HEAVY-TAILED DISTRIBUTION IN THE PRESENCE OF DEPENDENCE IN INSURANCE AND FINANCE

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by
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Abstract

In the past decade, the study of the renewal risk model in the presence of dependent insurance and financial risks and heavy-tailed claims is one of the key topics in modern risk theory. The purpose of this thesis is to study the renewal risk model with certain dependence structures. We also assume that claim sizes follow a heavy-tailed distribution, in particular, a subexponential distribution. We focus on studying the impact of heavy tails and dependence structures on ruin probabilities and the tail probabilities of aggregate claims.

For the study of dependence structure, we consider two assumptions here, namely, dependence between claims and inter-arrival times and dependence between insurance and financial risks, particular attention are paid for the dependent insurance and financial risks. In this case, an equation for the tail probability of maximal present value of aggregate net loss is derived, and hence some insights into the ruin probability can be obtained.

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Chapter 1

Introduction

1.1 Background

Risk theory is one of the main topics in actuarial science. It is particularly important for the analysis of the wealth of an insurance company. When considering the risk of a certain class of insurance business, actuarial researchers usually study the surplus process of an insurance portfolio. The surplus process is often modelled as a stochastic process $\{U_t, t \geq 0\}$ of a certain structure. Generally, the risk reserve U_t at time t involves three parts: the initial capital or initial reserve $u \geq 0$, the premium income rate $c \geq 0$, and the aggregate claims process $\{S_t, t \geq 0\}$. It describes the financial status of an insurance business and hence provides insights into the wealth of the company. In the continuous-time setting, the surplus process is given by

$$U_t = u + ct - S_t. \tag{1.1.1}$$

Usually, the aggregate claims process S_t is assumed to be a stochastic process which is given by the following random sums

$$S_t = \sum_{i=1}^{N_t} A_i, \quad t \geq 0,$$

where N_t is a counting process for claim numbers; and $\{A_i, i = 1, 2, ...\}$ is a sequence of independently and identically distributed (i.i.d.) random variables representing the sizes of individual claims. Hence, (1.1.1) is called the classical risk model.

Each insurance company has to face the risk of not having enough surplus to pay claims. Such a situation is referred to as ruin of the insurance company. A commonly-used tool to assess this risk is the probability of ruin, which may be treated as a guideline in pricing insurance policies. Let T denote the time of ruin, which is defined as

$$T = \inf_{t \geq 0}\{t : U_t < 0\},$$

with the convention that $\inf \emptyset = \infty$. By definition, it is the first time when the surplus falls below zero.

The ruin probability in finite time (or with finite horizon) is defined as

$$\psi(u, t) = \Pr(T \leq t | U_0 = u), \tag{1.1.2}$$

which denotes the probability that ruin will occur by time t. The ruin probability in infinite time (or with infinite horizon) is defined as

$$\psi(u) = \Pr(T < \infty | U_0 = u). \tag{1.1.3}$$

Note that the Model (1.1.1) is referred to as the continuous-time risk model. In practice one may only consider the surplus process at regular intervals of time. In this case, the discrete-time surplus process, in which we consider U_n only at discrete time points, can be written as

$$U_n = u + cn - S_n = u + cn - \sum_{i=1}^{n} A_i,$$

for $n = 0, 1, 2...$, where $U_0 = u$ is the initial surplus, c is the amount of premium per period, S_n is the claim process and A_i is the amount of claim in the ith period.

Denote the time of ruin by $\tilde{T} = \min\{n : U_n < 0\}$. Then the finite-time and ultimate ruin probabilities are give by

$$\tilde{\psi}(u, n) = \Pr(\tilde{T} < n | U_0 = u) \tag{1.1.4}$$

and

$$\tilde{\psi}(u) = \Pr(\tilde{T} < \infty | U_0 = u), \tag{1.1.5}$$

respectively.

The Cramér-Lundberg model, the foundation stone of the ruin theory approach, was introduced in the early 20th century. The main results of the Cramér-Lundberg model were developed by Lundberg (1903,1932) and Cramér (1930). In the classical risk model, claim size $\{A_i, i \in \mathbb{N}\}$ are positive i.i.d. with common distribution F and having finite mean μ. The inter-arrival times are denoted by $\{B_i, i \in \mathbb{N}\}$ which are i.i.d. exponentially distributed with finite mean $E = 1/\lambda$. The sequences $\{A_i, i \in \mathbb{N}\}$ and $\{B_i, i \in \mathbb{N}\}$ are independent of each other. Denote by $\{T_i = \sum_{k=1}^{i} B_k, i \in \mathbb{N}\}$ the claim arrival times, with $B_0 = 0$. The number of claims in the interval $[0, t]$ is denoted by $N_t = \sup\{n \geq 1 : T_n \leq t\}, t \geq 0$, which forms a homogeneous Poisson process with intensity $\lambda > 0$.

For the Cramér-Lundberg model,

$$EU_t = u + ct - \lambda\mu t$$

Therefore, $EU_t/t \to c - \lambda\mu$ and an obvious condition towards solvency is $c - \lambda\mu > 0$, implying (U_t) has a positive drift for large t. The net profit condition (the safety loading) is satisfied

$$\rho = \frac{c}{\lambda\mu} - 1 > 0.$$

Lundberg (1903,1932) and Cramér (1930) also obtained the Cramér-Lundberg approximation for $\psi(u)$ of (1.1.2). Further work based on the classical risk model can be found in Feller (1971), Grandell (1991), Rolski et al. (1999), Asmussen (2000), and Dickson and Willmot (2005). Their achievements include deriving a formula for $\psi(u)$ with $u = 0$ which depends only on the mean of the claim distribution and developing an exact form for $\psi(u)$ with exponential claim distributions.

Another well-known risk model, namely the renewal risk model (also referred to as the Sparre Anderson risk model) also has been extensively analyzed. Sparre Andersen (1957) extended the classical model by allowing claim inter-arrival times to have arbitrary distribution functions. Hence, the claim-number process is a renewal process. The renewal risk model has played a fundamental role in classical and modern risk theory as a natural generalization of the compound Poisson risk model. Recent works on the study of Sparre Anderson risk model includes Chen and Ng (2007), and Biard et al. (2008), they studied the ruin probability with heavy-tailed claim amounts when some independence assumptions are relaxed. Further works can be also found in Chen et al. (2010, 2012) as they studied the precise large deviations of aggregate claims for certain renewal models.

In addition to the ruin probability, researchers are also interested in other actuarial measures related to ruin. Recently, for practical reasons, precise large deviations of aggregate claims have received a remarkable amount of attention. Large deviation theory focuses on the asymptotic behaviour of the tail probability of random sums. The study of precise large deviations of random sums was initiated by Klüppelberg and Mikosh (1997), who presented several applications in insurance and finance. The results of precise large deviations for random sums make a great contribution to the evaluations of conditional tail expectation and value at risk of aggregate claims of a large insurance portfolio.

The classical compound Poisson risk model adopts simplifying assumptions to maintain nice properties and remain mathematically attractive. As it is too ideal to model the real word, researchers try to relax the assumptions of the classical model to make it more realistic. A brief review of variants of the classical risk model is given below.

The classical risk theory describing characteristics of the insurance surplus process usually relies on the assumption of independence of claim sizes and claim interoccurrence times. However, in many applications this assumption is too restrictive. For instance, for a line of business covering constructive damages due to earthquakes, more significant damages are expected with a longer period between claims. Therefore, generalizations to dependent scenarios are called for. In recent years, a number of results on ruin probabilities have been obtained for models that allow for specific types of dependence. The risk model with time-dependent claim sizes can be viewed as a more realistic model (than the classical compound Poisson risk model) to approximate the behavior of the aggregate claim process in a natural catastrophe context. Recent works include Boudereault et al. (2006), Cossette et al. (2008), Badescu et al. (2009), Asimit and Badescu (2010), Li et al. (2010), Chen et al. (2011).

Various different non-standard renewal risk models have been proposed in the recent literature. In the renewal risk model, several strong hypotheses may be found too restrictive to model accurately the complex evolution of the reserves of an insurance company. Biard et al. (2008) suggested that in the real world, the mutual independence among claim sizes and inter-arrival times is not realistic for a number of reasons. Firstly, the claim amounts

$X_k, k \geq 1$, are not independent in practice, and many present complex forms of positive dependence: some factors may have an impact on those amounts; some claims of a certain type may have identical severities depending on outcomes of trails at the court. Secondly, weather or economic conditions can create as well strong positive dependence on claim amounts, which can be weakly dependent and independent in the usual regime, and suddenly become strongly positively dependent if a so-called correlation crisis breaks out. In the case where the distributions of claim sizes are regular varying, Biard et al. (2008) assumed the claim sizes are positively dependent and non-stationary and they establish some asymptotic results on the finite-time ruin probability.

Recently, a vast amount of papers has been published on the issue of ruin of an insurer who is exposed to a stochastic economic environment. Such an environment has two kinds of risk, which were called by Norberg (1999) as insurance risk and financial risk, respectively. The first kind of risk is the traditional liability risk related to the insurance portfolio, and the second is the asset risk related to the investment portfolio. Nyrhinen (1999, 2001) investigated the asymptotic behaviour of the finite and infinite time ruin probabilities under a general assumption that both insurance and finance risks are independent, Nyrhinen (1999) employed large deviations techniques in the discrete time model and determined a crude estimate for the finite-time ruin probability. Following the works of Nyrhinen (1999, 2001), Tang and Tsitsiashvili (2003, 2004) derived precise estimates for the finite time ruin probability for a discrete time risk model, in which an insurer invested his surplus into both risk-free and risky assets, which may lead to negative returns. Goovaerts et al. (2005) investigated the tail probability of discounted aggregate sums of Pareto-like losses in insurance with stochastic discount factors which are mutually dependent. Since then, many authors including Tang and Vernic (2007), Zhang et al. (2009), Weng et al. (2009), and Chen (2011) have examined some ruin problems for the model.

As cited above, many researchers have taken into account the effect of heavy-tailed distributions in the risk model. The importance of subexponential class as a useful class of heavy-tailed distribution functions in the context of applied probability in general, and insurance mathematics in particular, was realized early by Teugels (1975). Embrechts et al. (1997) demonstrated that the class of subexponential distributions played a fundamental role in actuarial mathematics, especially modelling of large claims. The subexponential class contains many of popular heavy-tailed distributions, such as Pareto, Lognormal, heavy-tailed Weibull, and Loggamma distributions. Reviews of subexponential distributions can be found in Bingham et al. (1987), Embrechts et al. (1997), Rolski et al. (1999), Asmussen (2000), Rachev (2003), and Resnick (2007), among others.

In the following section, we are going to review the research on three aspects, namely risk models with dependent claim sizes and interarrival time, risk models with dependent insurance and financial risks and asymptotic behavior of aggregate claims in a renewal risk model.

1.2 Literature Review

1.2.1 Risk Models with Dependent Claims and Inter-Arrival Times

In recent years, the study of risk models with dependence between claim sizes and claim inter-occurrence times has been an important topic in the literature. Albrecher and Boxma (2004) considered a generalization of the classical ruin model to a dependent setting, where the distribution of the time between two claim occurrences depends on the previous clam size and they derived exact solution for the survival probability by means of Laplace transforms. See also Albrecher and Boxma (2005).

The time-dependent risk model was first proposed by Albrecher and Teugels (2006). In their paper, they described the surplus process as a random walk with independent increments over claim arrival times. It allowed the interclaim time and its subsequent claim size to be dependent according to an arbitrary copula structure. By employing the underlying random walk structure they derived explicit exponential estimates for both infinite- and finite-time ruin probabilities in the case of light-tailed claim sizes. However, their paper is just an attempt to obtain a clearer picture of the impact of dependence in risk theory, and many questions remain open to further study. For instance, one might attempt a similar study for heavy-tailed claims. Cossette et al. (2008) considered the family of risk models proposed by Albrecher and Teugels (2006) with a dependence structure between the claim amounts and the interclaim time defined with a generalized Farlie-Gumbel-Morgenster (FGM) copula where the individual claim follows an exponential distribution. In this framework, they derived the Laplace transform of the Gerber-Shiu discounted penalty function

Boudereault et al. (2006) examined several properties of an extension of the classical compound poisson risk model by assuming a dependence structure in which the claim size conditional on the interarrival time has a density function equal to a mixture of two arbitrary density functions, and they studied the Gerber-Shiu expected discounted penalty function and measured the impact of dependence structure on the ruin probability via the comparison of Lundberg coefficient. Badescu et al. (2009) considered an extension of the Sparre Andersen insurance risk model with the dependence structure that the joint distribution of the interclaim time and the subsequent claim size is bivariate phase-type (see e.g. Assaf et al. 1984 and Kulkarni 1986) and they employed the existing connection between risk processes and fluid flows to the analysis of various ruin-related quantities.

Recently, Asimit and Badescu (2010) introduced a general dependence structure for which the claim sizes and interarrival times correspondingly form a sequence of i.i.d. random pairs and that each pair obeys a dependence structure described via the conditional tail probability of a claim size given the interarrival time before the claim. They presented asymptotic tail probabilities for the discounted aggregate claims in the compound poisson risk model when the force of interest is constant and the claim amounts are heavy-tail distribution random variables. Further study of this model can be found in the literature. Li et al. (2010) used the same dependence structure as proposed by Asimit and Badescu (2010), to measure the impact of this dependence structure on the asymptotic tail probability of discounted aggregate claims in the continuous-time renewal risk model. Chen et al. (2011) used the size dependent model to study the large deviation of the aggregate amount of claims in a renewal risk model, and obtained a precise large-deviation formula for a heavy-tailed case.

1.2.2 Risk Models with Dependence in a Stochastic Economic Environment

It is a common practice in risk theory to assume that insurance risk and financial risk are independent of each other as well. Undoubtedly, this assumption of complete independence is far unrealistic. A recent new trend of study is to introduce various dependence structures to the risk model.

Tang and Tsitsiashvili (2003), Wang et al. (2005), Goovaerts et al. (2005), Wang and Tang (2006) further pointed out that the dependent assumption on financial risks is not necessary. Goovaerts et al. (2005) investigated the tail probability of discounted aggregate sums of Pareto-like losses in insurance. Here, the discounted sum of losses with finite or infinite time period can be described as a randomly weighted sum of a sequence of independent variables and the independent random variables represent the amounts of losses in successive development years, while the weights represent the stochastic discount factors which are mutually dependent.

