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The β-function of the Wess-Zumino model at $O(1/N^2)$

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Abstract. We extend the critical point self-consistency method used to solve field theories at their d-dimensional fixed point in the large N expansion to include superfields. As an application we compute the β-function of the Wess-Zumino model with an $O(N)$ symmetry to $O(1/N^2)$. This result is then used to study the effect the higher order corrections have on the radius of convergence of the 4-dimensional β-function at this order in $1/N$. The critical exponent relating to the wave function renormalization of the basic field is also computed to $O(1/N^2)$ and is shown to be the same as that for the corresponding field in the supersymmetric $O(N)$ σ model in d-dimensions. We discuss how the non-renormalization theorem prevents the full critical point equivalence between both models.

1 Introduction.

The Wess-Zumino model is the simplest four dimensional interacting quantum field theory which possesses supersymmetry, [1]. This bose-fermi symmetry is believed to play a role in the unification of the forces of nature beyond the energy scales accessible by the present generation of accelerators. Nevertheless theories with supersymmetry are also of interest for testing out and developing new ideas. One reason for this is that one can calculate much more easily in the quantum supersymmetric theory compared with say its bosonic sector, due to cancellations between graphs involving bosons and fermions. For example, there is no two loop term in the $\overline{\mathrm{MS}}$ β-function of the two dimensional supersymmetric σ model on an arbitary manifold, [2]. In the corresponding bosonic σ model the two loop contribution to the β-function is non-zero, [3]. Moreover, in certain situations, like the Wess-Zumino model, the renormalization is constrained in such a way that the calculation of the coupling constant renormalization is reduced to obtaining the wave function renormalization. In other words ensuring unbroken supersymmetry in the quantum theory restricts the form of possible counterterms in the superpotential and hence determines a renormalization constant, [1, 4]. This non-renormalization theorem allows one to compute higher order corrections rather easily. For instance, the four loop β-function for the pure model with one superfield is known, [5, 6, 7], as well as that for its extension to include superfields with general internal symmetries, [8]. Although the Wess-Zumino model was originally formulated in terms of ordinary spin-0 and $\frac{1}{2}$ fields, there is a further calculational tool which substantially reduces the amount of work needed for these higher order calculations. This is attained by grouping the component fields in the same multiplet into superfields where the Lorentz space-time is extended to include Grassmann coordinates. Consequently in this superspace the Feynman supergraphs group classes of component field diagrams together thereby reducing the number that need to be computed. Moreover performing the integrations over the Grassmann variables, known as the D-algebra, the resulting Feynman integrals are invariably simple scalar integrals. This property can be better appreciated in superfield calculations in supersymmetric gauge theories where the number of component fields is larger than in the Wess-Zumino model and high loop integrals involving the component gauge field become extremely tedious compared with a scalar integral.

Recently some insight into the structure of the Wess-Zumino model β-function beyond the present four loop level has been gained in [9] through using the conventional large N bubble sum method. The leading order corrections in $1/N$ were computed for the case where the Wess-Zumino model has a multiplet of $O(N)$ fields coupled to a separate scalar superfield. One of the motivations of that study was to ascertain the large coupling constant behaviour of the β-function in a four dimensional model and hence to understand some of the problems in its coupling constant resummation, as well as being the foundation for repeating the exercise for a supersymmetric gauge theory. As leading order calculations do not reveal a substantial amount of structure it would be interesting to push the $1/N$ calculations to a higher order in both the Wess-Zumino and supersymmetric gauge theories. However, a formalism does exist to achieve this and is based on the critical renormalization group and d-dimensional conformal field theory methods which were pioneered in the series of papers of ref. [10, 11] for the $O(N)$ σ model in two dimensions. Before attempting to apply such methods to these models there are several problems which need to be solved. Although the methods of [10, 11] have already been applied to the supersymmetric $O(N)$ σ model in [12, 13] to obtain the β-function at $O(1/N^2)$ and the supersymmetric $CP(N)$ σ model at $O(1/N)$, [14], these calculations were performed using the component lagrangians. Indeed the computation of 77 component Feynman diagrams in [13] to deduce the β-function at $O(1/N^2)$ only serves to indicate that the development of a superfield approach would yield a more compact approach. This is one of the aims of this paper and will be provided for the Wess-Zumino model. As an application we will then compute its β-function at a new order, $O(1/N^2)$, as a function of the space-time dimension, d. Briefly, the method involves determining the critical exponents of the theory at the phase transition defined by the non-trivial zero of the d-dimensional β-function. As there is no mass at such a point the propagators of all the fields, for instance, involve the momentum raised to some power which is known as a critical exponent. From a field theoretic point of view this is a fundamental object as it is renormalization group invariant and therefore physically measurable. It depends only on the space-time dimension and the parameters of any internal symmetries. For our purpose this will be N. Therefore it can be expanded in powers of $1/N$ when N is large and deduced order by order by examining the scaling behaviour of the particular Schwinger Dyson equations truncated at the appropriate order. Moreover, as the exponents will depend on $d = 4 - 2\epsilon$ they can be related through the critical renormalization group to the corresponding renormalization constants one ordinarily computes in perturbation theory. Hence one can deduce new coefficients in the perturbative series in successive orders in $1/N$ by performing the ϵ-expansion of the exponents beyond the order currently known.

Finally, another motivation for studying the Wess-Zumino model at this d-dimensional critical point is to understand the universality class to which it belongs. By this we mean the following. The bosonic sector of the $O(N)$ Wess-Zumino model is effectively ϕ^4 theory with an $O(N)$ symmetry. The field which is not an element of the multiplet $O(N)$ acts as auxiliary which if eliminated yields the usual quartic interaction. As is well known (see, for example, [15]) in d-dimensions $O(N)$ ϕ^4 theory lies in the same universality class as the lower dimensional $O(N)$ bosonic σ model and therefore the critical exponents one computes in either model are equivalent. Indeed these are of interest because both models underlie the physics of the phase transition of the Heisenberg ferromagnet. Therefore we are motivated to determine if there is an analogous lower dimensional equivalent theory for the Wess-Zumino model. Intuitively one would expect it to have some connection with the supersymmetric $O(N)$ σ model. To achieve this we need to compare the values for the Wess-Zumino exponents with those of [16, 12, 13]. Such an equivalence would therefore play a role in some supersymmetric extension of the Heisenberg ferromagnet in three dimensions, though of course it is not clear if such an object exists in the real physical world.

The paper is organised as follows. We review the relevant properties of the Wess-Zumino model in section 2 and introduce the superfield approach for the critical point large N method. This is used to reproduce the known results at leading order in $1/N$ in section 3 where the $O(1/N^2)$ correction is also derived for the wave function renormalization. The higher order correction to the β-function is determined in section 4 where the technical details of the computation of the three and four loop diagrams which occur are also discussed. Finally we collect our results in section 5 and produce several new coefficients that will appear in the $\overline{\mathrm{MS}}$ β-function at five and six loops.

