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Anomalous dimension of non-singlet quark currents at $O(1/N_f^2)$ in QCD

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Abstract. We compute the $O(1/N_f^2)$ corrections to the flavour non-singlet quark bilinear currents in QCD in arbitrary spacetime dimensions. Hence, the anomalous dimension of the QED current $\bar{\psi}\sigma^{\mu\nu}\psi$ is deduced at four loops in the $\overline{\mathrm{MS}}$ scheme up to one unknown parameter.

Recently, the large N_f method of computing the renormalization group functions of a quantum field theory has been extended to the determination of information in quantum chromodynamics, (QCD), at a *new* order of expansion, $O(1/N_f^2)$, [1]. In [1, 2] the critical exponents corresponding to the anomalous dimensions of the quark field and the quark mass were computed in d-dimensions by exploiting the structure of the field theory at the d-dimensional infrared stable fixed point of the β-function as well as using the relation between QCD and the reduced model known as the non-abelian Thirring model, (NATM), [3]. It was demonstrated in [1, 2] that the gluon field strength operator became irrelevant at the fixed point allowing one to perform calculations with only the quark gluon vertex and the ghost vertex. This important observation, [3], paved the way for the new $O(1/N_f^2)$ calculations in QCD. Whilst the quark anomalous dimension is a gauge dependent quantity, its evaluation in d-dimensions in the Landau gauge is necessary for other $O(1/N_f^2)$ computations, since like in explicit perturbative calculations the wave function renormalization needs to be performed first. However, in additionally providing the d-dimensional value of the quark mass anomalous dimension, which is a gauge independent quantity, it was possible not only to verify the correctness of the recent four loop $\overline{\mathrm{MS}}$ perturbative quark mass dimension, [4, 5], at $O(1/N_f^2)$ but also to determine several of the coefficients which will appear in the *five* loop $\overline{\mathrm{MS}}$ anomalous dimension. Indeed given the huge complexity of performing the very high order computations in QCD, insight into the large order structure of the renormalization group functions can currently, we believe, only be gained by approaches such as the large N_f expansion coupled with other techniques. For instance, a recent study by Broadhurst, [6], of QED in the quenched approximation involving the Schwinger Dyson equation has allowed several new coefficients to be determined in the three and four loop $\overline{\mathrm{MS}}$ anomalous dimensions of various operators. By quenched we mean that part of the renormalization group functions which does not depend on N_f and hence arises from Feynman diagrams with no electron (or quark in the case of QCD) bubbles. Clearly given the amount of resources required for large loop calculations it would seem that the construction of the high order terms could be achieved by the marriage of the large N_f approach which performs the electron or quark loop bubble sums with a method which focuses on the harder $N_f = 0$ graphs which have no subdiagrams. Therefore, it is the purpose of this article to extend the $O(1/N_f)$ calculation of [7] of the anomalous dimension of the (flavour non-singlet) quark currents $\bar{\psi}\mathcal{A}\psi$, where $\mathcal{A}$ represents a set of γ-matrices, to $O(1/N_f^2)$. Since the case $\mathcal{A} = 1$ corresponds to the quark mass operator, our calculation builds on the $O(1/N_f^2)$ d-dimensional computations of [1, 2]. As a consequence of the result we present here, it will be possible, for instance, to deduce three of the four $\overline{\mathrm{MS}}$ coefficients in the polynomial in N_f of the anomalous dimension of the QED operator $\bar{\psi}\sigma^{\mu\nu}\psi$, where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, with the quenched result of [6]. Moreover, since the three loop $\overline{\mathrm{MS}}$ anomalous dimension of the general set of operators $\bar{\psi}\mathcal{A}\psi$ has been provided in explicit perturbation theory in QCD in [8], there is a non-trivial perturbative check on our final result. A final motivation for considering the renormalization of these operators rests in the fact that they underly the running of various quantities in heavy quark effective field theory, [7], and it is therefore important to have information on their anomalous dimensions.