Tang (2006) considered a discrete-time insurance risk model with risky investments, and extended the results of Tang and Tsitsiashvili (2003) to the whole subexponential class and to nonstandard case with associated discount factors.

Following the recent works of Nyrhinen (1999, 2001) and Tang and Tsitashvili (2003, 2004), Tang and Vernic (2007) considered the discrete-time insurance risk model in which the financial risks constituted a stationary process with finite dimensional distributions of Farlie-Gumbel-Morgenstern type. They assumed the independence between insurance and financial risks and obtained an exact asymptotic formula for the ruin probability assuming the the insurance risks belonged to the class of regularly varying distributions.

Recently, Zhang et al. (2009), Shen et al. (2009), Gao and Wang (2010), Wang and Yin (2010) considered a special dependence structure, namely bivariate upper tail independence, for net loss, and obtained the asymptotic tail probabilities of randomly weighted sums and their maxima for the finite-time and ultimate cases. Simultaneously, Chen and Yuen (2009), Chen et al. (2010), and Yi et al. (2011) introduced similar dependence structures, which allow nonidentically distributed insurance risks, and derived corresponding asymptotic results for claim-size distributions belonging to the classes $\mathcal{C}$ and $\mathcal{D}$, respectively. In addition, Yang et al. (2012) investigated the same dependence structure as in Chen et al. (2010) and derived some further asymptotic and uniformly asymptotic results for the finite-time and ultimate ruin probabilities under assumptions that the insurance risks are heavy tailed (belong to the intersection of $\mathcal{L}$ and $\mathcal{D}$) and the financial risks satisfy some moment conditions.

However, there are few papers which take into account the dependence between insurance and financial risks, with the difficulty existing in describing the tail behaviour of the product of dependent random variables.

Chen (2011) considered a discrete-time insurance risk model in which the insurer makes both risk-free and risky investments, and derived a general asymptotic formula for the finite-time ruin probability by assuming that the net insurance losses follow a subexponential distribution. It is further assumed that the net insurance losses and the stochastic discount factors correspondingly form a sequence of i.i.d. random pairs following a common bivariate Farlie-Gumbel-Morgenstern distribution.

1.2.3 Research on the Asymptotic Behavior of Aggregate Claims

The study of precise large deviations of random sums was initiated by Klüppelberg and Mikosch (1997). They considered the classical risk model with claims following an Extended Regularly Varying (ERV) distribution. Tang et al. (2001) improved the result of Klüppelberg and Mikosch (1997) by introducing the compound renewal model which is more realistic than the classical one. In addition, Ng et al. (2004) studied the precise large deviations for sums of claims with consistently varying tails. For further applications of precise large deviations to insurance and finance, see Mikosch and Nagaev (1998) and Embrechts et al. (1997, Chapter 8), among others.

Recently, for practical reasons, precise large deviations of dependent random variables have received a remarkable amount of attention. Kaas and Tang (2005) investigated precise large deviation of a compound sum of claims in which the claims arrive in groups and the claim numbers in the group may follow a certain negative dependence structure, and they also built a platform both for the classical large deviation theory and for the modern stochastic ordering theory. Additionally, Tang (2006) studied the case of negatively dependent claim sizes and found the asymptotic behaviour of precise large deviations is insensitive to the negative dependence. By extending the negatively dependent structure Liu (2009) focused on precise large deviations of sums of extended negative dependence (END) random variables and found that the END structure has no effect on the asymptotic behavior of precise large deviations of partial sums and random sums either. Furthermore, Chen et al. (2011) extended the results of Tang (2006) and Liu (2009) to random sums in various situations and established a precise large deviation result for a nonstandard renewal risk model in which innovations, modelled as real-valued random variables, are negatively dependent with common consistently-varying-tailed distribution, and their inter-arrival times are also negatively dependent.

However, in the literature all large deviation results for loss processes have been studied for only one kind of claims. That is to say, the company provides only one kind of insurance contracts. In reality, this assumption is not correct, and, it is worthwhile to extend the study to the large deviation problem of multi-risk models. In this direction, Wang and Wang (2007) investigated large deviations for random sums of random variables with consistently varying tails in multi-risk models. Extentions of this work can be found in Lu (2012, 2013) who gave lower and upper bounds for the sums of subexponential claims.

1.3 Outline of the Thesis

In this thesis, we study some risk models with certain dependent structures. Specifically, we consider two types: i) the dependence between claims and their interarrival times, ii) the dependence between insurance risk and financial risk. We mainly consider using heavy-tailed distribution to model random variables.

In Chapter 2, we recall definitions of heavy-tailed distributions and study their properties. Special attention is paid to the subexponential class as one of the most useful and commonly used classes of heavy-tailed distributions. Some lemmas regarding heavy-tailed distributions are given. In addition, relevant extreme value distributions are also discussed.

In Chapter 3, we incorporate the heavy-tailed distributions into the renewal risk model based on the two dependent assumptions, namely, dependence among claim sizes and dependence between claims and their interarrivals. We give results on the ruin probability and large deviations of sums of random variables according to different dependent assumptions.

In Chapter 4, we introduce the renewal risk model with dependence structure and provides another approach to study ruin probability. In this case, an equation for the tail probability of maximal present value of aggregate net loss is derived, and hence some insights into the ruin probability can be obtained. Chapter 4 is based on the joint work of Chen et al. (2013).

Chapter 5 of the thesis contains some concluding remarks. Also, we discuss some possibilities for further study on the topic.

Chapter 2

Heavy-tailed Distributions

2.1 Notation and Conventions

Throughout this thesis we use the following conventions:

i) Without otherwise stated, the limit procedure is according to $x \to \infty$;

ii) We use $a_\vee = \max\{a, 0\} = a \vee 0$ and $a_\wedge = -\min\{a, 0\} = -(a \wedge 0)$ to denote the positive and negative parts of a real number a, respectively.

iii) For two positive functions $a(\cdot)$ and $b(\cdot)$ satisfying

$$l_1 = \liminf_{x\to\infty} \frac{a(x)}{b(x)} \leq \limsup_{x\to\infty} \frac{a(x)}{b(x)} = l_2, \tag{2.1.1}$$

for some $0 \leq l_1 \leq l_2 \leq \infty$, we write $a(x) = O(b(x))$ if $l_2 < \infty$; $a(x) = o(b(x))$ if $l_2 = 0$; $a(x) \asymp b(x)$ if $0 < l_1 \leq l_2 < \infty$; $a(x) \lesssim b(x)$ if $l_2 = 1$; $a(x) \gtrsim b(x)$ if $l_1 = 1$; and $a(x) \sim b(x)$ if both $l_1 = 1$ and $l_2 = 1$

Notation used is summarized below:

1_E	the indicator function of an event
a.s.	almost surely
$\mathcal{C}$	the class of distributions with consistently-varying tails
cdf	cumulative distribution function
$\mathcal{D}$	the class of distributions with dominatedly-varying tails
df	distribution function
$\stackrel{\mathrm{d}}{=}$	$X \stackrel{\mathrm{d}}{=} Y \Leftrightarrow \Pr(X > x) = \Pr(Y > x)$ for every x
$\mathbb{E}$	expectation
ERV	the class of distributions with extended-regularly-varying tails
$\overline{F}$	$1 - F$ for a distribution F
$F * G$	the convolution of distributions F and G
F^{n*}	the n-fold convolution of a distribution F
$F_+(x)$	$F(x)1_{(x\geq 0)}$ for a distribution F on $(-\infty, \infty)$
$F^{\leftarrow}(t)$	Quantile function $F^{\leftarrow}(t) = \inf\{x \in \mathbb{R}, F(x) \geq t\}$
iid	independently and identically distributed
$\mathbb{J}_F^{\pm}$	Matuszewska indices of a distribution F
$\mathcal{K}$	the class of heavy-tailed distributions
$\mathcal{L}$	the class of long-tailed distributions
pdf	probability density function
$\mathcal{R}$	the class of distributions with regularly-varying tails
$\mathcal{S}$	the class of distributions with subexponential tails
$\Phi(\cdot)$	the standard normal distribution
Φ_α	Fréchet distribution
Λ	Gumbel distribution
Ψ_α	Weibull distribution
MDA(Φ_α)	Maximum domain of attraction of the Fréchet distribution
MDA(Λ)	Maximum domain of attraction of the Gumbel distribution
MDA(Ψ_α)	Maximum domain of attraction of the Weibull distribution

2.2 Brief Review of Heavy-Tailed Distributions

Heavy-tailed distributions have been extensively used for modelling insurance risks for a long time. In probability theory, heavy-tailed distributions have heavier tails than that of the exponential distribution. In practice, most of commonly used heavy-tailed distributions belong to the subexponential class. In this thesis, we follow the style of Embrechts (1997) to define heavy-tailed distributions. That is, a random variable X or its distribution F is said to be heavy-tailed (on the right) if

$$Ee^{sX} = \int_{-\infty}^{\infty} e^{sx} F\,(\mathrm{d}x) = \infty, \quad \textit{for any } s > 0.$$

Here, we use $\mathcal{K}$ to denote the class of (right) heavy-tailed distributions. There are also other definitions in use. Some authors use the term to refer to those distributions which have finite moments up to a certain order; and some others to those distributions that do not have a variance.

On the contrary, a random variable X or its distribution F is light tailed (on the right) if there is some $s_0 > 0$ such that

$$Ee^{sX} < \infty, \qquad \textit{for all } 0 < s < s_o.$$

2.2.1 The Subexponential Class

We follow the definition of subexponentiality proposed by Embrechts et al. (1997).

Definition 2.2.1 *(Subexponential distribution function)*

A df F with support $(0,\infty)$ is subexponential, if for all $n \geq 2$

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n. \tag{2.2.1}$$

The class of subexponential dfs will be denoted by $\mathcal{S}$

More generally, a distribution F on $(-\infty,\infty)$ is also subexponential if $F_{+}(x) = F(x)1_{x\geq 0}$ is subexponential. In this case, relation (2.2.1) still holds.

In order to check the subexponentiality for a df, one does not have to show (2.2.1) for all $n \geq 2$. A sufficient condition for subexponentiality is:

$$\limsup_{x\to\infty} \frac{\overline{F^{2*}(x)}}{\overline{F}(x)} \leq 2 \tag{2.2.2}$$

Suppose now that $X_1, ..., X_n$ are iid with df $F \in \mathcal{S}$. Denote the partial sum of $X_1, ..., X_n$ by $S_n = X_1 + \cdots + X_n$ and their maximum by $M_n = \max\{X_1, ..., X_n\}$. Then the relations

$$\Pr(S_n > x) \sim n\overline{F}(x) \sim \Pr(M_n > x) \tag{2.2.3}$$

hold for each $n \geq 2$. Actually, the first relation in (2.2.3) is due to $\Pr(S_n > x) = \overline{F^{n*}}(x)$, and the second relation in (2.2.3) can be verified as follows:

$$\begin{aligned} \Pr(M_n > x) &= \overline{F^n}(x) \\ &= \overline{F}(x) \sum_{k=0}^{n-1} F^k(x) \\ &\sim n\overline{F}(x), \quad x \to \infty \end{aligned}$$

Thus, the tail probability of the sum and the maximum of the first n i.i.d. subexponential random variables are asymptotically of the same order, known as the principle of a single big jump. If we interpret the random variables as claim sizes, then relation (2.2.3) apparently indicates the strong influence of the largest claim on the total amount of claims. This feature explains the relevance of subexponential distributions in modelling heavy-tailed phenomena in insurance and finance.

Let F denote a distribution function and f denote a density function. The following examples of subexponential distributions are partially copied from Table 1.2.6 of Embrechts et al. (1997).

i) Benktander-type I ($\alpha > 0, \beta > 0$):

$$\overline{F}(x) = \left(1 + \frac{2\beta}{\alpha} \ln x\right) \exp\left(-\beta(\ln x)^2 - (\alpha + 1)\ln x\right);$$

ii) Benktander-type II ($\alpha > 0, 0 < \beta < 1$):

$$\overline{F}(x) = \exp(\alpha/\beta) x^{-(1-\beta)} \exp\left(-\frac{\alpha x^\beta}{\beta}\right);$$

iii) Burr ($\alpha > 0, \kappa > 0, \tau > 0$):

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x^{\tau}}\right)^{\alpha};$$

iv) Loggamma ($\alpha > 0, \beta > 0$):

$$f(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}.$$

v) Lognormal ($-\infty < \mu < \infty, \sigma > 0$):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\};$$

vi) Pareto ($\alpha > 0, \kappa > 0$):

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x}\right)^{\alpha};$$

vii) Weibull ($c > 0, 0 < \tau < 1$):

$$\overline{F}(x) = \exp(-cx^{\tau});$$

From the examples shown above we see that the class $\mathcal{S}$ essentially contains three kinds of distributions: Pareto-like, Logmormal-like, and heavy-tailed Weibull-like distributions.

Next, we will present several properties of the subexponential class.

Lemma 2.2.1 *(Lemma A 3.15 of Embrechts 1997). Suppose F and G are dfs on $(0, \infty)$. If $F \in \mathcal{S}$ and*

$$\lim_{x\to\infty}\frac{\overline{G}(x)}{\overline{F}(x)} = c \in (0, \infty),$$

then $G \in \mathcal{S}$.

Lemma 2.2.2 *(Lemma 4.4 of Tang 2004). Consider the convolution of two distributions F_1 and F_2 on $(-\infty, \infty)$. If $F_1 \in \mathcal{S}$ and $\overline{F_2}(x) \lesssim c\overline{F_1}(x)$ for some $c \geq 0$, then*

$$\overline{F_1 * F_2}(x) \lesssim (1 + c)\overline{F_1}(x).$$

In particular, with $c = 0$*, if* $F_1 \in \mathcal{S}$ *and* $\overline{F_2}(x) = o\left(\overline{F_1}(x)\right)$*, then*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x).$$

Cline and Samorodnitsky (1994) studied the subexponentiality of the product

$$Z = XY, \tag{2.2.4}$$

Lemma 2.2.3 *(Theorem 2.1 of Cline and Samorodnitsky 1994) Let* X *and* Y *be two independent random variables with distribution functions* F *on* $\mathbb{R}$ *and* G *on* $\mathbb{R}^+$*, respectively. Denote by* H *the distribution function of their product* XY*, We have the* $H \in \mathcal{S}$ *if* $F \in \mathcal{S}$ *and there is a function* $a(\cdot)$*:* $[0, \infty) \to [0, \infty)$ *satisfying:*

(a) $a(x) \uparrow \infty$,

(b) $a(x)/x \downarrow 0$,

(c) $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$*, and*

(d) $\overline{F}\left(x - a(x)\right) \sim \overline{F}(x)$.

