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Ultra-violet finiteness in noncommutative supersymmetric theories

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Abstract. We consider the ultra-violet divergence structure of general noncommutative supersymmetric $U(N_c)$ gauge theories, and seek theories which are all-orders finite.

In this paper we discuss noncommutative (NC) quantum field theories with spacetime dimension $d = 4$, $\mathcal{N} = 1$ supersymmetry and $U(N_c)$ local gauge invariance (for reviews and references see [1]). Our interest is in the ultra-violet (UV) divergence structure and in particular the identification of theories which are 'naturally' UV finite, where the meaning of 'natural' in this context will be explained later. Generally speaking in the following the term 'finite' will mean 'UV finite'.

The UV divergences in a NC theory are associated with the planar graph limit [2, 3], which, for a gauge theory with matter fields in the adjoint, fundamental and anti-fundamental representations (whether supersymmetric or not), can be obtained by taking both N_c and N_f (the number of matter multiplets) to be large [4]; a limit also called the Veneziano limit [5]. Moreover we showed in [4] that a particular set of NC theories were UV finite to all orders of perturbation theory, these theories being:

(1) $\mathcal{N} = 4$;
(2) one-loop finite $\mathcal{N} = 2$ theories;
(3) a specific one-loop finite $\mathcal{N} = 1$ theory.

As we shall see below (and from previous explicit calculations [2, 3, 6, 7]) the UV divergences of NC theories are well understood; however they suffer in general from singularities in the quantum effective action as $\theta \to 0$ ('UV/IR mixing' [3, 7]), where θ is the noncommutativity parameter. It was suggested in [7] that the NC $\mathcal{N} = 4$ effective action might in fact have a smooth limit as $\theta \to 0$. In this case the 'classical' $\theta \to 0$ limit (i.e. simply setting $\theta = 0$ in the Lagrangian) results in a finite commutative (C) theory, consisting of $SU(N_c)$ $\mathcal{N} = 4$ together with a free field $U(1)$ theory. However in cases (2) and (3) above the classical $\theta \to 0$ limit does *not* result in a UV finite theory (see later for more discussion) and therefore the $\theta \to 0$ limit of the effective action will not be smooth in these cases.

Cases (1) and (2) here are theories with only one independent coupling constant, the gauge coupling g, thanks to the $\mathcal{N} \geq 1$ supersymmetry; in case (3) one has at the outset two such couplings, g and a Yukawa coupling h, with the one-loop finiteness condition $h = g$. The fact that this *one-loop* condition suffices to render the theory UV finite to all orders is what we term *natural* UV finiteness. The corresponding class of theories (defined by the (h, g) parameter space) in the commutative $SU(N_c)$ case also contains a finite theory, but with the renormalisation-group (RG) trajectory defining the finite theory being an infinite power series of the form

$$h = a_1 g + a_5 g^5 + O(g^7) \tag{1}$$

where $a_1, a_5, \cdots$ are calculable constants. This leads us to our central question: are there any more naturally finite NC $\mathcal{N} = 1$ theories? (We can of course write down $\mathcal{N} = 1$ theories with Yukawa couplings $h \neq g$ which reduce to the $\mathcal{N} = 4$ and finite $\mathcal{N} = 2$ theories upon setting $h = g$, and so these theories are also naturally finite according to our definition.)

In order to address this question we begin by constructing a general renormalizable $\mathcal{N} = 1$ supersymmetric $U(N_c)$ gauge theory. We can consider theories with matter multiplets transforming as follows under gauge transformations:

$$\eta' = \eta \tag{2}$$

$$\chi' = U * \chi \tag{3}$$

$$\xi' = \xi * U^{-1} \tag{4}$$

$$\Phi' = U * \Phi * U^{-1} \tag{5}$$

where U is an element of $U(N_c)$, η, χ, ξ, Φ transform according to the singlet, fundamental, anti-fundamental and adjoint representations respectively, and $*$ denotes the standard noncommutative Moyal or $*$-product. The corresponding transformation on the gauge fields is

