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Three loop gauge β-function for the most general single gauge-coupling theory

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Abstract. We calculate the three loop contribution to the β-function of the gauge coupling constant in a general, anomaly-free, renormalisable gauge field theory involving a single gauge coupling using the background field method in the $\overline{\text{MS}}$ scheme.

The renormalisation group functions play an important role in governing the fundamental properties of quantum field theories. For example asymptotic freedom is a consequence of the sign of the one loop contribution to the gauge β-function, β_g, for a non-abelian gauge theory [1]. The variations with scale of the coupling constants of a given theory are established from knowledge of the β-functions, and so to relate physics at different scales in a given theory it is important to know the β-functions as accurately as possible. For QCD, there has been impressive progress recently with the provision of the four loop term for β_g in the $\overline{\text{MS}}$ scheme,[2]. This extended the scheme independent two loop results of [3] and the three loop calculation of [4, 5]. Evidently QCD does not, however, represent the most general renormalisable anomaly-free gauge field theory which one could consider in four dimensions; such a theory would involve both fermions and scalars, with Yukawa and quartic scalar interactions. Clearly this general gauge theory would contain the standard model (SM) as a particular case. The one loop β-functions for this most general theory, were first written down in [6], whilst the two loop corrections were calculated in the $\overline{\text{MS}}$ scheme in [7, 8].

Although the result of [2] represents substantial progress in refining the running of the strong coupling constant, it transpires that the renormalisation of the most general gauge theory has not yet been performed at *three* loops. In this letter, therefore, we extend the result of [4, 5] for $\beta_g^{(3)}$ to include Yukawa and quartic scalar interactions. We perform the calculation in such a way that the expressions for both the conventional dimensional regularisation MS (or $\overline{\text{MS}}$) scheme and (in the supersymmetric case) the dimensional reduction, (DRED), scheme [9] can be easily extracted. In the latter case we can compare our result with that of [10]. This is especially worthwhile since the result of [10] was arrived at indirectly in the non-abelian case. Therefore, in fact, our results will include the first *explicit* calculation of $\beta_g^{(3)\text{DRED}}$. We defer to a later publication [11], the full details of the calculation where we will also provide the extension to the multi-gauge coupling case, the three loop $\overline{\text{MS}}$ expressions of the remaining renormalisation group functions of the other couplings and the results in the special case of the SM.

Another motivation for undertaking this three loop renormalisation rests in recent exciting developments for constructing four dimensional conformal field theories, which are not necessarily supersymmetric, based on ideas from the Maldacena conjecture [12]. In [13], Kachru and Silverstein constructed four dimensional field theories on orbifolds so that (at least in the large N_c limit) their β-functions vanished, leading to (approximate) conformal symmetry. Inspired by this, Lawrence et al [14] conjectured that the orbifold approach could be refined so that the field theories were conformal for *finite* N_c with and without supersymmetry. Moreover, it was proposed in [15] that this framework could provide a non-supersymmetric resolution of the gauge hierarchy problem. In essence, above a certain energy scale of the order of $1-10$ TeV, a conformal field theory exists on an orbifold. The low energy theory (obtained by breaking the conformal invariance softly, by the addition of operators of dimension less than four) can be phenomenologically viable [16]. Subsequently [15]-[21], the abelian and non-abelian orbifold models satisfying these criteria were constructed. The method used was to impose that all the renormalisation group functions of the most general four dimensional gauge theory vanish, thereby ensuring conformality. For instance, the results for $\beta_g^{(2)}$ and the one loop scalar and Yukawa β-functions have been analysed. This is a systematic approach since it determines precisely the (non-supersymmetric) models which are not just conformal at large N_c but also when N_c is finite. A large set of possible theories emerged; however it is as yet unclear as to which is the strongest candidate for realistic physics beyond the SM. Moreover, it may be the case that the higher order corrections to the renormalisation group functions could restrict this set of such possible models further [18, 21]. Thus our provision of new results at three loops here is relevant.

We consider a general renormalisable quantum field theory defined by the following Lagrangian, $\mathcal{L}$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(G^A_{\mu\nu})^2 + i\,\psi_j\sigma^\mu D_\mu\bar{\psi}^j + \frac{1}{2}|D_\mu\phi|^2 - \frac{1}{2}(Y^{aij}\phi^a\psi_i\psi_j + \bar{Y}^a_{ij}\phi^a\bar{\psi}^i\bar{\psi}^j) \\ &\quad - \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d + \text{gauge fixing} + \text{ghosts} + \text{mass terms}\end{aligned} \tag{1}$$

where λ_{abcd} is totally symmetric, Y^{aij} and its complex conjugate $\bar{Y}^a_{ij}$ are symmetric on their ij indices and D_μ is the covariant derivative, $\partial_\mu - igA_\mu$.