2.2.2 Regular Variation

Another popular class of heavy-tailed distributions is the class of regularly-varying distributions, which is of also a subclass of subexponential distributions.

Definition 2.2.2 *(Regular Variation in Karamata's Sense)*

(a) A positive Lebesgue measurable function L *on* $(0, \infty)$ *is slowly varying at* ∞ *(we write* $L \in \mathcal{R}_0$*) if*

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad t > 0 \tag{2.2.5}$$

(b) A positive Lebesgue measurable function H on $(0,\infty)$ is regularly varying at ∞ of index $\alpha \in \mathbb{R}$ (we write $H \in \mathcal{R}_\alpha$) if

$$\lim_{x\to\infty} \frac{H(tx)}{H(x)} = t^\alpha, \quad t > 0 \tag{2.2.6}$$

Usually we write

$$\mathcal{R} = \{F \text{ } df \text{ } on \text{ } (0,\infty) : \lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \text{ } for \text{ } some \text{ } \alpha > 0\}. \tag{2.2.7}$$

We remark that the notion of regularly-varying distributions has been extensively investigated in the literature. For details, we refer the reader to Bingham et al. (1987) and Resnick (1987).

next we recall some properties of regular variation.

Lemma 2.2.4 *(Theorem A3.6 of Embrechts et al. (1997)) Let $f \in \mathcal{R}_\alpha$ for some $\alpha \in \mathbb{R}$ and f is locally bounded on $[0,\infty)$.*

(a) If $\alpha > -1$,

$$\lim_{x\to\infty} \frac{\int_0^x f(t)\mathrm{d}t}{xf(x)} = \frac{1}{\alpha+1};$$

if $\alpha < -1$,

$$\lim_{x\to\infty} \frac{\int_x^\infty f(t)\mathrm{d}t}{xf(x)} = -\frac{1}{\alpha+1}.$$

(b)Suppose F is absolutely continuous with density f such that for some $\alpha > 0$,

$$\lim_{x\to\infty} \frac{xf(x)}{F(x)} = \alpha,$$

then $f \in \mathcal{R}_{-1-\alpha}$ and consequently $\overline{F} \in \mathcal{R}_{-\alpha}$

The above results are usually referred to as Karamata's Theorem.

Lemma 2.2.5 *(Convolution Closure of regularly Varying Distributions). Let X and Y be two independent, regularly varying, non-negative random variables with index $\alpha \geq 0$. Then $X + Y$ is regularly varying with index α and*

$$\Pr(X + Y > x) \sim \Pr(X > x) + \Pr(Y > x), \quad x \to \infty.$$

These results can be found in Lemma 1.3.4 of Embrechts et al. (1997).

Lemma 2.2.6 *(Proposition 3 of Breiman (1956)). Let X and Y be two independent random variables distributed by F and G, respectively, where Y is nonnegative. If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and $E[Y^\beta] < \infty$ for some $\beta > \alpha$, then*

$$\lim_{x\to\infty} \frac{\Pr(XY > x)}{\Pr(X > x)} = E[Y^\alpha]$$

An extension of regular variation is rapid variation. By definition, a distribution F on $(-\infty, \infty)$ is said to be rapidly varying which denoted by $\mathcal{R}_{-\infty}$, if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = 0$$

holds for all $t > 1$. For example, lognormal, Benktander-type I and II, and Weibull distributions all belong to this class.

2.2.3 Other Related Classes of Heavy-Tailed Distributions

One useful subclass of $\mathcal{S}$ is $\mathcal{A}$, which was introduced by Konstantinides et al. (2002). By definition, a distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{A}$ if $F \in \mathcal{S}$, and for some $a > 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}(ax)}{\overline{F}(x)} < 1. \tag{2.2.8}$$

We remark that the class $\mathcal{A}$ almost coincides with $\mathcal{S}$ because it excludes only some very heavy-tailed distribution, such as slowly-varying distributions. Clearly, $\mathcal{R} \subset \mathcal{A}$, *and* $\mathcal{S} \cap \mathcal{R}_{-\infty} \subset \mathcal{A}$.

Indeed, relation (2.2.8) is satisfied by almost useful subexponential classes with unbounded supports on the right, including Pareto, Lognormal, and heavy-tailed Weibull distributions.

The class of long-tailed distributions, denoted by $\mathcal{L}$, and the class of dominatedly-varying distributions, denoted by $\mathcal{D}$, are closely related to the class $\mathcal{S}$.

A distribution F on $(-\infty, \infty)$ is said to belong to $\mathcal{L}$, if $\overline{F}(x) > 0$ for all x and

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1, \tag{2.2.9}$$

holds for all (or, equivalently, for some) $y \neq 0$.

A distribution F on $(-\infty, \infty)$ is said to belong to $\mathcal{D}$, if $\overline{F}(x) > 0$ for all x and

$$\limsup_{x\to\infty} \frac{\overline{F}(ax)}{\overline{F}(x)} < \infty, \tag{2.2.10}$$

holds for all (or, equivalently, for some) $0 < a < 1$.

A famous subclass of the intersection $\mathcal{L} \cap \mathcal{D}$ is ERV, which is the class of distributions with extended-regularly-varying tails. By definition, a distribution F on $(-\infty, \infty)$ is said to belong to ERV$(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ if $\overline{F}(x) > 0$ holds for all x and the relations:

$$y^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha}, \tag{2.2.11}$$

hold for all $y \geq 1$.

A larger class is the class $\mathcal{C}$ of distributions with consistently-varying tails defined by

$$\lim_{y\uparrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{y\downarrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1. \tag{2.2.12}$$

The following inclusions hold for these heavy-tailed distribution classes:

- $\mathcal{R} \subset \text{ERV} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{K}$,

- $\mathcal{D} \not\subseteq \mathcal{S}$ *and* $\mathcal{S} \not\subseteq \mathcal{D}$.

The inclusions $\mathcal{R} \subset \text{ERV} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L}$ can be easily verified by their definitions. For the other inclusions above, see Embrechts et al. (1997) for detailed discussions.

Next we present properties of the related heavy-tailed clsses in the following lemma.

Lemma 2.2.7 *(Lemma 3.2 of Tang and Tsitsiashvili 2003) Let F_1 and F_2 be two distribution functions on $\mathbb{R}$ and let $F = F_1 * F_2$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$ and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F \in \mathcal{S}$ and $\overline{F}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.*

2.2.4 Matuszewska Indices

In this subsection, we introduce Matuszewska indices of a distribution, which are connected with many useful properties of heavy-tailed classes. Following Theorem of 2.1.5 and Corollary 2.1.6 of Bingham et al. (1987), for a df F on $(-\infty, \infty)$ and for each $v > 0$, they defined

$$\mathbb{J}_F^+ = \inf_{v>1} \left\{ -\frac{\log \overline{F}_*(v)}{\log v} \right\}, \quad \mathbb{J}_F^- = \inf_{v>1} \left\{ -\frac{\log \overline{F}^*(v)}{\log v} \right\},$$

where

$$\overline{F}_*(v) = \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \quad \overline{F}^*(v) = \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}. \tag{2.2.13}$$

The quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ are called the upper and lower Matuszewska indices of the nonnegative and nondecreasing function $f = 1/\overline{F}$, respectively. In the work of Tang and Tsitsiashvili (2003), they simply called the quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ the upper and lower Matuszewska indices of the distribution F, respectively. For more details of Matuszewska indices, see Chapter 2.1 of Bingham et al. (1987) and Section 3 of Cline and Samorodnitsky (1994).

Clearly, $F \in \mathcal{D}$ if and only if $\mathbb{J}_F^+ < \infty$; $F \in \mathrm{ERV}(-\alpha, -\beta)$ then $\mathbb{J}_F^- \geq \alpha$ and $\mathbb{J}_F^+ \leq \beta$; and $F \in \mathcal{R}_\alpha$ then $\mathbb{J}_F^- = \mathbb{J}_F^+ = \alpha$.

Next, we give some useful lemmas.

Lemma 2.2.8 *(Proposition 2.2.1 of Bingham et al. 1987). Let F be a distribution on $(-\infty, \infty)$,*

(a) For every p_1, $0 < p_1 < \mathbb{J}_F^- \leq \infty$, there are positive constants C_1 and x_1 such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C_1 \left(\frac{y}{x}\right)^{-p_1} \tag{2.2.14}$$

holds whenever $y \geq x \geq x_1$

(b) For every p_2, $0 \leq \mathbb{J}_F^+ < p_2 < \infty$, there are positive constants C_2 and x_2 such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \geq C_2 \left(\frac{y}{x}\right)^{-p_2}, \tag{2.2.15}$$

holds whenever $y \geq x \geq x_2$.

Fixing the variable $x = x_1 \vee x_2$ in (2.2.14) and (2.2.15), we easily see that, for any $0 < p_1 < \mathbb{J}_F^-$

$$\overline{F}(x) = o(x^{-p_1}), \tag{2.2.16}$$

and for any $\mathbb{J}_F^+ < p_2 < \infty$

$$x^{-p_2} = o\left(\overline{F}(x)\right). \tag{2.2.17}$$

In particular, a distribution $F \in \mathrm{ERV}(-\alpha, -\beta)$, $\alpha < \mathbb{J}_F^-$ and $\beta > \mathbb{J}_F^+$, the inequalities in (2.2.14) and (2.2.15) apply to $F \in \mathrm{ERV}(-\alpha, -\beta)$.

Lemma 2.2.9 *(Theorem 1.5.6 of Bingham et al. (1987)). If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then for arbitrarily chosen $C > 1$ and $0 < \varepsilon < \alpha$, there exists $x_0 > 0$ such that for all $x, y \geq x_0$,*

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C \max\left(\left(\frac{y}{x}\right)^{-\alpha+\varepsilon}, \left(\frac{y}{x}\right)^{-\alpha-\varepsilon}\right). \tag{2.2.18}$$

This is the well-known Potter's bound for the class $\mathcal{R}$

2.2.5 Extreme Value Theory for Insurance and Finance

Extreme Value Theory is a classical topic in probability theory and mathematical statistics. In this chapter, we consider three families of possible limit laws which are known as extreme value distributions. Firstly, let us give a brief review of maximum domains of attraction.

A distribution function F belongs to the maximum domain of attraction of an extreme value distribution F_0, written $F \in \text{MDA}(F_0)$, if

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} |F^n(a_n x + b_n) - F_0(x)| = 0, \tag{2.2.19}$$

holds for some normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$.

In other words, for i.i.d. samples $\{X_1, X_2, \ldots\}$ from F, denoting by $M_n = \max\{X_1, \ldots, X_n\}$ the block maxima, we have

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} F_0.$$

This is the classical Fisher-Tippett theorem, see Embrechts et al. (1997). By the Fisher-Tippett theorem, only three choices for F_0 are possible, namely Fréchet distribution(Φ_α), Gumbel distribution(Λ), and Weibull distribution (Ψ_α).

The functional form of the Fréchet distribution is

$$\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}, \quad \alpha, x > 0.$$

$F \in \text{MDA}(\Phi_\alpha)$ if and only if $x_F = \infty$ and

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad y > 0.$$

This means that $\overline{F}$ is regularly varying at infinity with index $-\alpha$. There are some examples of $\text{MDA}(\Phi_\alpha)$, also see table 3.4.2 of Embrechts *et al*. (1997).

- Burr: $\overline{F}(x) = (\kappa/(\kappa + x^\tau))^\alpha$ for $\alpha > 0,\ \kappa > 0,\ \tau > 0$;

- Cauchy: $f(x) = (\pi(1 + x^2))^{-1}, x \in \mathbb{R}$;

- F-distribution: for $d_1 > 0$ and $d_2 > 0$,

$$f(x) = \frac{1}{B(d_1/2, d_2/2)} \left(\frac{d_1}{d_2}\right)^{d_1/2} x^{\frac{d_1}{2}-1} \left(1 + \frac{d_1}{d_2}x\right)^{-\frac{d_1+d_2}{2}};$$

- Loggamma: $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1} x^{-\alpha-1}$ for $\alpha > 0,\ \beta > 0$;

- Pareto: $\overline{F}(x) = (\kappa/(\kappa + x))^\alpha$ for $\alpha > 0,\ \kappa > 0$;

- Student's t: for $v > 0$,

$$f(x) = \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)} \left(1 + \frac{x^2}{v}\right)^{-(v+1)/2}.$$

The Gumbel distribution is given by

$$\Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}.$$

$F \in \text{MDA}(\Lambda)$ if and only if $x_F \leq \infty$ and there is some positive auxiliary function $a(\cdot)$ such that

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x + ya(x))}{\overline{F}(x)} = e^{-y}, \quad y \in \mathbb{R}.$$

Recall that the auxiliary function $a(\cdot)$ can be chosen to be the mean excess loss function:

$$a(x) = E(X - x|X > x).$$

There are some examples of MDA(Λ) also see Table 3.4.2 of Embrechts *et al.* (1997)

- Benktander-type I: for $\alpha > 0,\ \beta > 0$,

$$\overline{F}(x) = (1 + 2(\beta/\alpha)\ln x)\exp\{-\beta(\ln x)^2 - (\alpha+1)\ln x\};$$

- Benktander-type II: for $\alpha > 0,\ 0 < \beta < 1$,

$$\overline{F}(x) = \mathrm{e}^{\alpha/\beta} x^{-(1-\beta)} \exp\{-\alpha x^{\beta}/\beta\};$$

- Gamma: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $\alpha > 0$ and $\beta > 0$;

- Lognormal: for $-\infty < \mu < \infty$ and $\sigma > 0$,

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\{-(\ln x - \mu)^2/(2\sigma^2)\};$$

- Weibull-like: $\overline{F}(x) = \exp\{-cx^\tau\}$ for $c > 0, \tau > 0$

The functional form of the Weibull distribution is

$$\Psi_\alpha(x) = \exp\{-|x|^\alpha)\}, \quad \alpha > 0, x \le 0.$$

It is know that $F \in \mathrm{MDA}(\Psi_\alpha)$ if and only if $x_F < \infty$ and

$$\lim_{u\to\infty} \frac{\overline{F}(x_F - x/u)}{\overline{F}(x_F - 1/u)} = x^\alpha, \quad \forall x > 0.$$

Hence, MDA(Ψ_α) can only be used to model bounded risk variables. Examples for the Weibull case:

- Beta: $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ for $a > 0$ and $b > 0$;

- Uniform: on (a, b);

- The distribution of stochastic structure of the financial risk $Y = \frac{1}{c+R}$, which will be introduced in Chapter 4, belongs to MDA(Ψ_γ), ($c > 0$ is a constant and R is a random variable), if R is distributed by a gamma distribution: for $\alpha > 0$ and $\beta > 0$,

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

or an F-distribution: for $d_1 > 0$ and $d_2 > 0$,

$$f(x) = \frac{1}{B\left(d_1/2, d_2/2\right)} \left(\frac{d_1}{d_2}\right)^{d_1/2} x^{\frac{d_1}{2}-1} \left(1 + \frac{d_1}{d_2} x\right)^{-\frac{d_1+d_2}{2}}.$$

Chapter 3

A Risk Model with Dependent Claim Sizes or/and Inter-arrival Times

3.1 Introduction

In this chapter, we consider risk models in finite-time horizon allowing dependent claims and dependence between claim sizes and their inter-arrival times. In recent years, some researchers have considered risk models with dependence assumptions. Among them, Kaas and Tang (2005) inteoduced negative dependence for claim sizes and Albrecher and Teugels (2006) first proposed dependence between the waiting time for a claim and its actual size. In the latter, they described the surplus process as a random walk with independent increments over claim arrival times.