$$A'_\mu = U * A_\mu * U^{-1} + ig^{-1} U * \partial_\mu U^{-1}. \tag{6}$$

It is not clear how to construct gauge invariant theories with higher dimensional matter representations; if one considers, for example, a multiplet Ω such that under the gauge transformation

$$\Omega' = \tilde{U} * \Omega \tag{7}$$

where $\tilde{U}$ is a higher dimension $U(N_c)$ representation, then it is not obvious how to form the covariant derivative, since the transformation equation (6) is *not* equivalent to a similar expression with U replaced by $\tilde{U}$.

A general theory is then characterized by the superpotential

$$\begin{aligned} W = {} & r^i \eta_i + s^a \operatorname{Tr} \Phi_a + \frac{1}{2!} r^{ij} \eta_i \eta_j + \frac{1}{2!} s^{ab} \operatorname{Tr}(\Phi_a \Phi_b) + m^{\alpha\beta} \xi_\alpha \chi_\beta + \frac{1}{3!} r^{ijk} \eta_i * \eta_j * \eta_k \\ & + \frac{1}{3!} s^{abc} \operatorname{Tr}(\Phi_a * \Phi_b * \Phi_c) + \lambda^{\alpha a \beta} \xi_\alpha * \Phi_a * \chi_\beta + \rho^{\alpha\beta i} \xi_\alpha * \chi_\beta * \eta_i. \end{aligned} \tag{8}$$

Here $a : 1 \cdots N_\Phi$, $\alpha, \beta : 1 \cdots N_f$, $i, j, k : 1 \cdots N_\eta$. (We presume that $N_\xi = N_\chi = N_f$ in order to ensure anomaly cancellation [8].) Terms such as, for example, $\eta \operatorname{Tr} \Phi$ do not appear in equation

(8), because while $\int \mathrm{d}^4x\ \mathrm{Tr}\,\Phi$ is invariant under gauge transformations, $\mathrm{Tr}\,\Phi$ itself is not. Note also that, for example, $r^{ijk} = r^{jki} = r^{kij}$, but that r^{ijk} is not totally symmetric, and that quadratic terms are ordinary products (rather than $*$-products) within a spacetime integral.

The one-loop gauge β-function β_g is given by

$$16\pi^2\beta_g^{(1)} = [N_f + (N_\Phi - 3)N_c]\,g^3. \tag{9}$$

Note that this result remains valid in the Abelian case, i.e. for $N_c = 1$.

In the corresponding C $U(N_c) \equiv SU(N_c) \otimes U(1)$ theory, we would have

$$16\pi^2\beta_g^{(1)} = [N_f + (N_\Phi - 3)N_c]\,g^3 \quad \text{for } SU(N_c) \tag{10}$$

$$16\pi^2\beta_g^{(1)} = N_f g^3 \quad \text{for } U(1) \tag{11}$$

and of course equation (10) is valid for $N_c \geq 2$; in the case $N_c = 1$ we would have only equation (11). Notice that equations (9)–(11) become the same for $N_\Phi = 3$; this supports the conjecture [7] that the NC effective action is free of singularities as $\theta \to 0$ for $\mathcal{N} = 4$ theories. The fact that the condition $N_\Phi = 3$ renders the C and the NC β-functions identical is a one-loop result only; beyond one loop it is no longer sufficient, even when $N_f = 0$, as we shall show later. For $\mathcal{N} = 4$, however, the specific form of the Φ^3 interaction means that both C and NC β-functions vanish to all orders. For other NC finite theories such as finite $\mathcal{N} = 2$, with $N_f = 2N_c$, $N_\Phi = 1$, the situation is evidently different in that the corresponding C theories have a non-vanishing $U(1)$ β-function, and we therefore expect $\theta \to 0$ singularities (though the UV finiteness of $\mathcal{N} = 2$ theories beyond one loop suggests that for $\mathcal{N} = 2$ these singularities might be susceptible to summation).

