INVARIANTS OF GENUS 2 MUTANTS

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ABSTRACT

Pairs of genus 2 mutant knots can have different Homfly polynomials, for example some 3-string satellites of Conway mutant pairs. We give examples which have different Kauffman 2-variable polynomials, answering a question raised by Dunfield et al in their study of genus 2 mutants. While pairs of genus 2 mutant knots have the same Jones polynomial, given from the Homfly polynomial by setting $v = s^2$, we give examples whose Homfly polynomials differ when $v = s^3$. We also give examples which differ in a Vassiliev invariant of degree 7, in contrast to satellites of Conway mutant knots.

Keywords: Conway mutant, genus 2 mutant, Kauffman polynomial, Homfly polynomial, satellite, Vassiliev invariant.

Mathematics Subject Classifications 2000: 57M25

1. Introduction

Genus 2 mutation of knots was introduced by Ruberman [14] in a general 3-manifold. Cooper and Lickorish [1] give a nice account of an equivalent construction for knots in S^3, using genus 2 handlebodies, and it is this construction that we shall use here.

Genus 2 mutant knots provide a test-bed for comparing knot invariants, in the sense that they can be shown to share a certain collection of invariants, and so any invariant on which some mutant pair differs must be completely independent of the shared collection. This procedure can be refined by restricting further the class of genus 2 mutants under consideration, so as to increase the shared collection, and then looking for invariants which differ on some restricted mutants.

In a recent paper [2] Dunfield, Garoufalidis, Shumakovitch and Thistlethwaite survey some of the known results about shared invariants for genus 2 mutants, and show that Khovanov homology is not shared in general. They also give an example of a pair of genus 2 mutants with 75 crossings which differ on their Homfly polynomial. These are smaller examples than the known satellites of the Conway and Kinoshita-Teresaka knots [8]. They ask for examples of genus 2 mutants which don't share the 2-variable Kauffman polynomial, in the expectation that their 75 crossing knots, which are out of range of current programs for calculating the Kauffman polynomial, will indeed give such an example.

In this paper we give a number of smaller genus 2 mutant pairs with different Homfly polynomials, and show that they also have different 2-variable Kauffman polynomials. The smallest examples to date, shown in figure 17, have 55 crossings. The fact that their Kauffman polynomials are different can be detected without having to make a complete calculation. The difference in their Homfly polynomials persists in this example, and in some but not all of the other examples, after making the substitution $v = s^3$. This substitution calculates their quantum $sl(3)$ invariant when coloured by the fundamental 3-dimensional module.

We note too a distinction between general genus 2 mutants and those arising as satellites of Conway mutant knots, by exhibiting examples of a pair of genus 2 mutants which differ on a degree 7 Vassiliev invariant, while work of Duzhin [3] ensures that satellites of Conway mutants share all Vassiliev invariants of degree ≤ 8, extended to degree 10 more recently by Jun Murakami [12].

2. The general setting

The satellite knot $K * Q$ of a framed oriented knot K is constructed, as a framed oriented knot, by taking a framed oriented curve Q in the standard solid torus V. Embed V in $\mathbf{R}^3$ by following the knot K, using the embedding $h : V \to \mathbf{R}^3$ defined by regarding V as a thickened annulus and carrying the annulus to the framing annulus of K. Then $K * Q$ is the curve $h(Q) \subset \mathbf{R}^3$, with the induced orientation and framing.

In the illustration in figure 1 the framing of each curve is given implicitly by the blackboard framing.

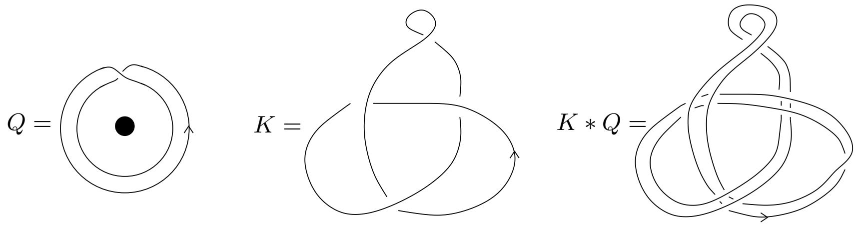


Fig. 1. Satellite construction

We can make a similar construction, starting from a framed oriented curve P in the standard genus 2 handlebody W.

The π-rotation $\tau : W \to W$, illustrated in figure 2, has 6 fixed points on ∂W, where it restricts to the *hyperelliptic involution* with quotient S^2. This lies in the centre of the mapping class group of ∂W and is unique up to conjugation by a homeomorphism isotopic to the identity.