2 Critical point formalism.

We will be working with an $O(N)$ symmetric Wess-Zumino model containing one chiral superfield σ and N chiral superfields Φ^i, $1 \leq i \leq N$. Its action is given by

$$S = \int d^dx \left[\int d^4\theta \left(\bar{\Phi}^i\Phi^i + \frac{\bar{\sigma}\sigma}{g^2} \right) - \frac{1}{2} \int d^2\theta \, \sigma\Phi^2 - \frac{1}{2} \int d^2\bar{\theta} \, \bar{\sigma}\bar{\Phi}^2 \right] \qquad (2.1)$$

The coupling constant g has been rescaled into the σ-field so that the 3-vertex is in the form which easily allows us to use the technique of uniqueness, [17, 11], to compute the higher order graphs which will arise. Moreover the chirality of the fields in (2.1) will prove to be useful in substantially reducing the number of possible diagrams which we will need to consider then. Before carrying out a dimensional analysis of the action to ascertain the canonical dimensions of the fields and the couplings, which is the first step in the critical point formalism, we recall some of the practicalities of the calculation of the perturbative β-function. If we define a new field σ' by $\sigma' = \sigma/g$ then one would recover the conventional form of the interaction term. The non-renormalization theorem of [1, 4] implies that the vertex is finite and so the overall vertex renormalization constant involves no infinities. Therefore the renormalization group functions of the rescaled interaction and its constituent fields satisfy the following simple relation

$$\beta(g) \;=\; [2\gamma_\Phi(g) \;+\; \gamma_{\sigma'}(g)]g \tag{2.2}$$

which implies the wave function renormalization purely determines how the coupling constant runs. The result (2.2) was used in [5, 6, 7] to deduce $\beta(g)$ for the model without any internal symmetries. As we are dealing with the model with an $O(N)$ symmetry we require the perturbative results as a function of N. To achieve this we have specialised the four loop result of [8] to the $O(N)$ model. Useful in checking the relevant symmetry factors in this case was the package QGRAF, [18]. Therefore, the $\overline{\mbox{MS}}$ results are

$$\begin{aligned} \gamma_\Phi(g) \;=\;& g \;-\; \frac{1}{2}(N+2)g^2 \;-\; \frac{1}{4}\left(N^2 - 10N - 4 - 24\zeta(3)\right) g^3 \\ &-\; \left(\frac{1}{24}(3N^3 + 16N^2 + 152N + 40) - \frac{\zeta(3)}{4}(N^2 - 4N - 36)(N+4) \right. \\ &\left. -\; 3\zeta(4)(N+4) + 20\zeta(5)(N+2) \right) g^4 \;+\; O(g^5) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \gamma_{\sigma'}(g) \;=\;& \frac{N}{2}g \;-\; Ng^2 \;+\; \frac{N}{2}(2N + 1 + 6\zeta(3))g^3 \\ &+ N\left(\frac{1}{12}(5N^2 - 56N - 16) - \frac{1}{2}\zeta(3)(N^2 + 6N + 44) \right. \\ &\left. +\; \frac{3}{2}\zeta(4)(N+4) - 10\zeta(5)(N+2) \right) g^4 \;+\; O(g^5) \end{aligned} \tag{2.4}$$

Hence, (2.2) implies

$$\begin{aligned} \beta(g) \;=\;& \frac{1}{2}(d-4)g \;+\; \frac{1}{2}(N+4)g^2 \;-\; 2(N+1)g^3 \;+\; \left(\frac{1}{2}(N^2 + 11N + 4) + 3(N+4)\zeta(3) \right) g^4 \\ &+\; \left(\frac{1}{6}(N^3 - 36N^2 - 84N - 20) - 3(N^2 + 16N + 24)\zeta(3) \right. \\ &\left. +\; \frac{3}{2}(N+4)^2\zeta(4) - 10(N+2)(N+4)\zeta(5) \right) g^5 \;+\; O(g^6) \end{aligned} \tag{2.5}$$

We note that for various reasons we have suppressed the usual factors of π associated with each power of the coupling in these expressions since, for example, they can readily be restored by a simple rescaling. Another reason is related to the critical point approach where we deduce critical exponents which correspond through the renormalization group to these renormalization group functions. The critical exponents are more fundamental quantities than the renormalization group functions in that they are renormalization group invariant. Therefore they take the same values in any mass independent renormalization scheme. This criterion arises from the fact that at a fixed point of the β-function there is scaling and conformal invariance which would be broken by any non-zero mass. Therefore since, for instance, MS and $\overline{\mbox{MS}}$ are two examples of mass independent renormalization schemes the critical exponents deduced in either scheme will be the same. The only difference is that the value of the critical couplings in each scheme will not be the same but they will be simply related by a constant rescaling of either of the couplings. However, throughout we will always have in mind that our results will relate to the $\overline{\rm MS}$ scheme.

With these renormalization group functions we can now develop the critical point formalism. We take the d-dimensional β-function of (2.5) and compute the location of the non-trivial d-dimensional fixed point as

$$\begin{aligned} g_c ~=~ & \frac{2\epsilon}{N} ~+~ \left(-8\epsilon + 16\epsilon^2 - 8\epsilon^3 - \frac{16}{3}\epsilon^4 + O(\epsilon^5) \right) \frac{1}{N^2} \\ & +~ \Bigg(32\epsilon - 176\epsilon^2 - 8(6\zeta(3) - 37)\epsilon^3 \\ & \quad + \frac{16}{3}(60\zeta(5) - 9\zeta(4) + 18\zeta(3) - 4)\epsilon^4 + O(\epsilon^5) \Bigg) \frac{1}{N^3} ~+~ O\left(\frac{\epsilon}{N^4}\right) \end{aligned} \qquad (2.6)$$

in powers of $1/N$ where $d = 4 - 2\epsilon$. Although supersymmetry is not easily defined in d-dimensions we note that this is a valid step since in perturbative $\overline{\rm MS}$ calculations with a dimensional regulator (2.5) would be produced as the penultimate step in determining the 4-dimensional β-function. Moreover it was shown in [12, 13] that the d-dimensional version of the supersymmetric σ model preserved certain features of the symmetry at its corresponding fixed point and this allows us to make similar assumptions about the Wess-Zumino model in d-dimensions. For instance, the component fields of each boson and fermion partner in a supermultiplet retained the *same* anomalous dimension in arbitrary dimensions. With (2.6) we can now dimensionally analyse the d-dimensional action (2.1) at g_c choosing to work with the σ-field. Unlike the component calculation of [12] we define the canonical dimensions of the fields in relation to the full superspace and take the dimension of the Grassmann coordinates as $-1/2$. As the renormalization group functions are non-zero we must also allow for the possibility of anomalous contributions which we will define with respect to the usual conventions of [10]. Defining the full dimension of the Φ superfield as α then the quadratic term of (2.1) implies we set

$$\alpha ~=~ \mu ~-~ 1 ~+~ \frac{\eta}{2} \qquad (2.7)$$

where η is the critical exponent corresponding to the wave function renormalization at criticality. In other words $\eta = \gamma_\Phi(g_c)/2$. From the interaction term, if we denote by β the full dimension of the σ superfield and by χ the vertex dimension, then

$$\beta ~=~ 2\mu ~-~ 1 ~-~ 2\alpha ~-~ \chi \qquad (2.8)$$

However, as the non-renormalization theorem implies the vertex is finite to all orders in perturbation theory then the (composite) operator $\sigma\Phi^2$ can have no anomalous piece. Therefore we set $\chi = 0$. This observation will have a simplifying effect in, for example, the computation of the $1/N^2$ corrections to the Schwinger Dyson equations we will solve. Hence

$$\beta ~=~ 1 ~-~ \eta \qquad (2.9)$$

and the vertices of all the Feynman diagrams will in principle be one step away from uniqueness. From the remaining term of the action we deduce a scaling law relating the dimension of the σ^2 composite operator to the β-function exponent. As $\beta(g_c) = 0$ this is defined to be $\omega = -\,\beta'(g_c)$. Therefore, we have

$$\omega ~=~ 2\mu ~-~ 2 ~-~ 2\beta ~+~ \eta_{\sigma^2} \qquad (2.10)$$

where η_{σ^2} is the anomalous dimension of the bare composite operator. Although one can use (2.10) to determine the β-function by calculating the anomalous dimension of σ^2 we have chosen to use the method of [10, 11] of solving the Dyson equations. It turns out that for scalar theories at least, both ways are related in that the Feynman diagrams where one inserts σ^2 in a σ 2-point function and examines the resulting diagrams, it is easy to see that they have a complete analogue with the diagrams which arise in the Schwinger Dyson approach. However, both methods require a similar amount of effort to determine the solution.