The one-loop anomalous dimensions of the various matter superfields are also closely related to those in the corresponding C $SU(N_c)$ case, which are given by the general formula

$$16\pi^2\gamma^{(1)i}{}_j = P^i{}_j, \tag{12}$$

where

$$P^i{}_j = \frac{1}{2}Y^{ikl}Y_{jkl} - 2g^2C(R)^i{}_j, \tag{13}$$

for a general cubic superpotential

$$W = \frac{1}{6}Y^{ijk}\phi_i\phi_j\phi_k \tag{14}$$

where $Y_{jkl} = (Y^{jkl})^*$, and

$$C(R)^i{}_j = (R^AR^A)^i{}_j, \tag{15}$$

for a multiplet ϕ_i transforming according to a representation R^A.

Thus in the C $SU(N_c)$ case we have

$$16\pi^2\gamma_\eta^{(1)i}{}_j = \frac{1}{2}r^{ilm}r_{jlm} + N_c\rho^{\alpha\beta i}\rho_{\alpha\beta j} \tag{16}$$

$$16\pi^2\gamma_\Phi^{(1)a}{}_b = \frac{N_c^2 - 2}{4N_c}s^{acd}s_{bcd} - \frac{1}{2N_c}s^{acd}s_{bdc} + \lambda^{\alpha a\beta}\lambda_{\alpha b\beta} - 2N_cg^2\delta^a{}_b \tag{17}$$

$$16\pi^2\gamma_\xi^{(1)\alpha}{}_{\alpha'} = 2C_F(\lambda^{\alpha a\beta}\lambda_{\alpha' a\beta} - g^2\delta^\alpha{}_{\alpha'}) + \rho^{\alpha\beta i}\rho_{\alpha'\beta i} \tag{18}$$

$$16\pi^2\gamma_\chi^{(1)\beta'}{}_\beta = 2C_F(\lambda^{\alpha a\beta'}\lambda_{\alpha a\beta} - g^2\delta^{\beta'}{}_\beta) + \rho^{\alpha\beta' i}\rho_{\alpha\beta i} \tag{19}$$

where $C_F = \frac{N_c^2-1}{2N_c}$, whereas in the NC $U(N_c)$ case we have

$$16\pi^2\gamma_\eta^{(1)i}{}_j = \frac{1}{4}r^{ilm}r_{jlm} + N_c\rho^{\alpha\beta i}\rho_{\alpha\beta j} \tag{20}$$

$$16\pi^2\gamma_\Phi^{(1)a}{}_b = \frac{1}{4}N_c s^{acd}s_{bcd} + \lambda^{\alpha a\beta}\lambda_{\alpha b\beta} - 2N_c g^2\delta^a{}_b \tag{21}$$

$$16\pi^2\gamma_\xi^{(1)\alpha}{}_{\alpha'} = N_c(\lambda^{\alpha a\beta}\lambda_{\alpha' a\beta} - g^2\delta^\alpha{}_{\alpha'}) + \rho^{\alpha\beta i}\rho_{\alpha'\beta i} \tag{22}$$

$$16\pi^2\gamma_\chi^{(1)\beta'}{}_\beta = N_c(\lambda^{\alpha a\beta'}\lambda_{\alpha a\beta} - g^2\delta^{\beta'}{}_\beta) + \rho^{\alpha\beta' i}\rho_{\alpha\beta i}. \tag{23}$$

The β-functions of all the parameters in the superpotential W are determined in terms of γ by the non-renormalization theorem, which continues to hold in the NC case. Thus for example

$$\beta_m^{\alpha\beta} = \gamma_\xi{}^\alpha{}_{\alpha'}m^{\alpha'\beta} + m^{\alpha\beta'}\gamma_\chi{}^\beta{}_{\beta'}. \tag{24}$$