We will determine the anomalous dimension of the set of operators

$$\mathcal{O}_{(n)} \;=\; \bar{\psi}\Gamma^{\mu_1\ldots\mu_n}_{(n)}\psi \tag{1}$$

where $\Gamma^{\mu_1\ldots\mu_n}_{(n)}$ is defined by

$$\Gamma^{\mu_1\ldots\mu_n}_{(n)} \;=\; \gamma^{[\mu_1}\ldots\gamma^{\mu_n]} \tag{2}$$

and is totally antisymmetric in the Lorentz indices for $n \geq 1$. This particular combination of γ-matrices is chosen since there will be no mixing under renormalization and also because the set $\{\Gamma_{(n)}\}$ corresponds to a complete γ-matrix basis in d-dimensions, [9]. The properties of the matrices $\Gamma_{(n)}$ are well established and given in [10, 11, 12]. To compute the large N_f contribution to the operator anomalous dimension one follows the standard critical point approach of [13] adopted for QCD, [14, 2]. This involves using the NATM where there is no triple or quartic gluon self interactions and the quark and gluon propagators are replaced by their leading d-dimensional critical point forms,

$$\psi(k) = \frac{A\not{k}}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) = \frac{B}{(k^2)^{\mu-\beta}} \left[\eta_{\mu\nu} - (1-b)\frac{k_\mu k_\nu}{k^2} \right] \tag{3}$$

where A and B are the momentum independent amplitudes, α and β are the respective dimensions of the quark and gluon fields and b is the covariant gauge fixing parameter. The field dimensions are related to their anomalous dimensions via

$$\alpha = \mu - 1 + \tfrac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi \tag{4}$$

where η is the quark anomalous dimension critical exponent, χ is the anomalous dimension of the quark gluon vertex and the spacetime dimension d is related to μ by $d = 2\mu$, [14, 2]. For the Feynman diagrams which arise at the order we consider none involve ghost fields which is why we have omitted its critical propagator. Ghost contributions are present indirectly in the final critical exponent for the operators $\mathcal{O}_{(n)}$ through the contribution from η and the combination $z = A^2B$ which arises from each vertex. Then the operators $\mathcal{O}_{(n)}$ are substituted into the two point Green's function $\langle \psi(p)\mathcal{O}_{(n)}\bar{\psi}(-p)\rangle$ and the scaling behaviour of the appropriate diagrams determined using (3). In particular a regularization with respect to Δ is introduced by shifting the anomalous dimension of the vertex, $\beta \to \beta - \Delta$, and the poles with respect to Δ are removed in a standard renormalization by the appropriate renormalization constants. If p is the external momentum then the resulting finite Green's function will involve a $\ln p^2$ term. For that Feynman diagram its coefficient contributes to the overall anomalous dimension critical exponent of the operator, [15]. By summing all such contributions at each order in $1/N_f$ one arrives at the full anomalous dimension. As we are interested in the $O(1/N_f^2)$ calculation there are only seven Feynman graphs to consider which are illustrated in figures 1 and 2 of [16] as well as the three loop graphs of figure 2 of [2] where the operator insertion is on a fermionic line which joins to an external vertex.

Although it was argued in [2] that when computing in the critical point large N_f approach, the Landau gauge was the only sensible choice of gauge, we have chosen to calculate the operator dimension in the Feynman gauge. The main reason for this is that since the operators $\mathcal{O}_{(n)}$ are gauge independent, their anomalous dimensions are therefore independent of b and hence can be determined in any gauge. We therefore set $b = 1$ in (3) to reduce the number of terms in the Feynman integrals and thereby minimise the amount of computation to be performed. Further, we will require η and z at $O(1/N_f^2)$ in this gauge. Since we are following the original skeleton Schwinger Dyson approach of [13], we record their respective values by first setting

$$z = \frac{z_1}{N_f} + \frac{z_2}{N_f^2} + O\left(\frac{1}{N_f^3}\right) \quad , \quad \eta = \frac{\eta_1}{N_f} + \frac{\eta_2}{N_f^2} + O\left(\frac{1}{N_f^3}\right) . \tag{5}$$

