Markov Denumerable Process and Queue Theory

Thesis submitted in accordance with the requirements of
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by

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DECLARATION

I certify that this work has not been accepted in substance for any degree, and is not concurrently submitted for any degree other than that of Doctor of Philosophy (PhD) of University of Liverpool. I also declare that this work is the result of my own research work except where otherwise stated.

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Abstract

In this thesis, we study a modified Markovian batch-arrival and bulk-service queue including finite states for dependent control. We first consider the stopped batch-arrival and bulk-service queue process Q^*, which is the process with the restriction of the state-dependent control. After we obtain the expression of the Q^*-resolvent, the extinction probability and the mean extinction time are explored. Then, we apply a decomposition theorem to resume the stopped queue process back to our initial queueing model, that is to find the expression of Q-resolvent. After that, the criteria for the recurrence and ergodicity are also explored, and then, the generating function of equilibrium distribution is obtained. Additionally, the Laplace transform of the mean queue length is presented. The hitting time behaviors including the hitting probability and the hitting time distribution are also established. Furthermore, the busy period distribution is also obtained by the expression of Laplace transform. To conclude the discussion of the queue properties, a special case that $m = 3$ for our queueing model is discussed.

Furthermore, we consider the decay parameter and decay properties of our initial queue process. First of all, similarly we consider the case of the stopped queue process Q^*. Based on this q-matrix, the exact value of the decay parameter λ_C is obtained theoretically. Then, we apply this result back to our initial queue model and find the decay parameter of our initial queueing model. More specifically, we prove that the decay parameter can be expressed accurately. After that, under the assumption of transient Q, the criteria for λ_C-recurrence are established. For λ_C-positive recurrent examples, the generating function of the λ_C-invariant measure and vector are explored. Finally, a simple example is provided to end this thesis.

Contents

Chapter 1

Introduction

The background of the continuous-time Markov chains, the Markov queue processes and the decay parameters will be introduced to start this thesis. Then, the main queueing model, which is mainly discussed throughout this thesis, will be presented. Additionally, I will briefly illustrate the structure of the thesis by the end of the chapter.

1.1 Background

How many fields of science or engineering can you enumerate? Which is unconcerned about probabilities and uncertainties? In fact, even everyday life cannot be explained clearly without randomness. The Markov process is such a variety of tool which clearly demonstrates most fundamental models of stochastic phenomenon. This significant concept was introduced by A. A. Markov in 1906, with only study on the simple cases of a finite number of states theoretically. The cases of countable infinite state spaces were launched by A. N. Kolmogorov in 1936. More and more general models such as queueing theory and branching process have been discussed by many mathematicians since then. Thus, the development of Markov process is playing a critical role in most scientific researches such as applied probabilities and statistics.

The continuous-time Markov chain (regarded as Markov chain throughout this thesis for simplicity) is one of the main parts of Markov processes and has a huge number of applications. For example, Brownian motion, which is the primary model applied in finance, is closely related to the Markov chain for they have many similar properties. The first former introducing of this concept was made by A. N. Kolmogorov (1931). In his study, he found that the transition probability law among the states of a Markov process follows either one of two differential equations, which are called the Kolmogorov backward and forward equations. This significant discovery promoted more and more mathematicians devoting themselves to the study of Markov chains. Recent two decades, Markov chains have shown their huge power in many areas of science and engineering with real practices and applications.

One of the most significant applications of Markov chains is the Markov queues. The first research of queueing theory was made by A. K. Erlang in 1909 regarding telephone conversations. Queueing theory is considered as an influential subject of mathematical sciences since the birth-death process is introduced by W. Feller in 1930's. In 1950's, D. G. Kendall applied the Markov chain into the queue behavior, which is a significant development of queueing theory. In fact, many real cases, i.e. bulk-internet block and the application of a season ticket of a football club, could be considered as a queueing process.

In recent years, Markovian queue is more and more important both in the general queueing theories (for instance, S. Asmussen (2003), D. Gross and C. M. Harris (1985), L. Kleinrock (1975) and J. Medhi (1991)) and in the application of Markov chains (for instance, W. Anderson (1991), K. L. Chung (1967), Z. T. Hou and Q. F. Guo (1988) and X. Yang (1990)). During their studies, the idea of state-dependent control of arrival and service was established. The negative arrival, which is to reduce the number of people in the system, is a quite useful concept introduced by E. Gelenbe (1991) and E. Gelenbe, P. Glynn and K. Sigman (1991). Since then, many other authors have followed, including N. Bayer and O. J. Boxma (1996), P.G. Harrison and E. Pitel (1993), W. Henderson (1993) and G. Jain and K. Sigman (1996). Furthermore, P. R. Parthasarathy and B. Krishna Kumar (1991) discussed the queueing model with the state dependent control at the state 0, which means that the arrival distribution is arbitrary when nobody is in the queue. Then A. Y. Chen and E. Renshaw (1997) (2004) established the possibility to let the queue system idle(i.e. nobody is in the queue) at any time. However, all the models discussed before are only the simple queue (i.e. only one person arrives or leaves at the same moment), which eliminates the cases such practical cases as the waiting for the lift and the arrival of the passengers in the aircraft. Since M. F. Neuts (1979) introduced versatile Markovian arrival processes by using several kinds of batch-arrival process and M. L. Chaudhry and J. G. C. Templeton (1983) discussed the first course of bulk queues, the theory of batch arrival and bulk service have been well developed until now. For example, we can see the most recent result from C. Armero and D. Conesa (2000), R. Arumuganathan and K. S. Ramaswami (2005), S. H. Chang, D. W. Choi and T. S. Kim (2004), D. Fakinos (1991), L. Srinivasan, N. Renganathan and R. Kalyanaraman (2002), U. Sumita and Y. Masuda (1997) and P. V. Ushakumari and A. Krishnamoorthy (1998).

The decay parameter corresponding with the invariant measures and vectors is one of the main theoretical part in Markov chains. A. M. Yaglom (1947) first established the idea to demonstrate the long-term behavior, by using a kind of conditional distribution, called quasi-stationary distribution. J. F. C. Kingman (1963) first showed the existence of decay parameter, which will be defined in Chapter 2. Since then, many scholars such as D. C. Flaspohler (1974), P. K. Pollett (1988), J. N. Darroch and E. Seneta (1967), M. Kijima (1993), M. G. Nair and P. K. Pollett (1993), R. L. Tweedie (1974) and E. A. Van Doorn (1985) (1991), have applied this significant discovery into many researches.

In view of this, this thesis is still only consider bulk queues, and the aim is to discuss both the basic properties and the decay properties of the model incorporating state-dependent control of any finite number (denote m ($m \geq 2$) in the thesis) of people in the system and no more than m people can be served at the same moment.

1.2 The Main Queueing Model of The Thesis

The model we discuss in this thesis is presented by a q-matrix denoted by $Q = \{q_{ij},\ i, j \geq 0\}$ of transition rates which is written as

$$Q = Q^* + Q^{(0)}, \tag{1.2.1}$$

where $Q^* = \{q^*_{ij},\ i, j \geq 0\}$ and $Q^{(0)} = \{q^{(0)}_{ij},\ i, j \geq 0\}$ are both conservative q-matrices, whose elements are given by

$$q^*_{ij} = \begin{cases} b_{j-i+m} & \text{if } j \geq i-m, i \geq m, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2.2}$$

and

$$q^{(0)}_{ij} = \begin{cases} q_{ij} & \text{if} \quad i < m, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2.3}$$

i.e.

$$Q = \begin{bmatrix} q_{00} & q_{01} & q_{02} & \cdots & q_{0m} & \cdots \\ q_{10} & q_{11} & q_{12} & \cdots & q_{1m} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ q_{m-1,0} & q_{m-1,1} & q_{m-1,2} & \cdots & q_{m-1,m} & \cdots \\ b_0 & b_1 & b_2 & \cdots & b_m & \cdots \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & \cdots \\ 0 & 0 & b_0 & \cdots & b_{m-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{1.2.4}$$

Here, $b_j (j \geq 0)$ must meet the conditions of

$$b_0 > 0,\ b_j \geq 0\ (j \neq m), \quad \sum_{j=m+1}^{\infty} b_j > 0, \text{ and } -b_m = \sum_{j \neq m} b_j < +\infty. \tag{1.2.5}$$

and $0 \leq -q_{ii} = \sum_{j \neq i} q_{ij} < +\infty$.

The sequence $\{b_{m+1}, b_{m+2}, \cdots\} \equiv \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$ are the arriving rates of a compound Poisson process. λ_1 means the rate of 1 person arriving at the same moment, λ_2 means the rate of 2 people arriving at the same time, and so on. $\{b_0, b_1, \cdots, b_{m-1}\}$ are the service rates with

b_{m-1} :	The service rate for a single customer;
b_{m-2} :	The service rate for a couple of customers (served together);
$\vdots$	$\vdots$
b_0 :	The service rate for m customers at the same time.

Thus, in this queueing model, the maximum of m people can be served at the same time but no limit for arrival. Q^*, which is named the stopped bulk-arrival and bulk-service generator, describes the general batch queueing model and Q_0 represents the state-dependent control.

It is recognized that the first m rows of the matrix, i.e. the so-called state-dependent controls, mean intuitively that when the queue length is less than the service capacity m, some manager may move arbitrary number of customers to the queue in order to increase the service efficiency. This assumption undoubtedly represents a large number of cases in the real world. For example, if the seats are still clear in a bus, then perhaps the manager will let some other customers without seats to come. Therefore, it is important to consider such kind of models. On the other hand, it can be seen that introducing such state-dependent control will make our queueing model far more interesting. More specifically, as a result of including state-dependent control, neither the arrival times nor the service times will still be independent and identically distributed random variables. More than that, we do not even know whether they are stationary or not. It is well known that these two conditions are the basic assumptions for most of the queueing models, since nearly all the techniques and tools in discussing the properties of queueing models are no longer well defined without these assumptions.

Hence, in order to discuss such kind of Markov queueing model with bulk arrival and bulk service incorporating state-dependent control, we should rely particularly on a useful method introduced by A. Y. Chen and E. Renshaw (1990) (1993a) (1993b), which has a close relationship with the excursion and large deviation theory. The methods, especially for Markov queueing processes, branching processes and birth-death processes, were discovered in A. Y. Chen and E. Renshaw (1995) (1997) (2000) and E. Renshaw and A. Y. Chen (1997). In the study of A. Y. Chen, P. Pollett, J. P. Li and H. J. Zhang (2010), state-dependent control was considered in the model, which means that arbitrary arrivals and services are allowed when the queue system is almost empty (actually less or equal 2 people). In this thesis, I will extend it to more generalized models and find some new methodology.

1.3 Outline of the Thesis

This thesis will be concentrating primarily on the basic and theoretical study of the Markov process and queueing theory. It is undoubtedly that these results are very useful in the applications of the actual world, but these connections will be considered in the future.

Chapter Two will show the basic concepts, theories and properties of Markov chains and queue theory. This should be known by most of specialists and students in the field, but it is obbligato in analyzing our model.

In Chapter Three, we develop a method to calculate the resolvent of the minimal Q^*-transition function and then we apply the construction theorem to find the transition function of the basic Q-matrix. Moreover, we will discuss its recurrence property, equilibrium distribution, the extinction probability, the mean extinction time and queue length distribution.

The decay parameters and decay properties incorporating the invariant measures and quasi-stationary distributions will be discussed in Chapter Four.

The last chapter, Chapter Five, summarizes the main conclusions of the thesis and some related problems and applications which need to be solved in the future.

Chapter 2

Basic Results of Markov Queue

In this Chapter, I will briefly introduce the basic concepts of continuous-time Markov chains and its useful properties and results, which are well known. More specifically, all the definitions, propositions, lemmas and theorems in section 2.1 come from the Chapter 1, 2, 5 and 6 of W. Anderson (1991), while those in section 2.2 come from the Chapter III of S. Asmussen (2003). Additionally, I will extend it to queue theory for the preparation of my main work.

2.1 Continuous-time Markov Chains

Definition 2.1.1. *A continuous-time stochastic process* $\{X(t), t \in [0, \infty)\}$*, which is defined on a probability space* $(\Omega, \mathscr{F}, P)$*, with values in a countable set* E *(called the state space of the stochastic process), is called a continuous-time Markov chain if for any finite set* $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1}$ *of times, and the corresponding set* $\{i_1, i_2, \cdots, i_{n-1}, i, j\} \subseteq E$ *of states such that* $P\{X(t_n) = i, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} > 0$*, we have*

$$\begin{aligned} & P\{X(t_{n+1}) = j | X(t_n) = i, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\ = \; & P\{X(t_{n+1}) = j | X(t_n) = i\}. \end{aligned} \tag{2.1.1}$$

(2.1.1) is called the *Markov property*, means that the future behaviour is independent of the past when the process is certain at present. If for all $s, t \geq 0$ and all $i, j \in E$, the conditional probability $P\{X(t+s) = j|X(s) = i\}$ only depends on t and is independent with s, we can say that the process $\{X(t), t \in [0, \infty)\}$ is *homogeneous* (in this thesis, this condition is automatically assumed), or has the *stationary transition probabilities*. Therefore, $P\{X(t+s) = j|X(s) = i\} = P\{X(t) = j|X(0) = i\}$, and we can define the transition function of the process $p_{ij}(t)$:

$$p_{ij}(t) =: P\{X(t) = j|X(0) = i\}, \; i, j \in E, \; t \geq 0.$$

It is easily known that the finite-dimensional probabilities of the process $\{X(t), t \geq 0\}$, which are probabilities of the form $P\{X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\}$, where $0 \leq t_1 < t_2 < \cdots < t_n$ and $i_1, i_2, \cdots, i_{n-1}, i_n \in E$ are all expressed by the transition function $p_{ij}(t)$ and the initial probability distribution $p_i = P(X(0) = i)$ $(i \in E)$ of $X(0)$. According to the stationary property of the transition function, we have

$$\begin{aligned}
& P\{X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
= \; & P\{X(t_n) = i_n|X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
\cdot \; & P\{X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
= \; & P\{X(t_n) = i_n|X(t_{n-1}) = i_{n-1}\} \\
\cdot \; & P\{X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
= \; & p_{i_{n-1},i_n} P\{X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1\} \\
= \; & \cdots \\
= \; & \sum_{i_0} p_{i_0} \Pi_{m=1}^{n} p_{i_{m-1},i_m}(t_m - t_{m-1}), \qquad (2.1.2)
\end{aligned}$$

where $t_0 = 0$. Furthermore, a continuous-time Markov chain is uniquely decided by its transition function and given initial probability distributions by applying the general theory of Markov chains. Hence, we can focus on the transition function to discuss the process. Now, we give the properties that the transition function must have the following properties:

(i) $p_{ij}(t) \geq 0$ for all $t \geq 0$ and $i, j \in E$ and

$$\sum_{j \in E} p_{ij}(t) = \sum_{j \in E} P\{X(t) = j | X(0) = i\} = P\{X(t) \in E | X(0) = i\} \leq 1 \tag{2.1.3}$$

for all $i \in E$;

(ii) For all $i, j \in E$,

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

(iii) For all $s, t \geq 0$ and $i, j \in E$, we have, according to the Markov property,

$$\begin{aligned} p_{ij}(s+t) &= P\{X(s+t) = j | X(0) = i\} \\ &= \sum_{k \in E} P\{X(s+t) = j, X(s) = k | X(0) = i\} \\ &= \sum_{k \in E} P\{X(s+t) = j | X(s) = k, X(0) = i\} \\ &\quad \cdot P\{X(s) = k | X(0) = i\} \\ &= \sum_{k \in E} p_{ik}(s) p_{kj}(t); \end{aligned} \tag{2.1.4}$$

(iv) $\lim_{t \to 0} p_{ii}(t) = 1$ for all $i \in E$.

If the equality in (2.1.3) holds true, then this transition function is honest. The equation (2.1.4) is called Chapman-Kolmogorov equation. (iv) means that the transition function is standard, which is the only assumption considered in the thesis. In fact, this kind of Markov chain is the Markov jump process with the parameter t. Based on the definition of the transition function, we now discuss some of its basic properties.

Proposition 2.1.1. *Let $p_{ij}(t), i, j \in E$ be a transition function, then*

(1) $p_{ii}(t) > 0$ for all $t \geq 0$ and $i \in E$. If $i \neq j$ and $i, j \in E$, and if $p_{ij}(t) > 0$ for some $t > 0$, then $p_{ij}(s) > 0$ for all $s > 0$;

(2) If $p_{ii}(t) = 1$ for some $t > 0$, then $p_{ii}(t) = 1$ for all $t \geq 0$.

Proposition 2.1.2. *Let $p_{ij}(t), i, j \in E$ be a standard transition function. We have*

(1) $q_i = \lim_{t \to 0}[1 - p_{ii}(t)]/t$ exists (or might be $+\infty$), i.e. $p'_{ii}(0) = -q_i$;

(2) $p_{ii}(t) \geq e^{-q_i t} \geq 1 - q_i t$ for all $t \geq 0$;

(3) $q_i = 0$ if and only if $p_{ii}(t) = 1$ for all $t \geq 0$.

If $q_i = -q_{ii} = 0$, then the state i is an *absorbing state*, which means that the process will no longer move whenever it is in the state i. If $q_i < +\infty$, then the state i is said to be stable.

Proposition 2.1.3. *Let $p_{ij}(t), i, j \in E$ be a transition function, and let i be a stable state. Then we can prove that $q_{ij} = p'_{ij}(0)$ exists and is finite for all $j \in E$.*

Now we can define Q-matrix of the transition function, which is a significant and useful method in the study of Markov chains.

Definition 2.1.2. *Let $p_{ij}(t), i, j \in E$ be a transition function. If the i, jth element of the matrix Q is $q_{ij} = p'_{ij}(0)$, then this matrix is called the q-matrix of the transition function $p_{ij}(t)$.*

Based on the basic properties of the transition function, the matrix Q must have the property that $q_{ij} \geq 0$ if $i \neq j$ and $\sum_{j \neq i} q_{ij} \leq -q_{ii}$. The matrix Q is a *conservative* matrix if $\sum_{j \neq i} q_{ij} = -q_{ii}$. If for all $i \in E$, $q_i < +\infty$, then the Q matrix is called a *stable* q-matrix.

Again, let $p_{ij}(t), i, j \in E$ be a transition function, define

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t) dt, \lambda > 0, i, j \in E. \tag{2.1.5}$$

In other word, $r_{ij}(\lambda)$ is the Laplace transform of $p_{ij}(t)$. Based on the properties of the transition function and the Laplace transform, we have

Proposition 2.1.4. *The Laplace transform of $p_{ij}(t), i, j \in E$ must have the following properties:*

(i) $r_{ij}(\lambda) \geq 0$ for all $i, j \in E$ and $\lambda > 0$;

(ii) $\lambda \sum_{k\in E} r_{ij}(\lambda) \le 1$ *for all* $i \in E$ *and* $\lambda > 0$*;*

(iii) $r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu)\sum_{k\in E} r_{ik}(\lambda) r_{kj}(\mu) = 0$ *for all* $i, j \in E$ *and* $\lambda, \mu > 0$*;*

(iv) $\lim_{\lambda\to\infty} \lambda r_{ii}(\lambda) = 1$ *for all* $i \in E$ *and* $\lim_{\lambda\to\infty} \lambda r_{ij}(\lambda) = 0$ *for all* $i \ne j$ *and* $i, j \in E$*.*

In fact, $r_{ij}(\lambda)$ is called a resolvent function if all the conditions of Proposition 2.1.4 hold true. That is the Laplace transform of a transition function is a resolvent function, and both $r_{ij}(\lambda)$ and $p_{ij}(t)$ are honest simultaneously.

Therefore, if $r_{ij}(\lambda), i, j \in E$ is a resolvent function, then there is a unique transition function $p_{ij}(t), i, j \in E$ that $r_{ij}(\lambda), i, j \in E$ is its Laplace transform. In other word, in order to find the transition function of a q-matrix, calculating the resolvent function is quite enough. The following proposition illustrates some other properties of a transition function.

Proposition 2.1.5. *Let* $p_{ij}(t), i, j \in E$ *be a transition function, and if* i *is a stable state, then*

$$p'_{ij}(t) \ge \sum_{k\in E} q_{ik} p_{kj}(t), t \ge 0, j \in E, \tag{2.1.6}$$

$$p'_{ij}(t) \ge \sum_{k\in E} p_{ik}(t) q_{kj}, t \ge 0, j \in E. \tag{2.1.7}$$

(2.1.6) is called the backward inequality and (2.1.7) is called the forward inequality. If the q-matrix Q is conservative, then the equality of (2.1.6) holds, that is

$$p'_{ij}(t) = \sum_{k\in E} q_{ik} p_{kj}(t), t \ge 0, j \in E. \tag{2.1.8}$$

The following equation is the equality case of (2.1.7):

$$p'_{ij}(t) = \sum_{k\in E} p_{ik}(t) q_{kj}, t \ge 0, j \in E. \tag{2.1.9}$$

Both (2.1.8) and (2.1.9) can be expressed by the matrix form, i.e.

$$P'(t) = QP(t), \tag{2.1.10}$$

$$P'(t) = P(t)Q. \tag{2.1.11}$$

Here, (2.1.8) and (2.1.10) are called the Kolmogorov backward equation and (2.1.9) and (2.1.11) are called the Kolmogorov forward equation. Moreover, both equations can be expressed by the resolvent versions

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k \in E} q_{ik} r_{kj}(\lambda), \lambda > 0, j \in E, \tag{2.1.12}$$

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k \in E} r_{ik}(\lambda) q_{kj}, \lambda > 0, j \in E. \tag{2.1.13}$$

That is

$$\lambda R(\lambda) = I + QR(\lambda), \lambda > 0, \tag{2.1.14}$$

$$\lambda R(\lambda) = I + R(\lambda)Q, \lambda > 0. \tag{2.1.15}$$

In this thesis, the Kolmogorov forward equation is a key method in calculating the transition function of our queueing model.

In nearly all cases, it is very hard to obtain the transition function. However, it is much easier to get its Q matrix. So the problem is for a given q-matrix, does its transition function exist? If so, is the transition function unique?

Theorem 2.1.1. *Let Q be a stable q-matrix (no requirement about conservative). Then there exists a (perhaps not honest) transition function $p_{ij}(t), i, j \in E$ such that both the Kolmogorov backward and forward equations hold. Furthermore, this $p_{ij}(t)$ is the minimal solution, which means that for any other transition function $\bar{p}_{ij}(t), i, j \in E$ of Q, $p_{ij}(t) \leq \bar{p}_{ij}(t), i, j \in E, t \geq 0$.*

This minimal solution was structured by W. Feller (1940) by using either the backward integral recursion or the forward integral recursion, that is

$$p_{ij}^{(n)}(t) = \begin{cases} \delta_{ij}e^{-q_i t} & \text{if } n = 0 \\ p_{ij}^{(0)}(t) + \int_0^t e^{-q_i t} \sum_{k\neq i} q_{ik} p_{kj}^{(n-1)}(t-s)ds & \text{if } n \geq 1 \end{cases} \tag{2.1.16}$$

and

$$p_{ij}^{(n)}(t) = \begin{cases} \delta_{ij}e^{-q_j t} & \text{if } n = 0 \\ p_{ij}^{(0)}(t) + \int_0^t e^{-q_j t} \sum_{k\neq j} p_{ik}^{(n-1)}(t-s) q_{kj} ds & \text{if } n \geq 1 \end{cases} \tag{2.1.17}$$

respectively.

In order to memorize this mathematician, this solution is called the Feller minimal Q-function.

Now we want to find the criterion of the uniqueness of the Q-function.

Theorem 2.1.2. *The following statements are equivalent:*

(1) The Feller minimal Q-function is the only solution of the Kolmogorov backward equation.

(2) The equation $Qx = \lambda x, 0 \leq x \leq 1$, that is to say,

$$\sum_{j\in E} q_{ij}x_j = \lambda x_i, 0 \leq x_i \leq 1, i \in E, \tag{2.1.18}$$

has only trivial solution for some (and hence for all) $\lambda > 0$.

(3) The inequality $Qx \geq \lambda x, 0 \leq x \leq 1$, that is to say,

$$\sum_{j\neq i} q_{ij}x_j \geq (\lambda + q_i)x_i, 0 \leq x_i \leq 1, i \in E, \tag{2.1.19}$$

has only trivial solution for some (and hence for all) $\lambda > 0$.

(4) The equation $Qx = \lambda x, -1 \leq x \leq 1$, that is to say,

$$\sum_{j\in E} q_{ij}x_j = \lambda x_i, -1 \leq x_i \leq 1, i \in E, \tag{2.1.20}$$

has only trivial solution for some (and hence for all) $\lambda > 0$.

If Q is a conservative q-matrix, then the Feller minimal Q-function is the unique transition function if and only if any one of (1)—(4) holds.

A conservative q-matrix Q is called regular if any condition of (1)—(4) in Theorem 2.1.2 is true. In this case, the Feller minimal Q-function is honest and thus is the unique Q-function. Here, (2) is the most commonly used condition to prove the uniqueness throughout the previous studies. There is a similar test for uniqueness of the solutions of the Kolmogorov forward equation.

Theorem 2.1.3. *Assume that the Feller minimal Q-function (Q is not necessarily conservative) is dishonest. Then the following two statements are equivalent:*

(1) The Feller minimal Q-function is the unique one following the Kolmogorov forward equation.

(2) The equation $yQ = \lambda y, y \in l^+$, that is to say,

$$\sum_{i \in E} y_i q_{ij} = \lambda y_j, y_j \geq 0, j \in E, \sum_{j \in E} y_j < +\infty, \tag{2.1.21}$$

has only trivial solution for some (and hence for all) $\lambda > 0$.

It is interested in discussing a regular q-matrix concerning the problems of absorption, recurrence and ergodicity. We have already said that i is an absorbing state if $q_i = 0$.

Definition 2.1.3. *Let $p_{ij}(t), i, j \in E$ be a standard transition function. Given any $i, j \in E$, we say that j can be reached from i if $p_{ij}(t) > 0$ for some (and hence for all) $t > 0$. We say that i and j communicate (denote $i \leftrightarrow j$) if i and j can be reached from each other. C is a communicating class if for any $i, j \in C$, $i \leftrightarrow j$.*

However, as generally it is only given by a q-matrix, it is necessarily to find the communicating class just in the q-matrix Q. In order to solve this problem, we first define the related jump chain (this is a discrete time

Markov chain) of Q:

$$p_{ij} = \begin{cases} q_{ij}/q_i & \text{if } i \neq j \text{ and } q_i > 0 \\ 0 & \text{if } i = j \text{ and } q_i > 0 \\ \delta_{ij} & \text{if } q_i = 0 \end{cases}.$$

Proposition 2.1.6. *Assume $i, j \in E, i \neq j$, then the following statements are equivalent.*

(1) j can be reached from i in the Feller minimal Q-function.

(2) j can be reached from i in the jump chain which is just defined.

Definition 2.1.4. *A state $i \in E$ is recurrent if*

$$\int_0^\infty p_{ii}(t)dt = +\infty,$$

and transient if

$$\int_0^\infty p_{ii}(t)dt < +\infty.$$

Moreover, the recurrent state i is positive recurrent if

$$\lim_{t \to +\infty} p_{ii}(t) > 0,$$

and null recurrent if

$$\lim_{t \to +\infty} p_{ii}(t) = 0.$$

It is easily proved that the recurrence and transience are class property, which means that if one state in a communicating class C is recurrent (positive or null) or transient, the same as any other state in C.

Proposition 2.1.7. *Let C be a communicating class. Suppose that Q is conservative on C. Then the following statements are equivalent.*

(1) C is recurrent for the Feller minimal Q-function $p_{ij}(t)$.

(2) For any state $k \in C$, the following equations

$$\sum_{i \in C} m_i q_{ij} \begin{cases} = 0 & \text{if } j \neq k, j \in C \\ \leq 0 & \text{if } j = k \end{cases} \quad m_i \geq 0, i \in C, m_k = 1, \tag{2.1.22}$$

have no solution except that the equivalent in (2.1.22) holds true.

(3) For any state $k \in C$, the following equations

$$\sum_{j \in C} q_{ij} y_j = 0, i \neq k$$

have no solution except that all the $y_j, j \in C$ are equal.

Theorem 2.1.4. *The limits $p_{ij}(\infty)$ exist for all $i, j \in E$. If $R = \{r_{ij}(\lambda), i, j \in E\}$ is the resolvent function of $p_{ij}(t)$, then we have*

$$p_{ij}(\infty) = lim_{\lambda \to 0} \lambda r_{ij}(\lambda). \tag{2.1.23}$$

Hence, if we can have some properties of the resolvent function, then it is a brief method by using (2.1.23) to calculate the extinction probability in many Markov models such as queue processes, branching processes and birth-death processes, which could avoid trying to find the transition function as it is quite difficult in most cases.

Definition 2.1.5. *Let $p_{ij}(t), i, j \in E$ be a transition function. The series $\{u_i : u_i \geq 0, i \in E\}$ is called an invariant measure if the following equation holds true.*

$$\sum_{i \in E} u_i p_{ij}(t) = u_j, \text{ for all } j \in E, t \geq 0, \tag{2.1.24}$$

or

$$uP(t) = u \text{ for all } t \geq 0 \tag{2.1.25}$$

in vector form. More specifically, $\boldsymbol{u}$ is regarded as an invariant distribution if $\sum_{i \in E} u_i = 1$.

Theorem 2.1.5. *Let $p_{ij}(t), i, j \in E$ be an irreducible transition function. Then this Markov process is positive recurrent if and only if there exists an invariant distribution of the transition function $p_{ij}(t)$.*

This invariant distribution (sometimes it might be called stationary distribution or steady-state distribution) is generally called the *limit distribution* or the *equilibrium distribution* (generally denoted as $\pi_j, j \in E$) of the Markov chain. Similarly, the most basic kind of invariant measure could also be defined.

