Aspects of the Gribov-Zwanziger Lagrangian in relation to Gluon Confinement

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor of Philosophy by

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Declaration

I hereby declare that all work described in this thesis is the result of my own research activities unless reference to others is given. None of the material has been previously submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences during the period October 2009 to September 2013.

Contribution from this work have been published in the following references:

J.A. Gracey & D. Thelan, Nuc. Phys. B (Proc.Suppl.), 205-206, (2010), 182.

J. A. Gracey & D. Thelan, Paper in preparation.

Abstract

Over the years there have been many developments in the field of Quantum Chromodynamics, however there still remains some unresolved issues with the theory. Central to these is the principle of gauge copies arising from the Gribov problem. In this work we shall try and address some these issues, in recent years there has been a shift in the lattice data for the low momentum behaviour of the propagators for the gluon and ghost. This change suggests there may now be disagreement with the Lagrangian proposed by Gribov and Zwanziger and the lattice data. It is possible to modify the Gribov-Zwanziger Lagrangian so that it fits the lattice data but these modifications are not unique. The aim of this work is to investigate what effect these modified solutions have to our understanding of the low momentum behaviour of QCD and to see which if any of these solutions is favourable. Further to see we shall explore many of the methods and techniques used in this work by performing a calculation involving polarized deep inelastic scattering operators in QCD, this work while not related to the Gribov problem will help provide a strong understanding of many of the problems we shall encounter while considering the low momentum behaviour in the Gribov case.

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Contents

Chapter 1

Introduction

1.1 Context and background

Over the years there have been many developments in the field of particle physics, these began in the mid 20th century and continue to the recent date. The current theory of particle physics is described by the Standard Model of Particle Physics. This combines three of the fundamental forces: electromagnetism, weak nuclear force and strong nuclear force. The final fundamental force of gravity has yet to be resolved, however there is consensus that its interactions are so small that they shall have little to no effect on the physics of the Standard Model. Each of the forces in the Standard Model is described by a separate theory, electromagnetism and the weak nuclear force are modelled using Glashow-Salam-Weinberg electroweak theory [1] [2] [3] and the strong nuclear force by Quantum Chromodynamics (QCD) [4] [5]. Between these they have been able to accurately describe most of the data arising from experimental results, this includes the recent discovery of a Higgs like object at the Large Hadron Collider in 2012 [6] [7]. This was a very important result as its existence has been predicted for years however it is only very recently that the experimental methods have been able to start to catch up with the theory. This is not to say there are not still unresolved issues within the current understanding of the Standard Model, we shall touch again on this later.

QCD and electroweak theory are both gauge theories, this means that the Lagrangian for these theories shall remain invariant under specific gauge transformations which are parametrised by the underlying elements of the groups they are derived from. It can be found from many of the text books [8] in the field that for QCD the gauge group is $SU(3)$, however it is $SU(2)$ for weak nuclear force and finally $U(1)$ for Quantum Electrodynamics (QED). The great achievement of Glashow-Salam-Weinberg in electroweak theory was to combine the gauge groups of QED and the weak nuclear force into $U(1)$ x $SU(2)$. The Standard Model further extends this to combine electroweak theory with the QCD gauge group to give $U(1)$ x $SU(2)$ x $SU(3)$. The focus of this work is within QCD so we shall be working within the $SU(3)$ region of the Standard Model.

The previously mentioned problems with the Standard Model mainly lie within the QCD section of the Standard Model, this is not because QCD predictions do not match the experimental data. Where it is possible to measure predictable results there has been good agreement between the two approaches, this can be seen from the data arising from deep inelastic scattering. The problem with QCD stems from the difficulty of performing calculations as well as a degree of difficulty getting comparable results from experimental data. This arises from some of the unique properties of QCD that differ from the other fundamental forces. The most important of these is the scaling of the strong coupling constant, this coupling only becomes small at very high energies. As QCD works within perturbation theory and this approach is only valid at high energies where the coupling constant is small, this greatly limits the range of possible calculations that are possible. This differs from QED where the coupling constant is sufficiently small across the whole energy scale that it is possible to use Perturbation theory for high and low energy calculations. Unfortunately there is a lot of interesting physics occurring towards the low energy scale of QCD and because of the large value of the coupling constant it is not possible to use perturbative methods to perform calculations at this scale. However there should be some connection between the low energy theory and QCD. The reason this is an important area of study is due to the problem of QCD confinement, that is the mechanism which relates the free high energy components of the theory, gluons and quarks, with the low energy hadronic bound states of the proton, neutron and pion. As the strong nuclear force is the driving force between all these interactions the theory of QCD should work well in both cases, however, this problem has not been fully resolved. With recent advances in computing power there has been an increasing use of lattice methods to carry out expansions using the inverse QCD coupling to try and observe the behaviour of the theory for large values of the coupling. The limits of these need to match with the expected results arising from bound states and with results arising from perturbative QCD in the higher energy limit. However, even with current computing technology it is still an expensive procedure to carry out and is used to match with data under specific conditions rather than trying to model the full QCD action. Even with progress here, we are still far from a full understanding of the strong nuclear force or the subtleties arising from it. Despite these problems it has generally been accepted that QCD is the correct model to describe the strong nuclear force and this should include the mechanism for confinement, the fact that some of the calculations are currently not possible or that the methods are not yet fully understood means these remain areas of active research. Of the viable calculations preformed there is agreement that some of the properties that make QCD difficult to use for calculation are present in the experimental data, the most noted example is the concept of asymptotic freedom. This is a way of describing the fact the coupling constant is inversely proportional to the amount of energy being applied, this is backed up empirical evidence showing a correlation between high energy perturbative QCD and high energy experimental data.

This work for the most part tries to deal with some of the unresolved problems related to QCD, in particular those arising from the Gribov problem and the effect this has on the dynamics of QCD. A qualitative overview of this shall be given in the following sections but for now we shall give a summary of what these issues are and why they are important in the context of QCD. The motivation behind us looking at the Gribov problem is two fold. Firstly the solution proposed by Gribov helps to resolve a problem resulting from the quantization of Yang-Mills theory for a gauge group. Secondly, in resolving the quantization issue there are additional effects from this that change the low energy behaviour of QCD and therefore confinement. What Gribov discovered in his seminal work [9] was that in the Faddeev-Popov method of quantizing Yang-Mills theory there is a problem of over counting gauge fields, this is due to gauge fields being indistinguishable from their copies. These gauge copies became known as Gribov copies and their presence is enough to affect the low energy infra-red region of the gauge dependent gluon and ghost propagators.

The problem arises as it is possible in gauge theories such as QCD to construct different gauge configurations that satisfy the same gauge condition: this equivalence is responsible for the over-counting in the path integral formalism used to describe QCD. The solution to this problem was to define Gribov copies in such a way where they could be identified and then restrict the action to a specific subspace free of copies. The space free from Gribov copies is known as the first Gribov region, this is bounded by a horizon and contains the origin. With this there remains the problem of how to actually use this information for calculations in QCD. In Gribov's original analysis of the problem he was able to discover many remarkable features, first that it was possible to modify the path integral to restrict it to the first Gribov region. Gribov also was able to show that introducing this cut off modified the gluon and ghost propagators at the zero momentum limit. Instead of their normal action of tending to infinity like $\frac{1}{p^2}$ as p tended to zero, however the new gluon propagator vanished at the zero momentum limit. The propagator of the ghost is also modified so that it now behaves as a dipole in the zero momentum limit. These effects are known respectively as a suppressed gluon and an enhanced ghost. The reason for this modification is that limiting the path integral introduces a new mass parameter known as the Gribov mass γ, this is not an independent quantity as it is expressed as a function of the QCD coupling g.

However, at this point it is not possible to use this Gribov Lagrangian to perform calculations using perturbative methods, this is because in the form derived by Gribov the horizon condition introduces a non-local operator into the Lagrangian. This is a problem as for any calculation to be physically valid we need to be working with a Lagrangian which is both local and renormalizable. Luckily Zwanziger over numerous articles [10] [11] [12] [13] [14] [15] [16] [17] was able to rewrite the Gribov Lagrangian in a way that it only includes a local term and also turned out to be fully renormalizable. That is not to say that the Zwanziger method is the only possible local renormalization possible, however to date it is the most extensively used one and the only one shown to be fully renormalizable. This localisation is achieved by introducing four additional ghost fields $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu, \omega^{ab}_\mu, \bar{\omega}^{ab}_\mu\}$ The first pair of these are Bosonic ghosts while the second two are Grassmann ghosts. These are defined in such a way where no extra renormalization constants are required, this is because the anomalous dimensions of the localizing ghosts are the same as the Faddeev-Popov ghost. Moreover the renormalization group function for the

Lagrangian is not affected therefore not affecting the asymptotic freedom of our theory. This localizing does not affect the form of the horizon condition introduced in Gribov's original work. This new local Lagrangian is referred to as the Gribov-Zwanziger Lagrangian and as it is local and renormalizable it can be used to perform calculations in a similar way to the original QCD Lagrangian. For instance the 1 and 2-loops corrections to the Gribov parameter γ have been calculated in the $\overline{\text{MS}}$ scheme [18]. It can also be used to perform many of the checks and tests relevant to QCD to see how the different low energy limits of the propagators affect the theory and if these changes lead any closer to resolving some of the problems related to QCD confinement. For a while these predictions agreed with those coming from the lattice for the low energy behaviour of the gluon propagator [19][20][21][22][23][24][25] and ghost propagator [23][24][25][26][27][28]. However, this has been thrown into doubt by recent results from lattice QCD data [29][30][31][32][33][34][35][36][37]. As computer power has advanced and techniques have become more refined, it has been possible for more complex studies to be carried out to gauge fix numerically on large lattices. This has been used to acquire a reasonable amount of data on the low momentum behaviour of the gluon and ghost propagator. The problem is this does not agree with the low momentum limit suggested by Gribov. On the lattice it appears that the zero momentum limit for the gluon is not zero, and the gluon propagator freezes to a non-zero value. Further it is also indicated that the Faddeev-Popov ghost does not enhance. What this means is that the gluon propagator appears to diverge from the standard Yang-Mills case as well as that proposed by Gribov while the ghost behaviour is the same as in Yang-Mills in the low momentum limit. The Gribov solution is referred to in the literature as the scaling solution while the newer results from lattice data are known as the decoupled solution. This presents a problem for the solution to the copy problem proposed by Gribov, as this does not currently match the lattice data being produced. The question is, is there a valid method to transform the Gribov-Zwanziger Lagrangian into something which matches the lattice prediction? This is a procedure worth considering as while the current Gribov-Zwanziger Lagrangian does not fit the numerical data it does satisfy and resolve many of the issues with QCD and without it we are still in a situation where we need to find a suitable Lagrangian which resolves the gauge copy problem while displaying the correct behaviour in the low energy limit. The answer to this may be in the proposed Refined Gribov-Zwanziger Lagrangian, this is a modified form of the Lagrangian where there is an extra BRST invariant dimension two operator included. This operator is constructed from a combination of the localizing ghost fields with the appropriate rank four colour tensor. In most theories this extra term would just add an additional mass for these fields only, however in a similar mechanism to the Gribov mass γ these extra terms affect the gluon and ghost propagators. It is possible to define this operator in such a way that the mass introduced to the gluon propagator does not go to zero in the zero momentum limit. It instead will freeze to a non-zero value, a similar change is also achieved in the Faddeev-Popov ghost which no longer enhances either. This modification brings the Gribov-Zwanziger Lagrangian back into accordance with the more recent lattice data.

This modification however, does not describe the full picture of what is happening with the extra BRST invariant dimension two operator. The operator included in the Refined Gribov-Zwanziger Lagrangian is contracted with a rank four colour tensor. However, this choice is not unique, in principle it is possible to choose from a number of rank four colour tensors or even the sum of many different ones. This idea was proposed by J.A.Gracey in 2010 [38], where he described the six potential colour channels that are possible for the operator. In this he shows that different colour channels produce different low energy behaviour, some of them producing the decoupled solution that matches the lattice data while others preserve the original scaling observed by Gribov. It is this alternative refined Gribov-Zwanziger Lagrangian that shall be the focus of much of the work presented within this thesis. This will cover two bodies of work. The first shall look at the one-loop effective potential of the most general sum of all possible colour channels using the local composite operator formalism, the second shall look at the the one-loop triple vertex correction at the symmetric point in 3-dimensions in the two colour channels that satisfy the conditions for the decoupled solution. Despite there being much work carried out in the Refined Gribov-Zwanziger Lagrangian most of this has looked at one colour channel without considering any of the others present. It is the aim of our first calculation for the effective potential to generalise this work to include all the colour channels and see from this if there is any one channel that dominates the final result. The motivation is that this should show us which of these results is the most energetically favourable result without arbitrarily imposing the channel which is most convenient for the calculation. To achieve this we shall closely follow the previous work of [39] where the effective potential was computed for the dimension two gluon operator $\frac{1}{2}A^2_\mu$ using the local composite operator (LCO) formalism. The advantage of this approach is that it allows us to use perturbative methods to study some of the non-perturbative aspects of the asymptotically free theory. The idea of using perturbation theory to compute non-perturbative quantities was originally raised in the Gross-Neveu model [40] which is a simpler model that is still an asymptotically free theory. In [41] it was explained how some non-perturbative quantities can be associated with local composite operators and following this how it is possible to find the effective potential in perturbation theory of these operators and from this describe non-perturbation results. For the Gross-Neveu model it can be seen that the non-perturbative mass gap is related to the choice of dimension two operator. By using the LCO method to calculate the effective potential it was possible to show that the operator condensed and that it was possible to work out the mass gap for the theory, this is a result not normally found by perturbation calculations. This was then checked against previous work on the lattice to show good agreement. Therefore our motivation of using this method on our alternative refined Gribov-Zwanziger Lagrangian is that it should show if our operator condenses or not and finally the non-perturbative value of the mass gap.

The second major part of this work looks at the power correction to the symmetric point 3-point vertex in the alternative refined Gribov-Zwanziger Lagrangian in 3-dimensions. As previously mentioned the alternative refined Gribov-Zwanziger Lagrangian matches the low momentum behaviour of the gluon and ghost to the lattice data however there are different colour channels that show the same behaviour in the limit but exhibit different behaviour between the low and high energy limits. So in principle each of these should give a different result which then could be compare to lattice data to try and determine which if any of these are correct. However while there has been much progress in the study of QCD on the lattice, there has not yet been any work looking into possible refinements of the Gribov-Zwanziger Lagrangian. Like the effective potential calculation the aim here is to produce measurable results using the perturbative approach to QCD that can eventually be checked against lattice data once it is produced. This brings us to why we shall perform this calculation in 3-dimensions instead of the more standard 4-dimensions, we have chosen this with consideration to the lattice in mind as well as for our own ease of calculation. Even with recent advances in computer power lattice QCD still remains an expensive procedure to perform. This expense scales with the lattice size and the number of dimensions used. By working in 3-dimensions instead of 4, we hope this choice will potentially motivate a suitable lattice calculation for comparison of results. In principle it should also be possible to perform the calculation in 4-dimensions however this will be more technically challenging as there are problems arising from some of the Feynman diagrams used. This is a further advantage of working in 3-dimensions rather than 4 as it occurs that the values of all the required integrals are known exactly, this prevents the problem of having to find suitable methods or approximations that could introduce errors into this work.

Finally, the last section of this work shall detail a calculation from the start of my PhD studies that while not directly linked to the Gribov problem covered in the majority of this work, it was a useful calculation for exploring many of the methods and techniques that were used in the later work. While not directly related to the work in previous sections this last section is very much motivated by the same aim, to provide results from perturbation theory that are measurable with appropriate lattice techniques. For this we have looked at deep inelastic scattering operators. The matching of lattice data in the continuum limit is one beset by many potential problems. A large number are related to the renormalization scheme and the method of gauge fixing employed by either the lattice or the perturbative calculation. The normal choice for working on lattice QCD is the RI$'$ scheme while in the continuum limit of perturbative QCD more commonly the results are given in terms of the $\overline{\text{MS}}$ scheme. For certain types of Green's functions there is a suitable conversion for the difference in scheme choice. A more fundamental problem is that of gauge fixing, while fixing the gauge does not present an unsolved problem in the perturbative limit it does however cause some problems on the lattice. Fixing the gauge on the lattice is a procedure which is far from straight forward and one which can lead to reliability issues surrounding the Gribov problem. A potential solution is to construct Green's function in such a way that they are completely gauge independent objects, this avoids the gauge problem altogether as it means the gauge never has to be fixed on the lattice. For this method to be accurate it is necessary to work to as high a loop order as possible as each loop order will increase the accuracy of the results. The appropriate choice here to is to work with correlation functions of operators $\langle \mathcal{O}(p)\mathcal{O}(-p)\rangle$ where p is the momentum flow through the correlation function. Our work builds on the previous work of using the quark current $\langle \psi(p)\mathcal{O}(0)\psi(-p)\rangle$ and some of these renormalization and anomalous dimensions shall be required in this work. While this is an area that has been the focus of much previous study, there are still some situations that have not been covered. We shall particularly focus on polarized operators containing γ^5 and study how this can be treated in the renormalized correlation function. Whilst our motivation is to provide the finite parts of these correlation functions, several technical issues need to be addressed to obtain the correct answers. To start with we shall re-derive some of the known results for correlation functions not containing the γ^5 term, this will provide us with a suitable check that our method is correct. We shall then need to look at how the γ^5 term factors into the correlation function and its renormalization. To do this we shall need to work out an extra renormalization counter term for γ^5 before then being able to give the finite part of the correlation function that satisfies the necessary constraints. The operators we study here shall only be the simple cases however the results shall be useful as they will show, how in principle, to handle operators involved in polarized deep inelastic scattering.

1.2 Conventions and notation

Before we can progress much further with any detailed calculation we must first stop to define some of the notations and conventions used in this work. In this work we use many different types of fields present in QCD and we have generalised to the gauge group $SU(N_C)$, this is a more general form of the standard $SU(3)$ associated with QCD. Where possible we shall try and stay within the more general gauge group, this may be useful for currently unforeseen calculations that require a different group. We shall only fix ourselves to $SU(3)$ to present specific results that require this or in cases where further progress is not possible without doing so. The fields present in our calculations are A^a_μ, ψ^{iI}, $\bar{\psi}^{iI}$, $\bar{c}^a$ and c^a which make up those found normally in QCD, however we also have ϕ^{ab}, $\bar{\phi}^{ab}$ and ω^{ab}, $\bar{\omega}^{ab}$ coming from the localizing of the Gribov Lagrangian. We also have the Gribov mass parameter γ which, while not a field, does play an important part in many of our interactions. We shall tackle each of these fields in turn and describe their notation. We start with the gauge field A^a_μ where $\mu \in \{0\,1\,2\,3\}$ are the space time components, with 0^{th} being the time one. $a \in \{1, ..., N_A\}$ is the colour index, where N_A is the dimension of the adjoint representation of the group. For $SU(N_c)$ it is given by

$$N_A = N_c^2 - 1 \tag{1.1}$$

Next is ψ^{iI}, $\bar{\psi}^{iI}$ which are the quark and antiquark fields respectively, each have N_c components labelled by the $I \in \{1, ..., N_c\}$ and the index i labels the quark flavour $i \in \{1, .., N_f\}$ with N_f being the number of quark flavours. Finally for QCD we have c^a and $\bar{c}^a$ which are the Faddeev-Popov ghost which are Grassman variables required to fix the gauge. Each of the ghost field has N_A components. Each of these fields has a covariant derivative associated with it.

$$D_\mu A^a_\nu = \partial_\mu A^a_\nu - g f^{abc} A^b_\mu A^c_\nu \tag{1.2}$$

$$D_\mu \psi^{iI} = \partial_\mu \psi^{iI} + ig A^a_\mu (T^a)^{IJ} \psi^{iJ} \tag{1.3}$$

$$D_\mu c^a = \partial_\mu c^a - g f^{abc} A^b_\mu c^a \tag{1.4}$$

With our fields defined, it is possible to construct the QCD Lagrangian with light quarks, in the absence of the gauge fixing term

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i \bar{\psi}^{iI} \not{D} \psi^{iI} \tag{1.5}$$

here we have made use of the slash notation for a Lorentz vector v which is defined as

$$\not{v} = \gamma^\mu v_\mu \tag{1.6}$$

Finally we can define the field strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{1.7}$$

where g is the QCD coupling constant and the quantities f^{abc} are the structure constant of our $SU(N_c)$ group. Considerably more detail on the structure constants and the role they play will be given in section 2.4. The Lagrangian (1.5) is gauge invariant, this means that under the special conditions of a gauge transformation it will remain unchanged. For the gluon field the gauge transformation is defined as

$$A'^a_\mu(x) = U(x) A^a_\mu U^\dagger(x) - \frac{i}{g} \left(\partial_\mu U(x)\right) U^\dagger(x) \tag{1.8}$$

Here $U(x)$ are functions of the generators of the adjoint group. Before we can continue any further with our definitions we must first tackle the issue of gauge fixing, this is a particularly important problem in the Gribov frame work. The Lagrangian (1.5) is gauge fixed by adding additional ghost fields as well as a new gauge parameter α for which there is a certain amount of freedom in its definition depending on the properties required from the action.

The gauge fixed Lagrangian for an arbitrary gauge parameter is given by

$$L^{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \tag{1.9}$$

This does not include the Gribov mass which we shall come to in a moment. Most of the work presented in this body of work shall in the Landau gauge for QCD, this means taking $\alpha = 0$ in the Lagrangian. This is not by any means the only possible choice for this, however, it is the one which allows for easiest comparison with other results. Adding the Gribov mass γ^4 to our Lagrangian modifies it in the following way.

$$\begin{aligned} L^{\text{Grib}} =& -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \\ &+ \frac{C_A\gamma^4}{2}A^{a\,\mu}\frac{1}{\partial^\nu D_\nu}A^a_\mu - \frac{dN_A\gamma^4}{2g^2} \end{aligned} \tag{1.10}$$

These extra terms are required to keep the action within in the horizon condition and ensure that there are no gauge copies in our theory. This was achieved by Gribov by imposing the horizon condition in such a way where it can be incorporated into the action. This is done by introducing the Gribov mass parameter γ into the Lagrangian, this is not an independent parameter and is a function of the coupling constant g^2. However (1.10) is not usable for practical calculation as it introduces a non-local term into the action. To solve this Zwanziger was able to remove the non-locality by introducing two extra sets of ghost fields into the action ϕ^{ab}_μ, $\bar{\phi}^{ab}_\mu$ and ω^{ab}_μ, $\bar{\omega}^{ab}_\mu$. These are Bose and Grassmann ghosts respectively. For our computation we shall be using the Bose ghost in terms of real and imaginary parts

$$\phi^{ab}_\mu = \frac{1}{\sqrt{2}}\left(\rho^{ab}_\mu + i\xi^{ab}_\mu\right) \tag{1.11}$$

$$\bar{\phi}^{ab}_\mu = \frac{1}{\sqrt{2}}\left(\rho^{ab}_\mu - i\xi^{ab}_\mu\right) \tag{1.12}$$

With these extra fields it is now possible to write (1.10) in such a way where it is not only local but was shown to be renormalizable.

$$
\begin{aligned}
L^{\mathrm{GZ}} = & -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \\
& + \frac{1}{2}\rho^{ab\,\mu}\partial^\nu\left(D_\nu\rho_\mu\right)^{ab} + \frac{i}{2}\rho^{ab\,\mu}\partial^\nu\left(D_\nu\xi_\mu\right)^{ab} - \frac{i}{2}\xi^{ab\,\mu}\partial^\nu\left(D_\nu\rho_\mu\right)^{ab} \\
& + \frac{1}{2}\xi^{ab\,\mu}\partial^\nu\left(D_\nu\xi_\mu\right)^{ab} - \bar{\omega}^{ab\,\mu}\partial^\nu\left(D_\nu\omega_\mu\right)^{ab} - \frac{1}{\sqrt{2}}gf^{abc}\partial^\nu\bar{\omega}^{ae}_\mu\left(D_\nu c\right)^b\rho^{ec\,\mu} \\
& - \frac{i}{\sqrt{2}}gf^{abc}\partial^\nu\bar{\omega}^{ae}_\mu\left(D_\nu c\right)^b\xi^{ec\,\mu} - i\gamma^2 f^{abc}A^{a\,\mu}\xi^{bc}_\mu - \frac{dN_A\gamma^4}{2g^2}\,. \qquad (1.13)
\end{aligned}
$$

An excellent review of the full procedure starting from Yang-Mills which covers gauge fixed QCD then extends this to the full Gribov Lagrangian can be found in [42]. For the purposes of carrying out calculations we require exact results we shall use perturbation theory and Feynman diagrams. While this is not the only method available for many of these calculations it is very useful in the sense that it is possible to program this type of calculation to be preformed automatically on a computer given the correct parameters. Further there exist a number of different packages and software already available which aid this task. For the Feynman diagram method to work, we start with the Lagrangian for our model, from this we define two sets of objects; propagators and vertices. These collectively define the Feynman rules for the model. Propagators are defined from quadratic terms of the action, these provide the lines that connect the vertices. All the other terms describe the interaction rules which are used for the possible vertices. These can contain any number of fields so that it is possible to have 3, 4 or higher number vertices. However in our models the most that arises is a 4 point vertex. In most of the calculations in question, we have evaluated n-point Green's functions, these are equivalent to a series of Feynman diagrams with n external legs connected by the relevant vertices and propagators for our model. Each of the legs has a fixed value of momentum associated with it, however these do not have to be independent and can be equal or zero. Each diagram is evaluated by substituting the Feynman rules for each propagator and vertex then integrating over the internal momentum. Rather than working with an infinite set of Feynman diagrams for each Green's functions which would prove impossible to do in most cases, we shall work with a perturbative expansion to m^{th} order. What this means is that we shall work with diagrams up to and including m closed loops. The higher the value of m, the more accurate the result becomes. This is similar for many other series approximations where the higher the order the series is worked out for, the better the approximation shall be. All Feynman diagrams must observe momentum conservation to be valid in this method.

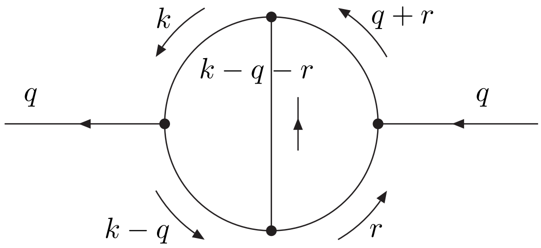

Figure 1.1: Example of momentum flow around a 2-loop diagram

In Figure 1.1 q is the external momentum and k and p are internal loop momentum. We shall define our momentum configuration so as to be compatible with MINCER [43], where calculations require its use. Each of the different Lagrangians (1.9), (1.10) and (1.13) has a different set of Feynman Rules, we shall give particular attention to the terms affected by the Gribov mass as these are of importance to our calculations. The results of many of the loop integrations is given in terms of fixed quantities

$$\gamma = \quad \text{the Euler Mascheroni constant} \tag{1.14}$$

$$\zeta_n = \quad \zeta(n), \text{ the Rienmann zeta function.} \tag{1.15}$$

These arise from the expansion of the counter term ϵ in the gamma function $\Gamma(x)$ which is used to evaluate many of the Feynman integral of loop momentum.

1.3 The Gribov Mass

As it will be central to our discussions later and it is an important fact when considering the problem of gauge copies, we shall now derive the exact form of the Gribov mass parameter. The Gribov mass parameter was Gribov's solution for restricting the Yang-Mills action to the region free of gauge copies. Gauge copies are defined as two equivalent fields A_μ and A'_μ linked by a gauge transformation

$$A'^a_\mu(x) = U(x)A^a_\mu U^\dagger(x) - \frac{i}{g}\left(\partial_\mu U(x)\right)U^\dagger(x) \tag{1.16}$$

that satisfy the condition $\partial_\mu A_\mu = 0$ and $\partial_\mu A'_\mu = 0$. This means that it is possible for a gauge orbit to intersect a gauge condition more than once. This multiple intersection gives rise to the idea of a gauge copy of the same gauge field. The solution to this proposed by Gribov [9] was to restrict the action to the so called Gribov region Ω.

$$\Omega = \{A^a_\mu\,,\, \partial_\mu A^a_\mu\,,\, \mathcal{M}^{ab} > 0\} \tag{1.17}$$

where $\mathcal{M}^{ab}$ is the Faddeev-Popov operator

$$\mathcal{M}^{ab}(x,y) = \partial_\mu D^{ab}_\mu \delta(x-y) = -\partial_\mu\left(\partial_\mu \delta_{ab} + f_{abc}A^c_\mu\right)\delta(x-y) > 0 \tag{1.18}$$

This is where the gauge field satisfies the Landau gauge and the Faddeev-Popov operator is always positive. Gribov was able to show [9] that this extra condition was enough to restrict the action to one which it was not possible for gauge copies to appear. A direct consequence of imposing this volume restriction on the action was the introduction of the Gribov mass parameter γ. This mass is not a new independent quantity as it is a function of the coupling constant g^2. Zwanziger was able over numerous works [11] [12] [44] to show that it is possible to write Gribov's volume restriction in the form

$$\mathrm{Tr}\mathcal{M}^{ab}(x,y) = \frac{C_A\gamma^4}{2} A^{a\,\mu} \frac{1}{\mathcal{M}^{ab}(x,y)} A^b_\mu - \frac{dN_A\gamma^4}{2g^2} \tag{1.19}$$

where the Gribov region is defined as

$$\mathrm{Tr}\mathcal{M}^{ab}(x,y) > 0 \tag{1.20}$$

Further to this we define the boundary of this region $\delta\Omega$ as the first Gribov horizon, this is the point where the Faddeev-Popov operator becomes zero. At this point (1.20) will become equal to zero, this enables us to rewrite (1.19) on the boundary

$$\frac{C_A\gamma^4}{2} A^{a\,\mu} \frac{1}{\mathcal{M}^{ab}(x,y)} A^b_\mu = \frac{dN_A\gamma^4}{2g^2} \tag{1.21}$$

These are the final two terms in the Gribov Lagrangian (1.10), therefore is possible to write this horizon condition in the form.

$$\frac{\partial\Gamma^{(1)}_\gamma}{\partial\gamma^4} = 0 \tag{1.22}$$

Now that we have a brief understanding of the Gribov region and the horizon condition it is possible to work out the form of the Gribov parameter in terms of the coupling constant g^2 This is preformed by computing the gluon propagators from the quadratic part of the action then imposing the horizon condition. While most of the time we shall choose to leave our results in terms of γ, these results are important for understanding the effect the horizon condition has on modifying our results in terms of standard measurable quantities.