Then for arbitrary b, [14, 2],

$$\eta_1^b = \frac{[(2\mu-1)(\mu-2)+b\mu]C_2(R)\eta_0}{(2\mu-1)(\mu-2)T(R)} \quad , \quad z_1 = \frac{\Gamma(\mu+1)\eta_0}{2(2\mu-1)(\mu-2)T(R)}$$

$$\eta_2^b = \Bigg[- \mu(2\mu^2+\mu b-5\mu+2)(\mu-1)\left[\hat{\Psi}^2(\mu)+\hat{\Phi}(\mu)\right] C_2(G)$$
$$+ \frac{(8\mu^5-92\mu^4+270\mu^3-301\mu^2+124\mu-12)(2\mu^2+\mu b-5\mu+2)\hat{\Psi}(\mu)C_2(G)}{2(2\mu-1)(2\mu-3)(\mu-2)}$$

$$
\begin{aligned}
&+\ 3\mu(\mu-1)[\mu b C_2(G) + 2(2\mu^2+\mu b-5\mu+2)C_2(R)]\hat{\Theta}(\mu) \\
&-\ [(32\mu^7 b - 96\mu^7 + 8\mu^6 b^2 - 224\mu^6 b + 912\mu^6 - 4\mu^5 b^3 - 84\mu^5 b^2 + 704\mu^5 b - 3360\mu^5 \\
&\quad + 16\mu^4 b^3 + 278\mu^4 b^2 - 1124\mu^4 b + 6240\mu^4 - 19\mu^3 b^3 - 387\mu^3 b^2 + 846\mu^3 b - 6292\mu^3 \\
&\quad + 6\mu^2 b^3 + 230\mu^2 b^2 - 222\mu^2 b + 3416\mu^2 - 48\mu b^2 - 4\mu b - 908\mu + 88)C_2(G)\mu \\
&\quad - 8(4\mu^5 + 4\mu^4 b - 32\mu^4 - 13\mu^3 b + 75\mu^3 + 8\mu^2 b - 70\mu^2 - 2\mu b \\
&\qquad + 32\mu - 6)(2\mu-1)(2\mu-3)(\mu-2)C_2(R)] \\
&\quad /[4(2\mu-1)(2\mu-3)(\mu-1)(\mu-2)\mu]\Bigg] \frac{C_2(R)\eta_0^2}{2(2\mu-1)^2(\mu-2)^2T^2(R)}
\end{aligned} \tag{6}
$$

where the second result derives from the $O(1/N_f^2)$ solution of the Schwinger Dyson 2-point equations for non-zero b, and

$$
\begin{aligned}
z_2 \;=\;& \frac{3\mu(\mu-1)\Gamma(\mu+1)C_2(R)\eta_0^2}{2(2\mu-1)^2(\mu-2)^2T^2(R)} \left[\hat{\Theta}(\mu) \;-\; \frac{1}{(\mu-1)^2}\right] \\
&+ \frac{\mu^2(\mu-1)\Gamma(\mu)C_2(G)\eta_0^2}{4(2\mu-1)^2(\mu-2)^2T^2(R)} \left[3\hat{\Theta}(\mu) \;-\; \hat{\Psi}^2(\mu) \;-\; \hat{\Phi}(\mu) \;+\; \frac{b^2}{4(\mu-1)^2}\right. \\
&\qquad + \frac{(8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)\hat{\Psi}(\mu)}{2\mu(\mu-1)(2\mu-1)(2\mu-3)(\mu-2)} \;+\; \frac{(\mu^2+2\mu-2)b}{2(\mu-1)^2\mu^2} \\
&\qquad \left. - \frac{(16\mu^7 - 120\mu^6 + 420\mu^5 - 776\mu^4 + 742\mu^3 - 349\mu^2 + 84\mu - 12)}{2(2\mu-1)(2\mu-3)(\mu-1)^2(\mu-2)\mu^2}\right]
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\eta_0 &= \frac{(2\mu-1)(\mu-2)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)} \\
\hat{\Psi}(\mu) &= \psi(2\mu-3) \;+\; \psi(3-\mu) \;-\; \psi(1) \;-\; \psi(\mu-1) \\
\hat{\Phi}(\mu) &= \psi'(2\mu-3) \;-\; \psi'(3-\mu) \;-\; \psi'(\mu-1) \;+\; \psi'(1) \\
\hat{\Theta}(\mu) &= \psi'(\mu-1) \;-\; \psi'(1)
\end{aligned} \tag{8}
$$

