CERN-TH/2001–287
CPT-2001/PE.4252
DESY 01–174
IFIC/01–58
INT-ACK 01–20
FTUV-011025
LTH 525

Scalar condensate and light quark masses from overlap fermions

Pilar Hernández[a] *, Karl Jansen [b] †, Laurent Lellouch[c] and Hartmut Wittig[d] [e] †

[a]CERN, Theory Division, 1211 Geneva 23, Switzerland

[b]NIC/DESY Zeuthen, Platanenallee 6, D-15738 Zeuthen, Germany

[c]Centre de Physique Théorique, Case 907, CNRS Luminy, F-13288 Marseille, France

[d]Division of Theoretical Physics, Department of Mathematical Sciences, University of Liverpool, UK

[e]DESY, Notkestr. 85, D-22603 Hamburg, Germany

We have studied pseudoscalar correlation functions computed using the overlap operator. Within the accuracy of our calculation we find that the quark mass dependence agrees with the prediction of lowest-order Chiral Perturbation Theory χPT for quark masses in the range of $m \sim m_{\rm s}/2 - 2m_{\rm s}$. We present the results of an analysis which assumes lowest-order χPT to be valid to extract the low-energy constants Σ and $f_{\rm P}$, as well as the strange quark mass. Non-perturbative renormalization is implemented via a matching procedure with data obtained using Wilson fermions in the Schrödinger functional set-up. We find that the scalar condensate computed here agrees with the one obtained previously through a finite-size scaling analysis.

1. INTRODUCTION

Testing for spontaneous chiral symmetry breaking in QCD by determining whether a scalar condensate is developed and computing its size is a challenge tailored for lattice field theory. One way to extract the scalar condensate is to look at the behaviour of the pseudoscalar mass $M_{\rm P}$ as a function of the quark mass m. Lowest-order chiral perturbation theory (χPT) predicts this behaviour to be linear [1,2],

$$(aM_{\rm P})^2 = B_M am \ \ , B_M = 4a\Sigma/f_\chi^2 \qquad (1)$$

where Σ and f_χ are the only constants of the lowest-order effective chiral Lagrangian. In the full theory they coincide with the quark condensate and the pseudoscalar decay constant in the chiral limit, respectively.

Within the numerical accuracy of our calculation in the quenched approximation (see below) it turns out that the linear behaviour of eq. (1) is indeed satisfied for a large range of quark masses, extending from $m \sim m_{\rm s}/2$ to $m \sim 2m_{\rm s}$. However, the linearity of the data is actually surprising, since it is observed in a range of quark masses where higher-order terms in the quenched chiral Lagrangian could be relevant. Furthermore, at one-loop order one expects logarithmic corrections to appear [1,2]. We suspect that the observed linearity is actually the result of a large cancellation between various higher-order effects. A detailed discussion into the rôle of these effects will be presented elsewhere [3]. In the preliminary analysis presented here, we restrict ourselves to lowest-order χPT.

Our assumption that lowest-order χPT gives a good description of the mass behaviour receives support from the fact that we have a completely independent method for determining Σ, which gives consistent results. Indeed, using overlap fermions it was shown that the bare subtracted scalar condensate could be obtained through finite-size scaling methods [4]. This analysis has now been repeated by other authors [5,6], who have obtained consistent results. Higher-order effects might be important for this method as well, though here the corrections will be different. Again, we restrict ourselves to lowest-order

*On leave of absence from Departamento de Física Teórica, Universidad de Valencia, Spain

†Presented by K. Jansen and H. Wittig at *Lattice 2001*

quenched χPT, which is consistent with the numerical data [4], being fully aware that this assumption must eventually be checked.

In order to be useful for phenomenological applications, the bare condensate has to be renormalized. Renormalization factors which relate matrix elements of local composite operators in lattice regularization to a standard continuum renormalization scheme such as $\overline{\mathrm{MS}}$ can be computed in lattice perturbation theory. However, it is well known that perturbation theory in the bare coupling g_0 does not converge very well, and although the situation can be improved by considering so-called "mean-field improvement" [7], it is preferable to determine these factors non-perturbatively. One method to compute the renormalization factor for the scalar condensate computed using the overlap operator has been described in a recent publication [8] and will be briefly reviewed in the following section.

