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Ultraviolet properties of noncommutative non-linear σ-models in two dimensions

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We discuss the ultra-violet properties of bosonic and supersymmetric noncommutative non-linear σ-models in two dimensions, both with and without a Wess-Zumino-Witten term.

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There has been a great deal of recent interest in noncommutative (NC) quantum field theories, stimulated by their connection with string theory and M-theory; for a review and comprehensive list of references see Ref. [1]. Most of this interest has focussed on four-dimensional theories. However, since two-dimensional theories have often been used as laboratories for investigating general properties of quantum field theories, it is natural to extend the discussion to this arena. Two-dimensional non-commutative non-linear σ-models have been discussed in Refs. [2][3]. A particularly interesting case to consider, by virtue of its conformal invariance properties, is the Wess-Zumino-Witten (WZW) model. This has been studied in the NC case in Refs. [2][4]. The NC WZW term is also discussed in Ref. [5] and the Kac-Moody algebra associated with the NC WZW model has been investigated in Ref. [6]. Moreover, its renormalisation has been carried out at one-loop order[7]. Our purpose in this paper is to continue the program of perturbative investigation of the NC WZW model, and also the NC version of the principal chiral model (i.e. the theory defined on a group manifold without the WZW term). We show how results for the NC U_N WZW, and also principal chiral, model may be obtained from the leading-N term in the corresponding result for the commutative SU_N theory.

Firstly we discuss the elements of the construction of NC field theories. The algebra of functions on a noncommutative space is isomorphic to the algebra of functions on a commutative space with coordinates x^μ, with the product $f*g(x)$ defined as follows

$$f*g(x) = e^{-i\Theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}} f(x+\xi)g(x+\eta)|_{\xi,\eta\to 0}, \tag{1}$$

where Θ is a real antisymmetric matrix. Quantum field theories analogous to the corresponding commuting theories are now straightforward to define, with $*$-products replacing ordinary products. In particular the noncommutative two-dimensional Wess-Zumino-Witten (WZW) model is defined by

$$S = -\frac{1}{4\lambda^2}\int_\Sigma d^2x \mathrm{Tr}\left(\partial_\mu g g^{-1}\partial^\mu g g^{-1}\right)_* + \frac{k}{24\pi}\int_B d^3x \epsilon^{\mu\nu\rho}\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right)_*, \tag{2}$$

where as usual Σ is the boundary of a three-dimensional manifold B, and g is a map from Σ (or its extension B) into U_N. (Note that SU_N is not a group under the $*$-product, whereas U_N is.) $\epsilon^{\mu\nu\rho}$ is the three-dimensional alternating symbol. A subscript $*$ in Eq. (2) indicates that every product of fields within the corresponding brackets is a $*$-product. We assume that the co-ordinates x^0, x^1 on the worldsheet are non-commutative, but the extended co-ordinate x^2 on the manifold B commutes with the others. The group-valued field g is defined as

$$g = \exp_*(i\phi) = 1 + i\phi - \frac{1}{2!}\phi * \phi + \ldots, \tag{3}$$

where ϕ is in the Lie algebra of U_N. ϕ can be expanded as

$$(\phi)^A_B = \phi_a (T_a)^A_B, \quad a = 0, 1, \ldots N^2 - 1, \quad A, B = 1, \ldots N \tag{4}$$

where T_a, $a = 1, \ldots N^2 - 1$ are the generators of SU_N, $T_0 = \sqrt{\frac{2}{N}} 1_N$, and with our conventions

$$\mathrm{Tr}(T_a T_b) = 2\delta_{ab}, \quad [T_a, T_b] = 2i f_{abc} T_c. \tag{5}$$

The U_N structure constants f_{abc} are totally antisymmetric, with $f_{0ab} = 0$ and f_{abc}, $a = 1, \ldots N^2 - 1$ being the structure constants of SU_N. The commutative version of the theory is the sum of the commutative SU_N theory together with a free scalar field. Later on we compare the β-function for λ in Eq. (2) with the corresponding β-function for the commutative SU_N theory.

