

# Mutant knots with symmetry

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## *Abstract*

Mutant knots, in the sense of Conway, are known to share the same Homfly polynomial. Their 2-string satellites also share the same Homfly polynomial, but in general their  $m$ -string satellites can have different Homfly polynomials for  $m > 2$ . We show that, under conditions of extra symmetry on the constituent 2-tangles, the directed  $m$ -string satellites of mutants share the same Homfly polynomial for  $m < 6$  in general, and for all choices of  $m$  when the satellite is based on a cable knot pattern.

We give examples of mutants with extra symmetry whose Homfly polynomials of some 6-string satellites are different, by comparing their quantum  $sl(3)$  invariants.

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## 1. Introduction

This paper has been inspired by recent observations of Ochiai and Jun Murakami about the Homfly skein theory of  $m$ -parallels of certain symmetrical 2-tangles. In [8] Ochiai remarks that the 3-parallels of the tangle  $AB$  in figure 1 and its mirror image  $\overline{AB} = BA$  are equal in the Homfly skein of 6-tangles, in other words, in the Hecke algebra  $H_6$ , [1].

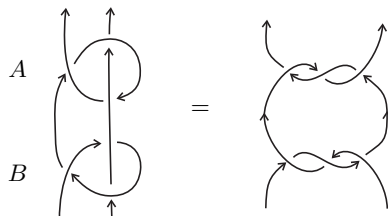


Fig. 1. The 2-tangle  $AB$  used by Ochiai

As a consequence, the 3-parallels of any mutant pair of knots given by composing the 2-tangles  $AB$  and  $BA$  with any other 2-tangle  $C$  and then closing, as in figure 3 will share the same Homfly polynomial. This is in contrast with the known fact that 3-parallels of mutant knots in general can have different Homfly polynomials, [7, 4].

There is interest in the extent to which the Homfly polynomial of  $m$ -parallels or other  $m$ -string satellites can distinguish mutants which are closures of  $ABC$  and  $BAC$  with  $A$  and  $B$  as above. Ochiai has found that the 4-parallels of  $AB$  and  $BA$  are different in the skein  $H_8$ .

The purpose of this paper is to show that if  $A$  and  $B$  are any two oriented 2-tangles which have the symmetry shown in figure 2 then the  $m$ -parallels, and indeed any directed  $m$ -string satellite, of knots  $\widehat{ABC}$  and  $\widehat{BAC}$  shown in figure 3 share the same Homfly polynomial provided that  $m < 6$ . In contrast there exist examples of  $A, B$  and  $C$  where  $A$  and  $B$  have the symmetry shown in figure 2, including Ochiai's case with

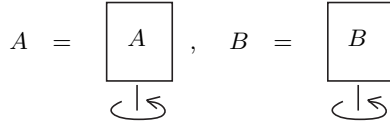
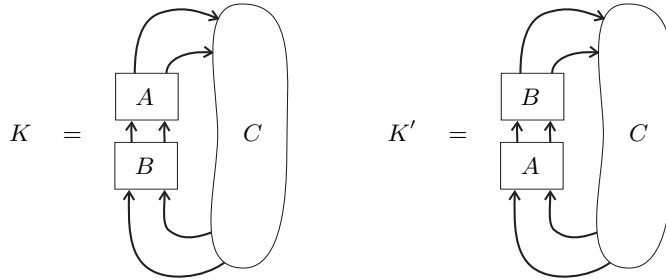
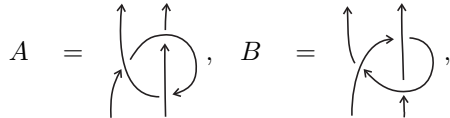
Fig. 2. The symmetry imposed on the tangles  $A$  and  $B$ 

Fig. 3. Tangle interchange



for which the Homfly polynomials of the 6-fold parallel of  $K$  and  $K'$  are different. The simplest such example, the pretzel knots shown in figure 8, uses Ochiai's choice of symmetric tangles  $A$  and  $B$ .

In an unexpected extension of the main result we show that the Homfly polynomial of a genuine connected cable, based on the  $(m, n)$  torus knot pattern, with  $m$  and  $n$  coprime, will not distinguish mutants with symmetry above, for *any* number of strings,  $m$ , although a more general connected satellite pattern can do so.

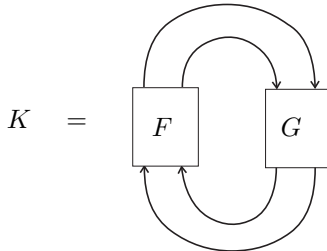
The examples which exhibit differences for the directly oriented 6-parallel can also be used to show that the 4-parallels with two pairs of reverse strands have distinct Homfly polynomials.

The proofs are based on the relation of the Homfly satellite invariants to quantum  $sl(N)$  invariants, and the techniques are an extension of work with Cromwell [4] and with H. Ryder [6]. The eventual calculations that exhibit the difference of invariants in the specific example depend on the 27 dimensional irreducible module over  $sl(3)$  corresponding to the partition 4, 2, and some Maple calculations following similar lines to those in [6].

## 2. Shared invariants of mutants

The term *mutant* was coined by Conway, and refers to the following general construction.