2. NON-PERTURBATIVE RENORMALIZATION

The chiral Ward identities imply that in lattice regularizations that preserve chiral symmetry the renormalization factor of the scalar density is the inverse of that for the quark mass (see e.g. [9])

$$Z_{\mathrm{P}} = Z_{\mathrm{S}} = \frac{1}{Z_{\mathrm{m}}}. \tag{2}$$

The non-perturbative renormalization of quark masses via an intermediate Schrödinger functional (SF) scheme has been studied by ALPHA [10], who have computed the relations between the running mass in the SF scheme, $\overline{m}_{\mathrm{SF}}$, and the renormalization group invariant (RGI) quark mass M, as well as the non-perturbative matching coefficient between $\overline{m}_{\mathrm{SF}}$ and the bare current quark mass m_{w} for $\mathrm{O}(a)$ improved Wilson fermions. Thus, in order to determine Z_{S} for overlap fermions, it is sufficient to compute the ratio of the bare mass in the overlap Lagrangian, m_{ov}, and $\overline{m}_{\mathrm{SF}}$. However, the formulation of the SF for overlap fermions is not straightforward, owing to the fact that the SF boundary conditions are incompatible with the Ginsparg-Wilson relation. We have therefore employed the following, more indirect approach.

The relation between the bare mass $m_{\mathrm{ov}}(g_0)$ and the RGI quark mass M is given by

$$M = Z_{\mathrm{M}}(g_0)\, m_{\mathrm{ov}}(g_0). \tag{3}$$

The ratio $M/m_{\mathrm{ov}}(g_0)$ can be rewritten as

$$\begin{aligned} \frac{M}{m_{\mathrm{ov}}(g_0)} &= \frac{M}{m_{\mathrm{w}}(g_0')} \cdot \frac{m_{\mathrm{w}}(g_0')}{m_{\mathrm{ov}}(g_0)} && (4) \\ &= Z_{\mathrm{M}}^{\mathrm{w}}(g_0') \cdot \frac{(r_0 m_{\mathrm{w}})(g_0')}{(r_0 m_{\mathrm{ov}})(g_0)}, && (5) \end{aligned}$$

where $g_0' \neq g_0$ in general, and the hadronic radius r_0 [11] is used to set the scale. The factor $Z_{\mathrm{M}}^{\mathrm{w}}$ has been computed in [10] for a wide range of couplings. The ratio $M/m_{\mathrm{ov}}(g_0)$ is then obtained by determining $(r_0 m_{\mathrm{ov}})$ and $(r_0 m_{\mathrm{w}})$ at a reference value x_{ref} of some observable, say $x_{\mathrm{ref}} = (r_0 M_{\mathrm{P}})^2$. Furthermore, the combination $Z_{\mathrm{M}}^{\mathrm{w}}(g_0')(r_0 m_{\mathrm{w}})(g_0')$ is a renormalized, dimensionless quantity. We can then define the universal factor U_{M} in the continuum limit as

$$U_{\mathrm{M}} = \lim_{g_0' \to 0} \left\{ Z_{\mathrm{M}}^{\mathrm{w}}(g_0')(r_0\, m_{\mathrm{w}})(g_0') \right\} \Big|_{(r_0 M_{\mathrm{P}})^2 = x_{\mathrm{ref}}}. \tag{6}$$

Since the renormalization of the scalar condensate is the inverse of that of the quark mass, we define the renormalization factor $\widehat{Z}_{\mathrm{S}}$, which relates the bare and RGI condensates as

$$\widehat{Z}_{\mathrm{S}}(g_0) = \frac{(r_0 m_{\mathrm{ov}})}{U_{\mathrm{M}}} \Big|_{(r_0\, M_{\mathrm{P}})^2 = x_{\mathrm{ref}}}. \tag{7}$$

At this point it is clear that all reference to the bare coupling g_0' has disappeared, and the only part that retains a dependence on the lattice regularization is $(r_0 m_{\mathrm{ov}})$.

