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Quark mass anomalous dimension at $O(1/N_f^2)$ in QCD

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Abstract. We compute the d-dimensional critical exponents corresponding to the wave function and mass renormalization of the quark in QCD in the Landau gauge at a new order, $O(1/N_f^2)$, in the large N_f expansion. The computations are simplified by the establishment in d-dimensions of the critical point equivalence of QCD and the non-abelian Thirring model beyond leading order. The form of the $O(1/N_f^2)$ coefficients in the $\overline{\rm MS}$ quark mass anomalous dimension at five loops is deduced and compared with the numerical asymptotic Padé approximant prediction.

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The renormalization group equation, (RGE), plays a fundamental role in our understanding of the properties of quantum field theories. Central to this equation are the renormalization group functions such as the β-function and the anomalous dimension, $\gamma(g)$, and for theories involving massive fields the mass anomalous dimension, $\gamma_m(g)$, also appears. Ordinarily such functions are computed order by order in the loop expansion. Though this clearly becomes more difficult at successive orders due to the increase in the number of Feynman integrals which need to be computed. Despite this problem it has been possible to determine the β-function and quark mass anomalous dimensions in quantum chromodynamics, (QCD), at four loops in the $\overline{\mathrm{MS}}$ renormalization scheme, [1-11]. Such calculations represent a significant achievement especially given that the order of 10^4 Feynman diagrams have been evaluated at four loops to deduce the renormalization constants. To proceed further in perturbation theory will clearly be a colossal if not impossible undertaking. However, there do exist methods which probe the perturbative structure of the RG functions from another direction. Given that they depend not only on the coupling constant, g, but other parameters in the theory such as the dimension of any internal symmetry groups, one can equally expand in one of these other variables. This will still involve the computation of Feynman integrals but not those associated with ordinary perturbation theory. In QCD, with N_f flavours of quarks, they can be expanded in the large N_f expansion where $1/N_f$ behaves as a bona fide perturbation parameter. Explicit details of the technique have been recorded in the literature but we emphasise that the extension of the $O(N)$ σ model methods of Vasil'ev et al, [12, 13], to four dimensional gauge theories, [14-18], offer the best and most efficient strategy to computing $1/N_f$ information in QCD. For example, the anomalous dimension of the twist-2 operator dimensions which are fundamental to the operator product expansion used in deep inelastic scattering have been computed at $O(1/N_f)$ and to all orders in the strong coupling constant, [19]. These results have been crucial in confirming the correctness of the explicit perturbative 3-loop results of [20] which play a key role in the full two loop evolution of the QCD structure functions. Clearly, given this important overlap with the current activity in explicit perturbative calculations, it is crucial that the large N_f method is developed to the next order, $O(1/N_f^2)$. However, before such a deep inelastic programme can proceed, various fundamental computations need to be performed. In any ordinary renormalization the wave function renormalization constant is always computed first before determining the renormalization constants of the other parameters of the field theory. Likewise in the large N_f programme, the wave function critical exponent, η, must be deduced prior to the calculation of any other operator or field dimension. Through the critical RGE, η is related to $\gamma(g)$ by $\eta = 2\gamma(g_*)$ where g_* is the value of the critical coupling at the d-dimensional fixed point of the QCD β-function.

Whilst η has been determined at $O(1/N_f^2)$ in an arbitrary covariant gauge as a function of d in QED, [16], the extension of that calculation to QCD has not yet been provided. This is one of the main aims of this letter where we will determine η in a simpler model in the same universality class as QCD, known as the non-abelian Thirring model. As we will argue it has a simpler form than QCD since the triple and quartic gluon vertices which are present in Yang-Mills theories are absent but the expression we obtain for η will correspond to the quark wave function renormalization constant in QCD itself. Clearly this reduces the number of Feynman diagrams which need to be considered. Indeed the $O(1/N_f)$ calculations of say, [19], were performed in the non-abelian Thirring model, (NATM). The connection with this model had previously been investigated at leading order in the large N_f expansion in [21]. Although the provision of η is fundamental to any future $O(1/N_f^2)$ calculation, we have also computed the critical exponent which relates to the quark mass dimension, $\gamma_m(g_*)$, at $O(1/N_f^2)$. There are several reasons for carrying out such a calculation. First, η is gauge dependent and is therefore not a fully meaningful physical quantity. On the contrary $\gamma_m(g_*)$ is known to be a gauge

independent (and scheme independent) exponent. This provides us with a non-trivial check on our computation, aside of course from the comparison with the explicit four loop $\overline{\rm MS}$ results of [10, 11]. In addition, the Feynman integrals with a $[\bar{\psi}\psi]$ insertion are closely related to those required to deduce η and so their computation does not in fact represent a significant amount of extra work. Moreover, as the calculation of $\gamma_m(g_*)$ involves the insertion of the mass operator $[\bar{\psi}\psi]$ in a 2-point quark Green function, it very much lays the foundation for the computation of the dimensions of similar operators which are bilinear in the quark fields such as those which arise in deep inelastic scattering.