Suppose that a knot  $K$  can be decomposed into two oriented 2-tangles  $F$  and  $G$



A new knot  $K'$  can be formed by replacing the tangle  $F$  with the tangle  $F' = \tau_i(F)$  given by rotating  $F$  through  $\pi$  in one of three ways,

$$\tau_1(F) = \boxed{F} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad \tau_2(F) = \boxed{F} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}, \quad \tau_3(F) = \boxed{F} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array},$$

reversing its string orientations if necessary. Any of the three knots

$$K' = \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \boxed{\tau_i(F)} \quad \boxed{G} \\ \curvearrowright \\ \curvearrowleft \end{array}$$

is called a *mutant* of  $K$ .

The two 11-crossing knots,  $C$  and  $KT$ , found by Conway and Kinoshita-Teresaka are the best-known example of mutant knots. These two knots are shown in figure 4.

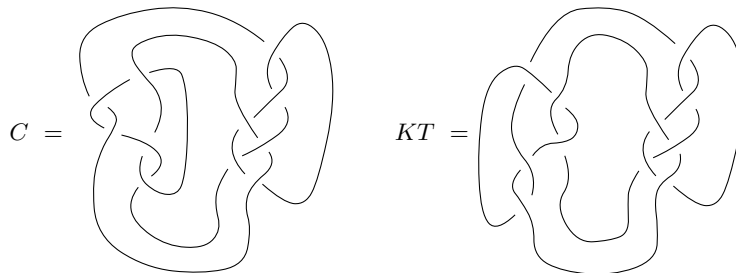


Fig. 4. The Conway and Kinoshita-Teresaka mutant pair

### 2.1. Satellites

A satellite of  $K$  is determined by choosing a diagram  $Q$  in the standard annulus, and then drawing  $Q$  on the annular neighbourhood of  $K$  determined by the framing, to give the satellite knot  $K * Q$ . We refer to this construction as *decorating  $K$  with the pattern  $Q$* , as shown in figure 5.

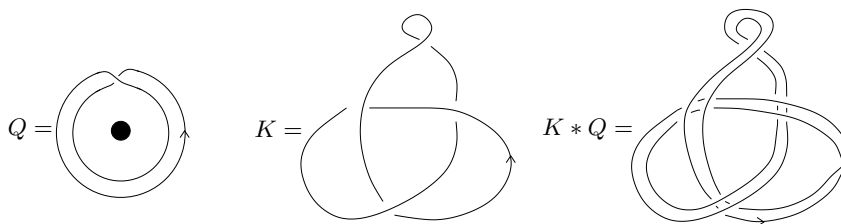


Fig. 5. Satellite construction

For fixed  $Q$  the Homfly polynomial  $P(K * Q)$  of the satellite is an invariant of the framed knot  $K$ . The invariants  $P(K * Q)$  as  $Q$  varies make up the *Homfly satellite invariants* of  $K$ . We use the alternate notation  $P(K; Q)$  in place of  $P(K * Q)$  when we want to emphasise the dependence on  $K$ .

The general symmetry result compares the invariants of two knots  $K$  and  $K'$  made up of 2-tangles  $A$ ,  $B$  and  $C$ , by interchanging  $A$  and  $B$  as in figure 3.

THEOREM 1. Suppose that  $A$  and  $B$  are both symmetric under the half-twist  $\tau_3$ , so that

$$A = \begin{array}{c} \boxed{A} \\ \underbrace{\quad} \end{array}, \quad B = \begin{array}{c} \boxed{B} \\ \underbrace{\quad} \end{array}$$

Let  $K$  and  $K'$  be knots which are the closure of  $ABC$  and  $BAC$  respectively for any tangle  $C$ , as in figure 3. Then  $P(K * Q) = P(K' * Q)$  for every closed braid pattern  $Q$  on  $m < 6$  strings.

REMARK 1. The proof applies equally to the case where  $Q$  is the closure of any directly oriented  $m$ -tangle with  $m < 6$ .

In order to prove the theorem we must rewrite the Homfly satellite invariants in terms of quantum  $sl(N)$  invariants, so we now give a brief summary of the relations between these invariants, originally established by Wenzl. Further details can be found in [1] and the thesis of Lukac, [3], including details of variant Homfly skeins with a framing correction factor,  $x$ . These are isomorphic to the skeins used here but the parameter allows a careful adjustment of the quadratic skein relation to agree directly with the natural relation arising from use of the quantum groups  $sl(N)$ .

## 2.2. Homfly skeins

For a surface  $F$  with some designated input and output boundary points the (linear) Homfly skein of  $F$  is defined as linear combinations of oriented diagrams in  $F$ , up to Reidemeister moves II and III, modulo the skein relations

$$(i) \quad \begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \swarrow \\ \searrow \\ \searrow \end{array} = (s - s^{-1}) \begin{array}{c} \nearrow \\ \searrow \end{array} \left( \begin{array}{c} \nearrow \\ \searrow \end{array} \right),$$

$$(ii) \quad \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = v^{-1} \left. \begin{array}{c} \uparrow \\ \left. \right\} \end{array} \right\}, \quad \begin{array}{c} \uparrow \\ \circlearrowright \end{array} = v \left. \begin{array}{c} \uparrow \\ \left. \right\} \end{array} \right\}.$$

It is an immediate consequence that

$$\begin{array}{c} \circlearrowleft \\ \left. \right\} \end{array} = \delta \left. \begin{array}{c} \uparrow \\ \left. \right\} \end{array} \right\},$$

where  $\delta = \frac{v^{-1} - v}{s - s^{-1}} \in \Lambda$ . The coefficient ring  $\Lambda$  is taken as  $Z[v^{\pm 1}, s^{\pm 1}]$ , with denominators  $s^r - s^{-r}$ ,  $r \geq 1$ .