The quantity U_{M} is in fact the RGI quark mass corresponding to $(r_0 M_{\mathrm{P}})^2 = x_{\mathrm{ref}}$. The results from ref. [12] are then easily converted into estimates for U_{M} at several values of x_{ref}. We stress that no additional calculations are required to determine the universal factor U_{M}, but that it can be derived easily from existing results in the literature. Furthermore, using the continuum value U_{M} ensures that cutoff effects of $\mathrm{O}(a^2)$ associated with the intermediate use of Wilson fermions at bare coupling g_0' are entirely removed from the renormalization condition. As described in detail in [8], additional cutoff effects in the case of non-zero g_0' can be quite substantial.

As indicated by eq. (2), the renormalization of the scalar condensate and the quark mass are inversely proportional if chiral symmetry is preserved. If eq. (7) is rewritten as

$$U_{\rm M}\big|_{x_{\rm ref}} = \frac{1}{\widehat{Z}_{\rm S}(g_0)}\,\left\{(r_0 m_{\rm ov})(g_0)\right\}_{x_{\rm ref}} \qquad (8)$$

it becomes clear that this condition fixes $\widehat{Z}_{\rm S}(g_0)$ by requiring that the renormalized quark mass reproduces the continuum result for Wilson fermions at *every* non-zero value of g_0. In other words, since the quark mass $m_{\rm ov}$ is required to fix $\widehat{Z}_{\rm S}$, the latter cannot be used to obtain independent predictions for renormalized quark masses for overlap fermions.

This problem can be circumvented after realizing that an alternative renormalization condition for $\widehat{Z}_{\rm S}$ is provided by the matrix element of the pseudoscalar density. The derivation is entirely analogous to the previous case. If we use the notation

$$G_{\rm P} = \langle 0|P|{\rm PS}\rangle \qquad (9)$$

as a shorthand for the matrix element of the pseudoscalar density P, and the superscripts "RGI", "ov" and "w" to distinguish the RGI matrix element from the one in the overlap and O(a) improved Wilson regularizations, we find

$$\frac{G_{\rm P}^{\rm RGI}}{G_{\rm P}^{\rm ov}(g_0)} = \frac{G_{\rm P}^{\rm RGI}}{G_{\rm P}^{\rm w}(g_0')}\cdot\frac{(r_0^2G_{\rm P}^{\rm w})(g_0')}{(r_0^2G_{\rm P}^{\rm ov})(g_0)}. \qquad (10)$$

The ratio $G_{\rm P}^{\rm RGI}/G_{\rm P}^{\rm w}(g_0')$ is given by [13,10]

$$\frac{G_{\rm P}^{\rm RGI}}{G_{\rm P}^{\rm w}(g_0')} = Z_{\rm P}^{\rm w}(g_0',\mu_0)\frac{\overline{m}_{\rm SF}(\mu_0)}{M}(1+b_{\rm P}am_{\rm q}), \quad (11)$$

where $m_{\rm q}$ is the bare subtracted quark mass for Wilson fermions. The improvement coefficient $b_{\rm P}$ is known in one-loop perturbation theory [14], and the combination $Z_{\rm P}^{\rm w}\overline{m}_{\rm SF}/M$ is known from ref. [10]. Again one can consider a reference point $x_{\rm ref}$ and define a universal factor in the continuum limit by

$$U_{\rm P} = \lim_{g_0'\to 0}\left\{Z_{\rm P}^{\rm w}(g_0',\mu_0)\frac{\overline{m}_{\rm SF}}{M} \times(1+b_{\rm P}am_{\rm q})(r_0^2G_{\rm P}^{\rm w})(g_0')\right\}\Big|_{(r_0M_{\rm P})^2=x_{\rm ref}}, \qquad (12)$$

Table 1
Results for $\widehat{Z}_{\rm S}$.