Apply τ to P to get another curve $\tau(P) \subset W$. For any embedding $h : W \to \mathbf{R}^3$

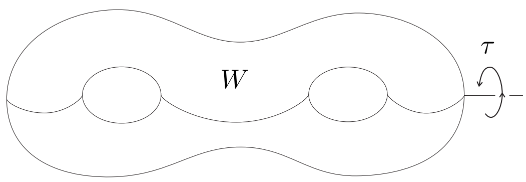


Fig. 2. The rotation τ

the pair of knots $h(P)$ and $h(\tau(P))$ are called *genus* 2 *mutants*.

Remark 2.1. The term 'genus 2 mutant' is used in [2], following Ruberman, for a construction involving an embedded genus 2 surface in a general 3-manifold. The construction detailed above gives rise to pairs of knots which could be termed 'genus 2 handlebody mutants' in general. However, any two knots in S^3 which are genus 2 mutants in the sense of [2] will also be genus 2 handlebody mutants.

The reason that we can restrict to knots in a handlebody when working in S^3 is that any embedded genus 2 surface F in S^3 either bounds a handlebody W or has a compressing disc on both sides.

In the first case the initial knot K must lie in W, otherwise mutation leaves it unchanged. Its mutant K' is then given by applying the involution τ within W.

In the second case K, which lies on one side of F, must avoid one of the two compressing discs. Surgery along this disc gives a solid torus V containing K, and genus 2 mutation amounts to reversing V along its longitude. The resulting knot K' is certainly a genus 2 handlebody mutant of K, based on a handlebody W where a second trivial handle is added to V.

Because we are only concerned in this paper with knots in S^3 we can then follow Lickorish in using the term 'genus 2 mutants' without conflicting with Ruberman's more general 3-manifold usage.

It should also be noted that for links, even in S^3 there are examples of genus 2 mutants which are not genus 2 handlebody mutants. Reversing the orientation of one component of a link can be realised by a genus 2 mutation, in which the two components of the link lie on opposite sides of the genus 2 surface F, so that even the most basic invariants such as linking number can be altered.

2.1. *Satellites of genus 2 mutants*

Theorem 2.2. *Satellites of genus* 2 *mutants are themselves genus* 2 *mutants.*

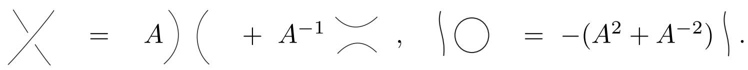
Proof. The satellite $h(P) * Q$ of the framed knot $h(P)$ using a pattern Q in the thickened annulus is the same as the knot constructed by taking the satellite $P * Q$ in W of the curve P and then applying h, since the framings correspond. Then $h(P) * Q = h(P * Q)$. Similarly $h(\tau(P)) * Q = h(\tau(P) * Q) = h(\tau(P * Q))$ with the matching framing and orientation. Hence the satellites $h(P) * Q$ and $h(\tau(P)) * Q$ of the genus 2 mutants $h(P)$ and $h(\tau(P))$ are genus 2 mutants. $\square$

It is easy to establish that genus 2 mutants have the same Jones polynomial, using essentially the argument of Morton and Traczyk [11] in establishing that satellites of Conway mutants have the same Jones polynomial.

This argument is given directly in [1] and [2] but we repeat it here for comparison with our extensions to some of the Homfly cases.

Theorem 2.3. *Genus 2 mutants have the same Jones polynomial.*

Proof. It is enough to work with the Kauffman bracket, defined by the usual skein relations



We can treat a framed curve P in W as an element in the Kauffman bracket skein of a surface S with $W \cong S \times I$ when calculating the Kauffman bracket of the genus 2 mutants $h(P)$ and $h(\tau(P))$. We take S to be a disc with 2 holes. The involution τ on W is induced by the involution on S which preserves the boundary components.

The Kauffman bracket skein of S is spanned by diagrams in S without crossings or null-homotopic curves. Such diagrams consist of unoriented curves parallel to the boundary components and are hence all unchanged by the involution τ on S. Then $\tau(P) = P$ as elements of the skein of S, and so $h(\tau(P)) = h(P)$ as elements of the skein of the plane. Since any diagram K in the plane represents $< K > \bigcirc$ in the skein of the plane, where $< K >$ is the Kauffman bracket of K, it follows that the genus 2 mutants $h(\tau(P))$ and $h(P)$ have the same Kauffman bracket. $\square$

Theorem 2.2 then shows that genus 2 mutants share all their satellite Jones invariants.

2.2. *Genus 2 embeddings following a 2-tangle*

We now show how to use a framed oriented 2-tangle F to define an embedding $h : W \to \mathbf{R}^3$ in such a way that we can readily compare the framed curves $h(P)$ and $h(\tau(P))$. This embedding is said to *follow* the tangle F.