3.2 Some Basic Results on the Risk Model with Dependent Claim sizes

The material presented in this subsection is mainly extracted from Tang (2006). Now assume that the claims, $A_1, A_2, ...$, form a sequence of identically distributed, not necessarily independent, and nonnegative random variables with common distribution function F, and their inter-arrival times, $B_1,\ B_2, ...$, form another sequence of i.i.d., nonnegative, and non-degenerate-at-zero random variables and are independent of the former sequence. Let $S_n = \sum_{i=1}^{n} A_i, n = 1, 2, ...,$ denote the partial sums. Here, we assume that the random variables $A_1, A_2, ...,$ are negatively dependent. The definition of negative dependence is as follows:

Definition 3.2.1 *We call random variables $\{A_k, k = 1, 2, ...\}$*

(1) Lower Negatively Dependent (LND) if for each $n = 1, 2, ...,$ and all $x_1, ..., x_n$,

$$\Pr(A_1 \leq x_1, ..., A_n \leq x_n) \leq \prod_{k=1}^{n} \Pr(A_k \leq x_k); \tag{3.2.1}$$

(2) Upper Negatively Dependent (UND) if for each $n = 1, 2, ...,$ and all $x_1, ..., x_n$,

$$\Pr(A_1 > x_1, ..., A_n > x_n) \leq \prod_{k=1}^{n} \Pr(A_k > x_k); \tag{3.2.2}$$

(3) Negatively Dependent (ND) if both (3.2.1) and (3.2.2) hold for each $n = 1, 2, ...,$ and all $x_1, ..., x_n$.

Recall that these random variables are called positively dependent (PD) if the inequalities in (3.2.1) and (3.2.2) hold both in the reverse direction. Roughly speaking, the negative dependence structure describes that the tails of finite-dimensional distributions of the random variables $A_1, A_2, ...$ in the lower left and upper right corners are dominated by the corresponding tails of the finite dimensional distributions of a sequence of independent random variables with the same marginal distributions. It is worthy mentioning that for $n = 2$, the LND, UND, and ND structure are equivalent; see, for example, Lehmann (1966). An extension of this study is pairwise negatively dependence. We say that a sequence of random variables $\{A_1, A_2, ...\}$ is pairwise negative dependent if for all positive integers $i \neq j$ and all real numbers x_i and x_j,

$$\Pr(A_i \leq x_i, A_j \leq x_j) \leq \Pr(A_i \leq x_i) \Pr(A_j \leq x_j), \tag{3.2.3}$$

or, equivalently,

$$\Pr(A_i > x_i, A_j > x_j) \leq \Pr(A_i > x_i) \Pr(A_j > x_j). \tag{3.2.4}$$

A simple example in which the underlying random variables are pairwise ND but not independent can be constructed in terms of the well-known Farlie-Gumbel-Morgenstern (FGM) distribution. An n-dimensional FGM distribution has the form

$$F_{A_1,...,A_n}(x_1,...,x_n) = \left(\prod_{i=1}^{n} F_i(x_i)\right) \left(1 + \sum_{1\leq j \leq k \leq n} a_{jk} \overline{F}_j(x_j) \overline{F}_k(x_k)\right), \tag{3.2.5}$$

where $F_1, ..., F_n$ are the corresponding marginal distributions of the random variables $A_1, ...A_n$, and a_{jk} are real numbers fulfilling certain requirements so that $F_{A_1,...,A_n}$ is a proper n-dimensional distribution. Clearly, the random variables $A_1, ..., A_n$ are pairwise ND if and only if the coefficients a_{jk} are all non-positive. We refer the reader to Kotz et al. (2000) for a general account on the multivariate FMG distributions. See also Proposition 2.1 of Li et al. (2006) for another simple construction of a sequence of pairwise ND but not independent random variables with arbitrarily prespecified marginal distributions.

We also remark that these notions of negative dependence are much more verifiable than the commonly used notion of negative association, the latter of which was introduced by Alam and Saxena (1981) and Joag-Dev and Proschan (1983). See also Bingham and Nilisani (2004) for a recent account and for a list of relevant references.

According to this model, Tang (2006) found that the asymptotic behavior of precise large deviations is insensitive to the negative dependence.

Theorem 3.2.1 *Let $\{A_k, k = 1, 2, ...\}$ be UND with commom distribution $F \in \mathcal{C}$ and mean 0. Then for each fixed $\gamma > 0$, the relation*

$$\Pr(S_n > x) \lesssim n\overline{F}(x), \tag{3.2.6}$$

holds uniformly for all $x \geq \gamma n$. That is

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{\Pr(S_n > x)}{n\overline{F}(x)} \leq 1. \tag{3.2.7}$$

Theorem 3.2.2 *Let $\{A_k, k = 1, 2, ...\}$ be LND with commom distribution $F \in \mathcal{C}$ and the mean 0 satisfying $xF(-x) = o\left(\overline{F}(x)\right), \quad x \to \infty$. Then for each fixed $\gamma > 0$, the relation*

$$\Pr(S_n > x) \gtrsim n\overline{F}(x), \tag{3.2.8}$$

holds uniformly for all $x \geq \gamma n$. That is

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{\Pr(S_n > x)}{n\overline{F}(x)} \geq 1. \tag{3.2.9}$$

Liu (2009) extended the results of Tang (2006) to the extended negatively dependent case. We call random variables $\{A_i, i \geq 1\}$ extended negatively dependent (END) if there exists $M > 0$ such that both

$$\Pr(A_1 \leq x_1, ..., A_n \leq x_n) \leq M \prod_{k=1}^{n} \Pr(A_k \leq x_k) \tag{3.2.10}$$

and

$$\Pr(A_1 > x_1, ..., A_n > x_n) \leq M \prod_{k=1}^{n} \Pr(A_k > x_k) \tag{3.2.11}$$

hold for each $n = 1, 2, ...$ and all $x_1, ..., x_n$.

The END structure is more general than the ND structure in that it can reflect not only a negative dependence but also a positive one, to some extent. Liu (2009) derived the following result.

Theorem 3.2.3 *Let $\{A_i, i \geq 1\}$ be END each with mean 0. If*

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^{n} \overline{F}_i(x)}{\overline{F}(x)} = 1, \quad and \quad \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^{n} \overline{F}_i(-x)}{\overline{F}(-x)} = 1 \tag{3.2.12}$$

hold, where $F \in \mathcal{C}$ satisfies $F(-x) = o\left(\overline{F}(x)\right), x \to \infty$ with a finite mean, then for large enough γ relation

$$\Pr(S_n > x) \sim n\overline{F}(x), \quad n \to \infty \tag{3.2.13}$$

holds uniformly for all $x \geq \gamma n$ as $n \to \infty$

Chen et al. (2010) extended the results of Tang (2006) and Liu (2009) to random sums. In particular, they established a precise large deviation result for a compound renewal risk model in which innovations modelled as real-valued random variables are ND with common consistently-varying-tailed distribution, and their inter-arrival times are also ND.

Theorem 3.2.4 *Let $\{A_k, k = 1, 2, ...,\}$ be a sequence of nonnegative and END random variables with common distribution $F \in \mathcal{C}$ and finite mean $\mu > 0$, and let $\{N_t,\ t > 0\}$ be a counting process (that is, a non-negative, non-decreasing, and integer-valued stochastic process) independent of $\{A_k, k = 1, 2, ...,\}$ and satisfying*

$$E\left(N_t^p 1_{(N_t > (1+\delta)\lambda_t)}\right) = O(\lambda_t), \quad t \to \infty, \tag{3.2.14}$$

for some $p > \mathbb{J}_F^+$ and all $\delta > 0$. Then, for every fixed $\gamma > 0$, the relation

$$\Pr\left(S_t - \mu\lambda_t > x\right) \sim \lambda_t \overline{F}(x), \quad t \to \infty, \tag{3.2.15}$$

holds uniformly for all $x \geq \gamma\lambda_t$.

Condition (3.2.14) first appeared in Tang et al. (2001) for weakening corresponding conditions in Klüppelberg and Mikosch (1997). This condition is fulfilled at least by commonly-used renewal counting process; see Lemma 3.5 of Tang et al. (2001). Other research on precise large deviation for random sums can be found in Ng et al. (2004), Kaas and Tang (2005), among others.

3.3 Some Basic Results on the Risk Model with Dependence between Claim Sizes and Interarrival Times

In this subsection dependence between claims and their interarrival times in the context of risk theory is discussed. Boudreault et al. (2006) introduced a risk model with time-dependent claim sizes and dependence structure between the claim amount A_k and the interclaim time B_k. They suppose the (conditional) density of $A_k|B_k$ to be defined as a special mixture of two arbitrary density functions f_1 and f_2 (with respective means μ_1 and μ_2), i.e.,

$$f_{A_k|B_k}(x|y) = e^{-\beta y} f_1(x) + (1 - e^{-\beta y}) f_2(x), \;\; x, y \geq 0, \tag{3.3.1}$$

for $k = 1, 2, ...$ From (3.3.1), the weight assigned to the c.d.f. f_1 is an exponential decreasing function (at rate β) of the time elapsed since the last claim B_k.

Note that this dependence structure can be linked to the one proposed by Albrecher and Boxma (2004). Next, consider a threshold structure where the threshold r.v.'s $\{C_j, j = 1, 2, ...\}$ from a sequence of i.i.d. exponentially distributed r.v.'s with mean $\frac{1}{\beta}$. We assume that the threshold r.v.'s are independent of all other sources of randomness in the proposed risk model. If the interclaim time B_j is larger (smaller) than the threshold r.v. C_j, then the density function of the claim amount A_j is $f_1(f_2)$. These assumptions lead to (3.3.1) for the density function of $A_k|B_k$. The risk model with time-dependent claim sizes and dependence structure (3.3.1) can be viewed as a more realistic model (than the classical compound Poisson risk model) to approximate the behavior of the aggregate claim process in a natural catastrophe context. Indeed, suppose B_j is the waiting time between the $(j-1)$th and jth catastrophes and such an event has two possible intensities, say $I_j = 1(usual), 2(severe)$. It results

$$\Pr(I_j = 1|B_j = y) = e^{-\beta y} = 1 - \Pr(I_j = 2|B_j = y), \tag{3.3.2}$$

and hence

$$\Pr(A_j \leq x|I_j = i) = F_i(x), \quad i = 1, 2. \tag{3.3.3}$$

For example, considering earthquakes, one can expect that the longer the time between two events is the larger the claim amount due to the next catastrophe will be. Hence, more weight should be assigned to the distribution F_2 which is chosen with a heavier tail than F_1.

Another dependent structure between claims and their intervals was proposed by Cossette et al. (2008) in the study of an extension of the classical risk model. They assumed that the joint distribution of (A, B) is defined with a generalized FGM copula. Copulas are referred as 'functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution function' and as 'distribution functions whose one-dimensional margins are uniform'. The FGM copula, which belongs to the family of copulas introduced and studied by Rodriguez-Lallena and Ubena-Flores (2004), is defined by

$$C(u, v) = uv + \theta h(u) g(v), \tag{3.3.4}$$

where h and g are two non-zero real functions with support $[0, 1]$. In this paper, they considered the special case where

$$h(u) = u^a(1-u)^b; \quad \text{and} \quad g(v) = v^c(1-v)^d. \tag{3.3.5}$$

with $a, b, c, d \geq 1$. Plugging (3.3.5) into (3.3.4), the expression of the copula is then given by

$$C(u, v) = uv + \theta u^a (1-u)^b v^c (1-v)^d. \tag{3.3.6}$$

It is obviously an extension to the classical FGM copula

$$C(u, v) = uv + \theta uv(1-u)(1-v), \;\; -1 \leq \theta \leq 1. \tag{3.3.7}$$

The pdf associated to (3.3.4) is given by

$$c(u, v) = 1 + \theta h^{'}(u) g^{'}(v). \tag{3.3.8}$$

The joint cdf $F_{A,B}$ is given by

$$\begin{aligned} F_{A,B}(x, t) &= C\left(F_A(x), F_B(t)\right) \qquad (3.3.9) \\ &= F_A(x)F_B(t) + \theta \left(F_A(x)\right)^a \left(1 - F_A(x)\right)^b \left(F_B(t)\right)^c \left(1 - F_B(t)\right)^d, \end{aligned}$$

and the joint pdf $f_{A,B}$ of (A, B) is

$$\begin{aligned} f_{A,B}(x, y) &= c\left(F_A(x), F_B(t)\right) f_A(x) f_B(t) \qquad (3.3.10) \\ &= f_A(x) f_B(t) + \theta h^{'}\left(F_A(x)\right) g^{'}\left(F_B(t)\right) f_A(x) f_B(t). \end{aligned}$$

Asimit and Badescu (2010) concerned with a different dependent structure. They assumed that the bivariate random vectors $(A_i, B_i), i = 1, 2, ...,$ are i.i.d. Moreover, there exists a positive and locally bounded function $g(\cdot)$ such that the relation

$$\Pr(A_1 > x | B_1 = y) \sim \Pr(A_1 > x) g(y) \tag{3.3.11}$$

holds uniformly for all $y \in (0, T]$ as $x \to \infty$, or equivalent,

$$\lim_{x \to \infty} \sup_{y \in (0,T]} \left| \frac{\Pr(A_1 > x | B_1 = y)}{\Pr(A_1 > x) g(y)} - 1 \right| = 0. \tag{3.3.12}$$

Relation (3.3.11) defines a general dependence structure which is easily verifiable for some commonly used bivariate copulas, and allows both positive and negative dependence. It is also very convenient when dealing with the tail behaviour of the sum or product of two dependent random variables.