The QCD Lagrangian in $d = 4 - 2\epsilon$ dimensional Euclidean space reads

$$S \;=\; \bar{\psi}^{iI} \not{D} \psi^{iI} \;+\; \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} \;+\; \frac{1}{2\xi g^2} (\partial \cdot A)^2 \;+\; \partial_\mu \bar{c}^a \, (D^\mu c)^a \;\; , \tag{1}$$

where ψ^{iI} is the quark field belonging to the fundamental representation of the colour group, $1 \leq I \leq N_f$, A^a_μ is the gluon field, c^a and $\bar{c}^a$ are the ghost fields in the adjoint representation of the colour group, ξ is the covariant gauge parameter and g is the coupling constant. The field strength tensor $F^a_{\mu\nu}$ and the covariant derivative are defined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^a_\mu A^c_\nu$ and $D_\mu = (\partial_\mu - i A^a_\mu T^a)$, where T^a are the group generators in the corresponding representation and f^{abc} are the structure constants with $[T^a, T^b] = i f^{abc} T^c$. To ensure the coupling constant, g, is dimensionless below four dimensions we rescale it in the standard way by setting $g \to M^\epsilon g$, where the parameter M has dimensions of mass.

The partition functions of the theory defined by the Lagrangian (1) read

$$\langle O_1(x_1) \ldots O_n(x_n) \rangle \;=\; Z^{-1} \int D\Phi \; O_1(x_1) \ldots O_n(x_n) \exp\{-S\} \;\; , \tag{2}$$

where $\Phi \equiv \{A, \bar{\psi}, \psi, \bar{c}, c\}$ is the set of fundamental fields and $O_i(x_i)$ represent a basic field or a composite operator. The divergences arising in the calculation of (2) are removed at each order of perturbation theory by the renormalization of the fields and parameters entering the QCD Lagrangian, as well as by renormalization of the composite operators. Hence the renormalized 1-particle irreducible n-point Green functions with the insertion of k multiplicatively renormalizable composite operators satisfy the RGE

$$\left(M\partial_M + \beta_g \partial_g + \beta_\xi \partial_\xi - n_\Phi \gamma_\Phi + \sum_{i=1}^{k} \gamma_{O_i} \right) \Gamma(x_1, \ldots, x_{n+k}, M, g, \xi) \;=\; 0 \;\; , \tag{3}$$

where $\gamma_{O_i}(g)$ is the anomalous dimension of the operator O_i, $n_\Phi \gamma_\Phi = n_A \gamma_A + n_\psi \gamma_\psi + n_c \gamma_c$ and the beta functions of the couplings g and ξ are given by $\beta_g = M\partial_M g$ and $\beta_\xi = M\partial_M \xi$. It should be noted here that the correlation functions of gauge invariant objects do not depend on the gauge fixing parameter, ξ, and in this case the term $\beta_\xi \partial_\xi$ drops out of Eq. (3).

Our analysis relies heavily on the existence of a non-trivial infra-red, (IR), stable fixed point g_* of the d-dimensional β-function, $\beta_g(g_*) = 0$, for large values of N_f. The β-function has been calculated in $\overline{\rm MS}$ using dimensional regularization and in the notation of the [22] with $a = (g/2\pi)^2$ is [1, 2, 3, 4, 5]

$$\begin{aligned} \beta_a(a) \;=\; & (d-4)a \;+\; \left[\frac{2}{3} T_F N_f - \frac{11}{6} C_A \right] a^2 \;+\; \left[\frac{1}{2} C_F T_F N_f + \frac{5}{6} C_A T_F N_f - \frac{17}{12} C_A^2 \right] a^3 \\ & - \left[\frac{11}{72} C_F T_F^2 N_f^2 + \frac{79}{432} C_A T_F^2 N_f^2 + \frac{1}{16} C_F^2 T_F N_f - \frac{205}{288} C_F C_A T_F N_f \right. \\ & \left. - \frac{1415}{864} C_A^2 T_F N_f + \frac{2857}{1728} C_A^3 \right] a^4 \;+\; O(a^5) \;\; , \end{aligned} \tag{4}$$

from which it follows that

$$a_* \;=\; \frac{3\epsilon}{T_F N_f} \;+\; \frac{1}{4T_F^2 N_f^2}\left(33C_A\epsilon - [27C_F + 45C_A]\,\epsilon^2 + O(\epsilon^3)\right) \;+\; O\left(\frac{1}{N_f^3}\right) . \tag{5}$$

The Casimirs for a general classical Lie group are defined by

$$\mathrm{Tr}\left(T^a T^b\right) \;=\; T_R\delta^{ab} \;, \quad T^a T^a \;=\; C_F I \;, \quad f^{acd}f^{bcd} \;=\; C_A\delta^{ab} \;. \tag{6}$$