Attaching the two thickened arcs of F to a solid ball results in a genus 2 handlebody as in figure 3 which is to be the image of h.

To specify h we assume that F has a framing, in other words each arc has a specified ribbon neighbourhood. Define a surface S_F in $\mathbf{R}^3$ consisting of a square plus two ribbons following the framing of F, illustrated in figure 4 using the tangle F from figure 11.

Regard W as the thickening, $S \times I$, of a standard surface S, and define h by thickening a map from S to S_F. Our choice of S, and hence the description of

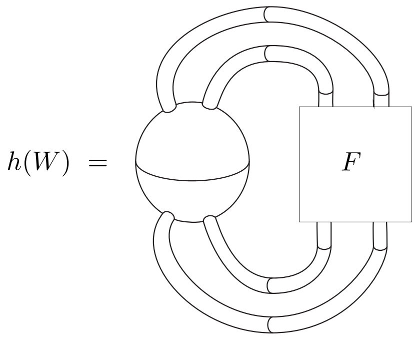


Fig. 3. The handlebody following a tangle F

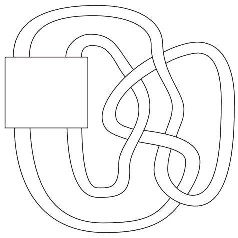


Fig. 4. The surface following a framed tangle

h, depends on the nature of the tangle F. We distinguish two types of oriented 2-tangle:

(1) A *pure* tangle, where the arcs join the two bottom points to the corresponding top points on the same side.
(2) A *transposing* tangle, where the arcs join the two bottom points to the top points on opposite sides.

Remark 2.4. The terms *parallel* and *diagonal* are used in [11] for the connections in these two types of tangle.

1. When F is a *pure* tangle the surface S_F is a disc with 2 holes. Take S to be the square with two ribbons in figure 5 and map S to S_F by taking the square to the square, and the two ribbons to the ribbons around the arcs of F.

2. When F is a *transposing* tangle the surface S_F is a torus with one hole. Take S to be the square with two ribbons in figure 6 and again map S to S_F by mapping the square to the square, and the ribbons around the arcs of F.

We say that h has been constructed by *following* the tangle F. An embedded handlebody in $\mathbf{R}^3$ always arises by following some tangle F, although the choice of F is not unique.

We can get a good view of the pair of mutants constructed from a curve $P \subset W$

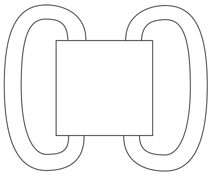


Fig. 5. The disc with 2 holes

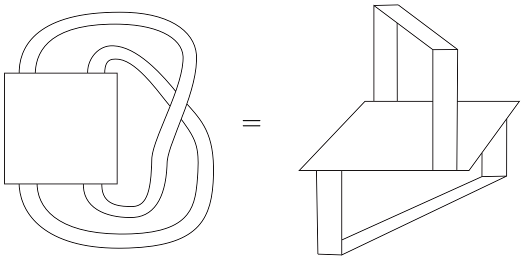


Fig. 6. The torus with one hole

by following a tangle F. The map $\tau : W \to W$ is a thickened map from S to S, which maps the square and each ribbon to itself. In case 1, τ is π-rotation about the horizontal x-axis, which we write as τ_1 when restricted to the square. In case 2, τ is π-rotation about the z-axis orthogonal to the plane of the square, and we write τ_2 for this rotation restricted to the square. These rotations are indicated in figure 7.

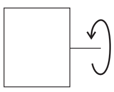
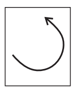
$\tau_1 =$  , $\tau_2 =$  .

Fig. 7. Rotations of the square

Draw P itself as a diagram on the surface S, so that its framing is the blackboard framing from S. We can assume that P runs through each ribbon of S in a number of parallel curves, possibly with different orientations. Suppose that there are m_1 curves in one ribbon and m_2 in the second, numbered from the attachment to the top edge of the square. The rest of the curve P determines a framed m-tangle T in the square, with $m = m_1 + m_2$.

In the case of a pure tangle F the knot $h(P)$ has a diagram as shown in figure 8, where $F^{(m_2,m_1)}$ is the (m_2, m_1) parallel of the framed tangle F with appropriate orientations, and the tangle T lies in the square. The mutant knot $\tau(h(P))$ has $\tau_1(T)$ in place of T, with all orientations in $F^{(m_2,m_1)}$ reversed.

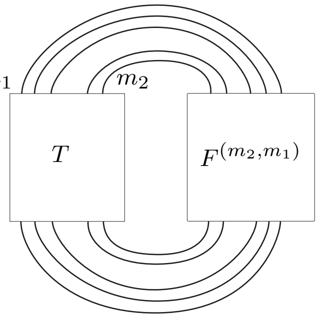


Fig. 8. The diagram for a knot following a pure tangle F

In the case of a transposing tangle the diagram is shown in figure 9, where $\tau(h(P))$ now has $\tau_2(T)$ in place of T.