β	$x_{\rm ref}$	eq. (7) $\widehat{Z}_{\rm S}$	eq. (13) $\widehat{Z}_{\rm S}$
5.85	1.5736	1.05(25)	1.03(15)
	3.0	1.04(8)	1.04(8)
	5.0	0.99(4)	0.99(6)
6.00	1.5736	0.98(17)	1.05(14)
	3.0	1.03(8)	1.07(8)
	5.0	1.00(5)	1.03(6)

such that the alternative definition for $\widehat{Z}_{\rm S}(g_0)$ is obtained from

$$\widehat{Z}_{\rm S}(g_0) = \frac{U_{\rm P}}{(r_0^2G_{\rm P}^{\rm ov})(g_0)}\Big|_{(r_0\,M_{\rm P})^2=x_{\rm ref}}. \qquad (13)$$

As in the previous case one can easily convert the results of [12] to obtain the continuum factor $U_{\rm P}$.

Our results for $(r_0m_{\rm ov})$ and $(r_0^2G_{\rm P})$ at several values of $x_{\rm ref}$ were extracted from the same pseudoscalar correlation functions used to determine the condensate according to eq. (1). In Table 1 we list our estimates for $\widehat{Z}_{\rm S}$ determined at two values of the lattice spacing for several reference points. As one can see, the results for the two different renormalization conditions are entirely consistent within errors. In the following we will use the results from eq. (13) for $x_{\rm ref} = 3.0$. We also note that the non-perturbative determination of $Z_{\rm S}$ in ref. [15], which is based on a different technique [16], agrees with our data at $\beta = 6.0$.

3. SCALAR CONDENSATE

We computed pseudoscalar propagators using overlap fermions [17] at two values of the lattice spacing. The parameters of the runs are listed in Table 2, and further simulation details can be found in [18,8]. Pseudoscalar meson masses were extracted by fitting the numerical data obtained for the zero momentum pseudoscalar propagator to a standard single cosh form. The masses $M_{\rm P}$ are shown as a function of the bare quark mass in Figs. 1 and 2. We observe that $M_{\rm P}^2$ is consistent with being a linear function of the quark mass in the whole region.

Table 2
Simulation parameters.

β	ρ	Vol.	cfgs	$am_{\rm ov}$
5.85	1.6	$10^3 \times 24$	50	0.047−0.188
6.00	1.4	$14^3 \times 24$	48	0.028−0.133

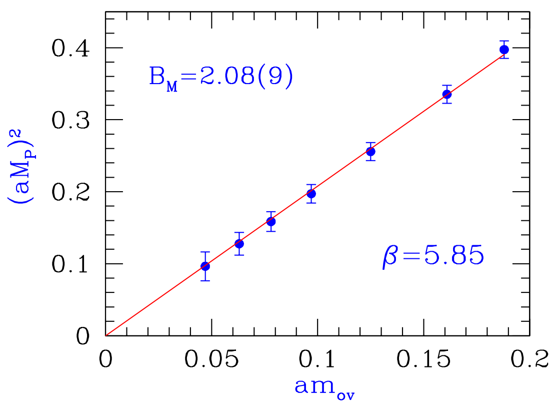

Figure 1. The pseudoscalar mass $(aM_{\rm P})^2$ as a function of the bare quark mass at $\beta = 5.85$.

Using eq. (1) an estimate for $a\Sigma/f_\chi^2$ can be extracted easily. Combining these results with the renormalization factor $\widehat{Z}_{\rm S}$ at the relevant lattice spacing and using r_0/a from [19] to set the scale, we obtain for the RGI condensate the following estimates:

$$\widehat{\Sigma} = \begin{cases} 0.0141(6)(11)\,\mathrm{GeV}^3, & \beta = 5.85 \\ 0.0144(6)(11)\,\mathrm{GeV}^3, & \beta = 6.0. \end{cases} \tag{14}$$

