THE JONES POLYNOMIAL OF SATELLITE LINKS AROUND MUTANTS

H. R. Morton and P. Traczyk

ABSTRACT. We prove that the (n,k)-cables around a mutant pair of knots C_1 and C_2 cannot be distinguished by the Jones polynomial V, for any (n,k). We prove further that the same result holds for any other satellite around C_1 and C_2.

1. INTRODUCTION.

Lickorish and Lipson, [LL], showed that if C_1 and C_2 are a mutant pair of knots then their 2-cables cannot be distinguished by the 2-variable polynomial P.

Here we prove that the (n,k)-cables around a mutant pair of knots C_1 and C_2 cannot be distinguished by the Jones polynomial V, for any (n,k). We prove further that the same result holds for any other satellite around C_1 and C_2.

The Jones polynomial for oriented links, originally defined via von Neumann algebras in [J], may be introduced in a simple combinatorial way using Kauffman's approach of regular isotopy, [K]. We recall his definition here.

The <u>bracket polynomial</u>, $\langle\ \rangle$, taking values in $Z[A^{\pm 1}]$, is an invariant of unoriented link diagrams which is defined up to regular isotopy (Reidemeister moves 2 and 3) by the following properties:

(1) $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$

where the diagrams differ only as shown.

(2) $\langle 0 \sqcup K \rangle = (-A^{-2} - A^{2}) \langle K \rangle$

where the diagram $0 \sqcup K$ is the distant union of a circle without crossings and the diagram K.

(3) $\langle 0 \rangle = 1$

From a diagram of an unoriented link K we set $f[K] = (-A)^{-3w(K)} \langle K \rangle$, where $w(K)$ is the twist number of the diagram used, i.e. the sum of the crossings with sign. Then $f[K]$ is invariant under all three Reidemeister moves, and the Jones polynomial $V_K(t)$ is given by $f[K](t^{-1/4})$.

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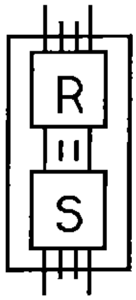
2. TANGLES AND MUTANTS.

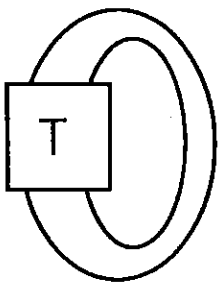
We shall refer to an n-tangle constructed from two n-tangles S and R as in figure 1 as their <u>product</u>, SR.

Figure 1.



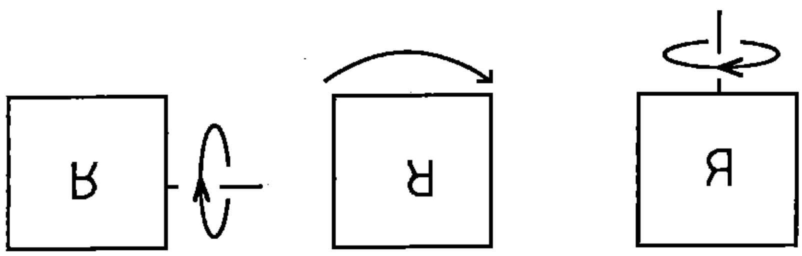
By analogy with braids we use the term <u>closure</u> of a tangle T, written $T\hat{}$, to mean the link formed by joining the n points at the top of T to those at the bottom without further crossings as shown in figure 2.

Figure 2.



In what follows we shall mainly be concerned with unoriented tangles whose diagrams are defined up to regular isotopy, so that we may refer unambiguously to the bracket polynomial of their closure.

Figure 3.



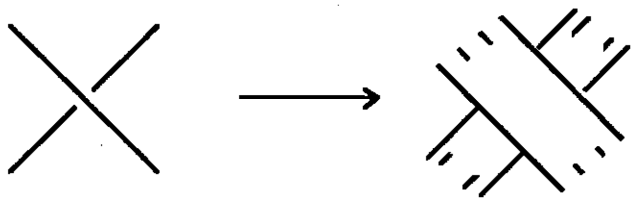
ρ τ σ

For an n-tangle R we define three rotated tangles ρR, σR and τR, by rotating R through π as shown in figure 3. If a link L has a diagram which can be written as $QT\hat{}$ where Q and T are 2-tangles then any link formed by rotating one of these tangles is called a <u>mutant</u> of L. Using the same rotation on both tangles yields a link isotopic to L, and so, since $\sigma = \rho\tau$, we can write the three possible mutants which arise from this decomposition as $(\rho Q)T\hat{}$, $Q(\tau T)\hat{}$ and $(\rho Q)(\tau T)\hat{}$.

3. BANDED TANGLES.

From a 2-tangle T in which the strings join the top points to the bottom points we shall construct an n-tangle S by <u>banding</u> about T. This consists of replacing each string by a band of m or k parallel strings, with $n = m + k$, so that crossings are replaced by band crossings as in figure 4. We allow the extreme cases $m = 0$ or $k = 0$.