Definition 2.1.6. *Let $p_{ij}(t), i, j \in C$ be the transition function, where C is a communicating class. If for all $t \geq 0$, two sets of strictly positive numbers $\{m_i, i \in C\}$ and $\{x_i, i \in C\}$ satisfy the following equations*

$$\sum_{i \in C} m_i p_{ij}(t) \leq e^{-\mu t} m_j, j \in C, \tag{2.1.26}$$

and

$$\sum_{j \in C} p_{ij}(t) x_j \leq e^{-\mu t} x_i, i \in C, \tag{2.1.27}$$

where μ is some fixed nonnegative number, then $\{m_i, i \in C\}$ (or $\{x_i, i \in C\}$) is called a μ-subinvariant measure (or vector) for the process $p_{ij}(t)$. Moreover, if the equality in (2.1.26) (or (2.1.27)) holds, then $\{m_i, i \in C\}$ (or $\{x_i, i \in C\}$) is called a μ-invariant measure (or vector). Furthermore, if $\mu = 0$, then they are generally called subinvariant or invariant measure (vector) respectively.

For a transient communicating class C of a Markov chain, it is quite normal to calculate the decay parameter and discuss its properties.

Theorem 2.1.6. *Assume that C is a communicating class of a Markov chain, denote $p_{ij}(t), i, j \in C$ be its transition function.*

(1) There exists a constant number $\lambda_C \geq 0$ such that for each $i, j \in C$,

$$\lim_{t \to +\infty} \frac{1}{t} log p_{ij}(t) = -\lambda_C. \tag{2.1.28}$$

(2) For each $i \in C$ and $t > 0$,

$$p_{ii}(t) \leq e^{-\lambda_C t}. \tag{2.1.29}$$

(3) There exists a constant positive number M_{ij} such that for each $i, j \in C$,

$$p_{ij}(t) \leq M_{ij} e^{-\lambda_C t}. \tag{2.1.30}$$

(4) $\lambda_C \leq inf_{i \in C} q_i$.

(5) The communicating class C is transient if $\lambda_C > 0$.

λ_C in Theorem 2.1.6 is often called the *decay parameter* of the communicating class C. Furthermore, λ_C is the minimum of the nonnegative numbers such that

$$\int_0^\infty p_{ii}(t) e^{at} dt = +\infty, i \in C.$$

In other words,

$$\int_0^\infty p_{ii}(t) e^{at} dt = \begin{cases} < +\infty & \text{if } a < \lambda_C \\ = +\infty & \text{if } a > \lambda_C \end{cases}. \tag{2.1.31}$$

If $a = \lambda_C$ in (2.1.31), then both cases might be happened.

Definition 2.1.7. *The state $i (i \in C)$ is λ_C-recurrent if*

$$\int_0^\infty p_{ii}(t) e^{\lambda_C t} dt = +\infty,$$

and λ_C-transient if

$$\int_0^\infty p_{ii}(t) e^{\lambda_C t} dt < +\infty.$$

The same as the positive and null recurrence, we can define λ_C-positive and λ_C-null recurrence.

Definition 2.1.8. *If the state i is λ_C-recurrent, then it is λ_C-positive recurrent if*

$$\lim_{t\to\infty} e^{\lambda_C t}p_{ii}(t) > 0,$$

and λ_C-null recurrent if

$$\lim_{t\to\infty} e^{\lambda_C t}p_{ii}(t) = 0.$$

Proposition 2.1.8. *If there exists a μ-subinvariant measure or μ-subinvariant vector for $p_{ij}(t), i, j \in C$, then $\mu \le \lambda_C$.*

Proposition 2.1.9. *Let $p_{ij}(t), i, j \in C, t \ge 0$ be the transition function, where C is a communicating class with decay parameter $\lambda_C \ge 0$. Then there exists λ_C-subinvariant measures and λ_C-subinvariant vectors for $p_{ij}(t)$ on C.*

Proposition 2.1.10. *Let $p_{ij}(t), i, j \in C, t \ge 0$ be the transition function and have the decay parameter λ_C on C. Assume that C is λ_C-recurrent, then both λ_C-subinvariant measure and λ_C-subinvariant vector are unique up to constant multiples. Moreover, both measure and vector are λ_C-invariant measure and vector.*

Similarly, since it is difficult to calculate the transition function, we define the invariant measure and vector by connecting with the q-matrix.

Definition 2.1.9. *A set of strictly positive numbers $\{m_i, i \in C\}$ (or $\{x_i, i \in C\}$) is called a μ-subinvariant measure (or vector) for Q on C if for all $j \in C$,*

$$\sum_{i\in C} m_i q_{ij} \le -\mu m_j, \tag{2.1.32}$$

or

$$\sum_{j\in C} q_{ij} x_j \le -\mu x_i. \tag{2.1.33}$$

Moreover, if the equality in (2.1.32) (or (2.1.33)) holds, then $\{m_i, i \in C\}$ (or $\{x_i, i \in C\}$) is called a μ-invariant measure (or vector).

Proposition 2.1.11. *Let Q be an q-matrix and $p_{ij}(t)$ be its Feller minimal Q-function. For a set of strictly positive numbers $\{m_i, i \in C\}$, then the following two statements are equivalent.*

(1) $\{m_i, i \in C\}$ is a μ-subinvariant measure (or vector) for $p_{ij}(t)$ on C;

(2) $\{m_i, i \in C\}$ is a μ-subinvariant measure (or vector) for Q on C.

Furthermore, the quasi-stationary distribution is needed to be calculated so as to discuss the long-term behaviors.

Definition 2.1.10. *Let $P(t) = \{p_{ij}(t) : i, j \geq 0\}$ be the transition function and C a communicating class of E. Define*

$$\bar{p}_{ij}(t) =: P\left\{X(t) = j | X(0) = i, X(t) \in C\right\} = \frac{p_{ij}(t)}{\sum_{k \in C} p_{ik}(t)}, \ i, j \in C,$$

If its decay parameter λ_C is strictly positive, then $\{\bar{p}_{ij}(\infty)\}$ exist and strictly positive, which is called a quasi-stationary distribution.

In general, an irreducible and positive recurrent transition function $p_{ij}(t)$, $i, j \in E$ is called ergodic. In other words, an ergodic transition function must possess an equilibrium distribution $\pi_j, j \in E$. Furthermore, an ergodic transition function must have

$$\lim_{t \to \infty} \sum_{j \in E} |p_{ij}(t) - \pi_j| = 0 \text{ for all } i \in E. \tag{2.1.34}$$

Definition 2.1.11. *An ergodic transition function $p_{ij}(t), i, j \in E$ is called strongly ergodic if*

$$\| P(t) - \Pi \| =: \sup_{i \in E} \sum_j |p_{ij} - \pi_j| \to 0 \ \textit{as}\ t \to \infty. \tag{2.1.35}$$

Definition 2.1.12. *An ergodic transition function $p_{ij}(t), i, j \in E$ is called exponentially ergodic if there exist a positive number α and C_{ij} such that*

$$|p_{ij}(t) - \pi_j| \leq C_{ij} e^{-\alpha t} \ \textit{for all}\ t \geq 0, i, j \in E. \tag{2.1.36}$$

For more details of the continuous-time Markov chains, a scholar can refer to the books, for example, W. J. Anderson (1991) and S. Asmussen (2003).

2.2 Markov Queue Theory

Whatever the queueing problems take, only three features can describe the simple structure of a queue: the *input* or the *arrival process* that is the process of the customers arriving to the queue; the *service numbers* and the *service time process*, that is the facilities to serve customers in the queue (These two features can be dealt with separately since they are totally independent); and the *queue discipline*, which decides the rules to run the queue system. However, to describe these features is a complicated and lengthy progress. In 1953, D. G. Kendall use a simple notation system to describe it. In this notation, a queueing process is simply denoted by $b/d/m$, where b denotes the format of the inter arrival distribution, d denotes the format of the service time distribution, and m denotes the number of servers. Generally, the values of b and d are:

M: The exponential distribution or the Poisson distribution. (M is the initial letter of Markovian.)

D: The distribution is certain at some point of $(0, +\infty)$. (D is the initial letter of deterministic. In the real world, usually $d = 1$.)

E_k: The Erlang distribution with k stages, that is the sum of kth independent exponential distributions with the same parameter.

H_k: The hyperexponential distribution with k stages, whose probability density function follows

$$f_X(x) = \sum_{i=1}^{k} f_{Y_i}(x) p_i,$$

where Y_i follows an exponential distribution with the parameter λ_i, and p_i is the probability that this hyperexponential distribution takes the form of Y_i.

PH: The generalized phase-type distribution.

G or GI: An arbitrary distribution. (G refers to general, GI refers to general independent).

For example, $M/D/1$ refers to the queue process with the Poisson inter-arrivals, the deterministic service time and only one server. There are some types of queue disciplines:

FIFO: That means, the first customer is first served and customers are served followed the order of the arrivals. This is the most common queue discipline. Hence, this discipline is the assumption in most literature. Our queueing model in the thesis follows this discipline. (FIFO=first in, first out.)

LIFO: That means, the last customer is first served and customers are served followed the reverse order of the arrivals. (LIFO=last in, first out.)

SIRO: That is a server chooses an arbitrary customer randomly. For instance, if we have some number of problems to solve, then we choose any one to serve first. (SIRO=service in random order.)

PS: That is when n customers in the queue system, the server serves each at the rate of $1/n$, (PS=processor sharing.)

RR: That is to say, each customer is served for at most a fixed time and goes to the back of the queueing line (if not having completed served). (RR=Round Robin.)

There are still some notations to be used in queue theory:

Q_t: That is the *queue length* in the system at time $t, t \geq 0$. In Markov chain, this is denoted $X_t, t \geq 0$.

W_n: That is to say, the *waiting time* of the nth customer in the queue, i.e. the time interval starting from the time of the arrival of the customer to the time that the customer is served.

V_t: The *workload* of the system at the time t, which means the total remaining time to make the queue system empty at the time t.

One of the most classical models of Markov processes is the *birth-death process*, whose q-matrix $\{q_{ij}, i, j \in E = N\}$ is given by

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ -\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & -\mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & -\mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (2.2.1)$$

where $\lambda_i, i \geq 0$ are the *birth intensities* and $\mu_i, i \geq 0$ are the *death intensities*. The most familiar example is the division and death of a group of cells. This example is a *linear birth-death process* with $\lambda_i = i\lambda$ and $\mu_i = i\mu$, which means that each cell splits independently of any other with rate λ and dies with rate μ and for i cells in the group, the rate of increase (or decrease) one are i times of λ (or μ).

If $\lambda_i \equiv \lambda$ and $\mu_i \equiv \mu$ $(i \geq 0)$ in (2.2.1), then this birth-death process is the simplest queueing process $M/M/1$, which is the most fundamental process to be discussed. Here, λ and μ refer to the inter-arrival rate and the service rate.

Proposition 2.2.1. *Let $\{X_t, t \geq 0\}$ be an $M/M/1$ queue process with the inter-arrival rate λ and the service rate μ. Then $\{X_t, t \geq 0\}$ is recurrent if and only if $\lambda/\mu \leq 1$. Moreover, $\{X_t, t \geq 0\}$ is ergodic if and only if $\lambda/\mu < 1$.*

In the ergodic case, there exists a equilibrium distribution $\pi_i, i \in N$ according to the Theorem 2.1.5. Furthermore, it is easily calculated that

$\pi_i, i \in N$ is geometric distributed, i.e.

$$\pi_i = \lim_{t \to \infty} P(X_t = i) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i, i = 0, 1, 2, \cdots$$

If $\lambda_i \equiv \lambda$ and $\mu_i = i\mu$ $(i \geq 0)$ in (2.2.1), then this process is the $M/M/\infty$ queue process. In this queueing model, the sojourn time in the queue system of each customer is independent with all other customers and identically exponentially distributed with the rate μ. One example for this process is the immigration-death process.

Proposition 2.2.2. *For any values of λ and μ in the $M/M/\infty$ queue, the process is ergodic. Moreover, the equilibrium distribution $\boldsymbol{\pi}$ follows the Poission distribution with the parameter λ/μ, that is*

$$\pi_i = e^{-\lambda/\mu}\frac{(\lambda/\mu)^i}{i!}, i = 0, 1, 2, \cdots$$

For details of $M/M/\infty$ models, please see the literatures, for example, J. Preater (2002).

If $\lambda_i \equiv \lambda$ and $\mu_i = (i \wedge m)\mu$ $(i \geq 0)$ in (2.2.1), then this process is called the $M/M/m$ queue process. In this process, $(i \wedge m)$ is the number of servers working when the process is at the state i.

Proposition 2.2.3. *Let $\{X_t, t \geq 0\}$ be an $M/M/m$ queue process with the inter-arrival rate λ and the service rate μ. Then $\{X_t, t \geq 0\}$ is ergodic if and only if $\lambda < m\mu$.*

Similarly, in the case of ergodic, the steady state distribution $\boldsymbol{\pi}$ is that

$$\pi_i = \begin{cases} \frac{1}{S}\frac{(\lambda/\mu)^i}{i!}, & i = 0, 1, \cdots, m \\ \frac{1}{S}\frac{(\lambda/\mu)^m}{m!}\left(\frac{\lambda}{m\mu}\right)^{i-m}, & i = m, m+1, \cdots \end{cases},$$

where

$$
\begin{aligned}
S &= 1+\sum_{i=1}^{\infty}\frac{\lambda_0\cdots\lambda_{i-1}}{\mu_1\cdots\mu_i} \\
&= \sum_{i=0}^{m-1}\frac{(\lambda/\mu)^i}{i!}+\frac{(\lambda/\mu)^m}{m!}\sum_{i=0}^{\infty}\left(\frac{\lambda}{m\mu}\right)^i \\
&= \sum_{i=0}^{m-1}\frac{(\lambda/\mu)^i}{i!}+\frac{(\lambda/\mu)^m}{m!}\left(1-\frac{\lambda}{m\mu}\right)^{-1}.
\end{aligned}
$$

Up to now, we have only discussed the infinite state space. However, in many cases the queue system has a limited number of customers, or it will be more than its capacity. For example, a barbershop cannot contain too many people and thus all new customers will not enter in the room when it is full. For a simple case, let $\{X_t, t \geq 0\}$ be an $M/M/1$ queue process with the inter-arrival rate λ and the service rate μ and the maximum of the waiting room size is K. Then $\lambda_i = \lambda$ if $i < K$, $\lambda_K = 0$ and $\mu_i = \mu, i \leq K$. In this model, it is still easily obtained that the equilibrium distribution exists, which is given by

$$\pi_i = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{K+1}}(\lambda/\mu)^i, i = 0, 1, \cdots, K.$$

For the arbitrary $M/M/1$ queue, the transition function $\{p_{ij}(t), i, j \in N, t \geq 0\}$, the busy-period distribution and the waiting times of the system can be obtained. Define $\rho =: \lambda/\mu$, $\iota_n = e^{-(\lambda+\mu)t}\rho^{n/2}I_n, n \in Z$, where I_n is Bessel function, i.e.

$$I_n = \sum_{k=0}^{\infty}\frac{(lt)^{n+2k}}{k!(n+k)!}, I_{-n} = I_n, n \in N, l = \sqrt{\lambda\mu}.$$

Theorem 2.2.1. *In the $M/M/1$ queue process, the transition function $\{p_{ij}(t), i, j \in N, t \geq 0\}$ is given by*

$$p_{ij}(t) = \iota_{j-i} + \rho^{-i-1}\iota_{i+j+1} + (1-\rho)(\rho)^j \sum_{n=-\infty}^{-j-i-2}\iota_n. \tag{2.2.2}$$

Theorem 2.2.2. *In the $M/M/1$ queue process, the busy-period distribution is given by the density function*

$$g(t) = \mu e^{-(\lambda+\mu)t}[I_0 - I_2] = \frac{\rho^{-0.5}}{t} e^{-(\lambda+\mu)t} I_1. \tag{2.2.3}$$

Theorem 2.2.3. *In the $M/M/1$ queue process, the stationary waiting time distribution is given by*

$$P(W_n \leq t) = 1 - \rho + \rho(1 - e^{(\lambda-\mu)t}) = 1 - \rho e^{(\lambda-\mu)t}, \tag{2.2.4}$$

where W_n denotes the the waiting time of the nth customer in the system.

For more details of Markov queue theory, one can check the literatures such as S. Asmussen (2003) and A. Y. Chen (2010). For the model (1.2.1)—(1.2.5) in the thesis, we will discuss its basic properties in the next section.

Chapter 3

Markovian Batch-arrival and Bulk-service Queues with Finite State-dependent Control

In this Chapter, we discuss the properties of the model incorporating bulk-arrival and bulk-service queue process with state-dependent control of any finite number m of people in the system and at most m people can be served at the same time.

3.1 Preliminaries

In order to find the transition function of our queueing model and its properties, we need some lemmas for preparation. Some lemmas in this section are the general cases of the lemmas in A. Y. Chen, P. Pollett, J. P. Li and H. J. Zhang (2010).

Define a generating function of $b_j, j \geq 0$

$$B(s) = \sum_{j=0}^{\infty} b_j s^j.$$

Since $\sum_{j=0}^{\infty} |b_j| < +\infty$, the convergence radius is no less than 1, that is to say, this generating function is valid at least for $s \in [-1, 1]$ and it is C^{∞} on

$(-1, 1)$. Further we define the expectations of birth and death rates

$$m_b =: \sum_{j=m+1}^{\infty} (j-m)b_j$$

and

$$m_d =: \sum_{j=0}^{m-1} (m-j)b_j,$$

respectively. It is clearly that $0 < m_b \leq \infty$ and $0 < m_d < +\infty$ as m is finite. In addition, we have

$$\begin{aligned} B'(1) &= \sum_{j=0}^{\infty} jb_j \\ &= \sum_{j \neq m} jb_j + mb_m \\ &= \sum_{j \neq m} jb_j + m(-\sum_{j \neq m} b_j) \\ &= \sum_{j=0}^{m-1} (j-m)b_j + \sum_{j=m+1}^{\infty} (j-m)b_j \\ &= m_b - m_d. \end{aligned} \tag{3.1.1}$$

Furthermore, for any $\lambda > 0$, define

$$B_\lambda(s) = B(s) - \lambda s^m. \tag{3.1.2}$$

Similarly as that of $B(s)$, it is still clearly seen that $B_\lambda(s) \in C^\infty(-1, 1)$. Moreover, both $B(s)$ and $B_\lambda(s)$ can be extended to the complex functions of the variable s. Obviously, both functions are well defined at least on the closed circle $\{s : |s| \leq 1\}$, and are analytic on the open circle $\{s : |s| < 1\}$. Here, we need to discuss the properties of $B_\lambda(s)$.

Lemma 3.1.1. *For any $\lambda > 0$, $B_\lambda(z)$ has exactly m zeros within the closed circle $\{z \in C; |z| \leq 1\}$, which are denoted as $u_1(\lambda), u_2(\lambda), \cdots, u_m(\lambda)$. Moreover, there is no zero on the edge of the circle $\{z \in C; |z| = 1\}$.*

Proof. In order to prove that there are exactly m roots in the open circle, the Rouché's Theorem will be applied. Separate $B_\lambda(z)$ as

$$B_\lambda(z) = g(z) + f(z),$$

where

$$f(z) = (b_m - \lambda)z^m,$$

and

$$\begin{aligned} g(z) &= B_\lambda(z) - f(z) \\ &= \sum_{j=0}^{m-1} b_j z^j + \sum_{j=m+1}^{\infty} b_j z^j. \end{aligned}$$

There is no doubt that both $f(z)$ and $g(z)$ are analytic functions within the open circle $\mathcal{C}_1^0 = \{z : |z| < 1\}$ and continuous on the edge of circle $\mathcal{C} = \{z : |z| = 1\}$. Moreover, on the circle $\mathcal{C}$, the following equations could be obtained

$$\begin{aligned} 0 &< |g(z)| \leq \sum_{j=0}^{m-1} b_j |z|^j + \sum_{j=m+1}^{\infty} b_j |z|^j \\ &= -b_m < -b_m + \lambda = |(-b_m + \lambda)z^m| \\ &= |f(z)|. \end{aligned}$$

Therefore, according to Rouché's Theorem, the two equations $f(z) = 0$ and $f(z) + g(z) = B_\lambda(z) = 0$ have the same number of roots inside $\mathcal{C}$. However, it is apparently that $f(z) = (b_m - \lambda)z^m = 0$ has exactly m zeros inside $\mathcal{C}$ and so does $B_\lambda(z)$.

Finally, on the edge of the circle $\mathcal{C} = \{z : |z| = 1\}$,

$$\begin{aligned} |B_\lambda(z)| &= |f(z) - g(z)| \geq |f(z)| - |g(z)| \\ &\geq (\lambda - b_m) - (-b_m) = \lambda > 0. \end{aligned}$$

Therefore, it is impossible that $B_\lambda(z) = 0$ when $|z| = 1$, that is $B_\lambda(z) = 0$ does not have any root on $\mathcal{C}$ for any $\lambda > 0$. $\square$

Remark 3.1.1. *The equation $B(s) = 0$ (just let λ=0 in the function*

(3.1.2)) has either m or $m+1$ roots within the closed circle $\{z : |z| \leq 1\}$. Additionally, $B(s) = 0$ has exactly $m+1$ roots if and only if $B'(1) > 0$ (in the case, $B'(1)$ can be $+\infty$). In this case, there are exactly two positive real roots q and 1 within the interval $[0,1]$ and in the former case (that is $B(s) = 0$ has exactly m roots), there exist only one positive root (it is trivial that $B(1) = 0$ as $\sum_{j=0}^{\infty} b_j = 0$) on the real interval $[0,1]$.

In fact, even if there are m roots of the function $B_\lambda(z) = 0$, the number of single real positive roots is just one.

Lemma 3.1.2. *For any $\lambda > 0$, the equation $B_\lambda(z) = 0$ has exactly one simple root $u_1(\lambda)$ (sometimes denoted by $u(\lambda)$ for simplicity in the thesis) in the open interval $(0,1)$, and the other $m-1$ roots $u_2(\lambda), \cdots, u_m(\lambda)$ in the open circle $\mathcal{C}_1^0 = \{z : |z| < 1\}$ are either complex or negative.*

Proof. Firstly, it is clear that $B_\lambda(0) = b_0 > 0$ and $B_\lambda(1) = -\lambda < 0$. Hence the equation $B_\lambda(z) = 0$ has at least one positive root in $(0,1)$. From Lemma 3.1.1, we have that $B_\lambda(z) = 0$ has no more than m roots in $(0,1)$. Thus, we could find the smallest (positive) root in $(0,1)$ and let $u(\lambda)$ denote this smallest positive root. Now define

$$\tilde{B}(z) = B_\lambda\left(u(\lambda)z\right).$$

There is no doubt that z_1 is a root of $\tilde{B}(z) = 0$ if and only if $u(\lambda)z_1$ is a root of $B_\lambda(z) = 0$. On the other hand, consider the equation $\tilde{B}(z) - \varepsilon z^m = 0$ for any $\varepsilon > 0$. According to the similar technique in Lemma 3.1.1 that applying Rouché's Theorem once again, it is easily obtained that $\tilde{B}(z) - \varepsilon z^m = 0$ has exactly m roots in $\mathcal{C}_1^o$. Let $\varepsilon \to 0$, we can get that each root of $\tilde{B}(z) - \varepsilon z^m = 0$ will tend to a root of $\tilde{B}(z) = 0$, and from previous lemmas we know that $\tilde{B}(z) = 0$ has at least m roots on $\mathcal{C}_1^o$. Therefore, $B_\lambda(z) = 0$ has exactly m roots in $\mathcal{C}_{u(\lambda)} = \{z : |z| \leq u(\lambda)\}$. In other word, $B_\lambda(z) = 0$ has no root in the interval $(u(\lambda), 1)$ (or equivalently, $\tilde{B}(z)$ has no zero in $(1, \frac{1}{u(\lambda)})$).

The remaining thing to do is to prove that $u(\lambda)$ is simple. Here, it is only required to show that $\tilde{B}'(1) < 0$. Suppose this is not true, then $\tilde{B}'(1) \geq 0$, we will prove that this will result in a contradiction. In fact, it is obvious that $\tilde{B}(z) = B_\lambda(u(\lambda)z) > 0$ for any $z \in (0,1)$. Hence the only possibility is that $\tilde{B}'(1) = 0$, which is equivalent to

$$\sum_{j=0}^{m-1}(m-j)\tilde{b}_j = \sum_{j=m+1}^{\infty}(j-m)\tilde{b}_j, \tag{3.1.3}$$

where $\tilde{b}_j = b_j(u(\lambda))^j$. However,

$$\begin{aligned}
\tilde{B}''(1) &= \sum_{j=2}^{\infty} j(j-1)\tilde{b}_j \\
&= \sum_{j=m+1}^{\infty}[j(j-1) - m(m-1)]\tilde{b}_j + \sum_{j=0}^{m-1}[j(j-1) - m(m-1)]\tilde{b}_j \\
&= \sum_{j=m+1}^{\infty}[j^2 - m^2]\tilde{b}_j - \sum_{j=0}^{m-1}[m^2 - j^2]\tilde{b}_j \\
&= \sum_{j=m+1}^{\infty}(j+m)(j-m)\tilde{b}_j - \sum_{j=0}^{m-1}(m+j)(m-j)\tilde{b}_j \\
&> 2m\sum_{j=m+1}^{\infty}(j-m)\tilde{b}_j - (2m-1)\sum_{j=0}^{m-1}(m-j)\tilde{b}_j \\
&= \sum_{j=0}^{m-1}(m-j)\tilde{b}_j > 0.
\end{aligned}$$

However, we have proved that $\tilde{B}(z) = 0$ has no root in $(1, 1/u(\lambda))$, so $\tilde{B}(z) > 0$ for any $z \in (1, 1/u(\lambda)]$, which contradicts with $\tilde{B}(1/u(\lambda)) = B_\lambda(1) = -\lambda < 0$.

Now, the whole lemma has been proved. $\square$

Lemma 3.1.3. *The root $u_l(\lambda)$ $(1 \le l \le m)$ defined in Lemma 3.1.2 has the following properties.*

(i) For all $1 \le l \le m$, the root $u_l(\lambda) \in C^\infty(0, +\infty)$;

(ii) $u(\lambda)$ is decreasing with $\lambda \to +\infty$;

(iii) $u(\lambda) \downarrow 0$ and $\lambda(u(\lambda))^m \to b_0$ with $\lambda \to +\infty$;

(iv) Let q be the smallest positive root of $B(s) = 0$ on $[0,1]$, then

$$u(\lambda) \uparrow q \begin{cases} = 1 & \textit{if} \;\; m_b \le m_d \\ < 1 & \textit{if} \;\; m_b > m_d \end{cases} \tag{3.1.4}$$

as $\lambda \to 0$*;*

(v) For any $i \in Z^+$*,*

$$\lim_{\lambda \to 0} \frac{1-(u(\lambda))^i}{\lambda} = \begin{cases} \infty & \textit{if } m_b \ge m_d \\ i/(m_d - m_b) & \textit{if } m_b < m_d\,; \end{cases} \tag{3.1.5}$$

(vi) If $m_b > m_d$ *(thus* $q < 1$ *and* $B'(q) < \infty$*), then, for any* $i \in Z^+$*,*

$$\lim_{\lambda \to 0} \frac{1-(u(\lambda)/q)^i}{\lambda} = \frac{-iq^{m-1}}{B'(q)} \tag{3.1.6}$$

and

$$\begin{aligned} & \lim_{\lambda \to 0} \frac{1}{\lambda} \left(\frac{-iq^{m-1}}{B'(q)} - \frac{1-(u(\lambda)/q)^i}{\lambda} \right) \\ = \;\; & \frac{i}{2} \frac{q^{2m-2}}{(B'(q))^3} \left((i+2m-1)B'(q) - qB''(q)\right); \end{aligned} \tag{3.1.7}$$

(vii) If $m_b = m_d$ *(thus* $q = 1$*) then, for any* $i \in Z^+$*,*

$$\lim_{\lambda \to 0} \frac{1-(u(\lambda))^i}{\sqrt{\lambda}} = \begin{cases} i\sqrt{\frac{2}{B''(1)}} & \textit{if } B''(1) < +\infty \\ 0 & \textit{if } B''(1) = +\infty \end{cases} \;. \tag{3.1.8}$$

(viii) If $m_b < m_d$ *(and hence* $q = 1$*), then, for any* $i \in Z^+$*,*

$$\begin{aligned} & \lim_{\lambda \to 0} \left(\frac{i}{\lambda(m_d - m_b)} - \frac{1-(u(\lambda))^i}{\lambda^2} \right) \\ = \;\; & \begin{cases} \frac{i((i+2m-1)(m_d-m_b)+B''(1))}{2(m_d-m_b)^3} & \textit{if } B''(1) < \infty \\ \infty & \textit{if } B''(1) = \infty. \end{cases} \end{aligned}$$

(ix) For any positive i *and for any* $l \in [1, m]$*,*

$$\lim_{\lambda \to 0} \frac{1 - (u_l(\lambda)/q_l)^i}{\lambda} = \frac{-iq_l^{m-1}}{B'(q_l)}, \tag{3.1.9}$$

and

$$\begin{aligned} & \lim_{\lambda \to 0} \frac{1}{\lambda} \left(\frac{-iq_l^{m-1}}{B'(q_l)} - \frac{1 - (u_l(\lambda)/q_l)^i}{\lambda} \right) \\ = \; & \frac{i}{2} \frac{q_l^{2m-2}}{(B'(q_l))^3} \left((i + 2m - 1) B'(q_l) - q_l B''(q_l) \right); \end{aligned} \tag{3.1.10}$$

where $q_l = \lim_{\lambda \to 0} u_l(\lambda)$.

Proof. Since $u(\lambda)$ is the unique positive root of $B_\lambda(s) = 0$ on $[0, 1]$, consider the curves of two functions $y = B(x)$ and $y = \lambda x^m$, then $u(\lambda)$ can be regarded as the x-coordinate of the intersection point between these two curves. Therefore, it is obvious that $u(\lambda)$ is decreasing with $\lambda \to +\infty$ as $y = \lambda x^m$ is increasing with λ in $[0, 1]$. So Parts (ii) has been proved.

In Part (iv), as it has just shown that $u(\lambda)$ is a decreasing function, it is trivial that $u(\lambda) \uparrow q$ when $\lambda \to 0$. From (3.1.1), we obtain that $B'(1) > 0$ if $m_b > m_d$. In this case, there exists a number $a \in [0, 1]$ such that $B(s)$ is strictly negative on $(a, 1)$, and hence $q < 1$. When $m_b \leq m_d$, $B'(1) \leq 0$. In this case, assume $q < 1$, then $B(z)$ is strictly positive on the open interval $(q, 1)$ since q is the unique zero of $B(z)$ within $(0, 1)$. Thus, we can obtain that $B'(q) > 0$. Similarly, it is apparently that there still exists a zero in $(0, q)$, which contradicts the uniqueness of the zero in $[0, 1]$. Thus, $q = 1$ must be satisfied.