Near g_c, we can define the asymptotic behaviour of the dressed propagators of Φ and σ in Euclidean space. As $p \to \infty$,

$$\begin{aligned} \langle\bar{\Phi}(-p,\theta)\Phi(p,\theta')\rangle &\sim \frac{A\delta^4(\theta-\theta')}{(p^2)^{\mu-\alpha}} \\ \langle\bar{\sigma}(-p,\theta)\sigma(p,\theta')\rangle &\sim \frac{B\delta^4(\theta-\theta')}{(p^2)^{\mu-\beta}} \end{aligned} \tag{2.11}$$

where we have set $d = 2\mu$ and A and B are unknown momentum independent amplitudes. Corrections to these scaling forms can also be defined. For example, we take

$$\begin{aligned} \langle\bar{\Phi}(-p,\theta)\Phi(p,\theta')\rangle &\sim \frac{A\delta^4(\theta-\theta')}{(p^2)^{\mu-\alpha}}\left[1 + \frac{A'}{(p^2)^{\omega}}\right] \\ \langle\bar{\sigma}(-p,\theta)\sigma(p,\theta')\rangle &\sim \frac{B\delta^4(\theta-\theta')}{(p^2)^{\mu-\beta}}\left[1 + \frac{B'}{(p^2)^{\omega}}\right] \end{aligned} \tag{2.12}$$

In principle one can include other correction terms involving other exponents with different canonical dimensions. However, renormalizability and hyperscaling laws ensure that knowledge of two independent exponents such as η and ν is sufficient to deduce the others. Here ω is the exponent which relates to corrections to scaling. The two pairs of amplitudes (A, B) and (A', B') are independent of one another. Following the methods of [10, 11], we will use (2.12) to solve the Dyson equations. The inclusion of a non-zero anomalous dimension in the exponents of the propagators means one considers only those diagrams with no dressing on the propagators which of course will further reduce the number of graphs which would need to be computed at high orders.

Ordinarily in the critical point method of [10, 11] the next stage is to determine the asymptotic scaling forms of the inverses to (2.11) and (2.12) by inverting them in momentum space. This is because they will appear in the Dyson equation as illustrated in fig. 1. However, the situation is different here and can be best introduced by first recalling the nature of perturbative calculations in superspace. In the calculation of the corrections to, say, the quadratic terms in the effective action one needs to retain the external momentum space superfields explicitly when representing the graph from the super-Feynman rules. The reason for this is that the interactions and propagators involve supercovariant derivatives which act on the Grassmann coordinates and hence the external fields since they too depend on θ. In ordinary non-supersymmetric calculations it is not necessary to include the external fields since it is the case that for (local) quantum field theories derivative couplings are defined in coordinate space. Therefore when performing computations in momentum space their Fourier transform corresponds to vectors of the conjugate variable which therefore do not act differentially on the external legs. After the manipulation of the supercovariant derivatives, known as performing the D-algebra, one is left with a simple momentum space Feynman integral which can in principle be computed. The final result is an integral over the full momentum superspace which includes the fields corresponding to the external legs. For a 2-point function this would be of the form, [19, 6],

$$\int d^dp \int d^4\theta\, f(p)\bar{\Phi}(-p,\theta)\Phi(p,\theta) \tag{2.13}$$

where Φ and $\bar{\Phi}$ are the external fields whose momentum is p. Therefore this would be regarded as the correction to the corresponding term in the effective action. From the point of view of perturbative renormalization the extraction of the pole part in $f(p)$ with respect to the regularization would then be absorbed by the appropriate counterterm. In light of this the critical point approach to the Dyson equations is adapted to be similar. One manipulates the graphs in momentum superspace using ordinary D-algebra but with the lines of the graphs of fig. 1 replaced by the critical propagators, (2.11) and (2.12), and the external fields assumed to be included. One again obtains an expression of the form (2.13) but the corresponding $f(p)$ will now depend on the exponents of the lines, α and β, of the original graph. The momentum dependence will be in scaling form with the exponent given by the dimension of the original integral. The remaining part of (2.13) would be regarded as the propagator. In the approach of [10, 11] in non-supersymmetric calculations this would not appear *here* since the fields of the external legs are not included which is the reason why one needs to invert the corresponding (2.11). So in the superspace approach this is not necessary since the propagator appears naturally with the scaling part of the graph. Although we have included the inverses in fig. 1, these are also meant to account for the situation when one has the corrections to scaling of (2.12). Therefore, we define

$$\Phi^{-1}(p) \sim \frac{1}{A}\left[1 - \frac{A'}{(p^2)^\omega}\right] \quad , \quad \sigma^{-1}(p) \sim \frac{1}{B}\left[1 - \frac{B'}{(p^2)^\omega}\right] \tag{2.14}$$

which is similar to the situation in [10, 11]. In practical calculations it will be a simple exercise to verify that the above argument leads to dimensionally consistent equations.

3 Solving the Dyson equations.

We are now in a position to determine expressions for the critical exponents and begin with the leading order analysis. With the propagators from (2.12), the Dyson equations, which are given in fig. 1 and have been truncated at leading order in $1/N$, may be written in the following integral form in the scaling region as

$$\begin{aligned} 0 &= \Phi^{-1}(p) + AB\mathcal{H}(\Phi,p)\int d^{2\mu}k\,\frac{1}{(k^2)^{\mu-\alpha}((k-p)^2)^{\mu-\beta}}\left[1 + \frac{A'}{(k^2)^\omega} + \frac{B'}{((k-p)^2)^\omega}\right] \\ 0 &= \sigma^{-1}(p) + \frac{1}{2}NA^2\mathcal{H}(\sigma,p)\int d^{2\mu}k\,\frac{1}{(k^2)^{\mu-\alpha}((k-p)^2)^{\mu-\alpha}}\left[1 + \frac{A'}{(k^2)^\omega} + \frac{A'}{((k-p)^2)^\omega}\right] \end{aligned} \tag{3.1}$$

where the functions $\mathcal{H}$ of the fields and external momentum represent the θ integral of (2.13) and we have omitted terms which are quadratic in A' and B'. We note that we use a form of the Dyson equations where the propagators are dressed but not the vertices. The factor of $\frac{1}{2}$ comes from the symmetry of the first graph in fig. 1. Substituting for α, β and ω and decoupling (3.1) into terms with the same momentum dimension, we obtain

$$\begin{aligned} 0 &= 1 + z\nu\,(\mu - \tfrac{1}{2}\eta, \mu - 1 + \eta, 1 - \tfrac{1}{2}\eta) \\ 0 &= 1 + \tfrac{1}{2}Nz\nu\,(2\mu - 2 + \eta, 1 - \tfrac{1}{2}\eta, 1 - \tfrac{1}{2}\eta) \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 = {} & [\nu\,(\mu - 1 + \omega' - \tfrac{1}{2}\eta, \mu - 1 + \eta, 2 - \tfrac{1}{2}\eta - \omega')\; z - 1]\; A' \\ & + \nu\,(2\mu - 3 + \omega' + \eta, 2 - \omega' - \tfrac{1}{2}\eta, 1 - \tfrac{1}{2}\eta)\, zB' \\ 0 = {} & zN\nu\,(\mu - 1 + \omega' - \tfrac{1}{2}\eta, 1 - \tfrac{1}{2}\eta, \mu + \eta - \omega')\, A' - B' \end{aligned} \tag{3.3}$$

where we have set $z = A^2B$ and defined the anomalous piece of ω by $\omega = \mu - 2 + \omega'$. The function ν is defined for arbitrary α, β and γ as $\nu(\alpha,\beta,\gamma) = a(\alpha)a(\beta)a(\gamma)$ with $a(\alpha) = \Gamma(\mu-\alpha)/\Gamma(\alpha)$. Eliminating z from (3.2) we obtain the relation that will give us η at leading order in $1/N$

$$\eta ~=~ \frac{4\Gamma\left(1+\eta/2\right)}{N\Gamma\left(\mu-\eta/2\right)}\nu\left(2-\mu-\eta,\mu-1+\tfrac{1}{2}\eta,\mu-1+\eta\right) \tag{3.4}$$

Setting $\eta = \sum_{i=1}^\infty \eta_i/N^i$ and expanding (3.4) in powers of $1/N$ we find η_1 is

$$\eta_1 ~=~ \frac{4\Gamma(2\mu-2)}{\Gamma(\mu)\Gamma^2(\mu-1)\Gamma(2-\mu)} \tag{3.5}$$