The ultra-violet properties of the NC WZW model may be investigated using the background field method. We expand the field g around a classical background g_c as $g = g_c * g_q$, and express g_q in terms of a quantum fluctuation π as

$$g_q = \exp_*(i\lambda\pi). \tag{6}$$

The expansion of the action may then be effected straightforwardly[8]; we readily obtain

$$\begin{aligned} S(g) =& S(g_c) \\ &+ \frac{1}{2\lambda^2} \int_\Sigma d^2x P^{\mu\nu} \mathrm{Tr} \left[e^{i\lambda\pi} \partial_\mu e^{-i\lambda\pi} g_c^{-1} \partial_\nu g_c - i\lambda \partial_\mu \pi \int_0^1 dt e^{-it\lambda\pi} \partial_\nu e^{it\lambda\pi} \right]_*, \end{aligned} \tag{7}$$

where

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{k\lambda^2}{4\pi} \epsilon^{\mu\nu}, \tag{8}$$

with $\epsilon^{\mu\nu}$ the two-dimensional alternating symbol, and then derive an expansion in terms of π by using

$$\exp_*(i\lambda\pi)_* \partial_\mu [\exp_*(-i\lambda\pi)] = -i\lambda \partial_\mu \pi + \frac{(-i\lambda)^2}{2!} [\partial_\mu \pi, \pi]_* + \frac{(-i\lambda)^3}{3!} [[\partial_\mu \pi, \pi], \pi]_* + \ldots \tag{9}$$

(together with a similar relation with $\lambda \to -t\lambda$). The difference between noncommutative and commutative theories at the level of Feynman diagrams is that in the NC case, Feynman diagrams can acquire momentum dependent phase factors arising from the $*$-product. If such a factor contains a loop momentum, the UV divergence for that loop is suppressed. Since the π are adjoint fields in U_N, the detailed discussion is simplified by using the diagrammatic notation originally introduced by 't Hooft [9], where we represent a π^A_B propagator by a double line as in Fig. 1, the arrow pointing towards the upper index.

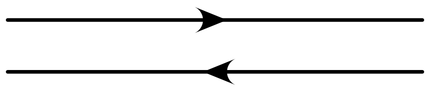

Fig. 1: The propagator for an adjoint U_N field

In terms of this notation, the phase factors cancel in planar graphs, and hence they give exactly the same contributions to the renormalisation-group (RG) functions (β-functions and anomalous dimensions) in the noncommutative U_N case as in the commutative SU_N case. In non-planar graphs, however, the phase factors do not cancel, so the corresponding Feynman integrals are UV convergent (after subtraction of subdivergences) and they do not contribute. This was first shown in the case of NC gauge theories in Ref. [10], but the same argument applies here. Now the planar contributions give the leading order in powers of N, and therefore the NC U_N result can be obtained from the commutative SU_N version by extracting the leading term in N. This simple connection between the NC and commutative cases makes it straightforward to extend results from the commutative to the NC case.

We start by considering conformal invariance properties of the WZW model. At the critical point

$$\lambda^2 = \frac{4\pi}{k} \tag{10}$$

the NC WZW model becomes conformally invariant, as discussed in Ref. [6]. In the commutative case the result can be derived straightforwardly starting from the commutative version of Eq. (7)[8](for the generalisation to an arbitrary parallelised manifold see Ref. [11]). We sketch the proof here. Feynman diagrams are constructed with vertices derived from the expansion of Eq. (7) in terms of π; the propagator, derived from the term in Eq. (7) quadratic in π, is simply $\frac{\eta^{\mu\nu}}{k^2}$. Note that there are two sorts of vertex; those with one derivative acting on a quantum field π and one factor of $g_c^{-1}\partial_\nu g_c$, (Type A) and those with two derivatives acting on π and no factors of $g_c^{-1}\partial_\nu g_c$ (Type B). Each vertex contains a factor of $P^{\mu\nu}$, although by symmetry only the $\eta^{\mu\nu}$ or the $\epsilon^{\mu\nu}$ in $P^{\mu\nu}$ contributes to the Type B vertices with even or odd numbers of πs respectively. The contributions to the renormalisation of λ arise from logarithmically divergent diagrams, which contain two Type A vertices and an arbitrary number of Type B vertices. (The WZW term is not renormalised [12].) If one is using dimensional regularisation, it is necessary to have a prescription for products of ϵ tensors, valid in $d \neq 2$ dimensions. The simplest is to define[13]

$$\epsilon^{\mu\rho}\epsilon_{\rho\nu} = \delta^{\mu}{}_{\nu}, \tag{11}$$

the contraction here being effected by the d-dimensional metric. (This definition leads to conformal invariance at the critical point (Eq. (10)) without the necessity of additional finite counter-terms.) Crucial is that we now have in d dimensions

$$P^{\mu\rho}P^{\nu}{}_{\rho} = \left[1 - \left(\frac{k\lambda^2}{4\pi}\right)^2\right]\eta^{\mu\nu}. \tag{12}$$