Part (iii) is obvious.

Next, since for all $l \in [1, m]$, $u_l(\lambda)$ is the root of the equation $\lambda = s^{-m} B(s)$, λ, which can be considered as a function of $s \in (0, 1]$, is C^∞. Therefore, the inverse function of λ, that is $u_l(\lambda)$, also belongs to $C^\infty[0, \infty)$. The proof of (i) has been completed. Up to now, we have proved (i)-(iv).

To complete the remaining of the proof, first of all substitute $u(0)$ for q in (3.1.4), then the following equation could be got by using the mean value theorem:

$$u(\lambda) - q = \lambda u'(\xi) \qquad (0 < \xi < \lambda), \tag{3.1.11}$$

since $u(\lambda)$ is a differentiable function for $\lambda > 0$. As from (3.1.4) $m_b > m_d$ implies $q < 1$, for any $i \in Z^+$, the equation

$$1 - (u(\lambda))^i > 0$$

is always right whenever λ is a sufficiently large number. Hence (3.1.5) holds true in this case. Then, consider the case of $m_b \le m_d$, i.e. $q = 1$. Now, (3.1.11) is equivalent to

$$(1 - u(\lambda))/\lambda = -u'(\xi) \qquad (0 < \xi < \lambda). \tag{3.1.12}$$

Recall that $u(\lambda)$ is the root of $B_\lambda(s) = 0$, we can write

$$B(u(\lambda)) = \lambda(u(\lambda))^m. \tag{3.1.13}$$

Differentiating on both sides of (3.1.13) will become

$$B'(u(\lambda))u'(\lambda) - m\lambda(u(\lambda))^{m-1}u'(\lambda) = (u(\lambda))^m. \tag{3.1.14}$$

Letting $\lambda \to 0$ on (3.1.14) and noting that both $u(\lambda)$ and $B'(u(\lambda))$ are continuous functions of λ on $[0, \infty)$ so that $\lambda u(\lambda)$ tends to 0 when $\lambda \to 0$ will lead us to conclude that

$$\lim_{\lambda \to 0} u'(\lambda) = \frac{q^m}{B'(q)}.$$

Next, since $m_b \le m_d$ is equivalent to $q = 1$, from (3.1.12) and recall that $B'(1) = m_b - m_d$, we have the result that

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(1 - u(\lambda)) = \lim_{\lambda \to 0} u'(\lambda) = \begin{cases} 1/(m_d - m_b) & \text{if } m_b < m_d \\ \infty & \text{if } m_b = m_d\,, \end{cases} \tag{3.1.15}$$

and so (3.1.5) holds true for $i = 1$. Moreover, when $m_b < m_d$, (3.1.15) could be rewritten as

$$u(\lambda) = 1 - \frac{\lambda}{m_d - m_b} + \circ(\lambda).$$

Hence, for any $i \in Z^+$,

$$\begin{aligned}(u(\lambda))^i &= \left(1 - \frac{\lambda}{m_d - m_b} + \circ(\lambda)\right)^i \\ &= 1 - \frac{i\lambda}{m_d - m_b} + \circ(\lambda)\end{aligned}$$

and thus

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\left(1 - (u(\lambda))^i\right) = \begin{cases} i/(m_d - m_b) & \text{if } m_b < m_d \\ \infty & \text{if } m_b = m_d \end{cases}.$$

Hence the proof of (v) is finished.

Then we need to prove (viii) in advance. Differentiating (3.1.14) will lead to

$$\begin{aligned}&u''(\lambda)\left(B'(u(\lambda)) - m\lambda(u(\lambda))^{m-1}\right) + (u'(\lambda))^2\left(B''(u(\lambda))\right. \\ &\quad \left.-m(m-1)\lambda(u(\lambda))^{m-2}\right) = 2m(u(\lambda))^{m-1}u'(\lambda),\end{aligned} \tag{3.1.16}$$

for all $\lambda > 0$. Letting $\lambda \to 0$, provided that both $B'(q)$ and $B''(q)$ are finite, then we have

$$\lim_{\lambda \to 0} u''(\lambda) = \left(\frac{q}{B'(q)}\right)^3 \left(2mq^{m-2}B'(q) - qB''(q)\right).$$

In particular, if $m_d > m_b$, recall on the assumption that $B''(1)$ is finite, then use the similar technique in the proof of (v) will result

$$u(\lambda) = 1 - \frac{\lambda}{m_d - m_b} + \frac{(2m(m_d - m_b) + B''(1))\lambda^2}{2(m_d - m_b)^3} + \circ(\lambda^2).$$

It can be then proved directly that, for any $i \in Z^+$,

$$\begin{aligned}(u(\lambda))^i &= 1 - \frac{i\lambda}{m_d - m_b} \\ &\quad + \frac{i\lambda^2((i + 2m - 1)(m_d - m_b) + B''(1))}{2(m_d - m_b)^3} + \circ(\lambda^2).\end{aligned}$$

Hence part (viii) has been proved.

Turn back to (vi), as $m_b > m_d$ (that is $q < 1$, both $B'(q)$ and $B''(q)$ are finite), it has been proved that $u(\lambda) \uparrow q < 1$ as $\lambda \to 0$. Then it follows from (3.1.14) that $\lim_{\lambda\to 0} u'(\lambda) = q^m/B'(q)$. Therefore, by using the same technique which has been used for (v), we obtain that for any $i \in Z^+$,

$$u(\lambda) = q + \frac{\lambda q^m}{B'(s)} + \circ(\lambda),$$

which leads to

$$(u(\lambda))^i = q^i + \frac{i\lambda q^{i+m-1}}{B'(q)} + \circ(\lambda).$$

Therefore, (3.1.6) follows. Similarly from (3.1.16) we get

$$\lim_{\lambda\to 0} u''(\lambda) = \frac{q^{2m-1}}{(B'(q))^3}(2mB'(q) - qB''(q)).$$

Therefore,

$$u(\lambda) = q + \frac{q^m}{B'(q)}\lambda + \frac{q^{2m-1}}{2(B'(q))^3}\left(2mB'(q) - qB''(q)\right)\lambda^2 + \circ(\lambda^2),$$

and then, for any positive integer i,

$$\begin{aligned}(u(\lambda))^i &= q^i + \frac{iq^{i+m-1}}{B'(q)}\lambda \\ &+ \frac{i\left((i+2m-1)B'(q) - qB''(q)\right)}{2(B'(q))^3}q^{2m+i-2}\lambda^2 + \circ(\lambda^2),\end{aligned}$$

and thus (3.1.7) follows.

To prove (vii), a little technique is needed. Note that as $m_d = m_b$, $q = 1$. Besides that $u(\lambda)$ is the root of $B_\lambda(s) = 0$, so $\lambda(u(\lambda))^m = B(u(\lambda))$.

Moreover, it is clear that

$$\begin{aligned}
B(u(\lambda)) &= (1-u(\lambda))^2\frac{1}{m}(\sum_{j=m+1}^{\infty}(j-m)b_j)(\sum_{k=0}^{m-2}(k+1)(u(\lambda)^k) \\
&- (u(\lambda))^m\sum_{j=m+1}^{\infty}b_j(\sum_{k=0}^{j-m-2}(k+1)s^k) \\
&+ \sum_{j=1}^{m-1}b_js^j(\sum_{k=0}^{m-j-2}(k+1)s^k) \\
&- \frac{1}{m}(\sum_{j=1}^{m-1}(m-j)b_j)(\sum_{k=0}^{m-2}(k+1)s^k) \\
&+ \sum_{j=m+1}^{\infty}(j-m)b_js^{m-1}(\sum_{k=0}^{j-m-1}s^k)
\end{aligned}$$

for the case that $m_d = m_b$. Recall $u(\lambda) \uparrow 1$ as $\lambda \to 0$. Therefore, according to the Monotone Convergence Theorem, we have

$$\begin{aligned}
\lim_{\lambda\to 0}\frac{\lambda}{(1-u(\lambda))^2} &= \lim_{\lambda\to 0}\frac{B(u(\lambda))}{u^m(\lambda)(1-u(\lambda))^2} \\
&= \frac{m-1}{2}\sum_{j=m+1}^{\infty}(j-m)b_j - \sum_{j=m+1}^{\infty}b_j\frac{(j-m)(j-m-1)}{2} \\
&\quad +\sum_{j=1}^{m-1}b_j\frac{(m-j)(m-j-1)}{2} - \frac{m-1}{2}\sum_{j=1}^{m-1}(m-j)b_j \\
&\quad +\sum_{j=m+1}^{\infty}(j-m)^2b_j,
\end{aligned}$$

which is indeed $\frac{B''(1)}{2}$.

Therefore,

$$\lim_{\lambda\to 0}\frac{1-u(\lambda)}{\sqrt{\lambda}} = \sqrt{\frac{2}{B''(1)}} \tag{3.1.17}$$

and then (3.1.8) will be obtained very easily.

Finally, we prove (ix). Since $u_l(\lambda)$ must be complex, just write

$$u_l(\lambda) = a_l(\lambda) + b_l(\lambda)i.$$

Then it is obvious that $a_l(0) = Re(q_l)$ and $b_l(0) = Im(q_l)$. Similarly, we have

$$\frac{a_l(0) - a_l(\lambda)}{\lambda} = -a_l'(\xi) \ (0 < \xi < \lambda) \tag{3.1.18}$$

and

$$\frac{b_l(0) - b_l(\lambda)}{\lambda} = -b_l'(\xi) \ (0 < \xi < \lambda). \tag{3.1.19}$$

Thus, (3.1.18) + (3.1.19)i yields

$$\frac{q_l - u_l(\lambda)}{\lambda} = -u_l'(\xi) \ (0 < \xi < \lambda), \tag{3.1.20}$$

and (3.1.9) could be obtained by the same method in proving (vi). □

Corollary 3.1.1. *For $u(\lambda)$ defined in Lemma 3.1.3, from previous proofs, we could conclude that*

(i)

$$\lim_{\lambda\to 0} u'(\lambda) = \begin{cases} 1/(m_b - m_d) & \text{if } m_b < m_d \\ -\infty & \text{if } m_b = m_d \\ q^m/B'(q) & \text{if } m_b > m_d\,; \end{cases}$$

(ii)

$$\lim_{\lambda\to 0} u''(\lambda) = \begin{cases} \left(2m(m_d - m_b) + B''(1)\right)/(m_d - m_b)^3 & \text{if } m_b < m_d \\ \frac{q^{2m-1}}{(B'(q))^3}\left(2mB'(q) - qB''(q)\right) & \text{if } m_b > m_d \end{cases}$$

(recall that in the case of $m_b < m_d$, $q = 1$ and $B'(1) = m_b - m_d$, so these two expressions are totally the same);

(iii) For any $i \in Z^+$,

$$\lim_{\lambda\to 0} \frac{d}{d\lambda}(u(\lambda))^i = \begin{cases} i/(m_b - m_d) & \text{if } m_b < m_d \\ iq^{i+m-1}/B'(q) & \text{if } m_b > m_d\,; \end{cases} \tag{3.1.21}$$

(iv)

$$\lim_{\lambda\to 0} \frac{d^2}{d\lambda^2}(u(\lambda))^i = \begin{cases} i\left((i + 2m - 1)(m_d - m_b) + B''(1)\right)/(m_d - m_b)^3 & \text{if } m_b < m_d \\ iq^{i+2m-2}\left((i + 2m - 1)B'(q) - qB''(q)\right)/(B'(q))^3 & \text{if } m_b > m_d\,. \end{cases} \tag{3.1.22}$$

Proof. The proofs of (i) and (ii) are in that of part (v) and part (viii) in Lemma 3.1.3 respectively. In part (iii), note that

$$\lim_{\lambda \to 0} u(\lambda) = q$$

and the following equation is obtained by using the common derivative

$$\frac{d}{d\lambda}(u(\lambda))^i = i(u(\lambda))^{i-1}u'(\lambda).$$

Then letting $\lambda \to 0$ on both sides will result in (3.1.21).

Finally, to prove (iv), again we write

$$\frac{d^2}{d\lambda^2}(u(\lambda))^i = i(i-1)(u(\lambda))^{i-2}(u'(\lambda))^2 + i(u(\lambda))^{i-1}u''(\lambda),$$

Then let $\lambda \to 0$ and use the conclusion of (i), (ii) and (iii), (3.1.22) could be obtained. □

We have already done some preparations for our models, now the transition function and other properties of our model can be explored.

3.2 The Way to Find The Resolvent of the Feller Minimal Q^*-transition Function

First of all, we want to discuss the model with m absorbing states $0, 1, 2, \cdots, m-1$ ($q_{ij} \equiv 0$ if $i < m$), whose q-matrix Q^* is given by (1.2.2) and (1.2.5), which is a much simper model. The practical meaning of this model is that the queue process stops forever whenever it enters the state $0, 1, 2, \cdots, m-1$. In this queueing model, all the parameters are only from the sequence $\{b_j,\, j \geq 1\}$.

Let $P^*(t) = \{p^*_{ij}(t),\, i, j \geq 0\}$ be the Feller minimal Q^*-transition function and let $\Phi^*(\lambda) = \{\phi^*_{ij}(\lambda),\, i, j \geq 0\}$ be its resolvent function. It is clear that the transition function of Q^* is unique.

Theorem 3.2.1. *Suppose all the roots of $B_\lambda(s) = 0$ are simple roots. For each $i \in N$, and $0 \leq j \leq m-1$, the resolvent function $\{\phi^*_{ij}(\lambda)\}$ is the unique* *solution of the m equations*

$$\sum_{k=0}^{m-1} \phi^*_{ik}(\lambda)(u_l(\lambda))^k = \frac{1}{\lambda}(u_l(\lambda))^i. \ (1 \leq l \leq m), \tag{3.2.1}$$

where $u_l(\lambda)(1 \leq l \leq m)$ are all the roots of the equation $B_\lambda(s) = 0$ (note in Lemma 3.1.1 that there are exactly m roots). For $j \geq m$, the generating function of $\{\phi^*_{ij}(\lambda)\}$ is given by

$$\sum_{j=0}^{\infty} \phi^*_{ij}(\lambda)s^j = \frac{B(s)\sum_{k=0}^{m-1}\phi^*_{ik}(\lambda)s^k - s^{m+i}}{B_\lambda(s)}. \tag{3.2.2}$$

More specifically, if $0 \leq i \leq m-1$, then

$$\phi^*_{ij}(\lambda) = \frac{\delta_{ij}}{\lambda}. \tag{3.2.3}$$

Proof. To prove it, the Kolmogorov Forward Equation (2.1.15) is the basic tool, which is certainly

$$\Phi^*(\lambda)(\lambda I - Q^*) = I.$$

This equation becomes

$$\begin{aligned} \lambda\phi^*_{ij}(\lambda) &= \sum_{k=0}^{\infty} \phi^*_{ik}(\lambda)q^*_{kj} + \delta_{ij} \\ &= \sum_{k=m}^{j+m} \phi^*_{ik}(\lambda)b_{j-k+m} + \delta_{ij}. \end{aligned}$$

For every $j(0 \leq j < +\infty)$, multiply s^j on both sides of the equation will result in

$$\lambda\sum_{j=0}^{\infty}\phi^*_{ij}(\lambda)s^j = \sum_{j=0}^{\infty}(\sum_{k=m}^{j+m}\phi^*_{ik}b_{j-k+m})s^j + s^i,$$

or

$$\begin{aligned}
& \lambda \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) s^j - s^i \\
= & \sum_{k=m}^{\infty} \sum_{j=k-m}^{\infty} \phi_{ik}^*(\lambda) b_{j-k+m} s^j \\
= & \sum_{k=m}^{\infty} \phi_{ik}^*(\lambda) \sum_{j=0}^{\infty} b_j s^{j+k-m}. \qquad (3.2.4)
\end{aligned}$$

Substitute $B(s)$ to $\sum_{j=0}^{\infty} b_j s^j$ in (3.2.4) will lead to

$$\begin{aligned}
& \lambda \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) s^j - s^i \\
= & B(s) \sum_{k=m}^{\infty} \phi_{ik}^*(\lambda) s^{k-m} \\
= & \frac{B(s)}{s^m} \left[\sum_{k=0}^{\infty} \phi_{ik}^*(\lambda) s^k - \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k \right].
\end{aligned}$$

Multiply s^m on both sides and sum all the $\lambda \phi_{ij}^* s^j$ by j will lead to

$$\lambda s^m \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) s^j - s^{m+i} = B(s) \sum_{k=0}^{\infty} \phi_{ik}^*(\lambda) s^k - B(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k,$$

or

$$(B(s) - \lambda s^m) \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) s^j = B(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k - s^{m+i}.$$

In other word, the following equation is obtained:

$$\sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) s^j = \frac{B(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k - s^{m+i}}{B_\lambda(s)}, \qquad (3.2.5)$$

which is just (3.2.2). Therefore, if for all $0 \leq j \leq m-1$, $\phi_{ij}^*(\lambda)$ is known, then the generating function of $\{\phi_{ij}^*(\lambda) : j \geq 0\}$ will have been got.

For any $u_l(\lambda)(1 \leq l \leq m)$, $B_\lambda(u_l(\lambda)) = 0$ is satisfied. Moreover, it is obviously that

$$\sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) \left(u_l(\lambda)\right)^j < +\infty$$

for each $l = 1, 2, \cdots, m$, since

$$|u_l(\lambda)| \le 1 \text{ and } \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) \le \frac{1}{\lambda}.$$

Therefore, in (3.2.5), by substituting each $u_l(\lambda)$ to s, it is apparent that the denominator of the right side, i.e. $B_\lambda(u_l(\lambda))$, is zero, and the left side of the equation, i.e. $\sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) \left(u_l(\lambda)\right)^j$, is a finite number. Thus, the only possibility is that the numerator of the right side must also be zero. In other word, for each $1 \le l \le m$, we have

$$B(u_l(\lambda)) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda)(u_l(\lambda))^k = (u_l(\lambda))^{m+i},$$

or

$$\sum_{k=0}^{m-1} (\phi_{ik}^*(\lambda))^k = \frac{(u_l(\lambda))^{m+i}}{B(u_l(\lambda))}. \tag{3.2.6}$$

Since $B(u_l(\lambda)) = \lambda(u_l(\lambda))^m$, for each $i \in N$ and $l \in \{1, 2, \cdots, m\}$, we have

$$\sum_{k=0}^{m-1} \phi_{ik}^*(\lambda)(u_l(\lambda))^k = \frac{1}{\lambda}(u_l(\lambda))^i,$$

which is just (3.2.1). The next step is to prove the uniqueness of the solution of the equation set.

If we denote $x_k = \phi_{i,k+1}^*(\lambda)$ for any fixed i, then (3.2.1) just means

$$\begin{cases} x_1 + u_1(\lambda)x_2 + u_1^2(\lambda)x_3 + \cdots + u_1^{m-1}(\lambda)x_m = \frac{1}{\lambda}(u_1(\lambda))^i \\ x_1 + u_2(\lambda)x_2 + u_2^2(\lambda)x_3 + \cdots + u_2^{m-1}(\lambda)x_m = \frac{1}{\lambda}(u_2(\lambda))^i \\ \cdots\cdots \\ x_1 + u_m(\lambda)x_2 + u_m^m(\lambda)x_3 + \cdots + u_m^{m-1}(\lambda)x_m = \frac{1}{\lambda}(u_m(\lambda))^i. \end{cases}$$

There are m unknown numbers in the equation set and its coefficient determinant is

$$\Delta = \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) & \cdots & u_1^{m-1}(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) & \cdots & u_2^{m-1}(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) & \cdots & u_3^{m-1}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_m(\lambda) & u_m^2(\lambda) & \cdots & u_m^{m-1}(\lambda) \end{vmatrix}. \tag{3.2.7}$$

However, (3.2.7) is indeed a Vandermonde matrix. Hence we have

$$\Delta = \Pi_{1\leq j<i\leq m}(u_i(\lambda) - u_j(\lambda)).$$

Thus, recall that none of any two roots of the equation $B_\lambda(s) = 0$ are the same (the same as the assumption in the theorem that all the roots $u_l(\lambda), l = 1, 2, \cdots, m$ are simple roots), then $\Delta \neq 0$, that is $\phi^*_{ik}(\lambda)(0 \leq k \leq m-1)$ are the unique solution of (3.2.1), which is given by

$$\phi^*_{i0}(\lambda) = \frac{\Delta_1^{(i)}}{\Delta}, \cdots, \phi^*_{i,m-1}(\lambda) = \frac{\Delta_m^{(i)}}{\Delta},$$

where Δ is given in (3.2.7), $\Delta_1^{(i)}$ is given by

$$\Delta_1^{(i)}(\lambda) = \frac{1}{\lambda}\begin{vmatrix} u_1^i(\lambda) & u_1(\lambda) & u_1^2(\lambda) & \cdots & u_1^{m-1}(\lambda) \\ u_2^i(\lambda) & u_2(\lambda) & u_2^2(\lambda) & \cdots & u_2^{m-1}(\lambda) \\ u_3^i(\lambda) & u_3(\lambda) & u_3^2(\lambda) & \cdots & u_3^{m-1}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m^i(\lambda) & u_m(\lambda) & u_m^2(\lambda) & \cdots & u_m^{m-1}(\lambda) \end{vmatrix}, \tag{3.2.8}$$

$$\Delta_2^{(i)}(\lambda) = \frac{1}{\lambda}\begin{vmatrix} 1 & u_1^i(\lambda) & u_1^2(\lambda) & \cdots & u_1^{m-1}(\lambda) \\ 1 & u_2^i(\lambda) & u_2^2(\lambda) & \cdots & u_2^{m-1}(\lambda) \\ 1 & u_3^i(\lambda) & u_3^2(\lambda) & \cdots & u_3^{m-1}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_m^i(\lambda) & u_m^2(\lambda) & \cdots & u_m^{m-1}(\lambda) \end{vmatrix}, \tag{3.2.9}$$

and $\Delta_k^{(i)}(1 \leq k \leq m)$ should be easily given similarly.

Furthermore, for any $i < m$, we obtain that

$$\Delta_j^{(i)} = \frac{\delta_{ij}}{\lambda}\Delta,$$

which just means

$$\phi_{ij}^*\lambda = \frac{\delta_{ij}}{\lambda}.$$

Now the whole theorem has been proved. $\square$

The way to calculate the Q^*-resolvent function for all $i, j \geq m$ is just solve a few equations. For example, for all $i \geq m$, (3.2.5) could be rewritten as

$$\begin{aligned}
\sum_{j=0}^{\infty} \phi_{ij}^*(\lambda)s^j &= \frac{B(s)\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k - s^{m+i}}{B_\lambda(s)} \\
&= \frac{(B(s)-\lambda s^m)\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k + \lambda s^m\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k - s^{m+i}}{B_\lambda(s)} \\
&= \sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k + \frac{\lambda s^m\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k - s^{m+i}}{B_\lambda(s)},
\end{aligned}$$

That is

$$\sum_{j=m}^{\infty}\phi_{ij}^*(\lambda)s^j(\sum_{j=0}^{\infty}b_js^j - \lambda s^m) = \lambda s^m\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)s^k - s^{m+i}.$$

Choose the parameter of s^m on both sides will lead to

$$\phi_{im}^*(\lambda)b_0 = \lambda\phi_{i0}^*(\lambda)$$

and thus

$$\phi_{im}^*(\lambda) = \frac{\lambda}{b_0}\phi_{i0}^*(\lambda).$$

Look at the term of s^{m+1}, that is

$$\phi_{im}^*(\lambda)b_1 + \phi_{i,m+1}^*(\lambda)b_0 = \lambda\phi_{i1}^*(\lambda),$$

so

$$\begin{aligned}\phi^*_{i,m+1}(\lambda) &= \frac{\lambda\phi^*_{i1}(\lambda)-\phi^*_{im}(\lambda)b_1}{b_0}\\ &= \frac{\lambda}{b_0^2}\cdot\left(b_0\phi^*_{i1}(\lambda)-b_1\phi^*_{i0}(\lambda)\right),\end{aligned}$$

and $\phi^*_{i,m+2}(\lambda), \phi^*_{i,m+3}(\lambda), \cdots$ could be obtained in the similar ways.

Even if we have calculated the whole resolvent of the Feller minimal Q^*-function by the expression of $u_l(\lambda), l = 1, 2, \cdots, m$, in fact, it is quite difficult to find all the roots $u_l(\lambda), l = 1, 2, \cdots, m$. Only some properties of the roots could be obtained. Hence, this problem needs to be solved in the future.

Recall that $\phi^*_{ij}(\lambda)$ is the Laplace transform of the transition probabilities $p^*_{ij}(t)$:

$$\phi^*_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p^*_{ij}(t)dt.$$

Therefore, the Feller minimal Q^*-transition function could be get by inverting this resolvent function.

Let $s = 1$ in (3.2.5) leads to

$$\sum_{j=0}^{\infty} \phi^*_{ij}(\lambda) = \frac{1}{\lambda},$$

and thus the Feller minimal Q^*-function is honest.

Remark 3.2.1. *In Theorem 3.2.1, it is assumed that all the roots $u_l(\lambda)$ are simple roots. However, sometimes $B_\lambda(s) = 0$ have multiple-roots. In this case, (3.2.5) does not have the unique solution since $\Delta = 0$ in this case. Therefore, we need some other techniques to solve it.*

From the basic equation (3.2.5), without loss of generality, we assume that $u_{m-1}(\lambda)$ is a double root of $B_\lambda(s)$ and all the other roots $u_1(\lambda), u_2(\lambda), \cdots, u_{m-2}(\lambda)$ are simple. That is

$$B_\lambda(u_{m-1}(\lambda)) = B'(u_{m-1}(\lambda)) = 0.$$

As (3.2.5) *is well-defined when $s = u_{m-1}(\lambda)$, $u_{m-1}(\lambda)$ is also a double root* *of the numerator of the fraction in the right side of* (3.2.5).

Let

$$D(s) = B(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k - s^{m+i}, \tag{3.2.10}$$

then

$$D(u_{m-1}(\lambda)) = D'(u_{m-1}(\lambda)) = 0.$$

Differentiating $D(s)$ in (3.2.10) leads to

$$D'(s) = B'(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) s^k + B(s) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) k s^{k-1} - (m+i) s^{m+i-1}.$$

Hence, we obtain two equations from this double root:

$$\begin{cases} B(u_{m-1}(\lambda)) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) u_{m-1}^k(\lambda) = (u_{m-1}(\lambda))^{m+i} \\ B'(u_{m-1}(\lambda)) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) u_{m-1}^k(\lambda) + B(u_{m-1}(\lambda)) \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) k u_{m-1}^{k-1}(\lambda) \\ = (m+i)(u_{m-1}(\lambda))^{m+i-1}. \end{cases}$$

Therefore, we can still have exact m number of equations, which are given by

$$\begin{cases} x_1 + u_1(\lambda) x_2 + u_1^2(\lambda) x_3 + \cdots + u_1^{m-1}(\lambda) x_m = \frac{1}{\lambda}(u_1(\lambda))^i \\ x_1 + u_2(\lambda) x_2 + u_2^2(\lambda) x_3 + \cdots + u_2^{m-1}(\lambda) x_m = \frac{1}{\lambda}(u_2(\lambda))^i \\ \cdots\cdots \\ x_1 + u_{m-1}(\lambda) x_2 + u_{m-1}^m(\lambda) x_3 + \cdots + u_{m-1}^{m-1}(\lambda) x_m = \frac{1}{\lambda}(u_{m-1}(\lambda))^i \\ m x_1 + (m-1) u_{m-1}(\lambda) x_2 + \cdots + u_{m-1}^{m-1}(\lambda) = \frac{m+i}{\lambda}(u_{m-1}(\lambda))^i \end{cases},$$

where $\phi_{i0}^*(\lambda), \phi_{i1}^*(\lambda), \cdots, \phi_{i,m-1}^*(\lambda)$ are its solution. For other kinds of multiple root, the similar technique can be used.

In the later part of the thesis, it is automatically assumed that all the m roots of $B_\lambda(s) = 0$, i.e. $u_1(\lambda), u_2(\lambda), \cdots, u_m(\lambda)$ are single and different with each other.

3.3 The Extinction Probability and the Mean Extinction Time

In our model, we regard the process ever entering the state $\{0, 1, 2, \cdots, m-1\}$ (starting from some other state $\{m, m+1, \cdots\}$) as extinction. In other word, the state-dependent part is not needed. Thus, Q^* is the only matrix to use in discussing the extinction property as the process will stop whenever it enters the first m states. Let $(a_{k0}, a_{k1}, \cdots, a_{k,m-1})$ $(k \geq m)$ be the extinction probability vector starting from k (for example, a_{k0} is the probability that the process will ever enter the state 0 starting from the state k, and the others are similar) and $\tau^* = (\tau_0^*, \tau_1^*, \cdots, \tau_{m-1}^*)$ be the related extinction time vector.

3.3.1 The Extinction Probability

Theorem 3.3.1. *Suppose all the roots of $B(s) = 0$ in $\{s : |s| \leq 1\}$ except the root $s = 1$ are simple roots. The extinction probability a_{ik} $(k \leq m-1, i \geq m)$ is the unique solution of the following equations:*

$$\begin{cases} a_{i0} + a_{i1}q_1 + a_{i2}q_1^2 + \cdots + a_{i,m-1}q_1^{m-1} = q_1^i \\ a_{i0} + a_{i1}q_2 + a_{i2}q_2^2 + \cdots + a_{i,m-1}q_2^{m-1} = q_2^i \\ \cdots \\ a_{i0} + a_{i1}q_m + a_{i2}q_m^2 + \cdots + a_{i,m-1}q_m^{m-1} = q_m^i \end{cases}, \tag{3.3.1}$$

where $q_1, q_2, \cdots, q_m$ are the m roots of the equation $B(s) = 0$, i.e. for all $1 \leq l \leq m$,

$$q_l = \lim_{\lambda \to 0} u_l(\lambda).$$

Proof. We start to prove this theorem by applying (3.2.1), i.e.

$$\sum_{k=0}^{m-1} \lambda \phi_{ik}^*(\lambda)(u_l(\lambda))^k = (u_l(\lambda))^i. \ (1 \leq l \leq m),$$

Let $\lambda \to 0$, and recall from Theorem 2.1.4 that

$$\lim_{\lambda \to 0} \lambda \phi_{ik}^*(\lambda) = a_{ik},$$

then (3.2.1) becomes

$$\sum_{k=0}^{m-1} a_{ik} q_l^k = q_l^i \ (\leq l \leq m), \tag{3.3.2}$$

which is certainly (3.3.1).

