

Banks-Zaks fixed point analysis in momentum subtraction schemes

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Abstract. We analyse the critical exponents relating to the quark mass anomalous dimension and β -function at the Banks-Zaks fixed point in Quantum Chromodynamics (QCD) in a variety of representations for the quark in the momentum subtraction (MOM) schemes of Celmaster and Gonsalves. For a specific range of values of the number of quark flavours, estimates of the exponents appear to be scheme independent. Using the recent five loop modified minimal subtraction ($\overline{\text{MS}}$) scheme quark mass anomalous dimension and estimates of the fixed point location we estimate the associated exponent as 0.263-0.268 for the $SU(3)$ colour group and 12 flavours when the quarks are in the fundamental representation.

1 Introduction.

Non-abelian gauge theories are known to be asymptotically free field theories due to the observations made in [1, 2], that for a certain range of the number of quark flavours, N_f , the one loop β -function is negative. The main theory where this fundamental property is relevant is Quantum Chromodynamics (QCD) which is believed to underpin our understanding of nature's strong nuclear force. Certainly at high energy where the quarks and gluons of QCD behave as effectively free fundamental particles, this asymptotic freedom feature has meant that the internal structure of hadrons can be probed experimentally. The range of N_f where asymptotic freedom is valid is limited since when N_f is sufficiently large the one loop β -function becomes positive and one in effect is in a theory with properties similar to Quantum Electrodynamics (QED). One immediate question which arose in light of the one loop QCD β -function's emergence is whether the only perturbative fixed point of the β -function was the one at the origin. Insight into this problem was given after the computation of the two loop correction to $\beta(a)$, [3, 4], where $a = g^2/(16\pi^2)$ is the perturbative coupling constant expressed in terms of the coupling constant g of the quark and gluon interaction in the QCD Lagrangian. As initially discussed in [3] and further developed in detail in [5], the β -function can exhibit a non-trivial zero for a range of N_f . This occurs when the one loop β -function is negative but the two loop coefficient is positive. Known now as the Banks-Zaks fixed point it has been studied since its discovery due to its potential connection with chiral symmetry breaking, for instance. In more recent years interest in this fixed point has in the main been due to the connection with physics beyond the Standard Model such as technicolor, [6, 7]. More specifically while the early focus was on QCD itself, taking colour groups other than $SU(3)$ with quarks in non-fundamental representations opened up the analysis to model building. This is primarily due to the need to understand where the conformal window is and the *true* range for which it exists. By conformal window we mean the range of N_f for which the non-trivial fixed point exists. The need to find the true range is not a trivial statement. The original observation of [5] used the two loop β -function and this implicitly assumed that the Banks-Zaks fixed point was accessible perturbatively which is not necessarily the case. The problem is that at the lower end of the conformal window, which for QCD is $N_f = 9$, [3], the location of the fixed point is beyond the range of perturbative reliability. So while there may be a formal non-trivial zero of $\beta(a)$, there is no rigorous evidence that it truly exists for relatively low values of N_f . Only a non-perturbative analysis could resolve this. In this respect there has been intense interest in the lattice community in studying this problem for relatively large values of N_f but which are on the limit of perturbative reliability. A non-exhaustive representation of such lattice analyses can be found in [8, 9, 10, 11, 12, 13, 14, 15], for example. Though studies have also been performed with Schwinger-Dyson methods, [16]. More specifically $N_f = 12$ QCD lattice measurements have been made [9, 14, 15]. Part of the motivation is to understand how to find the non-trivial fixed points non-perturbatively and from that knowledge endeavour to explore the fixed point structure for values of $N_f \leq 6$, if it exists, in order to tackle the relation to chiral symmetry breaking.

One of the main topics of current analyses is the measurement of critical exponents associated with the phase transition corresponding to the Banks-Zaks fixed point. These can be determined relatively accurately on the lattice. Indeed several recent studies, [9, 14, 15], show good agreement for the $N_f = 12$ quark mass anomalous dimension exponent. This exponent is of primary interest because of its relation to the definition of a conformal theory. Briefly the full dimension of the quark mass operator must be larger than unity. This places an upper bound of 2 on the contribution of the *anomalous* dimension to this for the theory to be conformal. (See, for example, the discussion in [17].) Determining the range of the conformal window for which a theory satisfies this condition is an indication of the whether conformal symmetry is present. However, the determination of critical exponents is not limited to the lattice. They can be computed from