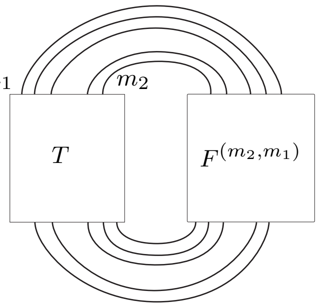
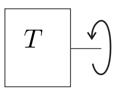
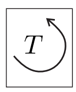


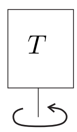
Fig. 9. The diagram for a knot following a transposing tangle F

2.3. *Conway mutants*

For an oriented tangle T write $\tau_1(T)$ and $\tau_2(T)$ for the π-rotations of T about the x-axis and z-axis respectively, as used above. Then $\tau_3(T) = \tau_1\tau_2(T)$ is the π-rotation of T about the y-axis, so that

$\tau_1(T) =$ 

, $\tau_2(T) =$ 

, $\tau_3(T) =$ 

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The term *mutant* was coined by Conway, and refers to the following general construction.

Suppose that a knot K can be decomposed into two oriented 2-tangles F and G as in figure 10. Any knot K' formed by replacing the tangle F with the tangle

$F' = \tau_i(F), i = 1, 2, 3$, reversing its string orientations if necessary is called a *(Conway) mutant* of K.

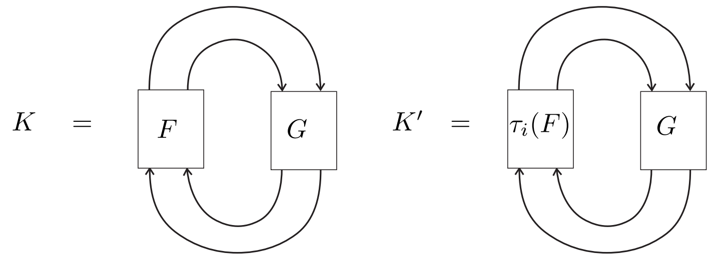


Fig. 10. A knot with mutants

The two 11-crossing knots in figure 11, found by Conway and Kinoshita-Teresaka, are the best-known example of a pair of mutant knots.

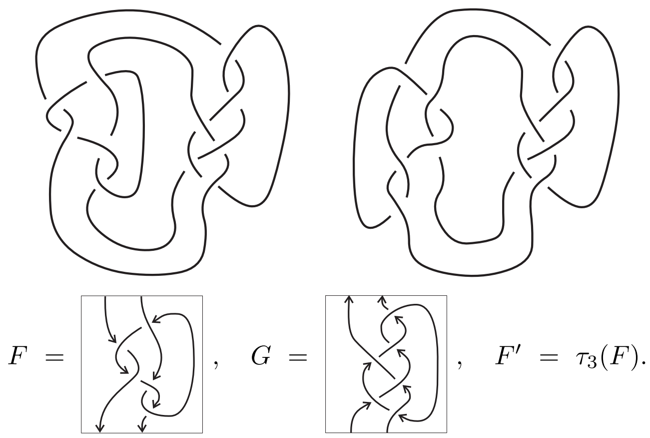


Fig. 11. The Conway and Kinoshita-Teresaka mutant pair, and their constituent tangles

2.4. *Conway mutants as genus 2 mutants*

Any knot K made up of two 2-tangles F and G as in figure 10 lies in two genus 2 handlebodies, one following F and the other following G. Each of these handlebodies defines a genus 2 mutant of K. We call them K_F and K_G respectively.

Since K is a *knot*, one of the tangles F, G is pure and the other is transposing. Let us suppose that F is pure. Then K_F and K_G have diagrams as shown in figure 12.

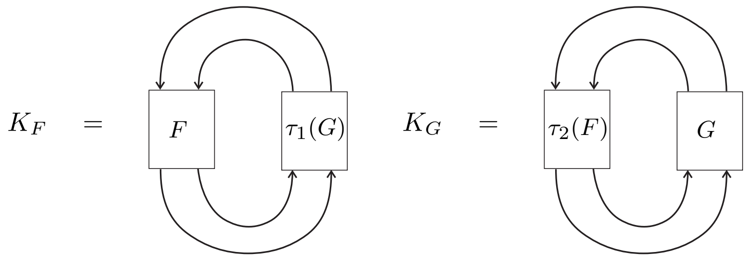


Fig. 12. Genus 2 mutants of K

We can repeat the construction on these knots. K_F lies in the handlebody following $\tau_1(G)$. Since $\tau_1(G)$ is transposing we get a genus 2 mutant $K_{F\tau_1(G)}$. The same knot $K_{G\tau_2(F)} = K_{F\tau_1(G)}$ arises as a genus 2 mutant of K_G from the handlebody following $\tau_2(F)$, shown in figure 13.