Here we have used $f_\chi = 128\,\mathrm{MeV}$ and $r_0 = 0.5\,\mathrm{fm}$ to convert into physical units. The first error is the statistical uncertainty in the determination of $a\Sigma/f_\chi^2$, whereas the second is due to the error in the renormalization factor. The results in eq. (14) can now be compared with the estimate obtained from the finite-size scaling (FSS) analysis at $\beta = 5.85$, which is

$$\widehat{\Sigma} = 0.0138(16)(10)\,\mathrm{GeV}^3, \quad \beta = 5.85. \tag{15}$$

This result is fully consistent with the above value. We note, however, that the scale r_0 enters as r_0^3 in eq. (15), and that the extraction of the FSS condensate in physical units does not depend at all on the value of f_χ. Thus, uncertainties in the lattice scale affect the two methods of extracting $\widehat{\Sigma}$ in different ways.

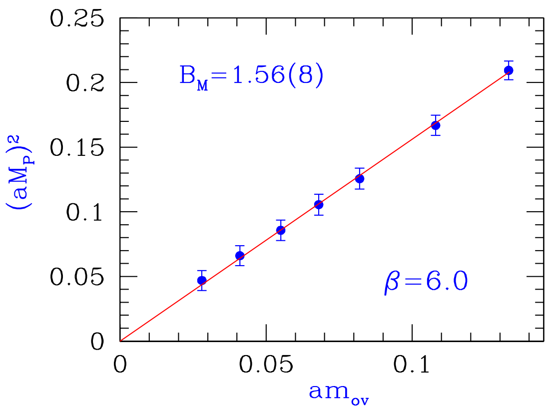

Figure 2. The pseudoscalar mass $(aM_{\rm P})^2$ as a function of the bare quark mass at $\beta = 6.0$.

Of course, eq. (1) can also be used to compute Σ from data obtained in simulations with $\mathrm{O}(a)$ improved Wilson fermions, despite the fact that chiral symmetry is explicitly broken in this formulation. What makes this strategy attractive after all is the fact that high statistics data at several values of β are available, allowing for a controlled continuum extrapolation. Combining our Wilson data at $\beta = 5.85$ with the data from the ALPHA Collaboration [12], we have extrapolated $r_0\widehat{\Sigma}/f_\chi^2$ to the continuum limit as a function of $(a/r_0)^2$ (see Fig. 3). The continuum results are entirely consistent with our results obtained using overlap fermions at the present level of accuracy. Fig. 3 also shows a compilation of results from various other determinations [20,21,15]. It demonstrates that there is not only broad agreement of the results but also that lattice spacing effects with overlap fermions seem to be rather small.

4. STRANGE QUARK MASS

The behaviour of the dependence of $(aM_{\rm P})^2$ on the quark mass (am) also allows us to calculate the renormalization group invariant strange quark mass M_s. Using again r_0 to set the scale and $m_K = 495\,\mathrm{MeV}$ to fix the quark mass we find

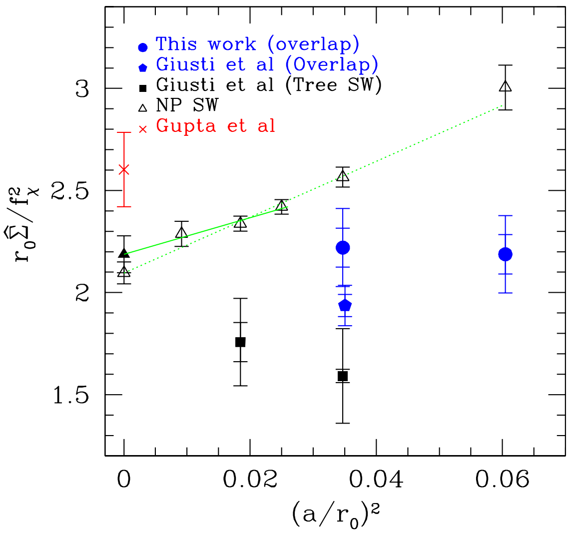

Figure 3. Comparison of different computations of the scalar condensate. We also show the result from Wilson fermions [12] extrapolated to the continuum. The double error bars denote the statistical and the quadratically combined statistical and systematic uncertainties.