knowledge of the renormalization group functions. As in [5] the explicit location of the fixed point can be deduced numerically and then the renormalization group functions are evaluated at that point to give estimates for the exponents. In the intervening years after the two loop work of [3, 4], the $\overline{\text{MS}}$ QCD β -function has been extended to four loops as has the quark mass anomalous dimension, [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. With this higher order information the location of the Banks-Zaks fixed point has been refined. See, for example, [28]. At a more technical level the work of [29, 30, 31, 32, 33] formally examines the dependence of the fixed point structure in various schemes and finds conditions on the relations between schemes which ensure credible results. More recently a comprehensive explicit study has been provided in [17]. There a range of colour groups has been examined with quarks in various representations which are relevant to several problems such as those underlying technicolor theories. One general feature of the results of [17] was that the exponent estimates were becoming more reliable when higher order perturbation theory was taken into account. Indeed there was an indication that a selection of estimates were converging. Although whether this was to a value which would be competitive with lattice estimates was not entirely clear for values of N_f in the low to mid-range of the conformal window. It would not be surprising if they did not since non-perturbative properties are present within lattice regularized theories. In more detail the perturbative analysis of [17] provided estimates for $N_f = 12$ on the edge of the error ranges given on the lattice, [14, 15]. One important guide to the credibility of exponent estimates using the renormalization group function approach was the analysis in schemes other than $\overline{\text{MS}}$, [17, 34, 35]. It is a property of the critical point renormalization group equation that critical exponents are renormalization group invariants. Therefore, the value one obtains for an exponent is independent of the renormalization scheme used to perform the computations. Of course, this is in the ideal scenario where one knows the renormalization group functions to all orders in various schemes. This is not the situation in general. So by computing in various schemes for *four* dimensional theories, as was carried out in [17], it may be the case that the convergence is faster than compared to another scheme. Although one never knows a priori which if any scheme would have this property. In [17] the schemes which were considered were the $\overline{\text{MS}}$, modified regularization invariant (RI'), [36, 37], and minimal momentum subtraction (mMOM) schemes, [38]. The renormalization group functions for the final two schemes are also known at four loops, [36, 37, 38, 39, 40, 41]. However, in some sense the three schemes are similar being defined with respect to Green's functions where there is a nullified external momentum. The mMOM scheme, for example, is based on the property that in the Landau gauge the ghost-gluon vertex is finite, [42], when one ghost external leg is nullified. This feature allows one to assign a scheme for an *arbitrary* linear covariant gauge. For a 3-point function this nullified external momentum configuration is termed an exceptional configuration and hence has potential infrared issues. While this ought not to be a problem for high energy analysis one has to be cautious in any low energy studies.

In [43, 44] an alternative set of renormalization schemes was introduced where the 3-point QCD vertices were renormalized at a non-exceptional external momentum configuration known as the symmetric point. Three momentum subtraction schemes (MOM) were defined based on the triple gluon, ghost-gluon and quark-gluon vertices and denoted respectively by MOM_{ggg}, MOM_h and MOM_q, [43, 44]. By their very nature they are physical schemes which are mass dependent. In [44] one hope which was expressed was that perturbative results in the MOM schemes would have a faster convergence than other schemes. What is perhaps more relevant, however, is that there is no doubt about infrared issues due to the non-exceptionality of the subtraction point. In light of this and interest in the Banks-Zaks fixed point the aim of this article is to extend the analysis of [17] to the three MOM schemes of [43, 44]. This is possible partly due to the provision of the three loop MOM β -functions, [45], for an arbitrary linear covariant gauge. Although our main interest here will be in the Landau gauge. At this loop order the scheme dependence first

appears in this latter gauge and so it is apt to study the convergence and scheme dependence of the Banks-Zaks critical exponents in another set of schemes. As will be evident from the explicit structure of the expressions for the renormalization group functions the MOM schemes are in a different class to those used in [17]. So these MOM schemes will offer non-trivial insight into properties of the Banks-Zaks fixed point discussed here. In order to carry out our study the first ingredient is to determine the quark mass anomalous dimension in the MOM schemes at three loops. These were not constructed in [45] and require the renormalization of the quark mass operator inserted in a two loop quark 2-point function where there is a non-zero external momentum flow through the inserted operator.

The article is organized as follows. We derive the three loop quark mass anomalous dimension in the three MOM schemes in section 2 by exploiting properties of the renormalization group equation. Properties of fixed points are reviewed in section 3 such as the renormalization group invariance of critical exponents. In particular we show that the critical exponents derived in the $\overline{\text{MS}}$ and MOM schemes at the Wilson-Fisher fixed point are the same to the various loop orders to which they are known. This clarifies why renormalization group functions, which in different schemes have different analytic structure, do produce renormalization group invariant Wilson-Fisher fixed point exponents. Our extensive Banks-Zaks analysis is provided in section 4. In the main the results are collected across various tables for ease of viewing. Conclusions are given in section 5.