Recently, Chen and Yuen (2012) consider a size dependent renewal risk model and obtained a result of precise large deviations of random sums. They assumed that there is a nonnegative random variable B^* such that B conditional on $(A > x)$ is stochastically bounded by B^* for all large $x > 0$; in other words, there is some $x_0 > 0$ such that it holds for all $x > x_0$ and $t \in [0, \infty)$ that

$$\Pr(B > t | A > x) \leq \Pr(B^* > t). \tag{3.3.13}$$

In contrast to (3.3.11), (3.3.13) describes a dependence structure via the conditional distribution of the inter-arrival time given the subsequent claim size being large. Hence, the model under (3.3.13) is termed as a size-dependent model. Roughly speaking, (3.3.13) means that A becoming large does not drag B to infinity. The size-dependent model seems more natural than the time-dependent model in view of the perception that the waiting time for a large claim is dependent on the claim size but not vice versa. Actually, let (3.3.11) be valid. As $x \to \infty$, it holds uniformly for all $t \in [0, \infty)$ that

$$\begin{aligned} \Pr(B > t | A > x) &= \frac{\Pr(A > x, B > t)}{Pr(A > x)} \\ &= \int_t^{\infty} \frac{\Pr(A > x | B = s)}{Pr(A > x)} \Pr(B \in \mathrm{d}s) \\ &\lesssim 2 \int_t^{\infty} h(s) \Pr(B \in \mathrm{d}s) \end{aligned}$$

Note that G_0 defined by $G_0(\mathrm{d}s) = h(s) \Pr(B \in \mathrm{d}s)$ is a proper distribution on $[0, \infty)$ since $E[h(B)] = 1$, and that the right-hand side above is equal to $2\overline{G}_0(t)$. Then, one can construct a nonnegative random variable B^* distributed by $G^* = (1-2\overline{G}_0)\vee 0$ to serve as the stochastic upper bound for Y conditional on $(A > x)$ for all large x.

Recall that two random variables A and B distributed by F and G, respectively, are called asymptotically independent (in the upper tail) if

$$\lim_{u\uparrow 1} \Pr\left(B > G^{\leftarrow}(u) | A > F^{\leftarrow}(u)\right) = 0, \tag{3.3.14}$$

see Section 5.2 of McNeil et al. (2005) for the definition. Clearly, (3.3.13) implies asymptotic independence of (A, B). Chen an Yuen (2012) derived the following result:

Theorem 3.3.1 *Consider the aggregate amount of claims* $S_t = \sum_{k=1}^{N_t} A_k, t \geq 0$. *In addition to (3.3.13), assume that* $F \in \mathcal{C}, E[A] = \mu$ *and* $E[B] = 1/\lambda \in (0, \infty)$. *Then, for arbitrarily given* $\gamma > 0$, *it holds uniformly for all* $x \geq \gamma t$ *that*

$$\Pr(S_t - \mu\lambda t > x) \sim \lambda t \overline{F}(x), \quad t \to \infty. \tag{3.3.15}$$

Chapter 4

A Risk Model with Dependent Insurance Risk or/and Financial Risk

4.1 Introduction

Consider an insurer in a discrete-time risk model with time horizon n. Within each period i the total premium income is denoted by C_i and the total claim amount plus other daily costs is denoted by A_i, both C_i and A_i are non-negative random variables. Suppose that the insurer positions himself in a stochastic economic environment, which leads to an overall stochastic accumulation factor W_i over each period i. Thus, with the initial wealth $U_0 = x$ the current wealth of the insurer at time n is

$$U_n = x \prod_{j=1}^{n} W_j + \sum_{i=1}^{n} (C_i - A_i) \prod_{j=i+1}^{n} W_j \tag{4.1.1}$$

Now introduce $X_i = A_i - C_i$ and $Y_i = W_i^{-1}$ which are respectively interpreted as the net loss and the overall stochastic discount factor over period i. Following Tang and Tsitsiashvili (2003), the random variables $\{X_i\}, i = 1, ..., 2$ are called insurance risks and $\{Y_i\}$ are called financial risk.

The discounted value of the insurer's wealth process at time n is

$$\begin{aligned}
U_n\left(\prod_{j=1}^{n} Y_j\right) &= \left(x\prod_{j=1}^{n} W_j + \sum_{i=1}^{n}(C_i - A_i)\prod_{j=i+1}^{n} W_j\right)\left(\prod_{j=1}^{n} Y_j\right) \\
&= x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \\
&= x - S_n. \qquad (4.1.2)
\end{aligned}$$

The last sum S_n above represents the stochastic present value of aggregate net loss up to time n. It is often written in the form

$$S_n = \sum_{i=1}^{n} \theta_i X_i, n = 1, 2, ..., \qquad (4.1.3)$$

with $\theta_i = \prod_{j=1}^{i} Y_i$ Then the probability of ruin by time n is equal to

$$\begin{aligned}
\psi(x;n) &= \Pr\left(\inf_{1\leq m\leq n} U_m < 0\right) \\
&= \Pr\left(\inf_{1\leq m\leq n} U_m \prod_{j=1}^{m} Y_j < 0\right) \\
&= \Pr\left(\inf_{1\leq m\leq n} (x - S_m) < 0\right) \\
&= \Pr\left(\max_{1\leq m\leq n} S_m > x\right) \qquad (4.1.4) \\
&= \Pr(M_n > x), \qquad (4.1.5)
\end{aligned}$$

where $M_n = \max_{1\leq m\leq n} S_m, n \geq 1$. Therefore, the finite-time ruin probability is the tail probability of the maximal present value of the aggregate net loss. Norberg (1999) first introduced the concepts of insurance and financial risks in modeling insurance business, the first kind of risk is the traditional liability related to the insurance portfolio, and the second is the asset risk related to the investment portfolio. Now, we give an example of financial risks. Consider a financial market consisting of a risk-free bond with constant interest rate $r \geq 0$ and a risky stock with stochastic return rate $R_i > -1$ over period i. Denote by p the proportion of the wealth invested in the stock. Note that the insurer may not be allowed to borrow additional money or take short positions. Thus, $p \in [0, 1]$. Denote by D_i the value process of this investment portfolio. It holds that

$$D_i = (1 - p)D_{i-1}(1 + r) + pD_{i-1}(1 + R_i).$$

Then the stochastic accumulation factor is

$$W_i = \frac{D_i}{D_{i-1}} = (1 - p)(1 + r) + p(1 + R_i)$$

and the corresponding financial risk is

$$Y_i = \frac{1}{(1 - p)(1 + r) + p(1 + R_i)}$$

4.2 Ruin with Insurance and Financial Risks Following a Special Dependence Structure

Recently, Chen (2011) studied the finite-time ruin probability in a discrete-time risk model in which insurance and financial risks form a sequence of independent and identically distributed random pairs following a common bivariate Farlie-Gumbel- Morgenstern distribution function with parameter $-1 \leq \theta \leq 1$ governing the strength of dependence. For the subexponential case, when $-1 < \theta \leq 1$, a general asymptotic formula for the finite-time ruin probability was derived. However, the derivation there is not valid for $\theta = -1$. In this section, we complete the study by extending Chen's work to $\theta = -1$. It turns out that the finite-time ruin probability behaves essentially differently for $-1 < \theta \leq 1$ and $\theta = -1$.

Consider a discrete-time insurance risk model. Within period i, the net insurance loss is denoted by a real-valued random variable $X_i, i \in \mathbb{N}$. Suppose that the insurer makes both risk-free and risky investments, leading to an overall stochastic discount factor, denoted by a nonnegative random variable Y_i, over the same time period. As usual, the probability of ruin by time n is defined to be

$$\psi(x;n) = \Pr\left(\max_{1\le m\le n}\sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x\right), \quad n \in \mathbb{N}, \tag{4.2.1}$$

where $x \geq 0$ is interpreted as the risk reserve of the insurer.

Recently, Chen (2011) studied the asymptotic behavior of the ruin probability $\psi(x;n)$ in (4.2.1) for the case with dependent insurance and financial risks. Precisely, it is assumed that $(X_i;Y_i), i \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) random pairs with a generic random pair $(X;Y)$ whose components are dependent. The dependence between X and Y is realized via a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution of the form

$$\prod(x,y) = F(x)G(y)\left(1+\theta\overline{F}(x)\overline{G}(y)\right), \tag{4.2.2}$$

where $F = 1 - \overline{F}$ on $\mathbb{R}$ and $G = 1 - \overline{G}$ on $\mathbb{R}^+$ are marginal distribution functions, and $\theta \in [-1,1]$ is a parameter governing the strength of dependence. Under the assumptions that F is a subexponential distribution function, G fulfills some constraints in order for the product convolution of F and G to be a subexponential distribution function too, and $\theta \in (-1,1]$, Chen (2011) derived a general asymptotic formula for $\psi(x;n)$. However, the assumption $\theta \neq -1$ is essentially applied there; see related discussions on Page 1041 of Chen (2011). Hence, the derivation of of Chen (2011) is not valid for the case $\theta = -1$.

The FGM distribution(4.2.2) describes an asymptotically independent scenario. Precisely, denote its corresponding copula by $C(u,v) = uv\,(1+\theta(1-u)(1-v))$ and the survival copula by $\hat{C}(u,v) = u+v-1+C(1-u,1-v)$ for $(u,v) \in (0,1)^2$. For the FGM case, it is clear that $\hat{C}(u,v) = C(u,v)$. For every $\theta \in [-1,1]$, the coefficient of upper tail dependence is

$$\chi = \lim_{u\downarrow 0} \frac{\hat{C}(u,u)}{u} = 0.$$

See Section 5.2 of Mcneil et al. (2005) for details of the concepts used here. Nonetheless, asymptotically independent random variables may still show different degrees of dependence. In this regard, Coles et al. (1999) proposed to use

$$\hat{\chi} = \lim_{u\downarrow 0} \frac{2\log u}{\log \hat{C}(u,u)} - 1.$$

to measure more subtly the strength of dependence in the asymptotic independence case. With a bit of calculation, we see that $\hat{\chi} = 0$ for $\theta \in (-1,1]$ while $\hat{\chi} = -1/3$ for $\theta = -1$. This illustrates the essential difference between the cases $-1 < \theta \leq 1$ and $\theta = -1$.

In this section we look at the same problem but for the case $\theta = -1$ and aim to complement the work of Chen (2011).

4.2.1 The Main Result

For simplicity, we say a function $a(.)$ define on $\mathbb{R}^+$ is an auxiliary function if it satisfies $0 \leq a(x) < x/2, a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$.

As in Yang et al. (2011), for a random variable X, we introduce $X^*_\vee$ and $X^*_\wedge$, which are identically distributed as $X^*_1 \vee X^*_2$ and $X^*_1 \wedge X^*_2$, respectively, and are independent of all other sources of randomness, where X^*_1 and X^*_2 are two i.i.d. copies of X. Trivially, if X is distributed by F, then $X^*_\vee$ is distributed by F^2 and the tail of $X^*_\wedge$ is $\overline{F}^2$.

For two independent random variables X^* and Y^* with distribution functions F on $\mathbb{R}$ and G on $\mathbb{R}^+$, respectively, denote by $H^* = F \otimes G$ the distribution function of the product

X^*Y^*. Thus,

$$\overline{H^*}(x) = \int_0^\infty \overline{F}\left(\frac{x}{y}\right) G(dy), \quad x > 0. \tag{4.2.3}$$

We recall here some facts, which will be used tacitly for a few times in this thesis. As usual, define the essential upper bound of Y^* as

$$\hat{y} = \sup\{y : G(y) < 1\}. \tag{4.2.4}$$

If $F \in \mathcal{L}$ and $0 < \hat{y} < \infty$, then by Theorem 2.2(iii) of Cline and Samorodnitsky (1994), we have $H^* \in \mathcal{L}$. Recently, Chen and Yuen (2013) also proved the inverse of this assertion; namely, if $H^* \in \mathcal{L}$ and $0 < \hat{y} < \infty$ then $F \in \mathcal{L}$. Moreover, if $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$, then by Theorem 2.1 of Cline and Samorodnitsky (1994), as recalled in Lemma 4.2.4 below, $H^* \in \mathcal{S}$. However, we still do not know yet if the inverse of this assertion is correct.

Recall that the dependence structure of (X, Y) is described by the joint distribution function 4.2.2 with $\theta = -1$; that is

$$\prod(x, y) = F(x)G(y)\left(1 - \overline{F}(x)\overline{G}(y)\right), \tag{4.2.5}$$

with F on $\mathbb{R}$ and G on $\mathbb{R}^+$. Introduce independent random variables X^*, Y^*, Y_1^*, Y_2^*, Y_3^*,..., with the first identically distributed as X and the other identically distributed as Y . Also recall $X_\vee^*$ and $Y_\wedge^*$ introduced in the beginning of this section. Denote by H the distribution function of the product XY and, as in (4.2.3), denote by H^* the distribution function of the product X^*Y^*. As before, $\hat{y}$ denotes the essential upper bound of the random variable Y.