and $\psi(x)$ is the derivative of the logarithm of the Euler Γ-function.

Since the evaluation of the $O(1/N_f^2)$ Feynman diagrams follows the standard methods of [2, 11, 14], we quote the final value of the critical exponent for the operator $\mathcal{O}_{(n)}$ for arbitrary n. We find

$$
\begin{aligned}
\eta_2^{(n)} \;=\;& \frac{\mu(\mu-1)(2\mu-n-1)(n-1)C_2(R)\eta_0^2}{(2\mu-1)^2(\mu-2)^2T^2(R)} \\
&\left[2C_2(R)\left(3\hat{\Theta}(\mu) + \frac{(4\mu^3 - 6\mu^2 n - 13\mu^2 + 3\mu n^2 + 2\mu n + 9\mu - n^2 - 3)}{\mu^2(\mu-1)^2}\right)\right. \\
&+ C_2(G)\left(\frac{(8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)\hat{\Psi}(\mu)}{2\mu(\mu-1)(2\mu-1)(2\mu-3)(\mu-2)} \;-\; \hat{\Psi}^2(\mu) \;-\; \hat{\Phi}(\mu)\right. \\
&\qquad - [16\mu^6 - 32\mu^5 n - 128\mu^5 + 16\mu^4 n^2 + 128\mu^4 n + 480\mu^4 - 64\mu^3 n^2 \\
&\qquad\quad - 152\mu^3 n - 900\mu^3 + 76\mu^2 n^2 + 48\mu^2 n + 831\mu^2 \\
&\qquad\quad \left.\left. - 24\mu n^2 - 344\mu + 44)]/[4\mu(2\mu-1)(2\mu-3)(\mu-1)^2(\mu-2)]\right)\right]
\end{aligned} \tag{9}
$$

in the notation of (5). The value for $\eta_1^{(n)}$ was given in [7] and we have correctly reproduced it in our leading order calculation.

There are various checks on the result, (9). First, since the operator $\bar{\psi}\gamma^\mu\psi$ is a conserved physical current, its anomalous dimension must vanish at all orders in perturbation theory. The overall factor of $(n-1)$ which naturally emerges in (9) ensures this. Second, the result for the quark mass anomalous dimension at $O(1/N_f^2)$, [1], is recovered when $n = 0$. The remaining checks are a consequence of comparing with explicit perturbative calculations since the three loop $\overline{\mbox{MS}}$ anomalous dimension for $\mathcal{O}_{(n)}$ has recently been provided in [8] and is