Remarkably the computation of η_2 requires only the expansion of (3.4) to the next order in $1/N$. Ordinarily one would have two and three loop corrections to each of the Dyson equations of fig. 1 but in the Wess-Zumino model the chirality condition on the vertex excludes these. Therefore the first non-zero corrections will arise at $O(1/N^3)$. In this respect the Hartree-Fock approximation at $O(1/N^2)$ determines the full value of η_2. Therefore we simply obtain

$$\eta_2 ~=~ \eta_1^2\left[\psi(2-\mu) ~+~ \psi(2\mu-2) ~-~ \psi(\mu-1) ~-~ \psi(1) ~+~ \frac{1}{2(\mu-1)}\right] \tag{3.6}$$

where $\psi(x)$ is the logarithmic derivative of the Euler Γ-function. Moreover the absence of these higher order contributions means that unlike the bosonic σ model calculation of [10, 11] the Dyson equation is finite to this order and does not need to be either regularized or renormalized. Regularization is usually necessary due to vertex subgraph divergences in the higher order graphs as the vertices of the Dyson equation are undressed. It is introduced by shifting the analogous σ-field exponent by an infinitesimal amount, Δ, which is equivalent to setting $\chi \to \chi + \Delta$. As there is no vertex renormalization in the Wess-Zumino model and since the critical large N method is effectively a perturbative expansion in the vertex anomalous dimension then we are led to reason that at the fixed point the Dyson equations do not require any renormalization at *any* order in $1/N$. For subsequent calculations we will require the value of z to two orders. Also setting $z = \sum_{i=1}^\infty z_i/N^i$ knowledge of η_1 and η_2 allows us to deduce

$$\begin{aligned} z_1 &~=~ -\tfrac{1}{2}\Gamma(\mu)\eta_1 \\ z_2 &~=~ \tfrac{1}{2}\Gamma(\mu)\left[\psi(2-\mu) ~+~ \psi(2\mu-2) ~-~ \psi(\mu-1) ~-~ \psi(1)\right]\eta_1^2 \end{aligned} \tag{3.7}$$

The second pair of equations, (3.3), determines corrections to the exponent ω. If one formally writes (3.3) in matrix form

$$\left(\begin{array}{cc} x & y \\ v & w \end{array}\right)\left(\begin{array}{c} A' \\ B' \end{array}\right) ~=~ 0 \tag{3.8}$$

then ω is determined by requiring that the determinant of this matrix vanishes, [11]. This gives

$$\begin{aligned} 1 ~=~ & z^2N\nu\left(2\mu-3+\omega'+\eta,1-\tfrac{1}{2}\eta,2-\omega'-\tfrac{1}{2}\eta\right)\nu\left(\mu-1+\omega'-\tfrac{1}{2}\eta,1-\tfrac{1}{2}\eta,\mu+\eta-\omega'\right) \\ & +~ z\nu\left(\mu-1+\omega'-\tfrac{1}{2}\eta,\mu-1+\eta,2-\tfrac{1}{2}\eta-\omega'\right) \end{aligned} \tag{3.9}$$

However, in deducing (3.9) one needs to bear in mind that the extra graphs at $O(1/N^2)$, which would usually play a role in the evaluation of η_2 but which are absent due to chirality, would also in principle need to be included here. Though they would only enter in one of the Dyson equations because the leading N-dependence of each of the entries of the matrix of (3.8), when considered in coordinate space, is not the same. Moreover this feature has already been observed and discussed in similar calculations in other models, [20]. Although the relevant graphs are absent in determining the first non-trivial term of ω certain topologies which cannot be ruled out by chirality occur at higher order and therefore will need to be included in the relevant Dyson equation of (3.8). Therefore setting $\omega' = \sum_{i=1}^{\infty} \omega_i/N^i$ and retaining only the leading terms in (3.9), we obtain

$$1 \;=\; -\;\frac{(2\mu-3)\eta_1}{(\omega_1-\eta_1)} \;-\; \frac{(\mu-1)\eta_1}{2(\mu-2)N} \tag{3.10}$$

As the second term is suppressed by a factor of $1/N$ compared with the first we can ignore it and rearrange (3.10) to find

$$\omega_1 \;=\; -\;2(\mu-2)\eta_1 \tag{3.11}$$

We need to compare these results for η_1 and ω_1 with the results already obtained via the conventional bubble sum methods, [9], and demonstrate their equivalence. To do so we must convert the critical exponents into the original renormalization group functions. From the definition of the critical coupling, (2.6), we can replace the μ-dependence in (3.5) at leading order by setting $\mu = 2 - Ng_c/2$ which determines $\gamma_{\Phi,1}(g_c)$ as an explicit function of g_c since $\gamma_{\Phi,1}(g_c)$ $= \eta_1/2$. Then it is easy to deduce for non-critical couplings that

$$\gamma_\Phi(g) \;=\; \frac{2\Gamma(2-y)}{N\Gamma(2-y/2)\Gamma^2(1-y/2)\Gamma(y/2)} \;+\; O\left(\frac{1}{N^2}\right) \tag{3.12}$$

where we have introduced the scaled coupling $y = Ng$ for convenience. For subsequent calculations it is also useful to define a new function $G(y/2)$ by

$$G\left(\frac{y}{2}\right) \;=\; \frac{\Gamma(2-y)}{\Gamma(2-y/2)\Gamma^2(1-y/2)\Gamma(1+y/2)} \tag{3.13}$$

Taking into account the symmetry factor of 1/2 present in each Φ loop, (3.12) is precisely the result obtained in [9]. We can perform a similar exercise for the β-function. Noting that $\omega_1 =$ $2y_c^2 G(y_c/2)$ and defining $\beta(g)$ at $O(1/N)$ as

$$\beta(g) \;=\; (\mu-2)g \;+\; \frac{1}{2}(N+4)g^2 \;+\; g^2 f_1\left(\frac{gN}{2}\right) \tag{3.14}$$

where $f_1(gN/2)$ is an arbitrary function we see that at the critical point we must have

$$0 \;=\; 2(\mu-2) \;+\; (N+4)g_c \;+\; 2g_c f_1\left(\frac{g_cN}{2}\right) \tag{3.15}$$

Using (3.15) to eliminate the $(N+4)g_c$ term in $\beta'(g_c)$ we find that

$$f_1'(\epsilon) \;=\; -\;\frac{\omega_1(\epsilon)}{2\epsilon^2} \tag{3.16}$$

Therefore we find in four dimensions

$$\beta(g) \;=\; \frac{1}{2}(N+4)g^2 \;-\; 4g^2\int_0^{gN/2} dx\, G(x) \;+\; O\left(\frac{1}{N^2}\right) \tag{3.17}$$

which was obtained originally in [9]. To compare with perturbation theory one computes the integral by first expanding the integrand in a Taylor series as a function of x before integrating each term individually. It is elementary and reassuring to check that the coefficients of (2.3) and (2.5) are reproduced at four loops respectively from (3.12) and (3.17). It is worthwhile comparing some of the features of both methods that lead to the same results. In [9] an infinite number of diagrams had to be summed which was only made possible by a seemingly miraculous cancellation of terms. By contrast one needs only to calculate two one-loop diagrams in the critical approach. The elimination of the amplitude factor z between both Dyson equations in effect reproduces the conventional bubble sum giving the same results in the end. However, the advantage of the critical method lies in performing $O(1/N^2)$ calculations. Indeed it is not always clear if the corresponding bubble sums at this order would prove to be easily calculable but would certainly rely on further remarkable cancellations. We can continue the exercise of converting the exponents into the renormalization group functions by considering η_2. By retaining the higher orders in the critical coupling we find that to $O(1/N^2)$

$$\begin{aligned}\gamma_\Phi(g) \; &= \; \frac{y}{N}G\left(\frac{y}{2}\right) \; + \; \frac{2y}{N^2}\left[\left(G\left(\frac{y}{2}\right) \; + \; \frac{y}{2}G'\left(\frac{y}{2}\right)\right)\beta_1\left(\frac{y}{2}\right)\right. \\ &\left. \quad + \; yG^2\left(\frac{y}{2}\right)\left(\Psi(y) - \frac{2}{y} - \frac{1}{(y-1)} - \frac{1}{(y-2)}\right)\right] \end{aligned} \quad (3.18)$$

where, for later convenience, we have introduced the function $\beta_1(gN/2)$ which is defined by

$$\beta(g) \; = \; (\mu - 2)g \; + \; \frac{Ng^2}{2} \; + \; g^2\beta_1\left(\frac{gN}{2}\right) \; + \; O\left(\frac{1}{N^2}\right) \quad (3.19)$$

and represents all the $O(1/N)$ corrections. Its explicit form can be deduced from (3.17). The function $\Psi(y)$ is given by

$$\Psi(y) \; = \; \psi(1-y) \; + \; \psi\left(1 + \frac{y}{2}\right) \; - \; \psi\left(1 - \frac{y}{2}\right) \; - \; \psi(1) \quad (3.20)$$

and its expansion in powers of y will only involve the Riemann zeta series. The higher order correction to (3.19) will be determined later. If we expand (3.18) to $O(g^4)$ we recover all the coefficients of (2.3) at $O(1/N^2)$ which leads us to believe the result for η_2 is correct.