In the first result below, the condition $0 < \hat{y} \leq 1$ indicates that there are risk-free investments only:

Theorem 4.2.1 *Let the random pair (X, Y) follow a bivariate FGM distribution function (4.2.5) with $F \in \mathcal{S}$ and $0 < \hat{y} \leq 1$. Then, it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_{\wedge}^* \prod_{j=2}^{i} Y_j^* > x \right), \tag{4.2.6}$$

where, and throughout the paper, the usual convention $\prod_{j=2}^{i} Y_j^* = 1$ is in force. In the second result below, the condition $1 \leq \hat{y} \leq \infty$ allows to include risky investments

Theorem 4.2.2 *Let the random pair (X, Y) follow a bivariate FGM distribution function (4.2.5) with $F \in \mathcal{L}$, $0 < \hat{y} \leq \infty$ and $H \in \mathcal{S}$. The relation*

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_{\wedge}^* \prod_{j=2}^{i} Y_j^* > x \right) + \sum_{i=1}^{n} \Pr\left(X_{\wedge}^* \prod_{j=1}^{i} Y_j^* > x \right), \tag{4.2.7}$$

holds for each $n \in \mathbb{N}$ under either of the following groups of conditions:

(i) there is an auxiliary function $a(.)$ such that $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$ and $\overline{H}\left(x - a(x)\right) \sim \overline{H}(x)$,

(ii) $J_F^- > 0$, and there is an auxiliary function $a(.)$ such that $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$.

Lemma 4.2.5 below gives an asymptotic expression for $\overline{H}(x)$ in terms of the tails of products of independent random variables,

$$\overline{H}(x) \sim P(X^* Y_{\wedge}^* > x) + P(X_{\wedge}^* Y^* > x).$$

This expression can help us verify $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$ and $\overline{H}(x - a(x)) \sim \overline{H}(x)$ in Theorem 4.2.1 in a given situation. We remark that the condition $\mathbb{J}_F^- > 0$ in part (ii) of Theorem 4.2.2 is so mild that, essentially, it does not exclude any distribution function of practical interest. On the other hand, it is usually troublesome to verify the condition $\overline{H}(x - a(x)) \sim \overline{H}(x)$ Therefore, part (ii) is more useful than part (i) for applications.

One naturally wonders if one sum on the right-hand side of (4.2.7) is negligible. The answer is diverse. Listed below are some important special cases, showing that sometimes the second sum on the right-hand side of (4.2.7) is negligible and, hence, relation (4.2.7) reduces to relation (4.2.6), but sometimes not.

Corollary 4.2.1 *Let the random pair (X, Y) follow a bivariate FGM distribution function* (4.2.2) *with $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$. Then relation* (4.2.6) *holds for each $n \in \mathbb{N}$.*

Corollary 4.2.1 extends Theorem 4.2.1 by loosing the restriction on Y from $0 < \hat{y} < 1$ to $0 < \hat{y} < \infty$.

Corollary 4.2.2 *Let the random pair (X, Y) follow a bivariate FGM distribution function* (4.2.2). *Relation* (4.2.6) *holds for each $n \in \mathbb{N}$ under either of the following groups of conditions:*

(i) $F \in \mathcal{C}$ and $EY^p < \infty$ for some $p > \mathbb{J}_F^+$.

(ii) $F \in \mathcal{L} \cap \mathcal{D}$ with $\mathbb{J}_F^- > 0$ and $EY^p < \infty$ for some $p > \mathbb{J}_F^+$.

In Corollaries 4.2.1 and 4.2.2, if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then applying Breiman's theorem (see Cline and Samorodnitsky 1994, who attributed it to Breiman 1965) to relation (4.2.7), we obtain

$$\psi(x; n) \sim E(Y_\wedge)^\alpha \frac{1 - (EY^\alpha)^n}{1 - EY^\alpha} \overline{F}(x), \tag{4.2.8}$$

where the ratio $\frac{1-(EY^\alpha)^n}{1-EY^\alpha}$ is understood as n if $\alpha = 0$. Relation (4.2.8) is identical to relation (3.2) of Chen (2011) with $\theta \neq -1$.

In the next two corollaries we look at a critical situation with the same heavy-tailed insurance and financial risks. The first one below addresses the regular variation case:

Corollary 4.2.3 *Let the random pair (X, Y) follow a bivariate FGM distribution function (4.2.2). If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, $\overline{F}(x) \sim c\overline{G}(x)$ for some $c > 0$, and $EY^\alpha = \infty$, then it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim (cE(Y_\wedge)^\alpha + E(X_\wedge^+)^\alpha) \Pr\left(\prod_{j=1}^{n} Y_j^* > x\right). \tag{4.2.9}$$

The second one below addresses the rapid variation case:

Corollary 4.2.4 *Let the random pair (X, Y) follow a bivariate FGM distribution function (4.2.2). If $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and $\overline{F}(x) \sim c\overline{G}(x)$ for some $c > 0$, then it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim (1 + c) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{n} Y_j^* > x\right). \tag{4.2.10}$$

4.2.2 Proof of Theorem 4.2.1

The following general derivations will be used in the proofs of both Theorems 4.2.1 and 4.2.2.

Notice the decomposition

$$\Pi = F^2 G + F G^2 - F^2 G^2. \tag{4.2.11}$$

and the facts that $X_\vee^*$ is distributed by F^2 and $Y_\vee^*$ by G^2. It follows that

$$
\begin{aligned}
& \Pr(XY > x) \\
= \;& \Pr(X_\vee^* Y^* > x) + \Pr(X^* Y_\vee^* > x) - \Pr(X_\vee^* Y_\vee^* > x) \\
= \;& 2\Pr(X^* Y^* > x) - \Pr(X_\wedge^* Y^* > x) \\
+ \;& 2\Pr(X^* Y^* > x) - \Pr(X^* Y_\wedge^* > x) \\
- \;& 4\Pr(X^* Y^* > x) + 2\Pr(X^* Y_\wedge^*) + 2\Pr(X_\wedge^* Y^* > x) - \Pr(X_\wedge^* Y_\wedge^* > x) \\
= \;& \Pr(X^* Y_\wedge^* > x) + \Pr(X_\wedge^* Y^* > x) - \Pr(X_\wedge^* Y_\wedge^* > x). \qquad (4.2.12)
\end{aligned}
$$

Define

$$T_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{n} Y_j, \quad n \in \mathbb{N}. \qquad (4.2.13)$$

Note that T_n is identically distributed as S_n in (4.1.3) due to the i.i.d. assumption on the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$, and that it fulfills the recursive formula

$$T_{n+1} = (T_n + X_{n+1})Y_{n+1}, \quad n \in \mathbb{N}. \qquad (4.2.14)$$

Similarly to in the derivation of (4.2.12), starting from (4.2.14) and applying the decomposition in (4.2.11) we have

$$
\begin{aligned}
\Pr(T_{n+1} > x) &= \Pr((T_n + X_\vee^*) Y^* > x) + \Pr((T_n + X^*) Y_\vee^* > x) - \Pr((T_n + X_\vee^*) Y_\vee^* > x) \\
&= \Pr((T_n + X^*) Y_\wedge^* > x) + \Pr((T_n + X_\wedge^*) Y^* > x) - \Pr((T_n + X_\wedge^*) Y_\wedge^* > x) \\
&= I_1(x) + I_2(x) - I_3(x). \qquad (4.2.15)
\end{aligned}
$$

The following lemma is well known and can be found in Embrechts and Goldie (1980), Cline (1986, Corollary 1) and Tang and Tsitsiashvili (2003, Lemma 3.2):

Lemma 4.2.1 *Let F_1 and F_2 be two distribution functions on $\mathbb{R}$ and let $F = F_1 * F_2$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$ and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F \in \mathcal{S}$ and $\overline{F}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.*

In the proof of Theorem (4.2.1) we need the lemmas (4.2.2) and (4.2.3):

Lemma 4.2.2 *Let (X, Y) follow a bivariate FGM distribution function* (4.2.5). *If $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$, then*

$$\Pr(X^*_\wedge Y^* > x) = o\left(\Pr(X^* Y^*_\wedge > x)\right), \tag{4.2.16}$$

$$\Pr(XY > x) \sim \Pr(X^* Y^*_\wedge > x). \tag{4.2.17}$$

Proof. We start from (4.2.12). Without loss of generality, assume $\hat{y} = 1$. We have

$$\frac{\Pr(X^*_\wedge Y^* > x)}{\Pr(X^* Y^*_\wedge > x)} \leq \frac{\overline{F}(x)^2}{\Pr(X^* > 2x, Y^*_\wedge > 1/2)} = \frac{1}{\overline{G}(1/2)^2} \frac{\overline{F}(x)^2}{\overline{F}(2x)}. \tag{4.2.18}$$

By Lemma 5.1 of Cai and Tang (2004), the right-hand side above converges to 0 as $x \to \infty$.

This proves relation (4.2.16). Looking at (4.2.12), relation (4.2.16) implies that the second term on the right-hand side of (4.2.12) and, hence, the third term there also, is negligible. Then relation (4.2.12) reduces to relation (4.2.17). The following lemma will enable us to conduct an induction procedure in the proof of Theorem (4.2.1): ■

Lemma 4.2.3 *In addition to the conditions in Lemma* 4.2.2, *assume $0 < \hat{y} \leq 1$. Then $XY + X^*$ follows a subexponential distribution with tail satisfying, such that*

$$\Pr(XY + X^* > x) \sim \Pr(XY > x) + \Pr(X > x). \tag{4.2.19}$$

Proof. As recalled at the end of previous section, the conditions $F \in \mathcal{L}$ and $0 < \hat{y} \leq 1$ imply that $\Pr(X^* Y^*_\wedge > x)$ is a long tail, and so is $\Pr(XY > x)$ by relation (4.2.17). The condition $0 < \hat{y} \leq 1$ implies that $\Pr(XY > x) \leq \Pr(X > x)$. The desired results follow from Lemma 4.2.1. ■

Proof of Theorem 4.2.1. As analyzed by Chen (2011), it suffices to prove the relation

$$\Pr(T_n > x) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.2.20}$$

Note that the first term on the right-hand side of (4.2.20) is a subexponential tail and the other terms have long tails and dominated by the first. Thus, as in Lemma 4.2.1, the right-hand side of (4.2.20) indeed gives a subexponential tail for T_n. We employ the method of induction to complete the proof of (4.2.20). Lemma 4.2.2 shows that relation (4.2.20) holds for $n = 1$. Now assume that relation (4.2.20) holds for n and we are going to prove it for $n+1$. For $I_1(x)$ in (4.2.15), since $F \in \mathcal{S}$ and $0 < \hat{y} \leq 1$, we have $P(T_n > x) \leq \Pr(\sum_{i=1}^{n} X_i > x) \sim n\overline{F}(x)$. By Lemma (4.2.1),

$$\begin{aligned} I_1(x) &= \int_0^1 \Pr\left(T_n + X^* > \frac{x}{y}\right) \Pr(Y_\wedge^* \in dy) \\ &\sim \int_0^1 \left(\Pr(T_n > \frac{x}{y}) + \Pr(X^* > \frac{x}{y})\right) \Pr(Y_\wedge^* \in dy) \\ &= \Pr(T_n Y_\wedge^* > x) + \Pr(X^* Y_\wedge^* > x). \end{aligned}$$

The second step is due to Lemma (4.2.1).

Now we turn to $I_2(x)$. Note that both $X_\wedge^* Y^*$ and $T_n Y^*$ are long tailed, one can choose some function $l(x)$ with $0 < l(x) \leq x/2$ and $l(x) \uparrow \infty$ such that

$$\Pr\left(X_\wedge^* Y^* > x \pm l(x)\right) \sim \Pr\left(X_\wedge^* Y^* > x\right) \quad \textit{and} \quad \Pr\left(T_n Y^* > x \pm l(x)\right) \sim \Pr(T_n Y^* > x)$$

Also note that, by relation (4.2.16) and relation (4.2.20) for $n = 1$

$$\Pr(X_\wedge^* Y^* > x) = o\left(\Pr(X^* Y_\wedge^* > x)\right) = o(1) \Pr(T_n > x). \tag{4.2.21}$$

We derive

$$
\begin{aligned}
I_2(x) &= \Pr\left((T_n + X_\wedge^*)Y^* > x\right) \\
&\le \Pr\left((T_n + X_\wedge^*)Y^* > x, T_nY^* \le l(x)\right) \\
&+ \Pr\left((T_n + X_\wedge^*)Y^* > x, X_\wedge^*Y^* \le l(x)\right) \\
&+ \Pr\left((T_n + X_\wedge^*)Y^* > x, T_nY^* > l(x), X_\wedge^*Y^* > l(x)\right) \\
&\le \Pr\left(X_\wedge^*Y^* > x - l(x)\right) + \Pr\left(T_nY^* > x - l(x)\right) \\
&+ \Pr\left((T_n + X_\wedge^*)Y^* > x, X_\wedge^*Y^* > l(x)\right) \\
&\le o(1)\Pr(T_n > x) + \left(1 + o(1)\right)\Pr(T_nY^* > x) + \Pr\left(T_n + X_\wedge^*Y^* > x, X_\wedge^*Y^* > l(x)\right),
\end{aligned}
$$

where in the last step we used (4.2.21). For the last term on the right-hand side above, we have

$$
\begin{aligned}
&\Pr\left(T_n + X_\wedge^*Y^* > x, X_\wedge^*Y^* > l(x)\right) \\
\le\ &\Pr\left(T_n + X_\wedge^*Y^* > x\right) - \Pr\left(T_n + X_\wedge^*Y^* > x, -l(x) \le X_\wedge^*Y^* \le l(x)\right) \\
\le\ &\left(1 + o(1)\right)\left(\Pr(T_n > x) + \Pr(X_\wedge^*Y^* > x)\right) - \left(1 + o(1)\right)\Pr\left(T_n > x + l(x)\right) \\
=\ &o(1)\Pr(T_n > x)
\end{aligned}
$$

where in the third step we used Lemma 4.2.1 and in the last step we used relation (4.2.21). It follows that

$$
I_2(x) \le \left(1 + 0(1)\right)\Pr(T_nY^* > x) + o(1)\Pr(T_n > x). \tag{4.2.22}
$$