After performing all the Feynman integrals and implementing all the resulting tensor algebra, the final result is proportional to $P^{\mu\rho}P^{\nu}{}_{\rho}\mathrm{Tr}[\partial_\mu g_c g_c^{-1}\partial_\nu g_c g_c^{-1}]_*$. At the critical point Eq. (10) we have $P^{\mu\rho}P^{\nu}{}_{\rho} = 0$, and therefore there are no corrections to λ. Clearly, since this proof relies only on the tensor structure of the vertices and not on the details of the Feynman diagrams, it is unaffected by the additional phase factors present in the NC case. Moreover, the proof in the commutative, and hence also the NC case, is equally valid for the supersymmetric theory (which will be defined explicitly later), as was emphasised in Ref. [14].

Another result for the commutative bosonic SU_N case, which can be proved using conformal field theory arguments, is [15]

$$\left.\frac{\partial\beta_\lambda}{\partial\lambda^2}\right|_{\lambda^2=\frac{4\pi}{k}} = \frac{4N}{k+2N}. \tag{13}$$

We can expand Eq. (13) as

$$\left.\frac{\partial\beta_\lambda}{\partial\lambda^2}\right|_{\lambda^2=\frac{4\pi}{k}} = \frac{\lambda^2 N}{\pi}\left[1 - \frac{\lambda^2 N}{2\pi} + \left(\frac{\lambda^2 N}{2\pi}\right)^2 - \ldots\right]. \tag{14}$$

This result can be interpreted as a perturbative loop expansion. Each term is leading order in N for the corresponding loop order. Therefore the result for the NC U_N theory will be identical, and we deduce that Eq. (13) is also valid for the U_N NC WZW model. The β-function β_λ for the commutative WZW model has been computed up to three loops in

Ref. [16]. After specialising to SU_N, the result is leading order in N at this order and hence the result is identical in the NC U_N case. For completeness, we quote it here:

$$\beta_\lambda = -\lambda^2(1-\eta^2)\left[2\rho + 2\rho^2(1-3\eta^2) + 3\rho^3(1-\tfrac{25}{3}\eta^2+10\eta^4)+\cdots\right], \tag{15}$$

where $\eta = \frac{k\lambda^2}{4\pi}$ and $\rho = \frac{\lambda^2 N}{4\pi}$. (In fact three-loop results have also been given in Ref. [17], but apparently in a different renormalisation scheme.) It is easy to verify that Eq. (15) is compatible with Eqs. (13), (14); notice that in taking the derivative with respect of λ^2 of Eq. (15), only the terms arising from hitting the $(1-\eta^2)$ factor survive because the result is to be evaluated at $\eta^2 = 1$.

Let us now turn to the supersymmetric case. The NC supersymmetric WZW model has the superspace action [18]

$$S_{\rm SUSY} = \frac{1}{4\lambda^2}\int d^2x d^2\theta \mathrm{Tr}\left[\overline{D}G^{-1}DG\right]_* + \frac{k}{16\pi}\int d^3x d^2\theta \mathrm{Tr}\left[G^{-1}\frac{dG}{dt}\overline{D}G^{+}\gamma_3 DG\right]_*, \tag{16}$$

where $t \equiv x^2$, and θ^α are the Grassman co-ordinates and G is now a superfield and a group element of U_N, defined in terms of a superfield Φ as $G = \exp_*(i\Phi)$. The supercovariant derivative D is defined by

$$D_\alpha = \frac{\partial}{\partial\overline{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu, \tag{17}$$

and $\gamma_3 = \gamma^0\gamma^1$. In the commutative supersymmetric SU_N case β_λ is given through three loops by