2 Mass operator anomalous dimension.

We begin our analysis by determining the quark mass anomalous dimension in the three MOM schemes using the same approach of others in a chiral theory, [25, 46]. Rather than renormalize the mass itself directly its anomalous dimension is deduced from the renormalization of the associated quark mass operator which is $\bar{\psi}\psi$. This is renormalized by inserting it into a quark 2-point function and ensuring that that Green's function is rendered finite with respect to the particular renormalization scheme of interest. For instance, in [25] the original three loop $\overline{\text{MS}}$ renormalization constant for the quark mass operator was inserted at zero momentum in this Green's function. This was the appropriate external momentum configuration for this particular scheme since one is only interested in the divergences with respect to the regularizing parameter. Throughout we will use dimensional regularization in $d = 4 - 2\epsilon$ dimensions with ϵ being the regularizing parameter. The advantage of this momentum configuration in the derivation of the results of [25, 46] is that the Green's function in effect reduces to the computation of massless 2-point Feynman diagrams which are readily calculable by standard techniques such as the MINCER algorithm, [47]. Although the original three loop results used the R operation and infrared rearrangement of [48, 49], later three loop computations of quark bilinear operator anomalous dimensions used MINCER, [46]. The subsequent extensions of the three loop result, [26, 27, 50], have used several different approaches. In [26] an adaptation of MINCER was developed which used a posteriori the four loop massless master 2-point functions of [51] while infrared rearrangement, [48, 49], together with the evaluation of four loop massive vacuum bubble graphs was used in [27]. While such methods of reducing the renormalization of the quark mass operator to 2-point functions allows access to higher order $\overline{\text{MS}}$ anomalous dimensions the particular external momentum configuration which was used, which is exceptional, cannot be exploited for the set of MOM schemes of [43, 44]. They require a momentum configuration where there is a non-zero momentum flowing through all external legs which means the configuration is non-exceptional. Hence it should suffer none of the infrared problems that could potentially arise in the MINCER approach. Though we need to qualify these remarks briefly. First, the computations of [25, 46] are perfectly infrared safe through the use

of infrared rearrangement, [48, 49]. Also the MINCER package has actually been used to study symmetric point vertex functions in [52]. However, this used MINCER to approximate the basic integrals numerically rather than analytically by an expansion method. Nevertheless compared to the exact three loop MOM β -functions which were determined in [45] there was agreement to a few percent. Therefore, to determine the quark mass operator in the MOM schemes of [43, 44] we have to consider the Green's function

$$\langle \psi(p) \bar{\psi}(q) [\bar{\psi}\psi](r) \rangle \Big|_{p^2=q^2=-\mu^2} \quad (2.1)$$

where

$$p + q + r = 0 \quad (2.2)$$

and p, q and r are the three external momenta. We will always take p and q as the two independent momenta. The restriction in (2.1) indicates evaluation at the symmetric point which is defined as

$$p^2 = q^2 = r^2 = -\mu^2 \quad (2.3)$$

implying

$$pq = \frac{1}{2}\mu^2. \quad (2.4)$$

Here μ is the mass scale which is introduced to ensure that with dimensional regularization the coupling constant, denoted by g here, is dimensionless in d -dimensions. In keeping with previous work we retain the same conventions here which were used in [45].

The evaluation of (2.1) requires some care since we will be using the same computational algorithm as [45] to determine the Green's function. First, (2.1) has to be decomposed into its Lorentz scalar components. For the symmetric point there are two possible independent Lorentz tensors in this basis which are

$$\mathcal{P}_{(1)}^{\bar{\psi}\psi}(p, q) = \Gamma_{(0)} \quad , \quad \mathcal{P}_{(2)}^{\bar{\psi}\psi}(p, q) = \Gamma_{(2)}^{pq} \quad (2.5)$$

where

$$\Gamma_{(n)}^{\mu_1 \dots \mu_n} = \gamma^{[\mu_1} \dots \gamma^{\mu_n]} \quad (2.6)$$

are totally antisymmetric generalized γ -matrices discussed in [53, 54, 55]. The normalization of $1/n!$ is included in the definition. This specific choice of γ -matrices means that the spinor space into which (2.1) decomposes partitions due to, [53, 54, 55],

$$\text{tr} \left(\Gamma_{(m)}^{\mu_1 \dots \mu_m} \Gamma_{(n)}^{\nu_1 \dots \nu_n} \right) \propto \delta_{mn} I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n} \quad (2.7)$$

where the unit matrix is denoted by $I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n}$. We use the convention that when a Lorentz index is contracted with a momentum then the dummy index is replaced by that momentum. Clearly for the momentum configuration which was used to derive the original $\overline{\text{MS}}$ high loop quark mass anomalous dimension one would have only one tensor in its decomposition basis since then p and q would be parallel. Therefore, for the symmetric point evaluation we define the projection by

$$\langle \psi(p) \bar{\psi}(q) [\bar{\psi}\psi](r) \rangle \Big|_{p^2=q^2=-\mu^2} = \sum_{k=1}^2 \mathcal{P}_{(k)}^{\bar{\psi}\psi}(p, q) \Sigma_{(k)}^{\bar{\psi}\psi}(p, q) \quad (2.8)$$

where $\Sigma_{(k)}^{\bar{\psi}\psi}(p, q)$ are values of the scalar amplitudes at the symmetric point. To determine these explicitly we use the projection method of [45] where formally

$$\Sigma_{(k)}^{\bar{\psi}\psi}(p, q) = \mathcal{M}_{kl}^{\bar{\psi}\psi} \left(\mathcal{P}_{(l)}^{\bar{\psi}\psi}(p, q) \langle \psi(p) \bar{\psi}(q) [\bar{\psi}\psi](r) \rangle \right) \Big|_{p^2=q^2=-\mu^2} \quad (2.9)$$

and

$$\mathcal{M}^{\bar{\psi}\psi} = \frac{1}{12} \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}. \quad (2.10)$$