On the other hand,

$$
\begin{aligned}
I_2(x) &\geq \Pr\Big((T_n - l(x))\, Y^* > x, -l(x) \leq X^*_\wedge \leq l(x)\Big) \\
&\geq \Pr\Big(T_n Y^* > x + l(x)\Big) \Pr\Big(- l(x) \leq X^*_\wedge \leq l(x)\Big) \\
&\geq \Big(1 + o(1)\Big) \Pr(T_n Y^* > x).
\end{aligned}
$$

For I_3, by going along the same lines of the derivation for I_2 and changing every Y^* to $Y^*_\wedge$, we obtain

$$
\Big(1 + o(1)\Big) \Pr(T_n Y^*_\wedge > x) \leq I_3(x) \leq \Big(1 + o(1)\Big) \Pr(T_n Y^*_\wedge > x) + o(1) \Pr(T_n > x).
$$

Plugging all these estimates for $I_1(x)$, $I_2(x)$ and $I_3(x)$ into (4.2.15), we obtain

$$
\begin{aligned}
&\Pr\left(T_{n+1} > x\right) \\
\lesssim\; &(1 + o(1)) \Pr\left(T_n Y^*_\wedge > x\right) + (1 + o(1)) \Pr\left(X^* Y^*_\wedge > x\right) \\
&+ (1 + o(1)) \Pr\left(T_n Y^* > x\right) + o(1) \Pr\left(T_n > x\right) \\
&- (1 + o(1)) \Pr\left(T_n Y^*_\wedge > x\right) \\
=\; &o(1) \left(\Pr\left(T_n Y^*_\wedge > x\right) + \Pr\left(T_n > x\right)\right) + (1 + o(1)) \left(\Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(T_n Y^* > x\right)\right) \\
=\; &o(1) \Pr\left(T_n > x\right) + (1 + o(1)) \left(\Pr\left(X^* Y^*_\wedge > x\right) + \sum_{i=1}^{n} \Pr\left(X^* Y^*_\wedge Y^* \prod_{j=2}^{i} Y^*_j > x\right)\right) \\
=\; &o(1) \sum_{i=1}^{n} \Pr\left(X^* Y^*_\wedge \prod_{j=2}^{i} Y^*_j > x\right) + (1 + o(1)) \sum_{i=1}^{n+1} \Pr\left(X^* Y^*_\wedge \prod_{j=2}^{i} Y^*_j > x\right) \\
\sim\; &\sum_{i=1}^{n+1} \Pr\left(X^* Y^*_\wedge \prod_{j=2}^{i} Y^*_j > x\right),
\end{aligned}
$$

where in the third and fourth steps we used (4.2.20) for n. The lower asymptotic bound is derived similarly. Hence, (4.2.20) holds for $n + 1$. ∎

4.2.3 Proofs of Theorem 4.2.2

The following first lemma is a restatement of Theorem 2.1 of Cline and Samorodnitsky (1994), which is crucial for establishing our Theorem 4.2.2(i):

Lemma 4.2.4 *Let F be a distribution on $\mathbb{R}$ and G be a distribution on $\mathbb{R}^+$. We have $H^* = F \otimes G \in \mathcal{S}$ if $F \in \mathcal{S}$ and there is an auxiliary function $a(\cdot)$ such that $\overline{G}\left(a(x)\right) = o\left(\overline{H^*}(x)\right)$ and $\overline{F}\left(x - a(x)\right) \sim \overline{F}(x)$.*

The lemma below dissolves the dependence between X and Y:

Lemma 4.2.5 *Let (X, Y) follow a bivariate FGM distribution function (4.2.5) with $\hat{y} = \infty$. If there is an auxiliary function $a(\cdot)$ such that $\overline{G}\left(a(x)\right) = o\left(\overline{H^*}(x)\right)$, then*

$$\Pr\left(XY > x\right) \sim \Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(X^*_\wedge Y^* > x\right). \tag{4.2.23}$$

Proof. We start from (4.2.12) and are going to prove that the last term on its right-hand side is negligible, namely,

$$\Pr\left(X^*_\wedge Y^*_\wedge > x\right) = o(1)\Bigg(\Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(X^*_\wedge Y^* > x\right) \Bigg). \tag{4.2.24}$$

For this purpose, we do the split

$$\Pr\left(X^*_\wedge Y^*_\wedge > x\right) = \Pr\left(X^*_\wedge Y^*_\wedge > x, \left(Y^*_\wedge \leq a(x)\right) \cup \left(Y^*_\wedge > a(x)\right) \right) = J_1(x) + J_2(x).$$

By conditioning on $Y^*_\wedge$, we have

$$J_1(x) \leq \overline{F}\left(\frac{x}{a(x)}\right) \int_0^{a(x)} \overline{F}\left(\frac{x}{y}\right) \Pr\left(Y^*_\wedge \in \mathrm{d}y\right) = o(1) \Pr\left(X^* Y^*_\wedge > x\right).$$

It is easy to see that

$$J_2(x) = o(1) \Pr\left(X^*_\wedge Y^* > x\right). \tag{4.2.25}$$

Actually, on the one hand, it is clear that

$$J_2(x) \leq \Pr\Big(Y_\wedge^* > a(x)\Big) = \overline{G}\Big(a(x)\Big)^2;$$

while on the other hand, by Jensen's inequality we have

$$\Pr\Big(X_\wedge^* Y^* > x\Big) = \int_0^\infty \overline{F}\left(\frac{x}{y}\right)^2 \Pr\left(Y^* \in \mathrm{d}y\right) \geq \overline{H^*}(x)^2.$$

Relation (4.2.25) follows since $\overline{G}\left(a(x)\right) = o\left(\overline{H^*}(x)\right)$. Thus, relation (4.2.24) holds. ■

With (X, Y) following a bivariate FGM distribution function (4.2.5), we see that $\overline{G}\Big(a(x)\Big) = o\left(\overline{H^*}(x)\right)$ is slightly weaker than $\overline{G}\Big(a(x)\Big) = o\Big(\overline{H}(x)\Big)$ since, by (4.2.23), $\overline{H}(x) \lesssim 2\overline{H^*}(x)$.

The following lemma is a counterpart of Lemma 4.2.3:

Lemma 4.2.6 *In addition to the conditions in Lemma 4.2.5, assume $F \in \mathcal{L}$ and $H \in \mathcal{S}$. Then $XY + X^*$ follows a subexponential distribution with tail satisfying*

$$\Pr\left(XY + X^* > x\right) \sim \Pr\left(XY > x\right) + \Pr\left(X > x\right).$$

Proof. By Lemma 4.2.5,

$$\Pr\left(XY > x\right) \gtrsim \Pr\Big(X^* Y_\wedge^* > x\Big) \geq \Pr\left(X^* > x\right) \Pr\Big(Y_\wedge^* \geq 1\Big).$$

Thus, the result follows from Lemma 4.2.1. ■

Define

$$V_n = \sum_{i=1}^n X_i Y_i \prod_{j=i+1}^n Y_j^*, \qquad n \in \mathbb{N}.$$

Lemma 4.2.7 *Let (X, Y) follow a bivariate FGM distribution function (4.2.5) with $\hat{y} = \infty$ and $H \in \mathcal{S}$.*

(i) If there is an auxiliary function $a(\cdot)$ such that $\overline{G}\Big(a(x)\Big) = o\left(\overline{H}(x)\right)$ and $\overline{H}\Big(x - a(x)\Big) \sim \overline{H}(x)$, then each V_n follows a subexponential distribution with tail satisfying

$$\Pr\left(V_n > x\right) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right);$$

(ii) If $J_F^- > 0$, then the restriction $\overline{H}\Big(x - a(x)\Big) \sim \overline{H}(x)$ on the auxiliary function $a(\cdot)$ is unnecessary.

Proof. For simplicity, write $Z_i = X_i Y_i$ for $i = 1, \ldots, n$. Notice that the sequence $\{V_n, n \in \mathbb{N}\}$ fulfills the recursive equation

$$V_{n+1} = V_n Y_{n+1}^* + Z_{n+1}.$$

Applying Lemmas 4.2.4 and 4.2.1, we can conduct a standard induction procedure to prove that for each $n \in \mathbb{N}$, the sum V_n follows a subexponential distribution with tail satisfying

$$\Pr\left(V_n > x\right) \sim \sum_{i=1}^{n} \Pr\left(Z_i \prod_{j=i+1}^{n} Y_j^* > x\right) = \sum_{i=1}^{n} \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right) \tag{4.2.26}$$

and that, for each $i = 2, \ldots, n$ and all $a > 0$,

$$\Pr\left(\prod_{j=2}^{i} Y_j^* > ax\right) = o(1) \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.2.27}$$

For case (i), we refer the reader to the proofs of Theorem 3.1 of Tang (2006b), Theorem 3.1 of Chen (2011) and, in particular, Theorem 1.1 of Zhou et al. (2012) for similar discussions. For case (ii), see Theorem 4.1 of Tang (2006a). Since relation (4.2.27) holds for all $a > 0$, it is easy to see that, for each $i = 2, \ldots, n$, there is some auxiliary function $a_i(\cdot)$ such that

$$\Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right) = o(1) \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.2.28}$$

For each $i = 2, \ldots, n$, we split

$$\Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right) = \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) + O(1) \Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right).$$

By conditioning on $\prod_{j=2}^{i} Y_j^*$ and applying Lemma 4.2.5, the first part is asymptotically equivalent to

$$\begin{aligned}
& \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) + \Pr\left(X_\wedge^* Y^* \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) \\
= \; & \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) + O(1) \Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right).
\end{aligned}$$

By (4.2.28), it follows that

$$\Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right) \sim \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right).$$

Substituting this into (4.2.26) leads to the desired result. ■

Proof of Theorem 4.2.2. Recall T_n introduced in (4.2.13) and the recursive formula (4.2.14). The same as before, it suffices to prove the relation

$$\Pr(T_n > x) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right). \tag{4.2.29}$$

Lemma 4.2.7 shows that the right-hand side of (4.2.29) indeed gives a subexponential tail for T_n.

Similarly to the proof of Theorem 4.2.1, we employ the method of induction to prove (4.2.29). Lemma 4.2.5 shows that relation (4.2.29) holds for $n = 1$. Now, assume that relation (4.2.29) holds for n and we are going to prove it for $n + 1$.

For this purpose, we still start from the decomposition in (4.2.15). For $I_1(x)$, since T_n is subexponential, X^* is long tailed, and $\Pr(X^* > x) = O\Big(\Pr(T_n > x)\Big)$, by conditioning on

$Y_\wedge^*$ and applying Lemma 4.2.1 we have

$$\begin{aligned} I_1(x) &= \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \Pr\left(T_n + X^* > \frac{x}{y}\right) \Pr\left(Y_\wedge^* \in \mathrm{d}y\right) \\ &= \Big(1+o(1)\Big) \int_0^{a(x)} \left(\Pr\left(T_n > \frac{x}{y}\right) + \Pr\left(X^* > \frac{x}{y}\right)\right) \Pr\left(Y_\wedge^* \in \mathrm{d}y\right) + O(1)\overline{G}\Big(a(x)\Big)^2 \\ &= \Big(1+o(1)\Big)\Big(\Pr\left(T_n Y_\wedge^* > x\right) + \Pr\left(X^* Y_\wedge^* > x\right) \Big) + O(1)\overline{G}\Big(a(x)\Big)^2. \end{aligned}$$

In the same way, we have

$$I_2(x) = \Big(1+o(1)\Big)\Big(\Pr\left(T_n Y^* > x\right) + \Pr\left(X_\wedge^* Y^* > x\right) \Big) + O(1)\overline{G}\Big(a(x)\Big)$$

and

$$I_3(x) = \Big(1+o(1)\Big)\Big(\Pr\left(T_n Y_\wedge^* > x\right) + \Pr\left(X_\wedge^* Y_\wedge^* > x\right) \Big) + O(1)\overline{G}\Big(a(x)\Big)^2.$$

Plugging these estimates into (4.2.15) and using the condition $\overline{G}\Big(a(x)\Big) = o\left(\overline{H}(x)\right)$ and relations (4.2.23)–(4.2.24), we obtain

$$\Pr\left(T_{n+1} > x\right) \sim \Pr\left(X^* Y_\wedge^* > x\right) + \Pr\left(X_\wedge^* Y^* > x\right) + \Pr\left(T_n Y^* > x\right).$$

For the last term above, by conditioning on Y^* and applying relation (4.2.29) for n it is easy to show that

$$\Pr\left(T_n Y^* > x\right) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i+1} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* \prod_{j=1}^{i+1} Y_j^* > x\right).$$

Thus, relation (4.2.29) holds for $n+1$. ■

4.2.4 Proofs of Corollary 4.2.1

When $0 < \hat{y} \le 1$, the result comes directly from Theorem 4.2.1. When $1 \le \hat{y} < \infty$, by the condition $F \in \mathcal{S}$ and Lemma 4.2.2, we have $\overline{H}(x) \sim \Pr\left(X^* Y_\wedge^* > x\right)$ and, hence, $H \in \mathcal{S}$. Furthermore, $H \in \mathcal{S} \subset \mathcal{L}$ implies the existence of an auxiliary function $a(\cdot)$ satisfying

$\overline{H}\Big(x-a(x)\Big) \sim \overline{H}(x)$. Since $\hat{y}<\infty$, the condition $\overline{G}\Big(a(x)\Big)=o\left(\overline{H}(x)\right)$ in Theorem 4.2.2(i) holds trivially for every such function $a(\cdot)$. Thus, all conditions of Theorem 4.2.2(i) are fulfilled.

For each $i=1,\ldots,n$, by conditioning on $\prod_{j=2}^{i} Y_j^*$ and using relation (4.2.16),

$$
\begin{aligned}
\Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) &= \int_0^{\hat{y}^{i-1}} \Pr\left(X_\wedge^* Y_1^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&= o(1) \int_0^{\hat{y}^{i-1}} \Pr\left(X^* Y_\wedge^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right).
\end{aligned}
$$

Substituting this into relation (4.2.7) leads to relation (4.2.6).