We extract divergences using dimensional regularisation, DREG, with modified minimal subtraction, $\overline{\text{MS}}$, in $d = 4 - 2\epsilon$ dimensions. The mass terms in the Lagrangian have been suppressed because they do not contribute to β_g in a mass-independent renormalisation scheme, such as $\overline{\text{MS}}$. Gauge fixing and ghosts will be discussed below.

Without loss of generality the scalar fields, ϕ^a, are taken to be real. As a result they transform in an antisymmetric representation, S^A, of the Lie algebra of the gauge group, G, where

$$S^A_{ab} = -S^A_{ba}, \qquad \left[S^A, S^B\right] = if^{ABC}S^C, \tag{2}$$

and f^{ABC} are the structure constants of G.

We work with two-component fermions, ψ_j, and their conjugates, $\bar{\psi}^j$, so chiral theories can be considered. The fermions transform in a representation, R^A, of the Lie algebra of G, where

$$R^A = R^{A\dagger}, \qquad \left[R^A, R^B\right] = if^{ABC}R^C. \tag{3}$$

For cancellation of gauge anomalies we require

$$\text{Tr}[R^A\{R^B, R^C\}] = 0. \tag{4}$$

In this paper, we restrict our attention to simple gauge groups and hence a single gauge coupling constant, g (of course the case of QED can easily be deduced as well). The extension to the case of several gauge couplings is non-trivial (unlike at one and two loops), and will be presented elsewhere [11] along with explicit results for the SM.

The β-function for the renormalised coupling, g, is defined as:

$$\beta_g = \mu\frac{dg}{d\mu}, \quad \text{with} \quad g_0 = \mu^\epsilon Z_g g \tag{5}$$

where g_0 is the bare coupling constant, and μ is the renormalisation scale. To find β_g, we must first calculate Z_g. One might consider calculating, for example, the 3-point gauge-particle vertex. The graphs which contribute to this can be generated by attaching three external gauge legs in all possible ways to vacuum graphs; at three loops this gives rise to an enormous number of graphs. In addition one would of course have to calculate the gauge-particle renormalization factor, Z_A. Evaluating this to three loops involves another significant calculation, this time of the 2-point gauge self-energy. Moreover, difficulties arise when we consider performing the 3-point momentum integrals required to three loop order. We will be using the integration package MINCER [22] as implemented in the symbolic manipulation language FORM [23] and described in [24], which has the ability to handle only one independent external momentum and therefore we must set the momentum on all but two of the external legs to zero. For graphs with more than two external legs, this leads to the potential introduction of spurious infrared divergences, which also show up as poles in ϵ in DREG and are therefore difficult to distinguish from ultraviolet infinities. Whilst it is no doubt possible to perform the calculation this way, significant simplifications result from the use of the background field method pioneered by DeWitt in [25] and later extended by 't Hooft [26], DeWitt [27], Boulware [28] and Abbott [29], who performed a two loop pure Yang-Mills calculation in the Landau gauge ($\alpha = 0$), later generalised to $\alpha \neq 0$ by Capper and Maclean [30]. More recently the method has been used to study the renormalisation of gauge theories with a non-semisimple colour group,[31].

The background field method consists of splitting the gauge field, A_μ into a quantum field, Q_μ, and a background field, B_μ, such that

$$A_\mu \;=\; Q_\mu + B_\mu. \tag{6}$$

Only Q_μ is integrated over in the path integral and we a choose a gauge-fixing term for Q_μ which is invariant under gauge transformations with respect to B_μ; the ghost vertices may then be derived from this in the usual way. Corrections to the effective action are gauge invariant with respect to B_μ and as a result it can be shown that the following relations hold

$$\begin{aligned} Z_\alpha &= Z_Q \\ \sqrt{Z_B}Z_g &= 1 \end{aligned} \tag{7}$$

Thus Z_g, which we need to obtain β_g, may be obtained from the background gauge-field renormalisation constant, Z_B, which is easier to compute. Similarly we can deduce the gauge parameter renormalisation constant, Z_α, from the quantum field renormalisation constant, Z_Q, and, as discussed below, this enables us to check our results by performing the alculation for arbitrary α. Consequently, both Z_g and Z_α can be found from calculations which only involve evaluating 2-point graphs. There are clearly significantly fewer of these graphs and they are more easily evaluated using MINCER as there is no need to nullify any external momenta.