We shall begin by taking the quadratic part of the Gribov Lagrangian (1.10) with a source term coupled to the gluon.

$$\int d^d x \left[\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{1}{2\alpha}\left(\partial_\mu A^a_\mu\right)^2 - N_A \gamma^4 A^a_\mu \frac{1}{\partial^2} A^a_\mu + J^a_\mu A^a_\mu + ...\right] \tag{1.23}$$

To obtain the gluon propagator from this we take the functional derivative of (1.23) in terms of the source J then set $J = 0$

$$\langle A^a_\mu(k) A^a_\nu(-k)\rangle = \frac{\delta^2}{\delta^a_\mu J(k)\delta^b_\nu J(-k)} \int [d\, A] \exp\left(-\int \frac{d^d\, k}{(2\pi)^d}\frac{1}{2} A^a_\mu(k) K^{ab}_{\mu\nu}(k) A^b_\nu(-k) + ...\right)\Bigg|_{J=0} \tag{1.24}$$

Here $K^{ab}_{\mu\nu}(k)$ is given by the function

$$K^{ab}_{\mu\nu}(k) = \delta^{ab}\left(\gamma^4 \delta_{\mu\nu}\frac{1}{k^2} + \delta_{\mu\nu}k^2 + \left(\frac{1}{\alpha} - 1\right) k_\mu k_\nu\right) \tag{1.25}$$

It is possible to use the Gaussian integral for a scalar field

$$\begin{aligned} I(A, J) = &\int [d\, \phi] \exp\left[-\frac{1}{2}\int d^d\, x d^d\, y \phi(x) A(x, y)\phi(y) + \int d^d x \phi(x) J(x)\right] \\ =&(\det A)^{-1/2} \exp\frac{1}{2}\int d^d\, x d^d\, y J(x) A^{-1}(x, y) J(y) \end{aligned} \tag{1.26}$$

to rewrite our generating functional in such a way where we require only our term $K^{ab}_{\mu\nu}(k)$ and the γ term from the (1.13) to form the 1-loop effective action

$$\Gamma^{(1)}_\gamma = -\frac{d N_A \gamma^4}{2g^2} + \mathrm{Tr}\ln\det K^{ab}_{\mu\nu}(k) \tag{1.27}$$

With this in mind, we need to evaluate $K^{ab}_{\mu\nu}(k)$ in such a way that we can take the limit $\alpha \to 0$ without causing a divergent term. Begin by taking the determinant of (1.25)

$$\operatorname{Tr}\ln K^{ab}_{\mu\nu}(k) = N_A \operatorname{Tr}\ln\left(\delta_{\mu k}\left(\frac{\gamma^4}{k^2}+k^2\right)\left(\delta_{k\nu}+\frac{1}{\frac{\gamma^4}{k^2}+k^2}\left(\frac{1}{\alpha}-1\right)k_\mu k_\nu\right)\right) \tag{1.28}$$

This produces a factor of N_A. Now using the properties of the logarithm term to split the product into two separate terms.

$$\begin{aligned}\operatorname{Tr}\ln K^{ab}_{\mu\nu}(k) =& N_A \operatorname{Tr}\ln\left(\delta_{\mu k}\left(\frac{\gamma^4}{k^2}+k^2\right)\right)\\ &+ N_A \operatorname{Tr}\ln\left(\delta_{k\nu}+\frac{k^2}{k^4+\gamma^4}\left(\frac{1}{\alpha}-1\right)k_\mu k_\nu\right)\end{aligned} \tag{1.29}$$

Next using the properties of the logarithmic series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \mathcal{O}(x^4) \tag{1.30}$$

it is possible to write the second term of equation (1.29)

$$\begin{aligned}\operatorname{Tr}\ln K^{ab}_{\mu\nu}(k) =& N_A \operatorname{Tr}\ln\left(\delta_{\mu k}\left(\frac{\gamma^4}{k^2}+k^2\right)\right)\\ &+ N_A \operatorname{Tr}\ln\left(1+\frac{k^2}{k^4+\gamma^4}\left(\frac{1}{\alpha}-1\right)k^2\right)\end{aligned} \tag{1.31}$$

Now taking the second term of this equation it is possible to put this over a common denominator.

$$\begin{aligned}\ln\left(1+\frac{k^2}{k^4+\gamma^4}\left(\frac{1}{\alpha}-1\right)k^2\right) &= \ln\left(\frac{\alpha(\gamma^4+k^4)+k^4(1-\alpha)}{(\gamma^4+k^4)\alpha}\right)\\ &= \ln\left(\frac{k^2}{k^4+\gamma^4}\left(\frac{\gamma^4}{k^2}+\frac{k^2}{\alpha}\right)\right)\end{aligned} \tag{1.32}$$

This can once again be split by using the product of logarithmic functions.

$$\ln\left(\frac{k^2}{k^4+\gamma^4}\left(\frac{\gamma^4}{k^2}+\frac{k^2}{\alpha}\right)\right) = \ln\left(\frac{k^2}{k^4+\gamma^4}\right) + \ln\left(\frac{\gamma^4}{k^2}+\frac{k^2}{\alpha}\right) \tag{1.33}$$

Finally it is possible to rewrite the α dependent term of this

$$\ln\left(\frac{\gamma^4}{k^2}+\frac{k^2}{\alpha}\right) = \ln\left(\gamma^4+\frac{k^4}{\alpha}\right) - \ln k^2 \tag{1.34}$$

To progress any further we must use the relation that taking the trace of our determinant is equivalent to integrating over the momentum k, this allows us to evaluate each of our logarithm terms. More information on the integrals found here can be seen in section 2.5 where the results are discussed in detail. It is now possible to deal with each of these in turn, we begin with the α dependent terms.

$$\int d^d\, k \ln\left(\gamma^4+\frac{k^4}{\alpha}\right) \sim \alpha^{d/4} \tag{1.35}$$

This is because there is a common pre-factor $\alpha^{d/4}$ in front of the other terms arising from the integral, this term dominates in the limit. Therefore as $\alpha \to 0$

$$\int d^d\, k \ln\left(\gamma^4+\frac{k^4}{\alpha}\right) = 0 \tag{1.36}$$

This can be used to evaluate all the terms containing α, as all such terms in the 1-loop effective action will be free from α terms. This is to be expected from working in the Landau gauge. Next we can evaluate the k^2 term.

$$\int d^d\, k \ln k^2 = 0 \tag{1.37}$$

This just leaves one final term in our action

$$\Gamma_{\gamma}^{(1)} = -dN_A\gamma^4 + N_A(d-1)\int \frac{d^d\,k}{(2\pi)^d}\ln(k^4+\gamma^4) \tag{1.38}$$

With the 1-loop correction to our action we can now impose the horizon condition (1.22)

$$\frac{\partial\Gamma_{\gamma}^{(1)}}{\partial\gamma^4} = 0 \tag{1.39}$$

Applying this to equation (1.38) gives

$$-2dN_A\gamma^2 + 2\gamma^2 N_A(d-1)\int \frac{d^d\,k}{(2\pi)^d}\frac{1}{k^4+\gamma^4} = 0 \tag{1.40}$$

This can then be rearranged to give the final form of the Gribov mass in d-dimensions from the quadratic part of the action.

$$\frac{(d-1)}{d}\int \frac{d^d\,k}{(2\pi)^d}\frac{1}{k^4+\gamma^4} = 1 \tag{1.41}$$

This is however only the starting point for our purposes, it is now possible to evaluate this using the methods of Section 2.5. As the integral (1.41) depends on the dimension of the theory we are working with we shall have different results for 3 and 4 dimensions. It is fairly straightforward to compute the 1-loop gap equation in 4 and 3 dimensions

$$1 = C_A\left[\frac{5}{8} - \frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right]\frac{g^2}{16\pi^2} + \mathcal{O}(g^4)\ d=4 \tag{1.42}$$

$$\frac{3}{4} = \frac{\sqrt{2}C_A^{3/4}g^2}{16\pi\gamma} + \mathcal{O}(g^4)\ d=3 \tag{1.43}$$

These are 4 and 3 dimensional relation between g^2 and γ respectively, it is clear to see from here that there is not always a simple relation between the two quantities. In (1.42) there is no simple way to substitute this exact form of γ into an equation without making it difficult to use, due to the logarithm term in the equation. In (1.43) the relation is far more simple, this makes the exact form of the Gribov mass possible to implement in theories in 3-dimensions. Also by looking at (1.43) it is possible to see what happens to the Gribov mass in the free non-interacting theory $g^2 = 0$. By setting g to zero it follows that γ also goes to zero. This can further be in seen from (1.40) as when working in the non-interacting theory both sides of this equation are equal to zero. This implies that in the free field theory there are no Gribov copies and Gribov Lagrangian reduces to the non-interacting QCD Lagrangian. The exact form of the Gribov mass in 3-dimensions shall be one of the motivations for the work in Chapter 3. Finally both the 3 and 4 dimensional Gribov masses have been worked out to 2-loops in the following [18] [45].

Chapter 2

BRST Invariant Operator

2.1 Introduction

While there has been extensive previous study of the effective potential in QCD as well as certain extensions arising from the Gribov mass there still remain areas that require further investigation. Much of this thesis has relied on the use of the local composite operator (LCO) formalism as applied by Verschelde in [46]. Despite the formalism being set up in perturbation theory it has been successfully used to calculate non-perturbative quantities. This was first done in the Gross-Neveu model [46] [47], an asymptotically free quark model. In an asymptotically free theory like the Gross-Neveu one, certain types of quantities cannot be accounted for in the perturbative expansions, this is related to unstable vacuum conditions. The LCO method uses a local composite operator to perturb away from the unstable vacuum and sees some of the stable vacua in the context of perturbation theory [26]. This has been previously used to compute the mass gap for the Gross-Neveu model which was then checked against non-perturbative results for the same quantity. These were found to be in good agreement [92]. Recent work [39] has focused on extending this method to QCD and the Gribov extension. This method was used to compute the effective potential for QCD containing massive quarks and the operator $\frac{1}{2}A^{a\mu}A^a_\mu$ and this was extended to the Gribov-Zwanziger Lagrangian approch [48]. However, since then recent work on the lattice [49] has shown that the propagator may not behave as shown in the Gribov-Zwanziger Lagrangian as it is believed that the gluon propagator freezes to a non-zero value. As such it has been necessary to modify the Gribov-Zwanziger Lagrangian to fit the lattice data. The aim of this thesis is to use the Local Composite Operator (LCO) formalism to work out some non-perturbative result which could be used to compare to the recent lattice data. A potential method for fitting the lattice data comes from the inclusion of a BRST invariant operator constructed of localizing ghost fields. Before defining our operator we shall give a brief overview of BRST symmetries and how they can be used to define our new operator. In [50] [51] [52] it was shows that while the Lagrangian (1.9) satisfies a global gauge transformation, it loses this property after being gauge fixed. However it was shown that it does satisfy a new symmetry called the BRST symmetry. We define the set of nilpotent, infinitesimal BRST transforms s as

$$sA^a_\mu = -(D_\mu c)^a \tag{2.1}$$

$$sc^a = \frac{1}{2}f^{abc}c^b c^c \tag{2.2}$$

$$s\psi^{iI} = igc^a(T^a)^{IJ}\psi^{iJ} \tag{2.3}$$

$$s\bar{\psi}^{iI} = -ig\bar{\psi}^{iJ}(T^a)^{JI} \tag{2.4}$$

$$\delta\bar{c}^a = b^a \tag{2.5}$$

$$\delta b = 0 \tag{2.6}$$

where we have introduced an extra field b [53] which is a non-interacting auxiliary field added to the Lagrangian.

$$
\begin{aligned}
L^{\mathrm{QCD}} =& -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \\
&+ b^a\partial A^a_\mu + \alpha\frac{(b^a)^2}{2}
\end{aligned}
\tag{2.7}
$$

Another important feature of our BRST transformation s is that it is anti-commuting with the ghost field, this property will be useful later on. As the b auxiliary field is free it is possible to integrate this field out to return to the Lagrangian (1.9). By applying (2.1) - (2.6) it is possible to show (2.7) action remains invariant after applying the BRST transformation s

$$
\begin{aligned}
sL^{\mathrm{QCD}} =& s\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI}\right. \\
&\left.+ b^a\partial A^a_\mu + \alpha\frac{(b^a)^2}{2}\right] = 0
\end{aligned}
\tag{2.8}
$$

The condition, that s nilpotent means that applying the transformation s twice to any object results in zero

$$
s^2 = 0 \tag{2.9}
$$

This can be shown for all the transformations (2.1) - (2.6), but is most obvious for (2.5) and (2.6). The reason this is important is because we would like to introduce our new dimension two operator in such a way that it is BRST invariant. We start by defining the BRST transformation for the localizing ghost fields

$$s\phi^{ab}_{\mu} = \omega^{ab}_{\mu} \tag{2.10}$$

$$s\omega^{ab}_{\mu} = 0 \tag{2.11}$$

$$s\bar{\phi}^{ab}_{\mu} = 0 \tag{2.12}$$

$$s\bar{\omega}^{ab}_{\mu} = \bar{\phi}^{ab}_{\mu} \tag{2.13}$$

From equation (2.10) - (2.13) it can quite easily be seen that these are nilpotent. So to add a BRST invariant operator to (1.13) we define it in such a way that our operator is BRST exact. This will then make use of the nilpotent property to remain invariant in the Lagrangian. We shall define our operator such that

$$\mathcal{O}^{abcd} \;=\; s(\bar{\omega}^{ab}\,\phi^{cd}) \;= (s\bar{\omega}^{ab})\phi^{cd} - \omega^{ab}(s\phi^{cd}) = \;\bar{\phi}^{ab}\phi^{cd} \;-\; \bar{\omega}^{ab}\omega^{cd} \tag{2.14}$$

This means that when this is added to the Lagrangian (1.13) it will remain invariant under as the object we are adding it itself nilpotent.

$$\begin{aligned} s^2(\bar{\omega}^{ab}\,\phi^{cd}) &= s\left(\bar{\phi}^{ab}\phi^{cd} \;-\; \bar{\omega}^{ab}\omega^{cd}\right) \\ &= (s\,\bar{\phi}^{ab})\phi^{cd} + \bar{\phi}^{ab}(s\,\phi^{cd}) - (s\,\bar{\omega}^{ab})\omega^{cd} + \bar{\omega}^{ab}(s\,\omega^{cd}) \\ &= 0 \end{aligned} \tag{2.15}$$

Both (2.14) and (2.15) make use of s and the ghost field anti-commuting. With this we have shown how to effectively add our operator in a way which does not affect the BRST symmetry of the model. However, due to the complicated way the interaction fields behave due to the Gribov mass we shall not be working with the standard QCD propagators. From this point on we shall define our

BRST invariant operator as

$$\mathcal{O}^{abcd} = \bar{\phi}^{ab}\phi^{cd} \; - \; \bar{\omega}^{ab}\omega^{cd} \tag{2.16}$$

Including this operator changes the structure of the gluon propagators rather than just adding a mass for these fields as would be normal for this type of interaction in the Lagrangian. This is due to the complicated nature in which the Gribov mass γ^4 affects the propagators. So that we can have a clear grasp of how this modification occurs, we shall look at one of the possible outcomes of including this operator into our Lagrangian and how it affects the exact form of the propagators. For this we shall use the Gribov-Zwanziger Lagrangian (1.13) with the operator (2.16) added in, this was first called the Alterative Refined Gribov-Zwanziger Lagrangian. To begin with we shall be working with the field ϕ and $\bar{\phi}$ rather than their real and imaginary parts as it shall make things clearer. We choose the notation $\mu_{\mathcal{Q}}^2$ with the appropriate accompanying group theory for our operator from [38]. We shall begin with the quadratic part of our action

$$\begin{aligned} S_{ARGZ} = \int d^4x \, \bigg[& \frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \bar{c}^a \partial^2 c^a + \bar{\phi}^a_i \partial^2_\mu \phi^a_i \\ & - \bar{\omega}_i \partial^2_\mu \omega_i - \gamma^2 g f^{abc} A^a_\mu \phi^a_\mu - \gamma^2 g f^{abc} A^a_\mu \bar{\phi}^a_\mu \\ & + \mu_{\mathcal{Q}}^2 \delta^{ac}\delta^{bd}\left(\bar{\phi}^{ab}\phi^{cd} \; - \; \bar{\omega}^{ab}\omega^{cd}\right)\bigg] \end{aligned} \tag{2.17}$$

This is the normal quadratic term with our BRST operator. These can be split into three different terms, the mixed A_μ and ϕ^{ab} terms, $\bar{\omega}\omega$ and finally $\bar{c}c$, we shall concentrate on the mixed term as these shall be modified by our extra operator.

$$S_{ARGZ} = \int d^4x \left[\frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \bar{\phi}^a_i \partial^2_\mu \phi^a_i - \gamma^2 g f^{abc} A^a_\mu \phi^a_\mu \right.$$
$$\left. - \gamma^2 g f^{abc} A^a_\mu \bar{\phi}^a_\mu + \mu^2_Q \delta^{ac} \delta^{bd} \bar{\phi}^{ab} \phi^{cd} \right] \tag{2.18}$$

By considering the generating functional in the presence of a source we have

$$Z(J) = \int [dX] \exp\Big[-\frac{1}{2} \int d^d x d^d y X(x) A(x,y) X^T(y)$$
$$+ \int d^d x X(x) J(x) \Big] \tag{2.19}$$

With $A(x,y)$ given by (2.17) and

$$X(x) = \left[A^a_\mu \,,\, \xi^{ab}_\mu(x) \,,\, \rho^{ab}_\mu(x) \right] \tag{2.20}$$

For this to be possible, $A(x,y)$ must be a matrix of terms that correctly describe the mixing, in (2.20) the Bose ghosts are written in the real and imaginary parts. As the generating functional is a standard Gaussian integral, it is possible to write this as

$$Z(J) = (\det A)^{1/2} \exp\left[\frac{1}{2} \int d^d x d^d y J(x) A^{-1}(x,y) J(y) \right] \tag{2.21}$$

From this it is possible to calculate the propagators in Fourier space by taking the functional derivative of the generator and setting the source to zero. However for this to be possible we need the correct form of $A^{-1}(x,y)$. First we define

$$A(x,y) = \begin{pmatrix} -\partial^2 P_{\mu\nu} \delta^{ab} & -\gamma^2 g f^{abc} \delta_{\mu\kappa} & 0 \\ -\gamma^2 g f^{bkl} \delta_{\alpha\nu} & \left(\partial^2 + \mu^2_Q \delta^{ac} \delta^{bd}\right) \delta^{\alpha\kappa} \delta^{kl} \delta^{ij} & 0 \\ 0 & 0 & \left(\partial^2 + \mu^2_Q \delta^{ac} \delta^{bd}\right) \delta^{\beta\lambda} \delta^{sp} \delta^{tq} \end{pmatrix} \tag{2.22}$$

Here we have made use of the short hand

$$P_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right) \tag{2.23}$$

We require $A^{-1}(x, y)$ such that

$$A(x, y)A^{-1}(x, y) = \begin{pmatrix} \delta^{ac}\delta_{\mu\tau} & 0 & 0 \\ 0 & \delta^{\kappa j}\delta^{ly}\delta_{aw} & 0 \\ 0 & 0 & \delta^{sq}\delta^{lh}\delta_{\beta\chi} \end{pmatrix} \tag{2.24}$$

These terms come from the general construction of $A^{-1}(x, y)$ with all the terms unknown. While this procedure seems simple it must be conducted carefully as it involves many group theory contractions. Also the $\xi\,\xi$ terms needs further treatment, as in $A^{-1}(x, y)$ it is not in the same form as the rest of the propagator terms. This is done via the use of some partial fractioning. With the exact form of $A^{-1}(x, y)$, it is possible to perform the functional derivative necessary to get each of the propagators. So far this calculation has been carried out in momentum space. Before it is possible to perform the functional derivative on the generating functional it is necessary to transform into Fourier space. In our notation we record the propagators with our operator included.

$$\begin{aligned}
\langle A^a_\mu(p) A^b_\nu(-p)\rangle_{\mathcal{Q}} &= -\frac{\delta^{ab}(p^2 + \mu^2_{\mathcal{Q}})}{[(p^2)^2 + p^2\mu^2_{\mathcal{Q}} + C_A\gamma^4]} P_{\mu\nu}(p) \\
\langle A^a_\mu(p) \xi^{bc}_\nu(-p)\rangle_{\mathcal{Q}} &= \frac{i f^{abc}\gamma^2}{[(p^2)^2 + p^2\mu^2_{\mathcal{Q}} + C_A\gamma^4]} P_{\mu\nu}(p) \\
\langle A^a_\mu(p) \rho^{bc}_\nu(-p)\rangle_{\mathcal{Q}} &= 0 \\
\langle \xi^{ab}_\mu(p) \xi^{cd}_\nu(-p)\rangle_{\mathcal{Q}} &= -\frac{\delta^{ac}\delta^{bd}}{(p^2 + \mu^2_{\mathcal{Q}})}\eta_{\mu\nu} \;+\; \frac{f^{abe}f^{cde}\gamma^4}{(p^2 + \mu^2_{\mathcal{Q}})[(p^2)^2 + p^2\mu^2_{\mathcal{Q}} + C_A\gamma^4]} P_{\mu\nu}(p) \\
\langle \xi^{ab}_\mu(p) \rho^{cd}_\nu(-p)\rangle_{\mathcal{Q}} &= 0 \\
\langle \rho^{ab}_\mu(p) \rho^{cd}_\nu(-p)\rangle_{\mathcal{Q}} &= \langle \omega^{ab}_\mu(p) \bar{\omega}^{cd}_\nu(-p)\rangle_{\mathcal{Q}} \;=\; -\frac{\delta^{ac}\delta^{bd}}{(p^2 + \mu^2_{\mathcal{Q}})}\eta_{\mu\nu}
\end{aligned} \tag{2.25}$$

The interesting thing to notice about this set is the gluon propagator does not go to zero at zero momentum any more and further to this it does not have a massless pole so does not blow up at the origin. It can easily be seen that the Gribov-Zwanziger Lagrangian propagators can be recovered by setting $\mu_{\mathcal{Q}}^2 = 0$. However this does not describe the full picture, in (2.17) we added a rank 4 tensor to contract the indices in our operator. However this is an arbitrary choice, as we are free to choose this as we see fit. In principle it could be any suitable rank 4 colour tensor or any combination of them. This is the motivation behind [38] where several possible tensors were looked at. This step is the cause of most of the difficulties in the calculation as it greatly increases not only the number of terms but also adds numerous complex group interactions. The most general colour structure is given by

$$\mathcal{O} = \left[\mu_{\mathcal{Q}}^2 \delta^{ac}\delta^{bd} + \mu_{\mathcal{W}}^2 f^{ace} f^{bde} + \frac{\mu_{\mathcal{R}}^2}{C_A} f^{abe} f^{cde} + \mu_{\mathcal{S}}^2 d_A^{abcd} + \frac{\mu_{\mathcal{P}}^2}{N_A}\delta^{ab}\delta^{cd} + \mu_{\mathcal{T}}^2 \delta^{ad}\delta^{bc}\right] \mathcal{O}^{abcd} \tag{2.26}$$

This is the sum of all the six possible rank four colour tensors, this combination was first proposed in [38] and takes into account all the feasible tensors. The μ_i^2 terms are the method of choice for tracing which if any of the colour terms occur in the final propagators. It is also worth noting that by setting all the $\mu_i^2 = 0$ we still recover the pure Gribov-Zwanziger propagators. The subscript notation comes from previous work in this area. The pre-factors on the $\mathcal{R}$ and $\mathcal{P}$ channels are not obvious additions at the moment but stem from wanting to achieve a uniform structure for all the different propagators. We shall aim to use the sum of all the possible colour terms in the coming calculation.

2.2 Local Composite Operator

We shall aim for the first time to apply the LCO method to the alternative refined Gribov-Zwanziger Lagrangian in the presence of all the possible colour channels. While partial results for this have been computed before [54] there has yet to be a full study without imposing some choice of colour channel. The aim here is to use the most general combination of colour channel to see which if any appear in the final result. This can then be used to define how potential modification to the propagators should be carried out. Before we begin with defining the LCO formalism there are still some further aspects of the action (1.13) we need to first consider. We shall begin the construction of our effective action in the most general way possible, for this we shall follow the methods presented in [39] [48] [54] [41] [55], however there will be some slight difference to our method which shall become apparent. We start with the general form of the actions for the Gribov-Zwanziger Lagrangian in the presence of both the $(A^a_\mu)^2$ and $(\bar{\phi}^{ab}\phi^{cd} \ - \ \bar{\omega}^{ab}\omega^{cd})$ operators.

$$S = S_0 + S_\gamma + S_{A^2} + S_{\bar{\phi}\phi} \tag{2.27}$$

The individual elements of this actions are given as follows

$$\begin{aligned} S_0 = S_{\text{QCD}} \quad &+ \quad \int d^4x \left(\bar{\phi}^{ab\,\mu}\partial^\nu \left(D_\nu\phi_\mu\right)^{ab} + \bar{\omega}^{ab\,\mu}\partial^\nu \left(D_\nu\omega_\mu\right)^{ab} \right. \\ &+ \quad \left. g f^{abc}\partial^\nu\bar{\omega}^{ae}_\mu \left(D_\nu c\right)^b \phi^{ec\,\mu} \right) \end{aligned} \tag{2.28}$$

the usual QCD action is

$$S_{\text{QCD}} = \int d^4x \left(\frac{1}{4}F^a_{\mu\nu}F^{a\ \mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a\partial^\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \right) \tag{2.29}$$

and the terms relating to the Gribov mass are the following

$$S_\gamma = -\gamma^2 \int d^4x \left(f^{abc} A^{a\,\mu} \phi^{bc}_\mu + f^{abc} A^{a\,\mu} \bar{\phi}^{bc}_\mu + \frac{d N_A \gamma^2}{2g^2} \right) \tag{2.30}$$

In this we have included the gague fixing term α however in this thesis we shall be working in the Landau gague where $\alpha = 0$. Lastly there are the extra terms for the operators, each of this has a source term attached to it and coupling with a quadratic source of the form

$$S_{A^2} = \int d^4x \left(\frac{\tau}{2} (A^a_\mu)^2 - \frac{\zeta}{2} \tau^2 \right) \tag{2.31}$$

$$S_{\bar{\phi}\phi} = \int d^4x \left(-J \mathcal{O}^{abcd} + \rho J \tau + \chi J^2 \right) \tag{2.32}$$

where we define J as a vector of sources

$$\begin{aligned} J \;=\; & \Big[J_{\mathcal{Q}} \delta^{ac} \delta^{bd} \;+\; J_{\mathcal{W}} f^{ace} f^{bde} \;+\; \frac{J_{\mathcal{R}}}{C_A} f^{abe} f^{cde} \\ & +\; J_{\mathcal{S}} d_A^{abcd} \;+\; \frac{J_{\mathcal{P}}}{N_A} \delta^{ab} \delta^{cd} \;+\; J_{\mathcal{T}} \delta^{ad} \delta^{bc} \Big] \end{aligned} \tag{2.33}$$

We define this in such a way that it is straight forward to discern which particular operator and action we are going to be working with at the time. Here is where our formalism takes its first departure from [54] as we have chosen to include the quadratic J^2 term in this part of the action. This is because now that we are working with all the colour channels, there could be divergent terms arising that are proportional to J^2. Also included in this action is a mixed source term that would in principle be required if both of the operators were included together. We shall be considering each of these operators individually. Previous work [39] [48] calculated the effective potential for A^2_μ in QCD and in the Gribov-Zwanziger Lagrangian. While we are no longer working with just the gluon operator this work will provide an important starting point for our work as we shall start by reproducing some of the results from it. So far our action is the unrenormalised bare action and to continue we shall need to consider the following set of renormalization parameters

$$
\begin{array}{rclcrcl}
A_{\rm o}^{a\,\mu} &=& \sqrt{Z_A}\,A^{a\,\mu} \ , \ c_{\rm o}^a &=& \sqrt{Z_c}\,c^a \ , \ \bar{c}_{\rm o}^a &=& \sqrt{Z_c}\,\bar{c}^a \ , \\
\psi_{\rm o} &=& \sqrt{Z_\psi}\psi & , & \bar{\psi}_{\rm o} &=& \sqrt{Z_\psi}\bar{\psi} \\
g_{\rm o} &=& \mu^\epsilon Z_g\, g & , & \alpha_{\rm o} &=& Z_\alpha^{-1} Z_A\,\alpha \\
\xi_{\rm o}^a &=& \sqrt{Z_\xi}\,\xi^a & , & \bar{\xi}_{\rm o}^a &=& \sqrt{Z_\xi}\,\bar{\xi}^a \ , \\
\rho_{\rm o}^a &=& \sqrt{Z_\rho}\,\rho^a & , & \bar{\rho}_{\rm o}^a &=& \sqrt{Z_\rho}\,\bar{\rho}^a \ , \\
\omega_{\rm o}^a &=& \sqrt{Z_\omega}\,\omega^a & , & \bar{\omega}_{\rm o}^a &=& \sqrt{Z_\omega}\,\bar{\omega}^a \ ,
\end{array}
\tag{2.34}
$$

These are the standard ones for QCD as well as extra ones arising from the localizing ghost fields [48] these are related by Slavnov-Taylor identities [54]

$$
Z_c \ = \ Z_\rho \ = \ Z_\xi \ = \ Z_\omega \ = \ \frac{1}{Z_g\sqrt{Z_A}} \quad , \quad Z_\gamma \ = \ (Z_A Z_c)^{-1/4} \tag{2.35}
$$

In addition to these we shall require some extra renormalization relating to our extra source terms.

$$
\tau_o = Z_m^2 \tau \qquad \zeta_o = \bar{\mu}^{-2\epsilon}\frac{Z_\zeta}{Z_m^4}\zeta \tag{2.36}
$$

$$
J_o = Z_J^2 J \qquad \chi_o = \bar{\mu}^{-2\epsilon}\frac{Z_\chi}{Z_J^4}\chi \tag{2.37}
$$

The work of [39], [41] has computed (2.36) in QCD, and [54] has worked out (2.37) using only one colour channel in the Gribov-Zwanziger Lagrangian. We shall begin by looking at the coupling of the source term ζ in pure QCD. This will allow us to construct the LCO formalism with known results to check our method before looking at the unknown case for the operator $(\bar{\phi}^{ab}\phi^{cd} - \bar{\omega}^{ab}\omega^{cd})$. We shall start by taking A^2_μ in the following action $S = S_{QCD} + S_{A^2}$

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i\bar{\psi}^{iI} \not{D} \psi^{iI} + \bar{c}^a \partial^\mu D_\mu c^a + \frac{\tau}{2} (A^a_\mu)^2 - \frac{\zeta}{2} \tau^2 \right) \quad (2.38)$$

This is the action presented in [39] and rewriting this as

$$S = S_{o\,QCD} + \int d^4x \left(\frac{1}{2} \tau_o A^a_{o\,\mu} A^{\mu\,a}_o - \frac{1}{2} \zeta_o \tau^2_o \right) \quad (2.39)$$

where S_{oQCD} is the bare QCD action. The addition of the τ^2 term as originally proposed in [41] was a means of removing the quadratic divergences arising from the coupling of the source term to the operator. However this does not solve all of the problems present with this action. The method used in [56] the new source term was coupled to a simple field and not a composite operator and further to this there is the quadratic source term. This can be resolved by introducing a Hubbard-Stratonovich transformation

$$1 = \int [d\sigma] \exp \left(\int d^4x \left[- \left(a\sigma + bA^a_{o\,\mu} A^{\mu\,a}_o + c\tau \right)^2 \right] \right) \quad (2.40)$$

This transformation relies on the properties of the Gaussian integral. The terms a, b and c here are arbitrary coefficients which will depend on the divergences arising from the τ^2 term. At this point it is best to continue with the renormalization of the action (2.39) before defining these terms. Our action now becomes

$$S = S_{QCD} + \int d^4x \left(\frac{1}{2} Z^2_m Z_A \tau A^a_\mu A^{\mu\,a} - \frac{1}{2} \bar{\mu}^{-2\epsilon} Z_\zeta \zeta \tau^2 \right) \quad (2.41)$$

The method used in [39] to work out Z_ζ is interesting to us, as in the case of the operator $(\bar{\phi}^{ab}\phi^{cd} - \bar{\omega}^{ab}\omega^{cd})$ its renormalization has already been worked out to

3-loops [38]. Z_ζ was worked out by using the fact that the divergences cancelled by it are proportional to τ^2, so it is possible to work out the renormalization by computing all the τ^2 terms to a required loop order. This can be done by considering a two point function of the source. This is possible as, by coupling the source to the operator, $\tau(A^a_\mu)^2$, leads to a source-operator interaction in the Feynman rules. This allows the problem to be viewed as the set of diagrams coming from a 2-point function of massless propagators with one external momentum flowing through. As such it is possible to use the FORM [57] version of MINCER [43] to help perform the calculation required. As the operator renormalization is already known we require only the J^2 divergent component as this is the calculation of interest to us here. Despite the Feynman rules changing for the two different setups the method used is exactly the same, so before continuing we shall re-derive the result of [39]. Re-defining Z_ζ

$$Z_\zeta \zeta = \zeta + \delta\zeta \tag{2.42}$$

this is in line with [41] and $\delta\zeta$ is the counter term arising from the divergences in the source two point function. To find the explicit value of this term, we have to compute all the necessary Feynman diagrams. For the A^2_μ there is a tree term, one 1-loop diagram, eight 2-loop diagrams and 154 3-loop diagrams that contribute to the source two point function. These have been first drawn using QGRAF [58] before being passed onto MINCER to reduce to masters integrals then evaluate. However this is not a straight forward procedure, a problem arises from our QGRAF setup that needs addressing before we can continue. Because of the way which QGRAF draws the diagrams for our given interactions the number of diagrams is more than is required for the two point function. The reason for this is as well as having a source leg as the ingoing and outgoing terms QGRAF has also added internal source propagators which should not be present in our final result. While in principle it would be possible to simply delete these from the QGRAF files generated this would be a time consuming process and not without potential error. This is especially important while dealing with the 3-loop case as the number of diagrams increases considerably. To solve this it is possible to enter a suitable substitution into FORM which sets all internal source propagators to zero, this is done by looking at the position of the source terms in the output of QGRAF, this makes it possible to set these to zero while leaving the incoming and outgoing legs non-zero. With this the number of contributing diagrams at 2-loops drops to six from the eight originally produced by QGRAF. With this in mind it is possible to calculate the final version of $\delta\zeta$ to 3-loops.