It immediately follows from Eq. (3) that the Green functions of gauge invariant operators are scale invariant at the critical point g_*. In other words $G(\lambda x_i) = \lambda^{D_i} G(x_i)$, where D_i is the scaling dimension of the corresponding Green function. Moreover, due to the IR nature of the fixed point, this index determines the power of the leading term of the IR asymptotic behaviour of the Green functions ($p_i \to 0$) near the critical points [23]. On the contrary Green functions of gauge dependent objects, such as the propagators of the basic fields which will in general depend on ξ, are not invariant under scale transformations. Although one may restrict attention from the outset to gauge independent quantities, since they have physical meaning, it is also possible and convenient to choose ξ so that *all* Green functions are scale invariant. Evidently, this is equivalent to the condition $\beta_\xi(g_*, \xi_*) = 0$. Since

$$\beta_\xi(g,\xi) = -2\xi(\epsilon + \gamma_A + \beta_g/g) \;, \tag{7}$$

one concludes that the equation $\beta_\xi(g_*, \xi_*) = 0$ has two solutions. One is $\xi_* = 0$ whilst the other is $\gamma_A(g_*, \xi_*) = -\,\epsilon$. Bearing in mind that our main aim is the development of the $1/N_f$ expansion we choose the first solution, $\xi = 0$, since the latter gives $\xi \sim N_f$, which leads to problems in the construction of the $1/N_f$ scheme. The origin of the above two solutions for ξ becomes more transparent if one tries to write down the most general form of the gluon propagator satisfying the requirements of both scale and gauge invariance. Indeed, scale invariance yields

$$G_{\mu\nu}(p) \;=\; \frac{M^{2\epsilon}}{p^{2\alpha}}\left(AP^{\perp}_{\mu\nu} + BP^{\parallel}_{\mu\nu}\right) \;, \tag{8}$$

where $P^{\perp}_{\mu\nu}$ and $P^{\parallel}_{\mu\nu}$ are the transverse and longitudinal projectors, respectively, and A and B are constants. As is well known, [23], radiative corrections do not contribute to the longitudinal part of gluon propagator. Hence, $G^{\parallel}_{\mu\nu} = \xi g^2 M^{2\epsilon} P^{\parallel}_{\mu\nu} p^{-2}$. This implies, that if $\alpha \neq 1$ then ξ must vanish, $\xi = 0$. On the other hand for $\xi \neq 0$ then one must have $\alpha = 1$ which is easy to check is equivalent to $\gamma_A = -\,\epsilon$ corresponding to the canonical dimension of the field. Earlier work concerning the relation of scaling and conformal symmetry in the context of gauge theories has been given in [24].

It is well known from the theory of the critical phenomena [23] that the critical properties of the system do not depend on the details of the interactions but is determined mainly by "global" characteristics such as symmetries and the dimension of spacetime. Thus different systems may exhibit the same behaviour at the critical point. An example of this universality is the fixed point relation between the Heisenberg ferromagnet and ϕ^4 field theory. In what follows we construct the theory which belongs to the same universality class as QCD but which has a simpler structure. We first develop the $1/N_f$ expansion for calculating correlators of the type given in (2). This can be achieved in the standard manner by integrating over the fermion fields in the functional integral which yields the following effective action for the gauge field,

$$S_A^{\mathrm{eff}} \;\equiv\; N_f\left(-\,\mathrm{tr}\ln(\partial\!\!\!/ - iA\!\!\!/^a T^a) + \frac{M^{-2\epsilon}}{4\bar{g}^2}(F^a_{\mu\nu})^2 + \frac{M^{-2\epsilon}}{2\xi\bar{g}^2}(\partial A)^2\right) + \partial_\mu\bar{c}^a\,(D^\mu c)^a \tag{9}$$

where bearing in mind that $g_*^2 \sim 1/N_f$ we have set $g^2 = \bar{g}^2/N_f$. The evaluation of the functional integral with action (9) by the saddle point method generates the systematic expansion for the correlators. If we now examine the IR asymptotic behaviour of the Green functions in this approach and first of all consider the gluon propagator to first order in $1/N_f$ in the Landau gauge, $\xi = 0$, it is

$$G_{\mu\nu}(p) \;=\; N_f^{-1} \left[a p^{d-2} + \frac{M^{-2\epsilon}}{2\bar{g}^2} p^2 \right]^{-1} P^{\perp}_{\mu\nu} \;, \tag{10}$$

where the first term in the brackets arises from the fermion loop. It is obvious that the contribution coming from the $(F^a_{\mu\nu})^2$ term is less singular in comparison with the contribution of the fermion loop in the asymptotic limit $p \to 0$. It can be easily checked that the diagrams with vertices contained in $(F^a_{\mu\nu})^2$ also do not contribute to the leading order of the IR limit. So one concludes that this term does not influence the critical properties of the theory and according to the general scheme should be excluded from action. Therefore we obtain the theory defined by the action