Applying this projection to each of the Feynman graphs comprising the Green's function produces scalar integrals involving scalar products of the external and internal momenta.

To evaluate these we used the Laporta approach, [56], where an intense amount of integration by parts produces a small set of basic master integrals. These have been computed explicitly over several years in [57, 58, 59, 60] but we use the notation of [61] where there is a summary of the master values in powers of ϵ to the order required to determine the finite part. In practical terms we used the version of the Laporta algorithm which was implemented in the REDUZE package, [62]. One useful feature of that package is that it creates a large database of relations between the integrals and solves them automatically in terms of the masters. The relations and results necessary for the computation at hand can be readily lifted from the database and converted into FORM notation. We use FORM and its threaded version TFORM, [63, 64], as the medium to handle the tedious amounts of large algebra which arise in the evaluation of the Green's function. Indeed this was the approach used in similar previous work, [45]. The Feynman diagrams contributing to (2.1) are generated with QGRAF, [65]. At one loop there is only one graph while at two loops there are 13 graphs. Once all the necessary components of this algorithm are assembled the calculation runs automatically. Included in this is the way we undertake the renormalization which follows the method of [20]. The Green's function is determined as a function of the bare coupling constant and gauge parameter but their respective counterterms are introduced by replacing the bare quantities by the renormalized parameters. The renormalization constant associated with each produces the canonical counterterms at each order in perturbation theory. The remaining overall divergences, as well as the appropriate finite part in the MOM case, are finally absorbed into the overall renormalization constant for the Green's function. In this case this will be the quark mass operator renormalization constant.

As part of this renormalization discussion it is worth defining the MOM schemes for the quark mass anomalous dimension, $\gamma_{\bar{\psi}\psi}(a, \alpha)$ where α is the gauge parameter of the canonical linear covariant gauge fixing. First, to carry out an $\overline{\text{MS}}$ determination of $\gamma_{\bar{\psi}\psi}(a, \alpha)$ for the symmetric point momentum configuration only the poles of the Green's function are important. However, the wave function renormalization of the external quark fields has to be included which will be the two loop $\overline{\text{MS}}$ ones of [18, 66]. Following this procedure we have verified that the two loop $\overline{\text{MS}}$ value of $\gamma_{\bar{\psi}\psi}(a, \alpha)$ is obtained. This is a check on our computer algebra set-up as the original two loop computation of [24], as well as that of [23], was performed by the direct evaluation of the quark 2-point function in the presence of *massive* quarks. Having verified this for (2.1) then we can repeat the computation for the various MOM schemes. This is similar in each case but requires not only the quark wave function renormalization constant but also the gauge parameter and coupling constant renormalization constants all in the same MOM scheme. The explicit values in each of the three schemes for these quantities are given in [43, 44, 45]. We note that in [45] the gauge parameter renormalization is performed in a MOM way. In some symmetric point analyses this parameter is renormalized in an $\overline{\text{MS}}$ fashion. However, as we are ultimately only interested in the expressions in the Landau gauge then the differences between the anomalous dimensions in both approaches would only be apparent in the α dependent terms. In other words they would be equivalent in the Landau gauge. Any expression we present here which depends explicitly on α will have used a MOM definition for the renormalization of α . The main reason we retain it within our computations is mainly as an internal check. For example, in the $\overline{\text{MS}}$ scheme the quark mass anomalous dimension is independent of α as the operator is gauge invariant. So we have checked that the two α independent mass operator renormalization constant correctly emerges when we compute in an arbitrary linear covariant gauge. We note that the full analytic

expressions for all main results here are provided in an attached electronic data file.

The main reason why we concentrate on the Landau gauge is due to the renormalization group. The gauge parameter of the linear covariant gauge fixing appears in the $\overline{\text{MS}}$ and MOM renormalization group functions and can be regarded as a second coupling constant albeit to a quadratic term in the gauge field. In this set of gauges the gauge parameter anomalous dimension can be thought of as the β -function of α . Thence it has in principle to be included in any fixed point analysis. Clearly from the high order loop anomalous dimension for α in the various schemes the anomalous dimension is proportional at α . So that $\alpha = 0$ is a fixed point and hence the focus on the Landau gauge. Of course, this is not the only solution since in principle there could be non-trivial Banks-Zaks type fixed points for α itself. We do not consider those here partly because the lattice analyses are in the Landau gauge. Some insight, though, into such additional fixed points has been given in [29, 30, 31, 32, 67].