$$\begin{aligned}
\delta\zeta \;=\; & N_A \left[-\frac{3}{2\epsilon} + \left(\left(\frac{35}{8} C_A - 2T_F N_f \right) \frac{1}{\epsilon^2} + \left(\frac{8}{3} T_F N_f - \frac{139}{12} C_A \right) \frac{1}{\epsilon} \right) \frac{g^2}{16\pi^2} \right. \\
& + \left(\left(\frac{73}{6} T_F N_f C_A - \frac{8}{3} T_F^2 N_f^2 - \frac{665}{48} C_A^2 \right) \frac{1}{\epsilon^3} \right. \\
& + \left(\frac{32}{9} T_F^2 N_f^2 - 4 T_F N_f C_F - \frac{535}{18} T_F N_f C_A + \frac{6629}{144} C_A^2 \right) \frac{1}{\epsilon^2} \\
& + \left(\frac{40}{27} T_F^2 N_f^2 + \left(\frac{115}{3} - 32\zeta(3) \right) T_F N_f C_F + \left(\frac{4381}{216} + 32\zeta(3) \right) T_F N_f C_A \right. \\
& \left. \left. \left. - \left(\frac{71551}{864} + \frac{231}{32} \zeta(3) \right) C_A^2 \right) \frac{1}{\epsilon} \right) \frac{g^4}{(16\pi^2)^2} \right] + \mathcal{O}(g^6) \, . \qquad (2.43)
\end{aligned}$$

This result agrees with [39] as would be expected by using the same method correctly. With this complete, this is as far as we need to progress with our reworking of the A_μ^2 operator, as we have successfully been able to reproduce the previous results and it is now possible to apply this method to our new operator $(\bar{\phi}^{ab}\phi^{cd} - \bar{\omega}^{ab}\omega^{cd})$. As we are just considering the effect this has without any mixing, we now take the action $S = S_0 + S_\gamma + S_{\bar{\phi}\phi}$ with $\tau = 0$ to remove the mixing term. Similar to (2.38) our action now becomes.

$$
\begin{aligned}
S = \int d^4x \Big[& \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i\bar{\psi}^{iI} \not{D} \psi^{iI} + (\bar{c}^a \partial^\mu D_\mu c^a) \\
& + \bar{\phi}^{ab\,\mu} \partial^\nu (D_\nu \phi_\mu)^{ab} + \bar{\omega}^{ab\,\mu} \partial^\nu (D_\nu \omega_\mu)^{ab} + g f^{abc} \partial^\nu \bar{\omega}^{ae}_\mu (D_\nu c)^b \phi^{ec\,\mu} \\
& - \gamma^2 \left(f^{abc} A^{a\,\mu} \phi^{bc}_\mu + f^{abc} A^{a\,\mu} \bar{\phi}^{bc}_\mu + \frac{d N_A \gamma^2}{2g^2} \right) \\
& - \Big[J_{\mathcal{Q}} \delta^{ac} \delta^{bd} \; + \; J_{\mathcal{W}} f^{ace} f^{bde} \; + \; \frac{J_{\mathcal{R}}}{C_A} f^{abe} f^{cde} \\
& + \; \mu^2_{\mathcal{S}} d_A^{abcd} \; + \; \frac{J_{\mathcal{P}}}{N_A} \delta^{ab} \delta^{cd} \; + \; J_{\mathcal{T}} \delta^{ad} \delta^{bc} \Big] + \chi J^2 \Big]
\end{aligned}
\tag{2.44}
$$

In this case the operator renormalization has previously been worked out to 2-loops in [38] which leaves us just the quadratic source renormalization requiring the counter terms as before.

$$
S = S_0 + S_\gamma + \int d^4x \left(Z_J^2 Z_{\mathcal{O}} J (\bar{\phi}^{ab} \phi^{cd} \; - \; \bar{\omega}^{ab} \omega^{cd}) - \bar{\mu}^{-2\epsilon} Z_\chi \chi J^2 \right) \tag{2.45}
$$

Despite now working in the Gribov-Zwanziger Lagrangian it is still possible to treat our renormalization using MINCER which works with massless propagators, this is because the divergent part of this calculation will stem from the momentum structure of the propagators which is similar to the massless cases. With a slight adjustment it is possible to use QGRAF to generate the new diagrams that we require. There are four at 1-loop, fifty eight at 2-loop and two thousand eight hundred and sixty at 3-loop. It thus becomes clear why it was necessary to automate the process of setting the internal source propagators to zero. The surprising result from this calculation is that up to 3-loops the quadratic source is completely finite, this is equivalent to setting $\delta\chi = 0$. This agrees in part with the result of [54] which states that the operator is finite in the one colour channel they have chosen to work in. Our result extends this to all possible colour channel at 3-loops. This means that without a counter term there should be no divergent terms arising from our sum of diagrams for the action other than those required to cancel divergences arising from γ^4 renormalization Z_γ. However this does present us with a small problem, the method of [39] relied on this counter term to help define the coupling parameter χ and its renormalization term. However in principle it is still possible that there could be a finite renormalization similar to those used in Section 4.4 coming from the LCO method. We shall now have to use alternative method to define this coupling. At this point in previous work the quadratic source was dropped from the action, however, as the quadratic source is finite does not mean this term is zero so in principle should remain in action. It is possible to rework the method of [41] without a counter term, to do this we shall try to construct the coupling term χ and only set this to zero if it can be shown that this is the only method to satisfy the LCO conditions.

Start off by taking the second term (2.37) and taking the derivative of its square with respect to the mass scale μ

$$0 = -2\epsilon + \mu \frac{\mathrm{d}\ln(Z_\chi)}{\mathrm{d}\mu} - 4\mu \frac{\mathrm{d}\ln(Z_J)}{\mathrm{d}\mu} + \frac{\mu}{\chi}\frac{\mathrm{d}\chi}{\mathrm{d}\mu} \tag{2.46}$$

This makes use of the fact that the unrenormalized terms are not dependent on the mass scale while the renormalised ones are. We can now define the anomalous dimension of our source term as [59]

$$\gamma_J(g) = -\frac{1}{2}\frac{\mathrm{d}\ln(Z_J)}{\mathrm{d}\mu} \tag{2.47}$$

This can be used to rewrite (2.46) in the following way [60]

$$\mu \frac{\mathrm{d}\chi}{\mathrm{d}\mu} = 2\epsilon\chi - 8\gamma_J(g^2)\chi - \mu\chi \frac{\mathrm{d}\ln(Z_\chi)}{\mathrm{d}\mu} \tag{2.48}$$

Now in line with (2.42) but for our new source

$$Z_\chi \chi = \chi + \delta\chi \tag{2.49}$$

however, as $\delta\chi = 0$ since the source is finite, this then becomes

$$Z_\chi = 1 \tag{2.50}$$

This would be expected from a term that did not need corrections to remove divergent terms. Now looking at the left hand side of (2.48) and applying the chain rule for differentiation

$$\mu\frac{\mathrm{d}\chi}{\mathrm{d}\mu} = \mu\frac{\mathrm{d}\chi}{\mathrm{d}g^2}\frac{\mathrm{d}g^2}{\mathrm{d}\mu} = 2\beta(g^2)\frac{\mathrm{d}\chi}{\mathrm{d}g^2} \tag{2.51}$$

Where $\beta(g^2)$ is the QCD beta function. Combining (2.50) (2.51) into (2.48) gives

$$2\beta(g^2)\frac{\mathrm{d}\chi}{\mathrm{d}g^2} + 8\gamma_J(g^2)\chi = 0 \tag{2.52}$$

The term $\epsilon\chi$ has also been removed as χ is finite and therefore will have no affect in the limit $\epsilon \to 0$ This result is different from the case of [39] where this expression had extra terms resulting from the source renormalization. We shall briefly consider this version now as it is the more general form of (2.52).

$$\beta(g^2)\frac{\mathrm{d}\zeta}{\mathrm{d}g^2} - 2\gamma_m(g^2)\zeta - \delta(g^2) = 0 \tag{2.53}$$

Here $\delta(g^2)$ is a function of the counter term $\delta\zeta(g^2)$. In [39] it was possible to construct ζ so that it takes the form.

$$\zeta(g^2) = \frac{c_{-1}}{g^2} + c_0 + c_1 g^2 + c_2 g^4 + \mathcal{O}(g^6) \tag{2.54}$$

Now it is possible to define $\beta(g^2)$, $\gamma_m(g^2)$ and $\delta(g^2)$ as function of g^2 in terms of their own coefficients.

$$\beta(g^2) = \beta_0 g^2 + \beta_1 g^4 + \beta_2 g^6 + \mathcal{O}(g^8) \tag{2.55}$$

$$\gamma_m(g^2) = \gamma_0 + \gamma_1 g^2 + \gamma_2 g^4 + \mathcal{O}(g^6) \tag{2.56}$$

$$\delta(g^2) = \delta_0 + \delta_1 g^2 + \delta_2 g^4 + \mathcal{O}(g^6) \tag{2.57}$$

Now with equations (2.54) - (2.57) it is possible to use (2.53) to write the coefficient of $\zeta(g^2)$.

$$c_{-1} = \frac{\delta_0}{(-\beta_0 - 2\gamma_0)} \tag{2.58}$$

$$c_0 = \frac{-\beta_1\, c_{-1} - \delta_1 - 2\gamma_1\, c_{-1}}{2\gamma_0} \tag{2.59}$$

and finally

$$c_1 = \frac{-\beta_2\, c_{-1} - \delta_2 - 2\gamma_1\, c_0 + 2\gamma_2\, c_{-1}}{2\gamma_0 - \beta_0} \tag{2.60}$$

From this it is possible to see that these equations will only be non-zero for non-zero values of $\delta(g^2)$. This is the result used to work out the values of the coupling in [39], however, in our case we have $\delta(g^2) = 0$ so it is not possible to find χ that is a function of g^2 in (2.52) that is non-zero. Fortunately the zero solution is valid, this is the result derived in [54] via the use of Ward identities. We have shown this result is accurate to 3-loops in the LCO formalism and satisfies the renormalization group equation for the action $W(J)$. This now presents us with a problem, as we have shown that $\chi^2 = 0$ in (2.44) we are now left unable to use a Hubbard-Stratonovich transformation to work out the effective potential. This is because without the quadratic source it is now possible to perform a transformation where we have a linear source term coupled to an auxiliary field σ. However, it is still possible to use the method of summing diagrams to compute the one-loop correction to $W(J)$

which include all the colour channels. This was one of the principle results produced in [54] and [38] where the authors were faced with a similar problem but using different methods.

2.3 Diagrams

It is now possible to compute the 1-loop correction to the action (2.44) as in the previous section we have fixed the value of the coupling term $\chi^2 = 0$. To compute the 1-loop correction we will use the method of summing the infinite number of Feynman diagrams suggested by [56], [40]. With a non-zero value of χ^2 it has been possible to perform a suitable inversion to work out the full effective potential however we can still investigate how including the extra colour channel affect the 1-loop correction. We define our 1-loop correction in the standard notation

$$\Gamma^1 = \int d^4\, x \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^{(n)}(k, ..., k_n)|_{k_1 = ... = k_n = 0} \tag{2.61}$$

This involves taking a infinite number of Feynman diagrams with zero incoming momentum with an increase number of incoming legs at 1-loop for the to give the one loop correction. There is also a tree level term contribution arising from our action, the tree term stems from the renormalized Gribov mass. This sum of diagrams is shown as the following

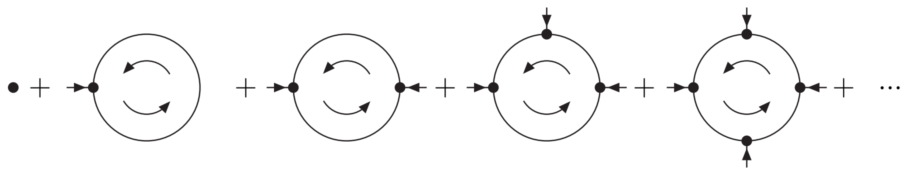

Figure 2.1: Sum of diagrams that form the one loop correction to the action $W(J)$

The ... here representing the infinite sum of diagrams, the aim shall be to find logarithmic terms that can be successfully expanded to form the infinite series. In the previous work [56] the trace of the logarithm of the determinant of the propagator matrix was used to compute the infinite series of diagrams required for the correction for the action. While in principle this method would still be usable in our action, we have chosen a method of automating the calculation so that it can cope with the large number of group contractions arising from the different colour channels. While in principle it would be possible to use this method to compute our correction we wish to perform an independent calculation to check the previous work and see what our changes make. While many of these terms would be zero while taking the trace of the determinant due to the group theory associated with them there would still be enough terms to make the calculation highly non-trivial, further to this there is not a straightforward way to automatically generate this matrix. This would mean that for each entry the group theory would more than likely need to be checked by hand and while the symmetry properties of the matrix would make this process slightly easier, it would not eliminate all of the problems. Finally the difficulty of error checking this approach is the last reason for us discounting it as a viable approach for our calculation. With the normal approach no longer available to us we shall have to compute the value of Feynman diagrams with increasing number of legs, luckily we already have many of the tools we shall require from calculating the renormalization of the source term. We shall be using our LCO modified action (2.44) to compute the one loop correction for the action in the presences of our BRST invariant operator. We shall use the fact that our operator is coupled to our source in a similar way to what we did for the renormalization, however, this time we shall have an increasing number of incoming legs with zero momentum. This is possible as we have shown that this will introduce no extra divergent terms into our action. As our source is coupled to the localizing ghost as described in (2.44) each leg order shall include all possible permutations of this before adding additional legs to the diagrams. As we are only computing the 1-loop correction to the action, we shall in fact only be considering loops of localizing ghost while there is an $A^a_\mu \xi^{cd}_\nu$ propagator the fact there is no $\sigma A^a_\mu A^b_\nu$ coupling in our action prevents this from occurring. If we consider the action with $\tau \neq 0$ then we would have to consider this mixing. For the purpose of doing this we shall begin by drawing our Feynman diagrams once again in QGRAF with the incoming field being our source J. For the purpose of making sure that the final sum accurately represents the sum of infinite diagrams we shall produce diagrams up to and including six incoming legs, there are two reasons for going to this high order in leg number. Firstly it will check that the method we are using is consistent at a high number of legs as well as low. This is important due to the large number of colour group theory contraction that shall need to be preformed. Secondly it allows our logarithm term to check to a high leg order that it accurately describes the Feynman diagrams and does not diverge from the results as they get more complicated. The trade off for this is that going up to six incoming legs will require a large number of Feynman diagrams.

Number of Legs	Number of Feynman Diagrams
1	3
2	4
3	10
4	54
5	408
6	3960

Table 2.1: Number of Feynman diagrams required to compute the 1-loop correction to the action

As can be seen from Table 2.1 the number of Feynman diagrams increases significantly at 5 and 6 incoming legs. As we are only working at 1-loop and with all external momentum set to zero we shall only have one loop momentum to contend with in our Feynman diagrams however with each increase in leg number we shall have to deal with an increasing amount of group colour theory.

2.4 Group Theory

Before we can progress any further we will need to deal with group thoery contractions, the group theory arises from the generators and structure constant of QCD. There have been several studies into the group theory that arise in Feynman diagrams previously [61] [62] [63] as well as other works on the subject [38] [64], we shall take much of our notation and conventions from these. Before going into the detail of the different contractions we shall require, let us first look at the way the colour structure is set out over our Feynman diagrams.

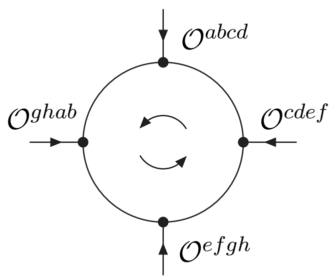

Figure 2.2: 4-leg diagram showing contracted colour indices on the operator

From this it is clear to see that we shall be working with fully contracted set of colour indices, this is important as without it we would not be working with a scalar object in the end. As each pair of operators shares a pair of indices it is necessary to include all the possible group contraction to deal with this in a systematic manner. Taking a 2-leg diagram as an example

$$\Big[\mu^2_{\mathcal{Q}}\delta^{ac}\delta^{bd} \; + \; \mu^2_{\mathcal{W}}f^{acq}f^{bdq} \; + \; \frac{\mu^2_{\mathcal{R}}}{C_A}f^{abq}f^{cdq} \; + \; \mu^2_{\mathcal{S}}d_A^{abcd} \\ + \; \frac{\mu^2_{\mathcal{P}}}{N_A}\delta^{ab}\delta^{cd} \; + \; \mu^2_{\mathcal{T}}\delta^{ad}\delta^{bc}\Big]\Big[\mu^2_{\mathcal{Q}}\delta^{ca}\delta^{db} \; + \; \mu^2_{\mathcal{W}}f^{caq}f^{dbq} \\ + \; \frac{\mu^2_{\mathcal{R}}}{C_A}f^{cdq}f^{abq} \; + \; \mu^2_{\mathcal{S}}d_A^{cdab} \; + \; \frac{\mu^2_{\mathcal{P}}}{N_A}\delta^{cd}\delta^{ab} \; + \; \mu^2_{\mathcal{T}}\delta^{cb}\delta^{da}\Big] \tag{2.62}$$

From (2.62) it can be seen that many of the contractions contain products of δ's. That shall not prove to be a problem, however there are a few terms here that shall require careful treatment. We shall write these in a general form here as in diagrams with more legs these term may not be fully contracted by themselves. Before turning our attention to any new terms arising from our contraction some of the more complex terms arising from the totally symmetric tensor d_a^{abcd} have been dealt with previously [64].

$$\begin{aligned} f^{ape}f^{bqe}d_A^{pqcd} \;&=\; b_1\delta^{ab}\delta^{cd} \; + \; b_2\left(\delta^{ac}\delta^{bd} \; + \; \delta^{ad}\delta^{bc}\right) \\ &+ \; b_3\left(f^{ace}f^{bde} \; + \; f^{ade}f^{bce}\right) \; + \; b_4 d_A^{abcd} \end{aligned} \tag{2.63}$$

$$
\begin{aligned}
d_A^{abpq} d_A^{pqcd} &= a_1 \delta^{ab}\delta^{cd} + a_2 \left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) \\
&+ a_3 \left(f^{ace} f^{bde} + f^{ade} f^{bce}\right) + a_4 d_A^{abcd} \qquad (2.64)
\end{aligned}
$$

These were worked out using the colour package for FORM which is based on the work [63]. While we shall not re-derive this result here, we shall use the methods in [63] to solve some of the other cases arising from our contraction. The coefficients a_1, a_2, a_3 and a_4 plus b_1, b_2, b_3, and b_4 can be worked out for different groups. However sticking to $SU(N_c)$ yields [64]

$$
\begin{aligned}
a_1 &= \frac{7}{12} N_c^2 \\
a_2 &= -\frac{1}{24} N_c^2 \\
a_3 &= \frac{7}{108} N_c (N_c^2 - 9) \\
a_4 &= \frac{1}{9} (N_c^2 + 9) \qquad (2.65)
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= \frac{1}{2} N_c \\
b_2 &= -\frac{1}{4} N_c \\
b_3 &= \frac{1}{36} N_c (N_c^2 + 18) \\
b_4 &= \frac{1}{3} N_c \qquad (2.66)
\end{aligned}
$$

In principle it would be possible to keep the results in an arbitrary group configuration, however doing so makes it impossible to reduce the sum of diagrams to a simple logarithm, this reduction is seen as a reasonable trade off for this to happen. Further to this if we use the Jacobi identity

$$
f^{abe} f^{pqe} = f^{ape} f^{bqe} - f^{bpe} f^{aqe} \qquad (2.67)
$$

and combine this with the result of (2.63) it is possible to see

$$f^{abe} f^{pqe} d_A^{pqcd} = 0 \tag{2.68}$$

Next if we use the identity

$$f^{acd} f^{bcd} = C_A \delta^{ab} \tag{2.69}$$

it is possible to rewrite

$$f^{abe} f^{pqe} f^{pqm} f^{cdm} = C_A \delta^{em} f^{abe} f^{cdm} = C_A f^{abe} f^{cde} \tag{2.70}$$

Further to this it is possible to use

$$f^{amn} f^{bmp} f^{cnp} = \frac{1}{2} C_A f^{abc} \tag{2.71}$$

to write

$$f^{ape} f^{bqe} f^{pqm} f^{cdm} = \frac{1}{2} C_A f^{abe} f^{cde} \tag{2.72}$$

Finally the last set of contraction that need defining is

$$f^{ape} f^{bqe} f^{cpe} f^{dqe} = d_A^{abcd} - \frac{1}{6} \left(2 f^{abe} f^{cdq} - f^{ace} f^{bdq} \right) \tag{2.73}$$

This comes from rewriting $f^{ape} f^{bqe} f^{cpm} f^{dqm}$ as the trace of generators T_A^i, this is possible as the structure constants in the adjoint representation are related to the generators by

$$(T_A)^a_{bc} = -i f^{abc} \tag{2.74}$$

This is also the source of the identity in equation (2.63), with this transformation it is possible to read off the result for $\mathrm{Tr}[T_A^a T_A^b T_A^c T_A^d]$ from the results given in [63]. This completes the identites we require in order to reduce the colour group theory terms we shall generate from our diagrams. The only remaining thing to consider is the case of repeated indices in f and d_A, these are given by

$$f^{aab} = 0 \tag{2.75}$$

$$d_A^{abcb} = \frac{5}{6} C_A^2 \delta ab \tag{2.76}$$

As a final note about the contraction here, it should be noted that f is anti-symmetric and d_A is totally symmetric so not only will we have to deal with the relations presented by their permutations, this is where a considerable amount of our technical difficulty arises. From our setup it is clear that adding another leg to our diagrams will add another set of group terms, however this shall be contracted with the incoming leg next to it. This will reduce the indices but increase the number of terms in our expression, this is repeated until we are left with only scalar terms. Luckily for us MINCER is ideally suited to this type of calculation, it was possible to perform this set of contractions for each of the diagrams before fixing the group to $SU(N_c)$ at the end. By doing this we fix the value of the quadratic Casimir

$$\begin{aligned} N_A &= N_c^2 - 1 \\ C_A &= N_c \end{aligned} \tag{2.77}$$

These relations are enough to reduce many of the algebraic relations which are built up over the course of summing the 1-loop diagrams, however they are not enough on their own to completely reduce our diagrams to something which is easy to turn into a series of logarithms. The final part of this process involves a number of partial fraction which help us to transform between propagator-like terms. The first step for our propagator is to reduce their exponents to that of single powers.

$$\frac{1}{((k^2)^2 + C_A\gamma^4)^n} = \left(\frac{1}{((k^2)^2 + C_A\gamma^4)}\right)^n \tag{2.78}$$

While on the surface this change does not achieve anything in terms of making our calculation simpler, it does help improve the speed at which FORM is able to handle the propagator like term. The reason for this is that each of the terms on the left hand side is treated as independent, for different values of n. Whereas on the right hand side we are working with up to n copies of the same object. The result of this is that instead of FORM having to perform the calculation with lots of different terms, potentially reducing the performance considerably it is able to work with only one term. Furthermore it allows all further relations to be written in the most general manner possible without the need to specify multiple copies of the same relations. This step is also preformed with the other propagator like term $\frac{1}{k^2}$. Of the relations we have used the first we define is

$$\frac{C_A\gamma^4}{(k^2)^2 + C_A\gamma^4} = 1 - \frac{(k^2)^2}{(k^2)^2 + C_A\gamma^4} \tag{2.79}$$

This is a useful relation as it allows for a factor of the integration momentum to be cancelled from our integral without the need to evaluate it. It can also be used as a relation to simplify the following terms

$$\frac{1}{(k^2)^2}\frac{C_A\gamma^4}{(k^2)^2 + C_A\gamma^4} = \frac{1}{(k^2)^2} - \frac{1}{(k^2)^2 + C_A\gamma^4} \tag{2.80}$$

It is also necessary to perform this reduction with higher order exponents in the integral

$$C_A\gamma^4\left(\frac{1}{(k^2)^2 + C_A\gamma^4}\right)^2 = \frac{1}{(k^2)^2 + C_A\gamma^4}\left(1 - \frac{(k^2)^2}{(k^2)^2 + C_A\gamma^4}\right) \tag{2.81}$$

It is also possible to perform a similar relation without the $C_A\gamma^4$ in the numerator in (2.79)

$$\frac{1}{(k^2)^2}\frac{1}{(k^2)+C_A\gamma^4} = \frac{1}{C_A\gamma^4}\left(\frac{1}{(k^2)^2} - \frac{1}{(k^2)+C_A\gamma^4}\right) \tag{2.82}$$

With these it is possible to reduce all the higher terms that we do not wish to appear in our final sum of diagrams, just as importantly it is possible to write all the propagator like terms as function of either $\frac{1}{k^2}$ or $\frac{1}{(k^2)+C_A\gamma^4}$ without any loose terms of $C_A\gamma^4$ appearing. This also greatly helps when it trying to find a function for summing sets of diagrams as will be seen in the next section.

2.5 Results

We now have everything in place to compute the sum of Feynman diagrams of increasing number of legs which include the sum of all the possible colour channels. Before we begin trying to create a series to describe our result we present a subsection of the results here. This is only a small part of the full result and covers only up to, and including two incoming legs. This section does not include any of our tree terms which are based on the classical part of the action. It is can be seen that there is a underlying pattern to the terms that are generated by FORM summing diagrams.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^{(n)}(k, ..., k_n)|_{k_1=...=k_n=0} = \\
&\quad + [d-1] J_{\mathcal{T}}^2 \left(+\frac{1}{2} \left(\frac{1}{k^2} \right) N_A - \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right) (k^2) N_A \right) \\
&\quad + [d-1] J_{\mathcal{T}}^4 \left(+\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_A - \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{R}}^2 \left(-\frac{1}{2} \left(\frac{1}{k^2} \right) N_A + \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right) (k^2) N_A \right) \\
&\quad + [d-1] J_{\mathcal{R}}^2 J_{\mathcal{T}}^2 \left(-\frac{1}{2} \left(\frac{1}{k^2} \right)^2 N_A + \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{R}}^4 \left(+\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_A - \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{W}}^2 \left(-\frac{1}{4} \left(\frac{1}{k^2} \right) N_c N_A + \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right) (k^2) N_c N_A \right) \\
&\quad + [d-1] J_{\mathcal{W}}^2 J_{\mathcal{T}}^2 \left(-\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_c N_A + \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_c N_A \right) \\
&\quad + [d-1] J_{\mathcal{W}}^2 J_{\mathcal{R}}^2 \left(+\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_c N_A - \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_c N_A \right) \\
&\quad + [d-1] J_{\mathcal{W}}^4 \left(+\frac{1}{16} \left(\frac{1}{k^2} \right)^2 N_c^2 N_A - \frac{1}{16} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_c^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{Q}}^2 \left(-\frac{1}{2} \left(\frac{1}{k^2} \right) N_A + \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right) (k^2) N_A \right) \\
&\quad + [d-1] J_{\mathcal{Q}}^2 J_{\mathcal{T}}^2 \left(-\frac{1}{2} \left(\frac{1}{k^2} \right)^2 N_A + \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{Q}}^2 J_{\mathcal{R}}^2 \left(+\frac{1}{2} \left(\frac{1}{k^2} \right)^2 N_A - \frac{1}{2} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + [d-1] J_{\mathcal{Q}}^2 J_{\mathcal{W}}^2 \left(+\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_c N_A - \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_c N_A \right) \\
&\quad + [d-1] J_{\mathcal{Q}}^4 \left(+\frac{1}{4} \left(\frac{1}{k^2} \right)^2 N_A - \frac{1}{4} \left(\frac{1}{(k^2)^2 + C_A \gamma^4} \right)^2 (k^2)^2 N_A \right) \\
&\quad + \mathcal{O}(J_i^6)
\end{aligned}
\tag{2.83}
$$

As there are two different propagator like terms here we shall try and construct a series expansion that includes two different logarithmic terms. The basic series we shall try and fit to our results is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \mathcal{O}(x^7) \qquad (2.84)$$

Looking at (2.83) it is apparent that a pre-factor of $(d-1)N_A$ will be required at all times. Also looking at the μ_i^2 in (2.83) which are the first order 1-leg terms, these all carry a factor of $\frac{1}{2}$. As the first order term in the expansion (2.84) do not have a pre-factor from the expansion this will need to be added as a global pre-factor. This expansion (2.84) needs to go to this high an order as we are working with up to six legs and each term represents the adding of an extra leg to our sum of diagrams. In contrast to previous results [54] it turns out that we require two logarithm terms to properly sum the diagrams required for our 1-loop correction. The series obtained by the sum of graphs with zero momentum external legs

$$\sum_{n=1}^{\infty} \frac{1}{n!}\Gamma^{(n)}(k,\ldots,k_n)|_{k_1=\ldots=k_n=0}$$
$$= \frac{N_A(d-1)}{2}\left[\ln\left(1+\frac{k^2M^2}{((k^2)^2+C_A\gamma^4)}\right) - \ln\left(1+\frac{M^2}{k^2}\right)\right] \qquad (2.85)$$

where

$$M^2 = \left[J_{\mathcal{R}} + J_{\mathcal{Q}} - J_{\mathcal{T}} + \frac{1}{2}C_A\, J_{\mathcal{W}}\right] \qquad (2.86)$$

For the most part we shall treat M^2 as a object itself without putting in the exact value unless it is necessary in the final result. From this it is possible to see that not all of the possible colour terms are present in the final result, also it would appear this term is different from the result given in [54], however looking closely at these terms

$$\ln\left(1+\frac{M^2}{k^2}\right) = \ln\left(k^2+M^2\right) - \ln\left(k^2\right) \tag{2.87}$$

$$\ln\left(1+\frac{k^2M^2}{((k^2)^2+C_A\gamma^4)}\right) = \ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right) - \ln\left((k^2)^2+C_A\gamma^4\right) \tag{2.88}$$

This is a similar to the results in [54] which were given in terms of $\ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right)$ and $\ln\left(k^2+M^2\right)$, taking the former of these and expanding it in powers of M^2

$$\ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right) = \ln\left(k^2+M^2\right) + \frac{k^2M^2}{(k^2)^2+k^2M^2+C_A\gamma^4} - \frac{1}{2}\left(\frac{k^2M^2}{(k^2)^2+k^2M^2+C_A\gamma^4}\right)^2 + \mathcal{O}(M^6) \tag{2.89}$$

This is exactly the result as derived in [54] however by a totally independent method. However there is a difference arising from the combination of colour channels used which is the most significant outcome of this calculation. Before going on to explain the significance of these results it is first necessary to check that this action is finite, this is an important check as without it there may be possible errors in our formalism. With our sum of diagrams our one-loop energy functional becomes

$$W(J) = -\frac{dN_A\gamma^4}{2g^2} + \frac{(d-1)N_A}{2}\int\frac{d^dk}{(2\pi^d)}\left[\ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right) - \ln\left(k^2+M^2\right)\right] + \mathcal{O}(g^2) \tag{2.90}$$

For our method to be consistent there should be no divergences arising from our integration that do not cancel. To check this we must first perform the integral to find their values. To begin with we shall look at our integrals independently.

$$\frac{(d-1)N_A}{2}\int\frac{d^dk}{(2\pi^d)}\left[\ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right)-\ln\left(k^2+M^2\right)\right] \qquad (2.91)$$

The first term of (2.91) can be divided into two terms by factorising it. Equation (2.91) can be then be factorised

$$\ln\left((k^2)^2+k^2M^2+C_A\gamma^4\right)=\ln\left(k^2+a_+^2\right)+\ln\left(k^2+a_-^2\right) \qquad (2.92)$$

Where

$$a_\pm^2=\frac{1}{2}\left(M^2\pm\sqrt{(M^2)^2-4C_A\gamma^4}\right) \qquad (2.93)$$

This can be done using the the text book integral for the vacuum bubble [8]

$$\int\frac{d^dk}{(k^2+m^2)^n}=\frac{1}{(16\pi^2)^{\frac{d}{4}}}\left(\frac{\Gamma(n-\frac{d}{2})(m^2)^{(\frac{d}{2}-n)}}{\Gamma(n)}\right) \qquad (2.94)$$

from this it is possible to expand the left hand side of this in powers of n

$$\int\frac{d^dk}{(k^2+m^2)^n}=\int d^dk\left(1-n\ln(k^2+m^2)+\mathcal{O}(n^2)\right) \qquad (2.95)$$