$$S \;=\; \bar{\psi} \left(\partial\!\!\!/ - i A\!\!\!/^a T^a \right) \psi \;+\; \frac{N_f}{2\xi} (\Box^{-\epsilon/2}\, \partial A)^2 \;+\; \partial_\mu \bar{c}^a \partial^\mu c \;+\; f^{abc} \partial^\mu \bar{c} A^b_\mu c^c \;, \tag{11}$$

which in the Landau gauge has the same critical behaviour as QCD. Of course, for gauge independent quantities it is true in any gauge. We have also modified the form of the gauge fixing condition, in order that the transverse and longitudinal parts of the gluon propagator have the same momentum dependence. In fact, to derive (11) one can start from the theory with manifest gauge invariance which is determined by the action $S = \bar{\psi}(\partial\!\!\!/ - iA\!\!\!/^a T^a)\psi$ with ghost and gauge fixing terms in turn arising from the application of the Faddeev-Popov procedure to the functional integral. Power counting shows that (11) is renormalizable within the $1/N_f$ expansion. Of course, we assume that a gauge invariant regularization is used. The renormalized action then takes the form

$$S_R \;=\; Z_1 \bar{\psi}\, \partial\!\!\!/ \psi \;-\; i Z_2 \bar{\psi} A\!\!\!/^a T^a \, \psi \;+\; \frac{N_f}{2\xi} (\Box^{-\epsilon/2}\, \partial A)^2 \;+\; Z_3 \partial_\mu \bar{c}^a \partial^\mu c \;+\; Z_4 f^{abc} \partial^\mu \bar{c} A^b_\mu c^c \;. \tag{12}$$

Due to the Slavnov-Taylor identities the renormalization constants Z_i are related by

$$Z_2 \, Z_1^{-1} \;=\; Z_4 \, Z_3^{-1} \;. \tag{13}$$

This was used in the exponent formulation to determine the ghost anomalous dimension at $O(1/N_f)$ in [25]. It should be noted that in the Landau gauge $Z_4 = 1$. As was proved above in the Landau gauge, the critical properties of QCD and this new theory which we shall refer to as the non-abelian Thirring model are identical. Therefore one can use the NATM model to deduce the QCD RG functions. This equivalence at leading order in $1/N_f$ was noted in [21] and used to deduce various exponents at $O(1/N_f)$, [14, 15, 16, 17, 19, 25]. The extension of these calculations to $O(1/N_f^2)$ requires special care. The main one is the necessity of using a gauge invariant regularization which was not crucial at $O(1/N_f)$. The conventional dimensional regularization is not applicable here, since the gluon propagator behaves as p^{2-d} and the theory remains logarithmically divergent in any dimension d. To our knowledge most other invariant regularizations such as higher derivatives spoil the masslessness of the propagators, which makes higher order calculations virtually impossible. Usually in $1/N$ calculations the analytical regularization of [12] is used. However, this breaks gauge invariance.

We now consider how we can reconcile gauge invariance with the calculational benefit of using massless propagators. First, we break gauge invariance of (11) from the beginning by introducing a new coupling, λ, for the ghost-gluon vertex in (11). For $\lambda = 1$ one recovers the original model but the theory remains renormalizable for arbitrary λ as well. The only effect

will be that the identity (13) will no longer hold. The bare coupling λ_0 is connected to the renormalized one, λ, by

$$\lambda_0 \;=\; Z_\lambda \lambda \;=\; Z_4\, Z_1\, Z_2^{-1} Z_3^{-1} \lambda \;, \tag{14}$$

where the Z_i now depend on λ. Let us suppose that we used an invariant method, such as regularization by higher derivatives [26], to regularize this extended theory. Then it immediately follows from (13) and (14) that the equality $\lambda = 1$ for the renormalized couplings implies that $\lambda_0 = 1$ as well. This leads us to the conclusion that $\lambda = 1$ is a fixed point, $\beta_\lambda(1) = 0$. The existence of this fixed point is the key point and it does not depend on the regularization used. So using any other regularization can only change the position of the fixed point with in general $\lambda_* = 1 + O(1/N_f)$. What is important, though, is that the anomalous dimensions calculated at the critical point, $\gamma(\lambda_*)$, are scheme independent and, hence, coincide with the anomalous dimensions deduced in the original model (11). Therefore one can use the regularization which is most convenient from the computational point of view. Moreover, since we do not need to consider diagrams with external ghost legs, then the only diagrams depending on λ are those with a ghost loop. As is evident from counting powers of $1/N_f$ these are themselves $O(1/N_f^2)$. So at this order it is sufficient to set $\lambda = 1$.