The skein of the annulus is denoted by  $\mathcal{C}$ . It becomes a commutative algebra with a product induced by placing one annulus outside another.

The skein of the rectangle with  $m$  inputs at the top and  $m$  outputs at the bottom is denoted by  $H_m$ . We define a product in  $H_m$  by stacking one rectangle above the other, obtaining the Hecke algebra  $H_m(z)$ , when  $z = s - s^{-1}$  and the coefficients are extended to  $\Lambda$ . The Hecke algebra  $H_m$  can also be regarded as the group algebra of Artin's braid group  $B_m$  generated by the elementary braids  $\sigma_i$ ,  $i = 1, \dots, m - 1$ , modulo the further quadratic relation  $\sigma_i^2 = z\sigma_i + 1$ .

The closure map from  $H_m$  to  $\mathcal{C}$  is the  $\Lambda$ -linear map induced by mapping a tangle  $T$  to its closure  $\widehat{T}$  in the annulus (see figure 6). We refer to a diagram  $Q = \widehat{T}$  as a *directly oriented* pattern.

The image of this map is denoted by  $\mathcal{C}_m$ , which has a useful interpretation as the space of symmetric polynomials of degree  $m$  in variables  $x_1, \dots, x_N$  for large enough  $N$ .

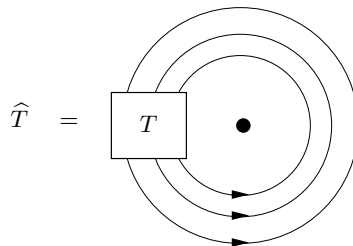


Fig. 6. The closure map

Moreover, the submodule  $\mathcal{C}_+ \subset \mathcal{C}$  spanned by the union  $\cup_{m \geq 0} \mathcal{C}_m$  is a subalgebra of  $\mathcal{C}$  isomorphic to the algebra of the symmetric functions.

### 2.3. Quantum invariants

A quantum group  $\mathcal{G}$  is an algebra over a formal power series ring  $\mathbf{Q}[[\hbar]]$ , typically a deformed version of a classical Lie algebra. We write  $q = e^\hbar, s = e^{\hbar/2}$  when working in  $sl(N)_q$ . A finite dimensional module over  $\mathcal{G}$  is a linear space on which  $\mathcal{G}$  acts.

Crucially,  $\mathcal{G}$  has a coproduct  $\Delta$  which ensures that the tensor product  $V \otimes W$  of two modules is also a module. It also has a *universal R-matrix* (in a completion of  $\mathcal{G} \otimes \mathcal{G}$ ) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \rightarrow W \otimes V.$$

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.

$$\begin{array}{c}
 W \otimes V \\
 \swarrow \quad \searrow \\
 R_{VW} \uparrow \\
 V \otimes W
 \end{array}$$

A braid  $\beta$  on  $m$  strings with permutation  $\pi \in S_m$  and a colouring of the strings by modules  $V_1, \dots, V_m$  leads to a module homomorphism

$$J_\beta : V_1 \otimes \dots \otimes V_m \rightarrow V_{\pi(1)} \otimes \dots \otimes V_{\pi(m)}$$

using  $R_{V_i, V_j}^{\pm 1}$  at each elementary braid crossing. The homomorphism  $J_\beta$  depends *only on the braid*  $\beta$  itself, not its decomposition into crossings, by the Yang-Baxter relation for the universal  $R$ -matrix.

When  $V_i = V$  for all  $i$  we get a module homomorphism  $J_\beta : W \rightarrow W$ , where  $W = V^{\otimes m}$ . Equally, a directed  $m$ -tangle  $T$  determines an endomorphism  $J_T$  of  $W = V^{\otimes m}$ . Now any  $sl(N)$  module  $W$  decomposes as a direct sum  $\bigoplus (W_\mu \otimes V_\mu^{(N)})$ , where  $W_\mu$  is the linear subspace consisting of the *highest weight vectors* of type  $\mu$  associated to the module  $V_\mu^{(N)}$ . Highest weight subspaces of each type are preserved by module homomorphisms, and so  $J_T$  determines (and is determined by) the restrictions  $J_T(\mu) : W_\mu \rightarrow W_\mu$  for each  $\mu$ .

If a knot  $K$  is decorated by a pattern  $Q$  which is the closure of an  $m$ -tangle  $T$  then its quantum invariant  $J(K * Q; V)$  can be found from the endomorphism  $J_T$  of  $W = V^{\otimes m}$  in terms of the quantum invariants of  $K$  and the highest weight maps  $J_T(\mu) : W_\mu \rightarrow W_\mu$  by the formula

$$J(K * Q; V) = \sum c_\mu J(K; V_\mu^{(N)}) \quad (2.1)$$

with  $c_\mu = \text{tr } J_T(\mu)$ . This formula follows from lemma II.4.4 in Turaev's book [11]. Here  $\mu$  runs over partitions with at most  $N$  parts when we are working with  $sl(N)$ , and we set  $c_\mu = 0$  when  $W$  has no highest weight vectors of type  $\mu$ .

