator is peculiar in the sequence of dimension three operators in that only its anomalous dimension *and* finite part are entwined with the Slavnov-Taylor identity. Repeating the same exercise for the RI$'$ scheme by introducing the definition

$$
\lim_{p^2 \rightarrow \mu^2} \left(Z^{\mathrm{RI}'}_\psi Z^{\mathrm{RI}'}_{\bar{\psi}\gamma^\mu\psi} \Sigma^{(1)}_{\bar{\psi}\gamma^\mu\psi}(p) \right) ~=~ 1 \qquad (3.16)
$$

we find

$$
Z^{\mathrm{RI}'}_{\bar{\psi}\gamma^\mu\psi} ~=~ 1 ~+~ O(a^4) ~. \qquad (3.17)
$$

This is consistent with the observation that if an anomalous dimension of a physical operator vanishes in one scheme it vanishes in any other scheme. Moreover, (3.16) is also consistent with the Slavnov-Taylor identity in the RI$'$ scheme since not only is it finite prior to renormalization but it is *also* unity which agrees with the finite part of the quark two-point function consistent with the nature of this renormalization scheme. Finally, to assist with lattice matching we record that the finite parts of the second component of the vector current Green's function are

$$
\begin{aligned}
\Sigma^{(2)~\overline{\mathrm{MS}}~\mathrm{finite}}_{\bar{\psi}\gamma^\mu\psi}(p) \Big|_{p^2 = \mu^2} ~=~ & - 2\alpha C_F a \\
& - C_F \left[\left(\frac{25}{2} + 7\alpha + \frac{3}{2}\alpha^2 \right) C_A - 4 T_F N_f + \left(2\alpha^2 - 3 \right) C_F \right] a^2 \\
& - C_F \left[\left(\frac{28}{3} - 11\alpha \right) T_F N_f C_F + \frac{208}{9} T_F^2 N_f^2 + \left(3 - \frac{17}{4}\alpha \right) C_F^2 \right. \\
& \quad + \left(16\zeta(3) + 8\zeta(3)\alpha - \frac{1528}{9} - \frac{175}{6}\alpha \right) T_F N_f C_A
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{19979}{72} + \frac{4393}{48}\alpha + \frac{295}{16}\alpha^2 + \frac{27}{8}\alpha^3 \right. \\
&\quad \left. - \left(\frac{245}{4} + \frac{59}{2}\alpha + \frac{9}{4}\alpha^2 \right) \zeta(3) \right) C_A^2 \\
&+ \left(\left(24 - 6\alpha - 6\alpha^2 \right) \zeta(3) - \frac{242}{3} + 33\alpha \right. \\
&\quad \left. \left. + \ 17\alpha^2 + \frac{11}{4}\alpha^3 \right) C_F C_A \right] a^3 \ + \ O(a^4) \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma^{(2)~\mathrm{RI}'~\mathrm{finite}}_{\bar{\psi}\gamma^\mu\psi}(p) \Big|_{p^2 \, = \, \mu^2} \; = \; & - \, 2\alpha C_F a \\
& - C_F \left[\left(\frac{25}{2} + \frac{223}{18}\alpha + \frac{5}{2}\alpha^2 + \frac{1}{2}\alpha^3 \right) C_A \right. \\
& \quad \left. - \left(4 + \frac{40}{9}\alpha \right) T_F N_f - 3C_F \right] a^2 \\
& - C_F \left[\left(\frac{28}{3} - \frac{110}{3}\alpha + 32\zeta(3)\alpha \right) T_F N_f C_F + \left(\frac{208}{9} + \frac{800}{81}\alpha \right) T_F^2 N_f^2 \right. \\
& \quad + \left(16\zeta(3) - 8\zeta(3)\alpha - \frac{1528}{9} - \frac{7952}{81}\alpha \right. \\
& \qquad \left. - \frac{100}{9}\alpha^2 - \frac{20}{9}\alpha^3 \right) T_F N_f C_A \\
& \quad + \left(\frac{19979}{72} + \frac{14135}{81}\alpha + \frac{314}{9}\alpha^2 + \frac{421}{36}\alpha^3 + \frac{17}{8}\alpha^4 + \frac{1}{4}\alpha^5 \right. \\
& \qquad \left. - \left(\frac{245}{4} + \frac{71}{2}\alpha - \frac{7}{4}\alpha^2 \right) \zeta(3) \right) C_A^2 + 3C_F^2 \\
& \quad \left. + \left(24\zeta(3) - \frac{242}{3} - 3\alpha^2 - \alpha^3 \right) C_F C_A \right] a^3 \ + \ O(a^4) \qquad (3.19)
\end{aligned}
$$