We can now address one of the issues raised in sect. 1 concerning the equivalence of the Wess-Zumino model with a lower dimensional theory. The criterion for determining which model this ought to be is dictated by the nature of symmetries and the interaction. At criticality the quadratic terms in an action primarily determine the canonical dimensions of the fields and couplings. Therefore in the case of the $O(N)$ Wess-Zumino model an obvious candidate is the $O(N)$ supersymmetric σ model which is defined in two dimensions and has the superspace action

$$S \; = \; - \, \frac{1}{4}\int d^dx \, d^2\theta \, \left[\frac{1}{2}\bar{D}\Phi D\Phi \; - \; \sigma\left(\Phi^2 - \frac{1}{\lambda}\right)\right] \quad (3.21)$$

Here the coupling constant λ differs from that of (2.1) in that it is dimensionless in two dimensions and its critical value is deduced from its β-function which will clearly be different from (2.5). As this model has also been studied to $O(1/N^2)$ we are in a position to compare the critical exponents of the fields in both models. (Comparing the exponents $\nu = - \, \beta'(\lambda_c)$ of the supersymmetric σ model with $\omega = - \, \beta'(g_c)$ of the Wess-Zumino model is not sensible as their respective canonical dimensions are $(\mu - 1)$ and $(\mu - 2)$.) From the results of [16, 12] it transpires that the Φ-superfield dimension has the same d-dimensional value to $O(1/N^2)$. Whilst this was expected at leading order because the diagrams are the same, the agreement at next order is remarkable when we compare the nature of the calculation of η_2 in both models. In the Wess-Zumino model η_2 is determined purely by iterating the equation which produces η_1 to the next order because there are no new contributing graphs at $O(1/N^2)$. By contrast in the σ model to obtain η_2 one has to compute the contribution from two and three loop topologies which are divergent and need to be regularized in the manner discussed earlier. In the component field calculation this equated to evaluating 24 graphs, [12]. Therefore it would appear that both (2.1) and (3.21) are equivalent at least at $O(1/N^2)$. In other words $\gamma_\Phi(g_c) \; = \; \gamma_\Phi(\lambda_c)$ to this order. However, the σ-field of both models does not have the same dimension as a direct consequence of the non-renormalization theorem. As noted earlier $\chi = 0$ in the Wess-Zumino model but in the σ

model χ is non-zero. For example, at leading order $\chi_1 = -\ \eta_1$. This property can be illustrated further if we consider calculating the exponent $\nu = \mu - 1 + O(1/N)$ in the Wess-Zumino model and compare it with the value in the σ model, [13]. To do this we can replace the exponent ω in the correction term in each of the scaling forms (2.12) with ν. Consequently the leading order value ν_1 can be deduced from the equation analogous to (3.9) where now both terms on the right side will contribute giving

$$\frac{\eta_1}{(\eta_1 - 2\nu_1)} ~=~ 1 \tag{3.22}$$

which implies $\nu_1 = 0$. This is the same as the value in the σ model, [16, 12], and would again have encouraged us to believe in a possible equivalence. However, unlike the situation with ω the coordinate space version of the matrix consistency equation which determines ν does not have the reordering problem and we can therefore also compute ν_2. This is essentially trivial due to the absence of the higher order two and three loop topologies due to chirality which again means there are no new higher order graphs which need to be included. Therefore ν_2 is deduced by expanding the consistency equation which gave ν_1 to the next order in $1/N$, in the same way that η_2 was deduced. We found that $\nu_2 = 0$. In the σ model ν has a non-zero contribution at $O(1/N^2)$, [13]. It is not clear whether ν will vanish to all orders in $1/N$ in the Wess-Zumino model since there will be new graphs at next order. Their contributions would have to cancel to have $\nu_3 = 0$. In light of these remarks it would certainly be very surprising if η_3 calculated in both models was the same since the effect of differing values of χ would almost certainly be important at $O(1/N^3)$. In all these observations concerning a possible equivalence the underlying reason for its failure after $O(1/N^2)$ is due to the chirality condition which implies $\chi = 0$. However, another way of understanding it is by considering supersymmetry in relation to the space time dimension. Clearly the Wess-Zumino model is invariant under $\mathcal{N}$ = 1 supersymmetry in four dimensions. However, theories in two dimensions which are built out of chiral superfields are invariant under $\mathcal{N}$ = 2 supersymmetry. Therefore if one wished to construct the lower dimensional equivalent theory to (2.1) it would need to have more than one supersymmetry. The $O(N)$ supersymmetric σ model unfortunately does not satisfy this criterion being invariant under $\mathcal{N} = 1$ supersymmetry which is why it is remarkable that its η_2 and that of the Wess-Zumino model are the same.

4 Calculation of ω_2.

We are now in a position to extend the result for ω_1 to the next order in $1/N$. Unlike the calculation for η_2 which simply involved the expansion of (3.4) to next order the reordering which occurs in (3.3) means we need to include the contributions from several higher order graphs. These are illustrated in fig. 2. In the analogous calculation of the β-function in ϕ^4 theory, [21, 22], there were 33 graphs which needed to be computed. In principle the same graphs would have to be considered here but the chirality condition excludes the majority of these to leave the topologies of fig. 2. Consequently if we denote their total value by Π_2, the solution of the Dyson equation, (3.9), can be expanded to the next order in $1/N$ to obtain

$$\omega_2 ~=~ -\left[4(\mu-1)(\mu-2)\hat{\Psi}(\mu) ~+~ \frac{(16\mu^4 - 96\mu^3 + 188\mu^2 - 136\mu + 25)}{2(2\mu-3)(\mu-1)} \right] \eta_1^2 ~+~ \Pi_2 \tag{4.1}$$

where $\hat{\Psi}(\mu) = \psi(2\mu - 3) + \psi(3-\mu) - \psi(\mu-1) - \psi(1)$. The terms not involving Π_2 correspond to the expansion of the a-functions of (3.9). We note that at $O(1/N^2)$ we can no longer neglect the second term of (3.10). Before we can substitute the lines with the critical propagators, we need to compute the D-algebra of each graph. Elementary application of the rules of [6] means that we are left with a $\Box$-operator acting on several lines which in momentum space will correspond to an extra factor of p^2 where p is the momentum flowing in that line. We note that in perturbation theory this invariably means that that line vanishes from the Feynman integral since massless propagators have unit exponent. In the critical large N case this will not always be the case as the $\Box$-operator can act on a σ-line whose canonical exponent is not unity. Before detailing the evaluation of each graph we record how the $\Box$-operators are distributed around each of the graphs of fig. 2. For the non-planar graph each of the upper Φ lines joining the external σ lines has a $\Box$-operator, which means in the light of the above remarks that they now have zero exponent. The analogous lines of the four loop graph also have one of these operators acting on them to also reduce them to zero exponent in momentum space, as does the completely central σ line.