*Proof of theorem 1* Take  $V = V^{(N)}$  as the fundamental module of dimension  $N$  for  $sl(N)$ . Then the only highest weight types  $\mu$  which occur in equation (2.1) are partitions of  $m$  with at most  $N$  rows. Because  $J(K * Q; V^{(N)}) = P(K * Q)$  when  $v = s^{-N}$  we can show that  $P(K * Q) = P(K' * Q)$  by showing that  $J(K * Q; V^{(N)}) = J(K' * Q; V^{(N)})$  for all  $N$ . By equation 2.1 it is then enough to show that  $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$  for all  $N$  and all partitions  $\mu \vdash m$ .

Now each tangle  $A$  and  $B$  determines an endomorphism  $J_A, J_B$  of  $V_\mu \otimes V_\mu$ . If  $J_A$  and  $J_B$  commute then  $J(K; V_\mu) = J(K'; V_\mu)$ . The endomorphisms  $J_A$  and  $J_B$  are determined by their restriction  $J_A(\nu), J_B(\nu)$  to the highest weight subspaces  $W_\nu$  in the decomposition  $V_\mu \otimes V_\mu = \sum W_\nu \otimes V_\nu$ , so it is enough to show that  $J_A(\nu)$  and  $J_B(\nu)$  commute where  $V_\nu$  is a summand of  $V_\mu \otimes V_\mu$ . This is certainly the case for all  $\nu$  where  $W_\nu$  is 1-dimensional, which includes the case of single row or column partitions  $\mu$ , [4].

As a special case of the work of Rosso and Jones, [9, 5], we know that the endomorphism of  $V_\mu \otimes V_\mu$  for the full twist  $\Delta^2$  on two strings operates as a scalar  $e^{f(\nu)}$  on each highest weight space  $W_\nu$ , while the half twist  $\Delta$ , represented by the  $R$ -matrix  $R_{V_\mu V_\mu}$ , operates on  $W_\nu$  with two eigenvalues  $\pm e^{\frac{1}{2}f(\nu)}$ .

The positive and negative eigenspaces correspond to the classical decomposition of the Schur function  $(s_\mu)^2$  into symmetric and skew-symmetric parts,  $h_2(s_\mu)$  and  $e_2(s_\mu)$ , and the dimension of each eigenspace of  $W_\nu$  is the multiplicity of  $s_\nu$  in  $h_2(s_\mu)$  and  $e_2(s_\mu)$  respectively.

Now  $A = \tau_3(A)$ , so that  $A\Delta = \Delta A$ . Hence the endomorphism  $J_A$ , and similarly  $J_B$ , preserves the positive and negative eigenspaces of each  $W_\nu$ . If these eigenspaces have dimension 1 or 0 then  $J_A$  and  $J_B$  will commute on  $W_\nu$ .

The theorem is then established by checking that no  $s_\nu$  occurs in  $h_2(s_\mu)$  or  $e_2(s_\mu)$  with multiplicity  $> 1$  for any  $\mu$  with  $|\mu| \leq 5$ . The decomposition of all of these can be quickly confirmed using the Maple program SF of Stembridge [10].  $\square$

**COROLLARY 2.** *Examples include the  $k$ -pretzel knots  $K(a_1, \dots, a_k)$  with odd  $a_i$  shown in figure 7, where the numbers of half-twists  $a_i$  can be permuted without changing the Homfly polynomial of any satellite with  $\leq 5$ -strings.*

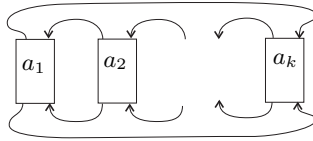


Fig. 7. The pretzel knot  $K(a_1, \dots, a_k)$

### 3. Satellites with different Homfly polynomials

A further check with the program SF when  $|\mu| = 6$  shows that there are just three partitions,  $\mu = 4, 2$ , its conjugate  $\mu = 2, 2, 1, 1$  and  $\mu = 3, 2, 1$  whose symmetric square  $h_2[s_\mu]$  contains summands with multiplicity  $> 1$ , as does the exterior squares of  $\mu = 3, 2, 1$ . Explicitly  $h_2[s_{4,2}] = s_{8,4} + s_{8,2,2} + s_{7,4,1} + s_{7,3,2} + s_{7,3,1,1} + s_{6,6} + s_{6,5,1} + 2s_{6,4,2} + s_{6,3,2,1} + s_{6,2,2,2} + s_{5,5,1,1} + s_{5,4,3} + s_{5,4,2,1} + s_{5,3,3,1} + s_{4,4,4} + s_{4,4,2,2}$ . This means that, although  $m$ -string satellites of  $K$  and  $K'$  must share the Homfly polynomial when  $m \leq 5$ , it is possible for the Homfly polynomials of some 6-string satellites to differ.

We give an example now where this does indeed happen.



Fig. 8. The pretzel knots  $K = K(1, 3, 3, -3, -3)$  and  $K' = K(1, 3, -3, 3, -3)$

**THEOREM 3.** *Let  $K$  and  $K'$  be the pretzel knots  $K = K(1, 3, 3, -3, -3)$  and  $K' = K(1, 3, -3, 3, -3)$  shown in figure 8. The 6-fold parallels  $K * Q$  and  $K' * Q$ , where  $Q$  is the closure of the identity braid on 6 strings, have different Homfly polynomials.*