In $\overline{\text{MS}}$, Z_g and hence Z_B may be written as a sum of poles in ϵ. We define

$$Z_B \;=\; 1 + \sum_{n=1}^{\infty} \frac{\Delta_n}{\epsilon^n}, \tag{8}$$

and it follows from Eqs. (5) and (7) that

$$g_0^2 = \mu^{2\epsilon} Z_B^{-1} g^2. \tag{9}$$

β_g is determined by Δ_1, the simple pole in Z_B; and at any given order in perturbation theory, the higher order poles Δ_n can be calculated from the simple pole at lower orders. We have:

$$\tilde{\beta_g} = -\frac{g}{2}\left(g\frac{\partial}{\partial g} + Y.\frac{\partial}{\partial Y} + \bar{Y}.\frac{\partial}{\partial \bar{Y}} + 2\lambda.\frac{\partial}{\partial \lambda}\right)\Delta_1 \tag{10}$$

and

$$\begin{aligned} &\tilde{\beta_g}\,\Delta_n + \frac{g}{2}\left(g\frac{\partial}{\partial g} + Y.\frac{\partial}{\partial Y} + \bar{Y}.\frac{\partial}{\partial \bar{Y}} + 2\lambda.\frac{\partial}{\partial \lambda}\right)\Delta_{n+1} \\ &\quad = \frac{g}{2}\left(\tilde{\beta_g}\frac{\partial}{\partial g} + \tilde{\beta}_Y.\frac{\partial}{\partial Y} + \tilde{\beta}_{\bar{Y}}.\frac{\partial}{\partial \bar{Y}} + \tilde{\beta}_\lambda.\frac{\partial}{\partial \lambda}\right)\Delta_n \end{aligned} \tag{11}$$

where the reduced β-functions, $\tilde{\beta}$ are independent of ϵ and

$$\tilde{\beta_g} \;=\; \beta_g + \epsilon g, \quad \tilde{\beta_Y} \;=\; \beta_Y + \epsilon Y, \quad \tilde{\beta_\lambda} \;=\; \beta_\lambda + 2\,\epsilon\lambda. \tag{12}$$

Eq. (11) can be used to verify that the higher order poles generated in the final result for β_g are consistent with the lower order results thus providing a strong check on any calculation.

The first step in the calculation of Z_B is to generate all of the relevant graphs along with their appropriate combinatoric factors. We found it useful to write a package in MATHEMATICA [32], called FEYNALYSE, which, among other things, is capable of generating all of the required graphs starting from the Lagrangian, $\mathcal{L}$. (As a check, we also generated the graphs using QGRAF [33].) By producing the graphs in a MATHEMATICA environment we are then able to manipulate them using other routines in FEYNALYSE, which can draw and label each graph systematically, calculate combinatoric factors and signs due to fermion loops, assign Feynman rules to the vertices and propagators and use these to generate the necessary mathematical expressions for each graph. Once the complete set of graphs, signs and combinatoric factors has been found, the remaining calculation can be factorised into two independent calculations: the group theoretical factor and the momentum integration. The only vertex for which this causes a problem is the 4-point gauge-particle interaction discussed in more detail in [11].

The group theory is dealt with in MATHEMATICA by using a set of specially written transformation rules which allow any group theory expression up to three loops to be reduced to a sum of linearly independent terms. It is important to construct such a set to ensure that all necessary cancellations between similar terms are carried out. We make use of the Lie algebra of the group generators, S^A and R^A, defined in Eqs. (2) and (3), the Jacobi relation for the structure constants, f^{ABC} and the following relations

$$R_k^{Aj}Y^{aik} + R_k^{Ai}Y^{akj} + Y^{bij}S_{ba}^A = 0, \tag{13}$$

$$\lambda_{ebcd}S_{ea}^A + \lambda_{aecd}S_{eb}^A + \lambda_{abed}S_{ec}^A + \lambda_{abce}S_{ed}^A = 0, \tag{14}$$

which follow from gauge invariance. Most of the group theoretical factors we encounter up to three loops can be expressed in terms of the quadratic Casimirs of the relevant representations of the Lie algebra of G. These are defined as follows:

$$\mathrm{Tr}(S^AS^B) = \delta^{AB}\,T(S) \qquad S_{ac}^AS_{cb}^A = C(S)_{ab} \tag{15}$$

$$\mathrm{Tr}(R^AR^B) = \delta^{AB}\,T(R) \qquad R_i^{Ak}R_k^{Aj} = C(R)_i{}^j \tag{16}$$

$$f^{ACD}f^{BCD} = \delta^{AB}\,C(G) \qquad \delta^{AA} = r. \tag{17}$$

As mentioned above, we employ the anomaly cancellation condition, Eq. (4). This justifies our use of the naïve γ_5 matrix in defining two component spinors, because, at three loops, graphs containing two one loop fermion triangles are the only graphs which can generate non-trivial corrections to the gauge propagator involving the $\epsilon^{\mu\nu\rho\sigma}$ tensor. As explained in detail in [11], Eq. (4) means that we can set $\epsilon^{\mu\nu\rho\sigma}$ to zero, thereby avoiding inconsistencies with the γ-algebra in d-dimensions.

The momentum integrations are performed using MINCER [22, 24]. We use it to calculate the divergent part of each diagram, expressed as poles in ϵ. Although we need only calculate the simple pole to deduce β_g, we keep higher order poles as a check on our results. By retaining all the poles and the appropriate finite terms for each diagram at one, two and three loops, we can perform the subtraction of subdivergences in one step, at the end of the calculation, as explained below. First, though, we must calculate the unsubtracted divergences for all graphs contributing to the gauge propagator up to three loops. In order to do this using MINCER it is essential to label the momenta in the graph according to its 'topology', exactly as defined in the MINCER documentation,[22]. Each 'topology' corresponds to a particular integration routine which assumes that the momenta are already correctly labelled when it is called. Due to the large number of graphs it is necessary to construct a program which can work out which integration package needs to be called to integrate a given graph and label its momenta accordingly. The program we have written uses the definition of the graph in MATHEMATICA, produced by our package FEYNALYSE, and the corresponding Feynman rules to produce a complete FORM file containing the necessary scalar integrals along with all header files and definitions needed by FORM and MINCER, as well as the correct call to the appropriate integration package for that topology. The file is completely self contained and can be read by FORM which performs the integrations using MINCER and stores the results on disk. These are then collected and read into MATHEMATICA by a further set of routines in FEYNALYSE, which combine the results with the simplified group theory factors and produce a result for the total contribution to the effective action from all graphs. By following this procedure we can evaluate the unsubtracted contributions to the gauge propagator at one, two and three loops as a function of the bare couplings. However, as we are calculating a three loop result we must ensure that the Lagrangian has been renormalised up to two loops. This means that there are additional one and two loop counterterms which, when inserted in graphs with one and two loop topologies, give rise to contributions at the three loop level. By multiplicative renormalisability, the counterterms have the same form as the tree level interactions and to calculate these contributions independently would be extremely inefficient. Therefore, we adopt the procedure used in [5] for the subtraction of subdivergences, which is summarised below.

We begin by introducing a loop-counting parameter h, by replacing

$$g_0 \rightarrow h\, g_0, \quad Y_0 \rightarrow h\, Y_0, \quad \lambda_0 \rightarrow h^2\, \lambda_0. \tag{18}$$

The corrections to the 2-point gauge-field interaction must be transverse, so that:

$$\Gamma^{BB}_{0\mu\nu} = \Gamma^{BB}_0 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \tag{19}$$

where q is the external momentum, and

$$\Gamma^{BB} = Z_B \Gamma^{BB}_0. \tag{20}$$

Here Γ^{BB} is finite (as $\epsilon \rightarrow 0$) when expressed in terms of the renormalised couplings. We have calculated

$$\Gamma^{BB}_0 = 1 + A_0\, h^2 + B_0\, h^4 + C_0\, h^6 + \ldots \tag{21}$$

where $A_0, B_0, C_0 \cdots$ are functions of the bare couplings g_0, Y_0, λ_0, α_0. First we expand Z_B in loops so that

$$Z_B = 1 + \sum_{n=1}^{\infty} \frac{\Delta_n}{\epsilon^n} = 1 + ah^2 + bh^4 + ch^6 + \ldots \tag{22}$$