While what we have described is the procedure to construct the two loop MOM operator renormalization constants, the *three* loop anomalous dimension in each of the MOM schemes can be determined with this information. This is possible due to a property of the renormalization group equation and knowledge of the three loop $\overline{\text{MS}}$ quark mass anomalous dimension, [25]. The construction requires the operator renormalization conversion function which is defined by

$$C_{\bar{\psi}\psi}^{\text{MOMi}}(a, \alpha) = Z_{\text{MOMi}}^{\bar{\psi}\psi} \left[Z_{\overline{\text{MS}}}^{\bar{\psi}\psi} \right]^{-1}. \quad (2.11)$$

In (2.11) the convention we use is that the function is expressed in terms of $\overline{\text{MS}}$ variables for the coupling constant and gauge parameter. We do not include the $\overline{\text{MS}}$ label on these variables. However, in computing the right hand side of (2.11) each renormalization constant is a function of the parameters defined in those respective schemes. In order to have a finite function in the $\epsilon \rightarrow 0$ limit the MOM variables have to be mapped to their $\overline{\text{MS}}$ versions before the perturbative expansion of $C_{\bar{\psi}\psi}^{\text{MOMi}}(a, \alpha)$ is deduced. For each of the MOM schemes we are interested in here these mappings are given in [45]. The full expressions for the quark mass conversion function is given in the associated data file. However, the numerical expression in each MOM scheme for $SU(3)$ is

$$C_{\bar{\psi}\psi}^{\text{MOMi}}(a, \alpha) = 1 + [0.229271\alpha - 0.645519]a + [0.568426\alpha^2 + 4.554664\alpha + 4.013539N_f - 22.607687]a^2 + O(a^3) \quad (2.12)$$

where in keeping with observations in previous work in the MOM schemes the same conversion function emerges in each scheme. Equipped with each conversion function then the renormalization group relation between the operator anomalous dimensions is given formally by

$$\gamma_{\bar{\psi}\psi}^{\text{MOMi}}(a_{\text{MOMi}}, \alpha_{\text{MOMi}}) = \left[\gamma_{\bar{\psi}\psi}(a) - \beta(a) \frac{\partial}{\partial a} \ln C_{\bar{\psi}\psi}^{\text{MOMi}}(a, \alpha) - \alpha \gamma_{\alpha}(a, \alpha) \frac{\partial}{\partial \alpha} \ln C_{\bar{\psi}\psi}^{\text{MOMi}}(a, \alpha) \right]_{\overline{\text{MS}} \rightarrow \text{MOMi}}. \quad (2.13)$$

Here the subscript mapping indicates that after the quantity in square brackets has been determined then that expression which is in $\overline{\text{MS}}$ variables is mapped to MOMi variables consistent with the arguments of the function on the left hand side. In (2.13) the $\overline{\text{MS}}$ quark mass anomalous dimension only depends on the coupling constant since that scheme is a mass independent one and it is known, [68], that in that case the anomalous dimension does not depend on α . By contrast, the MOM scheme is a mass dependent scheme and therefore anomalous dimensions of gauge invariant operators will depend on the gauge parameter. We again note that for our

purposes that although we include the gauge parameter throughout, our focus in analysing the critical exponents here will be solely on the Landau gauge.

Having described the method we have used to evaluate the quark mass anomalous dimension in each of the three MOM schemes we now record their explicit values for the Landau gauge. We have