*Proof.* Write  $K$  and  $K'$  as the closure of the products  $\Delta ABAB$  and  $\Delta BAAB$  respectively, where

$$A = \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}, \quad B = \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array},$$

are the partially closed 3-braids shown, and  $\Delta$  is the positive half-twist. We show that  $P(K * Q) \neq P(K' * Q)$  when  $v = s^{-3}$ . These values are given by the  $sl(3)$  quantum invariants  $J(K * Q; V^{(3)})$  and  $J(K' * Q; V^{(3)})$ , where  $V^{(3)}$  is the fundamental 3-dimensional module for  $sl(3)$ . Since  $Q$  is the closure of the identity braid on 6 strings it induces the identity endomorphism on the module  $(V^{(3)})^{\otimes 6}$ . This module decomposes as  $\bigoplus W_\mu \otimes V_\mu^{(3)}$  where  $\mu$  runs through partitions of 6 with at most 3 rows. The trace of the identity on  $W_\mu$  is just  $d_\mu = \dim W_\mu$ , giving

$$J(K * Q; V^{(3)}) = \sum d_\mu J(K; V_\mu^{(3)}).$$

The only partition  $\mu$  in this range for which the exterior or symmetric square contains highest weight vectors of multiplicity  $> 1$  is the partition  $\mu = 4, 2$ , since the partition  $\mu = 2, 2, 1, 1$  has 4 rows and the repeated factors for  $\mu = 3, 2, 1$  occur for partitions with more than 3 rows. Now  $J_A(\mu)J_B(\mu) = J_B(\mu)J_A(\mu)$  for all other  $\mu$  since  $A$  and  $B$  are symmetric up to altering the framing on both strings, while maintaining the writhe. Then

$$P(K * Q) - P(K' * Q) = d_\mu (J(K; V_\mu^{(3)}) - J(K'; V_\mu^{(3)}))$$

when  $v = s^{-3}$  and  $\mu = 4, 2$ . Since  $d_\mu \neq 0$  it is enough to show that  $J(K; V_\mu^{(3)}) \neq J(K'; V_\mu^{(3)})$ . The module  $V_\mu^{(3)}$  has dimension 27.

We now work in the quantum group  $sl(3)$  and drop the superscript (3) from the irreducible modules.

Decompose the module  $V_\mu \otimes V_\mu$  as  $\sum W_\nu \otimes V_\nu$  and compare the endomorphisms given by the tangles  $T = ABAB\Delta$  and  $T' = BAAB\Delta$ .

In this case just one of the invariant subspaces of highest weight vectors has dimension  $> 1$ . It can be shown that the corresponding  $2 \times 2$  matrices  $A_\mu$  and  $B_\mu$  arising from the two mirror-image tangles  $A$  and  $B$  with 3 crossings satisfy  $\text{tr}(A_\mu B_\mu A_\mu B_\mu - A_\mu A_\mu B_\mu B_\mu) \neq 0$ , which results in a difference in their  $sl(3)$  invariants  $J(K; V_\lambda)$ .

None of the other 6-cell invariants differ on the two knots. Consequently the 6-parallels have different  $sl(3)$  invariants. The  $sl(3)$  invariant of the 6-parallels of the two pretzel knots coloured with the fundamental module, and thus their Homfly polynomials, are then different.  $\square$

### 3.1. Use of the quantum group $sl(3)_q$

The calculation of the  $2 \times 2$  matrices  $A_\nu$  and  $B_\nu$  giving the effect of the two tangles on the highest weight vectors where there is a 2-dimensional highest weight subspace of the symmetric part of the module depends on finding the explicit action of the quantum group on the 27-dimensional module  $V_\mu^{(3)}$  with  $\mu = 4, 2$  and its tensor square, as well as the homomorphism representing its  $R$ -matrix. I used the linear algebra packages in Maple to handle the matrix working and subsequent polynomial factorisation, following fairly closely the techniques developed with H. Ryder in the paper [6].

In the interests of reproducibility I give an account of the methods used, and some of the checks applied during the calculations, to test against known properties.

We start from a presentation of the quantum group  $sl(3)_q$  as an algebra with six generators,  $X_1^\pm, X_2^\pm, H_1, H_2$ , and a description of the comultiplication and antipode.

Let  $M$  be any finite-dimensional left module over  $sl(3)_q$ . The action of any one of these six generators  $Y$  will determine a linear endomorphism  $Y_M$  of  $M$ . We build up explicit matrices for these endomorphisms on a selection of low-dimensional modules, using the comultiplication to deal with the tensor product of two known modules, and the antipode to construct the action on the linear dual of a known module. We must eventually determine the matrices  $Y_M$  for our module  $M = V_{\square\square\square}$ , and find the  $729 \times 729$   $R$ -matrix,  $R_{MM}$  which represents the endomorphism of  $M \otimes M$  needed for crossings.

We follow Kassel in the basic description of the quantum group from using generators  $H_1$  and  $H_2$  for the Cartan sub-algebra, but with generators  $X_i^\pm$  in place of  $X_i$  and  $Y_i$ . We use the notation  $K_i = \exp(hH_i/4)$ , and set  $a = \exp(h/4)$ ,  $s = \exp(h/2) = a^2$  and  $q = \exp(h) = s^2$ , unlike Kassel. The generators satisfy the commutation relations

$$[H_i, H_j] = 0, [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, [X_i^+, X_i^-] = (K_i^2 - K_i^{-2})/(s - s^{-1}),$$

where  $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  is the Cartan matrix for  $SU(3)$  (and also the Serre relations of degree 3 between  $X_1^\pm$  and  $X_2^\pm$ ).