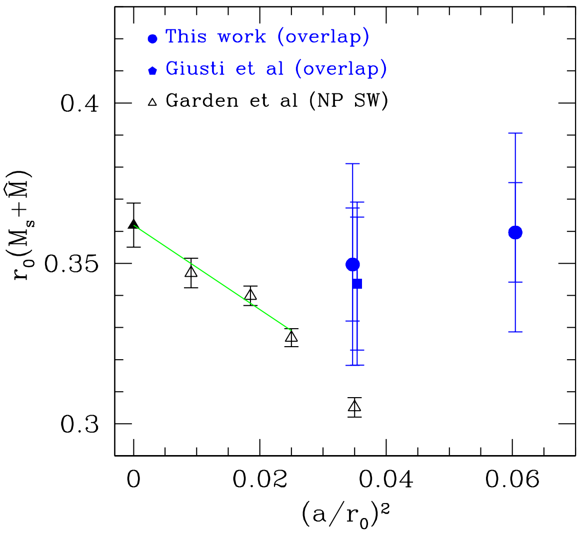

Figure 4. The renormalization group invariant strange quark mass as a function of the lattice spacing.

the results shown in Fig. 4 for $M_s + \hat{M}$, where $\hat{M}$ is the average of the RGI up and down quark masses. The agreement with the results obtained using $\mathrm{O}(a)$ improved Wilson fermions in the continuum limit is good, although the error bars in the overlap case are rather large due to our small statistics.

5. DECAY CONSTANT

Since the overlap formalism preserves chiral symmetry at non-zero lattice spacing, one can determine the pseudoscalar decay constant f_{P} without any further renormalization. By contrast, in simulations using Wilson fermions the renormalization factor Z_{A} of the axial current must be computed. In our work we have determined f_{P} through the PCAC relation, i.e.

$$f_{\mathrm{P}} = 2\frac{m_{\mathrm{ov}}}{M_{\mathrm{P}}^2}\langle 0|P|\mathrm{PS}\rangle, \tag{16}$$

where m_{ov} is an input parameter in the simulation, while M_{P} and $\langle 0|P|\mathrm{PS}\rangle$ are extracted from pseudoscalar correlation functions. In order to compare results from different fermionic discretizations we have plotted $r_0 f_{\mathrm{P}}$ versus $(r_0 M_{\mathrm{P}})^2$ in Fig. 5. One observes rather good agreement among the data employing either overlap (this work, [15]) or Domain Wall [22] fermions, as well as with the Wilson results extrapolated to the continuum limit with non-perturbative estimates for Z_{A} [23]. The fact that the overlap data at $\beta = 5.85$ are already close to the continuum Wilson results suggests that discretization errors for this quantity may be small. By contrast, $\mathrm{O}(a)$ improved Wilson results at $\beta = 5.85$ differ substantially from their continuum limit. The apparent smallness of cutoff effects in the overlap case is actually the reason for the excellent agreement of $\widehat{Z}_{\mathrm{S}}$ estimated either through eq. (7) or (13).

6. CONCLUSIONS

By employing overlap fermions at two values of the lattice spacing in the quenched approximation we tested whether or not the scalar condensate Σ obtained from a study of the behaviour of $(aM_{\mathrm{P}})^2$ as a function of the quark mass is compatible with