$$\begin{aligned}
\gamma_{\bar{\psi}\psi}^{\text{MOMq}}(a, 0) = & -3C_F a \\
& + \left[\left[2 + \frac{8}{9}\pi^2 - \frac{4}{3}\psi'(\frac{1}{3}) \right] N_f T_F C_F + \left[-\frac{13}{4} - \pi^2 + \frac{3}{2}\psi'(\frac{1}{3}) \right] C_F C_A \right. \\
& \left. + \left[-\frac{27}{2} - \frac{8}{9}\pi^2 + \frac{4}{3}\psi'(\frac{1}{3}) \right] C_F^2 \right] a^2 \\
& + \left[\left[41 - \frac{20}{3}\zeta(3) - \frac{16}{9}\pi^2 - \frac{8}{27}\pi^4 - 8s_2(\frac{\pi}{6}) + 16s_2(\frac{\pi}{2}) + \frac{40}{3}s_3(\frac{\pi}{6}) - \frac{32}{3}s_3(\frac{\pi}{2}) \right. \right. \\
& \left. \left. + \frac{8}{3}\psi'(\frac{1}{3}) + \frac{16}{9}\psi'(\frac{1}{3})\pi^2 - \frac{4}{3}(\psi'(\frac{1}{3}))^2 - \frac{1}{9}\psi'''(\frac{1}{3}) - \frac{1}{54}\ln^2(3)\sqrt{3}\pi \right. \right. \\
& \left. \left. + \frac{2}{9}\ln(3)\sqrt{3}\pi + \frac{29}{1458}\sqrt{3}\pi^3 \right] N_f T_F C_F C_A \right. \\
& + \left[\frac{130}{3} - \frac{32}{3}\zeta(3) + \frac{40}{9}\pi^2 - \frac{64}{81}\pi^4 + 64s_2(\frac{\pi}{6}) - 128s_2(\frac{\pi}{2}) - \frac{320}{3}s_3(\frac{\pi}{6}) \right. \\
& \left. + \frac{256}{3}s_3(\frac{\pi}{2}) - \frac{20}{3}\psi'(\frac{1}{3}) + \frac{16}{9}\psi'(\frac{1}{3})\pi^2 - \frac{4}{3}(\psi'(\frac{1}{3}))^2 + \frac{2}{27}\psi'''(\frac{1}{3}) \right. \\
& \left. + \frac{4}{27}\ln^2(3)\sqrt{3}\pi - \frac{16}{9}\ln(3)\sqrt{3}\pi - \frac{116}{729}\sqrt{3}\pi^3 \right] N_f T_F C_F^2 - 8N_f^2 T_F^2 C_F \\
& + \left[-\frac{249}{4} + \frac{2503}{48}\zeta(3) - \frac{1297}{72}\pi^2 - \frac{191}{486}\pi^4 + \frac{347}{2}s_2(\frac{\pi}{6}) - 347s_2(\frac{\pi}{2}) \right. \\
& \left. - \frac{1735}{6}s_3(\frac{\pi}{6}) + \frac{694}{3}s_3(\frac{\pi}{2}) + \frac{1297}{48}\psi'(\frac{1}{3}) + \frac{175}{324}\psi'(\frac{1}{3})\pi^2 - \frac{175}{432}(\psi'(\frac{1}{3}))^2 \right. \\
& \left. + \frac{23}{288}\psi'''(\frac{1}{3}) + \frac{347}{864}\ln^2(3)\sqrt{3}\pi - \frac{347}{72}\ln(3)\sqrt{3}\pi - \frac{10063}{23328}\sqrt{3}\pi^3 \right] C_F C_A^2 \\
& + \left[-\frac{467}{12} + \frac{106}{3}\zeta(3) + \frac{515}{9}\pi^2 + \frac{1216}{243}\pi^4 - 428s_2(\frac{\pi}{6}) + 856s_2(\frac{\pi}{2}) \right. \\
& \left. + \frac{2140}{3}s_3(\frac{\pi}{6}) - \frac{1712}{3}s_3(\frac{\pi}{2}) - \frac{515}{6}\psi'(\frac{1}{3}) - \frac{1192}{81}\psi'(\frac{1}{3})\pi^2 + \frac{298}{27}(\psi'(\frac{1}{3}))^2 \right. \\
& \left. - \frac{1}{27}\psi'''(\frac{1}{3}) - \frac{107}{108}\ln^2(3)\sqrt{3}\pi + \frac{107}{9}\ln(3)\sqrt{3}\pi + \frac{3103}{2916}\sqrt{3}\pi^3 \right] C_F^2 C_A \\
& + \left[-\frac{279}{2} + 56\zeta(3) - \frac{364}{9}\pi^2 + \frac{176}{243}\pi^4 - 48s_2(\frac{\pi}{6}) + 96s_2(\frac{\pi}{2}) + 80s_3(\frac{\pi}{6}) \right. \\
& \left. - 64s_3(\frac{\pi}{2}) + \frac{182}{3}\psi'(\frac{1}{3}) + \frac{400}{81}\psi'(\frac{1}{3})\pi^2 - \frac{100}{27}(\psi'(\frac{1}{3}))^2 - \frac{8}{9}\psi'''(\frac{1}{3}) \right. \\
& \left. - \frac{1}{9}\ln^2(3)\sqrt{3}\pi + \frac{4}{3}\ln(3)\sqrt{3}\pi + \frac{29}{243}\sqrt{3}\pi^3 \right] C_F^3 \Big] a^3 + O(a^4) \quad (2.14)
\end{aligned}$$