Comultiplication is given by

$$\begin{aligned} \Delta(H_i) &= H_i \otimes I + I \otimes H_i, \\ (\text{so } \Delta(K_i) &= K_i \otimes K_i,) \\ \Delta(X_i^\pm) &= X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm, \end{aligned}$$

and the antipode  $S$  by  $S(X_i^\pm) = -s^{\pm 1} X_i^\pm$ ,  $S(H_i) = -H_i$ ,  $S(K_i) = K_i^{-1}$ .

The fundamental 3-dimensional module, which we denote by  $E$ , has a basis in which the quantum group generators are represented by the matrices  $Y_E$  as listed here.

$$\begin{aligned} X_1^+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & X_2^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ X_1^- &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & X_2^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ H_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & & = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$



For calculations we keep track of the elements  $K_i$  rather than  $H_i$ , represented by

$$K_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$$

for the module  $E$ .

We can then write down the elements  $Y_{EE}$  for the actions of the generators  $Y$  on the module  $E \otimes E$ , from the comultiplication formulae. The  $R$ -matrix  $R_{EE}$  can be given, up to a scalar, by the prescription

$$\begin{aligned} R_{EE}(e_i \otimes e_j) &= e_j \otimes e_i, \text{ if } i > j, \\ &= s e_i \otimes e_i, \text{ if } i = j, \\ &= e_j \otimes e_i + (s - s^{-1})e_i \otimes e_j, \text{ if } i < j, \end{aligned}$$

for basis elements  $\{e_i\}$  of  $E$ .

The linear dual  $M^*$  of a module  $M$  becomes a module when the action of a generator  $Y$  on  $f \in M^*$  is defined by  $\langle Y_M^* f, v \rangle = \langle f, S(Y_M)v \rangle$ , for  $v \in M$ . For the dual module  $F = E^*$  we then have matrices for  $Y_F$ , relative to the dual basis, as follows.

$$\begin{aligned} X_1^+ &= \begin{pmatrix} 0 & 0 & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix} \\ X_1^- &= \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s^{-1} \\ 0 & 0 & 0 \end{pmatrix} \\ K_1 &= \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{pmatrix}. \end{aligned}$$

The most reliable way to work out the  $R$ -matrices  $R_{EF}, R_{FE}$  and  $R_{FF}$  is to combine  $R_{EE}$  with module homomorphisms  $\text{cup}_{EF}, \text{cup}_{FE}, \text{cap}_{EF}$  and  $\text{cap}_{FE}$  between the modules  $E \otimes F, F \otimes E$  and the trivial 1-dimensional module,  $I$ , on which  $X_i^\pm$  acts as zero and  $K_i$  as the identity. The matrices are determined up to a scalar by such considerations; a choice for one dictates the rest.

Once these matrices have been found they can be combined with the matrix  $R_{EE}^{-1}$  to construct the  $R$ -matrices  $R_{EF}, R_{FE}, R_{FF}$ , using the diagram shown in figure 9, for example, to determine  $R_{EF}$  as

$$R_{EF} = (1_F \otimes 1_E \otimes \text{cap}_{EF}) \circ (1_F \otimes R_{EE}^{-1} \otimes 1_F) \circ (\text{cup}_{FE} \otimes 1_E \otimes 1_F).$$

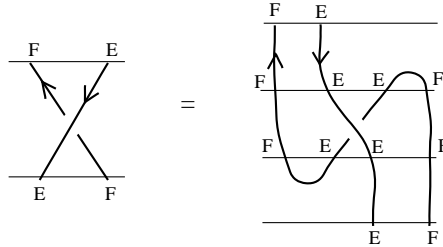


Fig. 9. Construction of the  $R$ -matrix  $R_{EF}$

The module structure of  $M = V_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$  can be found by identifying  $M$  as a 27-dimensional

submodule of  $V_{\square\square} \otimes V_{\square}$ , while the two 6-dimensional modules  $V_{\square}$  and  $V_{\square}$  are themselves submodules of  $E \otimes E$  and  $F \otimes F$  respectively.

We know, by the Pieri formula, that there is a direct sum decomposition of  $V_{\square\square} \otimes V_{\square}$  as  $M \oplus N$ , where  $M = V_{\square\square\square}$  and  $N$  is the sum of the 8-dimensional module  $V_{\square}$  and the 1-dimensional trivial module.

We first identify the module  $V_{\square}$  as a submodule of  $E \otimes E$ , knowing that  $E \otimes E$  is isomorphic to  $V_{\square} \otimes F$ . The full twist element on the two strings both coloured by  $E$  is represented by  $R_{EE}^2$  which acts on  $E \otimes E$  as a scalar on each of the two irreducible submodules  $V_{\square}$  and  $F$ .

Use Maple to find bases for the two eigenspaces of  $R_{EE}^2$ . Then we can identify  $V_{\square}$  with the 6-dimensional one, and write  $P$  and  $Q$  for the  $9 \times 6$  and  $9 \times 3$  matrices whose columns are these bases. The partitioned matrix  $(P|Q)$  is invertible, and its inverse, found by Maple, can be written as  $\begin{pmatrix} R \\ S \end{pmatrix}$ , where  $R$  is a  $6 \times 9$  matrix with  $RP = I_6$  and  $RQ = 0$ .

Regard  $P = \text{inj}M_1EE$  as the matrix representing the inclusion of the module  $V_{\square}$  into  $E \otimes E$ . Then  $R = \text{proj}EEM_1$  is the matrix, in the same basis, of the projection from  $E \otimes E$  to  $V_{\square}$ . For  $M_1 = V_{\square}$  the module generators  $Y_{M_1}$  are given by  $Y_{M_1} = RY_{EE}P$ , giving the explicit action of the quantum group on  $V_{\square}$ .