Then we re-express Γ^{BB}_0 in terms of renormalised couplings using Eq. (9) and similarly for the other bare couplings. (In fact we do not need to renormalise λ_{0abcd} as it appears for the first time at three loops and hence we can replace it with the renormalised value, λ_{abcd}. We will however require Z_α at two loops and Z_Y, for the Yukawa couplings, at one loop.) Note that from Eq. (22), a, b, c are functions of the renormalised couplings, g, Y, λ, α and contain only pole terms in ϵ. We thus obtain

$$\Gamma^{BB}_0 = 1 + A\, h^2 + B\, h^4 + C\, h^6 + \ldots \tag{23}$$

where $A, B, C \ldots$ are functions of the renormalised couplings. Now substituting Eqs. (22), (23) in Eq. (20), we simply impose that Γ^{BB} is finite as $\epsilon \rightarrow 0$ to deduce Z_B. We thus have:

$$\begin{aligned} a &= -\mathrm{Pole}\,[A] \\ b &= -\mathrm{Pole}\,[B + aA] \\ c &= -\mathrm{Pole}\,[C + Ba + Ab]\,, \end{aligned} \tag{24}$$

where Pole$[x]$ denotes the pole part of x with respect to ϵ. We stress that it is important to keep some of the finite corrections to the lower order one and two loop graphs (ie A and B) in order to calculate a, b and c correctly.

We define the loop expansion of the β-function to be

$$\tilde{\beta}_g = \tilde{\beta}_g^{(1)}h^2 + \tilde{\beta}_g^{(2)}h^4 + \tilde{\beta}_g^{(3)}h^6 + \ldots \tag{25}$$

Similarly, Eq. (22) implies that

$$\Delta_1 = a_1 h^2 + b_1 h^4 + c_1 h^6 + \ldots \tag{26}$$

where we would expect that

$$a = \frac{a_1}{\epsilon}, \quad b = \frac{b_1}{\epsilon} + \frac{b_2}{\epsilon^2} \quad \text{and} \quad c = \frac{c_1}{\epsilon} + \frac{c_2}{\epsilon^2} + \frac{c_3}{\epsilon^3}, \tag{27}$$

but it follows from Eq. (11) that $b_2 = c_3 = 0$. Using Eq. (10), we find that

$$\tilde{\beta}_g^{(1)} = -\,g a_1 \tag{28}$$
$$\tilde{\beta}_g^{(2)} = -\,2 g b_1 \tag{29}$$
$$\tilde{\beta}_g^{(3)} = -\,3 g c_1 \tag{30}$$

Our calculations verify that $b_2 = c_3 = 0$ and we find

$$(16\pi^2)a_1 = \frac{11\,C(G)\,g^2}{3} - \frac{2\,g^2\,T(R)}{3} - \frac{g^2\,T(S)}{6} \tag{31}$$

$$\begin{aligned}(16\pi^2)^2 b_1 = {} & \frac{17\,C(G)^2\,g^4}{3} - \frac{5\,C(G)\,g^4\,T(R)}{3} - \frac{C(G)\,g^4\,T(S)}{6} - \\ & \frac{g^4\,\mathrm{Tr}(C(S)^2)}{r} + \frac{g^2\,\mathrm{Tr}(C(R)\bar{Y}^a Y^a)}{2\,r} - \frac{g^4\,\mathrm{Tr}(C(R)^2)}{r}\end{aligned} \tag{32}$$