for the MOMq scheme and

$$\begin{aligned}
\gamma_{\bar{\psi}\psi}^{\text{MOMggg}}(a, 0) = & -3C_F a \\
& + \left[\left[\frac{2}{3} + \frac{88}{27}\pi^2 - \frac{44}{9}\psi'(\frac{1}{3}) \right] N_f T_F C_F + \left[-\frac{53}{6} - \frac{89}{27}\pi^2 + \frac{89}{18}\psi'(\frac{1}{3}) \right] C_F C_A \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{2}C_F^2] a^2 \\
& + \left[\left[\frac{2369}{54} - \frac{128}{3}\zeta(3) + \frac{226}{243}\pi^2 + \frac{12688}{2187}\pi^4 - \frac{377}{243\sqrt{3}}\pi^3 - \frac{52}{3\sqrt{3}}\ln(3)\pi \right. \right. \\
& \quad + \frac{13}{9\sqrt{3}}\ln^2(3)\pi + 208s_2(\frac{\pi}{6}) - 416s_2(\frac{\pi}{2}) - \frac{1040}{3}s_3(\frac{\pi}{6}) + \frac{832}{3}s_3(\frac{\pi}{2}) \\
& \quad \left. \left. - \frac{113}{81}\psi'(\frac{1}{3}) - \frac{15280}{729}\psi'(\frac{1}{3})\pi^2 + \frac{3820}{243}(\psi'(\frac{1}{3}))^2 + \frac{4}{9}\psi'''(\frac{1}{3}) \right] N_f T_F C_F C_A \right. \\
& + \left[18 - \frac{32}{3}\zeta(3) + \frac{104}{27}\pi^2 - \frac{320}{243}\pi^4 - \frac{116}{243\sqrt{3}}\pi^3 - \frac{16}{3\sqrt{3}}\ln(3)\pi \right. \\
& \quad + \frac{4}{9\sqrt{3}}\ln^2(3)\pi + 64s_2(\frac{\pi}{6}) - 128s_2(\frac{\pi}{2}) - \frac{320}{3}s_3(\frac{\pi}{6}) + \frac{256}{3}s_3(\frac{\pi}{2}) \\
& \quad \left. \left. - \frac{52}{9}\psi'(\frac{1}{3}) + \frac{272}{81}\psi'(\frac{1}{3})\pi^2 - \frac{68}{27}(\psi'(\frac{1}{3}))^2 + \frac{2}{27}\psi'''(\frac{1}{3}) \right] N_f T_F C_F^2 \right. \\
& + \left[-\frac{196}{27} + \frac{320}{243}\pi^2 - \frac{10240}{2187}\pi^4 - \frac{160}{81}\psi'(\frac{1}{3}) + \frac{10240}{729}\psi'(\frac{1}{3})\pi^2 \right. \\
& \quad \left. \left. - \frac{2560}{243}(\psi'(\frac{1}{3}))^2 \right] N_f^2 T_F^2 C_F \right. \\
& + \left[-\frac{220159}{1728} + \frac{6367}{48}\zeta(3) + \frac{1643}{243}\pi^2 - \frac{9779}{17496}\pi^4 + \frac{12499}{3888\sqrt{3}}\pi^3 \right. \\
& \quad + \frac{431}{12\sqrt{3}}\ln(3)\pi - \frac{431}{144\sqrt{3}}\ln^2(3)\pi - 431s_2(\frac{\pi}{6}) + 862s_2(\frac{\pi}{2}) \\
& \quad + \frac{2155}{3}s_3(\frac{\pi}{6}) - \frac{1724}{3}s_3(\frac{\pi}{2}) - \frac{1643}{162}\psi'(\frac{1}{3}) + \frac{22183}{2916}\psi'(\frac{1}{3})\pi^2 \\
& \quad \left. \left. - \frac{22183}{3888}(\psi'(\frac{1}{3}))^2 - \frac{427}{576}\psi'''(\frac{1}{3}) \right] C_F C_A^2 \right. \\
& + \left[13 + \frac{88}{3}\zeta(3) + \frac{593}{27}\pi^2 + \frac{880}{243}\pi^4 + \frac{319}{243\sqrt{3}}\pi^3 + \frac{44}{3\sqrt{3}}\ln(3)\pi \right. \\
& \quad - \frac{11}{9\sqrt{3}}\ln^2(3)\pi - 176s_2(\frac{\pi}{6}) + 352s_2(\frac{\pi}{2}) + \frac{880}{3}s_3(\frac{\pi}{6}) - \frac{704}{3}s_3(\frac{\pi}{2}) \\
& \quad \left. \left. - \frac{593}{18}\psi'(\frac{1}{3}) - \frac{748}{81}\psi'(\frac{1}{3})\pi^2 + \frac{187}{27}(\psi'(\frac{1}{3}))^2 - \frac{11}{54}\psi'''(\frac{1}{3}) \right] C_F^2 C_A \right. \\
& \left. - \frac{129}{2}C_F^3 \right] a^3 + O(a^4) \tag{2.15}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\bar{\psi}\psi}^{\text{MOMh}}(a, 0) & = -3C_F a \\
& + \left[\left[2 + \frac{8}{9}\pi^2 - \frac{4}{3}\psi'(\frac{1}{3}) \right] N_f T_F C_F + \left[-\frac{55}{4} - \frac{49}{18}\pi^2 + \frac{49}{12}\psi'(\frac{1}{3}) \right] C_F C_A \right. \\