Note that the coefficient determinant of the equation set (3.3.1) is

$$\Delta(q) = \begin{vmatrix} 1 & q_1 & \cdots & q_1^{m-1} \\ 1 & q_2 & \cdots & q_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_m & \cdots & q_m^{m-1} \end{vmatrix} = \Pi_{1 \leq j < i \leq m}(q_i - q_j).$$

It is assumed that $q_1, q_2, \cdots, q_m$ are all different (although the possibility that $\lim_{\lambda \to 0} u_i(\lambda) = \lim_{\lambda \to 0} u_j(\lambda)$ might exist for some $u_i(\lambda) \neq u_j(\lambda)$), then

$$\Delta(q) \neq 0.$$

Thus, (3.3.1) has the unique solution. $\square$

More specifically, we can express all the elements of the extinction vector by all the roots of $B(s) = 0$. The solution of the equation set (3.3.1) is given by

$$
\begin{aligned}
a_{k0} &= \begin{vmatrix} q_1^k & q_1 & \cdots & q_1^{m-1} \\ q_2^k & q_2 & \cdots & q_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_m^k & q_m & \cdots & q_m^{m-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & q_1 & \cdots & q_1^{m-1} \\ 1 & q_2 & \cdots & q_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_m & \cdots & q_m^{m-1} \end{vmatrix}^{-1}, \\
a_{k1} &= \begin{vmatrix} 1 & q_1^k & \cdots & q_1^{m-1} \\ 1 & q_2^k & \cdots & q_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_m^k & \cdots & q_m^{m-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & q_1 & \cdots & q_1^{m-1} \\ 1 & q_2 & \cdots & q_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_m & \cdots & q_m^{m-1} \end{vmatrix}^{-1},
\end{aligned}
$$

and $a_{k2}, \cdots, a_{k,m-1}$ could be obtained similarly.

Theorem 3.3.2. *The overall extinction probability is 1 if and only if $m_b \leq m_d$, i.e. $B'(1) \leq 0$.*

Proof. There is no doubt that the overall extinction probability is indeed $\sum_{k=0}^{m-1} a_{ik}$. Define $q = \lim_{\lambda \to 0} u(\lambda)$, where $u(\lambda)$ is that unique simple positive root of $B_\lambda(s) = 0$. From part (iv) of Lemma 3.1.3, $q = 1$ if $m_b \leq m_d$ and $q < 1$ if $m_b > m_d$. Obviously, q must follow one of the equations (3.3.2), that is

$$\sum_{k=0}^{m-1} a_{ik} q^k = q^i \tag{3.3.3}$$

If $q = 1$ (i.e. $m_b \leq m_d$), then the equation (3.3.3) will become

$$\sum_{k=0}^{m-1} a_{ik} = 1,$$

which means that the overall extinction probability is 1 in this case. On the other hand, If $q < 1$, then the equation (3.3.3) will become

$$\sum_{k=0}^{m-1} a_{ik} q^{k-i} = 1.$$

However, since $i \geq m > m-1 \geq k \geq 0$ and $q < 1$, thus $q^{k-i} > 1$, which ensures that

$$\sum_{k=0}^{m-1} a_{ik} < \sum_{k=0}^{m-1} a_{ik} q^{k-i} = 1.$$

Therefore, the overall extinction probability is 1 if and only if $m_b \leq m_d$, i.e. the expectation of the birth rate is no more than that of the death rate. $\square$

3.3.2 The Mean Extinction Time

For the problems of extinction, it is not enough just to discuss the extinction probability. The extinction time is also interesting. Denote $\bar{U}^{(k)}(\lambda) = \left(u_1^k(\lambda), u_2^k(\lambda), \cdots, u_m^k(\lambda)\right)$ and

$$W(\lambda) = \begin{pmatrix} \bar{U}^{(0)}(\lambda) \\ \bar{U}^{(1)}(\lambda) \\ \bar{U}^{(2)}(\lambda) \\ \vdots \\ \bar{U}^{(m-1)}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ u_1(\lambda) & u_2(\lambda) & u_3(\lambda) & \cdots & u_m(\lambda) \\ u_1^2(\lambda) & u_2^2(\lambda) & u_3^2(\lambda) & \cdots & u_m^2(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1^{m-1}(\lambda) & u_2^{m-1}(\lambda) & u_3^{m-1}(\lambda) & \cdots & u_m^{m-1}(\lambda) \end{pmatrix},$$

is an $m \times m$ matrix.

Similarly, denote $\bar{U}^{(k)}(0) = \lim_{\lambda \to 0} \bar{U}^{(k)}(\lambda)$ and $W(0) = \lim_{\lambda \to 0} W(\lambda)$. Recall the assumption that $u_1(\lambda), u_2(\lambda), \cdots, u_m(\lambda)$ are the m different roots of the equation $B_\lambda(s) = 0$, it is clearly that $W(\lambda)$ $(\forall \lambda \geq 0)$ is invertible since

$$det(W(\lambda)) = \Pi_{1 \leq j < i \leq m}(u_i(\lambda) - u_j(\lambda)) \neq 0.$$

Denote $A_{(k)} = (a_{k0}, a_{k1}, \cdots, a_{k,m-1})$ $(k \geq m)$ be the absorbing probability vector, and denote $A_{(k)}(\lambda) = (\lambda\phi_{k0}^*(\lambda), \lambda\phi_{k1}^*(\lambda), \cdots, \lambda\phi_{k,m-1}^*(\lambda))$.

Proposition 3.3.1. *The Laplace transform of the extinction time*

$$\tau^* = \left(\tau_0^*, \tau_1^*, \cdots, \tau_{m-1}^*\right)$$

is given by

$$\begin{array}{c}\int_0^\infty e^{-\lambda t} P(\tau_0^* \le t | X(0)=k)dt = \phi_{k0}^*(\lambda)\\ \int_0^\infty e^{-\lambda t} P(\tau_1^* \le t | X(0)=k)dt = \phi_{k1}^*(\lambda)\\ \cdots\\ \int_0^\infty e^{-\lambda t} P(\tau_{m-1}^* \le t | X(0)=k)dt = \phi_{k,m-1}^*(\lambda),\end{array}$$

where $\phi_{k0}^*(\lambda), \phi_{k1}^*(\lambda), \cdots, \phi_{k,m-1}^*(\lambda)$ are the resolvent of the Feller minimal Q^*-function.

The mean extinction time to the state j is that

$$E_k(\tau_j^* | X^*(\infty)=j) = \int_0^\infty P(\tau_j^* > t | X^*(\infty)=j)dt = \lim_{\lambda\to 0} \frac{a_{kj} - \lambda\phi_{kj}^*(\lambda)}{\lambda}, 0 \le j \le m-1.$$

That is the mean extinction time vector is just

$$\lim_{\lambda\to 0} \frac{1}{\lambda}(A_{(k)} - A_{(k)}(\lambda)),$$

which is indeed $(-1)A_k'(0)$. Therefore, the only thing to do is to calculate $A_{(k)}'(\lambda)$ and then let $\lambda = 0$. Moreover, the overall extinction probability is given by

$$P(\tau^* \le t) = P\left(\cup_{j=0}^{m-1}\{\tau_j^* \le t\}\right) = \sum_{j=0}^{m-1} P(\tau_j^* \le t).$$

Then, the mean extinction time starting from the state k is given by

$$\begin{aligned} E_k(\tau^*) &= \int_0^\infty P(\tau^* > t | X(0)=k)dt \\ &= \lim_{\lambda\to 0} \frac{1 - \sum_{j=0}^{m-1} \lambda\phi_{kj}^*(\lambda)}{\lambda} \\ &= \lim_{\lambda\to 0} \frac{\sum_{j=0}^{\infty} \lambda\phi_{kj}^*(\lambda) - \sum_{j=0}^{m-1} \lambda\phi_{kj}^*(\lambda)}{\lambda} \\ &= \lim_{\lambda\to 0} \sum_{j=m}^{\infty} \phi_{ij}^*(\lambda). \end{aligned}$$

Proposition 3.3.2. *The mean extinction time is finite if and only if $B'(1) < 0$.*

The proof could be seen in Theorem 3.6 of J. P. Li, A. Y. Chen and K. W. Ng (2012).

Theorem 3.3.3. *Again assume that all the q_l are different, then the mean extinction time exists and can be expressed by*

$$(-1)A'_{(k)}(0) = \left[\bar{U}^{(k)}(0)W^{-1}(0)W'(0) - \bar{U}^{(k)\prime}(0)\right] W^{-1}(0) \tag{3.3.4}$$

Proof. Note from (3.2.1) that

$$\sum_{k=0}^{m-1} \phi^*_{ik}(\lambda)(u_l(\lambda))^k = \frac{1}{\lambda}(u_l(\lambda))^i. \ (1 \leq l \leq m),$$

thus, express it by the matrix form will be

$$A_{(k)}(\lambda)W(\lambda) = \bar{U}^{(k)}(\lambda) \ (k \geq m),$$

and let $\lambda \to 0$ yields

$$A_{(k)}(0)W(0) = \bar{U}^{(k)}(0) \ (k \geq m).$$

Then we have

$$A_{(k)}(\lambda) = \bar{U}^{(k)}(\lambda)W^{-1}(\lambda), \tag{3.3.5}$$

because both $W(\lambda)$ and $W(0)$ are invertible matrices. Then differentiating both sides of the equation (3.3.5) leads to

$$A'_{(k)}(\lambda) = \frac{d\bar{U}^{(k)}(\lambda)}{d\lambda}W^{-1}(\lambda) + \bar{U}^{(k)}(\lambda)\frac{dW^{-1}(\lambda)}{d\lambda}. \tag{3.3.6}$$

Denote $C(\lambda) = W^{-1}(\lambda)$ for simplicity, which is still a $m \times m$ matrix, then (3.3.5) and (3.3.6) will be

$$A_{(k)}(\lambda) = \bar{U}^{(k)}(\lambda)C(\lambda),$$

and

$$A'_{(k)}(\lambda) = \bar{U}^{(k)\prime}(\lambda)C(\lambda) + \bar{U}^{(k)}(\lambda)C'(\lambda)$$

respectively. Now let $\lambda \to 0$ on both sides and use its continuous property leads to

$$A'_{(k)}(0) = \bar{U}'_{(k)}(0)C(0) + \bar{U}_{(k)}(0)C'(0).$$

In order to obtain $C'(0)$, $C'(\lambda)$ should be calculated. Obviously,

$$\begin{aligned} & W(\lambda)C(\lambda) = I \\ \Rightarrow\quad & W'(\lambda)C(\lambda) + W(\lambda)C'(\lambda) = 0 \\ \Rightarrow\quad & W'(0)C(0) + W(0)C'(0) = 0 \\ \Rightarrow\quad & C'(0) = -W^{-1}(0)W'(0)C(0) \\ i.e.\quad & C'(0) = -C(0)W'(0)C(0). \end{aligned}$$

Thus, it is easily obtained that

$$\begin{aligned} A'_{(k)}(0) &= \bar{U}'_k(0)C(0) + \bar{U}_k(0)[-C(0)W'(0)C(0)] \\ &= \bar{U}'_k(0)C(0) - \bar{U}_k(0)C(0)W'(0)C(0) \\ &= \left(\bar{U}'_k(0) - \bar{U}_k(0)C(0)W'(0)\right) C(0) \end{aligned} \tag{3.3.7}$$

As $C(\lambda) = W^{-1}(\lambda)$ and $C(0) = W^{-1}(0)$, (3.3.7) is just

$$A'_{(k)}(0) = \left(\bar{U}'_k(0) - \bar{U}_k(0)W^{-1}(0)W'(0)\right) W^{-1}(0), \tag{3.3.8}$$

and thus

$$(-1)A'_{(k)}(0) = \left(\bar{U}_k(0)W^{-1}(0)W'(0) - \bar{U}'_k(0)\right) W^{-1}(0).$$

which is the same as (3.3.4). □

To calculate the overall mean extinction time, just sum all the elements of $(-1)A'_{(k)}(0)$, i.e.

$$E_k(\tau^*) = (-1)A'_{(k)}(0)A^T_{(k)}. \tag{3.3.9}$$

Example 3.3.1. *Take $m = 2$ as an example, let $u(\lambda)$ and $v(\lambda)$ be the two roots of $B_\lambda(s) = 0$, from which $u(\lambda)$ is the unique positive root, then we* *have*

$$W(\lambda) = \begin{bmatrix} 1 & 1 \\ u(\lambda) & v(\lambda) \end{bmatrix}$$

and

$$W'(\lambda) = \begin{bmatrix} 0 & 0 \\ u'(\lambda) & v'(\lambda) \end{bmatrix}.$$

Hence, we have

$$W^{-1}(\lambda) = \frac{1}{v(\lambda) - u(\lambda)} \begin{bmatrix} v(\lambda) & -1 \\ -u(\lambda) & 1 \end{bmatrix},$$

and

$$W^{-1}(0) = \frac{1}{q_* - q} \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix},$$

where $q = \lim_{\lambda \to 0} u(\lambda)$ *and* $q_* = \lim_{\lambda \to 0} v(\lambda)$. *According to Lemma 3.1.3, we have*

$$W'(0) = \begin{bmatrix} 0 & 0 \\ u'(0) & v'(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ q^2/B'(q) & q_*^2/B'(q_*) \end{bmatrix},$$

where

$$\bar{U}^{(k)}(\lambda) = \left(u^k(\lambda), v^k(\lambda)\right)$$

and

$$\bar{U}^{(k)}(0) = \left(q^k, q_*^k\right).$$

Differentiating them results in

$$\bar{U}'^{(k)}(\lambda) = \begin{bmatrix} ku^{k-1}(\lambda)u'(\lambda) \\ kv^{k-1}(\lambda)u'(\lambda) \end{bmatrix}^T,$$

and

$$\bar{U}'^{(k)}(0) = \begin{bmatrix} kq^{k-1}\frac{q^2}{B'(q)} \\ kq_*^{k-1}\frac{q_*^2}{B'(q_*)} \end{bmatrix}^T.$$

Therefore, the mean extinction time vector is given by

$$
\begin{aligned}
(-1)A'_{(k)}(0) &= \left\{ (q^k, q_*^k) \frac{1}{q_* - q} \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ \frac{q^2}{B'(q)} & \frac{q_*^2}{B'(q_*)} \end{bmatrix} \right. \\
&\quad \left. - \left(\frac{kq^{k+1}}{B'(q)}, \frac{kq_*^{k+1}}{B'(q_*)} \right) \right\} \frac{1}{q_* - q} \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix} \\
&= \left\{ (q^k, q_*^k) \frac{1}{q_* - q} \begin{bmatrix} -\frac{q^2}{B'(q)} & -\frac{q_*^2}{B'(q_*)} \\ \frac{q^2}{B'(q)} & \frac{q_*^2}{B'(q_*)} \end{bmatrix} \right. \\
&\quad \left. - \left(\frac{kq^{k+1}}{B'(q)}, \frac{kq_*^{k+1}}{B'(q_*)} \right) \right\} \frac{1}{q_* - q} \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix} \\
&= \left\{ \frac{1}{q_* - q} \left[\frac{q_*^k q^2 - q^{k+2}}{B'(q)}, \frac{q_*^{k+2} - q_*^2 q^k}{B'(q_*)} \right] \right. \\
&\quad \left. - \left(\frac{kq^{k+1}}{B'(q)}, \frac{kq_*^{k+1}}{B'(q_*)} \right) \right\} \frac{1}{q_* - q} \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix} \\
&= \frac{1}{(q_* - q)^2} \begin{bmatrix} \frac{q_*^k q^2 + (k-1)q^{k+2} - kq^{k+1} q_*}{B'(q)} \\ \frac{(1-k)q_*^{k+2} - q_*^2 q^k + kq_*^{k+1} q}{B'(q_*)} \end{bmatrix}^T \cdot \begin{bmatrix} q_* & -1 \\ -q & 1 \end{bmatrix} \\
&= \frac{1}{(q_* - q)^2} \begin{bmatrix} \frac{q_*^{k+1} q^2 + (k-1)q^{k+2} q_* - kq^{k+1} q_*^2}{B'(q)} - \frac{(1-k)q_*^{k+2} q - q_*^2 q^k + kq_*^{k+1} q}{B'(q_*)} \\ -\frac{q_*^k q^2 + (k-1)q^{k+2} - kq^{k+1} q_*}{B'(q)} + \frac{(1-k)q_*^{k+2} - q_*^2 q^{k+1} + kq_*^{k+1} q^2}{B'(q_*)} \end{bmatrix}^T .
\end{aligned}
$$

Particularly, if $B'(1) \leq 0$, i.e. $q = 1$, then

$$E_k(\tau^*) = \frac{k - 1 - kq_* + q_*^k}{(q_* - 1)B'(1)}.$$

On the other hand, if $B'(1) > 0$, then the mean extinction time must be infinite since the overall extinction probability is strictly less than 1. Therefore,

$$E_k(\tau^*) = \begin{cases} \frac{k-1-kq_*+q_*^k}{(q_*-1)B'(1)} & \text{if } m_b < m_d \\ +\infty & \text{if } m_b \geq m_d \end{cases} .$$

Note that the assumption that $W(0)$ is invertible (*i.e.* $B(s) = 0$ has no multiple roots) is necessary when we obtain (3.3.4). From $A_{(k)}(\lambda)W(\lambda) =$

$\bar{U}^{(k)}(\lambda)$, we have

$$A_{(k)}(\lambda) = \bar{U}^{(k)}(\lambda)W^{-1}(\lambda),$$

where $A_{(k)}(\lambda) = (\lambda\phi_{k0}(\lambda), \lambda\phi_{k1}(\lambda), \cdots, \lambda\phi_{k,m-1}(\lambda))$. Afterwards, using the property that $\bar{U}^{(k)}(\lambda)$ and $W^{-1}(\lambda)$ are both infinite order differentiable functions on $(0, \infty)$, we can find the derivative of them and let $\lambda \to 0^+$. Finally, we can get the result by using the continuous property on 0.

3.4 Decomposition Theorem and Resuming the Process

Up to now, only the matrix Q^* with m absorbing states has been discussed. The resolvent and the transition function calculated were only regarding to Q^*. The aim of this subsection is to explore the resolvent of the whole q-matrix Q, which is the main model in the thesis by resuming the process from known $\phi^*_{ij}(\lambda), i, j \in E$. The method is to apply the decomposition theorem, which was initially studied by A. Y. Chen, H. J. Zhang and Z. T. Hou (2002).

Let Q be a regular q-matrix, which is defined on a countable (might be infinite) state space $E = \{0, 1, 2, \cdots\}$. Define $F = \{0, 1, 2, \cdots, m-1\}$ and $G = E \setminus F = \{m, m+1, \cdots\}$, then Q could be partitioned as

$$Q = \begin{pmatrix} Q^{FF} & Q^{FG} \\ Q^{GF} & Q^{GG} \end{pmatrix},$$

where

$$Q^{FF} = \begin{pmatrix} q_{00} & q_{01} & q_{02} & \cdots & q_{0,m-1} \\ q_{10} & q_{11} & q_{12} & \cdots & q_{1,m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{m-1,0} & q_{m-1,1} & q_{m-1,2} & \cdots & q_{m-1,m-1} \end{pmatrix},$$

$$Q^{FG} = \begin{pmatrix} q_{0m} & q_{0,m+1} & q_{0,m+2} & \cdots \\ q_{1m} & q_{1,m+1} & q_{1,m+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ q_{m-1,m} & q_{m-1,m+1} & q_{m-1,m+2} & \cdots \end{pmatrix},$$

$$Q^{GF} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-2} \\ 0 & 0 & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

and

$$Q^{GG} = \begin{pmatrix} b_m & b_{m+1} & b_{m+2} & \cdots \\ b_{m-1} & b_m & b_{m+1} & \cdots \\ b_{m-2} & b_{m-1} & b_m & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Furthermore, let $R(\lambda) = \{r_{ij}(\lambda),\, \lambda > 0, i, j \in E\}$ be the resolvent of the Feller minimal Q-function. Similarly, $R(\lambda)$ can be partitioned as the form

$$R(\lambda) = \begin{pmatrix} R^{FF}(\lambda) & R^{FG}(\lambda) \\ R^{GF}(\lambda) & R^{GG}(\lambda) \end{pmatrix}.$$

Proposition 3.4.1. *The resolvent of the Feller minimal Q-function $R(\lambda)$ could be written by the form*

$$R(\lambda) = \begin{pmatrix} A(\lambda) & A(\lambda)\eta(\lambda) \\ \xi(\lambda)A(\lambda) & \xi(\lambda)A(\lambda)\eta(\lambda) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Psi(\lambda) \end{pmatrix}, \qquad (3.4.1)$$

where

$$\Psi(\lambda) = \{\psi_{ij}(\lambda) : \psi_{ij}(\lambda) = \phi^*_{ij}(\lambda), i, j \in G\};$$

$$\eta(\lambda) = Q^{FG}\Psi(\lambda),$$

$$i.e.\ \eta_{aj}(\lambda) = \sum_{k=m}^{\infty} q_{ak}\phi^*_{kj}(\lambda)\ \ (a \in F, j \in G);$$

and

$$\xi(\lambda) = \Psi(\lambda) Q^{GF}.$$

That is

$$r_{ij}(\lambda) = \begin{cases} \phi^*_{ij}(\lambda) + \lambda \sum_{l=0}^{m-1} \sum_{k=0}^{m-1} \phi^*_{ik}(\lambda) f_{kl}(\lambda) \eta_{lj}(\lambda) & i \in G, j \in G \\ \sum_{k=0}^{m-1} f_{ik}(\lambda) \eta_{kj}(\lambda) & i \in F, j \in G \\ \sum_{k=0}^{m-1} \xi_{ik}(\lambda) f_{kj}(\lambda) & i \in G, j \in F \\ f_{ij}(\lambda) & i \in F, j \in F \end{cases},$$

where for any $i, j \in F$,

$$r_{ij}(\lambda) =: f_{ij}(\lambda) = \lambda \delta_{ij} + \lambda \sum_{k=m}^{\infty} \eta_{ik}(\lambda) \xi_{kj}.$$

Moreover, after writing the column vector

$$\eta(\lambda) = \left(\eta^{(0)}(\lambda), \eta^{(1)}(\lambda), \cdots, \eta^{(m-1)}(\lambda)\right)^T$$

and the row vector

$$\xi(\lambda) = \left(\xi^{(0)}(\lambda), \xi^{(1)}(\lambda), \cdots, \xi^{(m-1)}(\lambda)\right),$$

where for $i \in F$*, the row vector*

$$\eta^{(i)}(\lambda) = \{\eta_{ij}(\lambda), j \in G\},$$

and for $i \in G$*, the column vector*

$$\xi^{(j)}(\lambda) = \{\xi_{ij}(\lambda), i \in G\},$$

we have that

$$A(\lambda) = \left(C + \lambda I + \lambda \langle \eta(\lambda), \xi(\lambda) \rangle\right)^{-1}, \tag{3.4.2}$$

where

$$\langle \eta(\lambda), \xi(\lambda) \rangle = \left\{ d_{ij} : d_{ij} = \lambda \langle \eta^{(i)}(\lambda), \xi^{(j)}(\lambda) \rangle = \sum_{k \in G} \eta_{ik} \xi_{kj} \right\}$$

and $C = \{c_{ij}\}$ takes the form

$$c_{ij} = \begin{cases} q_{ij} + \lim_{\lambda \to +\infty} \lambda \langle \eta^{(i)}(\lambda), \xi^{(j)}(\lambda) \rangle & \textit{if } i \neq j \\ \sum_{j \neq i} c_{ij} + \lim_{\lambda \to +\infty} \lambda \langle \eta^{(i)}(\lambda), 1 - \sum_j \xi^{(j)}(\lambda) \rangle & \textit{if } i = j \end{cases} . \quad (3.4.3)$$

Note that E is stable, we further have

$$c_{ii} = q_i - \lim_{\lambda \to +\infty} \lambda \langle \eta^{(i)}(\lambda), \xi^{(i)}(\lambda) \rangle. \quad (3.4.4)$$

Now, each element of the resolvent $R(\lambda)$ has been expressed in Proposition 3.4.1. Therefore, the minimal Q-resolvent $R(\lambda)$ could be obtained by applying the Q^*-resolvent $\Phi^*(\lambda)$ analyzed in Section 3.2 to the proposition, and the transition function $p_{ij}(t)$ could be get again by inverting its resolvent. Hence, in theoretically, each element of the (unique) transition function of the $p_{ij}(t), i, j \in N$ can be expressed by all the roots of $B_\lambda(s) = 0$ even if it is hard to calculate the exact values of all the $u_l(\lambda)$.

Remark 3.4.1. *From Proposition 3.4.1, it is obvious that $\Psi(\lambda)$ is totally determined by both of the Q^{GF} and Q^{GG} parts, and then so does $\xi(\lambda)$. Return to our basic model and rewrite*

$$Q^* = \boldsymbol{0} + Q^*,$$

then we can still apply Proposition 3.4.1 to obtain the resolvent of Q^*-function. Since the Q^{GF} and Q^{GG} parts are equal between Q and Q^*, the $\xi(\lambda)$s of both q-matrix must also be equal. Note in Q^*, as the states $\{0, 1, \cdots, m\}$ are all absorbing states, it is clear that

$$A^*(\lambda) = \frac{1}{\lambda} I_m,$$

where $A^*(\lambda)$ is the corresponding $A(\lambda)$ of Q^*. Thus, it is obtained that

$$\frac{\xi(\lambda)}{\lambda} = \Phi^*_{GF}(\lambda),$$

$$i.e.\ \xi_{ij}(\lambda) = \lambda\phi^*_{ij}(\lambda), i \in G, j \in F.$$

3.5 Recurrence and Equilibrium Distribution

As the transition function $p_{ij}(t), i, j \in N$ of our queueing model has been discussed, what is interesting are the properties incorporating the equilibrium behaviour. Firstly, assume that Q is irreducible, then the minimal Q-transition function is also irreducible. According to Theorem 2.1.5, the equilibrium distribution exists iff the Markov chain is positive recurrent. Therefore, the recurrence property will be discussed first in this section.

3.5.1 Recurrence Properties of the Q-process

Theorem 3.5.1. *The Q-process is recurrent if and only if $B'(1) \leq 0$ and is, furthermore, positive recurrent if and only if $B'(1) < 0$ and $\sum_{j=0}^{\infty} jq_{ij} < +\infty$ for all $i \in \{0, 1, \cdots, m-1\}$.*

Based on Theorem 3.5.1, other important properties regarding our new (recurrent) queueing model, including equilibrium distribution, mean queueing length and mean busy period, etc. should be able to be obtained. In the remaining of the Chapter, the positive recurrent Q-process is only considered.

3.5.2 Equilibrium Distribution

The assumption that Q is irreducible and positive recurrent ensures that its equilibrium distribution exists. Denote $\Pi = (\pi_0, \pi_1, \cdots, \pi_m, \cdots)$ be the equilibrium distribution.

Theorem 3.5.2. *In order to avoid confusion, let $h_{ij} = q_{ij}$ for all $i < m$ here, i.e.*

$$Q = \begin{bmatrix} h_{00} & h_{01} & h_{02} & \cdots \\ h_{10} & h_{11} & h_{12} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ h_{m-1,0} & h_{m-1,1} & h_{m-1,2} & \cdots \\ b_0 & b_1 & b_2 & \cdots \\ 0 & b_0 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and define the generating functions $H_k(s) = \sum_{j=0}^{\infty} h_{kj}s^j$ $(0 \le k \le m-1)$. The equilibrium distribution of the first m states $\pi_0, \pi_1, \cdots, \pi_{m-1}$ is the solution of the following m equations:

$$1 = \sum_{k=0}^{m-1} \pi_k \frac{B'(1) - H_k'(1)}{B'(1)}, \tag{3.5.1}$$

$$\sum_{k=0}^{m-1} \pi_k H_k(q_l) = 0 \ (l = 1, 2, \cdots, m-1), \tag{3.5.2}$$

where q_l $(l = 1, 2, \cdots, m-1)$ are the roots of $B(s) = 0$ other than $s = 1$. The generating function of the remaining π_k $(k \ge m)$ is given by

$$\sum_{k=m}^{\infty} \pi_k s^k = -\frac{s^m \sum_{k=0}^{m-1} \pi_k H_k(s)}{B(s)}. \tag{3.5.3}$$

Proof. First of all, we use the same technique as Theorem 3.2.1 to prove the equilibrium incorporating the first m state. $\Pi Q = 0$ leads to

$$\sum_{k=0}^{\infty} \pi_k q_{kj} = \sum_{k=0}^{m-1} \pi_k h_{kj} + \sum_{k=m}^{j+m} \pi_k b_{j-k+m} = 0 \ (j \ge 0).$$

For every $j \in \{0, 1, \cdots\}$, multiply s^j on both sides of the equation and then sum up all these equations will result

$$
\begin{aligned}
0 &= \sum_{j=0}^{\infty}(\sum_{k=0}^{m-1}\pi_k h_{kj})s^j + \sum_{j=0}^{\infty}(\sum_{k=m}^{j+m}\pi_k b_{j-k+m})s^j \\
&= \sum_{j=0}^{\infty}[\sum_{i=0}^{m-1}\pi_i h_{ij} + \sum_{i=m}^{m+j}\pi_i b_{j-i+m}]s^j \\
&= \sum_{i=0}^{m-1}\pi_i \sum_{j=0}^{\infty} h_{ij}s^j + \sum_{j=0}^{\infty}(\sum_{i=m}^{m+j}\pi_i b_{j-i+m})s^j \\
&= \sum_{i=0}^{m-1}\pi_i \sum_{j=0}^{\infty} h_{ij}s^j + \sum_{i=m}^{\infty}\pi_i \sum_{j=i-m}^{\infty} b_{j-i+m}s^j \\
&= \sum_{i=0}^{m-1}\pi_i \sum_{j=0}^{\infty} h_{ij}s^j + \sum_{i=m}^{\infty}\pi_i s^{i-m} \sum_{j=0}^{\infty} b_j s^j. \qquad (3.5.4)
\end{aligned}
$$

Substitute $B(s)$ and $H_k(s)(0 \leq k \leq m-1)$ to $\sum_{j=0}^{\infty} b_j s^j$ and $\sum_{j=0}^{\infty} h_{kj}s^j$ respectively, then (3.5.4) will become

$$B(s)\Pi(s) - B(s)\sum_{k=0}^{m-1}\pi_k s^k + s^m \sum_{k=0}^{m-1}\pi_k H_k(s) = 0.$$

In other words,

$$
\begin{aligned}
\Pi(s)\frac{B(s)}{s^m} &= \frac{B(s)}{s^m}\sum_{k=0}^{m-1}\pi_k s^k - \sum_{k=0}^{m-1}\pi_k H_k(s), \\
\Rightarrow \quad \Pi(s) &= \sum_{k=0}^{m-1}\pi_k s^k - \frac{s^m \sum_{k=0}^{m-1}\pi_k H_k(s)}{B(s)},
\end{aligned}
$$

which is certainly (3.5.3). Therefore, checking the value of π_k for all $0 \leq k \leq m-1$ is quite enough to finish the proof.