which both agree in the Landau gauge and the variables in each expression correspond to those of the scheme indicated in the left hand side.

4 Tensor current in the RI$'$ scheme.

We now turn to the computation of the anomalous dimension for the flavour non-singlet tensor current at zero momentum insertion in the Landau gauge. This calculation is similar to the one for the vector current though the decomposition of the Green's function $G^{\mu\nu}_{\sigma^{\mu\nu}}(p)$ will have different Lorentz structures. In particular we have

$$
G^{\mu\nu}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \; = \; \langle \psi(p) \; [\bar{\psi}\sigma^{\mu\nu}\psi](0) \; \bar{\psi}(-p) \rangle \; = \; \Sigma^{(1)}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p)\sigma^{\mu\nu} \; + \; \Sigma^{(2)}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \left(\not{p}\gamma^\mu p^\nu \; - \; \not{p}\gamma^\nu p^\mu \right) \frac{1}{p^2} \qquad (4.1)
$$

since $\sigma^{\mu\nu}$ is antisymmetric in its Lorentz indices. The components are deduced from

$$
\begin{aligned}
\Sigma^{(1)}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \; &= \; - \, \frac{1}{4(d-1)(d-2)} \left[\mathrm{tr} \left(\sigma_{\mu\nu} G^{\mu\nu}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right) \; + \; \frac{1}{p^2} \mathrm{tr} \left((\not{p}\gamma_\mu p_\nu - \not{p}\gamma_\nu p_\mu) G^{\mu\nu}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right) \right] \\
\Sigma^{(2)}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \; &= \; - \, \frac{1}{4(d-1)(d-2)} \left[\mathrm{tr} \left(\sigma_{\mu\nu} G^{\mu\nu}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right) \; + \; \frac{d}{2p^2} \mathrm{tr} \left((\not{p}\gamma_\mu p_\nu - \not{p}\gamma_\nu p_\mu) G^{\mu\nu}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right) \right] .
\end{aligned} \qquad (4.2)
$$

As the anomalous dimension of the current had been computed for arbitrary covariant gauge parameter in the $\overline{\mathrm{MS}}$ scheme, [35, 36, 34], we note that the definition of the RI$'$ scheme renormalization constant is

$$\lim_{\epsilon \to 0} \left[Z^{\mathrm{RI}'}_\psi Z^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi} \Sigma^{(1)}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right] \Bigg|_{p^2 = \mu^2} ~=~ 1 ~. \tag{4.3}$$