$$\begin{aligned}(16\pi^2)^3 c_1 = {} & \frac{2857\,C(G)^3\,g^6}{162} - \frac{1415\,C(G)^2\,g^6\,T(R)}{162} + \frac{79\,C(G)\,g^6\,T(R)^2}{162} - \\ & \frac{545\,C(G)^2\,g^6\,T(S)}{648} + \frac{29\,C(G)\,g^6\,T(R)\,T(S)}{162} - \frac{C(G)\,g^6\,T(S)^2}{324} - \\ & \frac{1129\,C(G)\,g^6\,\mathrm{Tr}(C(S)^2)}{216\ r} + \frac{25\,g^6\,T(R)\,\mathrm{Tr}(C(S)^2)}{54\ r} + \\ & \frac{49\,g^6\,T(S)\,\mathrm{Tr}(C(S)^2)}{216\ r} - \frac{3\,C(G)\,g^4\,C(S)_{ab}\,\mathrm{Tr}(Y^a\bar{Y}^b)}{4\,r} + \\ & \frac{7\,g^4\,C(S)^2_{ab}\,\mathrm{Tr}(Y^a\bar{Y}^b)}{6\,r} + \frac{2\,C(G)\,g^4\,\mathrm{Tr}(C(R)\bar{Y}^a Y^a)}{r} + \\ & \frac{8\,g^4\,C(S)_{ab}\,\mathrm{Tr}(C(R)\bar{Y}^a Y^b)}{3\,r} + \frac{5\,g^4\,\mathrm{Tr}(C(R)^2\bar{Y}^a Y^b)}{12\,r} + \\ & \frac{g^4\,\mathrm{Tr}(C(R)\bar{Y}^a C(R) Y^a)}{12\,r} - \frac{g^2\,C(S)_{ab}\,\mathrm{Tr}(Y^c\bar{Y}^c Y^a\bar{Y}^b)}{6\,r} + \\ & \frac{g^2\,C(S)_{ab}\,\mathrm{Tr}(Y^c\bar{Y}^a Y^c\bar{Y}^b)}{6\,r} + \frac{g^2\,C(S)_{ab}\,\mathrm{Tr}(\bar{Y}^c Y^a\bar{Y}^c Y^b)}{6\,r} - \\ & \frac{g^2\,\mathrm{Tr}(C(R)\bar{Y}^a Y^a\bar{Y}^b Y^b)}{24\,r} - \frac{g^2\,\mathrm{Tr}(C(R)\,\bar{Y}^a Y^b\bar{Y}^a\,Y^b)}{2\,r} - \\ & \frac{g^2\,\mathrm{Tr}(R^A\bar{Y}^a Y^b R^A\bar{Y}^b Y^a)}{3\,r} - \frac{7\,g^2\,\mathrm{Tr}(Y^a\bar{Y}^b)\mathrm{Tr}(\bar{Y}^b Y^a C(R))}{12\,r} +\end{aligned}$$

$$
\begin{aligned}
&\frac{g^2\,\mathrm{Tr}(Y^a\bar{Y}^b)\mathrm{Tr}(Y^b\bar{Y}^c)C(S)_{ca}}{12\,r}-\frac{g^2\,\mathrm{Tr}(C(R)\,\bar{Y}^aY^b\bar{Y}^b\,Y^a)}{8\,r}-\\
&\frac{205\,C(G)\,g^6\,\mathrm{Tr}(C(R)^2)}{54\,r}+\frac{11\,g^6\,T(R)\,\mathrm{Tr}(C(R)^2)}{27\,r}+\\
&\frac{23\,g^6\,T(S)\,\mathrm{Tr}(C(R)^2)}{108\,r}+\frac{g^6\,\mathrm{Tr}(C(R)^3)}{3\,r}-\frac{29\,g^6\,\mathrm{Tr}(C(S)^3)}{12\,r}-\\
&\frac{g^4\,\lambda_{abcd}\,(S^AS^B)_{ab}\,(S^AS^B)_{cd}}{6\,r}+\frac{g^2\,C(S)_{ab}\lambda_{acde}\lambda_{bcde}}{72\,r}
\end{aligned}
\tag{33}
$$

where the quartic scalar couplings makes its first appearance in the final line [34]. We also find

$$
\begin{aligned}
(16\pi^2)^3c_2 \;=\;& -\frac{187\,C(G)^3\,g^6}{27}+\frac{89\,C(G)^2\,g^6\,T(R)}{27}-\frac{10\,C(G)\,g^6\,T(R)^2}{27}+\\
&\frac{14\,C(G)^2\,g^6\,T(S)}{27}-\frac{7\,C(G)\,g^6\,T(R)\,T(S)}{54}-\frac{C(G)\,g^6\,T(S)^2}{108}+\\
&\frac{11\,C(G)\,g^6\,\mathrm{Tr}(C(S)^2)}{9\,r}-\frac{2\,g^6\,T(R)\,\mathrm{Tr}(C(S)^2)}{9\,r}-\\
&\frac{g^4\,\mathrm{Tr}(C(R)^2\,\bar{Y}^a\,Y^a)}{2\,r}-\frac{g^4\,\mathrm{Tr}(C(R)\,\bar{Y}^a\,C(R)\,Y^a)}{2\,r}+\\
&\frac{g^2\,\mathrm{Tr}(C(R)\,\bar{Y}^a\,Y^a\,\bar{Y}^b\,Y^b)}{12\,r}+\frac{g^2\,\mathrm{Tr}(C(R)\,\bar{Y}^a\,Y^b\,\bar{Y}^a\,Y^b)}{3\,r}+\\
&\frac{g^2\,\mathrm{Tr}(C(R)\,\bar{Y}^a\,Y^b\,\bar{Y}^b\,Y^a)}{12\,r}-\frac{g^6\,T(S)\,\mathrm{Tr}(C(R)^2)}{18\,r}+\\
&\frac{g^2\,\mathrm{Tr}(Y^a\bar{Y}^b)\,\mathrm{Tr}(\bar{Y}^bY^aC(R))}{6\,r}-\frac{g^6\,T(S)\,\mathrm{Tr}(C(S)^2)}{18\,r}+\\
&\frac{11\,C(G)\,g^6\,\mathrm{Tr}(C(R)^2)}{9\,r}-\frac{2\,g^6\,T(R)\,\mathrm{Tr}(C(R)^2)}{9\,r}\,.
\end{aligned}
\tag{34}
$$