According to Theorem 3.5.1, $B'(1) < 0$ for the case of positive recurrence, which results in that $B(s) = 0$ has exactly m roots in $\{s : |s| \leq 1\}$ based on Lemma 3.1.1. Again from Theorem 3.5.1, we have that $H'(1) = \sum_{j=0}^{\infty} jh_{kj} < +\infty$. Note that $s = 1$ is the root of all the $H_k(s)$ and $B(s)$, the following equation is obvious by applying L'Hopital Law:

$$\sum_{k=m}^{\infty} \pi_k = -\sum_{k=0}^{m-1} \pi_k \lim_{s\to 1} \frac{H_k(s)}{B(s)} = -\sum_{k=0}^{m-1} \pi_k \frac{H_k'(1)}{B'(1)},$$

which brings about (3.5.1). Substitute the remaining $m-1$ roots q_l ($l = 1, 2, \cdots, m-1$) except the positive one to s in (3.5.3) from $B(s) = 0$, then, similarly, the numerator of the fraction must be zero since the denominator part is 0 in these cases, and the left side of the equation is finite. In other word, for all $l = 1, 2, \cdots, m-1$,

$$\sum_{k=0}^{m-1} \pi_k H_k(q_l) = 0,$$

which is indeed (3.5.2). $\square$

We do not need to show the existence of these roots for they are determined by the initial conditions. It is quite convenient to get some properties concerning the queue size in the equilibrium distribution theoretically.

Now we want the solve the m equations of (3.5.1) and (3.5.2) by similar way, then the coefficient determinant is

$$\Delta = \begin{vmatrix} 1 - \frac{H_0'(1)}{B'(1)} & 1 - \frac{H_1'(1)}{B'(1)} & \cdots & 1 - \frac{H_{m-1}'(1)}{B'(1)} \\ H_0(q_1) & H_1(q_1) & \cdots & H_{m-1}(q_1) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(q_{m-1}) & H_1(q_{m-1}) & \cdots & H_{m-1}(q_{m-1}) \end{vmatrix}.$$

Thus, the solution is

$$\pi_0 = \frac{1}{\Delta} \begin{vmatrix} H_1(q_1) & H_2(q_1) & \cdots & H_{m-1}(q_1) \\ H_1(q_2) & H_2(q_2) & \cdots & H_{m-1}(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ H_1(q_{m-1}) & H_2(q_{m-1}) & \cdots & H_{m-1}(q_{m-1}) \end{vmatrix},$$

$$\pi_1 = \frac{1}{\Delta}\begin{vmatrix} H_0(q_1) & H_2(q_1) & \cdots & H_{m-1}(q_1) \\ H_0(q_2) & H_2(q_2) & \cdots & H_{m-1}(q_2) \\ \vdots & \ddots & \vdots & \\ H_0(q_{m-1}) & H_2(q_{m-1}) & \cdots & H_{m-1}(q_{m-1}) \end{vmatrix},$$

and the other π_i could be expressed similarly.

3.6 The Mean Queue Length

In queue theory, the queue length distribution is also an interesting aspect. In this section, we want to discuss it. We also assume that at least one of q_{ij} for each $i \in \{0, 1, \cdots, m-1\}$ is nonzero to avoid the extreme case of absorbing states.

Define

$$Y_i(\lambda, s) = \sum_{k=m}^{\infty} q_{ik} \sum_{j=m}^{\infty} \phi^*_{kj}(\lambda) s^j. \ (0 \le i \le m-1) \tag{3.6.1}$$

Theorem 3.6.1. *The generating functions of the resolvent of the Q-function $R(\lambda) = \{r_{ij}(\lambda) : i, j \ge 0\}$ are given by*

$$\sum_{j=0}^{\infty} r_{ij}(\lambda) s^j = \sum_{j=0}^{m-1} r_{ij}(\lambda)\left(s^j + Y_j(\lambda, s)\right) \tag{3.6.2}$$

for $0 \le i < m$, and on the other hand, i.e. for $i \ge m$,

$$\begin{aligned} \sum_{j=0}^{\infty} r_{ij}(\lambda) s^j \quad &= \quad \sum_{j=m}^{\infty} \phi^*_{ij}(\lambda) s^j \\ &\quad + \lambda \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \phi^*_{ik}(\lambda) r_{kj}(\lambda)\left(s^j + Y_j(\lambda, s)\right), \end{aligned} \tag{3.6.3}$$

where each $Y_j(\lambda, s)$ has been defined in (3.6.1).

Proof. From Proposition 3.4.1, we have that the Q-resolvent $R(\lambda) = \{r_{ij}(\lambda)\}$

is given by

$$r_{ij}(\lambda) = \begin{cases} \phi^*_{ij}(\lambda) + \lambda \sum_{l=0}^{m-1} \sum_{k=0}^{m-1} \phi^*_{ik}(\lambda) f_{kl}(\lambda) \eta_{lj}(\lambda) & i \in G, j \in G \\ \sum_{k=0}^{m-1} f_{ik}(\lambda) \left(\sum_{t=m}^{\infty} q_{kt} \phi^*_{tj}(\lambda) \right) & i \in F, j \in G \\ \sum_{k=0}^{m-1} \lambda \phi^*_{ik}(\lambda) f_{kj}(\lambda) & i \in G, j \in F \\ f_{ij}(\lambda) & i \in F, j \in F \end{cases}, \tag{3.6.4}$$

where

$$f_{ij}(\lambda) = \lambda \delta_{ij} + \lambda \sum_{k=m}^{\infty} \left(\sum_{t=m}^{\infty} q_{it} \phi^*_{tk}(\lambda) \right) \lambda \phi^*_{kj}(\lambda).$$

Then, for each $0 \leq i \leq m-1$, it is satisfied that

$$\begin{aligned} \sum_{j=0}^{\infty} r_{ij}(\lambda) s^j &= \sum_{j=0}^{m-1} r_{ij}(\lambda) s^j + \sum_{j=m}^{\infty} \left[\sum_{k=0}^{m-1} r_{ik}(\lambda) \left(\sum_{t=m}^{\infty} q_{kt} \phi^*_{tj}(\lambda) \right) \right] s^j \\ &= \sum_{j=0}^{m-1} r_{ij}(\lambda) s^j + \sum_{j=m}^{\infty} \sum_{k=0}^{m-1} \sum_{t=m}^{\infty} r_{ik}(\lambda) q_{kt} \phi^*_{tj}(\lambda) s^j \\ &= \sum_{j=0}^{m-1} r_{ij}(\lambda) s^j + \sum_{k=0}^{m-1} r_{ik}(\lambda) \sum_{t=m}^{\infty} q_{kt} \sum_{j=m}^{\infty} \phi^*_{tj}(\lambda) s^j \\ &= \sum_{j=0}^{m-1} r_{ij}(\lambda) s^j + \sum_{k=0}^{m-1} r_{ik}(\lambda) Y_k(\lambda, s) \\ &= \sum_{j=0}^{m-1} r_{ij}(\lambda) \left(s^j + Y_j(\lambda, s) \right), \end{aligned}$$

which is indeed (3.6.2); and for each $i \geq m$,

$$\begin{aligned}
\sum_{j=0}^{\infty} r_{ij}(\lambda)s^j &= \sum_{j=0}^{m-1}\left(\sum_{k=0}^{m-1}\lambda\phi_{ik}^*(\lambda)f_{kj}(\lambda)\right)s^j \\
&+\sum_{j=m}^{\infty}\left[\phi_{ij}^*(\lambda)+\lambda\sum_{l=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kl}(\lambda)\left(\sum_{t=m}^{\infty}q_{lt}\phi_{tj}^*(\lambda)\right)\right]s^j \\
&= \sum_{j=m}^{\infty}\phi_{ij}^*(\lambda)s^j+\lambda\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kj}(\lambda)s^j \\
&+\lambda\sum_{j=m}^{\infty}\sum_{l=0}^{m-1}\sum_{k=0}^{m-1}\sum_{t=m}^{\infty}\phi_{ik}^*(\lambda)r_{kl}(\lambda)q_{lt}\phi_{tj}^*(\lambda)s^j \\
&= \sum_{j=m}^{\infty}\phi_{ij}^*(\lambda)s^j+\lambda\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kj}(\lambda)s^j \\
&+\lambda\sum_{l=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kl}(\lambda)\sum_{j=m}^{\infty}\sum_{t=m}^{\infty}q_{lt}\phi_{tj}^*(\lambda)s^j \\
&= \sum_{j=m}^{\infty}\phi_{ij}^*(\lambda)s^j+\lambda\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kj}(\lambda)s^j \\
&+\lambda\sum_{l=0}^{m-1}\sum_{k=0}^{m-1}\phi_{ik}^*(\lambda)r_{kl}(\lambda)Y_t(\lambda,s),
\end{aligned}$$

which is (3.6.3). Thus, the proof is complete. □

Now, denote $m_i(t)$ be the mean length of our Markov queueing process at the moment t ($t > 0$) under the condition that the process starts from the state i (the initial queue length is i), i.e.

$$m_i(t) = E[X_t|X_0 = i] = \sum_{j=0}^{\infty} jp_{ij}(t). \tag{3.6.5}$$

Furthermore, define $\varsigma_i(\lambda)$ as the Laplace transform of $m_i(t)$.

Theorem 3.6.2. *The Laplace transforms of the mean queueing length functions, $\varsigma_i(\lambda)$, are given by*

$$\varsigma_i(\lambda) = \sum_{j=0}^{m-1} r_{ij}(\lambda)\left(j + \frac{\partial Y_j(\lambda, s)}{\partial s}|_{s=1}\right) \tag{3.6.6}$$

for all $0 \leq i \leq m-1$*; and for* $i \geq m$*,*

$$\begin{aligned} \varsigma_i(\lambda) &= \sum_{j=m}^{\infty} j\phi_{ij}^*(\lambda) \\ &+ \lambda \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) r_{kj}(\lambda) \left(j + \frac{\partial Y_j(\lambda, s)}{\partial s}|_{s=1} \right), \end{aligned} \tag{3.6.7}$$

where each $Y_j(\lambda, s)$ *has been defined in (3.6.1).*

Proof. Since $\varsigma_i(\lambda)$ is the Laplace transform of $m_i(t)$, the following equation holds true because of (3.6.5):

$$\varsigma_i(\lambda) = \sum_{j=0}^{\infty} j r_{ij}(\lambda) = \frac{\partial \left(\sum_{j=0}^{\infty} r_{ij}(\lambda) s^j \right)}{\partial s}|_{s=1}. \tag{3.6.8}$$

As $0 \leq i \leq m-1$, combine (3.6.2) and (3.6.8) will yield

$$\varsigma_i(\lambda) = \sum_{j=0}^{m-1} r_{ij}(\lambda) \left(j + \frac{\partial Y_j(\lambda, s)}{\partial s}|_{s=1} \right),$$

which is (3.6.6); $i \geq m$, combine (3.6.3) and (3.6.8) will yield

$$\begin{aligned} \varsigma_i(\lambda) &= \sum_{j=m}^{\infty} j\phi_{ij}^*(\lambda) \\ &+ \lambda \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \phi_{ik}^*(\lambda) r_{kj}(\lambda) \left(j + \frac{\partial Y_j(\lambda, s)}{\partial s}|_{s=1} \right), \end{aligned}$$

which is (3.6.7) and thus the proof has been completed. □

3.7 Hitting Time Distribution

The hitting time (denotes τ), which is the time to make the queue system empty (that is $\tau = \inf\{t > 0 : X(t) = 0\}$), is interesting in Markov queue processes. In this section, it is meaningless whether the process will continue or not when it is at the state $i = 0$. Thus, the state 0 can be regarded as an absorbing state. Let $q_{0j} \equiv 0$ ($j \in E$) in our queueing model

(1.2.1)—(1.2.5) and denote the new q-matrix as $Q^{(b)}$, i.e.

$$Q^{(b)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ q_{10} & q_{11} & q_{12} & q_{13} & \cdots \\ q_{20} & q_{21} & q_{22} & q_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ q_{m-1,0} & q_{m-1,1} & q_{m-1,2} & q_{m-2,3} & \cdots \\ b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.7.1}$$

Before discussing the hitting behaviors, we need to calculate the resolvent of the Feller minimal $Q^{(b)}$-function similarly as that of Q^*.

Theorem 3.7.1. *Denote $\Psi_\lambda^{(b)} = \{\psi_{ij}(\lambda), i, j \in N\}$ as the resolvent of the Feller minimal $Q^{(b)}$-function. Then, for all $i \geq 0$ and $0 \leq j \leq m-1$, $\psi_{ij}(\lambda)$ is given by the following m equations:*

$$\lambda\psi_{i0}(\lambda) + \sum_{k=1}^{m-1} \psi_{ik}(\lambda)\left(\lambda(u_l(\lambda))^k - H_k(u_l(\lambda))\right) = (u_l(\lambda))^i, (l = 1, 2, \cdots, m) \tag{3.7.2}$$

where $u_l(\lambda)$ $(l = 1, 2, \cdots, m)$ are all the roots of the equation $B_\lambda(s) = 0$ within the unit circle $\{s : |s| < 1\}$ and for all $0 \leq k \leq m-1$, $H_k(s)$ is the generating function of $\{q_{kj} : j \geq 0\}$, i.e.

$$H_k(s) = \sum_{j=0}^{\infty} q_{kj} s^j.$$

For $j \geq m$, the generating function of the $Q^{(b)}$-resolvent takes the form of

$$\sum_{k=0}^{\infty} \psi_{ik}(\lambda)s^k = \frac{B(s)\sum_{k=0}^{m-1}\psi_{ik}(\lambda)s^k - s^{i+m} - s^m \sum_{k=1}^{m-1} H_k(s)\psi_{ik}(\lambda)}{B(s) - \lambda s^m}. \tag{3.7.3}$$

More specifically,

$$\psi_{0j}(\lambda) = \frac{\delta_{0j}}{\lambda}.$$

Proof. The last part of the theorem is trivial since state 0 is the absorbing state. For the remaining part, we use the similar technique incorporating the Kolmogorov Forward Equations to solve this problem, which is given by

$$\lambda\Psi^{(b)}(\lambda) - I = \Psi^{(b)}(\lambda)Q.$$

Then, for all $i \geq 1$ and j,

$$\begin{aligned}
\lambda\psi_{ij}(\lambda) &= \delta_{ij} + \sum_{k=0}^{\infty}\psi_{ik}(\lambda)q_{kj} = \delta_{ij} + \sum_{k=1}^{\infty}\psi_{ik}(\lambda)q_{kj} \\
&= \delta_{ij} + \sum_{k=1}^{m-1}\psi_{ik}(\lambda)q_{kj} + \sum_{k=m}^{j+m}\psi_{ik}(\lambda)b_{j-k+m}.
\end{aligned}$$

Multiply s^j on both sides and sum all the $\lambda\psi_{ij}s^j$ by j will lead to

$$\begin{aligned}
\Rightarrow \lambda\sum_{j=0}^{\infty}\psi_{ij}(\lambda)s^j &= s^i + \sum_{j=0}^{\infty}\left(\sum_{k=1}^{m-1}\psi_{ik}(\lambda)q_{kj}\right)s^j + \sum_{j=0}^{\infty}\left(\sum_{k=m}^{j+m}\psi_{ik}(\lambda)b_{j-k+m}\right)s^j \\
&= s^i + \sum_{k=1}^{m-1}\psi_{ik}(\lambda)\sum_{j=0}^{\infty}q_{kj}s^j + \sum_{k=m}^{\infty}\sum_{j=k-m}^{\infty}\psi_{ik}(\lambda)b_{j-k+m}s^j \\
&= s^i + \sum_{k=1}^{m-1}\psi_{ik}(\lambda)H_k(s) + \sum_{k=m}^{\infty}\psi_{ik}(\lambda)\sum_{j=0}^{\infty}b_j s^{j+k-m} \\
&= s^i + \sum_{k=1}^{m-1}H_k(s)\psi_{ik}(\lambda) + \left(\sum_{k=m}^{\infty}\psi_{ik}(\lambda)s^{k-m}\right)B(s) \\
&= s^i + \sum_{k=1}^{m-1}H_k(s)\psi_{ik}(\lambda) + \frac{1}{s^m}\left(\sum_{k=m}^{\infty}\psi_{ik}(\lambda)s^k\right)B(s).
\end{aligned}$$

In other words,

$$\begin{aligned}
&(B(s) - \lambda s^m)\sum_{k=0}^{\infty}\psi_{ik}(\lambda)s^k \\
= \;& B(s)\sum_{k=0}^{m-1}\psi_{ik}(\lambda)s^k - s^{i+m} - s^m\sum_{k=1}^{m-1}H_k(s)\psi_{ik}(\lambda),
\end{aligned}$$

So the generating function $\sum_{k=0}^{\infty}\psi_{ik}(\lambda)s^k$ is given by

$$\sum_{k=0}^{\infty}\psi_{ik}(\lambda)s^k = \frac{B(s)\sum_{k=0}^{m-1}\psi_{ik}(\lambda)s^k - s^{i+m} - s^m\sum_{k=1}^{m-1}H_k(s)\psi_{ik}(\lambda)}{B(s)-\lambda s^m}, \tag{3.7.4}$$

which is indeed (3.7.3).

Note that $u_1(\lambda), u_2(\lambda), \cdots, u_m(\lambda)$ are the roots of $B(s)-\lambda s^m = 0$. Thus, the numerator of the right side must be zero by letting $s = u_l(\lambda), 1 \le l \le m$ since the left side is finite at least inside the unit circle $\{s : |s| < 1\}$ and the denominator of the equation is 0 in these cases. That is for each $1 \le l \le m$,

$$B\left(u_l(\lambda)\right)\sum_{k=0}^{m-1}\psi_{ik}(\lambda)(u_l(\lambda))^k - (u_l(\lambda))^{i+m} - (u_l(\lambda))^m\sum_{k=1}^{m-1}H_k(u_l(\lambda))\psi_{ik}(\lambda) = 0. \tag{3.7.5}$$

Recall that

$$B(u_l(\lambda)) = \lambda(u_l(\lambda))^m. \tag{3.7.6}$$

Finally, (3.7.2) is obtained by combining (3.7.5) and (3.7.6). □

In the hitting time problems, firstly we want to know the probability that the process will hit the state 0.

Theorem 3.7.2. *Denote $a_{k0}^{(b)}$ be the probability that the Markov process with the q-matrix $Q^{(b)}$ will eventually hit the idle state 0 starting from the state k, then $a_{k0}^{(b)} = 1$ if and only if $m_b \le m_d$.*

Proof. If $m_b > m_d$, as it was discussed in section 3.3.1, starting from the initial state k, where $k \ge m$, the extinction probability of Q^* (the probability that the process will enter any of the states $\{0, 1, \cdots, m-1\}$) $\sum_{j=0}^{m-1} a_{kj}$ is less than 1, which is obviously no less than the hitting probability $a_{k0}^{(b)}$. That is

$$a_{k0}^{(b)} < 1.$$

Otherwise, from the equation set (3.7.2), it is apparent that

$$\psi_{k0}(\lambda) = \frac{|\Delta_0^{(b)}(\lambda, k)|}{|\Delta^{(b)}(\lambda)|}, \tag{3.7.7}$$

where

$$\Delta^{(b)}(\lambda) = \lambda \begin{bmatrix} 1 & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \cdots & \lambda u_1^{m-1}(\lambda) - H_{m-1}(u_1(\lambda)) \\ 1 & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \cdots & \lambda u_1^{m-1}(\lambda) - H_{m-1}(u_2(\lambda)) \\ \vdots & \vdots & \ddots & \cdots \\ 1 & \lambda u_m(\lambda) - H_1(u_m(\lambda)) & \cdots & \lambda u_m^{m-1}(\lambda) - H_{m-1}(u_m(\lambda)) \end{bmatrix}$$

and

$$\Delta_0^{(b)}(\lambda, k) = \begin{bmatrix} u_1^k(\lambda) & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \cdots & \lambda u_1^{m-1}(\lambda) - H_{m-1}(u_1(\lambda)) \\ u_2^k(\lambda) & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \cdots & \lambda u_1^{m-1}(\lambda) - H_{m-1}(u_2(\lambda)) \\ \vdots & \vdots & \ddots & \cdots \\ u_m^k(\lambda) & \lambda u_m(\lambda) - H_1(u_m(\lambda)) & \cdots & \lambda u_m^{m-1}(\lambda) - H_{m-1}(u_m(\lambda)) \end{bmatrix}.$$

Therefore, recall that $\lim_{\lambda\to 0} u_1(\lambda) = 1$ if $m_b \le m_d$ and $H_i(1) = 0$ $(1 \le i \le m-1)$, the hitting probability $a_{k0}^{(b)}$ is given by

$$\begin{aligned} a_{k0}^{(b)} &= \lim_{\lambda\to 0} \lambda\psi_{k0}(\lambda) = \lim_{\lambda\to 0} \frac{|\Delta_0^{(b)}(\lambda, k)|}{|\Delta^{(b)}(\lambda)|} \\ &= det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ q_2^k & -H_1(q_2) & -H_2(q_2) & \cdots & -H_{m-1}(q_2) \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ q_m^k & -H_1(q_m) & -H_2(q_m) & \cdots & -H_{m-1}(q_m) \end{pmatrix}. \\ & \quad det^{-1} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -H_1(q_2) & -H_2(q_2) & \cdots & -H_{m-1}(q_2) \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 1 & -H_1(q_m) & -H_2(q_m) & \cdots & -H_{m-1}(q_m) \end{pmatrix} \\ &= 1. \end{aligned}$$

Hence, the proof is completed. □

Corollary 3.7.1. *In Theorem 3.7.1, we can find* $\psi_{k0}(\lambda)$*, which is the Laplace transform of the Feller minimal transition function* $p_{k0}^{(b)}(t)$*. In fact, as* 0 *is the absorbing state,* $p_{k0}^{(b)}(t)$ *is exactly the hitting time distribution starting* *from the state k ($k \geq 1$), that is*

$$p_{k0}^{(b)}(t) = P(\tau \leq t | X(0) = k).$$

Moreover, it is clearly that the mean hitting time $E_k(\tau)$, which is given by

$$E_k(\tau) = \lim_{\lambda \to 0} \frac{1 - \lambda \psi_{k0}(\lambda)}{\lambda}$$

is finite if and only if $m_b < m_d$ and $\sum_{j=0}^{\infty} j q_{ij} < \infty$ for each $1 \leq i \leq m-1$.

3.8 Busy Period Distribution

After we discuss the hitting time distribution, the busy period distribution of our queueing process is now able to be considered. Without loss of generality, it is assumed that the initial queue in the queue system is empty, that is $X(0) = 0$. Denote this series of stopping times $\{\sigma_n : n \in N\}$ as

$$\sigma_0 = 0, \sigma_1 = \inf\{t : t \geq 0, X(t) \neq 0\},$$

and for all $n \geq 1$,

$$\sigma_{2n} = \inf\{t : t > \sigma_{2n-1}, X(t) = 0\}, \sigma_{2n+1} = \inf\{t : t > \sigma_{2n}, X(t) \neq 0\}.$$

Its practical meaning is that the queue system is empty from the time 0 to σ_1, and again empty at the time σ_2, and so on. Hence, these random variables $\{\sigma_{2n} - \sigma_{2n-1} : n \geq 1\}$ (generally named as the excursion times leaving the states 0) are described as the busy periods of our queueing model. According to *Itô*'s excursion law (see K. *Itô* (1971)), $\{\sigma_{2n} - \sigma_{2n-1} : n \geq 1\}$ are independent, identically distributed. By applying the strong Markov property, their common distribution is given by

$$
\begin{aligned}
& Pr(\sigma_{2n} - \sigma_{2n-1} \le t) \\
= & \sum_{k=1}^{\infty} P(X(\sigma_{2n-1}) = k) P(\sigma_{2n} - \sigma_{2n-1} \le t \mid X(\sigma_{2n-1}) = k) \\
= & \sum_{k=1}^{\infty} \frac{q_{0k}}{-q_{00}} P(\sigma_{2n} - \sigma_{2n-1} \le t \mid X(\sigma_{2n-1}) = k) \\
= & \sum_{k=1}^{\infty} \frac{q_{0k}}{-q_{00}} P(\sigma_2 - \sigma_1 \le t \mid X(\sigma_1) = k). \qquad (3.8.1)
\end{aligned}
$$

Recall from the last section that $p_{k0}^{(b)}(t)$ is the hitting time distribution starting from the state k ($k \ge 1$), it must be equivalent to the busy period distribution under the condition that $X(\sigma_1) = k$. Hence, the Laplace transform of the busy period distribution $g_T(\lambda)$ is given by

$$
\begin{aligned}
g_T(\lambda) &= \sum_{k=1}^{\infty} \frac{q_{0k}}{-q_{00}} \psi_{k0}(\lambda) \\
&= \sum_{k=1}^{m-1} \frac{q_{0k}}{-q_{00}} \psi_{k0}(\lambda) + \frac{\lambda}{-q_{00}} \sum_{t=1}^{m-1} \psi_{t0}(\lambda) \sum_{k=m}^{\infty} q_{0k} \phi_{kt}^{*}(\lambda). \qquad (3.8.2)
\end{aligned}
$$

Finally, the busy period distribution could be obtained again by the same method to reverse the Laplace transform $g_T(\lambda)$.

3.9 The Example that $m = 3$

In this section, we discuss all the properties in the case of $m = 3$.

Corollary 3.9.1. *According to Theorem 3.2.1, it is easily obtained that for all $i \ge 0$,*

$$
\begin{aligned}
\phi_{i0}^*(\lambda) &= \frac{1}{\lambda}\begin{vmatrix} u_1^i(\lambda) & u_1(\lambda) & u_1^2(\lambda) \\ u_2^i(\lambda) & u_2(\lambda) & u_2^2(\lambda) \\ u_3^i(\lambda) & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix}^{-1} \\
&= \left\{u_1^i(\lambda)u_2(\lambda)u_3(\lambda)\left[u_3(\lambda)-u_2(\lambda)\right]\right. \\
&\quad +u_1(\lambda)u_2^i(\lambda)u_3(\lambda)\left[u_1(\lambda)-u_3(\lambda)\right] \\
&\quad \left.+u_1(\lambda)u_2(\lambda)u_3^i(\lambda)\left[u_2(\lambda)-u_1(\lambda)\right]\right\} \\
&\quad \cdot\left[\lambda\left(u_3(\lambda)-u_2(\lambda)\right)\left(u_3(\lambda)-u_1(\lambda)\right)\left(u_2(\lambda)-u_1(\lambda)\right)\right]^{-1}
\end{aligned} \tag{3.9.1}
$$

$$
\begin{aligned}
\phi_{i1}^*(\lambda) &= \frac{1}{\lambda}\begin{vmatrix} 1 & u_1^i(\lambda) & u_1^2(\lambda) \\ 1 & u_2^i(\lambda) & u_2^2(\lambda) \\ 1 & u_3^i(\lambda) & u_3^2(\lambda) \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix}^{-1} \\
&= \left\{u_1^i(\lambda)\left[u_2^2(\lambda)-u_3^2(\lambda)\right]+u_2^i(\lambda)\left[u_3^2(\lambda)-u_1^2(\lambda)\right]\right. \\
&\quad \left.+u_3^i(\lambda)\left[u_1^2(\lambda)-u_2^2(\lambda)\right]\right\} \\
&\quad \cdot\left[\lambda\left(u_3(\lambda)-u_2(\lambda)\right)\left(u_3(\lambda)-u_1(\lambda)\right)\left(u_2(\lambda)-u_1(\lambda)\right)\right]^{-1}
\end{aligned} \tag{3.9.2}
$$

$$
\begin{aligned}
\phi_{i2}^*(\lambda) &= \frac{1}{\lambda}\begin{vmatrix} 1 & u_1(\lambda) & u_1^i(\lambda) \\ 1 & u_2(\lambda) & u_2^i(\lambda) \\ 1 & u_3(\lambda) & u_3^i(\lambda) \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix}^{-1} \\
&= \left\{u_1^i(\lambda)\left[u_3(\lambda)-u_2(\lambda)\right]+u_2^i(\lambda)\left[u_1(\lambda)-u_3(\lambda)\right]\right. \\
&\quad \left.+u_3^i(\lambda)\left[u_2(\lambda)-u_1(\lambda)\right]\right\} \\
&\quad \cdot\left[\lambda\left(u_3(\lambda)-u_2(\lambda)\right)\left(u_3(\lambda)-u_1(\lambda)\right)\left(u_2(\lambda)-u_1(\lambda)\right)\right]^{-1}
\end{aligned} \tag{3.9.3}
$$

where $u_1(\lambda), u_2(\lambda), u_3(\lambda)$ are the roots of $B_\lambda(s) = 0$ in $|s| < 1$. Particularly, if $i = 0, 1, 2$, then

$$
\phi_{ij}^*(\lambda) = \frac{\delta_{ij}}{\lambda}. \tag{3.9.4}
$$

Moreover, we can use (3.2.5) to find the other elements of the Q^*-resolvent.