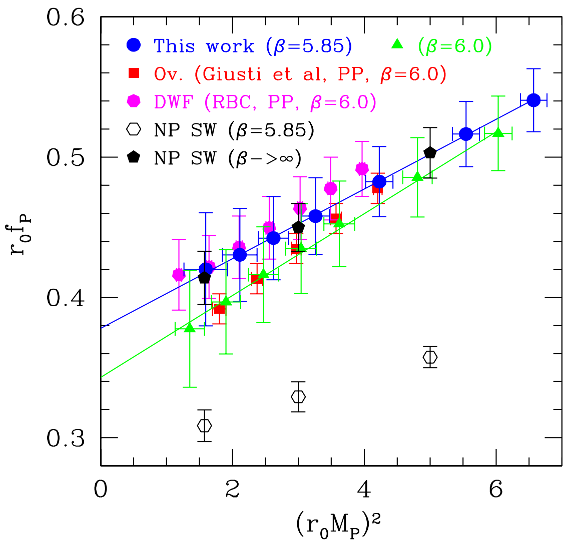

Figure 5. $r_0 f_{\rm P}$ versus $(r_0 M_{\rm P})^2$ for various discretizations. The lines are fits to our data at $\beta = 5.85$ (upper line) and $\beta = 6.0$ (lower line).

the one previously extracted from a completely independent finite-size scaling method.

We found this test to come out positive at leading order in quenched χPT and postpone the discussion of higher-order effects to future work [3]. We also determined the pseudoscalar decay constant, as well as the non-perturbatively renormalized strange quark mass. Within the relatively large statistical errors, the agreement with results obtained in the continuum limit using O(a) improved Wilson fermions [12], the overlap results of ref. [15] and also the Domain Wall formulation [22] at non-zero lattice spacing is excellent. There are strong indications that cutoff effects for overlap fermions are small, but more statistics as well as a larger range of lattice spacings is required to corroborate these findings.

ACKNOWLEDGMENTS

We are grateful to Jochen Heitger and Rainer Sommer for permission to use the ALPHA Collaboration data. LL thanks the INT at the University of Washington for its hospitality and the DOE for partial support during the completion of this work. This work was supported in part by EU-TMR project FMRX-CT98-0169 and EU-IHP project HPRN-CT-2000-00145.

REFERENCES

1. S.R. Sharpe, Phys. Rev. D46 (1992) 3146
2. C.W. Bernard and M.F.L. Golterman, Phys. Rev. D46 (1992) 853
3. P. Hernández, K. Jansen, L. Lellouch and H. Wittig, in preparation.
4. P. Hernández, K. Jansen and L. Lellouch, Phys. Lett. B469 (1999) 198
5. MILC Coll., T. DeGrand, hep-lat/0007046
6. P. Hasenfratz, S. Hauswirth, K. Holland, T. Jörg and F. Niedermayer, hep-lat/0109007,
7. G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48 (1993) 2250
8. P. Hernández, K. Jansen, L. Lellouch and H. Wittig, JHEP 07 (2001) 018
9. C. Alexandrou, E. Follana, H. Panagopoulos and E. Vicari, Nucl. Phys. B580 (2000) 394
10. S. Capitani, M. Lüscher, R. Sommer and H. Wittig, Nucl. Phys. B544 (1999) 669
11. R. Sommer, Nucl. Phys. B411 (1994) 839
12. J. Garden, J. Heitger, R. Sommer and H. Wittig, Nucl. Phys. B571 (2000) 237
13. M. Lüscher, S. Sint, R. Sommer and P. Weisz, Nucl. Phys. B478 (1996) 365
14. S. Sint and P. Weisz, Nucl. Phys. B502 (1997) 251
15. L. Giusti, C. Hoelbling and C. Rebbi, hep-lat/0108007 and hep-lat/0110184
16. G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, Nucl. Phys. B445 (1995) 81
17. H. Neuberger, Phys. Lett. B417 (1998) 141
18. P. Hernández, K. Jansen and L. Lellouch, hep-lat/0001008,
19. ALPHA Collab., M. Guagnelli, R. Sommer and H. Wittig, Nucl. Phys. B535 (1998) 389
20. R. Gupta and T. Bhattacharya, Phys. Rev. D55 (1997) 7203
21. L. Giusti, F. Rapuano, M. Talevi and A. Vladikas, Nucl. Phys. B538 (1999) 249
22. T. Blum et al., hep-lat/0007038,
23. M. Lüscher, S. Sint, R. Sommer and H. Wittig, Nucl. Phys. B491 (1997) 344