The strong coupling constant from lattice QCD with $N_f = 2$ dynamical quarks[*]

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We compute $\Lambda_{\overline{MS}}$ for two flavors of light dynamical quarks using non-perturbatively $O(a)$ improved Wilson fermions. We improve on a recent calculation by employing Padé-improved two-loop and three-loop perturbation theory to convert the lattice numbers to the $\overline{MS}$ scheme.

1. THE METHOD

The running of the strong coupling constant α_s is parameterized by the Λ parameter. In a recent letter [1] we have computed $\Lambda_{\overline{MS}}$ from the average plaquette and the scale parameter r_0. An essential step in our calculation was the conversion of the boosted coupling $g_\square$ to the renormalized coupling $g_{\overline{MS}}$. This was done in two-loop tadpole improved perturbation theory, which resulted in an (estimated) systematic uncertainty of the order of the statistical error.

We can do better now. In this talk we shall apply Padé techniques to approximate an unknown three-loop coefficient in the conversion formula. This reduces the systematic uncertainty to an insignificant amount, as we shall see.

The calculation of $\Lambda_{\overline{MS}}$ proceeds in three steps:

First we compute the average plaquette $P = (1/3) \langle \text{Tr} U_\square \rangle \equiv u_0^4$ from which we obtain $g_\square$,

$$\frac{1}{g_\square^2(a)} = \frac{P}{g^2(a)}, \tag{1}$$

where $g(a)$ is the bare lattice coupling. Note that the non-perturbative contribution to the plaquette, i.e. the difference between the lattice number and the perturbative series, is less than 10^{-3} [2].

In the second step we compute $g_{\overline{MS}}$ from $g_\square$. This is the main task. To three loops we obtain, following the notation of [1],

$$\begin{aligned}
\frac{1}{g_{\overline{MS}}^2(\mu)} = {} & \frac{1}{g_\square^2(a)} + 2b_0 \ln a\mu - t_1^\square \\
& + (2b_1 \ln a\mu - t_2^\square) \, g_\square^2(a) \\
& + \big(-2b_0 b_1 \ln^2 a\mu + 2(b_2^{\overline{MS}} + b_1 t_1^\square) \ln a\mu \\
& - t_3^\square + p_1 t_2^\square \big) \, g_\square^4(a),
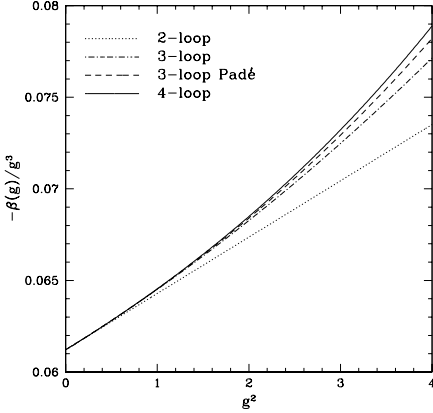\end{aligned} \tag{2}$$

Figure 1. The four-loop β function for the $\overline{MS}$ scheme compared with various approximations.

where $t_i^\square = t_i - p_i$. We know all coefficients except for $t_3^\square$. (See [1] for references.) The latter can be computed from the difference of the four-loop coefficients, $b_3^{\overline{MS}} - b_3^\square$, of the β function

$$\beta^{\mathcal{S}}(g_{\mathcal{S}}) = -g_{\mathcal{S}}^3 (b_0 + b_1 g_{\mathcal{S}}^2 + b_2^{\mathcal{S}} g_{\mathcal{S}}^4 + b_3^{\mathcal{S}} g_{\mathcal{S}}^4 + \cdots), \quad (3)$$

where $\mathcal{S} = \overline{MS}, \square$ denotes the scheme. We find

$$
\begin{aligned}
t_3^\square &= \big(b_3^{\overline{MS}} - b_3^\square + 2(b_2^{\overline{MS}} t_1 - b_2^\square p_1) \\
&+ b_1(t_1^2 - p_1^2) \big)/2b_0.
\end{aligned}
\quad (4)
$$