Corollary 3.9.2. *According to Theorem 3.3.3, the extinction time vector* *can be calculated. In this example,*

$$\bar{U}^{(k)}(0) = (q_1^k, q_2^k, q_3^k),$$

$$W(0) = \begin{bmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{bmatrix},$$

$$W'(0) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{q_1^3}{B'(q_1)} & \frac{q_2^3}{B'(q_2)} & \frac{q_3^3}{B'(q_3)} \\ \frac{2q_1^4}{B'(q_1)} & \frac{2q_2^4}{B'(q_2)} & \frac{2q_3^4}{B'(q_3)} \end{bmatrix},$$

and

$$W^{-1}(0) = \frac{1}{(q_3 - q_2)(q_3 - q_1)(q_2 - q_1)} \begin{bmatrix} q_2 q_3 & -(q_2 + q_3) & 1 \\ -q_1 q_3 & q_1 + q_3 & -1 \\ q_1 q_2 & -(q_1 + q_2) & 1 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} (-1)A'_{(k)}(0) \quad = \quad & \left\{ \left(q_1^k, q_2^k, q_3^k\right) \begin{bmatrix} q_2 q_3 & -(q_2 + q_3) & 1 \\ -q_1 q_3 & q_1 + q_3 & -1 \\ q_1 q_2 & -(q_1 + q_2) & 1 \end{bmatrix} ((q_3 - q_2)(q_3 - q_1)(q_2 - q_1))^{-1} \right. \\ & \left. \cdot \begin{bmatrix} 0 & 0 & 0 \\ \frac{q_1^3}{B'(q_1)} & \frac{q_2^3}{B'(q_2)} & \frac{q_3^3}{B'(q_3)} \\ \frac{2q_1^4}{B'(q_1)} & \frac{2q_2^4}{B'(q_2)} & \frac{2q_3^4}{B'(q_3)} \end{bmatrix} - \left(\frac{kq_1^{k+2}}{B'(q_1)}, \frac{kq_2^{k+2}}{B'(q_2)}, \frac{kq_3^{k+2}}{B'(q_3)} \right) \right\} \\ & \cdot \begin{bmatrix} q_2 q_3 & -(q_2 + q_3) & 1 \\ -q_1 q_3 & q_1 + q_3 & -1 \\ q_1 q_2 & -(q_1 + q_2) & 1 \end{bmatrix} ((q_3 - q_2)(q_3 - q_1)(q_2 - q_1))^{-1}. \end{aligned}$$

Moreover, if Q is recurrent, i.e. $B'(1) \leq 0$, then the mean extinction time *is that*

$$\begin{aligned}
E_k(\tau^*) &= \lim_{\lambda\to 0}\frac{1-\lambda\left(\phi^*_{k0}(\lambda)+\phi^*_{k1}(\lambda)+\phi^*_{k2}(\lambda)\right)}{\lambda}\\
&= \lim_{\lambda\to 0}\big\{(u_3(\lambda)-u_2(\lambda))u_3(\lambda)u_2(\lambda)(1-u_1(\lambda)^k)-(u_3^2(\lambda)-u_2^2(\lambda))(u_1(\lambda)-u_1^k(\lambda))\\
&\quad +(u_3(\lambda)-u_2(\lambda))(u_1^2(\lambda)-u_1^k(\lambda))+(u_2^k(\lambda)u_3^2(\lambda)-u_3^k(\lambda)u_2^2(\lambda))(1-u_1(\lambda)^2)\\
&\quad +(u_2^k(\lambda)-u_3^k(\lambda))(u_1(\lambda)-u_1^2(\lambda))-(u_2^k(\lambda)u_3(\lambda)-u_2(\lambda)u_3^k(\lambda))(1-u_1^2(\lambda))\big\}\\
&\quad \cdot\{\lambda(u_3(\lambda)-u_2(\lambda))(u_3(\lambda)-u_1(\lambda))(u_2(\lambda)-u_1(\lambda))\}^{-1}\\
&= \frac{k+((q_3-q_2)(q_2-1)(q_3-1))^{-1}\left[(q_3-1)^2(1+q_2)^k+(q_2-1)^2(1+q_3^k)\right]}{m_d-m_b}.
\end{aligned}$$

Corollary 3.9.3. *We now want to calculate the equilibrium distribution in this case, then* (3.5.1) *and* (3.5.2) *will become*

$$\begin{cases}
1=\pi_0\frac{B'(1)-H_0'(1)}{B'(1)}+\pi_1\frac{B'(1)-H_1'(1)}{B'(1)}+\pi_2\frac{B'(1)-H_2'(1)}{B'(1)}\\
0=\pi_0H_0(q_1)+\pi_1H_1(q_1)+\pi_2H_2(q_1)\\
0=\pi_0H_0(q_2)+\pi_1H_1(q_2)+\pi_2H_2(q_2)
\end{cases}$$

where q_1 and q_2 are the roots of $B(s)=0$ except the root $s=1$. Therefore, π_0,π_1,π_2 from the equilibrium distribution can be expressed by

$$\pi_0=\frac{1}{\mid\Delta\mid}\left(H_1(q_1)H_2(q_2)-H_1(q_2)H_2(q_1)\right),$$

$$\pi_1=\frac{1}{\mid\Delta\mid}\left(H_0(q_1)H_2(q_2)-H_0(q_2)H_2(q_1)\right),$$

$$\pi_2=\frac{1}{\mid\Delta\mid}\left(H_0(q_1)H_1(q_2)-H_0(q_2)H_1(q_1)\right),$$

where

$$\Delta=\begin{pmatrix}
\frac{B'(1)-H_0'(1)}{B'(1)} & \frac{B'(1)-H_1'(1)}{B'(1)} & \frac{B'(1)-H_2'(1)}{B'(1)}\\
H_0(q_1) & H_1(q_1) & H_2(q_1)\\
H_0(q_2) & H_1(q_2) & H_2(q_2)
\end{pmatrix}.$$

Corollary 3.9.4. *Note from (3.7.7) that the Laplace transform of the hitting time distribution is that*

$$
\begin{aligned}
\psi_{k0}(\lambda) &= \frac{|\Delta_0^{(b)}(\lambda, k)|}{|\Delta^{(b)}(\lambda)|} \\
&= \begin{vmatrix} u_1^k(\lambda) & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ u_2^k(\lambda) & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_2(\lambda)) \\ u_3^k(\lambda) & \lambda u_3(\lambda) - H_1(u_3(\lambda)) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix} \\
&\quad \cdot \frac{1}{\lambda} \begin{vmatrix} 1 & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ 1 & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \lambda u_2^2(\lambda) - H_2(u_2(\lambda)) \\ 1 & \lambda u_3(\lambda) - H_1(u_3(\lambda)) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix}^{-1}.
\end{aligned}
$$

Corollary 3.9.5. *According to (3.8.2), the Laplace transform of the busy period distribution is given by*

$$
\begin{aligned}
g_T(\lambda) \;=\; & \frac{1}{-\lambda q_{00}} \begin{vmatrix} u_1^k(\lambda) & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ u_2^k(\lambda) & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_2(\lambda)) \\ u_3^k(\lambda) & \lambda u_3(\lambda) - H_1(u_3(\lambda)) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix} \\
& \cdot \begin{vmatrix} 1 & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ 1 & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \lambda u_2^2(\lambda) - H_2(u_2(\lambda)) \\ 1 & \lambda u_3(\lambda) - H_1(u_3(\lambda)) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix}^{-1} \\
& \cdot \left\{ q_{01} + \sum_{k=3}^{\infty} q_{0k} \begin{vmatrix} 1 & u_1(\lambda) & u_1^k(\lambda) \\ 1 & u_2(\lambda) & u_2^k(\lambda) \\ 1 & u_3(\lambda) & u_3^k(\lambda) \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix}^{-1} \right\} \\
& + \frac{1}{-q_{00}} \begin{vmatrix} 1 & u_1^k(\lambda) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ 1 & u_2^k(\lambda) & \lambda u_1^2(\lambda) - H_2(u_2(\lambda)) \\ 1 & u_3^k(\lambda) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix} \\
& \cdot \begin{vmatrix} 1 & \lambda u_1(\lambda) - H_1(u_1(\lambda)) & \lambda u_1^2(\lambda) - H_2(u_1(\lambda)) \\ 1 & \lambda u_2(\lambda) - H_1(u_2(\lambda)) & \lambda u_2^2(\lambda) - H_2(u_2(\lambda)) \\ 1 & \lambda u_3(\lambda) - H_1(u_3(\lambda)) & \lambda u_3^2(\lambda) - H_2(u_3(\lambda)) \end{vmatrix}^{-1} \\
& \cdot \left\{ q_{02} + \sum_{k=3}^{\infty} q_{0k} \begin{vmatrix} 1 & u_1^k(\lambda) & u_1^2(\lambda) \\ 1 & u_2^k(\lambda) & u_2^2(\lambda) \\ 1 & u_3^k(\lambda) & u_3^2(\lambda) \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1(\lambda) & u_1^2(\lambda) \\ 1 & u_2(\lambda) & u_2^2(\lambda) \\ 1 & u_3(\lambda) & u_3^2(\lambda) \end{vmatrix}^{-1} \right\}.
\end{aligned}
$$

3.10 Notes

This queueing model has not been discussed before, even if both the theories of bulk queues and state-dependent control have already been developing. A. Y. Chen, E. Renshaw(2004) considered the special case of $m = 1$, while the case of $m = 2$ has been studied in A. Y. Chen, P. Pollett, J. P. Li, H. J. Zhang (2010). In section 3.1, two properties of the generating function $B(s)$ (Lemma 3.1.1 and Lemma 3.1.2) have been explored in J. P. Li, A. Y. Chen and K. W. Ng (2012), while Lemma 3.1.4 is new, which are appropriate for our more generalized queueing models. The initial idea of the decomposition theorem in section 3.4 comes from A. Y. Chen, H. J. Zhang and Z. T. Hou (2002).

Chapter 4

Decay Parameter and Decay Properties of Our Queueing Model

In the last Chapter, the primary properties of our queueing model have been explored. It was automatically assumed the (positive) recurrence condition of our queueing model Q, especially in the section 3.5—section 3.8. However, this assumption is not reasonable since there are still some practical cases (even rarely seen) that the number in the queue system is increasing. For example, if we regard the birth of each person as the time to arrive in the queue and the death as the time to leave the queue, then, the whole population could be considered as a queue system. In fact, it is impossible that the human will disappear without huge disasters. For such cases, the mean birth rate m_b must be larger than the average death rate m_d and thus such q-matrix is transient. In this Chapter, the decay parameter and the decay properties will be discussed. In order to avoid some trivial cases in analyzing, assume that $\sum_{j=m+1}^{\infty} b_j > 0$.

4.1 Preliminaries

In order to obtain the decay parameter and to discuss its properties, we need some preparations. Some lemmas in this section are the general cases of the lemmas in A. Y. Chen, J. P. Li, Z. T. Hou and K. W. Ng (2010).

First of all, we still discuss the property of the sequence $\{b_0, b_1, \cdots\}$. It has already been defined that

$$B_\lambda(s) = B(s) - \lambda s^m,$$

but at the moment λ can be arbitrary real number. Thus it is the same as

$$B_\lambda(s) = B(s) + \lambda s^m. \tag{4.1.1}$$

It has been proved that $B(s) = 0$ has at least 1 positive root in the interval $(0, 1]$. Denote q_S be the smallest positive root. Moreover, denote ρ be

$$\rho = \frac{1}{\limsup_{j\to+\infty}(b_j)^{\frac{1}{j}}}.$$

It is clear that ρ is the convergence radius, that is $B(s)$ is convergent if $s \in (-\rho, \rho)$.

Lemma 4.1.1. *The generating function $B(s) = 0$ has no more than two roots on $[0, \rho)$ if $\rho = +\infty$. Moreover, $B(s) = 0$ has the unique root if and only if $B'(1) = 0$. In other words, if $B'(1) \neq 0$, then $B(s) = 0$ has exactly two roots on $[0, \rho)$.*

Proof. Since for all $j \geq m + 1$, $b_j \geq 0$ and $\sum_{j=m+1}^{\infty} b_j > 0$, the $(m + 1)$th order derivative of $B(s)$, which is given by

$$B^{(m+1)}(s) = \sum_{j=m+1}^{\infty} \frac{j!}{(j - m - 1)!} b_j s^{j-m-1},$$

is strictly positive for all $s > 0$. Hence $B^{(m)}(s)$ is strictly increasing. It is clearly noted that

$$B^{(m)}(0) = m! \cdot b_m < 0$$

and for all $k = \{0, 1, \cdots, m\}$,

$$\lim_{s\to+\infty} B^{(k)}(s) = +\infty,$$

since the convergence radius is infinite. It is clear that $B^{(m)}(s) = 0$ has exactly 1 root $\eta^{(m)}$ on $(0, +\infty)$. More specifically, $B^{(m)}(s) < 0$ on $[0, \eta^{(m)})$ and $B^{(m)}(s) > 0$ on $(\eta^{(m)}, +\infty)$. Therefore, $B^{(m-1)}(s)$ is strictly decreasing on $[0, \eta^{(m)})$ and strictly increasing on $[\eta^{(m)}, +\infty)$.

Then we discuss the $(m-1)$th order derivative $B^{(m-1)}(s)$, which has two cases: either $B^{(m-1)}(0) = (m-1)! \cdot b_{m-1} = 0$ or $b_{m-1} > 0$. We discuss them separately.

Step (1) If $b_{m-1} = 0$, then obviously $B^{(m-1)}(s) = 0$ has a root 0, which is denoted by $\eta_1^{(m-1)}$. As $B^{(m-1)}(s)$ is strictly decreasing on $[0, \eta^{(m)})$ and strictly increasing on $[\eta^{(m)}, +\infty)$, it is clear that $B^{(m-1)}(\eta^{(m)}) < 0$. Similarly, note that $B^{(m-1)}(+\infty) = +\infty$, we can conclude that $B^{(m-1)}(s) = 0$ has another (unique) root $\eta_2^{(m-1)}$ on $[\eta^{(m)}, +\infty)$, which yields that $B^{(m-1)}(s) < 0$ on $(0, \eta_2^{(m-1)})$ and $B^{(m-1)}(s) > 0$ on $(\eta_2^{(m-1)}, +\infty)$. Therefore, $B^{(m-2)}(s)$ is strictly decreasing on $[0, \eta_2^{(m-1)})$ and strictly increasing on $[\eta_2^{(m-1)}, +\infty)$. Then we can discuss the $(m-2)$th order derivative in the same way. In details, If $b_{m-2} = 0$, then repeat this step, i.e. Step (1); otherwise, $b_{m-2} > 0$, then turn to Step (2) in the following.

Step (2) If $b_{m-1} > 0$, note that we still have that $B^{(m-1)}(s)$ is strictly decreasing on $[0, \eta^{(m)})$ and strictly increasing on $[\eta^{(m)}, +\infty)$. Then $B^{(m-1)}(\eta^{(m)})$ is the minimum, and it might be either nonnegative or negative:

(i) If $B^{(m-1)}(\eta^{(m)}) \geq 0$, then $B^{(m-1)}(s) \geq 0$ for all $s \geq 0$ and has no more than 1 zero (if the root exists, then it must be $\eta^{(m)}$) on the domain. Therefore, $B^{(m-2)}(s)$ is an increasing, strictly positive function on $(0, +\infty)$ and $B^{(m-2)}(0) = (m-2)! \cdot b_{m-2}$ is nonnegative. Repeat this step for several times, then we can find that $B(s)$ is also an increasing, strictly positive function. Recall that $B(0) = b_0 > 0$, it is easily proved that $B(s) = 0$ has no root. However, this is impossible because $s = 1$ must be a root of $B(s) = 0$.

(ii) Otherwise, i.e. $B^{(m-1)}(\eta^{(m)}) < 0$, then combine this assumption with the other two facts that $B^{(m-1)}(0) > 0$ and $B^{(m-1)}(+\infty) = +\infty$ will easily yield that $B^{(m-1)}(s) = 0$ has exactly one root $\eta_1^{(m-1)}$ on $[0, \eta^{(m)})$ and has another one $\eta_2^{(m-1)}$ on $[\eta^{(m)}, +\infty)$. Hence, $B^{(m-1)}(s)$ is strictly positive on $[0, \eta_1^{(m-1)}) \cup (\eta_2^{(m-1)}, +\infty)$ and strictly negative on $(\eta_1^{(m-1)}, \eta_2^{(m-1)})$. In other word, $B^{(m-2)}(s)$ is strictly increasing on $[0, \eta_1^{(m-1)}) \cup (\eta_2^{(m-1)}, +\infty)$ and strictly decreasing on $(\eta_1^{(m-1)}, \eta_2^{(m-1)})$. Apparently, $B^{(m-2)}(\eta_1^{(m-1)}) > B^{(m-2)}(\eta_2^{(m-1)})$. If $B^{(m-2)}(\eta_2^{(m-1)}) \geq 0$, then repeat Step (2i), which will lead to the contradiction. Otherwise, $B^{(m-2)}(s) = 0$ has exactly two roots $\eta_1^{(m-2)}$ and $\eta_2^{(m-2)}$, where $\eta_1^{(m-2)} \in (\eta_1^{(m-1)}, \eta_2^{(m-1)})$ and $\eta_2^{(m-2)} \in (\eta_2^{(m-1)}, +\infty)$. Similarly, $B^{(m-2)}(s)$ is strictly positive on $[0, \eta_1^{(m-2)}) \cup (\eta_2^{(m-2)}, +\infty)$ and strictly negative on $(\eta_1^{(m-2)}, \eta_2^{(m-2)})$ and hence $B^{(m-3)}(s)$ is strictly increasing on $(0, \eta_1^{(m-2)}] \cup [\eta_2^{(m-2)}, +\infty)$ and strictly decreasing on $[\eta_1^{(m-2)}, \eta_2^{(m-2)}]$. Then, we can go back to follow the step either $(2i)$ or $(2ii)$.

After repeating these steps and noting the fact that $B(0) > 0$, we can obtain that $B(s)$ is strictly increasing on $[0, \eta_1^{(1)}) \cup (\eta_2^{(1)}, +\infty)$ and strictly decreasing on $(\eta_1^{(1)}, \eta_2^{(1)})$, where $\eta_1^{(1)}, \eta_2^{(1)}$ are the two roots of $B'(1) = 0$ (note if $b_1 = 0$, then $\eta_1^{(1)} = 0$). Therefore, if $B'(1) \neq 0$, then $B(s)$ has exactly two roots q_S and q_L since $B(1) = 0$, where $0 < \eta_1^{(1)} < q_S < \eta_2^{(1)} < q_L < +\infty$ (here, q_L denote the larger root). It is obvious that $q_S = q_L = 1$ is the unique root If $B'(1) = 0$. □

Remark 4.1.1. *In Lemma 4.1.1, it is discussed just in the condition that $\rho = +\infty$. Assuming that $\rho < +\infty$ here, the following statements hold true.*

(i) If $0 < B(\rho) \leq +\infty$ or if $B(\rho) = 0$ and $\rho > 1$, then all the conclusions of Lemma 4.1.1 are still correct.

(ii) If $\rho = 1$, then $B(s)$ has one or two zeros on $[0, 1]$. More specifically, $B(s)$ has one zero if and only if $B'(1) \leq 0$.

(iii) If $B(\rho) < 0$, then $B(s) = 0$ has exactly one zero. (obviously in this case $B'(1) \neq 0$)

Remark 4.1.2. *Assuming that* $\sum_{j=0}^{\infty} b_j < 0$, *then all the conclusions in Lemma 4.1.1 still hold true.*

Lemma 4.1.2. *Suppose* $\lambda \geq 0$ *in (4.1.1). Then there exists a critical value* $\lambda_* \geq 0$, *such that if* $\lambda \leq \lambda_*$ *then* $B_\lambda(s) = B(s) + \lambda s^m$ *has positive zeros while for* $\lambda > \lambda_*$, *there exists no positive zero for* $B_\lambda(s)$. *Moreover,* $\lambda_* = 0$ *if and only if* $B'(1) = 0$.

Proof. It is clear that the root of $B_\lambda(s) = 0$ is the x-coordinate of the intersection point between the two functions $f(s) = B(s)$ and $g(s) = -\lambda s^m$. Since $B(1) = 0$, $B_\lambda(s) = 0$ has positive roots at least on $\lambda = 0$. Hence, there exists a value λ_*, where $0 \leq \lambda_* \leq +\infty$, such that for all $\lambda \leq \lambda_*$, $B_\lambda(s) = B(s) + \lambda s^m$ has at least a positive zero.

Additionally, note that $B(0) = b_0 > 0$ and q is the smallest positive root of $B(s) = 0$. Hence, $B(s) > 0$ on the interval $s \in [0, q)$. Moreover, for all $\lambda > 0$, the function $g(s) = -\lambda s^m$ is strictly negative at least on the interval $s \in (0, +\infty)$. Let λ tend to $+\infty$, the graph of the function $g(s)$ will approximate the y-axis. Thus, it is trivial that there exists a critical value $\lambda_* \geq 0$, such that if $\lambda > \lambda_*$ then $B_\lambda(s) = B(s) + \lambda s^m$ does not have any positive zeros. The first part of the theorem has been proved.

Finally, $\lambda_* = 0$ implies that $B(s) \geq 0$ for all $s \in [0, +\infty)$, which ensures that $B'(1) = 0$ since $B(1) = 0$. In the reverse, $B'(1) = 0$ yields that $B(s)$ is tangent to the s-axis at $s = 1$, and it is obtained from Lemma 4.1.1 that $B(s) = 0$ has only one root on $[0, \rho]$, i.e. $s = 1$. Hence $B(s) \geq 0$ for all $s \geq 0$ and thus $\lambda_* = 0$. □

Define $g(s)$ be

$$g(s) = sB'(s) - mB(s), \tag{4.1.2}$$

then the root of $g(s) = 0$ has the following properties.

Lemma 4.1.3. *The equation* $g(s) = 0$ *has a unique zero* s_* *on* $[0, \rho]$ *if one of the following conditions holds:*

(i) $\rho = +\infty$;

(ii) $\rho < +\infty$ *and* $B(\rho) = +\infty$;

(iii) $\rho < +\infty$, $B(\rho) < +\infty$ *and* $g(\rho) \geq 0$.

Otherwise, $g(s) = 0$ *has no zero on* $[0, \rho]$.

Proof. The proof is very similar to that of Lemma 4.1.1. Note that for the kth order derivative of $g(s)$, where $k = 0, 1, \cdots, m$,

$$g^{(k)}(s) = sB^{(k+1)}(s) - (m-k)B^{(k)}(s), \tag{4.1.3}$$

and

$$g^{(k)}(0) = (m-k)k! \cdot b_k \begin{cases} \leq 0 & \text{if } 0 \leq k \leq m-1 \\ = 0 & \text{if } k = m \end{cases}. \tag{4.1.4}$$

Hence, $g^{(m)}(s) = sB^{(m+1)}(s)$ is strictly positive on $(0, \rho)$. In other words, $g^{(m-1)}(s)$ is strictly increasing on $(0, \rho)$. Note that $g^{(m-1)}(0)$ could be either 0 or strictly negative, we need to discuss it separately.

Step (1) If $g^{(m-1)}(0) = 0$, then $g^{(m-1)}(s)$ is strictly positive on $(0, \rho)$ and hence $g^{(m-2)}(s)$ is strictly increasing on $(0, \rho)$. If $g^{(m-2)}(0) = 0$, then repeat this step; otherwise, turn to Step (2) or (3) in the following.

Step (2) If $g^{(m-1)}(0) < 0$ and $g^{(m-1)}(\rho) \leq 0$, then $g^{(m-1)}(s)$ is strictly negative on $(0, \rho)$ and therefore $g^{(m-2)}(s)$ is strictly decreasing on $(0, \rho)$. Hence, $g^{(m-2)}(s)$ is also strictly negative on $(0, \rho]$ and nonnegative on $s = 0$. Repeating this step yields that $g(s)$ is strictly negative on $[0, \rho]$ (since $g(0) = -mb_0 < 0$), which yields that $g(s)$ has no zero on the domain.

Step (3) If $g^{(m-1)}(0) < 0$ and $g^{(m-1)}(\rho) > 0$ (including $+\infty$), then $g^{(m-1)}(s) = 0$ has a unique root $s_*^{(m-1)}$ on $(0, \rho)$ and thus $g^{(m-2)}(s)$ is strictly decreasing on $(0, s_*^{(m-1)})$ and strictly increasing on $(s_*^{(m-1)}, \rho)$. Note that $g^{(m-2)}(0) \leq 0$.

(i) If $g^{(m-2)}(\rho) \leq 0$, then $g^{(m-2)}(s)$ is strictly negative on $(0, \rho)$ and then repeat Step (2).

(ii) If $g^{(m-2)}(\rho) > 0$, then $g^{(m-2)}(s) = 0$ has the unique root $s_*^{(m-2)}$ on $(s_*^{(m-1)}, \rho)$. and thus strictly negative on $(0, s_*^{(m-2)})$ and strictly positive on $(s_*^{(m-2)}, \rho)$. Then repeat Step (3).

After repeating these steps, we can easily conclude that there are only two possibilities for $g(s)$: $g(s)$ is strictly increasing on $(0, \rho)$, or $g(s)$ is strictly decreasing on $(0, s_*^{(1)})$ and strictly increasing on $(s_*^{(1)}, \rho)$. For both cases, since $g(0) < 0$, $g(s) = 0$ has unique root s_* on $[0, \rho]$ if $g(\rho) \geq 0$ and has no root on $[0, \rho]$ otherwise. $\square$

Lemma 4.1.4. *Let* $\{\tilde{b_j}, j \geq 0\}$ *be another set such that*

$$\tilde{b}_j = a^j b_j, \tag{4.1.5}$$

where a is a strictly positive number and it is assumed that $\sum_{j=0}^{\infty} \tilde{b}_j \leq 0$. Define $\tilde{Q}^* = \{\tilde{q}_{ij}\}$ be another q-matrix with the elements

$$\tilde{q}_{ij} = \begin{cases} \tilde{b}_{j-i+m} & \text{if } j \geq i-m, i \geq m \\ 0 & \text{otherwise} \end{cases}.$$

Obviously $C = \{m, m+1, m+2, \cdots\}$ is the common irreducible class of both Q^* and $\tilde{Q}^*$. Then the decay parameters of the two q-matrices, λ_C and $\tilde{\lambda}_C$, have the relationship

$$\tilde{\lambda}_C = a^m \lambda_C. \tag{4.1.6}$$

Proof. If we can prove both $\tilde{\lambda}_C \geq a^m \lambda_C$ and $\tilde{\lambda}_C \leq a^m \lambda_C$, then $\tilde{\lambda}_C = a^m \lambda_C$ is obvious. Firstly, we want to prove that

$$\tilde{\lambda}_C \geq a^m \lambda_C. \tag{4.1.7}$$

According to Proposition 2.1.9 and 2.1.11, since λ_C is the decay parameter of Q^* on C, there exists a set of strictly positive numbers $\{x_j : j \geq m\}$, which is a λ_C-subinvariant vector for Q^* on C, such that

$$\begin{cases} \sum_{j=m}^{\infty} b_j x_j \le -\lambda_C x_m, \\ \sum_{j=m-1}^{\infty} b_j x_{j+1} \le -\lambda_C x_{m+1}, \\ \vdots, \\ \sum_{j=0}^{\infty} b_j x_{j+i-m} \le -\lambda_C x_i, \qquad i \ge 2m. \end{cases} \tag{4.1.8}$$

Define $\tilde{x}_j = x_j/a^j$ $(j \ge m)$, then $\tilde{x}_j$ $(j \ge m)$ are also strictly positive numbers and the following inequalities could be obtained based on (4.1.8):

$$\begin{cases} \sum_{j=m}^{\infty} \tilde{b}_j \tilde{x}_j = \sum_{j=m}^{\infty} b_j x_j \le -\lambda_C x_m = -(a^m)\lambda_C \tilde{x}_m, \\ \sum_{j=m-1}^{\infty} \tilde{b}_j \tilde{x}_{j+1} = \frac{1}{a} \sum_{j=m-1}^{\infty} b_j x_{j+1} \le -\frac{1}{a}\lambda_C x_{m+1} = -(a^m)\lambda_C \tilde{x}_{m+1}, \\ \cdots\cdots \\ \sum_{j=0}^{\infty} \tilde{b}_j \tilde{x}_{j+i-m} = \dfrac{1}{a^{i-m}} \sum_{j=0}^{\infty} b_j x_{i+j-m} \le -\dfrac{1}{a^{i-m}}\lambda_C x_i = -(a^m)\lambda_C \tilde{x}_i, \ i \ge 2m. \end{cases} \tag{4.1.9}$$

Hence, $\{\tilde{x}_j : j \ge m\}$ is an $a^m\lambda_C$-subinvariant vector for $\tilde{Q}^*$ on C, and thus $\tilde{\lambda}_C \ge a^m \lambda_C$ based on Proposition 2.1.8.