$$\beta_\lambda = -2\rho\lambda^2(1-\eta^2), \tag{18}$$

i.e. the two[19] and three[20] loop contributions vanish[1]; this property clearly carries over to the NC case. The corresponding result to Eq. (13) in the supersymmetric case can be deduced from Ref. [23], namely

$$\left.\frac{\partial\beta_\lambda}{\partial\lambda^2}\right|_{\lambda^2=\frac{4\pi}{k}} = \frac{4N}{k}, \tag{19}$$

in other words the result for $\left.\frac{\partial\beta_\lambda}{\partial\lambda^2}\right|_{\lambda^2=\frac{4\pi}{k}}$ is one-loop exact. This result will clearly be equally valid in the NC case. Eq. (19) is consistent with the perturbative results through three loops, and predicts that β_λ at four loops and beyond should be proportional to $(1-\eta^2)^2$. Results have been presented at the 4-loop level[24] for a general $N = 1$ supersymmetric

[1] In the torsion-free case this was shown for a general manifold in Refs. [21], [22]

σ-model with torsion, but the specialisation to the group manifold case appears to be incorrect and we have been unable to verify Eq. (19) at this level.

In view of this uncertainty at four loops in the case of the WZW model, we now turn to the case of the NC σ-model defined on a group manifold without a WZW term–i.e. the NC version of the principal chiral model. Once again, we can obtain the NC results for the β-functions for the group U_N simply by picking out the leading N behaviour of the corresponding commutative results for SU_N. We start with the bosonic case. Results are available for the general bosonic σ model at two [25][21], three[26] and four[27] loops, expressed in terms of the Riemann tensor for the target space metric. The results for the SU_N case may be obtained by substituting the appropriate Riemannn tensor; in terms of general co-ordinates $\tilde{\phi}^k$ on the (commutative) group manifold, we have

$$R_{klmn} = e_k{}^a e_l{}^b e_m{}^c e_n{}^d f_{abe} f_{cde}, \tag{20}$$

where f_{abc} are the structure constants for SU_N and $e_k{}^a$ are the vielbeins for the metric on the group manifold, defined by

$$e_k{}^a e_l{}^a = g_{kl}, \quad g^{kl} e_k{}^a e_l{}^b = \delta^{ab}. \tag{21}$$

We find

$$\beta_\lambda = -\lambda^2 \left[2\rho + 2\rho^2 + 3\rho^3 + \rho^4 \left(\tfrac{19}{3} + \tfrac{12}{N^2}\zeta(3)\right) + \ldots\right]. \tag{22}$$

We deduce that the result in the NC U_N case is given by

$$\beta_\lambda = -\lambda^2 \left[2\rho + 2\rho^2 + 3\rho^3 + \tfrac{19}{3}\rho^4 + \ldots\right]. \tag{23}$$

Finally we turn to the case of the supersymmetric principal chiral model. As we already know from our earlier discussion of the WZW model, in this case the first non-zero contribution to the β-function beyond one loop appears at four loops [28]. The result in the commutative SU_N case may be obtained by substituting Eq. (20) into the general results given in Ref. [28], or, more easily, by recalling[27] that the four-loop $N = 1$ supersymmetric result is identical to the part of the four-loop bosonic result involving $\zeta(3)$. We then see from Eq. (22) that there is no leading contribution at four loops in the supersymmetric case. We deduce that β_λ for the NC $N = 1$ supersymmetric U_N principal chiral model vanishes from two through four loops.

In conclusion: we have established by perturbative arguments that the NC WZW U_N model (bosonic or supersymmetric) is all-orders finite at the critical point. We have pointed out that results for the NC U_N WZW or principal chiral model can be derived from the corresponding commutative SU_N result by extracting the leading-N term. This immediately led to Eq. (13) for the bosonic NC WZW U_N model and Eq. (19) for the supersymmetric NC WZW U_N model, together with the three-loop results Eq. (15) for the bosonic theory and Eq. (18) for the corresponding supersymmetric theory. In the case of the bosonic NC U_N principal chiral model we have given the β-function up to four loops in Eq. (23); and we have deduced that in the supersymmetric version of this theory, the β-function vanishes from two through four loops. This tempts us to speculate that β_λ may be one loop exact in this case, at least when using standard dimensional reduction. It is not clear however how to determine whether there are any general reasons why this result should persist at higher orders. The generally covariant methods used in the calculation of β-functions for general σ models as in Ref. [28] are not very well adapted for extracting the leading N behaviour; on the other hand, we have repeated the 4-loop $N = 1$ computation using the non-covariant expansion as in Eq. (7), and extracted the contributions which are planar in terms of 't Hooft's double line notation, but this does not seem to afford any general insights.