We perform a similar calculation on  $F \otimes F$  to identify the module  $M_2 = V_{\square}$  and the matrices  $\text{inj}M_2FF$  and  $\text{proj}FFM_2$ , giving the action of the quantum group on  $M_2 = V_{\square}$  in a similar way.

We use inclusion and projection further to find the four  $6^2 \times 6^2$   $R$ -matrices  $R_{M_iM_j}$ . For example, to construct  $R_{M_1M_2} : M_1 \otimes M_2 \rightarrow M_2 \otimes M_1$ , first map  $M_1 \otimes M_2$  to  $E \otimes E \otimes F \otimes F$  by  $\text{inj}M_1EE \otimes \text{inj}M_2FF$ . Then construct the  $R$ -matrix crossing two strings with  $E \otimes E$  and two with  $F \otimes F$  as the composite of  $1 \otimes R_{EF} \otimes 1$ ,  $R_{EF} \otimes R_{FE}$  and  $1 \otimes R_{FF} \otimes 1$ , and finally compose with the projections  $\text{proj}FFM_2 \otimes \text{proj}EEM_1$ , as indicated in figure 10. A similar calculation on the module  $M_1 \otimes M_2$  yields the submodule

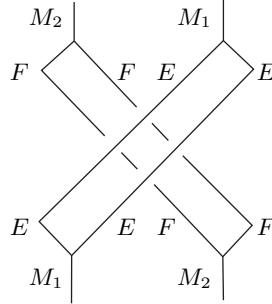


Fig. 10. Construction of the  $R$ -matrix  $R_{M_1M_2}$

$M = V_{\square\square\square}$ . The full twist on two strings, one coloured by  $M_1$  and one by  $M_2$ , is represented by the product  $R_{M_2M_1}R_{M_1M_2}$  and will have one 27-dimensional eigenspace  $M$  complemented by two other eigenspaces. Taking the bases of these eigenspaces in a partitioned  $36 \times 36$  matrix as above will determine a  $36 \times 27$  matrix  $P = \text{inj}MM_1M_2$  and a  $27 \times 36$  matrix  $R = \text{proj}M_1M_2M$ . The quantum group actions  $Y_{M_1M_2}$  on the tensor product are determined by the coproduct formulae, and the actions  $Y_M$  are then given from these using  $P$  and  $R$ . These in turn give rise to the quantum group actions  $Y_{MM}$  on  $M \otimes M$ .

We are also able to construct the  $27^2 \times 27^2$   $R$ -matrix  $R_{MM}$  using the same inclusion

and projection to map  $M \otimes M$  into  $M_1 \otimes M_2 \otimes M_1 \otimes M_2$ , followed by the matrix for crossing four strands, built up from the  $R$ -matrices  $R_{M_i M_j}$  and then the projections back to  $M \otimes M$ .

### 3.2. Completing the calculations

REMARK 2. *We can reach this stage directly if we know the six module generators  $Y_M$  and the  $R$ -matrix  $R_{MM}$  for the module  $M = V_{\square\square\square}$ . We can then calculate the module generators  $Y_{MM}$  using the coproduct, and the twisting element  $T_M = (K_{1M})^4(K_{2M})^4$ .*

Knowing the module generators  $Y_{MM}$  gives an immediate means of finding the highest weight vectors as common null-vectors of  $X_{iMM}^+$ , and their weights can be identified. All the submodules of  $M \otimes M$  occur with multiplicity 1 except  $V_\nu$  with partition  $\nu = 6, 4, 2$  whose highest weights are 2, 2. The 3-dimensional space  $W_\nu$  of highest weight vectors for  $\nu$  is found by solving the linear equations  $X_{1MM}^+ v = 0$ ,  $X_{2MM}^+ v = 0$ ,  $K_{1MM} v = a^2 v$  and  $K_{2MM} v = a^2 v$  for  $v$ . We then find the 2-dimensional positive eigenspace for  $R_{MM}$  on  $W_\nu$ . The endomorphisms  $J_A$  and  $J_B$  will preserve this eigenspace.

Represent the 3-braid  $\sigma_2 \sigma_1^{-1} \sigma_2$  in the 2-tangle  $A$  by an endomorphism  $F_A$  of  $M \otimes M \otimes M$ , using  $R_{MM}$  and its inverse. Then use  $T_M$  and the partial trace to close off one string, hence giving the endomorphism  $J_A$  of  $M \otimes M$  determined by  $A$ . Explicitly, choose a basis  $\{e_i\}$  of  $M$  and write

$$F_A(v \otimes T_M(e_i)) = \sum_j f_{ij}(v) \otimes e_j$$

with  $f_{ij}(v) \in M \otimes M$ . Then  $J_A(v) = \sum_i f_{ii}(v)$ . Applied to each of the two vectors in the highest weight space this determines a  $2 \times 2$  matrix  $A_\nu$  representing the restriction of  $J_A$  to this subspace. Similarly  $B_\nu$  is found using the mirror image braid  $\sigma_2^{-1} \sigma_1 \sigma_2^{-1}$ .

We know that  $R_{MM}$  acts as a scalar on the 2-dimensional space so  $J(K; V_\mu) - J(K'; V_\mu)$  is a non-zero scalar multiple of  $\text{tr}(A_\nu B_\nu A_\nu B_\nu - B_\nu A_\nu A_\nu B_\nu)$ .