Therefore, we find that the gauge dependent renormalization constant is

$$\begin{aligned}
Z^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi} ~=~ & 1 ~+~ \left(\frac{C_F}{\epsilon} + \alpha C_F \right) a ~+~ \left[\left(\frac{1}{2} C_F^2 - \frac{11}{6} C_F C_A + \frac{2}{3} C_F T_F N_f \right) \frac{1}{\epsilon^2} \right. \\
& +~ \left(\frac{257}{36} C_F C_A - \frac{13}{9} C_F T_F N_f - \left(\frac{19}{4} - \alpha \right) C_F^2 \right) \frac{1}{\epsilon} \\
& +~ \left(\left(\frac{5987}{216} + \frac{223\alpha}{36} + \frac{5\alpha^2}{4} + \frac{\alpha^3}{4} - 14\zeta(3) \right) C_F C_A - \left(\frac{313}{54} + \frac{20\alpha}{9} \right) C_F T_F N_f \right. \\
& \left. \left. ~~~+~ \left(\alpha^2 - \frac{535}{24} + 20\zeta(3) \right) C_F^2 \right) \right] a^2 \\
& +~ \left[\left(\frac{2}{3} C_F^2 T_F N_f - \frac{88}{27} C_F C_A T_F N_f + \frac{16}{27} C_F T_F^2 N_f^2 \right. \right. \\
& ~~~ \left. +~ \frac{121}{27} C_F C_A^2 - \frac{11}{6} C_A C_F^2 + \frac{1}{6} C_F^3 \right) \frac{1}{\epsilon^3} \\
& ~~~ +~ \left(\frac{980}{81} C_F C_A T_F N_f - \frac{104}{81} C_F T_F^2 N_f^2 + \left(\frac{2\alpha}{3} - \frac{13}{3} \right) C_F^2 T_F N_f - \frac{3439}{162} C_A^2 C_F \right. \\
& ~~~ \left. +~ \left(\frac{75}{4} - \frac{11\alpha}{6} \right) C_A C_F^2 + \left(\frac{\alpha}{2} - \frac{19}{4} \right) C_F^3 \right) \frac{1}{\epsilon^2} \\
& ~~~ +~ \left(\left(\frac{16}{3}\zeta(3) - \frac{13}{6} - \frac{11\alpha}{3} \right) C_F^2 T_F N_f - \left(\frac{16}{3}\zeta(3) + \frac{1004}{81} \right) C_F C_A T_F N_f \right. \\
& ~~~~~~ -~ \frac{4}{9} C_F T_F^2 N_f^2 + \left(\frac{13639}{324} - \frac{40}{3}\zeta(3) \right) C_A^2 C_F \\
& ~~~~~~ +~ \left(\frac{70}{3}\zeta(3) - \frac{851}{24} + \frac{40\alpha}{3} + \frac{5\alpha^2}{4} + \frac{\alpha^3}{4} \right) C_A C_F^2 \\
& ~~~~~~ \left. +~ \left(\alpha^2 - \frac{19\alpha^2}{4} - \frac{145}{72} - \frac{4}{3}\zeta(3) \right) C_F^3 \right) \frac{1}{\epsilon} \\
& ~~~ -~ \left(\left(\frac{186527}{729} + \frac{3976\alpha}{81} + \frac{50\alpha^2}{9} + \frac{10\alpha^3}{9} \right. \right. \\
& ~~~~~~~~ \left. +~ 4\alpha\zeta(3) - \frac{2200}{27}\zeta(3) + 8\zeta(4) \right) C_A C_F T_F N_f \\
& ~~~~~~ +~ \left(\frac{40\alpha^2}{9} + \frac{1267\alpha}{54} - \frac{10849}{81} + 96\zeta(3) - \frac{40\alpha}{3}\zeta(3) - 8\zeta(4) \right) C_F^2 T_F N_f \\
& ~~~~~~ -~ \left(\frac{13754}{729} + \frac{400\alpha}{81} + \frac{32}{27}\zeta(3) \right) C_F T_F^2 N_f^2 \\
& ~~~~~~ +~ \left(\left(\frac{72491}{216} + \frac{203\alpha}{12} - \frac{7\alpha^2}{8} \right) \zeta(3) + \frac{44}{3}\zeta(4) + \left(\frac{10\alpha}{3} - 30 \right) \zeta(5) \right. \\
& ~~~~~~~~~ \left. -~ \frac{6883865}{11664} - \frac{28621\alpha}{324} - \frac{157\alpha^2}{9} - \frac{421\alpha^3}{72} - \frac{17\alpha^4}{16} - \frac{\alpha^5}{8} \right) C_A^2 C_F \\
& ~~~~~~ +~ \left(\frac{495847}{648} - \frac{4673\alpha}{216} - \frac{89\alpha^2}{2} - 2\alpha^3 - \frac{\alpha^4}{2} \right.
\end{aligned}$$