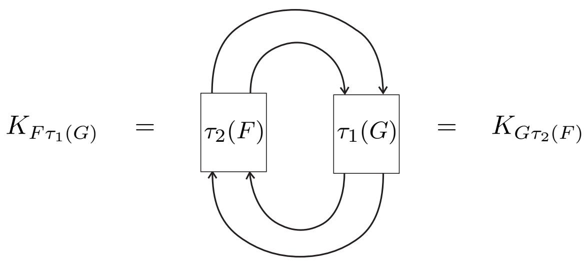


Fig. 13. A further genus 2 mutant, completing the Conway mutants of K

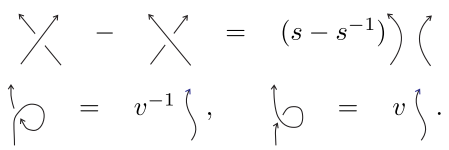
Rotation of the diagrams of K_F and $K_{F\tau_1(G)}$ about the x-axis shows that, up to a choice of string orientation, these three knots K_F, K_G and $K_{F\tau_1(G)}$ are the three Conway mutants of K given by replacing F with $\tau_1(F), \tau_2(F)$ or $\tau_3(F)$ respectively.

It follows that satellites of Conway mutants, with this orientation convention, are related by genus 2 mutation.

We have already seen that these must all share the same Jones polynomial. We now look at the Homfly polynomial of genus 2 mutants.

3. The Homfly polynomial of genus 2 mutants

We use the framed version of the Homfly polynomial based on the skein relations



Definition 3.1. As in earlier work, for example [7], the *Homfly skein* of a surface S consists of linear combinations of diagrams in $S \times I$, up to Reidemeister moves II and III, modulo the skein relations above. Diagrams are assumed to be framed, using the blackboard framing.

Remark 3.2. The Homfly skein of S is known by some authors as the Homfly *skein module* of S, or of $S \times I$.

The Homfly polynomial of a link in $\mathbf{R}^3$ is unchanged if the orientations of *all* its components are reversed. The map induced on the Homfly skein of the annulus when the annulus is rotated by π, reversing its core orientation, and at the same time reversing all string orientations, is the identity. To compare the Homfly polynomials of two genus 2 mutants $h(P)$ and $h(\tau(P))$, or indeed any satellite of them, it is enough to consider $h(\tau(P))$ with orientation reversed.

Given a framed oriented curve P in W we may then regard W as the thickened surface S which is the disc with 2 holes in figure 5, and compare P with $\tau(P)$ after reversing the orientation of $\tau(P)$. If we can present P as an (m_1+m_2)-tangle in the square with m_1 and m_2 curves following the two ribbons then we can write P in the skein of the twice-punctured disc S as a linear combination of simpler curves, each presented by a tangle with at most this number of curves in the ribbons.

Even if our curve P has originally been drawn in a picture following a transposing tangle, with m_1 and m_2 curves around the ribbons there, it can be redrawn as a curve following a pure tangle with the same numbers m_1 and m_2.

The first observation is that if $m_1 = m_2 = 1$ then the genus 2 mutants are Conway mutants, and their Homfly polynomials agree. This is because any 2-tangle can be reduced to a linear combination of 2-tangles which are unchanged under τ_1 plus string orientation reversal.

In the case $m_1, m_2 \leq 2$ the curve P again reduces in the skein of S to a combination of curves in the skein of S which are unchanged by the rotation τ with reversal of string orientation. This is essentially the result of Lickorish and Lipson [5]. There are a couple of cases depending on the relative orientation of the curves in the two ribbons. This argument then covers the case of any 2-string satellite of a pair of Conway mutants, as these can be presented as genus 2 mutants with $m_1 = m_2 = 2$.

The existence of 3-string satellite knots around the Conway and Kinoshita-Teresaka mutant pair with different Homfly polynomials, described in detail in [8], following the earlier calculations by Morton and Traczyk, shows that there are some genus 2 mutants with $m_1 = m_2 = 3$, constructed by following the constituent tangle G in figure 10, which have different Homfly polynomials. Take, for example, the tangle T to be the 3-parallel $F^{(3,3)}$ of the tangle F in figure 10 composed with the braid $\sigma_1\sigma_2$ and follow the tangle G to give a knot with 101 crossings. This is in fact a satellite of the Conway knot, whose genus 2 mutant has $\tau_2(T)$ in place of T.