4.2.5 Proofs of Corollary 4.2.2

(i) By Theorem 3.3(iv) of Cline and Samorodnitsky (1994), the conditions $F \in \mathcal{C}$ and $\mathrm{E}[Y^p] < \infty$ for some $p > J_F^+$ imply that $\overline{H^*}(x) \asymp \overline{F}(x)$. By Lemma 3.5 of Tang and Tsitsiashvili (2003), the relation $x^{-q} = o\left(\overline{F}(x)\right)$ holds for every $q > J_F^+$. Define an auxiliary function $a(x) = x^r$ for some $r \in (J_F^+/p, 1)$. We have

$$
\overline{G}\left(x^r\right) \le x^{-rp}\mathrm{E}[Y^p] = o\left(\overline{H^*}(x)\right).
$$

Thus, Lemma 4.2.5 is applicable and gives relation (4.2.23). For the two terms on the right-hand side of (4.2.23), we have, respectively, $\Pr\left(X^* Y_\wedge^* > x\right) \asymp \overline{F}(x)$ and

$$
\begin{aligned}
\Pr\left(X_\wedge^* Y^* > x\right) &\le \int_0^{x^r} \overline{F}\left(\frac{x}{y}\right)^2 G(\mathrm{d}y) + \overline{G}\left(x^r\right) \\
&\le \overline{F}\left(\frac{x}{x^r}\right) \overline{H^*}(x) + o\left(\overline{F}(x)\right) \\
&= o\left(\overline{F}(x)\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* > x\right). \qquad (4.2.30)
\end{aligned}
$$

It follows from (4.2.23) that

$$\overline{H}(x) \sim \Pr\left(X^* Y_\wedge^* > x\right) \asymp \overline{F}(x). \tag{4.2.31}$$

Thus, $a(x) = x^r$ satisfies the conditions $\overline{G}\Big(a(x)\Big) = o\left(\overline{H}(x)\right)$ and $\overline{H}\Big(x - a(x)\Big) \sim \overline{H}(x)$ in Theorem 4.2.2(i). By the first relation in (4.2.31) and Theorem 3.4(ii) of Cline and Samorodnitsky (1994), it is easy to see that $H \in \mathcal{C} \subset \mathcal{S}$. Thus, all conditions of Theorem 4.2.2(i) are satisfied and we have relation (4.2.7). For each $i = 2, \ldots, n$, similarly to (4.2.30), by conditioning on $\prod_{j=2}^{i} Y_j^*$ we obtain

$$\begin{aligned}
\Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) &\le \int_0^{x^r} \Pr\left(X_\wedge^* Y^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) + \Pr\left(\prod_{j=2}^{i} Y_j^* > x^r\right) \\
&= o(1) \int_0^{x^r} \Pr\left(X^* Y_\wedge^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) + o\left(\overline{F}(x)\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right). \qquad (4.2.32)
\end{aligned}$$

Thus, the second sum on the right-hand side of relation (4.2.7) is negligible and we finally obtain relation (4.2.6).

(ii) With $a(x) = x^r$ for some $r \in (J_F^+/p, 1)$, the verifications of the conditions of Lemma 4.2.5 and Theorem 4.2.2(ii) are similar to those in the proof of Corollary 4.2.2(i), and the proofs of relations (4.2.30)–(4.2.32) are also the same. A major difference is that we need to apply Theorems 2.2(iii) and 3.3(ii) of Cline and Samorodnitsky (1994) to the first relation in (4.2.31) to verify $H \in \mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$.

4.2.6 Proofs of Corollary 4.2.3

As in the proof of Corollary 2.1 of Chen and Xie (2005), by Fatou's lemma we have

$$\liminf_{x\to\infty} \frac{\overline{H^*}(x)}{\overline{G}(x)} \ge \int_0^\infty \liminf_{x\to\infty} \frac{\overline{G}\left(\frac{x}{y}\right)}{\overline{G}(x)} F(\mathrm{d}y) = \mathrm{E}[X_+^\alpha] = \infty. \tag{4.2.33}$$

It follows that $\overline{G}(x) = o(\overline{H^*}(x))$. By Lemma 3.2 of Chen and Xie (2005), there is an auxiliary function $a(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\overline{G}\Big(a(x)\Big) = o\left(\overline{H^*}(x)\right)$ of Lemma 4.2.5 hold simultaneously. By Lemma 4.2.5,

$$\begin{aligned}
\overline{H}(x) &\sim \Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(X^*_\wedge Y^* > x\right) \\
&\sim \mathrm{E}\left[(Y^*_\wedge)^\alpha\right] \overline{F}(x) + \mathrm{E}\left[\left(X^{*+}_\wedge\right)^\alpha\right] \overline{G}(x) \\
&\sim \Big(c\mathrm{E}\left[(Y^*_\wedge)^\alpha\right] + \mathrm{E}\left[\left(X^{*+}_\wedge\right)^\alpha\right] \Big)\overline{G}(x),
\end{aligned}$$

where the second step is due to Breiman's theorem. Hence, $H \in \mathcal{R}_{-\alpha}$ and the same auxiliary function $a(\cdot)$ satisfies $\overline{G}\Big(a(x)\Big) = o\left(\overline{H}(x)\right)$ of Theorem 4.2.2(ii). Thus, relation (4.2.7) holds.

Next, we simply (4.2.7) to (4.2.9). For each $i = 2, \ldots, n$, by the Corollary 1 of Embrechts and Goldie (1980), $\prod_{j=2}^i Y_j^*$ follows a distribution function belonging to the class $\mathcal{R}_{-\alpha}$. Based on the same reasoning as above we see that there is some auxiliary function $\tilde{a}(\cdot)$ such that

$$\Pr\left(\prod_{j=2}^i Y_j^* > \tilde{a}(x)\right) = o(1)\Pr\left(X^* Y^*_\wedge \prod_{j=2}^i Y_j^* > x\right).$$

We have

$$\begin{aligned}
\Pr\left(X^* Y^*_\wedge \prod_{j=2}^i Y_j^* > x\right) &\sim \int_0^{\tilde{a}(x)} \Pr\left(X^* Y^*_\wedge > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^i Y_j^* \in \mathrm{d}y\right) \\
&\sim c\mathrm{E}\left[(Y^*_\wedge)^\alpha\right] \int_0^{\tilde{a}(x)} \Pr\left(Y^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^i Y_j^* \in \mathrm{d}y\right) \\
&\sim c\mathrm{E}\left[(Y^*_\wedge)^\alpha\right] \Pr\left(\prod_{j=1}^i Y_j^* > x\right),
\end{aligned}$$

where the second step is due to Breiman's theorem. Similarly, for each $i = 2, \ldots, n$,

$$\Pr\left(X^*_\wedge \prod_{j=1}^i Y_j^* > x\right) \sim \mathrm{E}\left[\left(X^{*+}_\wedge\right)^\alpha\right] \Pr\left(\prod_{j=1}^i Y_j^* > x\right).$$

Substituting these asymptotic results into (4.2.7) gives

$$\psi(x; n) \sim \left(c\mathrm{E}\left[(Y^*_\wedge)^\alpha\right] + \mathrm{E}\left[\left(X^{*+}_\wedge\right)^\alpha\right]\right) \sum_{i=1}^n \Pr\left(\prod_{j=1}^i Y_j^* > x\right).$$

Similarly to (4.2.33), $\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right) = o(1)\Pr\left(\prod_{j=1}^{n} Y_j^* > x\right)$ for every $i = 1, \ldots, n-1$. Then relation (4.2.9) follows.

4.2.7 Proofs of Corollary 4.2.4

The following lemma will be needed in the proof of Corollary 4.2.4:

Lemma 4.2.8 *For two distribution functions F on $\mathbb{R}$ and G on $\mathbb{R}_+$, if $F \in \mathcal{R}_{-\infty}$ and $G \in \mathcal{R}_{-\infty}$, then $H^* = F \otimes G \in \mathcal{R}_{-\infty}$.*

Proof. By (4.2.3), it holds for every $a > 0$ that

$$\frac{\overline{G}(ax)}{\overline{H^*}(x)} \leq \frac{\overline{G}(ax)}{\overline{F}(2/a)\overline{G}(ax/2)} \to 0.$$

Thus, there is an auxiliary function $a(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $a(x) \uparrow \infty$, $a(x)/x \downarrow 0$, and $\overline{G}\left(a(x)\right) = o\left(\overline{H^*}(x)\right)$. By (4.2.3) again, for every $z > 1$,

$$\frac{\overline{H^*}(xz)}{\overline{H^*}(x)} \sim \frac{\int_0^{a(x)} \overline{F}\left(\frac{xz}{y}\right) G(\mathrm{d}y)}{\int_0^{a(x)} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y)} \leq \sup_{0<y\leq a(x)} \frac{\overline{F}\left(\frac{xz}{y}\right)}{\overline{F}\left(\frac{x}{y}\right)} \to 0.$$

Hence, $H^* \in \mathcal{R}_{-\infty}$. ■

Proof of Corollary 4.2.4. From the proof of Lemma 4.2.8, there is an auxiliary function $\tilde{a}(\cdot)$ such that $\overline{G}\Big(\tilde{a}(x)\Big) = o\left(\overline{H^*}(x)\right)$. Then, by Lemma 4.2.5, relation (4.2.23) holds. Since $\Pr\left(X^* > x\right) \sim c\Pr\left(Y^* > x\right)$ and $\Pr\left(X^*_\wedge > x\right) \sim c^2\Pr\left(Y^*_\wedge > x\right)$, by Lemma A.5 of Tang and Tsitsiashvili (2004) we have

$$\Pr\left(X^*_\wedge Y^* > x\right) \sim c^2\Pr\left(Y^*_\wedge Y^* > x\right) \sim c\Pr\left(X^* Y^*_\wedge > x\right). \tag{4.2.34}$$

It follows from relation (4.2.23) that

$$\Pr\left(XY > x\right) \sim \Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(X^*_\wedge Y^* > x\right) \sim (1+c)\Pr\left(X^* Y^*_\wedge > x\right).$$

By Corollary 2.1 of Tang (2006a), $\Pr(X^* Y^*_\wedge > x)$ is a subexponential tail and, hence, $H \in \mathcal{S}$. Again from the proof of Lemma 4.2.8, there is an auxiliary function $a(\cdot)$ such that $\overline{G}\Big(a(x)\Big) = o(1)\Pr(X^* Y^*_\wedge > x) = o\Big(\overline{H}(x)\Big)$. Thus, relation (4.2.7) holds.

Next we simplify (4.2.7) to (4.2.10). For each $i = 1, \ldots, n-1$, since $X^*_\wedge Y^* \prod_{j=2}^{i} Y^*_j$ is rapidly-varying tailed by Lemma 4.2.8, we have

$$\begin{aligned}
\Pr\left(X^*_\wedge Y^* \prod_{j=2}^{i} Y^*_j > x\right) &= o(1)\Pr\left(X^*_\wedge Y^* \prod_{j=2}^{i} Y^*_j > \frac{x}{2}\right) \\
&= o(1)\Pr\left(X^*_\wedge Y^* \prod_{j=2}^{i} Y^*_j > \frac{x}{2}, Y_n > 2\right) \\
&= o(1)\Pr\left(X^*_\wedge Y^* \prod_{j=2}^{n} Y^*_j > x\right).
\end{aligned}$$

Similarly, it holds for each $i = 1, \ldots, n-1$ that

$$\Pr\left(X^* Y^*_\wedge \prod_{j=2}^{i} Y^*_j > x\right) = o(1)\Pr\left(X^* Y^*_\wedge \prod_{j=2}^{n} Y^*_j > x\right).$$

It follows from relation (4.2.7) that

$$\psi(x;n) \sim \Pr\left(X^* Y^*_\wedge \prod_{j=2}^{n} Y^*_j > x\right) + \Pr\left(X^*_\wedge Y^* \prod_{j=2}^{n} Y^*_j > x\right).$$

Since $\prod_{j=2}^{n} Y^*_j$ is rapidly-varying tailed by Lemma 4.2.8, applying Lemma A.5 of Tang and Tsitsiashvili (2004) and relation (4.2.34) we obtain

$$\Pr\left(X^*_\wedge Y^* \prod_{j=2}^{n} Y^*_j > x\right) \sim c\Pr\left(X^* Y^*_\wedge \prod_{j=2}^{n} Y^*_j > x\right).$$

Relation (4.2.10) follows. ∎

Chapter 5

Concluding Remarks and Future Research

5.1 Conclusions

In the past two decades, the study of dependence has become a key topic in the actuarial literature. In this thesis, we investigated several risk models with claim frequencies or their interarrivals modeled with certain dependence structures and focused on the asymptotic behavior of the finite- and infinite-time ruin probabilities and the tail probabilities of aggregate claims in several nonstandard risk models in which claim sizes are heavy tailed.

In Chapters 2 and 3, we considered two types of dependence structure, namely dependence between claims and their interarrivals and dependence between insurance and financial risks. We incorporated the heavy-tailed distributions into these renewal risk models based and gave the relevant results on the ruin probabilities according to different dependent assumption.

5.2 Future Research Directions

5.2.1 Research Direction 1: Extremal Dependence

Recall two random variables X and Y with distributions F and G are called asymptotically independent if

$$\lim_{q \uparrow 1} \Pr\left(F(X) > q \mid G(Y) > q\right) = 0;$$

they are called asymptotically dependent if the limit exists and is positive.

Recent empirical studies often discover large joint movements of financial variables, exhibiting strong asymptotic dependence. However, our assumption of FGM distribution for (X, Y) corresponds to asymptotic independence between the insurance risk X and the financial risk Y. It is imperative to extend the study to asymptotically dependent (X, Y). The difficulty exists in establishing the subexponentiality for the product XY for dependent case.

5.2.2 Research Direction 2: Continuous-Time Models

As mentioned before, another trend of the study is to consider continuous-time models. Starting from Norberg (1999, SPA), a research trend is to consider a continuous-time model in which the wealth process is defined by

$$W_t = \mathrm{e}^{R_t} \int_0^t \mathrm{e}^{-R_s} \mathrm{d}P_s, \tag{5.2.1}$$

where $P_s = x + cs -$aggregate claims. The integral above represents the stochastic present value of aggregate net profit up to time t.

In the literature, it is often assumed that the two stochastic processes $\{P_s\}$ and $\{R_s\}$ are mutually independent. Such an assumption is obviously unrealistic. My thought is to construct a discrete-time model as an approximation of the continuous-time model. Importantly, this approach allows $\{P_s\}$ and $\{R_s\}$ to be dependent to a certain extent.

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