$$
\begin{aligned}
&- \left(530 - \frac{5\alpha}{3} + \alpha^2\right) \zeta(3) - 40\zeta(4) - 20\zeta(5)\Big) C_A C_F^2 \\
&+ \left(\left(\frac{742}{9}\zeta(3) - 18\alpha + 2\alpha^2\right) \zeta(3) + \frac{64}{3}\zeta(4) + 40\zeta(5) - \frac{17303}{108} \right.\\
&+ \left.\frac{523\alpha}{24} - 2\alpha^2 - \alpha^3\right) C_F^3\Big)\Big] a^3 ~+~ O(a^4) \qquad (4.4)
\end{aligned}
$$

where we have used the same symbolic manipulation programme to compute this as for the scalar and vector cases aside from changing the Feynman rule for the operator insertion. Moreover, given that the programme correctly reproduces the gauge independent $\overline{\mathrm{MS}}$ renormalization constant for all α, [35, 36, 34], we are confident that (4.4) is correct. Therefore, from

$$
\gamma^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi}(a) ~=~ -~ \beta(a) \frac{\partial \ln Z^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi}}{\partial a} ~-~ \alpha \gamma^{\mathrm{RI}'}_\alpha(a) \frac{\partial \ln Z^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi}}{\partial \alpha} \qquad (4.5)
$$

we find that, in four dimensions,

$$
\begin{aligned}
\gamma^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi}(a) ~=~ & C_F a ~+~ [(257 + 27\alpha + 9\alpha^2) C_A ~-~ 171 C_F ~-~ 52 T_F N_f] \frac{C_F a^2}{18} \\
& +~ \Big[(213548 + 16902\alpha + 5715\alpha^2 + 1215\alpha^3 + 162\alpha^4 - 92448\zeta(3)) C_A^2 \\
& \quad -~ (228744 - 972\alpha^2 - 324\alpha^3 - 167616\zeta(3)) C_A C_F \\
& \quad -~ (99536 + 6264\alpha + 1440\alpha^2 - 13824\zeta(3)) C_A T_F N_f \\
& \quad +~ (45576 - 24192\zeta(3)) C_F T_F N_f \\
& \quad +~ (39420 - 41472\zeta(3)) C_F^2 ~+~ 9152 T_F^2 N_f^2 \Big] \frac{C_F a^3}{648} ~+~ O(a^4) \qquad (4.6)
\end{aligned}
$$

which is one of the main results of this article. For completeness, we note that*, [35, 36, 34],

$$
\begin{aligned}
\gamma^{\overline{\mathrm{MS}}}_{\bar{\psi}\sigma^{\mu\nu}\psi}(a) ~=~ & C_F a ~+~ [257 C_A ~-~ 171 C_F ~-~ 52 T_F N_f] \frac{C_F a^2}{18} \\
& +~ \Big[13639 C_A^2 ~-~ 4320\zeta(3) C_A^2 ~+~ 12096\zeta(3) C_A C_F \\
& \quad -~ 20469 C_A C_F ~-~ 1728\zeta(3) C_A T_F N_f ~-~ 4016 C_A T_F N_f \\
& \quad -~ 6912\zeta(3) C_F^2 ~+~ 6570 C_F^2 ~+~ 1728\zeta(3) C_F T_F N_f \\
& \quad +~ 1176 C_F T_F N_f ~-~ 144 T_F^2 N_f^2) \Big] \frac{C_F a^3}{108} ~+~ O(a^4) \qquad (4.7)
\end{aligned}
$$

where the four loop expression of (4.7) is available for QED in the quenched approximation, [36]. Specifying the Landau gauge for the colour group $SU(3)$ we find