Notice that apart from the r^2 contribution to γ_η, the NC $U(N_c)$ and the C $SU(N_c)$ anomalous dimensions become identical if we drop $1/N_c$ terms. In the absence of singlets the general result is that the NC $U(N_c)$ anomalous dimensions can be precisely obtained as the Veneziano limit [5] of the corresponding C $SU(N_c)$ results: the Veneziano limit being large N_c and large N_f, with N_c/N_f fixed (notice that the λ^2 term in equation (21) is $O(N_f)$). This is because each Φ_a and each ξ, χ may be regarded as two-index objects, with two N_c-dimensional indices in the case of the Φ_a and one N_c-dimensional, one N_f-dimensional index in the case of the ξ, χ. Graphs are constructed using 't Hooft's double-line formalism [9]; the phase factors associated with the $*$-product then cancel, leaving a UV-divergent contribution only for planar graphs. These contain the maximum number of closed loops, corresponding to the maximum number of factors of N_c and/or N_f.

We shall be concentrating on the search for finite theories, and therefore (since clearly $\gamma_\eta^{(1)i}{}_j > 0$, unless η is a free field) we shall exclude singlet fields. The NC $U(N_c)$ anomalous dimensions can then be obtained to all orders as the Veneziano limit of the C $SU(N_c)$ ones. In this case one-loop finiteness requires (in addition to the vanishing of $\beta_g^{(1)}$ in equation (9))

$$\lambda^{\alpha' a\beta}\lambda_{\alpha a\beta} = g^2\delta^{\alpha'}{}_\alpha \tag{25}$$

$$\lambda^{\alpha a\beta}\lambda_{\alpha a\beta'} = g^2\delta^\beta{}_{\beta'} \tag{26}$$

$$\frac{1}{4}N_c s^{acd}s_{bcd} + \lambda^{\alpha a\beta}\lambda_{\alpha b\beta} = 2g^2N_c\delta^a{}_b. \tag{27}$$

We shall restrict ourselves to theories for which, in addition to equations (25) and (26), we also have

$$\lambda^{\alpha a\beta}\lambda_{\alpha b\beta} = N_f\lambda\delta^a{}_b, \tag{28}$$

$$s^{acd}s_{bcd} = s\delta^a{}_b. \tag{29}$$

Tracing equations (25), (26) and (28) we obtain $N_\Phi\lambda = g^2$, and hence from equations (27) and (29) that

$$s = \left(8 - \frac{4\sigma}{N_\Phi}\right)g^2. \tag{30}$$

where $\sigma = N_f/N_c$. Now any one-loop finite C theory is automatically two-loop finite, and it follows that the same will be true for our NC $U(N_c)$ theories. Moreover any two-loop finite C theory has vanishing $\beta_g^{(3)}$; once again it follows that the same will be true for these NC $U(N_c)$ theories. The check for higher-order finiteness thus starts with $\gamma^{(3)}$ for the NC $U(N_c)$ theory. This can be obtained (in the absence of singlets) as the large-N_c, large-N_f limit of $\gamma^{(3)}$ for the C $SU(N_c)$ theory. The result for $\gamma^{(3)}$ in a general commutative theory is [10]:

$$\begin{aligned}(16\pi^2)^3\gamma^{(3)} = (16\pi^2)^3\gamma_P^{(3)} \\ &+\kappa\{g^2\left[C(R)S_4 - 2S_5 - S_6\right] - g^4\left[PC(R)C(G) + 5PC(R)^2\right] \\ &+4g^6QC(G)C(R)\} + 2Y^*S_4Y - \frac{1}{2}S_7 - S_8 + g^2\left[4C(R)S_4 + 4S_5\right] \\ &+g^4\left[8C(R)^2P - 2QC(R)P - 4QS_1 - 10r^{-1}\mathrm{Tr}\left[PC(R)\right]C(R)\right] \\ &+g^6\left[2Q^2C(R) - 8C(R)^2Q + 10QC(R)C(G)\right]\end{aligned} \tag{31}$$