In what follows we shall use the Δ-regularization of [12, 13]. The propagators of the gluon, quark and ghost fields are obtained from (11) as

$$G^{ab}_{\nu\lambda}(p) \;=\; \frac{\delta^{ab}}{n} \frac{G}{(p^2)^{\mu-1}} \left(P^\perp_{\nu\lambda} + \tilde{\xi} P^{\parallel}_{\nu\lambda} \right) \;, \qquad D_\psi(p) \;=\; \frac{i\not{p}}{p^2} \;, \qquad D_c(p) \;=\; \frac{1}{p^2} \;, \tag{15}$$

where we define n to be the combination which arises naturally in the calculations as

$$n \;=\; N_f\, T_F\, \mathrm{Tr}_{\mathrm{spinor}}\, I \;. \tag{16}$$

The value of the amplidude G is derived from (11) as

$$G \;=\; (4\pi)^\mu \frac{\Gamma(2\mu)}{2\Gamma^2(\mu)\Gamma(2-\mu)} \;, \tag{17}$$

where we now set $\mu \equiv d/2$. The new gauge parameter $\tilde{\xi}$ is proportional to ξ but in what follows we shall omit the tilde. The regularization of the theory is carried out by shifting the index of the gluon propagator, by taking

$$G^{ab}_{\nu\lambda}(p) \;=\; \frac{\delta^{ab}}{n} \frac{G M^{2\Delta}}{(p^2)^{\mu-1+\Delta}} \left(P^\perp_{\nu\lambda} + \xi P^{\parallel}_{\nu\lambda} \right) \;, \tag{18}$$

where the factor $M^{2\Delta}$ is introduced to preserve the canonical dimension of the propagator. The divergences appearing in the diagrams as poles in Δ are removed by the renormalization procedure. Throughout we will adopt the minimal subtraction scheme. The technical details and subtleties which are inherent to this regularization can be found in the detailed discussion of [13, 27]. However, we record that up to $O(1/N_f^2)$ there exists a simple algorithm for the calculation of the anomalous dimensions, [27]. To write down the corresponding formula in a compact form we introduce a factor u for each the gluon propagator by setting

$$G_{\nu\lambda} \;\to\; u\, G_{\nu\lambda} \;, \tag{19}$$

so that each diagram with k-internal gluon lines acquires a factor u^k. Then the anomalous dimensions of the basic fields $\Phi = \{\psi, A_\mu, c\}$ can be expressed via the renormalization constants Z_Φ where $\Phi_0 = Z_\Phi \Phi$, as follows

$$\gamma_\Phi \;=\; -\; 2u\partial_u Z^{(1)}_\Phi \Big|_{u=1} \;, \tag{20}$$

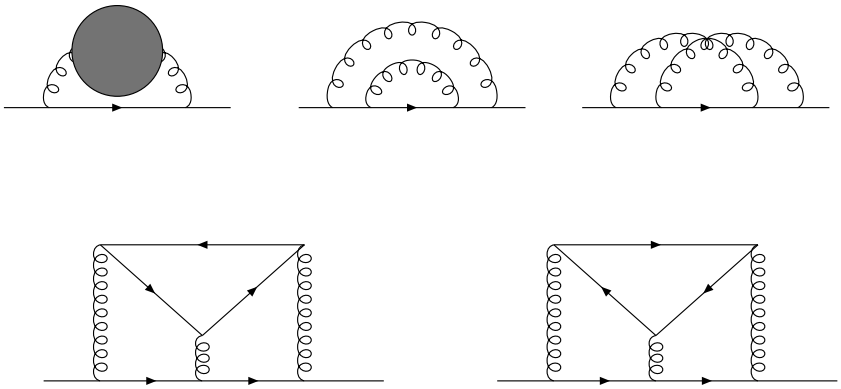

Figure 1: Diagrams contributing to the computation of η_2. The first graph represents the gluon self energy diagrams of Fig. 2.

where

$$Z_\Phi \;=\; 1 \;+\; Z_\Phi^{(1)}/\Delta \;+\; Z_\Phi^{(2)}/\Delta^2 \;+\; O\left(\frac{1}{\Delta^3}\right) \;. \tag{21}$$

The matrix of anomalous dimensions of the system of composite operators, $\{O_i\}$, which mix under renormalization, is given by

$$\gamma_{ik} \;=\; 2u\partial_u Z_{ik}^{(1)}\Big|_{u=1} \;+\; \delta_{ik} n_{k,\Phi}\gamma_\Phi \;, \tag{22}$$

where the mixing matrix, Z_{ik}, is defined in the standard way

$$O_i^R \;=\; Z_{ik} O_k \;, \tag{23}$$

so that all Green functions of the operator O_i^R are finite. Again, $Z_{ik}^{(1)}$ is the coefficient of the simple pole in Δ. The derivation of (20) and (22) can be found in [27]. Though it should be stressed that these formulæ are valid only up to $O(1/N_f^2)$. We note also the obvious resemblance of the formulæ (20) and (22) with those used in dimensional regularization.