3.1. *Genus 2 mutants with different Kauffman polynomials*

In [2] the authors exhibit a pair of genus 2 mutants with 75 crossings, which have different Homfly polynomials, and they ask whether genus 2 mutants can have different Kauffman polynomials. Although confident that this is the case they were unable to calculate the polynomials for their 75 crossing example, constructed following the pure 7-crossing tangle DG shown in figure 14.

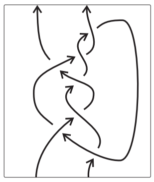
$DG =$ 

Fig. 14. The 7-crossing tangle DG

We give here a number of examples of genus 2 mutants with different Kauffman polynomials.

Theorem 3.3. *The genus* 2 *mutant pair of knots constructed by following the tangle DG, with $m_1 = m_2 = 3$, using the* 6*-string positive permutation braid $B = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4$, shown in figure 15, or its reverse $\tau_1(B)$ as the tangle T, have different Kauffman polynomials.*

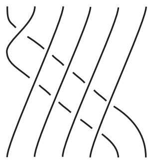


Fig. 15. The braid $B = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4$

Proof. The two knots are presented as closed 9-braids with 72 crossings, so it is quite easy to calculate their Homfly polynomials using the Morton-Short program [10] based on the Hecke algebras. When these are compared, as polynomials in $z = s - s^{-1}$ with coefficients in $\mathbf{Z}[v^{\pm 1}]$ they can be seen to differ in their constant term $P_0(v)$. Now Lickorish shows in [4] that $P_0(v)$ is also the constant term of the Kauffman polynomial when expanded similarly, and hence the Kauffman polynomials of the two knots are different. $\square$

Remark 3.4. This argument could not have been used for the 75 crossing knots in [2], since their Homfly polynomials have the same constant term $P_0(v)$.

3.2. *Vassiliev invariants*

We compared the Vassiliev invariants of the genus 2 mutants, by expanding the difference of their Homfly polynomials as a power series in h taking $s = e^{h/2}$ and $v = s^N = e^{Nh/2}$. In the 75 crossing examples from [2] the lowest degree term of the difference is

$$N\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(13\,N^2+51\right)h^{11},$$

while for our 72 crossing example it is

$$3N(N-1)(N-2)(N-3)(N+3)(N+2)(N+1)h^7.$$

This shows that the 72 crossing knots differ in a Vassiliev invariant of degree at most 7. Consequently satellites of Conway mutants share more Vassiliev invariants than general genus 2 mutants, since they have all Vassiliev invariants of degree ≤ 10 in common, using the result from [8] that Vassiliev invariants of degree $\leq k$ of a satellite $K*Q$ are Vassiliev invariants of K of the same degree, and Jun Murakami's result [12] about Vassiliev invariants of Conway mutants.

3.3. *The Homfly invariants with $v = s^3$*

In our 72 crossing examples the string orientations around each ribbon are all in the same sense $+++$, and as a result the knots have the same Homfly invariant after the substitution $v = s^3$. This is a general consequence of the analysis of the Kuperberg skein of the surface S in [9] for the case $m_1 = m_2 = 3$ in which all the orientations around the ribbons are $+$.

In contrast the 75 crossing examples in [2] use a 6-tangle T, again with $m_1 = m_2 = 3$, where the orientations of the three strands around one of the ribbons are $++-$ while around the other they are $+++$. In this case the Homfly polynomials remain different when $v = s^3$. The difference, as a Laurent polynomial in s, is:

$$\begin{aligned}&s^{-28}\left(s^4-s^2+1\right)\left(s^4+s^3+s^2+s+1\right)\left(s^4-s^3+s^2-s+1\right)\\&\quad\left(s^8+1\right)\left(s^6+s^5+s^4+s^3+s^2+s+1\right)\left(s^6-s^5+s^4-s^3+s^2-s+1\right)\\&\quad\left(s^2-s+1\right)^2\left(s^2+s+1\right)^2\left(s^4+1\right)^2\left(s^2+1\right)^3(s-1)^{11}(s+1)^{11}\end{aligned}$$

We had originally tried to make use of the difference when $v = s^3$ of the 75-crossing examples to show that the Kauffman polynomials are also different. We planned to argue through the comparison of the Homfly polynomials of a certain 2-string satellite at $v = s^4$, without actually calculating this Homfly polynomial, which would be well out of range. Our aim was to make use of a comparison in [6] between this evaluation of the satellite invariant and a different evaluation of the Kauffman polynomial of the original knots, knowing something of the evaluations of the satellite invariant at $v = s^3$. Unfortunately the difference in the invariants at $v = s^3$ contains a factor $\left(s^6 + s^5 + s^4 + s^3 + s^2 + s + 1\right)$ which means that the agreement of the evaluations of the satellite at $v = s^4$ can not be excluded.