In any calculation it is important to check the internal consistency of the computations as well as ensure that they agree with any results in the literature. Given that our calculation involves a large degree of automatic computation it is vital that we apply all possible checks to Eqs. (33) and (34). First we consider internal consistency. We have verified that the gauge self-energy is transverse, in other words that the sum of all the relevant Feynman diagrams yields zero when contracted with the external propagator momentum. We have also checked that β_g is independent of the gauge parameter, α. Notice also that there are no $\zeta(3)$-terms in Eq. (33); this non-trivial cancellation was to be expected, occurring as it does in the special case of QCD. Finally, we found $b_2 = c_3 = 0$, in accordance with Eq. (11) and, by calculating β_Y to one loop we have explicitly verified that c_2 is predicted to be exactly as given in Eq. (34) above. All this is evidence that we have performed the subtraction of subdivergences correctly.

In addition to these internal consistency checks there are also several existing calculations of the β-function up to and including some three loop calculations for specific cases. Many of these are concerned with supersymmetric theories, using DRED, and so we cannot obtain them easily from the DREG results given above, as this regularisation scheme violates supersymmetry. However, we in fact performed the computations [11], in such a manner that both DREG and DRED results are easily extracted. The γ-algebra was done in $d = D - 2C\epsilon$ dimensions and the momentum integrals in $d = 4 - 2\epsilon$ dimensions. Lack of space prevents us from reproducing the general result here. The DREG results (which we give) are obtained by choosing $D = 4$ and $C = 1$. Choosing instead $D = 4$ and $C = 0$ we can obtain the DRED result for a general $N = 1$ supersymmetric theory in four dimensions. (A full justification of this is provided in [11].) We need not reproduce this result here as it agrees entirely with that of [10]. As mentioned earlier, in [10] the non-abelian result was actually deduced from the explicit calculation for the *abelian* case. Therefore we have performed the first direct calculation of $\beta_g^{(3)}$ for a general $N=1$ supersymmetric non-abelian theory. Evidently this supersymmetric result includes the special cases of $N=2$ and $N=4$ theories; but as an additional check we can obtain these by setting $C=0$, with $D=6$ and $D=10$ respectively, thereby reproducing the results of [35] for these cases. (Of course in both cases β_g vanishes beyond one loop). Moreover, for the $N=4$ theory, but now using DREG, we obtain exactly the same non-zero result first obtained in [36]. Apart from supersymmetric results, we have also verified that both the one and two loop β-functions agree with those already given in the literature [7, 8]. In addition, by setting the Yukawa and scalar couplings to zero, we have checked that our results are in exact agreement with the calculation of the QCD $\overline{\text{MS}}$ β-function at three loops given in [4, 5], but not calculated using the background field method. Therefore our results pass a stringent series of tests and are in agreement with all existing results in the literature.

Our main result, Eq. (33), involves a lot of terms compared with the one and two loop expressions, Eqs. (31) and (32). This is, of course, dictated by the structure of the Feynman diagram topologies at three loops. For instance, one can have two distinct fermion loops giving rise to terms with two traces over the Yukawa indices. We have expressed the results as far as possible in terms of the usual colour group Casimirs, $C(S)$, $C(R)$ and $C(G)$; but at three loops two terms remain in which it is not possible to combine two generators, R^A, S^A, using their group properties, with one such term depending on the Yukawa couplings. It is interesting that this does not occur in the supersymmetric case. This might be a complication for the classification of non-supersymmetric conformal field theories [15]-[21].

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