Before proceeding to the results we underline an advantage of the approach of [13, 27] in comparison with the method of the self-consistency equations, (SE), of [12, 16] which has been more widely used in the $1/N$ computations. In the SE method to find the critical exponent η one has to calculate the corresponding renormalized Green functions and then solve the self-consistency equations. By contrast to calculate critical exponents using (20) and (22) one only needs to know the *divergent* part of the corresponding diagrams. This is much simpler from a computational point of view. As is well known progress in multiloop perturbative calculations relies heavily on the possibility of expressing the RG functions through the renormalization constants. For calculations of the latter there are various calculational shortcuts. We now demonstrate the effectiveness of this approach by calculating the exponent $\eta = 2\gamma(g_*)$ and the anomalous dimensions of the $[\bar\psi\psi]$ operator.

The diagrams which contribute to Z_ψ at $O(1/N_f^2)$ are illustrated in Figs. 1 and 2 where in the latter the dashed line corresponds to the ghost field. We now take $\lambda = 1$. Having calculated the contributions to Z_ψ from the diagrams in both figures, we find the following expression for $\eta = 2\gamma(g_*)$ at $O(1/N_f^2)$. If we set

$$\eta \;=\; \eta_1/n \;+\; \eta_2/n^2 \;+\; O(1/n^3) \;, \tag{24}$$

$$\eta_1 \;=\; C_F\,\eta_0 \;, \tag{25}$$

$$\eta_2 \;=\; C_F^2\,\eta_a \;+\; C_F\,C_A\,\eta_b \;, \tag{26}$$

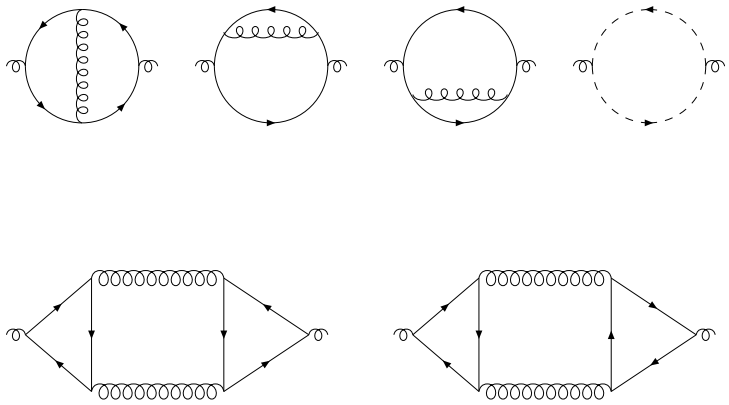

Figure 2: The diagrams contributing to the gluon self-energy at $O(1/N_f^2)$.

where

$$\eta_0 \;=\; \frac{(\mu-2)(2\mu-1)\Gamma(2\mu)}{\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)} \tag{27}$$

then in the Landau gauge

$$\eta_a \;=\; \frac{(\mu-1)\eta_0^2}{(\mu-2)(2\mu-1)}\left[\frac{2(\mu-1)(\mu-3)}{\mu(\mu-2)} \;+\; 3\mu\left(\Theta - \frac{1}{(\mu-1)^2}\right)\right] \tag{28}$$

which has been computed previously in [16], and

$$\begin{aligned}\eta_b \;=\; \eta_0^2\Bigg[&\frac{(12\mu^4-72\mu^3+126\mu^2-75\mu+11)}{2(2\mu-1)^2(2\mu-3)(\mu-2)^2} \;-\; \frac{\mu(\mu-1)}{2(2\mu-1)(\mu-2)}\left(\Psi^2+\Phi\right) \\ &+\; \frac{(8\mu^5-92\mu^4+270\mu^3-301\mu^2+124\mu-12)\Psi}{4(2\mu-1)^2(2\mu-3)(\mu-2)^2}\Bigg] \;.\end{aligned} \tag{29}$$

The functions Ψ, Φ and Θ are defined as

$$\begin{aligned}\Psi(\mu) &= \psi(2\mu-3) \;+\; \psi(3-\mu) \;-\; \psi(1) \;-\; \psi(\mu-1) \;, \\ \Phi(\mu) &= \psi'(2\mu-3) \;-\; \psi'(3-\mu) \;-\; \psi'(\mu-1) \;+\; \psi'(1) \;, \\ \Theta(\mu) &= \psi'(\mu-1) \;-\; \psi'(1) \;,\end{aligned} \tag{30}$$

where $\psi(x) = (\ln\Gamma(x))'$. The technical details of the calculations and the values for the individual graphs will be given elsewhere [28].