$$
\begin{aligned}
\gamma_{(n)}(a) &= - (n-1)(n-3)C_2(R)a \\
&+ \Big[4(n-15)T(R)N_f + (18n^3 - 126n^2 + 163n + 291)C_2(G) \\
&\quad - 9(n-3)(5n^2 - 20n + 1)C_2(R)\Big] \frac{(n-1)C_2(R)a^2}{18} \\
&+ \Big[\Big(144n^5 - 1584n^4 + 6810n^3 - 15846n^2 + 15933n + 11413 \\
&\quad - 216n(n-3)(n-4)(2n^2 - 8n + 13)\zeta(3)\Big) C_2^2(G) \\
&\quad + \Big(432n(n-3)(n-4)(3n^2 - 12n + 19)\zeta(3) \\
&\quad\quad - 3(72n^5 - 792n^4 + 3809n^3 - 11279n^2 + 15337n + 1161)\Big) C_2(G)C_2(R) \\
&\quad + \Big(1728(n-3)\zeta(3) + 8(3n^3 + 51n^2 - 226n - 278)\Big) C_2(G)T(R)N_f \\
&\quad - \Big(864n(n-3)(n-4)(n^2 - 4n + 6)\zeta(3) \\
&\quad\quad + 18(n-3)(17n^4 - 136n^3 + 281n^2 - 36n + 129)\Big) C_2^2(R) \\
&\quad - \Big(1728(n-3)\zeta(3) + 12(17n^3 + n^2 - 326n + 414)\Big) C_2(R)T(R)N_f \\
&\quad + 16(13n - 35)T^2(R)N_f^2\Big] \frac{(n-1)C_2(R)a^3}{108} ~+~ O(a^4)
\end{aligned}
\tag{10}
$$

where $\zeta(z)$ is the Riemann zeta function and a is related to the strong coupling constant, α_s, by $a = \alpha_s/(4\pi)$. The lower order coefficients for $n = 2$, for example, had been established in [7, 17] whilst the quenched term at three loop in QED had been given in [6]. To compare (9) with (10) we evaluate the latter at the critical coupling, a_c, in d-dimensions and expand in powers of ϵ and $1/N_f$, where $d = 4 - 2\epsilon$, with [14]

$$
\begin{aligned}
a_c &= \frac{3\epsilon}{T(R)N_f} + \frac{1}{T^2(R)N_f^2}\Big[\frac{33}{4}C_2(G)\epsilon - \left(\frac{27}{4}C_2(R) + \frac{45}{4}C_2(G)\right)\epsilon^2 \\
&+ \left(\frac{99}{16}C_2(R) + \frac{237}{32}C_2(G)\right)\epsilon^3 + \left(\frac{77}{16}C_2(R) + \frac{53}{32}C_2(G)\right)\epsilon^4 \\
&- \frac{3\epsilon^5}{256}\left[(288\zeta(3) + 214)C_2(R) + (480\zeta(3) - 229)C_2(G)\right] + O(\epsilon^6)\Big] + O\left(\frac{1}{N_f^3}\right).
\end{aligned}
\tag{11}
$$

To $O(\epsilon^3)$ we find exact agreement between both expansions of the critical exponent. Moreover, since (10) was computed in an arbitrary covariant gauge and (9) is in agreement with it, we have justified the choice of Feynman gauge in our calculations.

Having established the correctness of (9) with all regions of overlap, we can now determine new information on the higher order terms of (10). Writing the anomalous dimension in terms of its $O(1/N_f^2)$ part as

$$
\begin{aligned}
\gamma_{(n)}(a) &= - (n-1)(n-3)C_2(R)a \\
&+ \Big[4(n-15)T(R)N_f + (18n^3 - 126n^2 + 163n + 291)C_2(G)
\end{aligned}
$$

$$
\begin{aligned}
& - \; 9(n-3)(5n^2-20n+1)C_2(R)\Big] \, \frac{(n-1)C_2(R)a^2}{18} \\
& + \sum_{r=3}^{\infty} \Big(c_{r0}(T(R))^{r-1}N_f^{r-1} \; + \; c_{r1}(T(R))^{r-2}N_f^{r-2}C_2(R) \\
& \qquad + \; c_{r2}(T(R))^{r-2}N_f^{r-2}C_2(G) \Big) \, C_2(R)a^r \; + \; O\left(\frac{1}{N_f^3}\right)
\end{aligned} \tag{12}
$$