This has also proved to be the case in any other examples that we have found where the evaluations at $v = s^3$ are different, so there may be some underlying reason behind this in general.

3.4. *Smaller examples*

Inspired by the combinatorial interpretations of the $v = s^3$ substitution in leading to the Kuperberg skein of the twice-punctured disc we have found a pair of examples following DG with $m_1 = 3, m_2 = 2$ and orientations $++-$ and $+-$. The curve P is shown in figure 16 as a diagram in the disc with two holes, S, along with the resulting 5-tangle T.

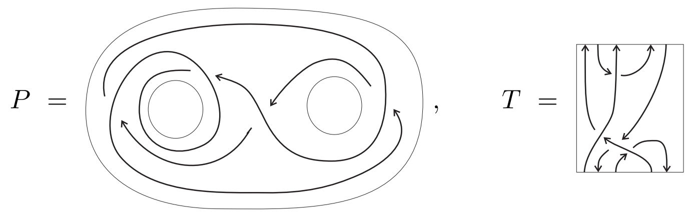


Fig. 16. The curve P in the standard handlebody, and related tangle T

We construct two 55-crossing genus 2 mutants from P by following the tangle DG, to give the knot S_{55}, shown in figure 17. Its mutant partner $S_{55}^{'}$ is given by applying the rotation τ_1 to the tangle T.

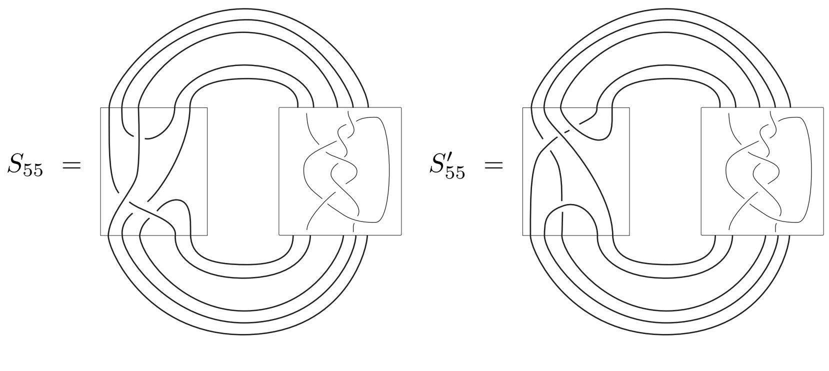


Fig. 17. Two 55-crossing genus 2 mutants with different Homfly and Kauffman polynomials

Theorem 3.5. *The knots S_{55} and S'_{55} shown in figure 17 have different Homfly and Kauffman polynomials. Their Homfly polynomials still differ after the substitution $v = s^3$.*

Proof. The coefficients for the Homfly polynomials of S_{55} and S'_{55} are shown below. They were calculated using Ochiai's program [13], since the knots are not readily expressed as closed braids. In the table the Lickorish-Millett variables l and m are used, with $l^2 = -v^2$ and $m^2 = -z^2$.

S_{55}	l^{-4}	l^{-2}	1	l^2	l^4	l^6	l^8	l^{10}	l^{12}
1	-36	-122	-143	-67	-23	-32	-23	-5	
m^2	276	986	1199	550	148	223	172	34	-3
m^4	-757	-3003	-3884	-1811	-345	-567	-478	-75	20
m^6	1048	4688	6531	3158	400	718	690	76	-45
m^8	-827	-4243	-6360	-3217	-253	-499	-585	-39	34
m^{10}	388	2355	3774	1985	87	192	302	10	-10
m^{12}	-107	-814	1386	-746	-15	-38	-92	-1	1
m^{14}	16	171	308	166	1	3	15		
m^{16}	-1	-20	-38	-20			-1		
m^{18}		1	2	1					

S'_{55}	l^{-4}	l^{-2}	1	l^2	l^4	l^6	l^8	l^{10}	l^{12}
1	-38	-135	-178	-116	-58	-39	-16		1
m^2	257	924	1171	662	288	209	60	-34	-16
m^4	-687	-2591	-3205	-1587	-562	-448	-72	142	54
m^6	964	3913	4779	2080	566	509	24	-226	-73
m^8	-782	-3530	-4260	-1623	-319	-334	10	172	43
m^{10}	377	1991	2356	766	100	126	-7	-67	-11
m^{12}	-106	-709	-814	-213	-16	-25	1	13	1
m^{14}	16	155	171	32	1	2		-1	
m^{16}	-1	-19	-20	-2					
m^{18}		1	1						

Immediately we can see that they have different Homfly polynomials. The first row of coefficients in each array is equivalent to $P_0(v)$, and so the result of Lickorish shows that S_{55} and S'_{55} must also have different Kauffman polynomials.