Next to determine the quark mass anomalous dimension at $O(1/N_f^2)$ one computes the anomalous dimension of the associated composite operator $[\bar{\psi}\psi]$. At $O(1/N_f)$ this was derived in [25] as well as the anomalous dimensions of the basic fields. The diagrams contributing to its renormalization constant, $Z_{\bar{\psi}\psi}$, are obtained from those for the quark propagator of Figs. 1 and 2 by the insertion of the operator in the fermion lines connected to external vertices. In fact, it is sufficient to calculate the diagrams arising from the two 2-loop diagrams in the first line of Fig. 1. The contributions to $Z_{\bar{\psi}\psi}$ from the other diagrams can be related to the contributions of the corresponding diagrams to Z_ψ by

$$\delta Z_{\bar{\psi}\psi} \;=\; -\;\frac{2\mu}{(\mu-2)}\cdot\delta Z_\psi \quad\Rightarrow\quad \delta\gamma_m \;=\; -\;\frac{2}{(\mu-2)}\cdot\delta\eta_2 \tag{31}$$

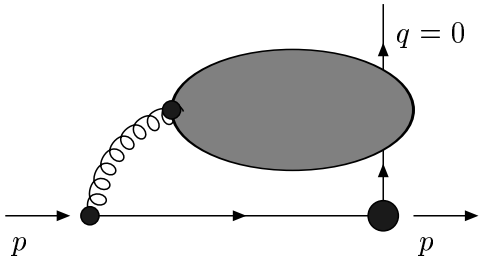

Figure 3: External momenta routing in the quark 2-point function with an operator insertion.

and

$$\gamma_m \;=\; \eta \;+\; \gamma_{\bar{\psi}\psi} \;. \tag{32}$$

This follows if we take the flow of the external momenta p in the propagator diagrams along the fermion lines and differentiate the corresponding integrals with respect to p_μ. Since

$$\frac{\partial}{\partial p^\mu}\,\frac{\not{p}}{p^2} \;=\; -\;\frac{\not{p}\gamma^\mu\not{p}}{(p^2)^2} \tag{33}$$

the resulting diagrams will have the same topology as those for the $[\bar{\psi}\psi]$ operator with the only difference being the presence of a new insertion containing γ^μ. Further, we use the fact that the pole part of the diagrams after the subtraction of the divergent subgraphs does not depend on the external momenta. So we choose the momenta flow as shown in Fig. 3. The big black dot in Fig. 3 denotes the insertion of the unit matrix I for the mass operator diagrams, and γ_μ for the propagator ones. Since the insertion of I or γ_μ in the vertex influences only the γ matrix structure of the diagram and does not touch the momentum integral, it can be easily checked that the identity (31) holds irrespective of the explicit structure of the coloured block on the Fig. 3. Eventually, using (22) we find for $\gamma_m(g_*)$

$$\gamma_m(g_*) \;=\; \gamma_{m,1}/n \;+\; \gamma_{m,2}/n^2 \;+\; O(1/n^3) \;, \tag{34}$$

where

$$\gamma_{m,1} \;=\; -\;\frac{2\,C_F\eta_0}{(\mu-2)} \tag{35}$$

$$\gamma_{m,2} \;=\; C_F^2\gamma_a \;+\; C_FC_A\gamma_b \;, \tag{36}$$

and γ_a and γ_b are given by

$$\gamma_a \;=\; -\;\frac{2\eta_a}{(\mu-2)} \;-\; \frac{2(2\mu^2-4\mu+1)\eta_0^2}{(\mu-2)^3(2\mu-1)} \tag{37}$$

$$\gamma_b \;=\; -\;\frac{2\eta_b}{(\mu-2)} \;+\; \frac{\mu^2(2\mu-3)^2\eta_0^2}{4(\mu-2)^3(\mu-1)(2\mu-1)} \;. \tag{38}$$

where the former was determined in [17]. Having derived a d-dimensional expression for the quark mass dimension, we make several remarks on its properties. First, if one sets $\mu = 2-\epsilon$ and expands in powers of ϵ, the coefficients to and including $O(\epsilon^4)$ agree with those determined from the explicit $\overline{\rm MS}$ four loop perturbative mass dimension of [7, 8, 9, 10, 11]. Given that we have only evaluated three loop diagrams agreement at this order represents a non-trivial check

on our analysis. In addition we have also computed the exponent in an arbitary covariant gauge and checked that in the final sum the gauge parameter cancels. These details are presented in [28]. Therefore, we are confident that our value for $\gamma_{m,2}$ is correct. This allows us now to produce new information on the mass anomalous dimension at 5-loops in $\overline{\rm MS}$. First, we write the $O(1/N_f^2)$ form of the mass anomalous dimension as

$$\begin{aligned}\gamma_m(a) &= -\frac{3}{2}C_F a + \left(\frac{10}{24}T_F N_f - \frac{3}{16}C_F - \frac{97}{48}C_A\right) C_F a^2 \\ &+ \sum_{r=3}^{\infty} \left(m_{r0} T_F^{r-1} N_f^{r-1} + m_{r1} T_F^{r-2} N_f^{r-2}\right) C_F a^r + O\left(\frac{1}{N_f^3}\right) ,\end{aligned} \tag{39}$$

where the order symbol means that we are ignoring contributions from $\gamma_{m,3}$ and six loop terms. To extract the coefficients m_{50} and m_{51} requires a_* at $O(\epsilon^5)$ and $O(1/N_f^2)$. Although the full 5-loop $\overline{\rm MS}$ QCD β-function is not yet available, the critical coupling which we require at $O(1/N_f^2)$ has already been computed in [18]. Hence, we find that