where $\kappa = 6\zeta(3)$, $16\pi^2\beta_g^{(1)} = Qg^3$, $C(G) = N_c$ for $SU(N_c)$,

$$S_{4j}^i = Y^{imn}P^p{}_mY_{jpn} \tag{32}$$

$$S_{5j}^i = Y^{imn}C(R)^p{}_mP^q{}_pY_{jnq} \tag{33}$$

$$S_{6j}^i = Y^{imn}C(R)^p{}_mP^q{}_nY_{jpq} \tag{34}$$

$$S_{7j}^i = Y^{imn}P^p{}_mP^q{}_nY_{jpq} \tag{35}$$

$$S_{8j}^i = Y^{imn}(P^2)^p{}_mY_{jpn} \tag{36}$$

$$Y^*S_4Y^i{}_j = Y^{imn}S_4{}^p{}_mY_{jpn}, \tag{37}$$

and where [11]:

$$\begin{aligned}(16\pi^2)^3\gamma_P^{(3)} = \kappa g^6\left[12C(R)C(G)^2 - 2C(R)^2C(G) - 10C(R)^3 - 4C(R)\Delta(R)\right] \\ +\kappa g^4\left[4C(R)S_1 - C(G)S_1 + S_2 - 5S_3\right] + \kappa g^2Y^*S_1Y + \kappa M/4\end{aligned} \tag{38}$$

where

$$S_{1j}^i = Y^{imn}C(R)^p{}_mY_{jpn} \tag{39}$$

$$Y^*S_1Y^i{}_j = Y^{imn}S_1{}^p{}_mY_{jpn} \tag{40}$$

$$S_{2j}^i = Y^{imn}C(R)^p{}_mC(R)^q{}_nY_{jpq} \tag{41}$$

$$S_{3j}^i = Y^{imn}(C(R)^2)^p{}_mY_{jpn} \tag{42}$$

$$M^i{}_j = Y^{ikl}Y_{kmn}Y_{lrs}Y^{pmr}Y^{qns}Y_{jpq} \tag{43}$$

$$\Delta(R) = \sum_\alpha C(R_\alpha)T(R_\alpha). \tag{44}$$

Note that in a one loop finite theory ($P = Q = 0$) $\gamma^{(3)}$ reduces to $\gamma_P^{(3)}$. In equation (44) the sum over α is a sum over irreducible representations. Thus whereas $C(R)$ is a matrix, $C(R_\alpha)$ and $\Delta(R)$ are numbers. To obtain the NC $U(N_c)$ result we need to specialize to $SU(N_c)$, and extract the leading terms in N_c, N_f. This involves replacing the Casimir $C_F = \frac{N_c^2-1}{2N_c}$ corresponding to the fundamental representation of $SU(N_c)$ by $C_F = \frac{1}{2}N_c$ corresponding to $U(N_c)$, and dropping the M-term, which is non-planar (and hence non-leading in N_c, N_f). Then upon using equations (25), (26), (28) and (29), we find for a one-loop finite theory

$$(16\pi^2)^3\gamma_\Phi^{(3)a}{}_b = \kappa N_c^3 g^2[-2\sigma g^4 - 4N_\Phi g^4 + 8g^4 + \frac{1}{8}s^2 - \frac{1}{2}sg^2]\delta^a{}_b, \tag{45}$$

$$(16\pi^2)^3\gamma_\xi^{(3)\alpha}{}_{\alpha'} = \frac{1}{4}\kappa N_c^3 g^4[16g^2 - 4\sigma g^2 - 8N_\Phi g^2 + s]\delta^\alpha{}_{\alpha'}, \tag{46}$$

$$(16\pi^2)^3\gamma_\chi^{(3)\beta'}{}_\beta = \frac{1}{4}\kappa N_c^3 g^4[16g^2 - 4\sigma g^2 - 8N_\Phi g^2 + s]\delta^\beta{}_{\beta'}, \tag{47}$$