On the other hand, note from (4.1.5) that $b_j = \tilde{b}_j/a^j$ for all $j \ge 0$, by using the same method, we have that

$$\lambda_C \ge \frac{\tilde{\lambda}_C}{a^m}. \tag{4.1.10}$$

Therefore, combining (4.1.7) and (4.1.10) will lead to $\tilde{\lambda}_C = a^m \lambda_C$. $\square$

Lemma 4.1.5. *Let $\{\tilde{b}_j : j \ge 0\}$ be the series such that*

$$\tilde{b}_j = \begin{cases} b_j \beta^j & \text{if } j \ne m \\ b_j \beta^j + \alpha & \text{if } j = m \end{cases}, \tag{4.1.11}$$

where $\alpha \ge 0$, $\beta > 0$ and it is assumed that $\sum_{j=0}^{\infty} \tilde{b}_j = 0$. Furthermore, define a q-matrix $\tilde{Q}^* = \{\tilde{q}_{ij}, i, j \ge 0\}$ (not necessarily conservative) with the elements

$$\tilde{q}_{ij} = \begin{cases} \tilde{b}_{j-i+m} & \text{if } j \ge i-m, i \ge m \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, $C = \{m, m+1, m+2, \cdots\}$ is a irreducible class for $\tilde{Q}^*$. Then the decay parameters λ_C and $\tilde{\lambda}_C$ of Q^* and $\tilde{Q}^*$ respectively, have the relation with

$$\tilde{\lambda}_C = \beta^m \lambda_C - \alpha\beta^m. \tag{4.1.12}$$

Proof. In this example, $\tilde{Q}^*$ could be written as:

$$\tilde{Q}^* = \bar{Q}^* + \alpha I,$$

where $\bar{Q}^* = \{\bar{q}_{ij}, i, j \geq 0\}$ is given by

$$\bar{q}_{ij} = \begin{cases} b_{j-i+m}\beta^{j-i+m} & \text{if } j \geq i-m, i \geq m \\ 0 & otherwise \end{cases},$$

and

$$I = \begin{cases} \delta_{ij} & i \in C \\ 0 & \text{otherwise} \end{cases}.$$

According to Lemma 4.1.4, the decay parameter $\bar{\lambda}_C$ of $\bar{Q}^*$ on C is given by

$$\bar{\lambda}_C = \beta^m \lambda_C.$$

Additionally, from Lemma 2.5 in A. Y. Chen, J. P. Li, Z. T. Hou and K. W. Ng (2010), (4.1.12) could be obtained automatically. □

4.2 Decay Parameter of the Q^*-Process

In this section, the decay parameter of the stopped queue process Q^* will be obtained. Recall that $B(1) = 0$, similarly in J. P. Li and A. Y. Chen (2008), we can define the largest s such that B(s) is nonpositive, that is

$$\rho_0 = sup\{s : s \geq 0, B(s) \leq 0\}.$$

Obviously, If $\rho = +\infty$, then $B(\rho_0) = 0$, otherwise, $B(\rho_0) \leq 0$. Moreover, define

$$\lambda_* = \sup\{\lambda : \lambda \geq 0; B(s) + \lambda s^m = 0 \text{ has a root in } [0, \rho_0]\}. \tag{4.2.1}$$

Since $B(s) > 0$ in $[0, q_S)$, (4.2.1) is equivalent to

$$\lambda_* = \sup\{\lambda : \lambda \geq 0; B(s) + \lambda s^m = 0 \text{ has a root in } [q_S, \rho_0]\}. \tag{4.2.2}$$

Define

$$\bar{\lambda} = \max\{-\frac{B(s)}{s^m} : s \in [q_S, \rho_0]\} = \max\{-\frac{B(s)}{s^m} : s \in [0, \rho_0].\}. \tag{4.2.3}$$

then $\lambda_* = \bar{\lambda}$ is easily proved.

Lemma 4.2.1. $\lambda_* = \bar{\lambda}$.

The proof is almost the same as that of Lemma 3.1 in A. Y. Chen, J. P. Li, Z. T. Hou and K. W. Ng (2010).

Define

$$f(s) = \frac{B(s)}{s^m},$$

where $s \in [q_S, \rho_0]$. Apparently, $f(s)$ is C^∞ on $[q_S, \rho_0]$ and therefore, by the properties of closed interval, there exists $s_* \in [q_S, \rho_0]$ such that

$$\lambda_* = \bar{\lambda} = -\frac{B(s_*)}{s_*^m}. \tag{4.2.4}$$

Lemma 4.2.2. *For the s_* in (4.2.4), if $s_* \in (q_S, \rho_0)$ (in this case, based on Lemma 4.1.3, s_* is the unique one), then*

$$\lambda_* = -\frac{B(s_*)}{s_*^m} = -\frac{B'(s_*)}{m s_*^{m-1}}; \tag{4.2.5}$$

otherwise (this will happen if and only if $g(\rho_0) = g(\rho) \leq 0$),

$$\lambda_* = -\frac{B(\rho_0)}{\rho_0^m}. \tag{4.2.6}$$

Proof. In this first case, the front equation is apparent based on the definition of $\bar{\lambda}$ and Lemma 4.2.1. Thus, $-\lambda_*$ is the minimum of $f(s)$ on $[q_S, \rho_0)$. In other words,

$$f'(s_*) = \frac{s_* B'(s_*) - mB(s_*)}{s_*^{m+1}} = 0.$$

Thus,

$$B(s_*) = \frac{s_* B'(s_*)}{m},$$

$$i.e. -\frac{B(s_*)}{s_*^m} = -\frac{B'(s_*)}{ms_*^{m-1}},$$

which is the latter equation of (4.2.5).

In the second case, since $g(s) = 0$ has no root on $[q_S, \rho_0)$, $g(s) < 0$ for all $0 \leq s < \rho_0$. That is $f'(s) < 0$ on $(0, \rho_0)$ and thus $f(s)$ is strictly decreasing on $(0, \rho_0)$. Therefore, $f(s)$ attains its minimum value on $s = \rho_0$, which results in (4.2.6). Moreover, if $g(\rho_0)$=0, then let $s_* = \rho_0$ in (4.2.5), the equation still holds. □

Now the main result concerning the decay parameter for the stopped queue process Q^* will be presented.

Theorem 4.2.1. *Unless the extreme case that $g(\rho_0) < 0$ happened, the decay parameter λ_C for the q-matrix Q^* defined in (1.2.2) on $C = \{m, m+1, m+2, \cdots\}$ is nothing else but λ_*, i.e.*

$$\lambda_C = \lambda_*. \tag{4.2.7}$$

Proof. Firstly, we consider the case that $m_b = m_d$. It is obtained from Lemma 4.1.2 that $\lambda_* = 0$ in this case. Assume on the contrary that $\lambda_C > 0$. Then based on the definition of decay parameter, we have that for all $\lambda \in (0, \lambda_C)$ and $i, j \in C$,

$$\int_0^\infty e^{\lambda t} p_{ij}(t) dt < +\infty.$$

Suppose $i = m$, then the following could be obtained according to the Kolmogorov forward equation that

$$\begin{cases} p'_{m0}(t) = p_{mm}(t)b_0 \\ p'_{m1}(t) = p_{mm}(t)b_1 + p_{m,m+1}(t)b_0 \\ \vdots \\ p'_{m,m-1}(t) = \sum_{j=0}^{m-1} p_{m,m+j}(t)b_{m-1-j}, \end{cases}$$

which leads to

$$\int_0^\infty e^{\lambda t} p'_{mk}(t)dt < +\infty, \tag{4.2.8}$$

for all $0 \leq k \leq m-1$. As it is well known that

$$e^{\lambda t} \geq 1 + \lambda t$$

for all $\lambda t \geq 0$, (4.2.8) guarantees that

$$\int_0^\infty t p'_{mk}(t)dt < +\infty. \tag{4.2.9}$$

Hence,

$$\int_0^\infty t \sum_{k=0}^{m-1} p'_{mk}(t)dt < +\infty. \tag{4.2.10}$$

However, the left part of (4.2.10) is nothing else but $E_m[\tau^*]$ in the section 3.3.2. In other word, (4.2.10) just means that starting from the state m, the mean extinction time is finite, which contradicts Proposition 3.3.2 with the case of $m_b = m_d$. Therefore,

$$\lambda_C = 0 = \lambda_*.$$

Now consider the general case $m_b \neq m_d$. Based on Lemma 4.2.2, if $s_* \in (q_S, \rho_0)$, then

$$B(s_*) = -\lambda_* s_*^m \text{ and } B'(s_*) = -m s_*^{m-1}\lambda_*. \tag{4.2.11}$$

Define the sequence $\{\tilde{b}_j : j \geq 0\}$, where

$$\tilde{b}_j = \begin{cases} b_j s_*^j & \text{if } j \neq m \\ b_j s_*^j + \lambda_* s^j & \text{if } j = m \end{cases} . \tag{4.2.12}$$

Certainly, $\tilde{b}_j \geq 0$ for all $j \neq m$ and

$$\sum_{j=0}^{\infty} \tilde{b}_j = B(s_*) + \lambda s_*^m = 0.$$

Define $\tilde{Q}^* = \{\tilde{q}_{ij} : i, j \geq 0\}$, where

$$\tilde{q}_{ij} = \begin{cases} b_{j-i+m} & \text{if } i \geq m, j \geq i - m \\ 0 & \text{otherwise} \end{cases} ,$$

then obviously $\tilde{Q}^*$ is exactly a stopped bulk-arrival and bulk-service conservative q-matrix. According to Lemma 4.1.5, the decay parameter $\tilde{\lambda}_C$ is given by

$$\tilde{\lambda}_C = \lambda_C s_*^m - \lambda_* s_*^m. \tag{4.2.13}$$

Now, define the generating function

$$\tilde{B}(s) = \sum_{j=0}^{\infty} \tilde{b}_j s^j.$$

Then, the following two properties are easily obtained:

(1)

$$\tilde{B}(1) = \sum_{j=0}^{\infty} \tilde{b}_j = 0;$$

(2)

$$\begin{aligned} \tilde{B}'(1) &= \sum_{j=0}^{\infty} j\tilde{b}_j = \sum_{j=0}^{\infty} j b_j s_*^j + m\lambda_* s_*^m \\ &= s_* B'(s_*) + m\lambda_* s_*^m = 0, \end{aligned}$$

by using (4.2.11). According to the previous proof in the case of $m_b = m_d$ (recall that this is equivalent to $B'(1) = 0$), we can conclude that

$$\tilde{\lambda}_C = 0. \tag{4.2.14}$$

Therefore, $\lambda_C = \lambda_*$ is obtained by combining (4.2.13) and (4.2.14) and the fact that $s_* > 0$. Noting that in this case, even if Q^* is not conservative, i.e. $B(1) < 0$, the result is still correct.

Finally, if $s_* \notin (q_S, \rho_0)$, then s_* is either q_S or ρ_0. If $s_* = q_S$, then note that $B(q_S) = 0$, thus $\lambda_* = 0$. According to Lemma 4.1.2, the only possibility of $\lambda_* = 0$ is that $B'(1) = 0$, which has been discussed in the beginning of the proof. If $s_* = \rho_0$ and $g(\rho_0) = 0$, then

$$\lambda_* = -\frac{B(\rho_0)}{\rho_0^m} = -\frac{B'(\rho_0)}{m\rho_0^{m-1}}.$$

Hence, we can still use the previous method involving the creation of another stopped queue process $\tilde{Q}^*$ to show that $\lambda_C = \lambda_*$. Otherwise, if $g(\rho_0) \neq 0$ and $s_* = \rho_0$, obviously $\rho_0 = \rho < +\infty$. The case that $g(\rho) > 0$ does not need to be considered since $g(0) = -mb_0 < 0$, $g(\rho_0) = 0$ must be followed. For the case $g(\rho) < 0$, this is the extreme case that mentioned in the theory. $\square$

Remark 4.2.1. *In the Theorem 4.2.1, Lemma 4.1.5 was applied, which did not require Q^* to be conservative. Therefore, if Q^* is not conservative, then the decay parameter λ_C is still λ_*.*

Lemma 4.2.3. *Assume that $\rho = 1$ and $B'(1) < 0$, then the decay parameter of the conservative matrix Q^* on $C = \{m, m+1, \cdots\}$ is given by*

$$\lambda_C = \lambda_* = 0$$

We can use the similar method as that in J. P. Li, A. Y. Chen, Z. T. Hou and K. W. Ng (2010) to prove this lemma.

Proof. Since $\rho = 1$ is the convergence radius, $B(s)$ is not well defined for all $s > 1$. Furthermore, $B'(1) < 0$ is equivalent to $q_S = 1$. Hence, it is clearly obtained that $\lambda_* = 0$. Recall that $\sum_{j=m+1}^{\infty} b_j > 0$, thus there exists $N_0 \geq m+1$ such that $b_{N_0} > 0$. For any $n \geq N_0$, define a new series $\{b_j^{(n)} : j \geq 0\}$, which is given by

$$b_j^{(n)} = \begin{cases} b_j & \text{if } j \leq n \\ 0 & \text{if } j > n \end{cases}, \tag{4.2.15}$$

and using this series to define a stopped bulk-arrival and bulk-service q-matrix $Q^{*(n)} = \{q^{*(n)}_{ij}, i, j \geq 0\}$ (not necessarily conservative) with the elements

$$q^{*(n)}_{ij} = \begin{cases} b_{j-i+m}^{(n)} & \text{if } j \geq i-m, i \geq m \\ 0 & \text{otherwise} \end{cases}. \tag{4.2.16}$$

For the new series $\{b_j^{(n)}\}$, we can also define the generating function

$$B_{(n)}(s) = \sum_{j=0}^{\infty} b_j^{(n)} s^j. \tag{4.2.17}$$

Similar as the definition of (4.2.2), define

$$\lambda_*^{(n)} = \sup\{\lambda : \lambda \geq 0, B_{(n)}(s) + \lambda s^m = 0 \text{ has a root in } [0, \rho_0^{(n)}]\}, \tag{4.2.18}$$

where $\rho_0^{(n)}$ is the largest positive s such that $B_{(n)}(s) \leq 0$. For each fixed n,

$$\limsup_{j \to +\infty} (b_j)^{\frac{1}{j}} = 0,$$

which means that the convergence radius of the series is $+\infty$. According to Theorem 4.2.1, the decay parameter $\lambda_C^{(n)}$ of $Q^{*(n)}$ for C is

$$\lambda_C^{(n)} = \lambda_*^{(n)}. \tag{4.2.19}$$

Note that for all $n \geq N_0$, $B_{(n)}(0) = b_0 > 0$, $B_{(n)}(1) = \sum_{j=0}^{n} b_j \leq 0$, $B'_{(n)}(1) = \sum_{j=1}^{n} j b_j \leq B'(1) < 0$ and $B_{(n)}(+\infty) = +\infty$. According to Remark 4.1.2, $B_{(n)}(s) = 0$ has exactly two positive roots (denoted by $q_S^{(n)}$, and $q_L^{(n)}$ here), which satisfies the following inequality:

$$0 < q_S^{(n)} \leq 1 < q_L^{(n)} < +\infty. \tag{4.2.20}$$

Moreover, since $q_S^{(n)} < q_L^{(n)}$ and $\rho^{(n)} = +\infty$, there exists $s_*^{(n)} \in (q_S^{(n)}, q_L^{(n)})$ such that

$$\lambda_C^{(n)} = \lambda_*^{(n)} = -\frac{B_{(n)}(s_*^{(n)})}{(s_*^{(n)})^m}. \tag{4.2.21}$$

It is trivial that

$$B_{(N_0)}(s) \leq B_{(N_0+1)}(s) \leq \cdots B(s). \tag{4.2.22}$$

Hence, for any $n \geq N_0$, we have $B_{(n)}(q_S^{(n)}) \leq B_{(n+1)}(q_S^{(n)})$, which leads to $q_S^{(n)} \leq q_S^{(n+1)}$ since $B'_{(n+1)}(q_S^{(n+1)}) < 0$. Similarly, $q_L^{(n)} \geq q_L^{(n+1)}$. The fact that both $\{q_S^{(n)}\}$ and $\{q_L^{(n)}\}$ are bounded and monotonous yields that both $\lim_{n\to\infty} q_S^{(n)}$ and $\lim_{n\to\infty} q_L^{(n)}$ exist and must have the following relation:

$$\lim_{n\to\infty} q_S^{(n)} \leq 1 \leq \lim_{n\to\infty} q_L^{(n)}. \tag{4.2.23}$$

For all $n \geq N_0$, $B_{(n)}(q_L^{(n)})$=0, i.e.

$$\sum_{j\neq m} b_j^{(n)} (q_L^{(n)})^j = -b_m (q_L^{(n)})^m. \tag{4.2.24}$$

Since $q_L^{(n)} \geq \lim_{n\to\infty} q_L^{(n)}$ and $b_j^{(n)} \geq 0$ for all $j \neq m$, it is obtained from (4.2.24) that

$$\sum_{j\neq m} b_j^{(n)} (\lim_{n\to\infty} q_L^{(n)})^j \leq -b_m (q_L^{(n)})^m.$$

Letting $n \to \infty$ on both sides will lead to

$$\sum_{j\neq m} b_j (\lim_{n\to\infty} q_L^{(n)})^j \leq -b_m (\lim_{n\to\infty} q_L^{(n)})^m < +\infty \tag{4.2.25}$$

and hence $B(\lim_{n\to\infty} q_L^{(n)}) < +\infty$. Recall the condition that $\rho = 1$, then $\lim_{n\to\infty} q_L^{(n)} \leq 1$. Combining this with (4.2.23) yields $\lim_{n\to\infty} q_L^{(n)} = 1$.

Similarly, we have for all $n \geq N_0$, $B_{(n)}(q_S^{(n)})$=0, i.e.

$$\sum_{j \neq m} b_j^{(n)} (q_S^{(n)})^j = -b_m (q_S^{(n)})^m.$$

Note that for a fixed j, both $\{b_j^{(n)} : n \geq N_0\}$ and $\{q_S^{(n)} : n \geq N_0\}$ are increasing series with n, then by applying the monotone convergence theorem, it is obtained that

$$\sum_{j \neq m} b_j (\lim_{n\to\infty} q_S^{(n)})^j = -b_m (\lim_{n\to\infty} q_S^{(n)})^m,$$

that is

$$B(\lim_{n\to\infty} q_S^{(n)}) = 0.$$

Since $s = 1$ is the unique root of $B(s) = 0$, it is clearly that $\lim_{n\to\infty} q_S^{(n)} = 1$, i.e.

$$\lim_{n\to\infty} q_S^{(n)} = \lim_{n\to\infty} q_L^{(n)} = 1, \tag{4.2.26}$$

which yields that $\lim_{n\to\infty} s_*^{(n)} = 1$ since $s_*^{(n)} \in (q_S^{(n)}, q_L^{(n)})$.

In order to find the decay parameter λ_C, the backward integral recursions (2.1.16) are applied. Denote ${}_{(k)}P^*(t) = \{{}_{(k)}p^*_{ij}(t) : i, j \geq 0\}$ and $P^*(t) = \{p^*_{ij}(t) : i, j \geq 0\}$ be the Feller minimal $Q^{*(k)}$-transition function and Q^*-transition function respectively, where $k \geq N_0$. In other words, ${}_{(k)}P^*(t)$ and $P^*(t)$ are given by the recursion formulas

$${}_{(k)}{p^*}_{ij}^{(n)}(t) = \begin{cases} \delta_{ij} e^{-q_i^* t} & \text{if } n = 0 \\ {}_{(k)}{p^*}_{ij}^{(0)}(t) + \int_0^t e^{-q_i^* t} \sum_{l \neq i} {q^*}_{il}^{(k)} \, {}_{(k)}{p^*}_{lj}^{(n-1)}(t-s) ds & \text{if } n \geq 1 \end{cases}, \tag{4.2.27}$$

and

$${p^*}_{ij}^{(n)}(t) = \begin{cases} \delta_{ij} e^{-q_i^* t} & \text{if } n = 0 \\ {p^*}_{ij}^{(0)}(t) + \int_0^t e^{-q_i^* l} \sum_{l \neq i} q^*_{il} {p^*}_{lj}^{(n-1)}(t-s) ds & \text{if } n \geq 1 \end{cases} \tag{4.2.28}$$

respectively. According to the properties of backward integral recursions, both ${}_{(k)}{p^*}_{ij}^{(n)}(t)$ and ${p^*}_{ij}^{(n)}(t)$ are increasing with n to their limits ${}_{(k)}p^*_{ij}(t)$ and

$p^*_{ij}(t)$ respectively. Moreover, for all $k \geq N_0$ and $n = 0$, apparently

$$ {}_{(k)}{p^*}_{ij}^{(0)}(t) \leq_{(k+1)} {p^*}_{ij}^{(0)}(t) \leq {p^*}_{ij}^{(0)}(t). $$

Suppose for the $(n-1)$th recursion, we have

$$ {}_{(k)}{p^*}_{ij}^{(n-1)}(t) \leq_{(k+1)} {p^*}_{ij}^{(n-1)}(t) \leq {p^*}_{ij}^{(n-1)}(t). \tag{4.2.29} $$

Then combine (4.2.29) with the fact that for all $i \neq j$, ${q^*}_{ij}^{(k)} \leq {q^*}_{ij}^{(k+1)} \leq {q^*}_{ij}$ leads to

$$ \begin{aligned} {}_{(k)}{p^*}_{ij}^{(n)}(t) &= \delta_{ij} e^{-q_i^* t} + \int_0^t e^{-q_i^* t} \sum_{l \neq i} {q^*}_{il}^{(k)} {}_{(k)}{p^*}_{lj}^{(n-1)}(t-s)ds \\ &\leq \delta_{ij} e^{-q_i^* t} + \int_0^t e^{-q_i^* t} \sum_{l \neq i} {q^*}_{il}^{(k+1)} {}_{(k+1)}{p^*}_{lj}^{(n-1)}(t-s)ds =_{(k+1)} {p^*}_{ij}^{(n)}(t) \\ &\leq \delta_{ij} e^{-q_i^* t} + \int_0^t e^{-q_i^* l} \sum_{l \neq i} q_{il}^* {p^*}_{lj}^{(n-1)}(t-s)ds = {p^*}_{ij}^{(n)}(t). \end{aligned} $$

Therefore, by applying the mathematical induction, we get that for all $n \geq 0, i, j \geq 0, t \geq 0$,

$$ {}_{(k)}{p^*}_{ij}^{(n)}(t) \leq_{(k+1)} {p^*}_{ij}^{(n)}(t) \leq {p^*}_{ij}^{(n)}(t), $$

which yields that for all $i, j \geq 0$ and $t \geq 0$,

$$ {}_{(k)}{p^*}_{ij}(t) \leq_{(k+1)} {p^*}_{ij}(t) \leq {p^*}_{ij}(t). \tag{4.2.30} $$

Recall from (2.1.31) that the decay parameter $\lambda_C^{(k)}$ is the nonnegative number such that for $i \in C$,

$$ \int_0^\infty {}_{(k)}{p^*}_{ii}(t) e^{at} dt = \begin{cases} < +\infty & \text{if } a < \lambda_C^{(k)} \\ = +\infty & \text{if } a > \lambda_C^{(k)} \end{cases}. $$

Hence, from (4.2.30), for all $a < \lambda_C^{(k+1)}$, we have

$$\int_0^\infty {}_{(k)}p^*{}_{ii}(t)e^{at}dt \le \int_0^\infty {}_{(k+1)}p^*{}_{ii}(t)e^{at}dt < +\infty,$$

which leads to

$$\lambda_C^{(k)} \ge \lambda_C^{(k+1)} \ge \lambda_C \ge 0. \tag{4.2.31}$$

Therefore, $\lim_{n\to\infty} \lambda_C^{(n)}$ exists and $\lim_{n\to\infty} \lambda_C^{(n)} \ge \lambda_C$. Recall from (4.2.31) and the fact that $\lim_{n\to\infty} s_*^{(n)} = 1$, then

$$\lim_{n\to\infty} \lambda_C^{(n)} = -B(1) = 0,$$

and thus $\lambda_C = 0$. The proof is completed. □

Now we can find the decay parameter of the stopped queue process Q^* even if $g(\rho_0) < 0$.

Theorem 4.2.2. *For our stopped bulk-arrival and bulk-service queueing model Q^*, the decay parameter λ_C for Q^* on $C = \{m, m+1, \cdots\}$ is exactly λ_* whatever $g(\rho_0)$ is.*

Proof. The case of $g(\rho_0) \ge 0$ has already been proved in Theorem 4.2.1. Hence, consider the case of $g(\rho_0) < 0$ is quite enough. In this case, $\rho_0 = \rho$ and it is obtained from (4.2.6) that

$$\lambda_* = -\frac{B(\rho)}{\rho^m}.$$

Denote $\{\tilde{b}_j : j \ge 0\}$ be the series such that

$$\tilde{b}_j = \begin{cases} b_j\rho^j & \text{if } j \ne m \\ b_j\rho^j + \lambda_*\rho^m & \text{if } j = m \end{cases}. \tag{4.2.32}$$

Moreover, define the generating function

$$\tilde{B}(s) = \sum_{j=0}^{\infty} \tilde{b}_j s^j.$$

Apparently, the convergence radius of $\tilde{B}(s)$ is 1, and we also have

$$\tilde{B}(1) = \sum_{j=0}^{\infty} b_j \rho^j + (-\frac{B(\rho)}{\rho^m}) \cdot \rho^m = B(\rho) - B(\rho) = 0$$

and

$$\begin{aligned}
\tilde{B}'(1) &= \sum_{j=1}^{\infty} j b_j \rho^j + m \cdot (-\frac{B(\rho)}{\rho^m}) \cdot \rho^m \\
&= \rho B'(\rho) - mB(\rho) \\
&= g(\rho) < 0.
\end{aligned}$$

Hence, $\tilde{B}(s)$ satisfies all the conditions of Lemma 4.2.3 and thus the decay parameter of $\tilde{Q}^*$ is that

$$\tilde{\lambda}_C = 0,$$

where $\tilde{Q}^* = \{\tilde{q}^*_{ij} : i, j \geq 0\}$ is given by

$$\tilde{q}^*_{ij} = \begin{cases} \tilde{b}_{j-i+m} & \text{if } j \geq i - m, i \geq m \\ 0 & \text{otherwise} \end{cases}.$$

Again, it is obtained from Lemma 4.1.5 that

$$\tilde{\lambda}_C = \rho^m \lambda_C - \lambda_* \rho^m,$$

which leads to $\lambda_C = \lambda_*$ since $\rho \geq 1$. □

Corollary 4.2.1. *We have shown that $\lambda_C = \lambda_*$. From Lemma 4.2.2, we have that*

$$\lambda_C = \lambda_* = \begin{cases} -\dfrac{B'(s_*)}{m s_*^{m-1}} & \text{if } g(\rho_0) \geq 0 \\ -\dfrac{B(\rho_0)}{\rho_0^m} & \text{if } g(\rho_0) < 0 \end{cases}, \tag{4.2.33}$$

where s_* is the unique positive root of $g(s_*) = 0$. In addition, $s_* = \rho_0$ if and only if $g(\rho_0) = 0$.

Remark 4.2.2. *In fact, ρ_0 is not difficult to find. There are only two possibilities for ρ_0: ρ_0 is indeed q_L, which is the largest root of $B(s) = 0$ (in* *this case the equation $B(s) = 0$ has exactly two roots); or $\rho_0 = \rho < +\infty$.*

Corollary 4.2.2. *Moreover, if $m_b > m_d$, i.e. $B'(1) > 0$, then the two positive roots q_S and q_L are both in the interval $(0, 1]$ (in this case, $q_L = 1$), and hence*

$$g(\rho_0) = g(q_L) = B'(1) - kB(1) > 0.$$

Thus, $q_S < s_* < 1$ and $\lambda_C > 0$.

If $m_b = m_d$, i.e. $B'(1) = 0$, then it is obtained that $s = 1$ is the unique root of $B(s) = 0$ by using Lemma 4.1.2. Hence, $s_* = \rho_0 = 1$ and $\lambda_C = 0$.

If $m_b < m_d$ and thus $q_S = 1$, then there are different results for the different kinds of ρ: (i)$\rho = +\infty$, or $1 < \rho < +\infty$ and $B(\rho) \geq 0$; (ii) $\rho = 1$; (iii) $1 < \rho < +\infty$, $B(\rho_0) < 0$ and $\rho B'(\rho) \geq mB(\rho)$; (iv) otherwise. In case (i), obviously $B(\rho_0) = 0$ and $B'(\rho_0) > 0$, which leads to $g(\rho_0) > 0$. Thus, $1 < s_* < \rho_0$ and $\lambda_C > 0$. In part (ii), the assumption $\rho = 1$ ensures that $B(s) > 0$ on $[0, 1)$ and $B(1) = 0$. Therefore, $s_* = 1$ and $\lambda_C = 0$. In the cases of (iii) and (iv), $\rho_0 = \rho$ and $B(\rho_0) < 0$. The assumption $\rho B'(\rho) \geq mB(\rho)$ just means that $g(\rho) \geq 0$ (in the reverse, $g(\rho) < 0$). It is obtained from Lemma 4.1.3 that $1 < s_* \leq \rho$ and $\lambda_C > 0$ in the case (iii) ($s_* = \rho$ if and only if $g(\rho) = 0$), while in the case (iv), s_* does not exist and $\lambda_C = -\frac{B'(\rho)}{\rho^m}$.

Up to now, the decay parameter can be determined in a very simple expression. What to do next is to discuss the decay properties.

4.3 Decay Properties of the transient Q-Process

In this section, we turn back to discuss the decay properties of our general queueing model Q, which is including the part of state-dependent controls. Since the decay parameter $\lambda_C > 0$ is the only significant one to discuss, the transient processes are only considered. For the matrix Q, it is known from Theorem 3.5.1 that Q is transient if and only if $B'(1) > 0$. It is automatically assumed that $B'(1) > 0$ and Q-transition function is irreducible in this section. Before the main theorem, we need to define some functions in advance.

Definition 4.3.1. *It has been defined that* $B(s) = \sum_{j=0}^{\infty} b_j s^j$ *and for all* $i = \{0, 1, \cdots, m-1\}$, $H_i(s) = \sum_{j=0}^{\infty} q_{ij} s^j$. *For all* $0 \le i \le m-1$, *Define* $g_i(s)$ *as*

$$g_i(s) = B(s) - s^{m-i} H_i(s). \tag{4.3.1}$$

It is obvious that $g_i(s) = 0$ *has at least one root on* $(0, 1]$, *since*

$$g_i(1) = B(1) - H_i(1) = 0.$$

Let s_i *be the smallest positive root of* $g_i(s) = 0$, *where* $0 < s_i \le 1$, *then* $g_i(s)$ *is strictly positive on* $[0, s_i)$ *since* $g_i(0) = b_0 > 0$. *We further define*

$$\tilde{s} = \min\{s_0, s_1, \cdots, s_{m-1}, s_*\}, \tag{4.3.2}$$

where s_* *is the unique root of* $g(s) = 0$ *on* $(q_S, 1)$ *(recall from Corollary 4.2.2 that* s_* *exists in the case of* $m_b > m_d$*).*

Theorem 4.3.1. *Assuming that* $B'(1) > 0$ *and letting* $C = \{0, 1, \cdots\}$ *be the communicating class, we have the following conclusions.*

(1) If $\tilde{s} = s_*$, *then the decay parameter* λ_C *of the* Q*-process on* C *is given by*

$$\lambda_C = -\frac{B(s_*)}{s_*^m}. \tag{4.3.3}$$

(2) If $\tilde{s} = s_k$, *where* $k \in \{0, 1, 2, \cdots, m-1\}$, *then*

$$\lambda_C = -\frac{B(s_k)}{s_k^m} = -\frac{H_k(s_k)}{s_k^k}. \tag{4.3.4}$$

Proof. Firstly, consider the case $\tilde{s} = s_*$, i.e. $s_* \le s_i$ holds true for all $i \in [0, m-1]$. Denote λ_G be the decay parameter of Q on the communicating class G, where $G = \{m, m+1, \cdots\}$. Apparently, λ_G is the decay parameter of the stopped bulk-arrival and bulk-service queueing model Q^*, thus based on Theorem 4.2.2, we have that

$$\lambda_G = \frac{B(s_*)}{s_*^m}.$$

$\lambda_C \le \lambda_G$ is always true, which has been proved in Lemma 2.2 of J. P. Li and A. Y. Chen (2011). Hence, just prove $\lambda_C \ge \lambda_G$ is quite enough. Define a series of $\{x_j : j \ge 0\}$, which is given by

$$x_j = s_*^j,$$

then for all $i \ge m$,

$$\begin{aligned} \sum_{j=0}^{\infty} q_{ij} x_j &= \sum_{j=i-m}^{\infty} b_{j-i+m} s_*^j \\ &= s_*^{i-m} B(s_*) = -\lambda_G s_*^i = -\lambda_G x_i; \end{aligned} \tag{4.3.5}$$

and for all $i < m$,

$$\begin{aligned} \sum_{j=0}^{\infty} q_{ij} x_j &= \sum_{j=0}^{\infty} q_{ij} s_*^j \\ &= H_i(s_*) \le \frac{B(s_*)}{s_*^{m-i}} = -\lambda_G s_*^i = -\lambda_G x_i. \end{aligned} \tag{4.3.6}$$

The inequality of (4.3.6) is always true, since $s_* \le s_i$ leads to

$$0 \le g_i(s_*) = B(s_*) - s_*^{m-i} H_i(s_*).$$

Therefore, according to (4.3.5) and (4.3.6), $\{x_j : j \ge 0\}$ is a λ_G-subinvariant vector and thus we have the conclusion that $\lambda_G \le \lambda_C$ by using Proposition 2.1.8. Part (1) has been completely proved.