We obtain Vassiliev invariants as the coefficients of powers of h in the power series given substituting $m = i(e^{\frac{h}{2}} - e^{-\frac{h}{2}})$, $l = ie^{\frac{Nh}{2}}$. The lowest term in the difference of the power series for S_{55} and S'_{55} is

$$3N(N-1)(N-2)(N-3)(N+3)(N+2)(N+1)h^7,$$

so again these differ in a Vassiliev invariant of degree at most 7. We can also look at $sl(3)$ invariant information as a Laurent polynomial in s by making the substitutions

$m = i(s - s^{-1})$, $l = is^3$. The difference is:

$$s^{-24}\left(s^4 - s^2 + 1\right)\left(s^4 + s^3 + s^2 + s + 1\right)\left(s^4 - s^3 + s^2 - s + 1\right)\left(s^8 + 1\right)$$
$$\left(s^6 + s^5 + s^4 + s^3 + s^2 + s + 1\right)\left(s^6 - s^5 + s^4 - s^3 + s^2 - s + 1\right)$$
$$\left(s^2 + s + 1\right)^2\left(s^2 - s + 1\right)^2\left(s^4 + 1\right)^2\left(s^2 + 1\right)^3(s - 1)^8(s + 1)^8$$

Here again there is a factor of $\left(s^6 + s^5 + s^4 + s^3 + s^2 + s + 1\right)$, as in the DGST case. The factor $(s - 1)^8$ shows that they differ in a Vassiliev invariant of degree 8 invariant arising from $sl(3)$. □

We have also constructed a pair of 56-crossing genus 2 mutants following the transposing Conway tangle G with 6 crossings, using the 6-braid $\sigma_2\sigma_3$ and its rotation $\tau_2(\sigma_1\sigma_2) = \sigma_3\sigma_4$, shown in figure 18, with $m_1 = m_2 = 3$. These are closed 9-braids, closely related to the original more complicated Conway and Kinoshita-Teresaka satellites. Like our 72-crossing examples in theorem 3.3 this pair have different Kauffman polynomials, because of $P_0(v)$, and also differ in a degree 7 Vassiliev invariant, but share the same value when $v = s^3$.

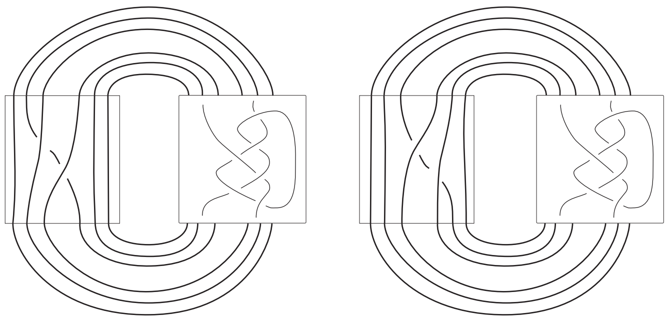


Fig. 18. Two closed 9-braid genus 2 mutants with different Homfly polynomial

3.5. *Other examples*

In [2] there are several nice examples with $m_1 = 2, m_2 = 1$, following the pure tangle AB in figure 19, which have different Khovanov homology. The simplest of these uses the curve P, shown in figure 20 as a diagram in the disc with two holes, S, along with the resulting 3-tangle T.

It is interesting to speculate whether satellites of Conway mutant *knots* can ever have different Khovanov homology, given that they have a greater range of shared invariants than the general genus 2 mutants.

There is a result of Wehrli [15] giving two Conway mutant links with different Khovanov homology, but unlike Conway mutant *knots* these two links are not related by genus 2 mutation.

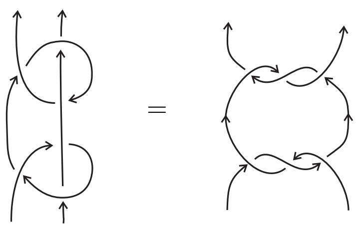


Fig. 19. The tangle AB used in [2]

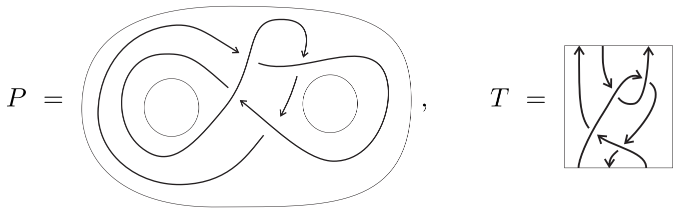


Fig. 20. A curve P, and related tangle T, giving mutants with different Khovanov homology

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