and expanding (9) to $O(\epsilon^5)$ using (11), we find the new $\overline{\mbox{MS}}$ coefficients

$$
\begin{aligned}
c_{40} &= 8(n-1)[45n-83-48\zeta(3)(n-3)]/81 \\
c_{41} &= 4(n-1)[143n^3-1205n^2+2292n+228+72(11n-45)\zeta(3)-648(n-3)\zeta(4)]/81 \\
c_{42} &= -\,2(n-1)[130n^3-958n^2+1683n-671+144(11n-45)\zeta(3)-1296(n-3)\zeta(4)]/81 \\
c_{50} &= 16(n-1)[16(n-15)\zeta(3)-144(n-3)\zeta(4)+5(25n-39)]/243 \\
c_{51} &= 2(n-1)[-144(60n^3-420n^2+731n-297)\zeta(3)+2592(11n-45)\zeta(4) \\
& \qquad -20736(n-3)\zeta(5)+(10308n^3-58428n^2+63695n+40347)]/729 \\
c_{52} &= (n-1)[96(72n^3-504n^2+17n+2013)\zeta(3)-3456(11n-51)\zeta(4) \\
& \qquad +110592(n-3)\zeta(5)-(7128n^3-37416n^2+64723n-56001)]/729 \; .
\end{aligned} \tag{13}
$$

As part of our motivation was to make use of other techniques aside from explicit perturbation theory to deduce higher order terms of the renormalization group functions, we can now examine the operator $\mathcal{O}_{(2)}$ in QED at four loops. The anomalous dimension in the quenched approximation is available from [6] and restricting (13) to QED, we find that

$$
\begin{aligned}
\gamma^{\mbox{QED}}_{(2)}(a) &= a \; - \; [171 \; + \; 52N_f]\frac{a^2}{18} \\
& + \Big[3285 \; - \; 3456\zeta(3) \; + \; 864\zeta(3)N_f \; + \; 588N_f \; - \; 72N_f^2\Big]\frac{a^3}{54} \\
& + \Big[\frac{3200}{3}\zeta(5) \; - \; \frac{2000}{3}\zeta(3) \; - \; \frac{10489}{24} \; + \; c_{43}N_f \\
& \quad + \left(\frac{4544}{81} \; - \; \frac{736}{9}\zeta(3) \; + \; 32\zeta(4)\right)N_f^2 \; + \; \left(\frac{56}{81} \; + \; \frac{128}{27}\zeta(3)\right)N_f^3\Big]\, a^4 \; + \; O(a^5)
\end{aligned} \tag{14}
$$

where only one unknown coefficient, c_{43}, remains to be determined. At present there is no next to quenched approximation technique which would allow for it to be deduced aside, of course, from an explicit calculation at four loops where one would isolate only those Feynman graphs which were linear in N_f.

Finally, we address the issue of the relation of the anomalous dimensions of $\mathcal{O}_{(n)}$ to their *four* dimensional counterparts. Due to the arbitrary dimensional nature of the large N_f method, we were forced to use the infinite dimensional $\Gamma_{(n)}$ basis which when restricted to four dimensions is related to the conventional γ-matrices. However, one has to be careful in treating γ^5 which is a strictly four dimensional object and never arises in the generalized basis. For example, setting $n = 0$, 1 and 2 in (9) we recover the correct perturbative structure of the respective flavour non-singlet currents. For $n = 3$ and 4 the anomalous dimensions $\eta^{(3)}$ and $\eta^{(4)}$ do not correspond to those of the axial vector and pseudoscalar currents, [18]. The reason for this is that the operator with $\Gamma_{(4)}$, for example, does not exactly project onto the operator with γ^5 in four dimensions and in some sense incorrectly retains an anticommuting γ^5 in d-dimensions. In the large N_f approach the method to properly account for the presence of γ^5 has been developed in [19] based on earlier observations of [20]. Like perturbation theory, [18], one has to add an additional critical exponent which would correspond to a finite renormalization constant in the perturbative approach. However, the result of [19] only concerned $O(1/N_f)$ calculations and therefore we extend the method here to the next order since it will also be important in future large N_f calculations of other operators which contain a γ^5 such as four quark operators. As in [19] we define a critical exponent corresponding to the finite renormalization by