We now begin to prove part (2). For simplicity, define

$$\mu = -\frac{B(s_k)}{s_k^m} = -\frac{H_k(s_k)}{s_k^k}$$

(the second equality follows just from the fact that $g_k(s_k) = 0$), then what we need to show is that $\lambda_C = \mu$. Again define the series $\{x_j : j \ge 0\}$ as

$x_j = s_k^j$. Then we have that for all $i \geq m$,

$$\begin{aligned}\sum_{j=0}^{\infty} q_{ij} x_j &= \sum_{j=i-m}^{\infty} b_{j-i+m} s_k^j \\ &= s_k^{i-m} B(s_k) = -\mu s_k^i = -\mu x_i;\end{aligned} \tag{4.3.7}$$

and for all $i < m$,

$$\begin{aligned}\sum_{j=0}^{\infty} q_{ij} x_j &= \sum_{j=0}^{\infty} q_{ij} s_k^j \\ &= H_i(s_k) \leq \frac{B(s_k)}{s_k^{m-i}} = -\mu s_k^i = -\mu x_i.\end{aligned} \tag{4.3.8}$$

Similarly, the inequality of (4.3.8) holds true, since $s_k \leq s_i$ leads to $0 \leq g_i(s_k) = B(s_k) - s^{m-i} H_i(s_k)$. Hence, $\{x_j : j \geq 0\}$ is a μ-subinvariant vector, which brings about $\lambda_C \geq \mu$. Suppose $\lambda_C > \mu$, and define new sequences $\{\tilde{b}_j : j \geq 0\}$ and $\{\tilde{h}_{ij} : j \geq 0\}$ for any fixed i $(0 \leq i \leq m-1)$, where

$$\tilde{b}_j = b_j s_k^{j-m} - \delta_{mj} \frac{H_k(s_k)}{s_k^k} \tag{4.3.9}$$

and

$$\tilde{h}_{ij} = q_{ij} s_k^{j-i} - \delta_{ij} \frac{H_k(s_k)}{s_k^k}. \tag{4.3.10}$$

Clearly, for all $i \neq m$,

$$\tilde{b}_j = b_j s_k^{j-m} \geq 0$$

and for all $i \neq j$,

$$\tilde{h}_{ij} = q_{ij} s_k^{j-i} \geq 0.$$

We also obtain from these sequences that

$$\sum_{j=0}^{\infty} \tilde{b}_j = \sum_{j=0}^{\infty} b_j s_k^{j-m} - \frac{H_k(s_k)}{s_k^k} = \frac{B(s_k)}{s_k^m} - \frac{H_k(s_k)}{s_k^k} = 0$$

and

$$\sum_{j=0}^{\infty} \tilde{h}_{ij} = \sum_{j=0}^{\infty} q_{ij} s_k^{j-i} - \frac{H_k(s_k)}{s_k^k} = \frac{H_i(s_k)}{s_k^i} - \frac{B(s_k)}{s_k^k} \leq 0.$$

Hence, we can define another q-matrix $\tilde{Q}$ with the elements

$$\tilde{q}_{ij} = \begin{cases} \tilde{h}_{ij} & \text{if } 0 \le i \le m-1, \\ \tilde{b}_{j-i+m} & \text{if } j \ge i-m, i \ge m, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.11}$$

Similarly, define the generating function of $\{\tilde{b}_j : j \ge 0\}$ as

$$\tilde{B}(s) = \sum_{j=0}^{\infty} \tilde{b}_j s^j.$$

Then, we have

$$\begin{aligned} \tilde{B}(s) &= \sum_{j=0}^{\infty} b_j s_k^{j-m} s^j - \frac{H_k(s_k)}{s_k^k} \cdot s^m \\ &= \frac{B(s_k s)}{s_k^m} - \frac{B(s_k) s^m}{s_k^m}, \end{aligned} \tag{4.3.12}$$

and thus

$$\tilde{B}'(s) = \frac{s_k B'(s_k s) - m s^{m-1} B(s_k)}{s_k^m}. \tag{4.3.13}$$

Hence, $\tilde{B}(1) = 0$ and according to the fact that $g(s_k) \le 0$ (since $s_k \le s_*$ and s_* is the unique root of $g(s) = 0$ on (q_S, ρ_0)), we have

$$\tilde{B}'(1) = \frac{s_k B'(s_k) - mB(s_k)}{s_k^m} = \frac{g(s_k)}{s_k^m} \le 0,$$

which proves that $\tilde{Q}$ is recurrent based on Theorem 3.5.1. Therefore, the decay parameter of $\tilde{Q}$ is that $\tilde{\lambda}_C = 0$. Recall that λ_C is the decay parameter for Q, then there exists a μ-subinvariant measure $\{m_i : i \ge 0\}$ for Q such that

$$\sum_{i=0}^{m-1} m_i q_{ij} + \sum_{i=m}^{j+m} m_i b_{j-i+m} \le -\lambda_C m_j \tag{4.3.14}$$

for all $j \ge 0$. Multiplying s_k^j on both sides of (4.3.14) and letting $\tilde{m}_i = m_i s_k^i$

will yield

$$\sum_{i=0}^{m-1} \tilde{m}_i q_{ij} s_k^{j-i} + \sum_{i=m}^{j+m} \tilde{m}_i b_{j-i+m} s_k^{j-i} \le -\lambda_C \tilde{m}_j, \ \forall j \in N. \tag{4.3.15}$$

Next, letting (4.3.15) minus $\tilde{m}_j \frac{H_k(s_k)}{s_k^k}$ will leads to

$$\sum_{i=0}^{m-1} \tilde{m}_i \tilde{h}_{ij} + \sum_{i=m}^{j+m} \tilde{m}_i \tilde{b}_{j-i+m} \le -\left(\lambda_C + \frac{H_k(s_k)}{s_k^k}\right) \tilde{m}_j, \ \forall j \in N. \tag{4.3.16}$$

Hence, $\tilde{m}_i$ is a $\left(\lambda_C + \frac{H_k(s_k)}{s_k^k}\right)$-subinvariant measure for $\tilde{Q}$ and therefore

$$0 = \tilde{\lambda}_C \ge \lambda_C + \frac{H_k(s_k)}{s_k^k} \ge 0,$$

which contradicts our assumption that $\lambda_C > \mu = -\frac{H_k(s_k)}{s_k^k}$. Therefore, $\lambda_C = -\frac{H_k(s_k)}{s_k^k}$, which completes the proof. $\square$

Now the decay parameter λ_C of our main queueing model has been obtained. Then, we can discuss its λ_C-recurrence property.

Theorem 4.3.2. *Suppose our basic Q-process has the decay parameter $\lambda_C > 0$ on $C = \{0, 1, \cdots\}$, that is Q is transient, then Q-process is λ_C-recurrent if and only if*

$$s_0 = s_1 = s_2 = \cdots, s_{m-1} \le s_*. \tag{4.3.17}$$

Furthermore, Q is λ_C-positive recurrent if and only if

$$s_0 = s_1 = s_2 = \cdots, s_{m-1} < s_*. \tag{4.3.18}$$

Proof. Suppose (4.3.17) does not hold true, then there are only two possibilities: (i) $\tilde{s} = s_*$ and there exists l ($l \in \{0, 1, \cdot, m-1\}$) such that $s_l > s_*$; (ii) or $\tilde{s} = s_k$ for some $k \in \{0, 1, \cdots, m-1\}$ and there exist l ($l \in \{0, 1, \cdots, m-1\}$)

such that $s_l > s_k$. In the case (i), it is obtained from Theorem 4.3.1 that

$$\lambda_C = -\frac{B(s_*)}{s_*^m}$$

and $\{x_j : x_j = s_*^j, j \in C\}$ is a λ_C-subinvariant vector for Q. Since $s_* < s_l$, we have that when $i = l$,

$$\begin{aligned} \sum_{j=0}^{\infty} q_{lj} x_j &= \sum_{j=0}^{\infty} q_{lj} s_*^j \\ = \quad H_l(s_*) &< \frac{B(s_*)}{s_*^{m-l}} = -\lambda_C x_l. \end{aligned} \tag{4.3.19}$$

Hence, $\{x_j\}$ is not a λ_C-invariant vector. However, according to Proposition 2.1.10, the λ_C-subinvariant vector $\{x_j\}$ is indeed a λ_C-invariant vector for a λ_C-recurrent process. Therefore, our queueing model Q is λ_C-transient. In the case (ii), similarly we have the decay parameter that

$$\lambda_C = -\frac{B(s_k)}{s_k^m} = -\frac{H_k(s_k)}{s_k^k}$$

and a λ_C-subinvariant vector $\{x_j : x_j = s_k^j, j \in C\}$ for Q. Similarly,

$$\sum_{j=0}^{\infty} q_{lj} x_j < -\lambda_C s_k^l = -\lambda_C x_l. \tag{4.3.20}$$

Hence, we can have the conclusion that Q is λ_C-transient by using the same theory.

Now suppose (4.3.17) holds true. Since all the $s_i (0 \le i \le m-1)$ are equal, we just let $\tilde{s} = s_0$ for simplicity. Now, the decay parameter

$$\lambda_C = -\frac{B(s_0)}{s_0^m} = -H_0(s_0)$$

and $\{x_j : x_j = s_0^j, j \in C\}$ is a λ_C-subinvariant vector. Let $\tilde{Q} = \{\tilde{q}_{ij} : i, j \in C\}$ be the same q-matrix defined in (4.3.9), (4.3.10) and (4.3.11) with $k = 0$, that is

$$\tilde{b}_j = b_j s_0^{j-m} + \delta_{mj} \lambda_C$$

and

$$\tilde{q}_{ij} = q_{ij} s_0^{j-i} + \delta_{ij}\lambda_C.$$

Denote $P(t) = \{p_{ij}(t) : i, j \in C\}$ and $\tilde{P}(t) = \{\tilde{p}_{ij}(t) : i, j \in C\}$ be the Feller minimal Q and $\tilde{Q}$-function respectively. From Lemma 5.4.2 in W. J. Anderson (1991), we have that for all $i, j \in C$,

$$p_{ij}(t)x_j = e^{-\lambda_C t}\tilde{p}_{ij}(t)x_i, \tag{4.3.21}$$

that is

$$\tilde{p}_{ij}(t) = e^{\lambda_C t} p_{ij}(t) s_0^{j-i}. \tag{4.3.22}$$

In the proof of Theorem 4.3.1, we have mentioned that $\tilde{Q}$ is recurrent, that is for all $i \in C$,

$$\int_0^\infty \tilde{p}_{ii}(t)dt = \int_0^\infty e^{\lambda_C t} p_{ii}(t)dt = +\infty, \tag{4.3.23}$$

which yields that Q is λ_C-recurrent. Therefore, we can conclude that Q is λ_C-recurrent if and only if $s_0 = s_1 = \cdots = s_{m-1} \leq s_*$.

If $s_0 = s_1 = \cdots = s_{m-1} = s_*$, then

$$\tilde{B}'(1) = \frac{g(s_0)}{s_0^m} = 0.$$

Hence, according to Theorem 3.5.1, $\tilde{Q}$ is null recurrent and thus

$$\lim_{t\to+\infty} \tilde{p}_{ii}(t) = \lim_{t\to+\infty} e^{\lambda_C t} p_{ii}(t) = 0,$$

which yields that Q is λ_C-null recurrent. If $s_0 = s_1 = \cdots = s_{m-1} < s_*$, then then $\tilde{B}'(1) < 0$. To prove that $\tilde{Q}$ is positive recurrent, it is enough just to prove that $\sum_{j=0}^{\infty} j\tilde{q}_{ij} < +\infty$, i.e. $H_i'(1) < +\infty$ for all $0 \leq i \leq m-1$. It is easily calculated that

$$\tilde{H}_i'(s) = \frac{H_i'(s_0 s)s_0}{s_0^i} + i\lambda_C s^{i-1}$$

and hence

$$\tilde{H}_i'(1) = \frac{H_i'(s_0)s_0}{s_0^i} + i\lambda_C.$$

So we just need to verify $H_i'(s_0) < +\infty$. Note that

$$H_i(s_0) = H_i(s_i) = B(s_i)s_i^{m-i} = B(s_0)s_0^{m-i},$$

and define $f(s) = B(s)s^{m-i}$. Then since $0 < s_0 < s_*$, we have

$$H_i'(s_0) = f'(s_0) = B'(s_0)s_0^{m-i} + (m-i)s_0^{m-i-1}B(s_0) < +\infty.$$

Therefore, $\tilde{Q}$ is positive recurrent and thus

$$\lim_{t\to\infty} e^{\lambda_C t}p_{ii}(t) > 0,$$

which yields that Q is λ_C-positive recurrent. The proof is now finished. □

The next theorem is to calculate the λ_C-invariant measure and vector for the λ_C-positive recurrent queue process.

Theorem 4.3.3. *Suppose Q is λ_C-positive recurrent, then Q has the unique λ_C-invariant measure $\{m_i : i \geq 0\}$ and λ_C-invariant vector $\{x_j : j \geq 0\}$ up to constant multiples. Moreover, the λ_C-invariant vector is given by*

$$x_j = s_0^j, \ \forall j \geq 0. \tag{4.3.24}$$

For all $0 \leq i \leq m-1$, m_i is the solution of the equation set

$$\begin{cases} \sum_{i=0}^{m-1} m_i \left(s_0^i - s_0^m \dfrac{s_0 H_i'(s_0) - mH_i(s_0)}{s_0 B'(s_0) - mB(s_0)} \right) = 1 \\ \sum_{i=0}^{m-1} m_i \left(H_i(s_0\tilde{q}_1) - H_i(s_0)\tilde{q}_1^m \right) = 0 \\ \sum_{i=0}^{m-1} m_i \left(H_i(s_0\tilde{q}_2) - H_i(s_0)\tilde{q}_2^m \right) = 0 \\ \cdots \\ \sum_{i=0}^{m-1} m_i \left(H_i(s_0\tilde{q}_{m-1}) - H_i(s_0)\tilde{q}_{m-1}^m \right) = 0 \end{cases}, \tag{4.3.25}$$

where $\tilde{q}_l$ $(1 \leq l \leq m-1)$ are the other $m-1$ roots of $B(s_0 s) - B(s_0)s^m = 0$

except that $s = 1$ and s_0 is defined in Definition 4.3.1; and the generating function of the λ_C-invariant measure m_i is given by

$$M(s) = \sum_{i=0}^{\infty} m_i s^i = \sum_{i=0}^{m-1} m_i s^i - \frac{s^m \sum_{i=0}^{m-1} m_i \left(H_i(s) s_0^m - H_i(s_0) s^m\right)}{s_0^m B(s) - B(s_0) s^m}. \tag{4.3.26}$$

Proof. According to Proposition 2.1.9 and Proposition 2.1.10, there exists the unique λ_C-invariant measure and vector for Q since Q is λ_C-recurrent. It is obtained from (4.3.7) and (4.3.8) that $\{x_j : x_j = s_0^j, j \geq 0\}$ is a λ_C-subinvariant vector for Q and thus invariant.

Now we prove regarding the λ_C-invariant measure. Similarly, Let $\tilde{Q} = \{\tilde{q}_{ij} : i, j \in C\}$ be the same q-matrix defined in (4.3.9), (4.3.10) and (4.3.11) with $k = 0$, that is

$$\tilde{b}_j = b_j s_0^{j-m} + \delta_{mj} \lambda_C$$

and

$$\tilde{q}_{ij} = q_{ij} s_0^{j-i} + \delta_{ij} \lambda_C.$$

Denote $\tilde{B}(s)$ and $\tilde{H}_i(s)$ be the generating function of $\{\tilde{b}_j\}$ and $\{\tilde{q}_{ij}\}$ respectively, i.e.

$$\tilde{B}(s) = \frac{B(s_0 s) - B(s_0) s^m}{s_0^m}$$

and

$$\tilde{H}_i(s) = \frac{H_i(s_0 s) - H_i(s_0) s^m}{s_0^i}.$$

Obviously, $\tilde{B}(1) = 0$ and $\tilde{B}'(1) < 0$ and thus $\tilde{Q}$ is positive recurrent. Clearly, $\tilde{Q}$ is another bulk-arrival and bulk-service queue process with state-dependent control. Hence, by using Theorem 3.5.2, $\tilde{Q}$ has the equilibrium distribution (it is certainly an invariant measure) $\{\pi_i : i \geq 0\}$. More specifically, for all $0 \leq i \leq m-1$, π_i is the unique solution of

$$\sum_{i=0}^{m-1} \pi_i \frac{\tilde{B}'(1) - \tilde{H}_i'(1)}{\tilde{B}'(1)} = 1, \tag{4.3.27}$$

and

$$\sum_{i=0}^{m-1} \pi_i \tilde{H}_i(\tilde{q}_l) = 0 \ (l = 1, 2, \cdots, m-1), \tag{4.3.28}$$

where $\tilde{q}_l$ $(l = 1, 2, \cdots, m-1)$ are the roots of $\tilde{B}(s) = 0$ except that $s = 1$ ($\tilde{B}(s) = 0$ refers to $B(s_0 s) - B(s_0)s^m = 0$). For all $i \geq 0$, the generating function of $\{\pi_i\}$ is expressed by

$$\sum_{i=0}^{\infty} \pi_i s^i = \sum_{i=0}^{m-1} \pi_i s^i - \frac{s^m \sum_{i=0}^{m-1} \pi_i \tilde{H}_i(s)}{\tilde{B}(s)} \tag{4.3.29}$$

Let $m_i = \pi_i / s_0^i$, then (4.3.27) will become

$$\sum_{i=0}^{m-1} m_i s_0^i \left(1 - \frac{(s_0 H_i'(s_0) - m H_i(s_0))/s_0^i}{(s_0 B'(s_0) - m B(s_0))/s_0^m}\right) = 1,$$

which is exactly the first equation of (4.3.25); (4.3.28) will be

$$\sum_{i=0}^{m-1} m_i s_0^i \frac{H_i(s_0 \tilde{q}_l) - H_i(s_0)\tilde{q}_l^m}{s_0^i} = 0, \ (l = 1, 2, \cdots, m-1),$$

which is indeed the remaining $m-1$ equations of (4.3.25); and (4.3.29) yields

$$\sum_{i=m}^{\infty} m_i (s_0 s)^i = -\frac{s^m \sum_{i=0}^{m-1} m_i s_0^i \left(H_i(s_0 s) - H_i(s_0) s^m\right)/s_0^i}{\left(B(s_0 s) - B(s_0) s^m\right)/s_0^m} \tag{4.3.30}$$

Hence, (4.3.26) can be obtained by using s/s_0 to replace s in (4.3.30).

The remaining thing to prove is that this constructed $\{m_i : i \geq 0\}$ is indeed a λ_C-invariant measure. Recall that π_i is an invariant measure for $\tilde{Q}$, that is for all $j \geq 0$,

$$\sum_{i=0}^{m-1} \pi_i \tilde{q}_{ij} + \sum_{i=m}^{j+m} \pi_i \tilde{b}_{j-i+m} = 0,$$

which yields that

$$\sum_{i=0}^{m-1}(m_i s_0^i)(q_{ij} s_0^{j-i}) + \sum_{i=m}^{j+m}(m_i s_0^i)(b_{j-i+m} s_0^{j-i}) = -\lambda_C m_j s_0^j. \qquad (4.3.31)$$

Then, by dividing s_0^j on both sides, (4.3.31) will become

$$\sum_{i=0}^{m-1} m_i q_{ij} + \sum_{i=m}^{j+m} m_i b_{j-i+m} = -\lambda_C m_j,$$

which indicates that $\{m_i : i \geq 0\}$ is a λ_C-invariant measure on Q and hence is the unique one. The proof is completed. $\square$

4.4 An Example

In this section, a simple example will be presented. Suppose $m = 2$ and b_j is given by

$$b_j = \begin{cases} 2 & \text{if } j = 0, \\ -7 & \text{if } j = 2, \\ 5 & \text{if } j = 3, \\ 0 & \text{otherwise.} \end{cases},$$

that is

$$B(s) = 2 - 7s^2 + 5s^3.$$

Since $B'(1) = 15 - 14 = 1 > 0$, our queueing model is transient and hence we can discuss the decay properties. Moreover, assume for $i = 0, 1$, q_{ij} is given by

$$q_{ij} = \begin{cases} -0.25 & \text{if } j = i, \\ 0.25 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases},$$

i.e.

$$H_1(s) = -0.25s + 0.25s^2 = sH_0(s).$$

Hence, our queue process could be expressed by a q-matrix

$$Q = \begin{bmatrix} -0.25 & 0.25 & 0 & 0 & 0 & \cdots \\ 0 & -0.25 & 0.25 & 0 & 0 & \cdots \\ 2 & 0 & -7 & 5 & 0 & \cdots \\ 0 & 2 & 0 & -7 & 5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Corollary 4.4.1. *It is given that*

$$g(s_*) = s_* B'(s_*) - 2B(s_*) = 5s_*^3 - 4 = 0.$$

Hence, $s_* = \left(\frac{4}{5}\right)^{\frac{1}{3}}$*, and for the communicating class* $G = \{2, 3, \cdots\}$*, the decay parameter* λ_G *is given by*

$$\lambda_G = -\frac{B(s_*)}{s_*^2} = \frac{7 \cdot 4^{2/3} - 6 \cdot 5^{2/3}}{4^{2/3}}.$$

Then we want to calculate the decay parameter λ_C *for the whole state* $C = \{0, 1, \cdots\}$ *(it is indeed a communicating class). It is easily obtained that*

$$\begin{aligned} g_0(s) = g_1(s) &= B(s) - s^2 H_0(s) \\ &= 2 - 6.75s^2 + 4.75s^3 \\ &= (s-1)(4.75s^2 - 2s - 2), \end{aligned}$$

which results that

$$s_0 = s_1 = \frac{1 + \sqrt{10.5}}{4.75} \approx 0.8927 < s_*.$$

Therefore, it is clear that:

(1) the decay parameter λ_C *is given by*

$$\lambda_C = -\frac{B(s_0)}{s_0^2} = -H_0(s_0) = \frac{3.75 - \sqrt{10.5}}{19} \approx 0.02682;$$

(2) the Q-process is λ_C-positive recurrent.

Corollary 4.4.2. *To calculate the λ_C-invariant measure, first of all, define $\{\tilde{b}_j : j \geq 0\}$ and $\{\tilde{q}_{ij} : j \geq 0\}$ be*

$$\tilde{b}_j = b_j s_0^{j-2} + \delta_{2j}\lambda_C$$

and

$$\tilde{q}_{ij} = q_{ij} s_0^{j-i} + \delta_{ij}\lambda_C$$

respectively, i.e.

$$\tilde{B}(s) = 2s_0^{-2} - (7 - \lambda_C)s^2 + 5s_0 s^3.$$

Thus, the two roots of $\tilde{B}(s) = 0$ are

$$\tilde{q} = 1 \text{ and } \tilde{q}_* \approx -0.5608$$

respectively. We also have that the limit distribution of $\tilde{Q}$ follows

$$\begin{cases} \pi_0 \frac{\tilde{B}'(1) - \tilde{H}_0'(1)}{\tilde{B}'(1)} + \pi_1 \frac{\tilde{B}'(1) - \tilde{H}_1'(1)}{\tilde{B}'(1)} = 1 \\ \pi_0 \tilde{H}_0(\tilde{q}_*) + \pi_1 \tilde{H}_1(\tilde{q}_*) = 0. \end{cases}$$

i.e.

$$\begin{cases} 1.4015\pi_0 + \pi_1 = 1 \\ -0.3483\pi_0 + 0.1953\pi_1 = 0 \end{cases}$$

Hence,

$$\pi_0 \approx 0.3140 \text{ and } \pi_1 = 0.5600.$$

According to Theorem 4.3.3, the λ_C-invariant measure is given by

$$m_0 = \pi_0 \approx 0.3140, m_1 = \frac{\pi_1}{s_0} \approx 0.6273,$$

and

$$M(s) \approx 0.3140 + 0.6273s - \frac{-0.06256s^2 - 0.06244s^3 + 0.1279s^4}{1.5938 - 5.5570s^2 + 3.9846s^3}.$$

4.5 Notes

Since the queueing model we are discussing is new, all the theories in this chapter are not appropriate for those developed models. Section 4.2 mainly discusses the stopped queueing model for any finite m, whose specific case $m = 1$ has been viewed in J. P. Li and A. Y. Chen (2008) and $m = 2$ has been explored by A. Y. Chen, J. P. Li, Z. T. Hou and K. W. Ng (2010). Section 4.3 shows the decay properties for our main queue process Q, whose specific case $m = 1$ has been considered by J. P. Li and A. Y. Chen (2011).

Chapter 5

Conclusions and Future works

5.1 Conclusions

During this thesis, we have explored a practical queueing model which includes batch-arrival and bulk-service with the most general form of state-dependent control in Markovian languages, especially in the q-matrix. An acceptable method has been given to get the Feller minimal Q-function. In addition, some significant properties for our queue process have been discussed in the thesis. Moreover, the decay parameter and the decay properties have also been researched. Let us start to summarize our main results now.

5.1.1 Markovian Batch-arrival and Bulk-service Queues with Finite State-dependent Control

In Chapter 3, we first presented a method to calculate the Resolvent function of the stopped batch-arrival and bulk-service queue process Q^*, whose practical meaning is that the process stopped whenever it enters any of these states $\{0, 1, \cdots, m-1\}$. Then we applied the decomposition theorem to resume back to our initial model Q and thus obtained the Feller minimal Q-transition function. Meanwhile, the probability that the process will end finally and the mean time to stop the process (this was called extinction in thesis) were also investigated. It is revealed that the extinction probability is exactly one if and only if the mean arrival rate is no more than the mean service rate in the general queue part. More specifically, the mean extinction time is finite if that mean arrival rate is strictly less than that mean service rate.

Additionally, the condition for positive recurrence was obtained and under this condition, the equilibrium distribution, the mean queue length distribution, the hitting time distribution to the idle state 0 and the busy period distribution of our queue process were presented by the expressions of the Q-resolvent.

Finally, an example regarding the case $m = 3$ associated with some of its properties was discussed.

5.1.2 Decay Parameter and Decay Properties of Our Queueing Model

In Chapter 4, it is assumed that our queueing model is transient on its relative communicating class so that the discussion of the decay properties is valuable. Firstly, the decay parameters of both the stopped queueing model Q^* and the main queueing model Q were presented and proved in detail.

After we obtain the decay parameter λ_C, the λ_C-recurrence property was discussed and proved by different kinds of Q. Then, under the condition of λ_C-positive recurrence, the unique λ_C-invariant measure and vector were also presented by applying the result of the limit distribution in Chapter 3.

Finally, a very simple example was presented. During this special model, all the properties and related results of decay parameter were calculated in detail.

5.2 Future works

This queueing model considered in the thesis is a classical model. It is playing an important role in real world. Many applications have close relationship with this model. However, there still exist some problems which have not been solved. Let me state them in detail.

Question 1. We have already proved that $B_\lambda(s) = 0$ has exactly m root in the open circle $\{|s| < 1\}$. However, the only property which is always true is that only 1 positive root exists in the interval $[0, 1]$. It is difficult for us to calculate all the roots. We even have no idea to obtain the number of real roots. Hence, for most results involving the Markovian batch-arrival and bulk-service queues with finite state-dependent control, we can only present the method to obtain them. Therefore, one of the most significant future work is to explore more properties of all the roots to allow the results look like much simpler.

Question 2. In our queueing model, it is assumed that the maximum number of people can be served at the same moment is m, which is a finite number. However, it is quite common that the queue system will vanish at any state. Thus, it is somehow unreasonable to restrict the service since there is no restriction in arrival. Hence, we need to extend this model to more generalized cases even if it is difficult to discuss without any fresh ideas.

Question 3. Since it is well known that for any Q-matrix, there exists a unique λ_C-invariant measure if Q is λ_C-recurrent. In this thesis, we have obtained the λ_C-invariant measure for the λ_C-positive recurrent case. We must calculate it in the λ_C-null recurrent case. For the special case of $m = 1$, it has been discussed by J. P. Li and A. Y. Chen (2011), while A. Y. Chen, J. P. Li, Z. T. Hou and K. W. Ng (2010) have developed that in the stopped queue process Q^* of the case $m = 2$. Hence, it is necessary to extend it in more generalized cases.

Finally, we have to say that most discussions in this thesis are theoretical. Hence, the application is of importance. In the near future, we will try to consider the actual cases as some different kinds of queue processes and apply these results to classic problems in other fields, especially in finance and risk aspects.

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