It is also possible to express the right hand side (2.94) by expanding it in powers of n, however before doing so it is necessary to transform the denominator first

$$\Gamma(n)=\frac{1}{n}\Gamma(n+1) \qquad (2.96)$$

the right hand side of (2.94) now becomes

$$\frac{1}{(16\pi^2)^{\frac{d}{4}}}\left(\frac{n\Gamma(n-\frac{d}{2})(m^2)^{(\frac{d}{2}-n)}}{\Gamma(n+1)}\right)=\frac{1}{(16\pi^2)^{\frac{d}{4}}}\Gamma\left(-\frac{d}{2}\right)(m^2)^{\frac{d}{2}}n+\mathcal{O}(n^2) \qquad (2.97)$$

With both sides of (2.94) expanded it is possible to match the coefficient n to give the final result

$$\int d^d k \ln(k^2 + m^2) = \frac{1}{(16\pi^2)^{\frac{d}{4}}} \Gamma\left(-\frac{d}{2}\right) (m^2)^{\frac{d}{2}} \tag{2.98}$$

It is now possible to evaluate this using dimensional regularization, with $d = 4 - 2\epsilon$

$$\frac{1}{(16\pi^2)^{1-\frac{\epsilon}{2}}} \Gamma(\epsilon - 2)(m^2)^{2-\epsilon} = -\frac{m^4}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(m^2) + \frac{3}{2}\right) + \mathcal{O}(\epsilon^2) \tag{2.99}$$

With this it is possible to compute the value of the integrals in equation (2.91)

$$\ln\left(k^2 + M^2\right) = -\frac{m^4}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(M^2) + \frac{3}{2}\right) \tag{2.100}$$

$$\ln\left(k^2 + a_+^2\right) = -\frac{(a_+^2)^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(a_+^2) + \frac{3}{2}\right) \tag{2.101}$$

and

$$\ln\left(k^2 + a_-^2\right) = -\frac{(a_-^2)^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(a_-^2) + \frac{3}{2}\right) \tag{2.102}$$

We shall continue by considering the divergent and finite parts of (2.100) - (2.102) separately. As there are no renormalisation counter terms arising for the local composite operator formalism then there should not be any remaining divergence in our integral other than those required to cancel the divergence from Z_γ in our action. This provides a strong check on the earlier calculation. Taking the divergent parts of (2.100) - (2.102) and including the correct sign for each from equation (2.90)

$$\begin{aligned}
&\frac{1}{16\pi^2\epsilon}\left[+(M^2)^2-(a_+^2)^2-(a_-^2)^2\right]= \\
&\frac{1}{16\pi^2\epsilon}\Bigg[+M^4-\frac{1}{4}\left(M^4+2M^2\sqrt{M^4-4C_A\gamma^4}+M^4\right. \\
&\left.-\ 4C_A\gamma^4\right)-\frac{1}{4}\left(M^4-2M^2\sqrt{M^4-4C_A\gamma^4}+M^4-4C_A\gamma^4\right)\Bigg] \\
&=\frac{2}{16\pi^2\epsilon}\left[C_A\gamma^4\right]
\end{aligned} \tag{2.103}$$

This is the divergent part of the integral, however, it does not take into account our pre-factor, with this the divergent term becomes

$$2\frac{(d-1)N_A}{2}\frac{C_A\gamma^4}{16\pi^2\epsilon} \qquad =3N_A\frac{C_A\gamma^4}{16\pi^2}\frac{1}{\epsilon}-2N_A\frac{C_A\gamma^4}{16\pi^2} \tag{2.104}$$

Next taking our Z_γ renormalization counter term from [54]

$$Z_\gamma^2=1+\frac{3}{2}\frac{g^2C_A}{16\pi^2}\frac{1}{\epsilon} \tag{2.105}$$

applying this to our action

$$\begin{aligned}
-\frac{dN_AZ_\gamma^2\gamma^4}{2g^2} \qquad &=-\frac{dN_A\gamma^4}{2g^2}\left(1+\frac{3}{2}\frac{g^2C_A}{16\pi^2}\frac{1}{\epsilon}\right) \\
&=-\frac{dN_A\gamma^4}{2g^2}-\frac{3}{2}\frac{(4-2\epsilon)N_A\gamma^4}{2g^2}\frac{g^2N}{16\pi^2}\frac{1}{\epsilon} \\
&=-\frac{dN_A\gamma^4}{2g^2}-3N_A\frac{C_A\gamma^4}{16\pi^2}\frac{1}{\epsilon}+\frac{3}{2}N_A\frac{C_A\gamma^4}{16\pi^2}
\end{aligned} \tag{2.106}$$

Finally combining (2.104) and (2.106) cancels a remaining divergent terms. This means that the form of the one loop correction to the action from the summing of a diagrams agrees with the fact that there are no divergent terms in our local composite operator formalism. We can now look at the remaining finite part of the integral by looking at the remaining parts of equations (2.100)

- (2.102)

$$\frac{3}{2}N_A\frac{C_A\gamma^4}{16\pi^2} - 4N_A\frac{C_A\gamma^4}{32\pi^2} + \frac{3}{2}N_A\frac{1}{32\pi^2}\left[-\frac{3}{2}M^4 + M^4\ln(M^2) + \frac{3}{2}(a_+^2)^2 - (a_+^2)^2\ln(a_+^2) + \frac{3}{2}(a_-^2)^2 - (a_-^2)^2\ln(a_-^2)\right] \tag{2.107}$$

in a similar way to (2.103)

$$\begin{aligned}
&-N_A\frac{C_A\gamma^4}{32\pi^2} + \frac{(d-1)N_A}{2}\frac{1}{32\pi^2}\frac{3}{2}\left[+(M^2)^2 - (a_+^2)^2 - (a_-^2)^2\right] = \\
&-N_A\frac{C_A\gamma^4}{32\pi^2} + \frac{(d-1)N_A}{2}\frac{1}{32\pi^2}\frac{3}{2}\Big[+M^4 - \frac{1}{4}\Big(M^4 + 2M^2\sqrt{M^4 - 4C_A\gamma^4} + M^4 \\
&- 4C_A\gamma^4\Big) - \frac{1}{4}\Big(M^4 - 2M^2\sqrt{M^4 - 4C_A\gamma^4} + M^4 - 4C_A\gamma^4\Big)\Big] \\
&= -N_A\frac{C_A\gamma^4}{32\pi^2} + N_A\frac{3}{64\pi^2}\left[3C_A\gamma^4\right] \\
&= \frac{1}{64\pi^2}N_A\left(-2+9\right)C_A\gamma^4 = \frac{1}{64\pi^2}N_A\left(7C_A\gamma^4\right)
\end{aligned} \tag{2.108}$$

Finally adding this result to the logarithm terms from (2.107)

$$\frac{1}{64\pi^2}N_A\left[7C_A\gamma^4 + 3M^4\ln(M^2) - 3(a_+^2)^2\ln(a_+^2) - 3(a_-^2)^2\ln(a_-^2)\right] \tag{2.109}$$

This is the final form of our 1-loop correction to the effective potential, finally combining this with our tree terms give the full value for the 1-loop effective potential

$$W(J) = -\frac{dN_A\gamma^4}{2g^2} + \frac{1}{64\pi^2}N_A\left[7C_A\gamma^4 + 3M^4\ln(M^2) - 3(a_+^2)^2\ln(a_+^2) - 3(a_-^2)^2\ln(a_-^2)\right] \tag{2.110}$$

It is possible and advantageous to simplify the logarithm terms in (2.110) this will help when looking at its properties, this is done by using the properties of the logarithm terms.

$$
\begin{aligned}
&3\left(M^4\,\ln(M^2)-(a_+^2)^2\ln(a_+^2)-(a_+^2)^2\ln(a_+^2)\right)=\\
&3\Big(M^4\,\ln(M^2)-\frac{1}{4}(M^4-2M^2\sqrt{M^4-4C_A\gamma^4}+M^4-4C_A\gamma^4)\ln(a_+^2)\\
&\qquad-\frac{1}{4}(M^4+2M^2\sqrt{M^4-4C_A\gamma^4}+M^4-4C_A\gamma^4)\ln(a_-^2)\Big)=\\
&3\Big(M^4\,\ln(M^2)-\frac{1}{4}(M^4-4C_A\gamma^4)(\ln(a_+^2)+\ln(a_-^2))\\
&\qquad+\frac{1}{2}M^2\sqrt{M^4-4C_A\gamma^4}((\ln(a_+^2)-\ln(a_-^2))\Big)
\end{aligned}
\tag{2.111}
$$

It is now possible to combine the sum and difference of logarithm terms

$$
\begin{aligned}
&\ln(a_+^2)+\ln(a_-^2)=\ln(a_+^2a_-^2)\\
&=\ln\left(\frac{1}{4}(-M^2+\sqrt{M^4-4C_A\gamma^4})(-M^2-\sqrt{M^4-4C_A\gamma^4})\right)\\
&=\ln\left(\frac{1}{4}(M^4-M^4+4C_A\gamma^4)\right)=\ln(C_A\gamma^4)
\end{aligned}
\tag{2.112}
$$

Thus we are able to remove a M^2 dependence from one of our logarithm terms, continuing on we get

$$
\ln(a_+^2)-\ln(a_-^2)=\ln\left(\frac{a_+^2}{a_-^2}\right)
\tag{2.113}
$$

This can be simplified by using the method of removing surds from the denominator of a fraction.

$$
\frac{1}{a+\sqrt{b}}=\frac{1}{a+\sqrt{b}}\frac{a-\sqrt{b}}{a-\sqrt{b}}=\frac{a-\sqrt{b}}{a^2-b}
\tag{2.114}
$$

Applying this relation to (2.113)

$$\ln\left(\frac{a_+^2}{a_-^2}\right) = \ln\left(\frac{-M^2+\sqrt{M^4-4C_A\gamma^4}}{-M^2-\sqrt{M^4-4C_A\gamma^4}}\right)$$
$$= \ln\left(\left(\frac{-M^2+\sqrt{M^4-4C_A\gamma^4}}{-M^2-\sqrt{M^4-4C_A\gamma^4}}\right)\left(\frac{-M^2+\sqrt{M^4-4C_A\gamma^4}}{-M^2+\sqrt{M^4-4C_A\gamma^4}}\right)\right) \tag{2.115}$$

From equation (2.112) it possible to see that the denominator of (2.115) is equal to $C_A\gamma^4$, it is also clear the numerator can be written as $(a_+^2)^2$ giving

$$\ln\left(\frac{a_+^2}{a_-^2}\right) = \ln\left(\frac{(a_+^2)^2}{C_A\gamma^4}\right) \tag{2.116}$$

Finally this can be reduced to

$$\ln\left(\frac{(a_+^2)^2}{C_A\gamma^4}\right) = 2\ln\left(a_+^2\right) - \ln\left(C_A\gamma^4\right) \tag{2.117}$$

Now using (2.112) and (2.117) we rewrite (2.111)

$$3M^4\ln(M^2) - 3(a_+^2)^2\ln(a_+^2) - 3(a_+^2)^2\ln(a_+^2) =$$
$$3\left(M^4\ln(M^2) - \frac{1}{4}(M^4-4C_A\gamma^4)\ln(C_A\gamma^4)\right.$$
$$\left. + \frac{1}{2}M^2\sqrt{M^4-4C_A\gamma^4}\ln\left((a_+^2)^2\right) - \frac{1}{2}M^2\sqrt{M^4-4C_A\gamma^4}\ln\left(C_A\gamma^4\right)\right) \tag{2.118}$$

We have successfully taken the finite part of our integral containing three logarithm terms and reduced that to two, also in the same process we have simplified our pre-factor terms, this makes determining the properties of our potential considerably easier. Using (2.118) to rewrite our one loop energy functional (2.110) as

$$\begin{aligned} W(J) = &- \frac{dN_A\gamma^4}{2g^2} + \frac{1}{64\pi^2}N_A \left[7C_A\gamma^4 + M^4 \ln(M^2) \right. \\ &- \frac{3}{4}(M^4 - 4C_A\gamma^4)\ln(C_A\gamma^4) + 3M^2\sqrt{M^4 - 4C_A\gamma^4}\ln\left(a_+^2\right) \\ &\left. - \frac{3}{2}M^2\sqrt{M^4 - 4C_A\gamma^4}\ln\left(C_A\gamma^4\right) \right] \end{aligned} \tag{2.119}$$

This gives the final simplified version of our loop energy functional derived using the LCO formalism for our dimension two BRST invariant composite operator. To proceed to the effective potential it is necessary to introduce a field $\sigma(x)$ which is the field which couples linearly to the source [41] [65] [66], and in effect corresponds to the original composite operator. As such the effective potential for σ emerges from the the constant field value of the effective action for the operator given by

$$\Sigma[\sigma] = W[J] - \int d^4x J(x)\sigma(x) \tag{2.120}$$

after a Legendre tranformation [41] [65] [66]. In the current context the effective potential will depend on the combination arsing from (2.86) given by

$$m^2 = \left[\mu_{\mathcal{R}}^2 + \mu_{\mathcal{Q}}^2 - \mu_{\mathcal{T}}^2 + \frac{1}{2}C_A\mu_{\mathcal{W}}^2 \right] \tag{2.121}$$

where m represents the constant field value of the corresponding σ. The potential has an absolute minimum at a particular value of m^2. However, when considering the overall situation this means not only does the operator $\mathcal{O}^{abcd}$ condense, it does so in a particular colour direction. This is seen by computing the different combinations of $\mu_{\mathcal{I}}^2$ which emerge when the colour tensor

$$\mathcal{T}^{abcd} = \Bigg[\mu_{\mathcal{Q}}^2 \delta^{ac}\delta^{bd} + \mu_{\mathcal{W}}^2 f^{ace} f^{bde} + \frac{\mu_{\mathcal{R}}^2}{C_A} f^{abe} f^{cde} + \mu_{\mathcal{S}}^2 d_A^{abcd} + \frac{\mu_{\mathcal{P}}^2}{N_A}\delta^{ab}\delta^{cd} + \mu_{\mathcal{T}}^2 \delta^{ad}\delta^{bc}\Bigg] \quad (2.122)$$

is multiplied by each constituent tensor in turn. From this it is possible to see

$$\mathcal{T}^{abcd} f^{ace} f^{bde} = C_A \left[\mu_{\mathcal{R}}^2 + \mu_{\mathcal{Q}}^2 - \mu_{\mathcal{T}}^2 + \frac{1}{2} C_A \mu_{\mathcal{W}}^2\right] \quad (2.123)$$

Therefore at one-loop from the LCO effective potential the general operator condensation is in the $\mathcal{R}$ colour direction since

$$\langle \mathcal{O}^{abcd} \rangle \propto f^{ace} f^{bde} \quad (2.124)$$

This differs from the one used in [54] and first proposed in [38] however, at the time the full calculation was not preformed to prove this.

2.6 Discussion

While it has been possible to some extent to use the local composite operator method to look at the properties of the alternative refined Gribov-Zwanziger Lagrangian there have been unforeseen setbacks that were not expected at the start of the calculation. By using this method of direct calculation rather than those tried before [48] [38] [54] [55] [67] we had hoped to be able to progress to a full calculation of the one loop correction to the one loop effective potential. However, as we have shown in Section 2.2 if you compute the exact value of the divergences arising from the J^2 part of the action it is possible to show there are no contributing terms. This results in there being no need for the δJ^2, this has considerable implications for the rest of the calculations. Even taking into account the potential for a finite renormalization term, it is still not possible to find a coupling term χ as a function of g^2 that is non-zero and satisfies the renormalization group equation which is used in [41] and [39]. This lack of coupling prevents us from performing the effective potential calculation in the same way as previously done [39]. However, this is not to say that we have not gained anything from this calculation, aside from having a powerful set of tools capable of evaluating next to next to leading order (NNLO) loop diagrams in the Gribov-Zwanziger Lagrangian. We have also been able to suggest that there is evidence for not naively taking the $\mathcal{Q}$ colour as the rank 4 tensor of choice when working in the refined case. This was first suggested in [38] however that paper mainly focused on the implications of the other colour channels rather than an investigation into which colour channel was the most energetically favourable. While computing the one-loop energy functional we have found strong evidence to support the idea that it is the $\mathcal{R}$ channel rather than the $\mathcal{Q}$ channel that has a dominant effect. This is also backed up by the idea proposed in [38] where the BRST operator was looked at in terms of the affect it had on the localizing ghost propagators. From this point of view our results here are in good agreement with this previous work. A further point that is favourable to this result is that it is self consistent, by this we mean the properties we find during the course of the LCO calculation are borne out by our final result. As our operator insertion is totally finite we require that we do not obtain any pole in ϵ which cannot be cancelled via terms already in our action. In this case the only divergences arising from our integral sum of diagrams is cancelled by the renormalization term related to the Gribov mass Z_γ. This check is important as it shows our methods are valid in the context we have used them. With our results in mind it is now possible to write the exact form of the of the propagators for the $\mathcal{R}$ channel. This is done by using the methods used in Section 2.1 but starting with a different group theory term for our operator $f^{abe}f^{cde}$, this gives

$$\begin{aligned}
\langle A^a_\mu(p) A^b_\nu(-p)\rangle_{\mathcal{R}} &= -\frac{\delta^{ab}[p^2+\mu^2_{\mathcal{R}}]}{[(p^2)^2+\mu^2_{\mathcal{R}}p^2+C_A\gamma^4]}P_{\mu\nu}(p) \\
\langle A^a_\mu(p) \xi^{bc}_\nu(-p)\rangle_{\mathcal{R}} &= \frac{if^{abc}\gamma^2}{[(p^2)^2+\mu^2_{\mathcal{R}}p^2+C_A\gamma^4]}P_{\mu\nu}(p) \\
\langle A^a_\mu(p) \rho^{bc}_\nu(-p)\rangle_{\mathcal{R}} &= 0 \\
\langle \xi^{ab}_\mu(p) \xi^{cd}_\nu(-p)\rangle_{\mathcal{R}} &= -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}[\mu^2_{\mathcal{R}}p^2+C_A\gamma^4]}{C_Ap^2[(p^2)^2+\mu^2_{\mathcal{R}}p^2+C_A\gamma^4]}P_{\mu\nu}(p) \\
&\quad + \frac{f^{abe}f^{cde}\mu^2_{\mathcal{R}}}{C_Ap^2[p^2+\mu^2_{\mathcal{R}}]}L_{\mu\nu}(p) \\
\langle \xi^{ab}_\mu(p) \rho^{cd}_\nu(-p)\rangle_{\mathcal{R}} &= 0 \\
\langle \rho^{ab}_\mu(p) \rho^{cd}_\nu(-p)\rangle_{\mathcal{R}} &= -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\mu^2_{\mathcal{R}}}{C_Ap^2[p^2+\mu^2_{\mathcal{R}}]}\eta_{\mu\nu} \\
\langle \omega^{ab}_\mu(p) \bar{\omega}^{cd}_\nu(-p)\rangle_{\mathcal{R}} &= -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\mu^2_{\mathcal{R}}}{C_Ap^2[p^2+\mu^2_{\mathcal{R}}]}\eta_{\mu\nu}
\end{aligned} \tag{2.125}$$

where

$$P_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \quad , \quad L_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} \tag{2.126}$$

Here it is clear to see that in the limit $p \to 0$ that the gluon propagators freezes to a non-zero value. In fact the gluon propagator for both the $\mathcal{Q}$ and $\mathcal{R}$ channels are both identical. There is however a difference in limit on the localising ghost terms, with the $\mathcal{R}$ channel tending to infinity as $p \to 0$. This potentially does not tell the full story in terms of modifications to the Gribov-Zwanziger Lagrangian that could affect the low momentum behaviour of the propagators. In principle, in the same way that we have tried to include the sum of all the possible colour channels it is also possible to include a massive gluon. The massive insertion has been widely studied in QCD and in the Gribov-Zwanziger Lagrangian [41] [39] [48]. However, so far there has not been a study of the possibility of including all seven potential channels at once in the action and how this would change the low momentum limit of the theory. It may even provide a solution to the problem related to our BRST operator being finite, as there is a mixing element between the localizing ghost operator and massive gluon. This term may allow for a Hubbard-Stratonovich transformation to be used to couple the source to auxiliary fields. However before considering this there are numerous other problems that would need to be dealt with first. While these problems should not in principle be insurmountable they do at least prevent it being possible to perform this calculation at the current time. Finally for a true picture of how this BRST operator affects the low energy limit one would need to calculate the effective potential to two-loops. However, to do this would require the use of the full propagator structure to be worked out for all the colour channels active at once then use these to perform the two calculations. While in principle this should be possible it would be so vastly expensive in terms of computer power and time that like many calculations in QFT there is little justification for it at the current time.

Chapter 3

Symmetric point

3.1 Introduction

Previous work on the gluon triple vertex has shown there is a discrepancy between the expected behaviour and the observed behaviour in the intermediate energy range in calculations involving lattice QCD [68] [69] [70] [71] [72]. In the Landau gauge the effective coupling constant appears to deviate from the results that had been expected. While such deviations are not without explanation or previous study being related to a power correction to the theory. It was however, found that the correction corresponded to a dimension 2 object rather than a dimension 4 one. While in most cases the normal choice for a dimension 2 object would be the massive gluon operator $\frac{1}{2}A^{a\,2}_{\mu}$ this is normally associated with the gluon condensate rather than a power correction. However there is no reason to assume that the power correction must come from the gluon operator and could not be generated by any potential dimension 2 operator. Considering the recent investigation involving the alternative refined Gribov-Zwanziger Lagrangian this present an option that not only contains a dimension 2 operator it also matches with the most recent lattice data for the low energy behaviour. The aim of this work then is to derive a power series for the gluon triple vertex, ghost gluon vertex and the quark gluon vertex using the alternative refined Gribov-Zwanziger Lagrangian. As can be seen in Section 2 the propagators for the $\mathcal{Q}$ (2.25) and $\mathcal{R}$ (2.125) give a non-zero value for the gluon propagator at zero momentum. While in previous section we discovered that the $\mathcal{R}$ was the colour channel that the operator condensed in, however our aim is to produce results that are sensitive to their difference. We choose these two channels from the six (2.26) as $\mathcal{R}$ and $\mathcal{Q}$ are the only ones that gives the required zero momentum behaviour. A full study of all the different potential propagators for each of the channels is given in [38]. Both of the $\mathcal{R}$ and $\mathcal{Q}$ give the required modified form of the gluon propagator at low energy so in principle either could generate the power correction. Finally to motivate study in this area from the lattice as well as our own ease of calculations we shall work in 3 rather than 4 dimensions. This will not only allow for less expensive lattice results to be generated but will mean we can implement the exact value of the Gribov mass should it be desired.

Our definition of the symmetric point uses a very specific momentum configuration to avoid creating problems in the low energy limit either in the form of non-cancelling poles in ϵ or terms containing logarithmic divergences. These problems are known to exist in the asymmetric point calculation which is known as the exceptional configuration. The main benefit of this approach is at the intermediate energy scale and there are known approaches for dealing with some of the issues. For most cases one would need to calculate each of the integrals for the Feynman diagrams exactly then expand them in the mass scale to get the mass power correction. However for the symmetric point it is possible to use the method of Nickel [73] to expand the integrals without the need to work out their exact form first. This not only simplifies the result but should make the calculation simpler and take less computation time to perform.

The reason for the importance of this work is as follows, lattice work has discovered that a power correction exists in dimension two. In principle there is no reason why the Gribov parameter γ^2 could not be the source of this correction in the pure Gribov-Zwanziger case. It is possible to relate γ^2 to the vacuum value of $\frac{1}{2}A_\mu^{a2}$ as γ^2 is introduced in the gluon propagator. As $\left\langle\frac{1}{2}A_\mu^{a2}\right\rangle$ has the same dimension as γ^2 so when reading the power correction on the lattice either term could contribute to the final result.

The aim of this work is to see what effect including the pure Gribov-Zwanziger case has, but the result should give some detail as to the nature of the correction arising from the Gribov-Zwanziger Lagrangian. Once we have established the pure Gribov-Zwanziger case it should be possible to extend our Lagrangian to include the extra BRST invariant operator associated with the alternative refined Gribov-Zwanziger Lagrangian. We have shown that there is a naturally forming condensate that occurs from the effective potential. Here the $\mathcal{Q}$ and $\mathcal{R}$ solutions shall be modelled, as while the $\mathcal{R}$ solution is the one which arises from the effective potential there is very little difference between the two sets of propagators. This will also provide us with some checks on our work as by taking this operator to zero should reproduce the results of the pure Gribov-Zwanziger case.

Finally as the aim of this calculation is the motivation of potential lattice study of the symmetric point we shall carry out this calculation in 3 dimension. This is because study of the symmetric point in 3 dimension will be less expensive in terms of computer time than the full 4 dimension. Also it may give some checks on the structure of the 4 dimensional results, the final coefficient of the result would be expected to be different however they may have the same underlying structure.

This calculation will be broken down into several steps, and we shall looks at three different vertex corrections. The gluon triple vertex, the ghost gluon vertex and finally the quark gluon vertex. This means for the Q and R solutions three different sets of Feynman diagrams are needed to be completed, couple this with the pure Gribov-Zwanziger case and this will bring the number of diagrams up to nine for a full set of results. Once again we shall be working in the Landau gauge as in the previous work and as such much of the methods and code can be reused from before.

This calculation will be arranged as followed. First we shall generate the diagrams using QGRAF as in the previous calculations, these will then have our Feynman rules applied. The necessary group theory will be carried out, this shall use the same set of contractions as previously used. Once this is done we shall deal with the projection we need to apply to these different vertex corrections. We will then outline the method of reducing our diagrams to master ones using integration by parts in REDUZE. Finally these master integrals can then be expanded using the method of Nickel [73], finally the results can be expanded in the $\overline{\text{MS}}$ scheme.

These steps take place for each of the vertex corrections in each of the different models. This work brings together nine different calculations to provide a comprehensive collection of results.

3.2 Symmetric Point Configuration

For our work on the symmetric point we shall follow the work of [74] which outlines many of the methods which have been used, this shall also be the source of much of our notation. We now define the specific Green's function which we shall compute for the symmetric point, for our one loop correction we have 3 incoming vertices with two independent momenta p and q. The third momentum shall be the sum of these $r = -p-q$, none of these momentum will be nullified as these would leave us in an exceptional momenta configuration.

We further impose

$$p^2 = q^2 = r^2 = -\mu^2 \tag{3.1}$$

From this we can see

$$p \cdot q = \frac{1}{2}\mu^2 \tag{3.2}$$

where μ is the mass scale introduced to ensure the coupling constant remains dimensionless. These relations shall become important when helping reduce our integrals to master ones. Finally we define our loop momentum as k, with this we are in a situation to define the momentum around our Feynman diagrams.

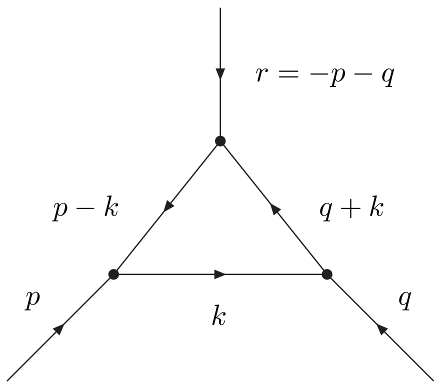

Figure 3.1: One loop correction to triple vertex

We need to compute three separate Green's functions, one for each of the three vertices we shall be studying: the gluon triple vertex, the quark gluon vertex and finally the ghost gluon vertex. We formally define these via

$$\begin{aligned}
\left\langle A^a_\mu(p) A^b_\nu(q) A^c_\sigma(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= f^{abc}\, \Sigma^{\text{ggg}}_{\mu\nu\sigma}(p,q,\gamma^2,\mu_I^2) \Big|_{p^2=q^2=-\mu^2} \\
\left\langle \psi^i(p) \bar{\psi}^j(q) A^c_\sigma(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= T^c_{ij}\, \Sigma^{\text{qqg}}_{\sigma}(p,q,\gamma^2,\mu_I^2) \Big|_{p^2=q^2=-\mu^2} \\
\left\langle c^a(p) \bar{c}^b(q) A^c_\sigma(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= f^{abc}\, \Sigma^{\text{ccg}}_{\sigma}(p,q,\gamma^2,\mu_I^2) \Big|_{p^2=q^2=-\mu^2}
\end{aligned}$$

where the notation ggg, qqg and ccg defines each of these respectively. Each of these carry both colour and Lorentz indices, these come from the tree terms for each of the vertices in the standard notation for QCD. As these Green's functions carry these indices we shall need to decompose these into scalar amplitudes so that we have fully contracted object with which to perform the final calculation. The method for this is outlined in [74], however more detail shall be given in the next section.