It turns out that for the non-planar graph this D-algebra immediately reduces it to the simple two loop topology given in fig. 3 where we use the coordinate space convention of integrating over vertices in the language of [11]. This graph can be reduced to an integral whose value is well known if one applies the transformation $\swarrow$ of [11]. The result is an integral originally computed by Chetyrkin and Tkachov, [23], denoted by $ChT(1,1)$ in the notation of [11]. Hence the full contribution of the non-planar graph to Π_2 is

$$- \frac{3}{2}(\mu-1)\hat{\Theta}(\mu)\eta_1^2 \tag{4.2}$$

where $\hat{\Theta}(\mu) = \psi'(\mu-1) - \psi'(1)$. We note that each term of the ϵ-expansion of $\hat{\Theta}(\mu)$, with $\mu = 2-\epsilon$, is proportional to the Riemann ζ-function beginning with $\zeta(3)$.

The treatment of the remaining graphs is more involved since there are three σ-lines where the correction in the asymptotic scaling functions can be inserted and the distribution of the $\Box$'s around the graph introduce an asymmetry. Otherwise two of the insertions would be equivalent under a symmetry transformation of the integral. Therefore we cannot assume a priori that their contributing values would be the same. Of the three possibilities it transpires that that graph where the σ-line insertion is on the bottom line of the graph in fig. 2 (with the $\Box$'s on the upper Φ-lines) has already been computed in the bosonic $O(N)$ σ model calculation of its two dimensional β-function, [11]. For completeness, we note that the elementary rules of integration by uniqueness, [11], after transforming to the momentum representation leaves the three loop integral illustrated in the first graph of fig. 4. Therefore, we merely quote the contribution of this graph is

$$- \frac{(\mu-1)(2\mu-3)}{2(\mu-2)}\left[\frac{\hat{\Psi}^2(\mu)}{2} + \frac{\hat{\Phi}(\mu)}{2} + \frac{\hat{\Psi}(\mu)}{(\mu-2)} - 3\hat{\Theta}(\mu)\right]\eta_1^2 \tag{4.3}$$

where $\hat{\Phi}(\mu) = \psi'(2\mu-3) - \psi'(3-\mu) - \psi'(\mu-1) + \psi'(1)$. Like (4.2) the ϵ-expansion of the function of μ within the brackets also only involves the Riemann ζ series and no rationals. However, here the first term involves $\zeta(5)$.

It turns out in fact that the graph where the σ-insertion is on the top σ-line relative to the $\Box$'s is equivalent in value to (4.3), at this order in $1/N$. This can be established either by explicit evaluation or by a series of conformal manipulations on the three loop integral which results after the same elementary steps which produced the first graph of fig. 4. Instead the second graph of fig. 4 is obtained. To relate it to the former we first perform a conformal transformation with the origin on the right which produces an integral with the same topology but with different exponents on the lines. Replacing the $(y-u)$ line by a chain integral, [11], and choosing the exponents of these two lines in such a way that the top right internal vertex is now unique, one obtains a three loop graph with the same topology but now with exponents $(\mu - 1)$ on all the lines except for the $(y-u)$ and $(x-z)$ lines whose exponents are $(3-\mu)$. Finally, by attaching a propagator of exponent $(2\mu - 2)$ to the x external point using the chain integration rule of [11], the final integration over this unique vertex yields the first graph of fig. 4. Moreover the factors associated with each of these integration steps reduce to unity. Therefore both these insertions are equal in value.

The remaining integral with the insertion on the central σ-line cannot immediately be reduced to one whose computation has been given previously. Therefore we will give its evaluation in detail. Transforming to the momentum representation yields the first graph of fig. 5. After carrying out a conformal left transformation on it the resulting integral has a unique triangle, so that when it is completed the subsequent integral has a unique vertex. Performing this and a simple chain integral, before undoing the original conformal left transformation, one is left with the second integral of fig. 5. The associated factor is $a^4(\mu-1)/a(2\mu-4)$ relative to the first graph of fig. 5. Then applying the successive transformations $\rightarrow$ and $\leftarrow$ in the language of [11], before applying the momentum representation transformation yields the final graph of fig. 5. Ordinarily in perturbation theory when such a topology is encountered where all the lines have unit exponent, the way to proceed is to integrate by parts. Although the same strategy applies here, it turns out that the set of graphs which result from this are individually divergent, though the overall result is clearly finite. To handle this one has to introduce a temporary (analytic) regulator δ in the power of various lines. The final result will of course be independent of any choice, but it is judicious to choose the distribution of δ's in such a way that subsequent integrations can be carried out. To this end we replace the exponents of the lines with $(2\mu-3)$ by $(2\mu - 3 - \delta)$ and the central lines with exponent 1 by $(1+\delta)$. This symmetric distribution means that two integrals result after integration by parts, each of which can be immediately reduced to a two loop integral where there is a common factor proportional to $1/\delta$. After several elementary transformations each of these has the $\delta = 0$ form $ChT(3-\mu, \mu-1)$. However, the $1/\delta$ pole means that each $\delta \neq 0$ integral needs to be determined to the $O(\delta)$ term. Unfortunately this is not possible for each case. Instead it turns out that due to the distribution of δ's in each integral we can compute the *difference* in the values of the integrals to the order we require and it is merely given by the value of $ChT(3-\mu, \mu-1-\delta)$ to $O(\delta)$. This is achieved by comparing the corresponding exponents in each graph. In other words if the lines have the same exponent for non-zero δ, then to $O(\delta)$ they will not influence the value of the difference. Therefore in these lines we can effectively set $\delta = 0$. Only for those lines where there is a mismatch in the exponent in corresponding lines must one not set $\delta = 0$ in them. After matching in this way, one essentially reduces the calculation to the evaluation of $[ChT(3-\mu, \mu-1-\delta) - ChT(3-\mu,\mu-1)]/\delta$ as $\delta \rightarrow 0$. Thus piecing the steps in this summary together, the total contribution to Π_2 from this integral is, including the amplitude factors,

$$- \frac{(\mu-1)(2\mu-3)}{2(\mu-2)} \left[\hat{\Phi}(\mu) + \hat{\Psi}^2(\mu) - \frac{8(2\mu-3)}{(\mu-1)(\mu-2)\eta_1} - \frac{2}{(\mu-2)^2} \right] \eta_1^2 \tag{4.4}$$

Although the function within the brackets is different from (4.3) its ϵ-expansion is similar in that it begins with $\zeta(5)$.

Finally, it is an elementary exercise now to substitute the values for these integrals into (4.1) to find

$$\begin{aligned} \omega_2 \ = \ \eta_1^2 \Bigg[& \frac{3(3\mu-4)(\mu-1)\hat{\Theta}(\mu)}{2(\mu-2)} - \frac{(2\mu-3)(\mu-1)\hat{\Psi}^2(\mu)}{(\mu-2)} + \frac{4(2\mu-3)^2}{(\mu-2)^2\eta_1} \\ & - \frac{(4\mu^3 - 24\mu^2 + 50\mu - 35)(\mu-1)\hat{\Psi}(\mu)}{(\mu-2)^2} - \frac{(2\mu-3)(\mu-1)\hat{\Phi}(\mu)}{(\mu-2)} \end{aligned}$$

$$+ \quad \frac{(16\mu^7 - 192\mu^6 + 956\mu^5 - 2536\mu^4 + 3825\mu^3 - 3212\mu^2 + 1328\mu - 182)}{2(\mu-1)(\mu-2)^3(2\mu-3)}\Bigg] \tag{4.5}$$

We draw attention to the fact that as in other models where there is a reordering of the Dyson equation to produce the β-function, [20], there is a term linear in η_1 which prevents the $O(1/N^2)$ correction being proportional to η_1^2.