Figure 4.



We distinguish two types of 2-tangle T, depending on whether the top points in T are eventually joined to the corresponding bottom points, giving <u>parallel</u> connection, or to the diagonally opposite points, giving <u>diagonal</u> connection.

In considering diagrams for a link L and any mutants we may assume, after a simple redrawing, if necessary, that neither of the 2-tangles from which L has been constructed contains a string joining its two top points, so that the tangles each have parallel or diagonal connections and may be banded as above.

THEOREM 1. Let S be an n-tangle given by banding a 2-tangle T, and let R be any n-tangle. Write $L = SR^\wedge$.

(a) Suppose that T has parallel connections. Write $\rho L = S(\rho R)^\wedge$, and suppose that L and ρL are oriented so that the orientation on S in ρL is the same as that in L, while the orientation on ρR is the reverse of that induced by rotating R. Then $V(\rho L) = V(L)$.

(b) Suppose that T has diagonal connections. Under the assumptions of (a) with ρ replaced by τ we have $V(\tau L) = V(L)$.

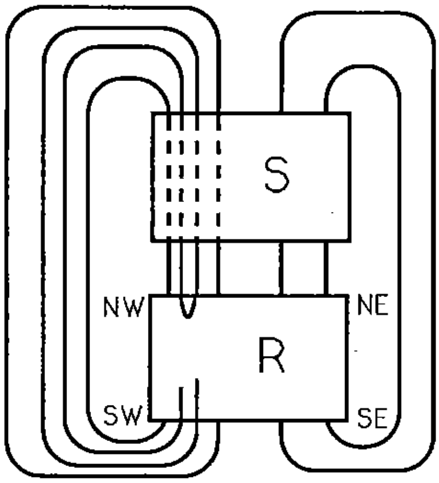
PROOF. We shall prove (a). The proof of (b) is analogous.

It is immediate that the diagrams for L and ρL have the same twist number, $w(L) = w(\rho L)$. It is then enough to forget the orientation and prove that $\langle L \rangle = \langle \rho L \rangle$.

We can begin to construct a binary tree for computing $\langle L \rangle$ by applying the formula $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$ to a crossing c in the tangle R. Simultaneously we compute $\langle \rho L \rangle$ by applying this formula to the crossing ρc in ρR. Induction on the number of crossings in R reduces the problem to the case when R is an n-tangle with no crossing points. We may assume further that R has no free circles.

We now proceed by induction on n. The end points of R lie in one of four quadrants, distinguished by which of the bands in S is attached to them. Since T has parallel connections the band attached to the NW quadrant of R will eventually emerge from S and attach to the SW quadrant, as in figure 5.

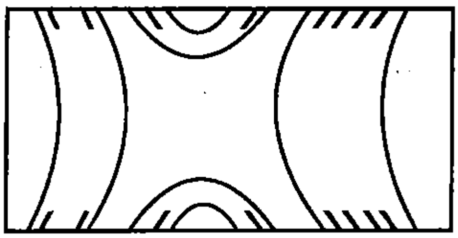
Figure 5.



(i) Suppose that there is an arc in R whose endpoints are adjacent and lie in the same quadrant, say the NW quadrant. We may isotop this arc by regular isotopy through the band in S until it reappears in the SW quadrant, altering the diagram SR to S_1R_1, where S_1 is banded from T using two string less on one of the bands than for S. Then $\langle SR^\wedge \rangle = \langle S_1R_1{}^\wedge \rangle$. In R the corresponding arc has endpoints in the SW quadrant. Isotopy through the same band of S will replace $S(\rho R)$ by $S_1(\rho R_1)$. The required result, that $\langle SR^\wedge \rangle = \langle S(\rho R)^\wedge \rangle$, then follows by induction on n.

(ii) If no arcs in R have adjacent endpoints in the same quadrant it follows readily that R has the form shown in figure 6. Then $\rho R = R$ and there is nothing further to prove.

Figure 6.



The following result, proved in a more general setting by Lickorish, [L], is an immediate corollary.