3.3 Projections

For each of our Green's functions we need to defined a scalar amplitude, these are needed to contract the Lorentz indices as we need fully contracted object to perform our final set of integration.

$$
\begin{aligned}
\Sigma^{\rm ggg}_{\mu\nu\sigma}(p,q,\gamma^2,\mu_I^2)\Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^{3} \mathcal{P}^{\rm ggg}_{(k)\,\mu\nu\sigma}(p,q)\,\Sigma^{\rm ggg}_{(k)}(p,q,\gamma^2,\mu_I^2) \\
\Sigma^{\rm qqg}_{\sigma}(p,q,\gamma^2,\mu_I^2)\Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^{6} \mathcal{P}^{\rm qqg}_{(k)\,\sigma}(p,q)\,\Sigma^{\rm qqg}_{(k)}(p,q,\gamma^2,\mu_I^2) \\
\Sigma^{\rm ccg}_{\sigma}(p,q,\gamma^2,\mu_I^2)\Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^{2} \mathcal{P}^{\rm qqg}_{(k)\,\sigma}(p,q)\,\Sigma^{\rm ccg}_{(k)}(p,q,\gamma^2,\mu_I^2)
\end{aligned}
$$

Each vertex has a different Lorentz and scalar amplitude, these decompositions are not unique as there could be many possible ways to define them. One of the further benefits of working at the symmetric point is that it is possible to use the relations $p^2 = q^2 = -\mu^2$ to reduce the number of independent terms in each amplitude. For the Lorentz sector we need to construct a tensor basis built from p_μ, q_μ and $\eta_{\mu\nu}$, this is because these are the objects used to construct the original vertex terms. This is further complicated for the quark gluon vertex as this also includes spinors, this means that our tensor basis has to be constructed from p_μ, q_μ, $\eta_{\mu\nu}$ and γ_μ. This also means we need to consider that it is possible to use a set of γ-matrices to form our tensor basis. As we are working in dimensional regularization we need a definition of γ_μ that is compatible with our working in d-dimensions. In d-dimensions we define the object $\Gamma_{(n)}^{\mu_1 ... \mu_n}$ where n is a positive integer as

$$\Gamma_{(n)}^{\mu_1 ... \mu_n} = \gamma^{[\mu_1} ... \gamma^{\mu_n]} \tag{3.3}$$

this is the full set of antisymmetrized contracted γ-matrices. With this defined we have everything we need to compute the basis tensor for each of our projections. Individual amplitudes can be isolated by multiplying the Green's function by a linear combination of the basis tensors. For the following example we shall choose the gluon triple vertex but the same procedure is repeated in the same fashion for the other two vertices. Starting with

$$\left\langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} = \sum_{k=1}^{3} \mathcal{P}_{(k)\,\mu\nu\sigma}(p,q)\, \Sigma_{(k)}(p,q,\gamma^2,\mu_I^2) \tag{3.4}$$

we define a new matrix $\mathcal{N}_{lk}(p,q)$ such that

$$\mathcal{P}_{(l)}^{\mu\nu\sigma}(p,q) \mathcal{P}_{(k)\,\mu\nu\sigma}(p,q) = \mathcal{N}_{lk}(p,q) \tag{3.5}$$

This is the matrix constructed from each element of the tensor structure for the vertex. This results in a $k \times k$ matrix in polynomials of d as it is formed of Lorenz contractions in our d-dimensional dimensional regularization. From this it possible to define its inverse such that

$$\mathcal{M}_{kl}(p,q) \mathcal{N}_{lk'}(p,q) = \delta_{kk'} \tag{3.6}$$

This means that $\mathcal{M}_{kl}(p,q)$ shall be the inverse of the matrix formed from the contracted tensor basis of our vectors. It is now possible to multiply both sides of (3.4) from the left by

$$\mathcal{M}_{kl}(p,q)\mathcal{P}^{\mu\nu\sigma}_{(l)}(p,q) \tag{3.7}$$

this gives

$$\begin{aligned} &\mathcal{M}_{kl}(p,q)\mathcal{P}^{\mu\nu\sigma}_{(l)}(p,q)\left\langle A^a_\mu(p)A^b_\nu(q)A^c_\sigma(-p-q)\right\rangle\Big|_{p^2=q^2=-\mu^2} \\ &\quad = \mathcal{M}_{kl}(p,q)\mathcal{P}^{\mu\nu\sigma}_{(l)}(p,q)\mathcal{P}_{(k')\,\mu\nu\sigma}(p,q)\,\Sigma_{(k')}(p,q,\gamma^2,\mu_I^2) \end{aligned} \tag{3.8}$$

From our definitions of $\mathcal{M}_{kl}(p,q)$ and $\mathcal{N}_{kl}(p,q)$ it is clear to see that the terms on the right hand side of (3.8) reduce as we are left with a scalar object. We shall work out the explicit form of both $\mathcal{M}_{kl}(p,q)$ and $\mathcal{N}_{kl}(p,q)$ as these will be needed for later parts of the calculation. The projections have been previously calculated [74] [75] and we choose the more compact notation of [74] for the gluon triple vertex. This compact tensor basis makes this process simpler as otherwise we would be left trying to invert a 14×14 matrix of polynomials. The full tensor basis for each of the vertices is given below.

$$\begin{aligned} \mathcal{P}^{ggg}_{(1)\mu\nu\sigma}(p,q) &= \eta_{\mu\nu}p_\sigma - \eta_{\mu\nu}q_\sigma - 2\eta_{\mu\sigma}p_\nu - \eta_{\sigma\mu}q_\nu + \eta_{\nu\sigma}p_\mu + 2\eta_{\nu\sigma}q_\mu \\ \mathcal{P}^{ggg}_{(2)\mu\nu\sigma}(p,q) &= [2p_\mu p_\nu p_\sigma + p_\mu q_\nu p_\sigma - p_\mu q_\nu q_\sigma + 2q_\mu p_\nu p_\sigma - 2q_\mu p_\nu q_\sigma - 2q_\mu q_\nu q_\sigma]\frac{1}{2\mu^2} \\ \mathcal{P}^{ggg}_{(3)\mu\nu\sigma}(p,q) &= [p_\mu p_\nu q_\sigma - q_\mu p_\nu p_\sigma + q_\mu p_\nu q_\sigma - q_\mu q_\nu p_\sigma]\frac{1}{\mu^2}\ . \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathcal{P}^{\text{qqg}}_{(1)\sigma}(p,q) &= \gamma_\sigma \quad , \quad \mathcal{P}^{\text{qqg}}_{(2)\sigma}(p,q) = \frac{p_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}^{\text{qqg}}_{(3)\sigma}(p,q) = \frac{p_\sigma \not{q}}{\mu^2}\ , \\ \mathcal{P}^{\text{qqg}}_{(4)\sigma}(p,q) &= \frac{q_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}^{\text{qqg}}_{(5)\sigma}(p,q) = \frac{q_\sigma \not{q}}{\mu^2} \quad , \quad \mathcal{P}^{\text{qqg}}_{(6)\sigma}(p,q) = \frac{1}{\mu^2}\Gamma_{(3)\sigma p q} \end{aligned} \tag{3.10}$$

$$\mathcal{P}^{\text{ccg}}_{(1)\sigma}(p,q) = p_\sigma \quad , \quad \mathcal{P}^{\text{ccg}}_{(2)\sigma}(p,q) = q_\sigma \tag{3.11}$$

Here the $\frac{1}{2}\mu^2$ has been added where needed to give the tensor basis the correct dimension. We now list the full form of the matrices $\mathcal{N}$ and $\mathcal{M}$, to perform the indices contraction. We have used FORM as, while this would be possible by hand it prevents unnecessary errors occurring.

$$\mathcal{N}^{\mathrm{ggg}} = -\frac{9}{8}\begin{pmatrix} -8(d-1) & 2 & 4 \\ 2 & -2 & 1 \\ -4 & 1 & -2 \end{pmatrix}. \tag{3.12}$$

$$\mathcal{N}^{\mathrm{qqg}} = \begin{pmatrix} 4d & -4 & 2 & 2 & -4 & 0 \\ -4 & 4 & -2 & -2 & 1 & 0 \\ 2 & -2 & 4 & 1 & -2 & 0 \\ 2 & -2 & 1 & 4 & -2 & 0 \\ -4 & 1 & -2 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3(d-2) \end{pmatrix}. \tag{3.13}$$

$$\mathcal{N}^{\mathrm{ccg}} = \frac{1}{2}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.14}$$

With these matrices defined, it is possible to invert them using REDUCE to give the final form of $\mathcal{M}$ we require

$$\mathcal{M}^{\mathrm{ggg}} = -\frac{1}{27(d-2)}\begin{pmatrix} 3 & 0 & -6 \\ 0 & 16(d-2) & 8(d-2) \\ -6 & 8(d-2) & 4(4d-5) \end{pmatrix}. \tag{3.15}$$

$$\mathcal{M}^{\mathrm{qqg}} = \frac{1}{36(d-2)}\begin{pmatrix} 9 & 12 & 6 & 6 & 12 & 0 \\ 12 & 16(d-1) & 8(d-1) & 8(d-1) & 4(d+2) & 0 \\ 6 & 8(d-1) & 4(4d-7) & 4(d-1) & 8(d-1) & 0 \\ 6 & 8(d-1) & 4(d-1) & 4(4d-7) & 8(d-1) & 0 \\ 12 & 4(d+2) & 8(d-1) & 8(d-1) & 16(d-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix}. \tag{3.16}$$

$$\mathcal{M}^{\mathrm{ccg}} = -\frac{2}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} . \tag{3.17}$$

The gluon triple vertex agrees with [74] so this provides a check on the method used here. The final result for the power correction shall be split between the different projection for each of the different vertices.

3.4 Auxiliary Mapping

The aim now is to reduce our current set of propagator terms into something that can be used to master integrals, we call this process auxiliary mapping. For which we shall use REDUZE. Details of this will be given in the next section. The current aim is to generate integrals of the form

$$\mathcal{I} = \int d^3k_1 \frac{1}{(k_1^2+m_1^2)^{\alpha}((k_1-p_1)^2+m_2^2)^{\beta}((k_1+p_2)^2+m_3^2)^{\gamma}} \tag{3.18}$$

Where α, β and γ can either be positive or negative, this is the form for the integrals that can be reduced to masters. The first step is to solve all the scalar products in the numerator. Luckily in our model it is possible to turn these terms into propagator like terms. We use the relations.

$$\begin{aligned} k\cdot p &= \frac{1}{2}\left(k^2 + p\cdot p - (k-p)^2\right) \\ k\cdot q &= \frac{1}{2}\left(-k^2 + q\cdot q - (k+q)^2\right) \end{aligned} \tag{3.19}$$

The final terms in this are of the form of the massless propagator and the second term is equal to μ^2, this just leaves the k^2 that needs to be reduced. Before we can continue we lastly address the Gribov propagator as in its current form it is not to possible integrate. If we take the R channel propagator it is possible to define two new mass terms m_+ and m_-, these are defined as

$$m_\pm^2 = \frac{1}{2}\left[-\mu_i^2 \pm \sqrt{\mu_i^4 - 4C_A\gamma^4}\right] \tag{3.20}$$

The Gribov propagator now becomes

$$\frac{1}{(k^2)^2 + \mu_R^2 k^2 + C_A\gamma^4} = \frac{1}{(k^2+m_+^2)}\frac{1}{(k^2+m_-^2)} \tag{3.21}$$

Now we have all the components required to start the reductions we require. The following is a break down of the steps required, these are implemented in FORM and done in such a way that we build the most general possible construction of propagator terms. To do this multiply each of the terms by the following

$$\begin{aligned} 1 = \frac{1}{(k^2)^0}\frac{1}{((k-p)^2)^0}\frac{1}{((k_1+q)^2)^0}\frac{1}{(k^2+m_x^2)^0} \\ \frac{1}{((k-p)^2+m_x^2)^0}\frac{1}{((k+q)^2+m_x^2)^0}\frac{1}{(k^2+m_+^2)^0} \\ \frac{1}{((k-p)^2+m_+^2)^0}\frac{1}{((k+q)^2+m_+^2)^0}\frac{1}{(k^2+m_-^2)^0} \\ \frac{1}{((k-p)^2+m_-^2)^0}\frac{1}{((k+q)^2+m_-^2)^0} \end{aligned} \tag{3.22}$$

This term is equal to 1 as all the exponents are zero. However as this is multiplying each terms we have from the Feynman rules some of these will either gain or lose powers. This gives the most general case possible and is necessary as reducing some of the exponents may raise others. This may look cumbersome however it is straight forward to implement in FORM with careful use of a vector with twelve indices. This vector $\text{int}(a,b,c,d,e,f,g,h,i,j,k,l)$ where each of the letters corresponds to the one of the exponents in (3.22) allows every integral in to be expressed in the same manor. This allows the following operation to be dealt with without introducing a large amount of additional terms into our program which would slow its operation considerably. The next step is to deal with the higher order numerator terms that arise, this is done by breaking terms down to the product of a string of the terms to the power of one, An example of this is as follows

$$(k^2 + m_+^2)^3 = (k^2 + m_+^2)(k^2 + m_+^2)(k^2 + m_+^2) \tag{3.23}$$

The need for this becomes clear when you consider the following possible transformation

$$(k^2 + m_+^2)^1 = (k^2)^1 + (m_+^2)(k^2)^0 \tag{3.24}$$

As we are looking to integrate over the loop momentum this effectively reduces the propagator like terms from a massive one to a massless one. This generates more terms but reduces the power of the exponent. A similar step can be preformed with massless loop momentum in the numerator. These can be used to perform the following partial fraction

$$\frac{k^2}{(k^2 + m^2)} = \left(1 - \frac{m^2}{(k^2 + m^2)}\right)(k^2)^0 \tag{3.25}$$

here k^2 and m^2 could be any of the loop momenta or masses. This once again increases the overall number of terms, however in doing so we reduce a power of one of the loop momenta. This step is repeated until there are no eligible terms left to perform this reduction. Next look at terms with the same loop momentum and different non-zero exponents

$$\frac{1}{(k^2 + m_1^2)^{h1}} \frac{1}{(k^2 + m_2^2)^{h2}} = \frac{-1}{(m_1^2 - m_2^2)} \left[\frac{1}{(k^2 + m_1^2)^{h1}} \frac{1}{(k^2 + m_2^2)^{h2-1}} - \frac{1}{(k^2 + m_1^2)^{h1-1}} \frac{1}{(k^2 + m_2^2)^{h2}} \right] \tag{3.26}$$

This step is then repeated until either h_1 or h_2 is equal to zero. As we have up to three masses for each of the loop momentum it is possible to reduce the number of masses per loop momentum down to one. With this in mind what we now should have for our twelve termed vector with only one mass for each of the loop momentum with non-zero exponent. This can be either positive or negative. Now that we have our integrals in the REDUZE format, however our current setup is not readable by REDUZE so there is still some work required in order to export the output we have just generated. Our aim is to define a vector which describes the exponent of each loop momentum and the mass associated with it. Start by defining a new vector from our function of 12 propagator terms with each component in the vector coming from the exponents of the powers of loop momentum. This vector is divided into 3 groups of 4 each representing a particular loop momentum. In each of these groups three of the four indices will be zero and the non-zero ones correspond to only the masses to be tackled by REDUZE. With these non-zero terms labelled they are combined into a new set of vectors with six components. These six terms are the exponents of the loop momentum and the mass associated with it. As this is not particularly clear from the description the following is one possible example.

$$\int_k \frac{1}{(k^2+m_1^2)^\alpha((k-p)^2+m_2^2)^\beta((k+q)^2+m_3^2)^\gamma}$$
$$= \mathrm{int1abc}(m_1, m_2, m_3, \alpha, \beta, \gamma) \tag{3.27}$$

This approach taken may not be the most effective way of tackling these problem however it is a simple method making it easier to implement within FORM. The final step is to preform a slight change in our notation to help speed up REDUZE. REDUZE defines a set of reductions based on the number of masses and which loop momentum they are attached to, to save on the large amount of computation time required for the most general case of three masses we shall split our integrals into the following forms

$$\text{int1abc}(m_1, 0, 0, \alpha, \beta, \gamma) = \text{int1a00}(m_1, 0, 0, \alpha, 0, 0) \tag{3.28}$$

This is the triangle graph with one non-zero mass but with still 3 possible propagator like terms.

3.5 Reduction to Master Integrals

Unfortunately despite the best effort of applying the different partial fraction rules and scalar product reductions, it is not always possible to reduce the integral to a master one by hand. By this we mean of the form

$$\begin{aligned} &\text{int1abc}(m_1, m_2, m_3, \alpha, \beta, \gamma) \\ &\qquad = \int_k \frac{1}{(k^2 + m_1^2)^\alpha ((k-p)^2 + m_2^2)^\beta ((k+q)^2 + m_3^2)^\gamma} \end{aligned} \tag{3.29}$$

From this point on, these shall be called irreducible master integrals if α, β, γ are either one or zero, as these are the simplest form of the integrals. Fortunately this is a common problem in loop calculations so there are several different computer packages available for solving these problems. The one which has been chosen in this case is Reduze 2.0 which uses GINAC symbolic manipulation system which is written in C++. This program has many additional features, which luckily were not necessary for this calculation as we are working with triangle diagrams or simpler graphs. This allows us to skip the normal step of having to define auxiliary topologies as we are not reducing complex interactions with multiple incoming and outgoing momenta. The following is a brief description of how Reduze has been applied to the current problem. We start by introducing the notation used in the program and how it is used to uniquely label each of the different integrals. From here we shall explain the lexicographical ordering which is used for the reductions. This is what is used to define which of the integrals are simplest. Finally we shall explain how this is then used with the different integration by parts routines to generate the data base of reductions required. To start with we introduce the integral

$$\mathcal{I} = \int [dk] \frac{N_{j_{t+1}}^{s_1} ... N_{j_\nu}^{s_{\nu-t}}}{N_{j_1}^{r_1} ... N_{j_t}^{r_t}} \tag{3.30}$$

Here $\nu - t$ is the number of numerator terms with non-negative exponents, this can be greater than or equal to zero as there may or may not be terms in the numerator. t is the number of denominator terms with a positive exponent, negative exponents here would be in the numerator so these are not counted in this measure, $r_i \geq 1$ as there should be at least one denominator term for REDUZE to work. For REDUZE to work, it requires a set of labels that uniquely describe each of the possible integrals, these are defined from the information available in each of the integrals. The first thing to define is the sector ID for each of the auxiliary topologies we are working with. This is not always required in REDUZE as it is possible to do reduction in some configuration without needing this. The sector ID is a function of the denominator terms of the integral. If we define an example of our integral as follows

$$\mathcal{I} = \int \frac{N_3^2}{N_1 N_2^2 N_4^0 N_5^3} \tag{3.31}$$

The sector ID for this problem has the following equation

$$\text{ID} = \sum_{k=1}^{t} 2^{j_k - 1} \tag{3.32}$$

From the example above this would be $\text{ID} = 2^{1-1} + 2^{2-1} + 2^{5-1} = 19$. This label could then be used to relate this integral to an auxiliary topology if required.

The next label required is the sum of the denominator exponents. This sum should be greater than or equal to the number of terms in the denominator, this is obvious as each of the denominator terms should at least carry a power of 1.

$$r = \sum_{i=1}^{t} r_i \tag{3.33}$$

In a similar way it is possible to sum the exponents in the numerator. This should be greater than or equal to zero, this can be zero as it would represent no terms in the numerator.

$$s = \sum_{i=1}^{\nu - t} s_i \tag{3.34}$$

Lastly one needs to form a vector that describes the information of each of the exponents, in this the denominator exponents shall be positive and the numerator ones negative

$$\mathbf{v} = [v_1 \quad v_2 \ldots \quad v_\nu] \tag{3.35}$$

With these it is possible to write a unique label for every possible integral $\mathrm{INT}\,[t\,,\,r\,,\,s\,,\,\mathrm{ID}\,,\,\mathbf{v}]$ so taking the example (3.31). $t = 3$, $r = 6$, $s = 2$, $\mathrm{ID} = 19$ and $\mathbf{v} = [1\,,\,2\,,\,-2\,,\,0\,,\,3]$ this gives the full label

$$\mathrm{INT}\,[3\,,\,6\,,\,2\,,\,19\,,\,[1\,,\,2\,,\,-2\,,\,0\,,\,3]] \tag{3.36}$$

This integral is more complicated than those arising from the symmetric point as we have to contend with integrals with three propagator like terms in them. Each possible integral has been described in this way and it is possible to assign each an order, REDUZE does this so that there is a fixed path of reductions required to get to the irreducible masters. The approach used is one of the key features of the Laporta algorithm [76] which is implemented by REDUZE.

The first step is to define each of the integrals as a vector

$$\tilde{\mathbf{V}}_I = [t\,,\, r\,,\, s\,,\, \mathrm{ID}\,,\, \mathbf{v}] \tag{3.37}$$

A vector I is said to be simpler than J if there exists $I < J$ and only if there exists $m \in \{1\,,\, \ldots\,,\, \nu + 4\}$ such that

$$\left|\tilde{\mathbf{V}}_I(m)\right| < \left|\tilde{\mathbf{V}}_J(m)\right|$$

and

$$\left|\tilde{\mathbf{V}}_I(k)\right| = \left|\tilde{\mathbf{V}}_J(k)\right|$$

for all $k < m$. What this does is it looks at the t, r and s labels for each of the integrals and compares each of them in turn to define which is simpler. So if integral I had $s = 1$ and integral J had $s = 2$ but both had t and r equal then I would be the simpler integral. If I had $r = 2$ and integral J had $r = 3$ with both the same value of t then once again I would be simpler. Finally integrals with lower values of t are considered simplest as these have the fewest terms in the denominator. This leads to the potential problem where terms could be taken from the denominator and moved to the numerator, however, these terms are then further reduced to leave no irreducible scalar pre-factors in the numerator terms. The ID term was not part of the original Laporta algorithm and as such is not used in these reductions, it does however play a role in speeding up the calculation as each of the different ID's are dealt with independently. This helps break down larger ones to smaller ones. Before we can perform the integral reduction for the symmetric point Reduze needs to be provided with information relating to the momentum configuration we are using as well as the definition of any scalar products. We also need to define which of our propagator terms carry masses and finally the dimension of the kinematic invariant so that the overall reductions carry the correct mass dimension. Our momentum configuration has already been defined in section 3.2 and the scalar products we require are given by (3.1) and (3.2). As usual μ is defined as having dimension one so that the solution remains a dimensional quantity overall once the mass terms are included. For each of our six possible mass configuration int1a00$(m_1, 0, 0, \alpha, 0, 0)$, int1aa0$(m_1, m_1, 0, \alpha, \beta, 0)$, int1ab0$(m_1, m_2, m_0, \alpha, \beta, 0)$, int1aaa$(m_1, m_1, m_1, \alpha, \beta, \gamma)$, int1abb$(m_1, m_2, m_2, \alpha, \beta, \gamma)$, and int1abc$(m_1, m_2, m_3, \alpha, \beta, \gamma)$ the mass carrying terms are defined. The convention used is that the ordering starts with the first mass on the k propagator, then the $(k - p)$ and finally $(q + k)$ carrying the third mass if it is used in the integral. With this data it is possible for REDUZE to carry out the reduction to masters by constructing a set of integration by parts relations for each the integrals in a specific range of r and s. These are defined at the start of the program and the caveat being the larger the given range the longer the computation will take to complete. However there are some mechanisms in place in REDUZE to speed these calculations up. The most important one in our case is the use of symmetry relations, as it can be shown that there is a high degree of symmetry in some of the relations that only carry one or two masses. This works by shifting the loop momentum within the integration. This is possible as in dimensional regularisation the integration remains invariant under a shift of loop momentum k. This allows many of the different integrals to be defined as equal saving the need to calculate many independent terms. For example the following shift is possible int1aa0$(m_1, 0, m_1, 1, 1, 1)$ to int1aa0$(m_1, m_1, 0, 1, 1, 1)$. This step is only possible as we are working at the symmetric point and can make use of the relations arsing from our momentum configuration

$$p^2 = q^2 = r^2 = -\mu^2 \tag{3.38}$$

This means that we can use the rotational symmetry of our three point diagrams to reduce the number of independent integrals we need to perform. For the sake of convenience and clear notation we shall shift all the masses to the front of our defined integrals. To make this process easier it is possible to have REDUZE export the reduction in our preferred format for FORM, this will save time and prevent possible error while needing to convert the code. As previously mentioned REDUZE performs the actual reductions by constructing a database of integration by parts routines. This takes advantage of the fact that in dimensional regularisation integration over a total derivative is equal to zero.

$$\int \frac{\partial}{\partial k_i^\mu} \left[q^\mu I'(p_1, ..., p_m, k_1, ...k_l) \right] = 0 \tag{3.39}$$

Here the index μ is summed over and q^μ could be any loop or external momentum. Also m and l are the number of internal and external momenta respectively. This equation allows us to write relations between integrals with different exponents and numerator terms. These recursive relations allow any integral to be reduced to a master one by repeatedly applying different reductions.

The problem with this method is that it can be computationally quite expensive. REDUZE starts the reduction process by building a database for all integrals up to and including the most complex in the set defined by r and s. This may at first not seem like the most efficient way of solving this problem, however it has several advantages. First of all it allows any integral less then r and s to be reduced towards a master integral, this saves individually programing in each of the integrals which require reducing. Further, this approach allows each reduction to a master integral to use the simplest set of integration by parts routines. This is an important feature as there may be a non-unique of performing reduction and this enables the use of routines with the fewest possible steps and therefore generates the least amount of additional algebra.

The final step in this process is the tidying up of the output so that it once again conforms to our notation, in our notation it would be possible for the integral int1aaa$(m_1, m_1, m_1, \alpha, \beta, \gamma)$ to be reduced to int1aaa$(m_1, m_1, m_1, 1, 0, 0)$ as this new integral is a vacuum as the $\beta = \gamma = 0$ this just leaves one terms in the denominator. In the notation we are using this can then be rewritten as int1a00$(m_1, 0, 0, 1, 0, 0)$. Preforming this at this point will save time when working out the value of each of these integrals, as it will prevent FORM from generating unnecessary amount of algebra when performing expansions. There are several of these that need defining as our notations and conventions are different from those native to REDUZE, however FORM is well suited for this sort of task.

3.6 3D-Integrals

In this section we shall describe the method for dealing with our newly calculated master integrals. These now need to be evaluated in 3-dimensions to give the final results for the power correction to the symmetric point. In this regard we are lucky that it is possible to use the work of [73] to evaluate the two and three point one loop integrals. We shall briefly define how these integrals are evaluated before going on to give the expansion used for the power corrections. We define our master integral as the following

$$\begin{aligned} &\text{int1abc}(m_1, m_2, m_3, 1, 1, 1) \\ &\qquad = \int_k \frac{1}{(k^2 + m_1^2)((k-p)^2 + m_2^2)((k+q)^2 + m_3^2)} \end{aligned} \tag{3.40}$$

here m_1, m_2 and m_3 represent the three potential masses. To allow us to easily follow [73] we shall define our different denominator terms as

$$K_1 = k^2$$
$$K_2 = (k-p)^2$$
$$K_3 = (k+q)^2 \tag{3.41}$$

This has been set up in the most general way possible to ensure this set up work in all possible cases. We also define the object

$$\mathbf{k_{ij}} = \mathbf{K_i} - \mathbf{K_j} \tag{3.42}$$

which shall be equal to our external momentum. At the symmetric point we are able to use our previously defined relations

$$k_{ij}^2 = p^2 = q^2 = r^2 = -\mu^2 \tag{3.43}$$

to greatly simplify the problem. The first step towards being able to solve this integral is to separate this into three separate terms via the uses of partial fractions.

$$\begin{aligned} I = \frac{1}{\pi^2}\int_k \Big[& \left(m_2^2 - m_1^2 + K_2^2 - K_1^2\right)^{-1}\left(m_3^2 - m_1^2 + K_3^2 - K_1^2\right)^{-1}\left(m_1^2 + K_1^2\right)^{-1} \\ & + \left(m_1^2 - m_2^2 + K_1^2 - K_2^2\right)^{-1}\left(m_3^2 - m_2^2 + K_3^2 - K_2^2\right)^{-1}\left(m_2^2 + K_2^2\right)^{-1} \\ & + \left(m_1^2 - m_3^2 + K_1^2 - K_3^2\right)^{-1}\left(m_2^2 - m_3^2 + K_2^2 - K_3^2\right)^{-1}\left(m_3^2 + K_3^2\right)^{-1}\Big] \end{aligned} \tag{3.44}$$

The method followed for each of these terms is the same but shall be preformed on them individually then summed at the end to give the final result. To continue with this process, it is necessary to shift to cylindrical coordinates, this is used to describe the position of K_1 in terms of new coordinates k, θ and k_z. The axis k_z is chosen in such a way that it is perpendicular to $\mathbf{k_{12}}$ and $\mathbf{k_{13}}$, further the angle θ is now the angle between $\mathbf{k_{12}}$ and $\mathbf{k_{13}}$.

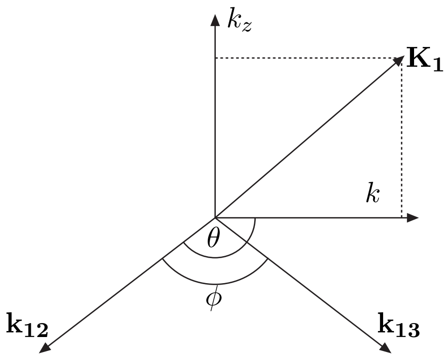

Figure 3.2: Redefined cylindrical coordinated system

This also changes our integration term in the normal way for cylindrical coordinate system

$$d^3K_1 = k\,dk\,d\theta\,dk_z \tag{3.45}$$

We can now define the following relations

$$\mathbf{k_{12}} \cdot \mathbf{K_1} = k_{12}\,k\,\cos\theta \qquad \mathbf{k_{13}} \cdot \mathbf{K_1} = k_{13}\,k\,\cos(\theta - \phi) \tag{3.46}$$

We can use our redefined coordinate system to write K_1^2 in terms of the new variables $K_1^2 = k^2 + k_z^2$. Further to this we can also rewrite K_1^2 in terms of $K_1^2 = k_{12}^2 + 2k_{12} \cdot K_2 + K_2^2$. This allows us to remove all independent K_1 terms from the first term in (3.44).

$$\frac{1}{\pi^2}\int k\,dk\,d\theta\,dk_z \qquad \Big[\big(m_2^2-m_1^2+k_{12}^2-k_{12}\,k\,\cos\theta\big)^{-1} \big(m_3^2-m_1^2+k_{13}^2-2k_{13}\,k\,\cos(\theta-\phi)\big)^{-1} \big(m_1^2+k^2+k_z^2\big)^{-1}\Big] \tag{3.47}$$

We can now integrate this with respect to k_z.

$$\int_{-\infty}^{\infty} dk_z\,\big(m_1^2+k^2+k_z^2\big)^{-1} = \left[\frac{\arctan(\frac{k_z}{(m_1^2+k^2)^{1/2}})}{(m_1^2+k^2)^{1/2}}\right]_{-\infty}^{\infty} = \pi \tag{3.48}$$

This leaves us with the integration over the last two variables,

$$2\int \frac{k\,dk}{\big(m_1^2+k^2\big)^{1/2}}\frac{d\theta}{2\pi}\Big[\big(m_2^2-m_1^2+k_{12}^2-k_{12}\,k\,\cos\theta\big)^{-1} \big(m_3^2-m_1^2+k_{13}^2-2k_{13}\,k\,\cos(\theta-\phi)\big)^{-1}\,\big(m_1^2+k^2\big)^{-1/2}\Big] \tag{3.49}$$

the integration over θ is preformed using a complex set of contour integration in the complex plane. We take the results and the final form of this from [73].

$$2\int \frac{k\,dk}{\big(m_1^2+k^2\big)^{1/2}}\left[\frac{1}{S_{12}}\frac{k_{12}}{\alpha_3-iS_{12}k_{13}\sin\phi}+\frac{1}{S_{13}}\frac{k_{13}}{\alpha_2-iS_{13}k_{12}\sin\phi}\right] \tag{3.50}$$

where

$$S_{12} = \big[\sigma_{12}-4k_{12}^2(m_1^2+k^2)\big]^{1/2} \tag{3.51}$$

$$\sigma_{12} = \big(m_2^2-m_1^2+k_{12}^2\big)^2+4m_1^2k_{12}^2 \tag{3.52}$$

$$\alpha_3 = k_{12}\big(m_3^2-m_1^2+k_{13}^2\big)-k_{13}\big(m_2^2-m_1^2+k_{12}^2\big)\cos\phi \tag{3.53}$$

and likewise for the second terms, exchanging the indices 2 and 3. As useful as this procedure seems, so far there is still the need to define the problem in terms of objects which are workable in a practical sense, for this reason it is advantageous to relate these objects to a set of dimensionless invariants.

$$y_{ij} = \frac{(m_i^2 + m_j^2 + k_{ij}^2)}{2m_i\, m_j} \tag{3.54}$$

$$D^{(3)} = \det|y_{ij}| \tag{3.55}$$

$$F_i^{(3)} = \frac{\partial}{\partial m_i}\left(m_i^2 D^{(3)}\right) \tag{3.56}$$

y_{ij} is the symmetric matrix defined with $i, j = 1, 2, 3$. The full expansions of (3.55) and (3.56) were produced using REDUCE for the purpose of saving time and minimising errors. As these invariants are defined in terms of terms used to construct the geometrical objects in our integrand it is possible to write (3.51), (3.52) and (3.53) in terms of (3.54), (3.55) and (3.56).

$$\sigma_{12} = \left(m_2^2 - m_1^2 + k_{12}^2\right)^2 + 4m_1^2 k_{12}^2 = 4m_1^2 m_2^2 y_{12}^2 - 4m_1^2 m_2^2 = m_1^2 m_2^2 (y_{12}^2 - 1) \tag{3.57}$$

Similarly for

$$\alpha_3 k_{12} = 2m_1^2 m_2^2 (F_3^{(3)})$$

$$\alpha_3^2 \sigma_{12} \sin^2\theta = 4m_1^2 m_2^2 m_3^2 D^{(3)}$$

These relations have made extensive use of the cosine formula to expand the scalar products and removing the ϕ dependent terms.

$$2\mathbf{k_{12}} \cdot \mathbf{k_{13}} = 2k_{12}k_{13}\cos\phi = k_{12}^2 + k_{13}^2 - k_{23}^2$$

$$4\, k_{12}^2 k_{13}^2 \sin^2\phi = 2k_{12}^2 k_{13}^2 + 2k_{12}^2 k_{23}^2 + 2k_{13}^2 k_{23}^2 - k_{12}^4 - k_{13}^4 - k_{23}^4$$

It is now possible to look at performing the final integration over k, for this it is necessary to introduce a change of variables. Taking the first part of equation (3.50)

$$\int \frac{k\,dk}{\left(m_1^2 + k^2\right)^{1/2}} \tag{3.58}$$

and defining the denominator such that

$$m_1^2 + k^2 = x^2 \tag{3.59}$$

This not only changes our integration variable

$$dx = \frac{k\,dk}{x} \tag{3.60}$$

but also modifies the parameter S_{12}

$$S_{12} = \left[\sigma_{12} - 4k_{12}^2(x^2)\right]^{1/2} \tag{3.61}$$

While this change goes some of the way to making this problem more solvable it does not fully eliminate the problem terms from (3.50), as S_{12} is still defined as the square root of the integration variable x. This can be dealt with by introducing a further change in variables

$$2\,k_{12}\,x = z + \frac{\sigma_{12}}{4z} \tag{3.62}$$

With z being the new and final integration parameter. Taking the square of (3.62) and substituting this into S_{12} yields

$$\begin{aligned} S_{12} &= \left[\sigma_{12} - z^2 - \frac{\sigma_{12}}{2} - \frac{\sigma_{12}^2}{(4z)^2}\right] \\ &= \left[-\left(z - \frac{\sigma_{12}}{4z}\right)^2\right]^{1/2} = i\left(z - \frac{\sigma_{12}}{4z}\right) \end{aligned} \tag{3.63}$$

Further we can write the integral dx in terms of the new variable z

$$\int dx \frac{k_{12}}{S_{12}} = \frac{-i}{2} \int \frac{dz}{z} \tag{3.64}$$

Taking the first term of (3.50) then yields

$$\int \frac{dz}{i} \frac{2k_{13} \sin\phi}{((2zk_{13} \sin\phi + \alpha_3) - 2m_1m_2m_3(D^3)^{1/2})((2zk_{13} \sin\phi + \alpha_3) + 2m_1m_2m_3(D^3)^{1/2})} \tag{3.65}$$

Finally this integration gives

$$\frac{i}{2(D^3)^{1/2}} \ln\left(\frac{2zk_{13} \sin\phi + \alpha_3 - 2m_1m_2m_3(D^3)^{1/2}}{2zk_{13} \sin\phi + \alpha_3 + 2m_1m_2m_3(D^3)^{1/2}}\right)\Bigg|_{z=z_0} \tag{3.66}$$

This can be split into real and imaginary parts by use of the identities for arctan

$$\frac{i}{2(D^3)^{1/2}} \arctan\left(\frac{m_3(m_2^2 - m_1^2 + k_{12}^2)}{2m_1^2m_2} \frac{(D^3)^{1/2}}{F^{(3)_3}}\right) \tag{3.67}$$

This gives the form of one of the the integrals, however this does not cover the different permutations that arise during the calculation. For these it is necessary to use the relations for the sum of arctan with different arguments but similar pre-factors.