5 Results and discussion.

Following the same procedure we used to determine $\gamma_\Phi(g)$ to $O(1/N^2)$, we can write down the β-function in a similar way. If we define the $O(1/N^2)$ β-function by

$$\beta(g) \;=\; (\mu-2)g \;+\; \frac{Ng^2}{2} \;+\; g^2\beta_1\left(\frac{gN}{2}\right) \;+\; \frac{g^2}{N}\beta_2\left(\frac{gN}{2}\right) \;+\; O\left(\frac{1}{N^3}\right) \tag{5.1}$$

then a similar derivation to that which produced (3.17) gives

$$\beta_2\left(\frac{gN}{2}\right) \;=\; 2\beta_1^2\left(\frac{gN}{2}\right) \;-\; 8 \;+\; \frac{1}{2}\int_0^{gN} dy \left[y\beta_1\left(\frac{y}{2}\right)\beta_1''\left(\frac{y}{2}\right) \;-\; \frac{2\Omega_2(y)}{y^2}\right] \tag{5.2}$$

where

$$\begin{aligned}
\Omega_2(y) \;=\; & 4\Bigg[(y-2)(y-1)y\left(\Psi^2(y) + \Phi(y) - \left(\frac{2}{y} + \frac{y^2}{(y-1)}\right)\Psi(y) - \frac{3}{4}\left(3 - \frac{1}{(y-1)}\right)\Theta(y)\right) \\
& - \frac{y(4y^2-7y+4)}{(y-2)} - \frac{y^2}{(y-1)} \;-\; \frac{8}{y} + (y-2)(y^3+y^2-6)\Bigg] G^2\left(\frac{y}{2}\right) \\
& + \frac{32(y-1)^2}{y} G\left(\frac{y}{2}\right)
\end{aligned} \tag{5.3}$$

and the new functions $\Phi(y)$ and $\Theta(y)$ are given by

$$\begin{aligned}
\Phi(y) \;&=\; \psi'(1-y) \;-\; \psi'\left(1+\frac{y}{2}\right) \;-\; \psi'\left(1-\frac{y}{2}\right) \;+\; \psi'(1) \\
\Theta(y) \;&=\; \psi'\left(1-\frac{y}{2}\right) \;-\; \psi'(1)
\end{aligned} \tag{5.4}$$

After expanding the integrand in powers of y and completing the integral, it is easy to verify that (5.3) reproduces the first four coefficients of (2.5) at the appropriate order in $1/N$ which gives us confidence that (4.5) is correct. Having therefore verified that our result is consistent with all known information we can now provide some new coefficients in the perturbative series in four dimensions beyond (2.5). For instance, if we represent the unknown $O(1/N^2)$ higher order contributions to the β-function by

$$\beta(g) \;=\; \frac{1}{2}(d-4)g \;+\; \frac{1}{2}(N+4)g^2 \;+\; \sum_{r=2}^{\infty}(a_r N + b_r)N^{r-2}g^{r+1} \tag{5.5}$$

then we can deduce

$$\begin{aligned}
a_5 \;&=\; \frac{1}{16} \;-\; \frac{\zeta(3)}{8} \\
a_6 \;&=\; \frac{1}{40} \;+\; \frac{\zeta(3)}{20} \;-\; \frac{3\zeta(4)}{40}
\end{aligned}$$

$$b_5 = \frac{25}{24} + \left(\frac{13}{8} - \frac{9}{4}\zeta(3)\right)\zeta(3) - \frac{27}{16}\zeta(4) + \frac{39}{2}\zeta(5) - \frac{75}{8}\zeta(6)$$

$$b_6 = \frac{33}{40} - \left(\frac{11}{20} - \frac{51}{10}\zeta(3) + \frac{27}{10}\zeta(4)\right)\zeta(3) + \frac{27}{40}\zeta(4) - \frac{57}{5}\zeta(5) - \frac{153}{8}\zeta(6) - 11\zeta(7) \tag{5.6}$$

The five loop coefficients, which are new, will provide a useful crosscheck on future full perturbative calculations in $\overline{\mathrm{MS}}$ in this model.

Ordinarily once an exponent has been checked for consistency with perturbation theory one can gain additional information about the same exponent in the three dimensional theory since the expression is usually valid in d-dimensions. However, attempting this for (4.5) one finds that ω_2 diverges as $\mu \to 3/2$. This is due to non-cancelling singularities arising from the functions $\hat{\Phi}(\mu)$, $\hat{\Psi}^2(\mu)$ and the final term. If one compares this to the situation in $O(N)$ ϕ^4 theory this is somewhat unexpected since its ω_2 is well behaved in three dimensions, [21, 22]. For (4.5) one can trace the source of the terms which are divergent in three dimensions to the original Feynman integrals. It turns out that they arise solely from the four loop integral of fig. 2. In the calculation of [21] there is an additional singularity from the graph which is a non-planar version of the four loop graph of fig. 2. Its contribution in $O(N)$ ϕ^4 theory precisely cancels the singularity from the planar four loop graph. As we have already noted the non-planar topology does not arise in the Wess-Zumino model due to chirality and it is its absence here which leads to a singular ω_2. In making this comparison it should be pointed out that whilst the explicit μ-dependent values of the analogous four loop integral of fig. 2 will in general be different in both models, each will have the same combination of divergent functions, $\hat{\Phi}(\mu)$ and $\hat{\Psi}^2(\mu)$, with the same residues as $\mu \to 3/2$.

In producing expressions for the β-function like (3.17) and (5.2) we have so far assumed that they are always evaluated by first expanding the integrand in a power series about the lower end of the integration range. Each term of the series is then integrated separately. However, we can also regard these expressions as being a summation of part of the full (unknown) perturbative series for $\beta(g)$. Although the explicit integration cannot be performed even for $\beta_1(gN/2)$ we can at least determine some information on the range of validity of these expressions and how the subsequent corrections affect it. As was pointed out in [9] the obstruction to the resummation at $O(1/N)$ is given by the first singularity in the integrand of $\beta_1(gN/2)$. This was found to be at $g = 3/N$ due to the singularity in the numerator Γ-function of $G(y/2)$, (3.13). Although this is also divergent at $y = 2$ there is a compensating pole in the denominator of $G(y/2)$ at the same point. In other words,

$$G(y) \underset{y \to 3/2}{\sim} \frac{1}{3\pi^2}\frac{1}{(2y-3)} \tag{5.7}$$

Therefore to $O(1/N)$ $\beta(g)$ diverges logarithmically to $+\infty$ as $g \to 3/N$. Moreover, it is also positive in this interval. Repeating this analysis now when the explicit expression for $\beta_2(gN/2)$ is taken into account we observe that the first singularity is now at $g = 1/N$ It arises purely from the $\Psi^2(y)$ and $\Phi(y)$ terms of $\Omega_2(y)$ and in particular

$$\Omega_2(y) \underset{y \to 1}{\sim} -\frac{128}{\pi^4}\frac{1}{(y-1)} \tag{5.8}$$

which again leads to a logarithmic singularity in $\beta(g)$ but now at $g = 1/N$ and tending to negative infinity. Therefore the effect of including the higher order corrections is that the obstruction to the resummation moves closer to the origin. Moreover, this singularity is also related to the absence of a finite value for ω_2 in the three dimensional model. This is simply because the leading order relation between the coupling and d is $\mu = 2 - y/2$. It is also interesting to compare these results with the analogous ones for the $O(N)$ ϕ^4 β-function at $O(1/N^2)$. From the expressions for ω_1 and ω_2 given in [21, 22] and the perturbative β-function of [24], we find that β_1 has a pole for $y = 15$, whereas β_2 is singular for $y = 9$. Again, the pole in β_2 stems from the (analogous) $[\Psi^2(y) + \Phi(y)]$ term. The reason why the coupling range is so much bigger in the ϕ^4 model is simply that the coefficients of the one-loop terms (which determine the arguments of the functions G, Ψ, Φ and Θ) are $1/2$ and $1/6$ in the Wess-Zumino and ϕ^4 β-functions respectively. Moreover the signs of corresponding integrand poles are the same as (5.7) and (5.8). Therefore in both models the radius of convergence of the large N β-function decreases when higher order corrections are included. Further, at $O(1/N^2)$ a spurious fixed point emerges since both models are non-asymptotically free and hence their β-functions are positive for very small couplings. In explicit perturbative calculations similar zeros can also occur when higher order terms are successively added. Since their appearance fluctuates with the number of terms included in the series we would expect that the zero at $O(1/N^2)$ would disappear if the $O(1/N^3)$ correction was included. In this case it also would be interesting to ascertain where the first integrand pole occurs.