$$\eta^{\mathrm{fin}} ~=~ \eta^{(4-n)} ~-~ \eta^{(n)} ~. \qquad (15)$$

This choice is motivated by [18, 8] in that since the currents are non-singlet, and hence anomaly free, this condition will restore the four dimensional properties of γ^5 in the renormalization of the operators. Hence, from (9) we find,

$$\begin{aligned} \eta^{\mathrm{fin}} ~=~ & - ~ \frac{8(n-2)C_2(R)\eta_0}{(2\mu-1)T(R)N_f} \\ & - ~ \left[24\mu(\mu-1)\hat{\Theta}(\mu)C_2(R) ~-~ 4\mu(\mu-1)\left(\hat{\Psi}^2(\mu) + \hat{\Phi}(\mu)\right) C_2(G) \right. \\ & ~~+~ \frac{2(8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)\hat{\Psi}(\mu)C_2(G)}{(2\mu-1)(2\mu-3)(\mu-2)} \\ & ~~-~ [16\mu^6 - 224\mu^5 + 32\mu^4n^2 - 128\mu^4n + 1104\mu^4 - 128\mu^3n^2 + 512\mu^3n \\ & ~~~~ - 2316\mu^3 + 152\mu^2n^2 - 608\mu^2n + 2115\mu^2 - 48\mu n^2 + 192\mu n \\ & ~~~~ - 704\mu + 44]C_2(G)/[(2\mu-1)(2\mu-3)(\mu-1)(\mu-2)] \\ & ~~+~ 8[4\mu^3 - 31\mu^2 + 6\mu n^2 - 24\mu n + 60\mu - 2n^2 \\ & ~~~~~ \left. + 8n - 18]C_2(R)/[\mu(\mu-1)] \right] \frac{(n-2)C_2(R)\eta_0^2}{(2\mu-1)^2(\mu-2)T^2(R)N_f^2} ~+~ O\left(\frac{1}{N_f^3}\right) \end{aligned} \qquad (16)$$

which is only relevant when $n = 3$ or 4. Adding (16) to (9) for these values one correctly recovers the critical exponents for the axial vector and pseudoscalar currents which agree with the known four dimensional perturbative results, [18].

We conclude with various remarks. First, we have demonstrated that it is possible to marry results from several techniques aside from perturbation theory to determine the structure of a renormalization group function at four loops in $\overline{\mathrm{MS}}$, up to one unknown parameter. Second, we focused on the flavour non-singlet currents. For even n the results are also valid for the flavour singlet currents since the extra graphs which arise when the operator $\mathcal{O}_{(n)}$ is inserted in a closed quark loop trivially vanish as they involve a trace over an odd number of γ-matrices. The study of the flavour singlet currents at $O(1/N_f^2)$ for odd n is more involved than the non-singlet computation. Due to the nature of the $1/N_f$ expansion the largest loop order one would have to consider is *five* loops. However, similar diagrams arise in the computation of the quark gluon vertex anomalous dimension at $O(1/N_f^2)$ and since that more fundamental calculation has not yet been performed, the formalism and *integration* techniques necessary to handle the five and lower loop diagrams in the critical point approach using the propagators (3) are not yet available. Therefore, to extend (9) to the singlet case, prior to treating the problem of correctly accounting for the axial vector anomaly, which is an issue in its own right, a substantial amount of additional calculations would be required which is beyond the scope of the present article.

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