where s is determined by equation (30). We are seeking 'naturally' finite $\mathcal{N} = 1$ theories; those for which one-loop finiteness implies all-orders finiteness. The obvious strategy is firstly to choose the field content to ensure vanishing $\beta_g^{(1)}$ in equation (9), then to choose the Yukawa couplings to make $\gamma^{(1)} = 0$ in equations (20)–(23), and finally to check for vanishing of the higher-order RG functions. From equation (9) we see that to achieve vanishing $\beta_g^{(1)}$ we need to take either $N_f = 0$, $N_\Phi = 3$; $N_f = N_c$, $N_\Phi = 2$; $N_f = 2N_c$, $N_\Phi = 1$; or $N_f = 3N_c$, $N_\Phi = 0$. However, in the last case, in the absence of singlet interactions it is clearly impossible to arrange $\gamma_\xi^{(1)} = \gamma_\chi^{(1)} = 0$. We shall consider each remaining case in turn.

The first class of theories ($N_f = 0$, $N_\Phi = 3$) includes NC $\mathcal{N} = 4$, which, as we showed in [4], is all-orders finite. The superpotential for NC $\mathcal{N} = 4$ is

$$W_1 = g\,\mathrm{Tr}\,(\Phi_1 * [\Phi_2, \Phi_3]_*) = g(W_a - W_b) \tag{48}$$

where $W_a = \mathrm{Tr}(\Phi_1 * \Phi_2 * \Phi_3)$ and $W_b = \mathrm{Tr}(\Phi_1 * \Phi_3 * \Phi_2)$; surprisingly, we were also able to show that the theory with superpotential

$$W_2 = g\,\mathrm{Tr}\,(\Phi_1 * \{\Phi_2, \Phi_3\}_*) = g(W_a + W_b) \tag{49}$$

is also all-orders finite, and hence is naturally finite according to our definition. Both these theories are special cases of the general three-adjoint case defined by

$$W = \frac{1}{3!}s^{abc}\,\mathrm{Tr}\,(\Phi_a * \Phi_b * \Phi_c)\,, \quad a, b, c : 1 \cdots 3. \tag{50}$$

According to equation (27), these theories are one-loop finite if

$$s^{acd}s_{bcd} = 8g^2\delta^a{}_b. \tag{51}$$

Are all such theories finite to all orders? We will now show that, unlike in the C case, the class of theories defined by equations (50) and (51) is indeed naturally finite through three loops. It is clear that for the NC $U(N_c)$ theory with the three-adjoint superpotential equation (50), if equation (51) holds, i.e. $s = 8g^2$ and $\sigma = 0$, then in equation (45), $\gamma^{(3)a}{}_b = 0$. The M-term, which does not contribute in the NC case as it is non-planar, is indeed solely responsible for the non-vanishing of $\gamma^{(3)}$ in the C case in, for example the two-loop finite $SU(N_c)$ theory [12]

$$W = \frac{\sqrt{2}gN_c}{\sqrt{N_c^2 - 4}}d^{abc}\phi_1^a\phi_2^b\phi_3^c. \tag{52}$$

This theory, in fact, is closely related to an example of the class of commutative theories which can be made finite by defining a Yukawa coupling as a power series in the gauge coupling. Thus if we replace the superpotential W by

$$W = \sqrt{2}hd^{abc}\phi_1^a\phi_2^b\phi_3^c, \tag{53}$$

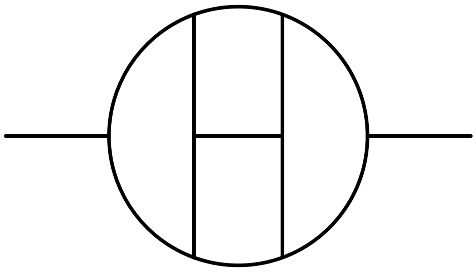

Figure 1. Graph giving irreducible contribution to $\gamma^{(4)}$.