We conclude with several comments. Although as we have discussed there is not a full equivalence between the $O(N)$ Wess-Zumino model and the $O(N)$ supersymmetric σ model we can make some observations on the numerology of the wave function exponent in both models and examine the effect supersymmetry has on η_2. By this we mean the presence of rational and irrational coefficients that appear in the corresponding renormalization group function. For example, the β-function of the Wess-Zumino model clearly contains rationals and the Riemann zeta series. Likewise if one expands η_2 to higher order rational numbers will occur. Although this may not appear surprising, the ϵ-expansion of η_2 for the $O(N)$ supersymmetric σ model near $d = 2 - \epsilon$ dimensions, which is the same as the result for the Wess-Zumino model, yields only the series $\zeta(n)$, $n \geq 3$, beyond two loops to all orders in perturbation theory. The β-function at $O(1/N^2)$ has the same property, [13]. By contrast the bosonic $O(N)$ σ model has rational coefficients at two and higher loops in its renormalization group functions, [3, 2], and the imposition of supersymmetry renders the cancellation of the rationals by fermionic partner graphs. Naively one might expect a similar property for four dimensions. However, in rewriting the $\psi(x)$ functions of (3.6) to ensure their ϵ-expansion has only $\zeta(n)$ terms now with respect to $d = 4 - 2\epsilon$, one introduces functions whose expansion will produce rationals. This arises from the iteration of the Hartree Fock approximation and, at least for the Wess-Zumino β-function, we have shown at $O(1/N^2)$ that the rational coefficients in the four dimensional result also arise from a similar iteration. The higher order graphs included here involve only $\zeta(n)$'s in their expansion as we emphasised in their computation. Indeed in light of these remarks several questions arise as to whether the imposition of another supersymmetry on this model would reduce the renormalization group functions of that model to rational at one loop only and irrational thereafter. Also, it is natural to question whether *all* the higher order graphs which remain in the Wess-Zumino model after the chirality condition has been imposed and the D-algebra performed yield integrals whose ϵ-expansion involve irrationals or non-zeta transcendentals as occurs in the $O(N)$ ϕ^4 theory, [25, 21]. One area where such issues could be addressed further is in the calculation of higher order corrections to some of the other critical exponents. We would hope to return to this issue later.

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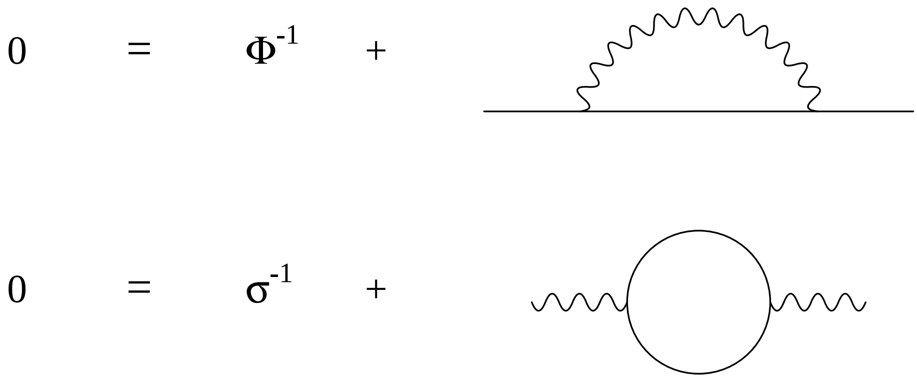

Fig. 1. Leading order Schwinger Dyson equations.

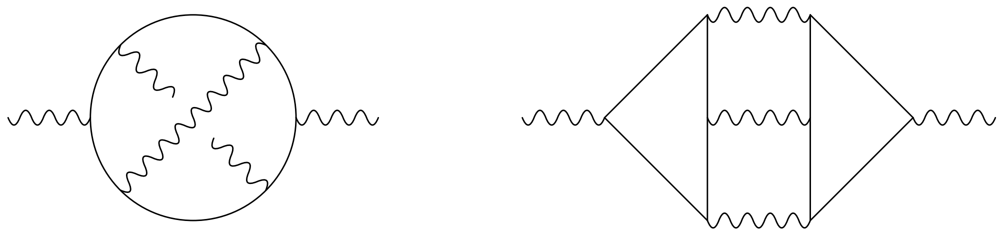

Fig. 2. Additional graphs for the σ Dyson equation.

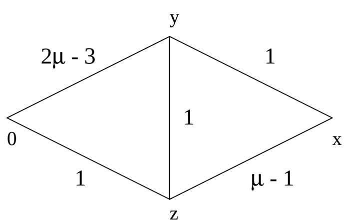

Fig. 3. Intermediate integral in the calculation of the non-planar graph.

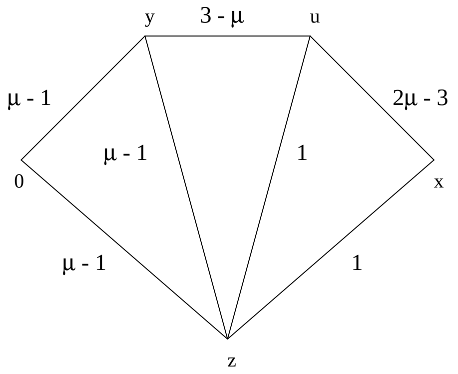

Fig. 4. Equivalent three loop integrals.

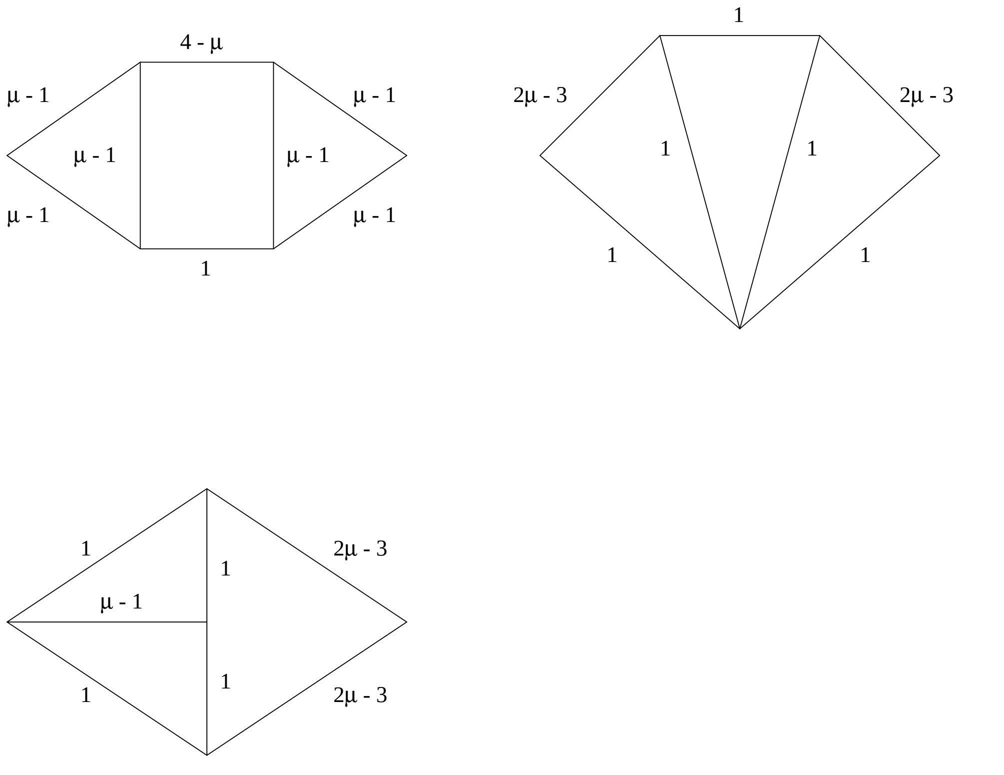

Fig. 5. Intermediate integrals in the calculation of the four graph with insertion on the central line.