This difference is  $2(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^4 + 1)(q^6 + q^3 + 1)^2(q^4 - q^2 + 1)^2(q^4 + q^3 + q^2 + q + 1)^3(q^2 + 1)^4(q^2 + q + 1)^4(q^2 - q + 1)^4(q + 1)^{10}(q - 1)^{18}$ , up to a power of  $q = s^2$  and the quantum dimension of  $V_\nu$ .

### 3.3. Further examples of difference

Using the same matrices  $A_\nu$  and  $B_\nu$  it is possible to find further pretzel knot examples based on sequences of the tangles  $A$  and  $B$  where the 6-parallel have different Homfly polynomial, such as the knots  $K(3, 3, 3, -3, -3)$  and  $K(3, 3, -3, 3, -3)$ . The difference here is the same as for the first example multiplied by the factor  $2q^{32} - q^{31} - 3q^{30} + 5q^{29} + 3q^{28} - 10q^{27} + q^{26} + 14q^{25} - 6q^{24} - 19q^{23} + 21q^{22} + 20q^{21} - 46q^{20} + 2q^{19} + 61q^{18} - 48q^{17} - 35q^{16} + 83q^{15} - 27q^{14} - 66q^{13} + 72q^{12} + 3q^{11} - 57q^{10} + 40q^9 + 10q^8 - 33q^7 + 16q^6 + 7q^5 - 12q^4 + 7q^3 - 4q + 2$ . The same calculations guarantee that satellites based on any closed 6-tangle  $Q = \widehat{T}$  will have different Homfly polynomial, provided that the trace  $c_\mu$  of the endomorphism  $J_{\widehat{T}}$  on the highest weight space  $W_\mu$  of  $V^{\otimes 6}$  is non-zero, where  $\mu$  is the partition 4, 2. This will be the case for most, but not all, patterns  $Q$ , and certainly will be the case for many satellites which are knots rather than links.

The calculations in section 3.2 also show that the 4-parallel of the two pretzel knots  $K(1, 3, 3, -3, -3)$  and  $K(1, 3, -3, 3, -3)$  with two strings oriented in one direction and two in the opposite direction will have different Homfly polynomials, by using the decomposition of the corresponding  $sl(3)_q$  module  $W = V \otimes V \otimes V_{\square} \otimes V_{\square}$  into a sum of

irreducible  $sl(3)_q$  modules. The only module to figure in this decomposition with any multiplicity in its symmetric or exterior square is again  $V_{\square\square\square}$ . The calculations above, using the fact that Homfly with  $v = s^{-3}$  can be calculated by colouring strings with reverse orientation by the dual module  $V^*$  to the fundamental module, and that this is  $V_{\square}$  for  $sl(3)_q$ .

#### 4. Cable patterns

By way of contrast, if the pattern  $Q$  is a cable on any number of strings then  $K * Q$  and  $K' * Q$  share the same Homfly polynomial, where  $K$  and  $K'$  have the same symmetry as in theorem 1.

**THEOREM 4.** *Suppose that  $A$  and  $B$  are both symmetric under the half-twist  $\tau_3$ , so that*

$$A = \begin{array}{c} \boxed{A} \\ \underbrace{\quad \quad} \end{array}, \quad B = \begin{array}{c} \boxed{B} \\ \underbrace{\quad \quad} \end{array}$$

*Let  $K$  and  $K'$  be knots which are the closure of  $ABC$  and  $BAC$  respectively for any tangle  $C$ , as in figure 3. Then  $P(K * Q) = P(K' * Q)$  for every  $(m, n)$  cable pattern  $Q$  where  $m$  and  $n$  are coprime.*

*Proof.* As in the proof of theorem 1 we show that  $J(K * Q; V^{(N)}) = J(K' * Q; V^{(N)})$  for all  $N$ . By equation 2.1 it is then enough to show that  $J(K; V_{\mu}^{(N)}) = J(K'; V_{\mu}^{(N)})$  for all  $N$  and all partitions  $\mu \vdash m$  for which the coefficient  $c_{\mu} \neq 0$ . The coefficients  $c_{\mu}$  depend on the pattern  $Q$  and arise as the trace of the endomorphism  $J_T$  when restricted to the highest weight space  $W_{\mu} \subset V^{\otimes m}$ , where  $Q$  is the closure of the  $m$ -braid  $T = (\sigma_1 \sigma_2 \cdots \sigma_{m-1})^n$ .

It is shown in [9], (see also [5]), that for any such cable  $Q$  the only non-zero coefficients  $c_{\mu}$  occur when the partition  $\mu$  is a *hook*, if  $m$  and  $n$  are coprime. It is then enough to show that  $J(K; V_{\mu}^{(N)}) = J(K'; V_{\mu}^{(N)})$  for all hook partitions  $\mu$ .

Using the same argument as in theorem 1 it remains to check that no Schur function  $s_{\nu}$  occurs with multiplicity  $> 1$  in the decomposition of either the symmetric or exterior squares,  $h_2(s_{\mu})$  or  $e_2(s_{\mu})$ , for any hook partition  $\mu$ . This fact has been established by Carbonara, Rimmel and Yang in theorem 3 of [2], and so the proof is complete.  $\square$

**REMARK 3.** *Theorem 4 highlights the importance of a precise terminology for different types of satellite. The term cable is sometimes used to mean any satellite, while there is a clear distinction here between the behaviour of cables and of parallels or other satellites, which is not primarily a matter of the number of components of the satellite.*

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