$$m_{50} = \frac{5\zeta(3)}{162} - \frac{\zeta(4)}{18} + \frac{65}{2592} \tag{40}$$

$$\begin{aligned}m_{51} &= \left[\frac{5\zeta(4)}{8} - \frac{\zeta(5)}{3} - \frac{11\zeta(3)}{48} - \frac{4483}{20736}\right] C_F \\ &+ \left[\frac{8\zeta(5)}{9} - \frac{17\zeta(4)}{36} - \frac{671\zeta(3)}{1296} - \frac{18667}{124416}\right] C_A\end{aligned} \tag{41}$$

where $\zeta(n)$ is the Riemann zeta function.

Whilst this gives an indication of the exact form of these coefficients at 5-loops, we have another motivation for computing it. Recently, there has been activity in trying to estimate the higher order coefficients of perturbative functions in four dimensional field theories from knowledge of the known lowest orders and the asymptotic behaviour at high orders, [29, 30]. This technique known generally as the asymptotic Padé approximant method, (APAP), has had varying degree of success. For instance, the 4-loop β-function of $\mathcal{N} = 1$ supersymmetric QCD had been determined analytically in [31] up to one unknown parameter. By applying APAP methods and its refinement, WAPAP, this parameter was determined numerically. Subsequent explicit calculations in [32] produced this parameter analytically and it was in good agreement with the APAP estimate for it. On the other hand the original application of the APAP method to the 4-loop $\overline{\rm MS}$ QCD β-function, [30], did not yield as accurate a prediction. In this case the coefficients of the polynomial in N_f which appears at 4-loops were estimated with the leading coefficient fixed from the known result of the $1/N_f$ expansion, [18]. One of the reasons for a less accurate prediction was the appearance of new colour group Casimirs at four loops which were absent in lower orders and which therefore had not been built into the estimating procedure. Taking account of these issues the method was refined in [33] when the full 4-loop $\overline{\rm MS}$ result became available and new estimates were provided instead for the 5-loop β-function and quark mass dimension. In particular the coefficients of the N_f-polynomials were estimated. Whilst we believe it will be some time before the full five loop $\overline{\rm MS}$ renormalization of QCD will be performed, if the APAP method for higher order estimation is to serve any useful purpose in the interim it is important to determine its reliability. The determination of (41) for QCD therefore provides us with that test. In [33] the coefficient of the dominant N_f term in the 5-loop N_f polynomial was fixed by the leading order large N_f coefficient, m_{50}. The results were quoted for various values of N_c for two cases. One was where the effect of the new 4-loop colour Casimirs, Q_4, was accounted for and the other was the case where their presence was ignored. We have reproduced the values of these coefficients for the choices of N_c given in [33] together with the

numerical values of (41). In order to facilitate comparison we note the relation between our notation and that of [33] is

$$E_4 \;=\; -\;\tfrac{1}{2}T_F^4\,C_F\,m_{50} \quad , \quad D_4 \;=\; -\;\tfrac{1}{2}T_F^3\,C_F\,m_{51} \;. \tag{42}$$

From the table it would appear that the WAPAP estimates are not in agreement with (41).

N_c	D_4 (w Q_4)	D_4 (w/o Q_4)	$D_4^{\rm exact}$
2	8.12×10^{-3}	8.88×10^{-3}	0.0396
3	0.037	0.037	0.1083
4	0.0891	0.0831	0.2049
5	0.165	0.148	0.3292
20	4.31	3.48	5.5113

Table 1. Comparison of APAP results for D_4 with the numerical value of the exact coefficient.

However, it is worth putting the estimates in the context of the other coefficients. For $N_c = 3$, for instance, the leading coefficient of the N_f polynomial is $O(10^{-5})$ whilst the constant term has a WAPAP prediction of 530 which is roughly seven orders of magnitude larger. That the WAPAP prediction for m_{51} is within an order of magnitude is perhaps remarkable particularly given the nature of the estimation method. By fitting to the large order asymptotic behaviour of the perturbation series one is essentially ensuring that the constant term of the polynomial in N_f is close to the correct value in the WAPAP as this will always be the dominant contribution for a range of values of N_f where N_f is relatively small. Therefore, one would not fully expect that the first few coefficients to be reliably predicted. Moreover, in light of our exact evaluation of m_{51}, it would seem worthwhile to refine the WAPAP estimate of the other coefficients by taking our value either as an extra normalization or another constraining number to fit to.

To conclude our article, we emphasise that we have computed the critical exponents corresponding to the wave function and mass anomalous dimension in d-dimensions at a new order in the $1/N_f$ expansion in QCD. Not only have we provided new information on the perturbative structure of these RG functions but we have also demonstrated the viability of the large N_f procedure to compute information on the other important quantities such as those which relate to operators in deep inelastic scattering.

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