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) \tag{3.68}$$

these are derived from the same logarithm functions that are used in the simplification of (3.66). These are used to combine the terms generated changing the co-ordinates to cylindrical polar form of the integral. Using this to combine both of the terms (3.50), yields

$$\frac{i}{2(D^3)^{1/2}} \arctan\left(\frac{(D^3)^{1/2}}{y_{23} - y_{12}y_{13}}\right) \tag{3.69}$$

This is one third of the final result, however it necessary to combine this with the final two terms of (3.44) to give full form of the integral.

$$\frac{1}{2(D^3)^{1/2}} \arctan\left(\frac{2(D^3)^{1/2}(1+y_{12}+y_{13}+y_{23})}{(1+y_{12}+y_{13}+y_{23})^2 - D^{(3)}}\right) \tag{3.70}$$

This can be further reduced with the double angle rule for arctan

$$\arctan\left(\frac{2x}{1-x^2}\right) = 2\arctan(x) \tag{3.71}$$

This allows the final result to be written in terms of the invariant terms we have defined earlier (3.54) - (3.56)

$$\text{int1abc}(m_1, m_2, m_3, 1, 1, 1) = \frac{1}{(D^3)^{1/2}} \arctan\left(\frac{(D^3)^{1/2}}{1+y_{12}+y_{13}+y_{23}}\right) \tag{3.72}$$

This result is surprising as it is in no way obvious from the start that this shall be the result for a 3-dimensional triangle diagram. While this result proves useful for our calculation, it is far from simple in its current form. We shall start by defining the matrix y_{ij} in our notation

$$y_{ij} = \begin{pmatrix} 1 & \frac{m_1^2+m_2^2-\mu^2}{2m_1m_2} & \frac{m_1^2+m_3^2-\mu^2}{2m_1m_3} \\ \frac{m_2^2+m_1^2-\mu^2}{2m_2m_1} & 1 & \frac{m_2^2+m_3^2-\mu^2}{2m_2m_3} \\ \frac{m_3^2+m_1^2-\mu^2}{2m_3m_1} & \frac{m_3^2+m_2^2-\mu^2}{2m_3m_2} & 1 \end{pmatrix} \tag{3.73}$$

here we have used the momentum relations for the symmetric point. The diagonal terms being one as $k_{ii} = 0$. It is possible to work out D^3 by taking the determinant of this matrix

$$\begin{aligned} D^3 \;=\; & \left(-m_1^4 + m_1^2m_2^2 + m_1^2m_3^2 + m_1^2\mu^2 - m_2^4 + m_2^2m_3^2\right. \\ & \left. + m_2^2\mu^2 - m_3^4 + m_3^2\mu^2 - \mu^4\right)\mu^2 \frac{1}{4m_1^2m_2^2m_3^2} \end{aligned} \tag{3.74}$$

As we do not wish to have the final result expressed as a function of arctan the final form shall be the Taylor expansion of (3.72) in powers of μ, this will give a power expansion in terms of the momentum for the symmetric point. There are similar possible expressions for the 3-dimensional 2-point function as well as the vacuum bubble [73] [77]. The 2-point function uses very similar method to the 3-point graphs, however having one fewer term in its denominator it is simpler.

$$\begin{aligned} \text{int1ab0}(m_1, m_2, 0, 1, 1, 0) &= \frac{1}{\pi} \int_k \frac{1}{(k^2 + m_1^2)((k-p)^2 + m_2^2)} \\ &= \frac{1}{\sqrt{-\mu^2}} \frac{1}{\pi} \arctan\left(\frac{\sqrt{-\mu^2}}{m_1 + m_2}\right) \end{aligned} \tag{3.75}$$

Once again as we do not wish to have our results as a function of arctan we shall expand these as a Taylor series. Finally the vacuum bubble can be worked out using methods similar to those used in Section 2.5, however with the dimension of the integral changed to $d = 3 - 2\epsilon$

$$\begin{aligned} \text{int1a00}(m_1, 0, 0, 1, 0, 0) &= \frac{1}{\pi^2} \int_k \frac{1}{(k^2 + m_1^2)} \\ &= \frac{m_1}{4\pi} \left(1 + \epsilon \left(2 + 2\ln(\frac{1}{2m_1}\right)\right) \end{aligned} \tag{3.76}$$

As we are going to be working in dimensional regularisation we shall be using the limit $\epsilon \to 0$ as such

$$\frac{1}{\pi^2} \int_k \frac{1}{(k^2 + m_1^2)} = \frac{m_1}{4\pi} \tag{3.77}$$

Using all of this it is possible to write the expanded form of each of our mass configurations.

Vacuum Bubble

$$\text{int1a00}(m_1, 0, 0, 1, 0, 0) = \frac{m_1}{4\pi} \left(1 + \epsilon \left(2 + 2\ln(\frac{1}{2m_1}\right)\right) \tag{3.78}$$

2-Point Diagram

$$\text{int1a00}(m_1,0,0,1,1,0) = \frac{1}{4m_1\pi} + \frac{1}{12m_1^3\pi}\mu^2 + \frac{1}{20m_1^5\pi}\mu^4 + \mathcal{O}(\mu^6) \quad (3.79)$$

$$\text{int1aa0}(m_1,m_1,0,1,1,0) = \frac{1}{8m_1\pi} + \frac{1}{96m_1^3\pi}\mu^2 + \frac{1}{640m_1^5\pi}\mu^4 + \mathcal{O}(\mu^6) \quad (3.80)$$

$$\begin{aligned}\text{int1ab0}(m_1,m_2,0,1,1,0) = {} & \frac{1}{4\pi}\frac{1}{m_1+m_2} + \frac{1}{12\pi}\frac{1}{(m_1+m_2)^3}\mu^2 \\ & + \frac{1}{20\pi}\frac{1}{(m_1+m_2)^5}\mu^4 + \mathcal{O}(\mu^6)\end{aligned} \quad (3.81)$$

3-Point Diagram

$$\begin{aligned}\text{int1a00}(m_1,0,0,1,1,1) = {} & \ln(-1)\frac{1}{8m_1^2\pi}\frac{1}{\mu} - \frac{1}{4m_1^3\pi} + \ln(-1)\frac{1}{16m_1^4\pi}\mu \\ & - \frac{1}{3m_1^5\pi}\mu^2 - \ln(-1)\frac{1}{64m_1^6\pi}\mu^3 - \frac{13}{60m_1^7\pi}\mu^4 + \mathcal{O}(\mu^6)\end{aligned} \quad (3.82)$$

$$\text{int1aa0}(m_1,m_1,0,1,1,1) = \frac{1}{8m_1^3\pi} + \frac{13}{96m_1^5}\pi\mu^2 + \frac{263}{1920m_1^7\pi}\mu^4 + \mathcal{O}(\mu^6) \quad (3.83)$$

$$\begin{aligned}\text{int1aa0}(m_1,m_2,0,1,1,1) = {} & \frac{1}{4\pi}\frac{1}{m_1m_2}\frac{1}{(m_1+m_2)} \\ & + \frac{1}{12\pi}\left(m_1^4 + 3m_1^3m_2 + 5m_1^2m_2^2 + 3m_1m_2^3 + m_2^4\right)\frac{1}{m_1^3m_2^3}\frac{1}{(m_1+m_2)^3}\mu^2 \\ & + \frac{1}{60\pi}\left(3m_1^8 + 15m_1^7m_2 + 34m_1^6m_2^2 + 50m_1^5m_2^3\right. \\ & + 59m_1^4m_2^4 + 50m_1^3m_2^5 + 34m_1^2m_2^6 + 15m_1m_2^7 \\ & \left.+ 3m_2^8\right)\frac{1}{(m_1+m_2)^5}\frac{1}{m_1^5m_2^5}\mu^4 + \mathcal{O}(\mu^6)\end{aligned} \quad (3.84)$$

$$\text{int1aaa}(m_1, m_1, m_1, 1, 1, 1) = \frac{1}{32m_1^3\pi} + \frac{1}{256m_1^5\pi}\mu^2 + \frac{1}{256m_1^7\pi}\mu^4 + \mathcal{O}(\mu^6) \tag{3.85}$$

$$\begin{aligned}\text{int1abb}(m_1, m_2, m_2, 1, 1, 1) = {} & \frac{1}{8\pi}\frac{1}{(m_1+m_2)^2 m_1} \\ & + \frac{1}{96\pi}\left(13m_1^2 + 4m_1m_2 + m_2^2\right)\frac{1}{(m_1+m_2)^4 m_1^3}\mu^2 \\ & + \frac{1}{1920\pi}\left(263m_1^4 + 138m_1^3m_2 + 58m_1^2m_2^2 + 18m_1m_2^3\right. \\ & \left. + 3m_2^4\right)\frac{1}{(m_1+m_2)^6 m_1^5}\mu^4 + \mathcal{O}(\mu^6)\end{aligned} \tag{3.86}$$

$$
\begin{aligned}
\text{int1abc}(m_1, m_2, m_3, 1, 1, 1) = {} & \frac{1}{4\pi} \frac{1}{((m_1+m_2)(m_1+m_3)(m_2+m_3))} \\
& + \frac{1}{12\pi}\left(m_1^4 + 3m_1^3 m_2 + 3m_1^3 m_3 + 5m_1^2 m_2^2\right. \\
& + 12m_1^2 m_2 m_3 + 5m_1^2 m_3^2 + 3m_1 m_2^3 + 12m_1 m_2^2 m_3 \\
& + 12m_1 m_2 m_3^2 + 3m_1 m_3^3 + m_2^4 + 3m_2^3 m_3 + 5m_2^2 m_3^2 \\
& \left. + 3m_2 m_3^3 + m_3^4\right) \frac{1}{(m_1+m_2)^3(m_1+m_3)^3(m_2+m_3)^3}\mu^2 \\
& + \frac{60}{\pi}\left(3m_1^8 + 15m_1^7 m_2 + 15m_1^7 m_3 + 34m_1^6 m_2^2\right. \\
& + 80m_1^6 m_2 m_3 + 34m_1^6 m_3^2 + 50m_1^5 m_2^3 + 195m_1^5 m_2^2 m_3 \\
& + 195m_1^5 m_2 m_3^2 + 50m_1^5 m_3^3 + 59m_1^4 m_2^4 + 305m_1^4 m_2^3 m_3 \\
& + 510m_1^4 m_2^2 m_3^2 + 305m_1^4 m_2 m_3^3 + 59m_1^4 m_3^4 + 50m_1^3 m_2^5 \\
& + 305m_1^3 m_2^4 m_3 + 710m_1^3 m_2^3 m_3^2 + 710m_1^3 m_2^2 m_3^3 + 305m_1^3 m_2 m_3^4 \\
& + 710m_1^2 m_2^3 m_3^3 + 510m_1^2 m_2^2 m_3^4 + 195m_1^2 m_2 m_3^5 + 34m_1^2 m_3^6 \\
& + 15m_1 m_2^7 + 80m_1 m_2^6 m_3 + 195m_1 m_2^5 m_3^2 + 305m_1 m_2^4 m_3^3 \\
& + 305m_1 m_2^3 m_3^4 + 195m_1 m_2^2 m_3^5 + 80m_1 m_2 m_3^6 + 15m_1 m_3^7 \\
& + 3m_2^8 + 15m_2^7 m_3 + 34m_2^6 m_3^2 + 50m_2^5 m_3^3 \\
& + 59m_2^4 m_3^4 + 50m_2^3 m_3^5 + 34m_2^2 m_3^6 + 15m_2 m_3^7 \\
& \left. + 3m_3^8\right) \frac{1}{(m_1+m_2)^5(m_1+m_3)^5(m_2+m_3)^5}\mu^4 + \mathcal{O}(\mu^6) \quad (3.87)
\end{aligned}
$$

These expressions have been given here up to order $\mathcal{O}(\mu^4)$ for the sake of compactness, however to compute the correction to the symmetric point to order $\mathcal{O}(\mu^6)$ these expressions have to be expressed to order $\mathcal{O}(\mu^{12})$. This is because of denominator factors introduced while doing some of the simplifications. Clearly computing these expressions to such a high order introduced a large number of terms into each expression and made the calculation rather cumbersome to perform, however thanks to the process of automating the calculation this did not prove to be an insurmountable problem.

3.7 Simplifications

Before being able to use the methods of Sections 3.5 and 3.6 to express the final scalar amplitudes, there are still some algebraic simplifications that need to take place, most of them involve removing the sums of mass from the denominator or expressing terms in an irreducible manner. In Section 3.6 we begin this process by the way in which the denominator terms in the integral expansions are defined. By keeping these as the sum of two different mass terms we have saved ourselves the effort of having to re-sum these. Instead these can be dealt with in their current form.

$$\frac{1}{m_- + m_+} = \frac{m_+ - m_-}{m_+^2 - m_-^2}; \tag{3.88}$$

$$\frac{1}{m_+ + \mu_i} = \frac{m_+ - \mu_i}{m_+^2 - \mu_i^2}; \tag{3.89}$$

$$\frac{1}{m_- + \mu_i} = \frac{m_- - \mu_i}{m_-^2 - \mu_i^2} \tag{3.90}$$

In these μ_i can be the mass term associated with either of the $\mu_{\mathcal{R}}$ or $\mu_{\mathcal{Q}}$ colour channels. The purpose of this is that it removes the some of objects which otherwise be the square root of a square root from the denominator of our expansions. The new denominator terms can now be reduced using the relations between our masses.

$$m_+^2 - \mu_i^2 = -m_-^2 \tag{3.91}$$

$$m_-^2 - \mu_i^2 = -m_+^2 \tag{3.92}$$

These relations are implemented in such a way that they work for powers of $(m_+^2 - \mu_i^2)^a$ where a could be either positive or negative. These denominator terms of single masses can now be used to cancel masses in the numerator coming from the integral expansions. Next it's possible to use partial fractions, to rewrite the product of denominator terms as their sum

$$\frac{1}{m_+^2 + m_-^2} \frac{1}{m_+^2 - m_-^2} = \frac{1}{2}\left(\frac{1}{m_+^2}\frac{1}{m_+^2 + m_-^2} + \frac{1}{m_+^2}\frac{1}{m_+^2 - m_-^2}\right) \tag{3.93}$$

Further we wish to remove all quadratic power of mass m_+ and m_- from the denominator.

$$\frac{1}{m_+^2} = \frac{m_-^2}{C_A \gamma^4} \tag{3.94}$$

$$\frac{1}{m_-^2} = \frac{m_+^2}{C_A \gamma^4} \tag{3.95}$$

This is achieved by using the difference of two squares and multiplying through by a factor of one.

$$\begin{aligned} \frac{1}{m_-^2} &= \frac{1}{\mu_i^2 - m_+^2} = \frac{1}{\mu_i^2 - m_+^2}\frac{\mu_i^2 + m_-^2}{\mu_i^2 + m_-^2} = \frac{\mu_i^2 + m_-^2}{\mu_i^4 - m_+^2 m_-^2} \\ &= \frac{\mu_i^2 + m_+^2}{\mu_i^4 - \mu_i^4 + C_A \gamma^4} = \frac{m_+^2}{C_A \gamma^4} \end{aligned} \tag{3.96}$$

This formula can also be used to remove single powers of m_+ and m_- from the denominator

$$\frac{1}{m_+} = \frac{m_+ m_-^2}{C_A \gamma^4} \tag{3.97}$$

$$\frac{1}{m_-} = \frac{m_- m_+^2}{C_A \gamma^4} \tag{3.98}$$

They work by simply multiplying through by whichever mass term requires cancelling. The final reduction we require is

$$\frac{1}{m_+^2 - m_-^2} = \frac{\sqrt{\mu_i^4 - 4C_A \gamma^4}}{\mu_i^4 - 4C_A \gamma^4} \tag{3.99}$$

This is simply removing the square root from the denominator. The aim of these reductions is to have all the terms expressed as irreducible terms in both the numerator and denominator. By doing this, the final results for the amplitudes should be given in terms μ^i, m_+ and m_- where i can be any real number. The factors of m_+ and m_- should only appear in the numerator and should not have any exponent. The order of reduction given here is the order in which they are applied in, deviating from this changes how the reductions are carried out and in some cases does not reduce the amplitudes down to the irreducible point.

3.8 Results

We can now give the power correction for the systemic point for each amplitude in up to order $\mathcal{O}(\mu^4)$ the full results have been computed to $\mathcal{O}(\mu^8)$ however these results are far too numerous to list here. These results all have a common factor of g^2 which would be expected at 1-loop. These result are given as series of μ which is our external momentum. Also there is extensive use of formula (3.20) to help to simplify these results, we give this again for ease of reference.

$$m_\pm^2 = \frac{1}{2}\left[-\mu_i^2 \pm \sqrt{\mu_i^4 - 4C_A\gamma^4}\right] \tag{3.100}$$

These results agree with the pure Gribov-Zwanziger case in the limit $\mu_i \to 0$ we have omitted these limit in this article as they can be inferred from those here. The following is only a section of the full result, the full results are contained in Appendix A

Q Channel

$$
\begin{aligned}
\Sigma^{qqv}_{(1)}(p,q,\gamma^2,\mu_Q) \;=\;& 1 + g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{13}{960}[m_+ + m_-]\right) \\
&+g^2\pi^{-1}C_A\left(+\frac{1}{32}\mu_Q^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
&+g^2\pi^{-1}C_A\mu^2\left(+\frac{11}{240}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
&+g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{20}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
&+g^2\pi^{-1}C_FC_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{24}[m_+ + m_-]\right) \\
&+ig^2\mu^3\gamma^{-4}\left(+\frac{1}{384}\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{13}{960}\mu_Q^{-2}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\left(+\frac{13}{96}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{13}{240}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{41}{480}\mu_Q^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\gamma^4\left(-\frac{1}{4}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\mu_Q^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_FC_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{24}\mu_Q^{-2}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F\left(-\frac{1}{6}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F\mu^2\left(+\frac{1}{6}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
&+\mathcal{O}(\mu^4)
\end{aligned} \tag{3.101}
$$

It may at first seem surprising that these results agree with the pure Gribov-Zwanziger case in the $\mu_i \to 0$ as it would appears there will be divergent terms from dividing by zero. However looking closer at (3.101) the four possible divergent terms will cancel each other in the $\mu_i \to 0$ limit. This provides us with a strong check on our work as without this cancellation our result would not be well defined. There is another possible check on this result from dimensional analysis, as our scalar amplitudes should be dimensionless. It is possible to see this once we define the dimension of our terms in 3-dimensions.

$$[g] = \frac{1}{2}\,,\ [\gamma] = 1\,,\ [\mu_i] = 1\,,\ [\mu] = 1 \text{ and } [m_{\pm}] = 1 \tag{3.102}$$

From this it is possible to see that each of the terms in (3.101) are dimensionless, this provides us with another important self consistency check. While it is difficult to work with the results due to the complex nature, it is possible to describe some trends in the results by looking at the behaviour at the low momentum limit. This limit of $\mu \to 0$ shows different effects for each of the vertex functions. Both of the different colour channels produce the same effect in the limit of $\mu \to 0$. However comparing the like for like terms it is possible to see that while the structure is the same they do in fact have different coefficients. This difference means that while the behaviour at the limit will be the same, the path they take to reach there shall be different. This hopefully will allow for comparison between the channel to be possible.

Separate from this is the value that each of the vertex function take at the low momentum limit. From our results it would appear that the low momentum for the gluon triple vertex is divergent, the quark gluon vertex freezes to a non-zero value and finally the ghost gluon vertex freezes to a zero value. These results are consistent for both of our colour channels. The result for the gluon triple vertex is the most surprising and at this point it is unclear if this is the genuine nature of the underling physics or a consequence of preforming this calculation in 3-dimensions. It can be seen that the integral int1a00$(m_1, 0, 0, 1, 1, 1)$ (3.82) produces terms that are both divergent in the low momentum limit of μ and containing imaginary parts. Pending results of this calculation in 4-dimensions should hopefully show if this feature is to be expected or if there is something not fully understood in our model. The remaining two vertex function give results that seem more reasonable. However with data from ether a 4-dimensional model or lattice data there is very little that can be achieved with this data along. hopefully in coming time more work in this field will allow for a greater understanding of the results produced here.

3.9 Discussion

The results shown in section 3.8 are the one loop corrections to each of the three vertices at the symmetric point. These can be seen to agree in the limit $\mu_i \to 0$ which covers the pure Gribov-Zwanziger case and it also provides an important check on our work. Further to this it is possible to see some difference between the solutions for the $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{Q}}$ channels. While this difference is not very noticeable, differing mainly in coefficients rather than the structure of the solutions. This however is not surprising given the similar nature of the propagator. However, this does mean that the symmetric point is sensitive to which if any of the alternative refined colour channel does appear. In principle it should be possible to compare with a suitable lattice calculation should one be preformed. Also by choosing to work in 3-dimensions we can also easily implement the exact value of the Gribov mass in terms of g^2 into our results

$$\frac{3}{4} = \frac{\sqrt{2} C_A^{3/4} g^2}{16\pi\gamma} + \mathcal{O}(g^4) \tag{3.103}$$

This has is a distinct advantage over the case in 4-dimensions, where the relation between the two quantities is dependent on logarithmic terms. To further increase the accuracy of these results and potentially see bigger difference between the colour channels it should be possible to perform this calculation to next to leading order at 2-loops [75]. However the trade off with this is that the calculation would be difficult and time consuming to perform. As part of this careful consideration would have to be paid to non-planar diagrams as well as the sheer number of diagrams overall. The number of Feynman diagrams required for the 2-loop triple gluon vertex is 3626 and this would be for each colour channel. This is an increase from 30 at 1-loop. This would means in excess of over 10,000 diagrams to look at the possible situation we have examined here for the gluon triple vertex without considering the other vertex corrections. While it would be possible to perform this calculation in the more standard four dimensions this is not without its problems. Firstly looking to any potential lattice calculation, performing the calculation in 4-dimensions will be more computationally expensive and more difficult. This is not the only problem as there are some technical issues to in perturbation theory as well, foremost of these is dealing with integrals with 3 independent mass parameters. While a large number of these are know for the setup we would require, there are still some configurations that are not know. This problem is exacerbated should the need to perform calculation at 2-loops arise. This is not to say there is no chance of this calculation being carried out in 4-dimensions, however at the moment the situation does not look as promising as it does for our 3-dimensional case. The final aim for this project is to make the full data set available along with the publication, this will allow other research groups to perform comprehensive studies using our data.

Chapter 4

Operator Correlation Function

4.1 Introduction

The last part of this work shall focus on a calculation unrelated to the previous ones in the Gribov-Zwanziger Lagrangian however, it was actually preformed first as a introduction to many methods and concepts used when considering the more complex cases involving the mass gap. This chapter looks at correlation functions of gauge invariant operators, this can be expressed as gauge invariant Green's functions. This method was a evolution of the previous method of using gauge fixed operators inside a quark two point function which has been used on the lattice, while it was possible to perform this type of calculation on the lattice it is also not without potential problems arising from the gauge fixing [78][79] [80][81][82][83]. The aim therefore was to provide results comparable with the lattice data in a way that does not have the same issues with gauge fixing. This was then further extended to operators which could include a γ^5 term. The reason for doing this is to test if it is possible to construct a suitable theory in case it should be needed when looking at something more complex like polarized scattering. We shall begin by first reviewing the method of using a quark two point function before moving on to consider the correlation function. Let us consider the quark two point function with an operator insertion.

$$\langle \psi(p_1) \mathcal{O}^i(-p_1 - p_2) \bar{\psi}(p_2) \rangle \tag{4.1}$$

Here ψ is the quark field. In this p is momentum and when $p_2 = -p_1$ the operator can take the form

$$\text{Scalar } S = \bar{\psi}\psi \tag{4.2}$$

$$\text{Vector } V = \bar{\psi}\gamma^\mu\psi \tag{4.3}$$

$$\text{Tensor } T = \bar{\psi}\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\psi \tag{4.4}$$

this is not the full list of operators that are possible as there are higher Wilson operators containing derivative terms. We shall not be considering these here as our interest in adding the γ^5 is a test to see if this is feasible in a simpler case of the model. For this calculation of the correlation function we shall be working in the standard QCD Lagrangian with massless quarks. Our Lagrangian includes the normal gauge fixing term however this shall not appear in our final result, although this is not the same as having set $\alpha = 0$ at any point during this work. This is the prime motivation from moving away from the previous method of quark currents to this new set up. This natural cancelling of the gauge parameter α provides a strong error check on our methods and results as it checks the consistency of the Feynman rules and the operator renormalization. With this in mind we define our correlation function schematically in Figure 4.1.

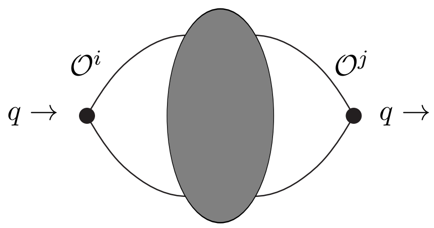

Figure 4.1: Operator correlation function $\left\langle \mathcal{O}^i(q)\mathcal{O}^j(-q) \right\rangle$

The notation here requires explaining before we can continue, the subscripts i and j denote the left and right operators $\mathcal{O}$ of the correlation function with the momentum flowing in through the left one and out of the right one. The flavour aspects of this equation will be discussed later on. Each of these operators may carry its own set of Lorentz indices $\{\mu_i\}$ and $\{\nu_j\}$ which when present will require a suitable projection operator, this is discussed in more detail later on. The formal definition of the correlation function is as follows

$$\Pi^{ij}_{\mu_1 \dots \mu_{n_i}, \nu_1 \dots \nu_{n_j}}(q^2) = (4\pi)^2 i \int d^4x e^{iqx} \langle 0 | O^i_{\mu_1 \dots \mu_{n_i}}(x) O^j_{\nu_1 \dots \nu_{n_j}}(0) | 0 \rangle \quad (4.5)$$

In this calculation we shall be closely following the approach of [84] [85] [86] in computing the correlation function as well as extending it to new operators involving γ^5. In principle it is possible to consider the case where i and j are different operators then there would need to be careful consideration of any mixing involved in the renormalization procedure used, fortunately this is not the case for this calculation as we shall be considering the two new operators

$$\text{Pseudo Scalar } P = \bar{\psi}\gamma^5\psi \quad (4.6)$$

$$\text{Axial Vector } A = \bar{\psi}\gamma^\mu\gamma^5\psi \quad (4.7)$$

these are both modified versions of the scalar and vector operators with the γ^5

included. For the purposes of this calculation we shall begin with recalculating the results for the scalar, vector and tensor operators. Not only shall we require some of these results while working on the pseudo scalar and axial vector cases but it shall provide important checks on our method so we know that our method is correct before continuing with new results.

4.2 Projection

We shall begin by calculating unrenormalized scalar amplitudes using the appropriate projections, before going on to discuss the renormalization for each of these. The method of projection is similar to those carried out in Section 3.3 but with a different set of amplitudes. The next step of the calculation is to break the correlation function down into a scalar amplitude $\Pi^{ij}(q)$.

$$\Pi^{ij}_{\mu_1....\mu_{n_i},\nu_1....\nu_{n_j}}(q^2) = \sum_{k=1}^{n_{ij}} P^{ij}_{(k)(\mu_1....\mu_{n_i}|\nu_1....\nu_{n_j})}(q)\Pi^{ij}_{(k)}(q) \tag{4.8}$$

This is achieved by the use of Lorentz projectors $P^{ij}_{(k)(\mu_1....\mu_{n_i}|\nu_1....\nu_{n_j})}(q)$, the indice $\mu_1....\mu_{n_i}$ and $\nu_1....\nu_{n_j}$ come from the left and right operators respectively. Here i j label the number of indices in each of the operators. The motivation for using the method of projections in this case is that it allows for the speeding up of the calculation when working with the higher order Wilson operators. While not strictly necessary for the operators we are working with in this case, due to their simple nature they have been used as the method has been used extensively for other areas of study and this case provides a suitable introduction to the method. As before define the matrix $\mathcal{N}^{ij}_{kl}$ where k and l label the projectors

$$\mathcal{N}^{ij}_{kl} = P^{ij}_{(k)(\mu_1....\mu_{n_i}|\nu_1....\nu_{n_j})}(q)P^{ij\ (\mu_1....\mu_{n_i}|\nu_1....\nu_{n_j})}_{(l)}(q) \tag{4.9}$$

here there is no sum over i and j, the elements in the matrix. The method used to construct the tensor basis is to write down the complete set of tensors built from the metric $\eta_{\mu\nu}$ and the momentum q_μ, these need to have the same number of free indices as the operator in question. The mass scalar and pseudo scalar have no free indices, where the vector and tensor operators have 2 and 4 respectively. The explicit form of the tensor basis we shall use is the following for the vector

$$\mathcal{P}^{V,V}_{(1)\{\mu|\nu\}}(q) = \eta_{\nu\mu} - \frac{q_\mu q_\nu}{q^2} \quad , \quad \mathcal{P}^{V,V}_{(2)\{\mu|\nu\}}(q) = \frac{q_\mu q_\nu}{q^2} \tag{4.10}$$

and for the tensor operator

$$\begin{aligned}
\mathcal{P}^{T,T}_{(1)\{\mu\nu|\sigma\rho\}}(q) &= \eta_{\nu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma} \\
\mathcal{P}^{T,T}_{(2)\{\mu\nu|\sigma\rho\}}(q) &= \eta_{\nu\sigma}\frac{q_\nu q_\rho}{q^2} - \eta_{\nu\rho}\frac{q_\nu q_\sigma}{q^2} - \eta_{\mu\sigma}\frac{q_\mu q_\rho}{q^2} + \eta_{\nu\rho}\frac{q_\mu q_\sigma}{q^2}
\end{aligned} \tag{4.11}$$

Our matrix $\mathcal{N}^{ij}_{kl}$ is then formed by taking the i and j components of these elements, in our cases the matrix shall be a 2 x 2 matrix of polynomials in d as these are formed of Lorenz contractions in our d-dimensional dimensional regularization. As before we define $\mathcal{M}_{kl}(p,q)$ such that (3.6)

$$\mathcal{M}_{kl}(p,q)\mathcal{N}_{lk'}(p,q) = \delta_{kk'} \tag{4.12}$$

this means that $\mathcal{M}_{kl}(p,q)$ shall be in the inverse of our matrix of tensor basis. We shall now define $\mathcal{N}^{ij}_{kl}$ and use this to work out the exact form of the matrix $\mathcal{M}_{kl}(p,q)$. For our cases

$$\mathcal{N}^{V,V} = \begin{pmatrix} d-1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.13}$$

and

$$\mathcal{N}^{T,T} = \begin{pmatrix} 2(d-1)d & 4(d-1) \\ 4(d-1) & 4(d-1) \end{pmatrix} \tag{4.14}$$

Taking the inverse of these gives

$$\mathcal{M}^{V,V} = \frac{1}{(d-1)} \begin{pmatrix} 1 & 0 \\ 0 & d-1 \end{pmatrix} \tag{4.15}$$

and

$$\mathcal{M}^{T,T} = \frac{1}{4(d-1)(d-2)} \begin{pmatrix} 2 & -2 \\ -2 & d \end{pmatrix} \tag{4.16}$$

With these done it is possible to take the final form of the scalar amplitude by multiplying both sides of equation (4.8) by $\mathcal{M}_{kl}\mathcal{P}^{ij}_{(l)}$ to give

$$\mathcal{M}_{kl}\mathcal{P}^{ij}_{(l)}\Pi^{ij}(q^2) \quad = \quad \mathcal{M}_{kl}\mathcal{P}^{ij}_{(l)}P^{ij}_{(k')}(q)\Pi^{ij}_{(k')}(q) = \Pi^{ij}_{(k)}(q) \tag{4.17}$$

Here we have dropped some of the subscripts to make the notation clearer. This is similar to the form used in (3.8), it breaks down correlation function into the sum of each projection. From this point onwards each of the projections will be dealt with individually. The renormalization for each of the projections will be the same but they will be run on the own within Form. The renormalization of the operator correlation is undertaken via the method of multiplicative renormalization