$$
\begin{aligned}
\gamma^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}\psi}(a)\Big|^{SU(3)}_{\alpha=0} ~=~ & \frac{4}{3} a ~-~ \frac{2}{27}[26 N_f - 543] a^2 \\
& + \frac{2}{243}\Big[572 N_f^2 + (1152\zeta(3) - 29730) N_f - 58824\zeta(3) + 269259\Big] a^3 ~+~ O(a^4)
\end{aligned}
\qquad (4.8)
$$

where we have set $T_f = 1/2$, $C_F = 4/3$ and $C_A = 3$. Finally, we note

$$
\Sigma^{(1)~\overline{\mathrm{MS}}~\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p)\Big|_{p^2=\mu^2} ~=~ 1 ~-~ \left[\left(\frac{3773}{216} - 3\alpha - \frac{3}{8}\alpha^2 - 11\zeta(3) + 3\zeta(3)\alpha\right) C_F C_A\right.
$$

*In [34] the Casimir of the final coefficient should have been $T_F^2 N_f^2$ and not $T_F^2 C_F^2$.

$$
\begin{aligned}
& - \frac{62}{27} T_F N_f C_F + \left(\alpha^2 - \frac{65}{3} + 20\zeta(3) \right) C_F^2 \Big] a^2 \\
- & \Big[\left(\frac{23831}{162} - \frac{4}{3}\alpha - 112\zeta(3) - \frac{8}{3}\zeta(3)\alpha + 8\zeta(4) \right) T_F N_f C_F^2 \\
& + \Big(\frac{673}{72}\alpha - \frac{79544}{729} + \frac{1732}{27}\zeta(3) \\
& \quad - 4\zeta(3)\alpha - 8\zeta(4) \Big) T_F N_f C_A C_F \\
& + \left(\frac{32}{27}\zeta(3) - \frac{376}{729} \right) T_F^2 N_f^2 C_F \\
& + \Big(\frac{4017239}{11664} - \frac{12817}{576}\alpha - \frac{197}{64}\alpha^2 - \frac{29}{48}\alpha^3 \\
& \quad - \left(\frac{5530}{27} - \frac{253}{12}\alpha - \frac{15}{4}\alpha^2 - \frac{1}{3}\alpha^3 \right) \zeta(3) \\
& \quad - \left(\frac{497}{48} + \frac{3}{8}\alpha + \frac{3}{16}\alpha^2 \right) \zeta(4) \\
& \quad - \left(\frac{45}{4} + \frac{35}{6}\alpha + \frac{5}{4}\alpha^2 \right) \zeta(5) \Big) C_A^2 C_F \\
& + \Big(\left(486 + \frac{79}{3}\alpha - 2\alpha^2 - \alpha^3 \right) \zeta(3) + 34\zeta(4) \\
& \quad + (40 - 20\alpha)\zeta(5) - \frac{58616}{81} \\
& \quad + \frac{1}{6}\alpha + 6\alpha^2 + \frac{3}{2}\alpha^3 \Big) C_F^2 C_A \\
& + \Big(\frac{4490}{27} - \alpha + 2\alpha^2 - \frac{64}{3}\zeta(4) - 40\zeta(5) \\
& \quad - \left(\frac{742}{9} + 2\alpha + 2\alpha^2 - \frac{2}{3}\alpha^3 \right) \zeta(3) \Big) C_F^3 \Big] a^3 \\
+ & O(a^4)
\end{aligned}
\tag{4.9}
$$

which implies

$$
\begin{aligned}
\left. \Sigma^{(1)~\overline{\mathrm{MS}}~\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right|^{SU(3)\,,\,\alpha\,=\,0}_{p^2\,=\,\mu^2} &= 1 - \left(\frac{1693}{54} - \frac{76}{9}\zeta(3) - \frac{124}{81}N_f \right) a^2 \\
& - \Big(\frac{1946885}{2916} - \frac{14872}{243}\zeta(3) + \frac{2111}{324}\zeta(4) - \frac{445}{27}\zeta(5) \\
& \quad + \left(\frac{776}{27}\zeta(3) - \frac{80}{9}\zeta(4) - \frac{63764}{729} \right) N_f \\
& \quad + \left(\frac{32}{81}\zeta(3) - \frac{376}{2187} \right) N_f^2 \Big) a^3 + O(a^4) .
\end{aligned}
\tag{4.10}
$$