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References

[1] M.R. Douglas and N.A. Nekrasov, hep-th/0106048

[2] L. Dabrowski, T. Krajewski and G. Landi, Int. J. Mod. Phys. B**14** (2000) 2367

[3] H.O. Girotti, M. Gomes, V.O. Rivelles and A.J. da Silva, hep-th/0102101

[4] E.F. Moreno and F.A. Schaposnik, JHEP **0003** (2000) 032; Nucl. Phys. B**596** (2001) 439;
C. Nunez, K. Olsen and R. Schiappa, JHEP **0007** (2000) 030

[5] C. Chu, Nucl. Phys. B**580** (2000) 352

[6] A.M. Ghezelbash and S. Parvizi, Nucl. Phys. B**592** (2001) 408

[7] K. Furuta and T. Inami, Mod. Phys. Lett. A**15** (2000) 997

[8] M. Bos, Ann. Phys. **181** (1988) 177

[9] G. 't Hooft, Nucl. Phys. B**72** (1974) 461;
P. Cvitanovic, P.G. Lauwers and P.N. Scharbach, Nucl. Phys. B**203** (1982) 385

[10] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP **0002** (2000) 020

[11] S. Mukhi, Phys. Lett. B **162** (1985) 345

[12] S. Mukhi, Nucl. Phys. B **264** (1986) 640

[13] C.M. Hull and P.K. Townsend, Phys.Lett. B**191** (1987) 115;
R.R. Metsaev and A.A. Tseytlin, Phys. Lett. B**191** (1987) 354;
H. Osborn, Ann. Phys. **200** (1990) 1

[14] R.W. Allen, I. Jack and D.R.T. Jones, Z. Phys. C**41** (1988) 323

[15] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B**247** (1984) 83

[16] Z-M. Xi, Phys.Lett. B**214** (1988) 204; Nucl. Phys. B**314** (1989) 112

[17] S.V. Ketov, Theor. Math. Phys. **80** (1989) 709 [Teor. Mat. Fiz. **80** (1989) 56];
S.V. Ketov, A.A. Deriglazov and Y.S. Prager, Nucl. Phys. B**332** (1990) 447

[18] T.L. Curtright and C.K. Zachos, Phys. Rev. Lett. **53** (1984) 1799 ;
S.J. Gates, C.M. Hull and M. Rocek, Nucl. Phys. B **248** (1984) 157 ;
E. Abdalla and M.C. Abdalla, Phys. Lett. B **152** (1985) 59;
P. Di Vecchia et al, Nucl. Phys. B**253** (1985) 701;
E. Braaten, T.L. Curtright and C.K. Zachos, Nucl. Phys. B **260** (1985) 630

[19] B.E. Fridling and A.E. van de Ven, Nucl. Phys. B **268** (1986) 719;
D.R.T. Jones, Phys. Lett. B**192** (1987) 391

[20] S.V. Ketov, Phys. Lett. B**207** (1988) 140

[21] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, Ann. Phys. **134** (1981) 85

[22] L. Alvarez-Gaumé, Nucl. Phys. B **184** (1981) 180

[23] J. Fuchs, Nucl. Phys. B**286** (1987) 455

[24] A.A. Deriglazov and S.V. Ketov, Nucl. Phys. B**359** (1991) 498

[25] D. Friedan, Phys. Rev. Lett. **45** (1980) 1057

[26] S.J. Graham, Phys. Lett. B**197** (1987) 543;
A.P. Foakes and N. Mohammedi, Phys. Lett. B**198** (1987) 359; Nucl. Phys. B**306** (1988) 343

[27] I. Jack, D.R.T. Jones and N. Mohammedi, Nucl. Phys. B**322** (1989) 431

[28] M.T. Grisaru, A.E. van de Ven and D. Zanon, Phys. Lett. B**173** (1986) 423; Nucl. Phys. B**277** (1986) 388; Nucl. Phys. B**277** (1986) 409