4.3 Renormalization

The first step towards Renormalizing our operator is to perform the calculation with bare parameters then these are replaced with renormalized ones to remove the divergences. This also includes introducing an arbitrary scaling constant too because of the dimensional regularization. We shall use the normal renormalization parameters for QCD

$$
\begin{aligned}
A_{\rm o}^{a\,\mu} &= \sqrt{Z_A}\,A^{a\,\mu} \;\;,\;\; c_{\rm o}^a \;=\; \sqrt{Z_c}\,c^a \;\;,\;\; \bar{c}_{\rm o}^a \;=\; \sqrt{Z_c}\,\bar{c}^a \;\;, \\
\psi_{\rm o} &= \sqrt{Z_\psi}\psi \qquad , \qquad \bar{\psi}_{\rm o} \;=\; \sqrt{Z_\psi}\bar{\psi} \\
g_{\rm o} &= \mu^\epsilon Z_g\, g_r \qquad , \qquad \alpha_{\rm o} \;=\; Z_\alpha^{-1} Z_A\,\alpha
\end{aligned}
\tag{4.18}
$$

These are the same as the ones previously used in (2.34) without those defined for the localizing ghosts. The arbitrary scaling constant is introduced on the renormalization for the coupling constant g. We define the renormalization of our correlation function as follows

$$
\Pi^{i,j}_{(k)}(q) = Z^{i,j}_{(k)} q^2 + \mu^{2\epsilon} \left(Z^i\right)^2 \Pi^{i,j}_{O(k)}(q) \tag{4.19}
$$

here the indices i and j are the left and right operators respectively and k is the projection. $Z^{ij}_{(k)}$ is a new term introduced for the contact renormalization and Z^i is the renormalization associated with the operator itself. These have been worked out from the quark current renormalization of earlier work [87]. The first term on the right hand side of (4.19) arises from the divergent term of the operator mixing. When working with i and j different operators then there is a mixing matrix that deals with how the counter terms arise, however in our case we shall not need to worry about this as we are dealing with repeated operators. We can now begin by calculating the value of the operator renormalization, to do so we require the bare amplitude and for this we require the complete set of Feynman diagrams. Our tool of choice for this is the program QGRAF which we used to draw the Feynman diagrams in symbolic notation. Our QGRAF setup shall include all the normal interactions for QCD as well as have the extra incoming and outgoing operators on the external legs. This can be set up in such a way so that the operators do not appear as internal propagator like terms. As each of the correlation functions uses the same Feynman diagrams, we shall generate a general set of diagrams which can then be tailored to our required operator setup. For the correlation function we require one 1-loop diagram, eight 2-loop diagrams and finally one hundred and nine 3-loop diagrams. With this done we pass these diagrams onto FORM in which we use the package MINCER to reduce these diagrams to master ones. We are able to use the powerful MINCER package as we are working in a two point massless theory. This is a further advantage of working with correlations function as opposed to the previous method of computing quark currents. It is now possible to work out the exact value of the contact renormalization for each of the different operators. To do this we take a general form of the renormalization term in the following form

$$Z^{ii} = \frac{Z_1^{ii}}{\epsilon} + \left(\frac{Z_{21}^{ii}}{\epsilon} + \frac{Z_{22}^{ii}}{\epsilon^2} \right) g_r^2 + \left(\frac{Z_{31}^{ii}}{\epsilon} + \frac{Z_{32}^{ii}}{\epsilon^2} + \frac{Z_{33}^{ii}}{\epsilon^3} \right) g_r^4 + \mathcal{O}(g_r^6) \qquad (4.20)$$

where Z_1^{ii}, Z_{21}^{ii}, Z_{22}^{ii}, Z_{31}^{ii}, Z_{32}^{ii} and Z_{33}^{ii} are the quantities we fix in order to cancel the divergences in the theory, these parameters should be simple numbers and should not contain the gauge parameter α as our correlation function must be independent of α. With this it is possible to work out the counter terms for each of the operators. For the scalar we obtain

$$\begin{aligned} Z_{(1)}^{SS} &= -2N_cN_f\frac{1}{\epsilon} + \left(-5N_cN_fC_F\frac{1}{\epsilon} + 6N_cN_fC_F\frac{1}{\epsilon^2} \right) g^2 \\ &+ \left(\left(-\frac{22}{3}N_cN_fC_FC_A - 12N_cN_fC_F^2 + \frac{8}{3}N_f^2N_cT_fC_F \right) \frac{1}{\epsilon^3} \right. \\ &+ \left(+\frac{83}{3}N_cN_fC_FC_A + 12N_cN_fC_F^2 - \frac{20}{3}N_f^2N_cT_fC_F \right) \frac{1}{\epsilon^2} \\ &+ \left(\left(-\frac{77}{3} + 12\zeta(3) \right) N_cN_fC_FC_A + \left(\frac{119}{6} - 24\zeta(3)N_cN_fC_F^2 \right) \right. \\ &\left. \left. + \frac{16}{3}N_f^2N_cT_fC_F \right) \frac{1}{\epsilon} \right) g^4 + \mathcal{O}(g^6) \end{aligned} \qquad (4.21)$$

For the vector

$$
\begin{aligned}
Z_{(1)}^{VV} &= \frac{4}{3}N_f N_c \frac{1}{\epsilon} + 2N_f N_c C_F \frac{1}{\epsilon} g^2 \\
&+ \left(\left(-\frac{22}{9} N_f N_c C_F C_A + \frac{8}{9} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon^2} \right. \\
&+ \left(\frac{133}{27} N_f N_c C_F C_A - \frac{2}{3} N_f N_c C_F^2 \right. \\
&- \left. \left. \frac{44}{27} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon} \right) g^4 + \mathcal{O}(g^6) \qquad (4.22)
\end{aligned}
$$

and

$$
Z_{(2)}^{VV} = \mathcal{O}(g^6) \qquad (4.23)
$$

Finally for the tensor operator

$$
\begin{aligned}
Z_{(1)}^{TT} &= +\frac{2}{3} N_f N_c \frac{1}{\epsilon} + \left(\frac{2}{3} N_f N_c C_F \frac{1}{\epsilon^2} + \frac{11}{9} N_f N_c C_F \frac{1}{\epsilon} \right) g^2 \\
&+ \left(\left(-\frac{22}{27} N_f N_c C_F C_A + \frac{4}{9} N_f N_c C_F^2 + \frac{8}{27} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon^3} \right. \\
&+ \left(\frac{131}{27} N_f N_c C_F C_A - \frac{92}{27} N_f N_c C_F^2 - \frac{20}{27} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon^2} \\
&+ \left(\left(-\frac{787}{243} + \frac{28}{9} \zeta 3 \right) N_f N_c C_F C_A + \left(+\frac{1601}{162} - \frac{56}{9} \zeta 3 \right) N_f N_c C_F^2 \right. \\
&+ \left. \left. \frac{8}{243} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon} \right) g^4 + \mathcal{O}(g^6) \qquad (4.24)
\end{aligned}
$$

and

$$
\begin{aligned}
Z_{(2)}^{TT} &= -\frac{4}{3}N_f N_c \frac{1}{\epsilon} + \left(-\frac{4}{3}N_f N_c C_F \frac{1}{\epsilon^2} - \frac{34}{9}N_f N_c C_F \frac{1}{\epsilon}\right) g^2 \\
&+ \left(\left(\frac{44}{27}N_f N_c C_F C_A - \frac{8}{9}N_f N_c C_F^2 - \frac{16}{27}N_f^2 N_c T_f C_F\right)\frac{1}{\epsilon^3}\right. \\
&+ \left(-\frac{218}{27}N_f N_c C_F C_A + \frac{160}{27}N_f N_c C_F^2 + \frac{8}{9}N_f^2 N_c T_f C_F\right)\frac{1}{\epsilon^2} \\
&+ \left(\left(-\frac{388}{243} - \frac{56}{9}\zeta 3\right) N_f N_c C_F C_A + \left(-\frac{1121}{81} + \frac{112}{9}\zeta 3\right) N_f N_c C_F^2\right. \\
&+ \left.\left.\frac{200}{243}N_f^2 N_c T_f C_F\right)\frac{1}{\epsilon}\right) g^4 + \mathcal{O}(g^6) \qquad (4.25)
\end{aligned}
$$

It is now time to address the flavour non-singlet issue. In the previous formula given for the Green's function and the correlation function there has been a flavour matrix left out. This is not to say that is it something that is being completely ignored. For each of the operators there is a flavour matrix λ^i where the index denotes the right and left operators. To exclude the flavour singlet terms we impose the condition $\mathrm{tr}(\lambda^i)\mathrm{tr}(\lambda^j) = 0$. These would be the terms that would contribute to the flavour non-singlet part of the formalism. This leaves only terms which contain $\mathrm{tr}(\lambda^i \lambda^j)$. These then disappear as they are set to unity, removing them from any subsequent equation. It can clearly be seen from these, that each of the projections gives a different result for the contact renormalization. As in the previous case it is also possible to construct the related renormalization group equation for each of the correlation functions [88].

$$0 = \mu \frac{d}{d\mu}\Pi_{(k)}^{i,i}(q) + 2\gamma^i(a)\Pi_{(k)}^{i,i}(q) - q^2\gamma_{(k)}^{i,i}(a) \qquad (4.26)$$

here the index i is our repeated operator. To simplify our notation we have written the coupling constant in terms of

$$a = \frac{g^2}{16\pi^2} \qquad (4.27)$$

The contact anomalous dimension is given by

$$\gamma_k^{i,i}(a) = \left[-\epsilon + \beta(a)\frac{\partial}{\partial\alpha} + 2\gamma^i\right] Z_k^{i,i} \tag{4.28}$$

these equation are similar to those we have used in Section 2.2 while building the renormalization effective potential. For each of projections it is possible to work out the explicit form of the anomalous dimension. For the scalar we find

$$\begin{aligned}\gamma^{S,S}(a) =& d(R)\left[2 \ + \ 10C_F a \ + \ \frac{C_F}{2}\left[(154 - 72\zeta(3))C_A\right.\right.\\ &\left.\left. + \ (144\zeta(3) - 119)C_F - 32T_F N_f\right] a^2\right] \ + \ \mathcal{O}(a^3) \ . \end{aligned}\tag{4.29}$$

For the vector

$$\gamma^{V,V}(a) \ = \ d(R)\left[-\frac{4}{3} \ - \ 4C_F a \ + \ \frac{C_F}{9}\left[18C_F - 133C_A + 44T_F N_f\right] a^2\right] \ + \ \mathcal{O}(a^3) \ . \tag{4.30}$$

and

$$\gamma_{(2)}^{VV} = \mathcal{O}(a^3) \tag{4.31}$$

Finally for the tensor operator

$$\begin{aligned}\gamma_{(1)}^{T,T}(a) =& d(R)\left[-\frac{2}{3} \ - \ \frac{22}{9}C_F a \ + \ \frac{C_F}{162}\left[(3024\zeta(3) - 4803)C_F\right.\right.\\ &\left.\left. + \ (1574 - 1512\zeta(3))C_A - 16T_F N_f\right] a^2\right] \ + \ \mathcal{O}(a^3)\end{aligned}\tag{4.32}$$

and

$$\gamma^{T,T}_{(2)}(a) = d(R)\left[\frac{4}{3} \ + \ \frac{68}{9}C_F a \ + \ \frac{C_F}{81}\left[(1512\zeta(3)+388)C_A \right.\right.$$
$$\left.\left. + \ (3363-3024\zeta(3))C_F - 200T_F N_f\right]a^2\right] \ + \ \mathcal{O}(a^3) \ . \qquad (4.33)$$

Finally all that is left is to use these results to calculate the final forms for the correlation functions. In each case a common factor of q^2 is removed. Also in the following equations $d(R)$ is the dimension of the representation used and

$$\ell = \ln\left(\frac{\mu^2}{q^2}\right) \qquad (4.34)$$

$$\Pi^{S,S}(a) = d(R)\left[4+2\ell \ + \ C_F\left[\frac{131}{2} - 24\zeta(3) + 34\ell + 6\ell^2\right]a \right.$$
$$+ \ C_F\left[\left(64\zeta(3) - \frac{2044}{9} - 130\ell + 32\zeta(3)\ell - \frac{88}{3}\ell^2 - \frac{8}{3}\ell^3\right)T_F N_f\right.$$
$$+ \ \left(\frac{14419}{18} - 300\zeta(3) - 18\zeta(4) - 40\zeta(5) + \frac{893}{2}\ell\right.$$
$$\left. - \ 124\zeta(3)\ell + \frac{284}{3}\ell^2 + \frac{22}{3}\ell^3\right)C_A$$
$$+ \ \left(\frac{1613}{4} - 384\zeta(3) + 36\zeta(4) + 240\zeta(5) + \frac{691}{2}\ell\right.$$
$$\left.\left.\left. - \ 72\zeta(3)\ell + 105\ell^2 + 12\ell^3\right)C_F\right]a^2\right] \ + \ \mathcal{O}(a^3) \qquad (4.35)$$

$$\Pi^{V,V}_{(1)}(a) = d(R)\left[-\frac{20}{9} - \frac{4}{3}\ell \ + \ C_F\left[16\zeta(3) - \frac{55}{3} - 4\ell\right]a\right.$$
$$+ \ C_F\left[\left(\frac{7402}{81} - \frac{608}{9}\zeta(3) + \frac{88}{3}\ell - \frac{64}{3}\zeta(3)\ell + \frac{8}{3}\ell^2\right)T_F N_f\right.$$
$$+ \ \left(\frac{1816}{9}\zeta(3) + \frac{80}{3}\zeta(5) - \frac{44215}{162} - 82\ell + \frac{176}{3}\zeta(3)\ell - \frac{22}{3}\ell^2\right)C_A$$
$$\left.\left. + \ \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5) + 2\ell\right)C_F\right]a^2\right] \ + \ \mathcal{O}(a^3) \qquad (4.36)$$

$$\Pi^{V,V}_{(2)}(a) = \mathcal{O}(a^3) \tag{4.37}$$

$$\begin{aligned}\Pi^{T,T}_{(1)}(a) = d(R)\Bigg[&-\frac{4}{9} - \frac{2}{3}\ell \ + \ C_F\left[8\zeta(3) - \frac{491}{54} - \frac{14}{9}\ell + \frac{2}{3}\ell^2\right] a \\ &+ \ C_F\Bigg[\left(\frac{10672}{243} - \frac{1024}{27}\zeta(3) + \frac{766}{81}\ell - \frac{32}{3}\zeta(3)\ell - \frac{8}{9}\ell^2 - \frac{8}{27}\ell^3\right) T_F N_f \\ &+ \ \Bigg(\frac{2732}{27}\zeta(3) - \frac{14}{3}\zeta(4) + \frac{40}{3}\zeta(5) - \frac{19427}{162} \\ &- \ \frac{1771}{162}\ell + 20\zeta(3)\ell + \frac{20}{3}\ell^2 + \frac{22}{27}\ell^3\Bigg) C_A \\ &+ \ \Bigg(\frac{608}{9}\zeta(3) + \frac{28}{3}\zeta(4) - 80\zeta(5) - \frac{15973}{972} - \frac{1075}{54}\ell \\ &+ \ \frac{8}{3}\zeta(3)\ell - \frac{43}{9}\ell^2 - \frac{4}{9}\ell^3\Bigg) C_F\Bigg] a^2\Bigg] \ + \ \mathcal{O}(a^3)\end{aligned} \tag{4.38}$$

$$\begin{aligned}\Pi^{T,T}_{(2)}(a) = d(R)\Bigg[&\frac{20}{9} + \frac{4}{3}\ell \ + \ C_F\left[\frac{593}{27} - 16\zeta(3) + \frac{28}{9}\ell - \frac{4}{3}\ell^2\right] a \\ &+ \ C_F\Bigg[\left(\frac{2048}{27}\zeta(3) - \frac{21328}{243} - \frac{1532}{81}\ell + \frac{64}{3}\zeta(3)\ell + \frac{16}{9}\ell^2 + \frac{16}{27}\ell^3\right) T_F N_f \\ &+ \ \Bigg(\frac{58075}{243} - \frac{5296}{27}\zeta(3) + \frac{28}{3}\zeta(4) - \frac{80}{3}\zeta(5) \\ &+ \ \frac{1771}{81}\ell - 40\zeta(3)\ell - \frac{40}{3}\ell^2 - \frac{44}{27}\ell^3\Bigg) C_A \\ &+ \ \Bigg(\frac{22051}{486} - \frac{1328}{9}\zeta(3) - \frac{56}{3}\zeta(4) + 160\zeta(5) + \frac{1075}{27}\ell \\ &- \ \frac{16}{3}\zeta(3)\ell + \frac{86}{9}\ell^2 + \frac{8}{9}\ell^3\Bigg) C_F\Bigg] a^2\Bigg] \ + \ \mathcal{O}(a^3)\end{aligned} \tag{4.39}$$

4.4 Polarized Operators

Now that we have been able to reproduce the results within [86] we can now move on to the new case where the polarised operator including γ^5. While these results are useful as a check that our procedure works correctly they shall also be useful for checking the limits of our new result as the condition we are about to impose shall rely on these. It is now time to explain in more detail the problems relating to including γ^5 in this problem. The operators without it correspond to the unpolarized scattering case, whereas including it enables us to study in polarisation. The approach that will be used is the one developed in [88] where

$$\gamma^5 = i\frac{1}{4!}\epsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4} \tag{4.40}$$

Then the product of of two ϵ tensors are replaced by the sum of twenty-four combinations of the form $g_{\nu_1\nu_2}g_{\nu_3\nu_4}$ which are treated as d-dimensional. This extends the normally 4-dimensional objects to d-dimensions, and successfully preserves the anti-commutativity properties of γ^5. With this it is possible to obtain a scalar expression inside correlation function containing only d-dimensional objects. In principle it may for some operators be possible to anticommute the γ^5 of the operator through the quark loop and use $(\gamma^5)^2 = 1$ to remove the γ^5 from the operator. However this approach is not certain to work for all the possible types of operators, as such we shall try to define a more general method. However in redefining γ^5 in doing so it no longer preserves the chiral symmetry of the system. This is an important part of QCD which will need to be treated in the forthcoming formula. To take this into account we introduce an extra renormalization Z_5 which acts on the finite part of the formula. The effect the renormalization has on the Green's function is that it equal to the function with the γ^5 commuted out. We shall apply this in the similar way to before on the quark current

$$Z_5\langle\psi(p)\left[\bar{\psi}\gamma^5\psi\right](0)\bar{\psi}(-p)\rangle = \gamma^5\langle\psi(p)\left[\bar{\psi}\psi\right](0)\bar{\psi}(-p)\rangle \tag{4.41}$$

What this means is that the value of the renormalized polarised quark current is the same as the unpolarised current with the γ^5 commuted out. We define

$Z^{i}_{5\,(k)}$ in a similar way to before

$$Z^i_5 = Z^i_{5\,1} + \left(Z^i_{5\,21} + Z^i_{5\,22}\right) g_r^2 + \left(Z^i_{5\,31} + Z^i_{5\,32} + Z^i_{5\,33}\right) g_r^4 + \mathcal{O}(g_r^6) \qquad (4.42)$$

Here we have dropped the projection index (k) to make things slightly clearer. With this in mind it is possible to modify our previous renormalized correlation function, as we are working with correlation functions with operators each containing a γ^5 it will be necessary to introduce a factor of $(\gamma^5)^2$ to our renormalization of our bare function

$$\Pi^{i,i}_{(k)}(q) = Z^{i,i}_{(k)} q^2 + \mu^{2\epsilon} \left(Z^i Z_5\right)^2 \Pi^{i,i}_{(k)\,O}(q) \qquad (4.43)$$

This modifies our bare correlation function so that the right hand terms of (4.19) and (4.46) give the same result, however this is not enough to fix the overall value of the correlation function due to the contact term. There is one final condition that is required to restore Chiral symmetry, this involves the fixing of a contact term. This takes a slightly different form for the pseudo scalar and axial vector cases

$$\Pi^{S,S}(q) = -\Pi^{P,P}(q) \qquad (4.44)$$

$$\Pi^{V,V}(q) = \Pi^{A,A}(q) \qquad (4.45)$$

The difference here arises from the extra γ-matrices in the Axial Vector operator. What this requires is a γ^5 renormalization on the contact term also $Z^{ii}_{5\,(k)}$, this is not to be unexpected as there will be some γ^5 dependence in our contact renormalization term. This again is a finite renormalization imposed to satisfy (4.44) and (4.45), as each of the operators will require a different finite correction there shall be a new counter term for each of the operators and projections. The final form of the correlation now becomes

$$\Pi^{i,i}_{(k)}(q) = Z^{i,i}_{5\,(k)} Z^{i,i}_{(k)} q^2 + \mu^{2\epsilon} \left(Z^i Z^i_{5\,(k)} \right)^2 \Pi^{i,i}_{(k)\,O}(q) \tag{4.46}$$

Before working out the correlation function for the polarised operators, we first need the appropriate renormalization terms. For each correlation function we have defined 3 new renormalization which need to be computed, $Z^i_{5\,(k)}$, $Z^{i,i}_{(k)}$ and $Z^{ii}_{5\,(k)}$ respectively. The ordering here is important, first of all the correlation function is worked out with the renormalization $(Z^i_{(k)})^2$ applied, this then allows for the finite renormalization $Z^i_{5\,(k)}$ to be fixed. After this divergences arising from the contact terms are removed by adding $Z^{i,i}_{(k)}$ into our equation, finally the finite terms are then once again fixed by applying the last finite renormalization $Z^{ii}_{5\,(k)}$. This approach allows for a systematic way of dealing with each step of the process. For the pseudo scalar

$$Z^P_5 = -8C_F + \left(\frac{2}{9} C_F C_A + \frac{8}{9} T_f N_f C_F \right) g^2 + \mathcal{O}(g^6) \tag{4.47}$$

$$\begin{aligned} Z^{PP}_{(1)} \;=\; & -2N_f N_c \frac{1}{\epsilon} + \left(6N_f N_c C_F \frac{1}{\epsilon^2} - 8N_f N_c C_F \frac{1}{\epsilon} \right) g^2 \\ & + \left(\left(-\frac{22}{3} N_f N_c C_F C_A - 12 N_f N_c C_F^2 + \frac{8}{3} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon^3} \right. \\ & + \left(\frac{94}{3} N_f N_c C_F C_A 14 N_f N_c C_F^2 - 8 N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon^2} \\ & + \left(\left(-\frac{869}{18} + 12\zeta(3) \right) N_f N_c C_F C_A + \left(+\frac{1303}{6} - 24\zeta(3) \right) N_f N_c C_F^2 \right. \\ & \left. \left. + \frac{2}{9} N_f^2 N_c T_f C_F \right) \frac{1}{\epsilon} \right) g^4 + \mathcal{O}(g^6) \end{aligned} \tag{4.48}$$

$$
\begin{aligned}
Z^{PP}_{5\,(1)} &= +\frac{19}{3}N_fN_c - \frac{227}{6}N_fN_cC_Fg^2 \\
&+ \left(\left(-38\zeta(3) + \frac{2245}{27}\right)N_fN_cC_FC_A \right.\\
&+ \left(+76\zeta(3) - \frac{5999}{12}\right)N_fN_cC_F^2 \\
&\left. + \frac{562}{27}N_f^2N_cT_fC_F\right)g^4 + \mathcal{O}(g^6) \qquad (4.49)
\end{aligned}
$$

And the axial vector

$$
\begin{aligned}
Z^A_5 &= -4C_F + \left(-\frac{107}{9}C_FC_A + \frac{4}{9}T_fN_fC_F + 22C_F^2\right)g^2 \\
&+ \left(\left(\frac{64}{3}\zeta(3) + \frac{712}{81}\right)T_fN_fC_FC_A + \left(-\frac{64}{3}\zeta(3) - \frac{124}{27}\right)T_fN_fC_F^2 \right.\\
&+ \frac{208}{81}T_f^2N_f^2C_F + \left(56\zeta(3) - \frac{2147}{27}\right)C_FC_A^2 \\
&+ \left(-160\zeta(3) + \frac{5834}{27}\right)C_F^2C_A \\
&\left. + \left(96\zeta(3) - \frac{370}{3}\right)C_F^3\right)g^4 + \mathcal{O}(g^6) \qquad (4.50)
\end{aligned}
$$

$$
\begin{aligned}
Z^{AA}_{(1)} &= -\frac{4}{3}N_fN_c\frac{1}{\epsilon} + \frac{10}{3}N_fN_cC_F\frac{1}{\epsilon}g^2 \\
&+ \left(\left(-\frac{110}{27}N_fN_cC_FC_A + \frac{40}{27}N_f^2N_cT_fC_F\right)\frac{1}{\epsilon^2} \right.\\
&+ \left(\frac{863}{27}N_fN_cC_FC_A - \frac{298}{9}N_fN_cC_F^2 \right.\\
&\left.\left. - \frac{44}{9}N_f^2N_cT_fC_F\right)\frac{1}{\epsilon}\right)g^4 + \mathcal{O}(g^6) \qquad (4.51)
\end{aligned}
$$

$$
\begin{aligned}
Z^{AA}_{5\,(1)} &= +\frac{8}{3}N_fN_c - \frac{116}{9}N_fN_cC_Fg^2 + \left(\left(\frac{64}{3}\zeta(3) \right.\right.\\
&\left. - \frac{18494}{243}\right)N_fN_cC_FC_A + \left(-\frac{128}{3}\zeta(3) + \frac{3404}{27}\right)N_fN_cC_F^2 \\
&\left. + \frac{1096}{243}N_f^2N_cT_fC_F\right)g^4 + \mathcal{O}(g^6) \qquad (4.52)
\end{aligned}
$$

$$Z^{AA}_{(2)} = \left(-\frac{22}{27}N_fN_cC_FC_A + \frac{8}{27}N_f^2N_cT_fC_F\right)\frac{1}{\epsilon}g^4 + \mathcal{O}(g^6) \tag{4.53}$$

$$\begin{aligned} Z^{AA}_{5\,(2)} &= -\frac{4}{3}N_fN_c + \frac{2}{3}N_fN_cC_Fg^2 \\ &+ \left(\frac{1073}{81}N_fN_cC_FC_A - \frac{170}{9}N_fN_cC_F^2 \right. \\ &\left. - \frac{124}{81}N_f^2N_cT_fC_F\right)g^4 + \mathcal{O}(g^6) \end{aligned} \tag{4.54}$$

4.5 Results

With this it is now possible to start to work out the anomalous dimension and the correlation function. These have been computed using the symbolic manipulation program FORM and within this MINCER as before Given here is the correlation function for the pseudo scalar and axial vector which due to the condition we have imposed is the same for the scalar and vector operators. In each case a common factor of q^2 removed. Once again in the following equations $d(R)$ is the dimension of the representation used and

$$\ell = \ln\left(\frac{\mu^2}{q^2}\right) \tag{4.55}$$

as before in 4.34. Finally all calculations are given to order $\mathcal{O}(a^3)$

$$\begin{aligned} \Pi^{P,P}(a) &= d(R)\left[(+4+2\ell) + C_F\left[+\frac{131}{2} + 34\ell + 6\ell^2 - 24\zeta(3)\right]a \right. \\ &+ C_F\left[N_fT_f\left(-\frac{2044}{9} + 32\ell\zeta(3) - 130\ell - \frac{88}{3}\ell^2 - \frac{8}{3}\ell^3 + 64\zeta(3)\right)\right. \\ &+ C_A\left(+\frac{14419}{18} - 124\ell\zeta(3) + \frac{893}{2}\ell + \frac{284}{3}\ell^2 + \frac{22}{3}\ell^3 \right. \\ &- 300\zeta(3) - 18\zeta(4) - 40\zeta(5)) \\ &+ C_F\left(+\frac{1613}{4} - 72\ell\zeta(3) + \frac{691}{2}\ell + 105\ell^2 + 12\ell^3 \right. \\ &- \left. 384\zeta(3) + 36\zeta(4) + 240\zeta(5))\right]a^2\Big] + \mathcal{O}(a^3) \end{aligned} \tag{4.56}$$

$$
\begin{aligned}
\Pi^{A,A}_{(1)} \quad &= d(R)\left[(+\frac{20}{9}+\frac{4}{3}\ell) + C_F\left[+\frac{55}{3}+4\ell-16\zeta(3)\right]a\right. \\
&+C_F\left[N_fT_f\left(-\frac{7402}{81}+\frac{64}{3}\ell\zeta(3)-\frac{88}{3}\ell-\frac{8}{3}\ell^2+\frac{608}{9}\zeta(3)\right)\right. \\
&+C_A\left(+\frac{44215}{162}-\frac{176}{3}\ell\zeta(3)+82\ell+\frac{22}{3}\ell^2-\frac{1816}{9}\zeta(3)-\frac{80}{3}\zeta(5)\right) \\
&\left.\left.+\ C_F\left(-\frac{286}{9}-2\ell-\frac{296}{3}\zeta(3)+160\zeta(5)\right)\right]a^2\right]+\mathcal{O}(a^3) \qquad (4.57)
\end{aligned}
$$

$$
\Pi^{A,A}_{(2)} \quad = \mathcal{O}(a^3) \qquad (4.58)
$$

With these it is then possible to work out the anomalous dimension of the pseudo scalar and axial vector. The results here are ones in d-dimensions computed in the $\overline{\text{MS}}$ scheme and once again given to order $\mathcal{O}(a^3)$

$$
\begin{aligned}
\gamma^{P,P}(a) \quad &= d(R)\left[-\frac{19}{3}\epsilon+2+C_F\left[-\frac{77}{3}\epsilon+10\right]a\right. \qquad (4.59) \\
&+[12312C_AC_F\epsilon\zeta(3)-26332C_AC_F\epsilon-3888C_AC_F\zeta(3)+8316C_AC_F \\
&-24624C_F^2\epsilon\zeta(3)+139797C_F^2\epsilon+7776C_F^2\zeta(3)-6426C_F^2 \\
&\left.-\ 4312C_F\epsilon N_fT_f-1728C_FN_fT_f)\frac{1}{108}\right]a^2\Big]+\mathcal{O}(a^3)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{A,A}_{(1)}(a) \quad &= d(R)\left[-\frac{8}{3}\epsilon+\frac{4}{3}+C_F\left[\frac{40}{9}\epsilon+4\right]\right. \qquad (4.60) \\
&+\left[64C_AC_F\epsilon\zeta(3)+\frac{8222}{81}C_AC_F\epsilon+\frac{1197}{81}C_AC_F+128C_F^2\epsilon\zeta(3)\right] \\
&\left.-\ \frac{1132}{9}C_F^2\epsilon-2C_F^2-\frac{712}{81}C_F\epsilon N_fT_f-\frac{396}{81}C_FN_fT_f\right]a^2\Big]+\mathcal{O}(a^3)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{A,A}_{(2)}(a) \quad &= d(R)\left[\frac{4}{3}\epsilon a+\frac{28}{3}C_F\epsilon+\left[\frac{213}{9}C_AC_F\epsilon\right.\right. \\
&\left.\left.-\ \frac{210}{9}C_F^2\epsilon+\frac{20}{9}C_F\epsilon N_fT_f\right]a^2\right]+\mathcal{O}(a^3) \qquad (4.61)
\end{aligned}
$$

A good check on these results is that they coincide with the scalar and the vector case respectively in the limit of $\epsilon \to 0$. The vector case being non-zero in d-dimensions but vanishes when limited to four dimensions.

4.6 Discussion

What follows shall be a brief discussion of the result present above as we have achieved our aim of calculating finite parts of various operator correlation functions including γ^5 to $\mathcal{O}(a^2)$ thus concluding the work started in [86]. This work shows it is possible to extend the previous method used in [86] to a situation where it is possible to include further operators that contain γ^5's. While in principle we have developed a method of introducing a finite renormalization that preserves the renormalization group and chiral symmetry our solution is far from ideal for the idea of automatic calculations. The reason for this is that this finite renormalization condition will change depending on the operators being used. This does not present a problem in simple cases such that we have studied here, however if wishing to use this method to work out more complex operators then careful consideration will be required for each one. This prevents it being possible to fully automate this type of calculation as before this is possible the finite renormalization condition will need to be checked for every possible combination of operators. This shows in principle that it should be able to look at higher dimensional operators including γ^5. The problem with this is usual problem of increases complexity and computation time required in doing this. Therefore while we have shown it is possible to include γ^5 in correlation function of two operators, the process is far from universal to all potential operators.