and define by h as follows:

$$h = gN_c/\sqrt{N_c{}^2 - 4} + a_5 g^5 + O(g^7) \tag{54}$$

then it is possible to choose $a_5, \cdots$ to achieve finiteness [13]. This C theory, though finite, is not *naturally* finite. Notice that if we set $\theta = 0$ in equation (49) we obtain $W_2^C = \sqrt{2} g d^{abc} \phi_1^a \phi_2^b \phi_3^c$, which is *not* finite. Thus although the NC theory defined by equation (49) is UV finite, we expect its effective action to develop singularities as $\theta \to 0$; moreover, since the theory does not have $\mathcal{N} = 2$, we expect the structure of these singularities to be more involved beyond one loop.

Returning to the NC case, does the natural finiteness persist beyond three loops, given equation (51)? Consider the $O(s^8)$ graph shown in figure 1. Now clearly this graph, being planar, contributes to the NC $U(N_c)$ result for $\gamma^{(4)}$. However the condition equation (51) is not sufficient to reduce the tensor expression $s^{imn} s^{qrs} s^{puw} s^{tvx} s_{mpq} s_{nst} s_{ruv} s_{wxj}$ and therefore in the general case there will be an $O(s^8)$ contribution to the NC $\gamma^{(4)}$; thus this class of theories is not in general naturally finite beyond three loops.

We now turn to the case $N_f = N_c$ and $N_\Phi = 2$. From equation (30) we have $s = 6g^2$, and it is easy to show from equations (45)–(47) that neither $\gamma_\Phi^{(3)}$ nor $\gamma_{\xi,\chi}^{(3)}$ vanish. There are therefore no naturally finite theories in the case $N_f = N_c$ and $N_\Phi = 2$.

Finally we turn to the case $N_f = 2N_c$ and $N_\Phi = 1$. Note that in this case, writing $\lambda^{\alpha 1 \beta} = \Lambda^{\alpha\beta}$, we can redefine ξ and χ to diagonalize Λ. Then equations (25) and (26) imply $\Lambda\Lambda^\dagger = g^2 1$, and so in this diagonal basis we can write $\Lambda = g1$. We also have from equation (30) that $s = 0$. However, this is now simply the $\mathcal{N} = 2$ theory†. Thus there are no new naturally finite NC theories with $N_f = 2N_c$.

We thus find no evidence of any additional naturally finite supersymmetric theories in the NC case beyond the one already discovered in [10], at least under the assumptions of equations (28) and (29). However, since our arguments are founded on the impossibility of reducing complex tensor expressions in the general case, we cannot rule out the existence of further isolated examples of naturally finite supersymmetric theories. Indeed, it appears likely that other naturally finite theories must exist, as it has been argued [14] that theories obtained by orbifold truncation from NC $\mathcal{N} = 4$ supersymmetry, whose planar graphs may be evaluated

† This is UV finite beyond one loop (because of $\mathcal{N} = 2$ supersymmetry) in both C and NC cases. The $N_f = 2N_c$ condition renders both the NC $U(N_c)$ theory and the C $SU(N_c)$ theory finite at one loop as well; in the C $U(N_c)$ theory, however, the additional $U(1)$ gauge coupling has a non-zero one-loop (and one-loop only) β-function, unless there are no matter (ξ, χ) hypermultiplets, in which case this β-function is also zero. Thus (as we remarked earlier) for a $\mathcal{N} = 2$ theory with hypermultiplets we would expect singularities to occur in the effective action in the limit $\theta \to 0$.

using the corresponding graphs of the original NC $\mathcal{N} = 4$ theory [15], are naturally finite; such theories may have $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry or indeed may be non-supersymmetric. These theories are highly constrained in their field content, interactions and also in their gauge group, which is typically a product of $U(N_c)$ factors. At present our only example of an all-orders finite NC supersymmetric gauge theory with a $U(N_c)$ gauge group and which is *not* finite by virtue of finiteness of the corresponding $\mathcal{N} > 1$ commutative theory is the theory defined by equation (49).

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