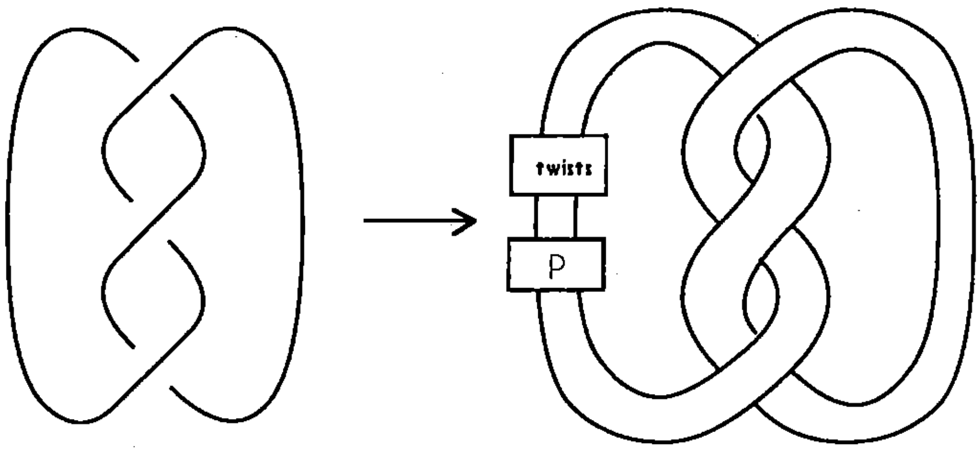
COROLLARY. Let S be given by banding a 1-tangle, and let L be represented by $SR^\wedge$, for some tangle R. Then R may be replaced by ρR or τR without altering the Jones polynomial, so long as the orientations for ρR or τR can be chosen consistently as above.

PROOF. Regard S as given by banding a 2-tangle, using no strings on one of the bands, and apply Theorem 1.

4. SATELLITES.

We now apply Theorem 1 to satellites around mutant knots. A diagram for an r-strand satellite around a knot C, with pattern P, can be easily described, starting from a diagram of C and an r-tangle P. To compare this with other descriptions of satellites, e.g. [MS], we must view $P^\wedge$ as lying in an unknotted solid torus, with the top of the tangle forming a meridian disk. Then we band the diagram of C with an r-band, and modify the band by inserting P, as shown in figure 7, together with $-w(C)$ full twists, where $w(C)$ is the twist number of the given diagram for C.

Figure 7



THEOREM 2. Let C_1 be a knot with diagram $QT^\wedge$, where Q and T are 2-tangles, and let C_2 be any mutant of C_1. Let K_1 and K_2 be the satellites of C_1 and C_2 respectively, constructed with the same pattern P. Then $V(K_1) = V(K_2)$.

PROOF. Since C_1 is a knot we may assume that one of Q and T has parallel connections, T say, and that the other has diagonal connections. Then a diagram for C_2 can be chosen as either $(\rho Q)T^\wedge$, $Q(\tau T)^\wedge$ or $(\rho Q)(\tau T)^\wedge$. Let $T^{(r)}$ (and similarly $Q^{(r)}$) be the $2r$-tangle formed by banding each string of T with an r-band. Then the satellite K_1 has a diagram $\overline{P}Q^{(r)}T^{(r)\wedge}$, where $\overline{P}$ is the $2r$-tangle constructed using P and a number of full twists on the first r strands only. The number of twists required will be the same for C_1 as for any of the possible diagrams for C_2, since the diagrams have the same twist number.

Apply Theorem 1, with $S = T^{(r)}$ and $R = \overline{P}Q^{(r)}$ to show that K_1 and $(\rho R)S^\wedge$ have the same polynomial. Now $\rho R = (\rho Q^{(r)})(\rho\overline{P})$ so the new link is the satellite of $C_2 = (\rho Q)T^\wedge$ constructed with pattern ρP, since the twists in the band may be inserted at any point. By the corollary to Theorem 1 we may replace P by ρP (or

τP) in forming a satellite, without altering V. Hence the satellite K_2 of C_2 with pattern P has $V(K_1) = V(K_2)$.

A similar argument, taking $S = Q^{(r)}$, with τ in place of ρ, will work for the second possible diagram for C_2. This same argument, applied now to $(\rho R)T^{(r)}$ above, taking $S = \rho Q^{(r)}$, deals with the remaining possibility.

5. FURTHER CALCULATIONS.

The result presented here was suggested by an earlier algebraic approach to the problem, which in a natural sense represents a dual attack, [MT]. This appraoch also suggested that the corresponding stronger result for the 2-variable polynomial P, proved in [LL] for 2-cables around mutants, was unlikely to hold for 3-cables around mutants.

We have subsequently compared P for corresponding 3-cables K_1 and K_2 around the Conway and Kinoshita-Teresaka 11-crossing knots and found that the polynomials are indeed different.

So as to make the computations technically practicable we calculated the difference $P_{K_1}(v,z) - P_{K_2}(v,z)$ with $z = 2$, and the coefficients reduced modulo 23, and repeated the calculations modulo 19 with $z = 1$. In each case the difference was non-zero, as a polynomial in v, but became zero when the value of v required to give the Jones polynomial was substituted. This gave us some confidence that the computer algorithm used was correct.

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DEPARTMENT OF PURE MATHEMATICS
THE UNIVERSITY, PO BOX 147,
LIVERPOOL L69 3BX ENGLAND.

INSTYTUT MATEMATYKI
PKiN IXp
00-901 WARSZAWA POLAND

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