We know $b_3^{\overline{MS}}$, but we do not know $b_3^\square$. In the $\overline{MS}$ scheme the four-loop β function turns out to be very well reproduced by the Padé-improved three-loop β function

$$\beta_{[1,1]}^{\mathcal{S}}(g_{\mathcal{S}}) = -g_{\mathcal{S}}^3 \frac{b_0 b_1 + (b_1^2 - b_0 b_2^{\mathcal{S}}) g_{\mathcal{S}}^2}{b_1 - b_2^{\mathcal{S}} g_{\mathcal{S}}^2}. \quad (5)$$

In Fig. 1 we compare (5) with the four-loop expression. We see that in the region $g_{\overline{MS}}^2 \lesssim 2$, which is the region relevant to our calculation, the Padé-improved three-loop result differs by less than 0.02% from the four-loop result. We expect that the same is true for the boosted β function $\beta^\square$, which converges at a similar rate as $\beta^{\overline{MS}}$. Thus we may approximate $\beta^\square$ by $\beta_{[1,1]}^\square$. Expanding (5) gives us then

$$b_3^\square = \frac{b_2^{\square\,2}}{b_1}. \quad (6)$$

We can improve the convergence of the series (2) further by re-expressing it in terms of the tadpole improved coefficients

$$\widetilde{c}_{SW} = c_{SW} u_0^3, \; a\widetilde{m} = am/u_0. \quad (7)$$

This amounts, for example, to replacing every c_{SW} by

$$\widetilde{c}_{SW}\Big(1 + \frac{3}{4} p_1 \, g_\square^2 + \frac{3}{4}\big(\widetilde{p}_2 - \frac{1}{6} p_1^2\big) \, g_\square^4\Big). \quad (8)$$

We have already seen in [1] that tadpole improvement reduces the coefficients of the two-loop fermionic contribution. We find that tadpole improvement is equally successful in reducing the coefficients of the three-loop contributions.

In the final step we compute $\Lambda_{\overline{MS}}$ from $g_{\overline{MS}}(\mu)$ using the renormalization group equation

$$
\begin{aligned}
\frac{\mu}{\Lambda_{\overline{MS}}} &= \big(b_0 g_{\overline{MS}}^2\big)^{\frac{b_1}{2b_0^2}} \exp\Big(\frac{1}{2b_0 g_{\overline{MS}}^2} \\
&+ \int_0^{g_{\overline{MS}}} d\xi \big(\frac{1}{\beta^{\overline{MS}}(\xi)} + \frac{1}{b_0 \xi^3} - \frac{b_1}{b_0^2 \xi}\big)\Big).
\end{aligned}
\quad (9)
$$

For $\beta^{\overline{MS}}$ we use the Padé-improved four-loop β function

$$
\begin{aligned}
\beta_{[1,2]}^{\overline{MS}}(g_{\overline{MS}}) &= -g_{\overline{MS}}^3 \Big(b_0\,(b_1^2 - b_0 b_2^{\overline{MS}}) + (b_1^3 \\
&- 2b_0 b_1 b_2^{\overline{MS}} + b_0^2 b_3^{\overline{MS}}) \, g_{\overline{MS}}^2 \Big)\Big/\Big(b_1^2 \\
&- b_0 b_2^{\overline{MS}} + (b_0 b_3^{\overline{MS}} - b_1 b_2) \, g_{\overline{MS}}^2 \\
&+ (b_2^{\overline{MS}\,2} - b_1 b_3^{\overline{MS}}) \, g_{\overline{MS}}^4 \Big), \quad (10)
\end{aligned}
$$

which is a better approximation than the four-loop β function. The result should be independent of μ if our approximations of $\beta^{\overline{MS}}$ in (2) and (9) are consistent.

2. RESULTS

We have varied μ between 4 and 10 GeV and found indeed that $\Lambda_{\overline{MS}}$ changes by a fraction of a percent only. Finally we have chosen $a\mu = 2.5$, which corresponds to $\mu \approx 5$ GeV. We may estimate the error arising from our approximation (6) by replacing $b_3^{\overline{MS}}$ by $b_2^{\overline{MS}\,2}/b_1$ as well. We found that this changes our results by at most 0.05%.