Chapter 5

Summary and Outlook

The main aim of this thesis has been to look at the Gribov-Zwanziger Lagrangian and look at some of the implication of how possible modification can make it relevant to current lattice data being produced around the world. Having given a brief overview of its formalism in Section 1 and why the Gribov-Zwanziger Lagrangian is important for proposing a potential solution to the problem of gauge copies. It would seem to be unfortunate considering many nice properties of the action that it should not describe the expected low momentum of Quantum Chromodynamics accurately. While it may be possible to construct other actions that satisfy the lattice data, there would still exist the problem of gauge copies. Finding a solution that simultaneously satisfies both of these problems has motivated us to look at the possible modification of the Gribov-Zwanziger Lagrangian.

While over the course of this work we have made a substantial amount of progress in understanding what effect introducing a new operator $\mathcal{O}^{abcd}$ into the Gribov-Zwanziger Lagrangian we are still far from a comprehensive knowledge on the subject. The work of Section 2 has allowed us to show that out of the six colour channels possible for a dimension two operator there is a preferred colour channel in the $\mathcal{R}$ direction. This solution arises naturally from our LCO formalism.

This modification to the Gribov-Zwanziger Lagrangian allows us to successfully model the behaviour now seen for zero energy gluon on the lattice. This means that with the dimension two localizing ghost operator we are able to achieve a gluon propagator which is non-zero at zero momentum. With this there is no reason just yet to suggest that the Gribov-Zwanziger Lagrangian in not still a potential candidate for modelling Quantum Chromodynamics and describing the strong nuclear force. This study is still far from the most comprehensive as it is only one possible dimension two operator. It is possible that a similar effect could be produced by the massive gluon $\frac{1}{2}m^2 A^a_\mu A^{a\,\nu}$. It is possible to see [38] that adding this term to the Gribov-Zwanziger Lagrangian modifies the propagators in a similar fashion to that of our proposed solution to the $\mathcal{R}$. Therefore in a similar principle for including all the different colour channels before applying our LCO mechanism it should be possible to construct an action that includes both the massive gluon and the extra localizing ghost field operator. While a study of the Gribov-Zwanziger Lagrangian with the dimension two massive gluon has been conducted before [89], there has yet to be a study which looks at the full potential action. While many of the tools required to do this would be similar to those used in previous work and by us in Section 2, this full action is beset by technical difficulties. Many of these arise due to the mixing of the gluon and the localising ghost in the propagators in the Gribov-Zwanziger Lagrangian. Therefore extreme care is required when computing Feynman diagrams as there will be additional diagrams arising from this mixed term which need handling differently. These extra diagrams also present a challenge relating to the understanding of the RGE for this model, and any potential mixing that may occur here. Therefore while our study in Section 2 presents the most compressive study of the localizing ghost operator thus far it is by no means the complete possible picture.

Further to this, all the studies looking at modification to the Gribov-Zwanziger Lagrangian looking at including extra dimension two operators have only looked at the action to one-loop. Previous work looking at the effective potential for the gluon in QCD [39] had previously been able to do this to two-loops. The reason this has yet to be possible while working with the Gribov-Zwanziger Lagrangian is because of the extra difficulty associated with mass like terms that appear in the propagators. A successful two-loop calculation would give a further understanding of the underlying physics behind the process rather than just knowledge dimension two operators are feasible. An idea calculation on the subject would be the effective potential of the Gribov-Zwanziger Lagrangian including the massive gluon and the localizing ghost with all of the six colour channels performed to at least two-loops. Only a study of this form would be able to fully grasp what significance there would be in the most general possible modification. Unfortunately while there is potential in this for future work, it is currently beyond the means of current methods and techniques.

With more understanding of localizing ghost modifications to the Gribov-Zwanziger Lagrangian given in Section 2, Section 3 looks at developing a test aside from the low momentum behaviour of the propagators that could be used to look at what happens at the low momentum limit of the theory. To this end we have looked at the one-loop symmetric point for three different vertices. The aim here being to measure in perturbation theory something which can also be computed on the lattice, from this it should be possible to compare the low momentum limits of the different models. While the ideas suggested in Section 2 have been studied previously, the idea of using the symmetric point to look at the low momentum behaviour of a theory is a more recent one. This has some important implications for the work we have produced here.

First there is currently less literature available on the subject with which to guide our work, secondly much of the data we would like to compare our work with has not been produced. It is this second factor that has motivated us to work in 3-dimensions rather than the more common 4-dimensions. The logic behind this is that a 3-dimensional computation should be less expensive to produce on the lattice. This will hopefully provide motivation for people working in lattice QCD to perform this type of calculation. The comparative lack of literature on the subject has prevented us from performing many of the checks we would normally choose to carry out on our work. In Section 2 we started constructing our LCO formalism by checking our method against known results, this lack of equivalent results in this case means we have only been able to perform self consistence checks. While these provide us with some means of error checking our results we await further study in this area and forthcoming results from a full 4-dimensional symmetric point calculation.

As in Section 2, our work in Section 3 is by no means a complete study of all the possible symmetric point configurations. In this section we have chosen to look at the one-loop symmetric vertex correction for the $\mathcal{Q}$ and $\mathcal{R}$ channels of our localizing ghost operator $\mathcal{O}^{abcd}$. Despite the work of the previous section showing that there is reason to believe that the $\mathcal{R}$ channel is the more important one, we have calculated both of these as there is a high degree of similarity between the propagators of each. However in principle there is no reason to exclude the sum of both of these channels or further to this the massive gluon which also modifies the low energy behaviour of the gluon propagator in a similar way. Also, due to how similar the $\mathcal{Q}$ and $\mathcal{R}$ channel propagators are, the results for the two channels have a lot of common factors. As such it would be advantageous to try and perform the symmetric vertex correction to two-loops to see if there is more of a difference arising between the two sets of results. Each of these as described here has the potential for further study, so there is considerably more work on the area possible before we can claim to have a full understanding of the low momentum behaviour of the Gribov-Zwanziger Lagrangian and gluon behaviour.

Appendix A

Symmetric Point Results

Here we present the full results from Section 3.8, these are arranged with the $\mathcal{Q}$ first then $\mathcal{R}$ channel. Each of these is then divided into sections for each of the different vertex functions.

Q Channel

$$
\begin{aligned}
\Sigma^{qqv}_{(1)}(p,q,\gamma^2,\mu_Q) \quad = \quad & 1+g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{13}{960}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\left(+\frac{1}{32}\mu_Q^2\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\mu^2\left(+\frac{11}{240}\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{20}\left[\mu_Q^4-4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_FC_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{24}[m_+ + m_-]\right) \\
& +ig^2\mu^3\gamma^{-4}\left(+\frac{1}{384}\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{13}{960}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_A\left(+\frac{13}{96}\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{13}{240}\mu_Q^{-2}\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{41}{480}\mu_Q^2\left[\mu_Q^4-4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\gamma^4\left(-\frac{1}{4}\left[\mu_Q^4-4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\mu_Q^2\left[\mu_Q^4-4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_FC_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{24}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_F\left(-\frac{1}{6}\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4-4C_A\gamma^4}\,g^2\pi^{-1}C_F\mu^2\left(+\frac{1}{6}\mu_Q^{-2}\left[\mu_Q^4-4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{qqv}_{(2)}(p,q,\gamma^2,\mu_Q) \quad = \quad & +g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{3}{160}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\mu^2\left(+\frac{7}{120}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{30}[m_+ + m_-]\right) \\
& +ig^2\mu^3\gamma^{-4}\left(+\frac{35}{4608}\right) \\
& +ig^2 C_F C_A^{-1}\mu^3\gamma^{-4}\left(-\frac{1}{64}\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{3}{160}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(+\frac{3}{40}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{31}{240}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{5}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{30}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F\mu^2\left(-\frac{2}{15}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{qqv}_{(3)}(p,q,\gamma^2,\mu_Q) \quad = \quad & +g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{96}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\mu^2\left(+\frac{1}{12}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{60}[m_+ + m_-]\right) \\
& +ig^2\mu^3\gamma^{-4}\left(-\frac{7}{576}\right) \\
& +ig^2 C_F C_A^{-1}\mu^3\gamma^{-4}\left(+\frac{1}{64}\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{96}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{1}{24}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{43}{240}\mu_{\mathcal{Q}}^{2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{5}\mu_{\mathcal{Q}}^{2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{60}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F\mu^2\left(+\frac{1}{15}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{qqv}_{(4)}(p,q,\gamma^2,\mu_Q) \quad = \quad & +g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{96}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\mu^2\left(+\frac{1}{12}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{60}[m_+ + m_-]\right) \\
& +ig^2\mu^3\gamma^{-4}\left(-\frac{7}{576}\right) \\
& +ig^2C_F C_A^{-1}\mu^3\gamma^{-4}\left(+\frac{1}{64}\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{96}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{1}{24}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A\mu^2\left(-\frac{43}{240}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{5}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{60}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,g^2\pi^{-1}C_F\mu^2\left(+\frac{1}{15}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{qqv}_{(5)}(p,q,\gamma^2,\mu_Q) \quad = \quad & g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{3}{160}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A\mu^2\left(+\frac{7}{120}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(-\frac{1}{10}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ + m_-]\right) \\
& +g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{30}[m_+ + m_-]\right) \\
& +ig^2\mu^3\gamma^{-4}\left(+\frac{35}{4608}\right) \\
& +ig^2 C_F C_A^{-1}\mu^3\gamma^{-4}\left(-\frac{1}{64}\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{3}{160}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}C_A\mu^2\left(+\frac{3}{40}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}C_A\mu^2\left(-\frac{31}{240}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}C_A^2\mu^2\gamma^4\left(+\frac{1}{5}\mu_Q^{2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{30}\mu_Q^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_Q^4 - 4C_A\gamma^4}\, g^2\pi^{-1}C_F\mu^2\left(-\frac{2}{15}\mu_Q^{-2}\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.5)}
\end{aligned}
$$

$$\Sigma^{qqv}_{(6)}(p,q,\gamma^2,\mu_Q) = +g^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{3}{64}[m_+ + m_-]\right)$$
$$+g^2\pi^{-1}C_A\mu^2\left(+\frac{5}{96}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right)$$
$$+g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(+\frac{1}{12}[m_+ + m_-]\right)$$
$$+ig^2\mu^3\gamma^{-4}\left(+\frac{29}{768}\right)$$
$$+ig^2C_F C_A^{-1}\mu^3\gamma^{-4}\left(-\frac{1}{16}\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}g^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{3}{64}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}g^2\pi^{-1}C_A\mu^2\left(+\frac{3}{16}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}g^2\pi^{-1}C_A\mu^2\left(-\frac{5}{48}\mu_{\mathcal{Q}}^{2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}g^2\pi^{-1}C_F C_A^{-1}\mu^2\gamma^{-4}\left(-\frac{1}{12}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}g^2\pi^{-1}C_F\mu^2\left(-\frac{1}{3}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.6}$$

$$\Sigma^{acc}_{(1)}(p,q,\gamma^2,\mu_Q) = -i + g^2C_A^{-1}\mu^3\gamma^{-8}\left(+\frac{13}{3072}\mu_{\mathcal{Q}}^4\right)$$
$$+g^2mu\gamma^{-4}\left(+\frac{11}{1536}\mu_{\mathcal{Q}}^2\right)$$
$$+g^2\mu^3\gamma^{-4}\left(+\frac{11}{1536}\right)$$
$$+ig^2\pi^{-1}C_A^{-1}\mu^2\gamma^{-8}\left(+\frac{1}{960}\mu_{\mathcal{Q}}^4[m_+ + m_-]\right)$$
$$+ig^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{19}{1280}[m_+ + m_-]\right)$$
$$+ig^2\pi^{-1}C_A\mu^2\left(-\frac{61}{1920}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ + m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}ig^2\pi^{-1}C_A^{-1}\mu^2\gamma^{-8}\left(-\frac{1}{960}\mu_{\mathcal{Q}}^2[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}ig^2\pi^{-1}\mu^2\gamma^{-4}\left(-\frac{13}{768}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}ig^2\pi^{-1}C_A\mu^2\left(-\frac{13}{192}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}ig^2\pi^{-1}C_A\mu^2\left(+\frac{61}{960}\mu_{\mathcal{Q}}^{2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.7}$$

$$
\begin{aligned}
\Sigma^{acc}_{(2)}(p,q,\gamma^2,\mu_Q) \quad = \quad & +g^2 C_A^{-1}\mu^3\gamma^{-8}\left(-\frac{55}{3072}\mu_{\mathcal{Q}}^4\right) \\
& +g^2 mu\gamma^{-4}\left(+\frac{1}{1536}\mu_{\mathcal{Q}}^2\right) \\
& +g^2\mu^3\gamma^{-4}\left(+\frac{5}{768}\right) \\
& +ig^2\pi^{-1}C_A^{-1}\mu^2\gamma^{-8}\left(-\frac{7}{320}muq^4[m_+ + m_-]\right) \\
& +ig^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{23}{3840}[m_+ + m_-]\right) \\
& +ig^2\pi^{-1}C_A\mu^2\left(+\frac{61}{1920}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\, ig^2\pi^{-1}C_A^{-1}\mu^2\gamma^{-8}\left(+\frac{7}{320}\mu_{\mathcal{Q}}^2[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\, ig^2\pi^{-1}\mu^2\gamma^{-4}\left(+\frac{29}{768}\mu_{\mathcal{Q}}^{-2}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\, ig^2\pi^{-1}C_A\mu^2\left(+\frac{29}{192}\mu_{\mathcal{Q}}^{-2}\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}[m_+ - m_-]\right) \\
& +\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\, ig^2\pi^{-1}C_A\mu^2\left(-\frac{61}{960}\mu_{\mathcal{Q}}^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}[m_+ - m_-]\right) \\
& +\mathcal{O}(\mu^4) \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
&\Sigma^{aaa}_{(1)}(p,q,\gamma^2,\mu_Q) = +\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(-\frac{1}{48}i\mu_Q^{12}\left[m_+ + m_-\right] + \frac{1}{24}i\mu_Q^{13}\right)\\
&+\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{9}{128}i\mu_Q^{8}\left[m_+ + m_-\right] - \frac{31}{192}i\mu_Q^{9}\right)\\
&+\pi^{-1}C_A^{-1}\gamma^{-8}\left(-\frac{1}{96}i\mu_Q^{6}\left[m_+ + m_-\right] + \frac{1}{48}i\mu_Q^{7}\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{53}{1536}i\mu_Q^{4}\left[m_+ + m_-\right] + \frac{563}{3840}i\mu_Q^{5}\right)\\
&+\pi^{-1}\gamma^{-4}\left(+\frac{23}{256}i\mu_Q^{2}\left[m_+ + m_-\right] - \frac{17}{384}i\mu_Q^{3}\right) + \pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{37}{5120}i\left[m_+ + m_-\right] - \frac{73}{7680}i\mu_Q\right)\\
&+\pi^{-1}C_A\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{57}{1024}i\mu_Q^{2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{4499}{40960}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2\gamma^4\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{9}{256}i\mu_Q^{2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{667}{1920}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{9}{64}i\left[m_+ + m_-\right]\right)\\
&+C_A^{-1}g^2\mu^3\gamma^{-8}\left(-\frac{3}{256}\mu_Q^{4}\right) + g^2\mu^3\gamma^{-4}\left(-\frac{3}{102}\right) + T_f N_f g^2\mu^{-1}\left(-\frac{1}{36}\right)\\
&+C_A g^2\mu^{-1}\left(+\frac{1}{288}\right) + \sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(-\frac{1}{48}i\mu_Q^{10}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{11}{384}i\mu_Q^{6}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-1}\gamma^{-8}\left(-\frac{1}{96}i\mu_Q^{4}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{401}{7680}i\mu_Q^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}\gamma^{-4}\left(-\frac{59}{768}i\left[m_+ + m_-\right] + \frac{59}{384}im_-\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{449}{3072}i\mu_Q^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{733}{3072}i\left[m_+ + m_-\right]\right) - i\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{22063}{122880}i\mu_Q^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{449}{768}i\mu_Q^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^2\gamma^4\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{201}{512}i\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{5201}{7680}i\mu_Q^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^3\gamma^8\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{9}{32}i\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_Q^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_Q^4 - 4C_A\gamma^4\right]^{-4}\left(+\frac{9}{32}i\mu_Q^{2}\left[m_+ + m_-\right]\right)
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
&\Sigma^{aaa}_{(2)}(p,q,\gamma^2,\mu_{\mathcal{Q}}) = \pi^{-1}C_A^{-4}g^2\mu^2\gamma^{-20}\left(-\frac{3}{35}i\mu_{\mathcal{Q}}^{16}\left[m_+ + m_-\right] + \frac{6}{35}i\mu_{\mathcal{Q}}^{17}\right)\\
&+\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(+\frac{193}{420}i\mu_{\mathcal{Q}}^{12}\left[m_+ + m_-\right] - \frac{211}{210}i\mu_{\mathcal{Q}}^{13}\right)\\
&+\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(-\frac{17}{30}i\mu_{\mathcal{Q}}^{8}\left[m_+ + m_-\right] + \frac{52}{35}i\mu_{\mathcal{Q}}^{9}\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{201}{4480}i\mu_{\mathcal{Q}}^{4}\left[m_+ + m_-\right] - \frac{479}{1344}i\mu_{\mathcal{Q}}^{5}\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{239}{53760}i\left[m_+ + m_-\right] - \frac{899}{26880}i\mu_{\mathcal{Q}}\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{10257}{71680}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{5471}{26880}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{27}{280}i\left[m_+ + m_-\right]\right)\\
&+T_f N_f g^2\mu^{-1}\left(-\frac{5}{54}\right)\\
&+C_A g^2\mu^{-1}\left(-\frac{5}{864}\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-4}g^2\mu^2\gamma^{-20}\left(-\frac{3}{35}i\mu_{\mathcal{Q}}^{14}\left[m_+ + m_-\right] + \frac{6}{35}i\mu_{\mathcal{Q}}^{14}m_-\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(+\frac{121}{420}i\mu_{\mathcal{Q}}^{10}\left[m_+ + m_-\right] - \frac{121}{210}i\mu_{\mathcal{Q}}^{10}m_-\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(-\frac{17}{105}i\mu_{\mathcal{Q}}^{6}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{17}{4480}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{529}{53760}i\mu_{\mathcal{Q}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{56557}{215040}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{529}{13440}i\mu_{\mathcal{Q}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{5309}{13440}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-4}\left(-\frac{27}{140}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)\\
&+\mathcal{O}(\mu^4) \qquad \text{(A.10)}
\end{aligned}
$$

$$\Sigma^{aaa}_{(3)}(p,q,\gamma^2,\mu_{\mathcal{Q}}) = +\pi^{-1}C_A^{-4}g^2\mu^2\gamma^{-20}\left(-\frac{3}{70}i\mu_{\mathcal{Q}}^{16}\left[m_+ + m_-\right] + \frac{3}{35}i\mu_{\mathcal{Q}}^{17}\right)$$
$$+\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(+\frac{79}{420}i\mu_{\mathcal{Q}}^{12}\left[m_+ + m_-\right] - \frac{44}{105}i\mu_{\mathcal{Q}}^{13}\right)$$
$$+\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(-\frac{127}{960}i\mu_{\mathcal{Q}}^{8}\left[m_+ + m_-\right] + \frac{447}{1120}i\mu_{\mathcal{Q}}^{9}\right)$$
$$+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{599}{5376}i\mu_{\mathcal{Q}}^{4}\left[m_+ + m_-\right] + \frac{2197}{13440}i\mu_{\mathcal{Q}}^{5}\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{53}{6720}i\left[m_+ + m_-\right] - \frac{479}{6720}i\mu_{\mathcal{Q}}\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{21219}{143360}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{1903}{5376}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{261}{1120}i\left[m_+ + m_-\right]\right)$$
$$+T_fN_fg^2\mu^{-1}\left(-\frac{11}{108}\right)$$
$$+C_Ag^2\mu^{-1}\left(-\frac{11}{1728}\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-4}g^2\mu^2\gamma^{-20}\left(-\frac{3}{70}i\mu_{\mathcal{Q}}^{14}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-3}g^2\mu^2\gamma^{-16}\left(+\frac{43}{420}i\mu_{\mathcal{Q}}^{10}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(-\frac{89}{6720}i\mu_{\mathcal{Q}}^{6}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{295}{5376}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{817}{6720}i\mu_{\mathcal{Q}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{21187}{86016}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{817}{1680}iq\mu_{\mathcal{Q}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{19813}{26880}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{Q}}^4 - 4C_A\gamma^4\right]^{-4}\left(+\frac{261}{560}i\mu_{\mathcal{Q}}^{2}\left[m_+ + m_-\right]\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.11}$$

R Colour Channel

$$\Sigma^{aaa}_{(1)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{1}{60}i\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}\gamma^{-4}\left(+\frac{7}{96}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right) - i$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{17}{1920}i\left[m_+ + m_-\right] + \frac{13}{1920}i\mu_{\mathcal{R}}\right)$$
$$+\pi^{-1}C_A\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{65}{1024}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{12313}{122880}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{9}{256}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{643}{1920}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{9}{64}i\left[m_+ + m_-\right]\right)$$
$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(-\frac{3}{256}\mu_{\mathcal{R}}^4\right) + g^2\mu^3\gamma^{-4}\left(-\frac{5}{3072}\right)$$
$$+T_fN_fg^2\mu^{-1}\left(-\frac{1}{36}\right) + C_Ag^2\mu^{-1}\left(+\frac{1}{288}\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1}{60}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}\gamma^{-4}\left(-\frac{7}{96}i\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{47}{960}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{733}{3072}i\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{6629}{40960}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{47}{240}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{233}{512}i\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{5009}{7680}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^3\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{9}{32}i\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-4}\left(+\frac{9}{32}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right) \tag{A.12}$$

$$\Sigma^{aaa}_{(2)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{47}{3360}i\mu_{\mathcal{R}}^8\left[m_+ + m_-\right] - \frac{47}{1680}i\mu_{\mathcal{R}}^9\right)$$
$$+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1037}{13440}i\mu_{\mathcal{R}}^4\left[m_+ + m_-\right] + \frac{799}{6720}i\mu_{\mathcal{R}}^5\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{167}{8960}i\left[m_+ + m_-\right] - \frac{617}{13440}i\mu_{\mathcal{R}}\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{21251}{215040}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{1363}{5376}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(+\frac{27}{280}i\left[m_+ + m_-\right]\right)$$
$$+g^2\mu^3\gamma^{-4}\left(+\frac{77}{18432}\right) + T_fN_fg^2\mu^{-1}\left(-\frac{5}{54}\right) + C_Ag^2\mu^{-1}\left(-\frac{5}{864}\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{47}{3360}i\mu_{\mathcal{R}}^6\left[m_+ + m_-\right] - \frac{47}{1680}i\mu_{\mathcal{R}}^6 m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{1}{4480}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right] - \frac{1}{2240}i\mu_{\mathcal{R}}^2 m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1559}{26880}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right] - \frac{1559}{13440}i\mu_{\mathcal{R}}^{-2}\, m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{36173}{215040}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right] + \frac{36173}{107520}i\mu_{\mathcal{R}}^2 m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{1559}{6720}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right] - \frac{1559}{3360}i\mu_{\mathcal{R}}^{-2} m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{6653}{13440}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right] + \frac{6653}{6720}i\mu_{\mathcal{R}}^2 m_-\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^3g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-4}\left(-\frac{27}{140}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right] + \frac{27}{70}i\mu_{\mathcal{R}}^2 m_-\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.13}$$

$$
\begin{aligned}
&\Sigma^{aaa}_{(3)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{41}{3360}i\mu_{\mathcal{R}}^8\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{949}{13440}i\mu_{\mathcal{R}}^4\left[m_+ + m_-\right] + \frac{697}{6720}i\mu_{\mathcal{R}}^5\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{683}{53760}i\left[m_+ + m_-\right] - \frac{1387}{26880}i\mu_{\mathcal{R}}\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{12149}{86016}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{10859}{26880}i\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{261}{1120}i\left[m_+ + m_-\right]\right) + g^2\mu^3\gamma^{-4}\left(+\frac{5}{576}\right)\\
&+T_f N_f g^2\mu^{-1}\left(-\frac{11}{108}\right) + C_A g^2\mu^{-1}\left(-\frac{11}{1728}\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-2}g^2\mu^2\gamma^{-12}\left(+\frac{41}{3360}i\mu_{\mathcal{R}}^6\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{47}{13440}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{701}{10752}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{97423}{430080}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{701}{2688}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\left(-\frac{22501}{26880}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\,\pi^{-1}C_A^3 g^2\mu^2\gamma^8\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-4}\left(+\frac{261}{560}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\mathcal{O}(\mu^4)
\end{aligned}
\tag{A.14}
$$

$$\Sigma^{acc}_{(1)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = \pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{1}{960}i\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{19}{1280}i\left[m_+ + m_-\right]\right) - i$$
$$+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{61}{1920}i\left[m_+ + m_-\right]\right)$$
$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{13}{3072}\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{11}{1536}\mu_{\mathcal{R}}^2\right)$$
$$+g^2\mu^3\gamma^{-4}\left(+\frac{11}{1536}\right) + \sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1}{960}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{13}{768}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(+\frac{61}{960}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(-\frac{13}{192}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.15}$$

$$\Sigma^{acc}_{(2)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{7}{320}i\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{23}{3840}i\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{61}{1920}i\left[m_+ + m_-\right]\right)$$
$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(-\frac{55}{3072}\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{1}{1536}\mu_{\mathcal{R}}^2\right) + g^2\mu^3\gamma^{-4}\left(+\frac{5}{768}\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{7}{320}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{29}{768}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{61}{960}i\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{29}{192}i\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.16}$$

$$
\begin{aligned}
&\Sigma^{qqv}_{(1)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = \pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1}{30}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(+\frac{1}{30}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}\gamma^{-4}\left(+\frac{1}{12}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{12}\left[m_+ + m_-\right]\right)+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{13}{960}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(+\frac{17}{320}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\left(+\frac{1}{32}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{19}{96}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{41}{480}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\left(+\frac{11}{240}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(+\frac{17}{80}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2\gamma^4\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{4}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-3}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{10}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-2}\left(+\frac{1}{20}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\left(+\frac{1}{24}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{24}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}\gamma^{-4}\left(-\frac{1}{12}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(+\frac{1}{12}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{1}{24}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{24}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(+\frac{1}{6}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Fg^2\mu^2\left[\mu_{\mathcal{R}}^4-4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4-4C_A\gamma^4}\left(-\frac{1}{6}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)+1\\
&+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{1}{96}i\mu_{\mathcal{R}}^4\right)\\
&+\mu\gamma^{-4}\left(-\frac{5}{192}i\mu_{\mathcal{R}}^2\right)+g^2\mu^3\gamma^{-4}\left(+\frac{1}{384}i\right)+C_FC_A^{-1}\mu\gamma^{-4}\left(+\frac{5}{96}i\mu_{\mathcal{R}}^2\right)\\
&+\mathcal{O}(\mu^4)
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
&\Sigma^{qqv}_{(2)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1}{60}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{60}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{3}{160}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{5}{96}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{31}{240}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{7}{120}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{5}{24}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{5}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{1}{10}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\left(-\frac{1}{30}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{30}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1}{30}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{30}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_Fg^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{2}{15}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{1}{144}i\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{5}{576}i\mu_{\mathcal{R}}^2\right)\\
&+g^2\mu^3\gamma^{-4}\left(+\frac{35}{4608}i\right) + C_FC_A^{-2}g^2\mu^3\gamma^{-8}\left(+\frac{1}{64}i\mu_{\mathcal{R}}^4\right) + C_FC_A^{-1}\mu\gamma^{-4}\left(-\frac{5}{288}i\mu_{\mathcal{R}}^2\right)\\
&+C_FC_A^{-1}g^2\mu^3\gamma^{-4}\left(-\frac{1}{64}i\right)\\
&+\mathcal{O}(\mu^4)
\end{aligned}
\tag{A.18}
$$

$$\Sigma^{qqv}_{(3)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{7}{120}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{7}{120}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1}{96}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{17}{160}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{43}{240}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{1}{12}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{17}{40}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{5}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{1}{10}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\left(+\frac{1}{60}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{60}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{1}{60}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{60}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Fg^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{15}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{61}{1152}i\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{1}{576}i\mu_{\mathcal{R}}^2\right) + g^2\mu^3\gamma^{-4}\left(-\frac{7}{576}i\right)$$
$$+C_FC_A^{-2}g^2\mu^3\gamma^{-8}\left(-\frac{1}{64}i\mu_{\mathcal{R}}^4\right) + C_FC_A^{-1}\mu\gamma^{-4}\left(-\frac{1}{288}i\mu_{\mathcal{R}}^2\right)$$
$$+C_FC_A^{-1}g^2\mu^3\gamma^{-4}\left(+\frac{1}{64}i\right)$$
$$+\mathcal{O}(\mu^4) \tag{A.19}$$

$$\Sigma^{qqv}_{(4)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{7}{120}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{7}{120}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1}{96}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{17}{160}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{43}{240}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{1}{12}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{17}{40}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{5}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_A^2g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{1}{10}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\left(+\frac{1}{60}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{60}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{1}{60}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{60}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$

$$+\pi^{-1}C_Fg^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{15}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$

$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{61}{1152}i\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{1}{576}i\mu_{\mathcal{R}}^2\right) + g^2\mu^3\gamma^{-4}\left(-\frac{7}{576}i\right)$$

$$+C_FC_A^{-2}g^2\mu^3\gamma^{-8}\left(-\frac{1}{64}i\mu_{\mathcal{R}}^4\right) + C_FC_A^{-1}\mu\gamma^{-4}\left(-\frac{1}{288}i\mu_{\mathcal{R}}^2\right)$$

$$+C_FC_A^{-1}g^2\mu^3\gamma^{-4}\left(+\frac{1}{64}i\right)$$

$$+\mathcal{O}(\mu^4) \tag{A.20}$$

$$
\begin{aligned}
&\Sigma^{qqv}_{(5)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(-\frac{1}{60}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{60}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{3}{160}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{5}{96}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{31}{240}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{7}{120}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{5}{24}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-3}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{5}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_A^2 g^2\mu^2\gamma^4\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\left(-\frac{1}{10}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F C_A^{-2}g^2\mu^2\gamma^{-8}\left(-\frac{1}{30}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F C_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{30}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F C_A^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1}{30}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F C_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{30}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+\pi^{-1}C_F g^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{2}{15}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)\\
&+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{1}{144}i\mu_{\mathcal{R}}^4\right) + \mu\gamma^{-4}\left(+\frac{5}{576}i\mu_{\mathcal{R}}^2\right) + g^2\mu^3\gamma^{-4}\left(+\frac{35}{4608}i\right)\\
&+C_F C_A^{-2}g^2\mu^3\gamma^{-8}\left(+\frac{1}{64}i\mu_{\mathcal{R}}^4\right) + C_F C_A^{-1}\mu\gamma^{-4}\left(-\frac{5}{288}i\mu_{\mathcal{R}}^2\right)\\
&+C_F C_A^{-1}g^2\mu^3\gamma^{-4}\left(-\frac{1}{64}i\right)\\
&+\mathcal{O}(\mu^4)
\end{aligned}
\tag{A.21}
$$

$$\Sigma^{qqv}_{(6)}(p,q,\gamma^2,\mu_{\mathcal{R}}) = +\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\left(+\frac{1}{48}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_A^{-1}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{1}{48}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\left(-\frac{3}{64}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{192}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-2}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(-\frac{5}{48}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\left(+\frac{5}{96}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Ag^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{48}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\left(-\frac{1}{12}\mu_{\mathcal{R}}^4\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-2}g^2\mu^2\gamma^{-8}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{12}\mu_{\mathcal{R}}^2\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\left(+\frac{1}{12}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_FC_A^{-1}g^2\mu^2\gamma^{-4}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{12}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+\pi^{-1}C_Fg^2\mu^2\left[\mu_{\mathcal{R}}^4 - 4C_A\gamma^4\right]^{-1}\sqrt{\mu_{\mathcal{R}}^4 - 4C_A\gamma^4}\left(+\frac{1}{3}\mu_{\mathcal{R}}^{-2}\left[m_+ + m_-\right]\right)$$
$$+C_A^{-1}g^2\mu^3\gamma^{-8}\left(+\frac{5}{192}i\mu_{\mathcal{R}}^4\right) + g^2\mu^3\gamma^{-4}\left(+\frac{29}{768}i\right)$$
$$+C_FC_A^{-2}g^2\mu^3\gamma^{-8}\left(+\frac{1}{16}i\mu_{\mathcal{R}}^4\right) + C_FC_A^{-1}g^2\mu^3\gamma^{-4}\left(-\frac{1}{16}i\right)$$
$$+\mathcal{O}(\mu^4) \qquad \text{(A.22)}$$

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