For $\left. \Sigma^{(2)~\overline{\mathrm{MS}}~\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right|_{p^2\,=\,\mu^2}$ and $\left. \Sigma^{(2)~\mathrm{RI}'~\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}\psi}(p) \right|_{p^2\,=\,\mu^2}$ it transpires that there are no contributions to the finite part to and including three loops. This is consistent with the one loop evaluation of the same quantity in [37] when the chiral limit is taken. For completeness we have again checked our RI′ scheme anomalous dimension by constructing the conversion function $C_{\bar{\psi}\sigma^{\mu\nu}\psi}(a,\alpha)$ explicitly. From

$$
\begin{aligned}
C_{\bar{\psi}\sigma^{\mu\nu}\psi}(a,\alpha) &= 1 + \alpha C_F a + \Big[\left(216\alpha^2 + 4320\zeta(3) - 4815 \right) C_F - 1252 T_F N_f \\
& \quad + \left(162\alpha^2 + 756\alpha - 3024\zeta(3) + 5987 \right) C_A \Big] \frac{C_F a^2}{216}
\end{aligned}
$$

$$
\begin{aligned}
+ & \Big[\Big(23328\alpha^3 + 104976\alpha^2 + 23328\alpha^2\zeta(3) + 504684\alpha - 38880\alpha\zeta(3) \\
& + 12363840\zeta(3) + 933120\zeta(4) + 466560\zeta(5) - 17850492\Big) C_A C_F \\
& + \Big(39366\alpha^3 - 26244\alpha^2\zeta(3) + 215055\alpha^2 - 324648\alpha\zeta(3) - 77760\alpha\zeta(5) \\
& \quad + 1092771\alpha - 7829028\zeta(3) - 342144\zeta(4) \\
& \quad + 699840\zeta(5) + 13767730\Big) C_A^2 \\
& + (93312\alpha\zeta(3) - 340200\alpha + 1900800\zeta(3) \\
& \quad - 186624\zeta(4) - 5968864) C_A T_F N_f \\
& - (62208\alpha\zeta(3) + 119664\alpha + 2239488\zeta(3) \\
& \quad - 186624\zeta(4) - 3124512) C_F T_F N_f \\
& + \Big(23328\alpha^3 - 46656\alpha^2\zeta(3) + 46656\alpha^2 + 419904\alpha\zeta(3) - 508356\alpha \\
& \quad - 1923264\zeta(3) - 497664\zeta(4) - 933120\zeta(5) + 3737448\Big) C_F^2 \\
& + (27648\zeta(3) + 440128) T_F^2 N_f^2\Big] \frac{C_F a^3}{23328} + O(a^4)
\end{aligned}
\tag{4.11}
$$

we again have exact agreement.

5 Discussion.

To conclude we have first renormalized QCD to three loops in arbitrary covariant gauge in the RI$'$ scheme. The full renormalization was necessary since, for example, the anomalous dimension of the gauge parameter is required when converting the renormalization constants to renormalization group functions for non-zero α. Although in practice one only requires information in the Landau gauge, computing for $\alpha \neq 0$ provides important internal checks on the calculation such as for comparing with established $\overline{\mathrm{MS}}$ results. Further, we have extended the machinery of [12] to compute the anomalous dimensions of the tensor operator $\bar{\psi}\sigma^{\mu\nu}\psi$ at three loops in the chiral limit in the Landau gauge in a scheme which is natural in lattice regularization. Given this approach it would be interesting to examine other operators whose anomalous dimensions are required in the RI$'$ scheme such as the low moments of the twist-2 Wilson operators which occur in the operator product expansion in deep inelastic scattering and those relating to transversity, in order to provide the foundation to improve lattice estimates of matrix elements.

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