To convert our numbers to a physical scale we have used the force parameter r_0. The results are

β	$\kappa_{\rm sea}$	V	c_{SW}	P	r_0/a	am	$\Lambda_{\overline{MS}}r_0$
5.20	0.1355	16^332	2.0171	0.536294(9)	5.041(40)	0.02364(16)	0.4860(39)
5.20	0.1350	16^332	"	0.533676(9)	4.754(40)	0.04586(19)	0.4704(40)
5.25	0.1352	16^332	1.9603	0.541135(24)	5.137(49)	0.04268(17)	0.4757(46)
5.26	0.1345	16^332	1.9497	0.539732(9)	4.708(52)	0.07196(20)	0.4430(49)
5.29	0.1355	24^348	1.9192	0.547081(26)	5.62(9)	0.03495(12)	0.4917(79)
5.29	0.1350	16^332	"	0.545520(29)	5.26(7)	0.05348(19)	0.4679(62)
5.29	0.1340	16^332	"	0.542410(9)	4.813(45)	0.09272(29)	0.4428(42)

Table 1
The results for $\Lambda_{\overline{MS}}r_0$, together with r_0/a, P and the quark masses am, and the parameters of the simulation.

presented in Table 1. The numbers have increased by $\approx 2\%$ relative to our previous values [1], which is well inside the systematic error that was estimated in that paper. We fit our data by
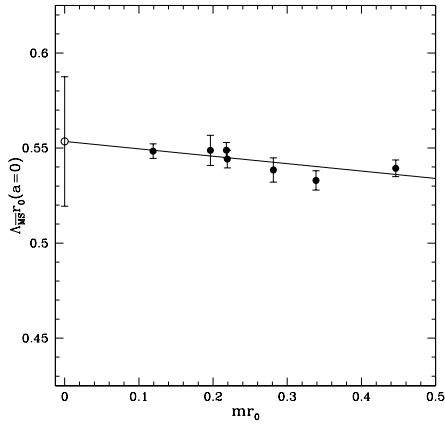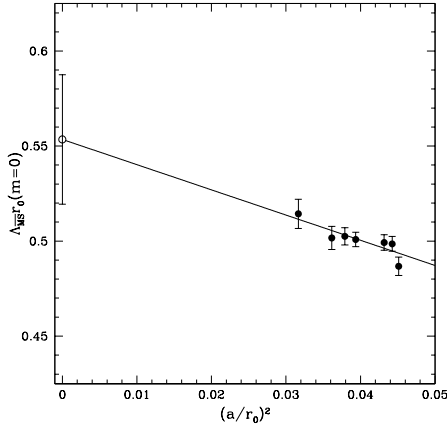

Figure 2. $\Lambda_{\overline{MS}}r_0(m=0)$ (top) and $\Lambda_{\overline{MS}}r_0(a=0)$ (bottom) together with the fit.

$$\Lambda_{\overline{MS}}r_0 = A(1+Bam)(1+Cmr_0)+D(a/r_0)^2. \quad (11)$$

(For details of the fit see [1].) In Fig. 2 we show $\Lambda_{\overline{MS}}r_0(m=0) = \Lambda_{\overline{MS}}r_0 - A(Bam + Cmr_0 + BCam\,mr_0)$ and $\Lambda_{\overline{MS}}r_0(a=0) = (\Lambda_{\overline{MS}}r_0 - D(a/r_0)^2)/(1+Bam)$ which, if the fit is successful, should collapse the data points onto a single line each. We see that this is indeed the case. We also see that the largest uncertainty arises from the continuum extrapolation, while the data depend only weakly on the quark mass. In the chiral and continuum limit our fit gives

$$\Lambda_{\overline{MS}}r_0 = 0.553(34) \quad (12)$$

$(= A)$. Using $r_0 = 0.5$ fm, we finally obtain

$$\Lambda_{\overline{MS}}^{N_f=2} = 218(13)\,{\rm MeV}. \quad (13)$$

This is to be compared with our previous result $\Lambda_{\overline{MS}}^{N_f=2} = 217(16)(11)\,{\rm MeV}$.

3. CONCLUSIONS

Our calculation faces two problems. One is the conversion from the boosted to the $\overline{MS}$ scheme. The other is the extrapolation to the chiral and continuum limit. We have solved the first problem, and with time and more data we hope to improve on the extrapolation.

REFERENCES

1. S. Booth et al., Phys. Lett. B **519** (2001) 229 (hep-lat/0103023).
2. R. Horsley, P.E.L. Rakow and G. Schierholz, hep-lat/0110210.