& \quad \left. - \frac{3}{2}C_F^2 \right] a^2 \\
& + \left[\left[\frac{157}{2} - \frac{32}{3}\zeta(3) + \frac{313}{27}\pi^2 + \frac{104}{243}\pi^4 - \frac{29}{243\sqrt{3}}\pi^3 - \frac{4}{3\sqrt{3}}\ln(3)\pi \right. \right. \\
& \quad + \frac{1}{9\sqrt{3}}\ln^2(3)\pi + 16s_2(\frac{\pi}{6}) - 32s_2(\frac{\pi}{2}) - \frac{80}{3}s_3(\frac{\pi}{6}) + \frac{64}{3}s_3(\frac{\pi}{2}) - \frac{313}{18}\psi'(\frac{1}{3}) \\
& \quad \left. \left. - \frac{104}{81}\psi'(\frac{1}{3})\pi^2 + \frac{26}{27}(\psi'(\frac{1}{3}))^2 \right] N_f T_F C_F C_A \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{46}{3} - \frac{32}{3} \zeta(3) - \frac{152}{27} \pi^2 - \frac{320}{243} \pi^4 - \frac{116}{243\sqrt{3}} \pi^3 - \frac{16}{3\sqrt{3}} \ln(3)\pi \right. \\
& \quad + \frac{4}{9\sqrt{3}} \ln^2(3)\pi + 64s_2\left(\frac{\pi}{6}\right) - 128s_2\left(\frac{\pi}{2}\right) - \frac{320}{3}s_3\left(\frac{\pi}{6}\right) + \frac{256}{3}s_3\left(\frac{\pi}{2}\right) \\
& \quad \left. + \frac{76}{9}\psi'\left(\frac{1}{3}\right) + \frac{272}{81}\psi'\left(\frac{1}{3}\right)\pi^2 - \frac{68}{27}\left(\psi'\left(\frac{1}{3}\right)\right)^2 + \frac{2}{27}\psi'''\left(\frac{1}{3}\right) \right] N_f T_F C_F^2 \\
& + \left[-\frac{1419}{8} + \frac{3757}{48} \zeta(3) - \frac{3383}{108} \pi^2 - \frac{469}{486} \pi^4 + \frac{1015}{3888\sqrt{3}} \pi^3 + \frac{35}{12\sqrt{3}} \ln(3)\pi \right. \\
& \quad - \frac{35}{144\sqrt{3}} \ln^2(3)\pi - 35s_2\left(\frac{\pi}{6}\right) + 70s_2\left(\frac{\pi}{2}\right) + \frac{175}{3}s_3\left(\frac{\pi}{6}\right) - \frac{140}{3}s_3\left(\frac{\pi}{2}\right) \\
& \quad \left. + \frac{3383}{72}\psi'\left(\frac{1}{3}\right) + \frac{4751}{1296}\psi'\left(\frac{1}{3}\right)\pi^2 - \frac{4751}{1728}\left(\psi'\left(\frac{1}{3}\right)\right)^2 - \frac{37}{384}\psi'''\left(\frac{1}{3}\right) \right] C_F C_A^2 \\
& + \left[\frac{97}{12} + \frac{88}{3} \zeta(3) + \frac{1217}{54} \pi^2 + \frac{880}{243} \pi^4 + \frac{319}{243\sqrt{3}} \pi^3 + \frac{44}{3\sqrt{3}} \ln(3)\pi \right. \\
& \quad - \frac{11}{9\sqrt{3}} \ln^2(3)\pi - 176s_2\left(\frac{\pi}{6}\right) + 352s_2\left(\frac{\pi}{2}\right) + \frac{880}{3}s_3\left(\frac{\pi}{6}\right) - \frac{704}{3}s_3\left(\frac{\pi}{2}\right) \\
& \quad \left. - \frac{1217}{36}\psi'\left(\frac{1}{3}\right) - \frac{748}{81}\psi'\left(\frac{1}{3}\right)\pi^2 + \frac{187}{27}\left(\psi'\left(\frac{1}{3}\right)\right)^2 - \frac{11}{54}\psi'''\left(\frac{1}{3}\right) \right] C_F^2 C_A \\
& - 8N_f^2 T_F^2 C_F - \frac{129}{2} C_F^3 \Big] a^3 + O(a^4) \tag{2.16}
\end{aligned}$$

for MOMggg and MOMh respectively. Here C_F and C_A are the usual rank 2 Casimirs in the fundamental and adjoint representations respectively which have dimensions N_F and N_A . The Dynkin index is T_F and N_f is the number of massless quark flavours. Various numbers arise through the values of the underlying masters. The function $\psi(z)$ is the Euler ψ -function and $\zeta(z)$ is the Riemann zeta function. Various specific values of the polylogarithm function, $\text{Li}_n(z)$, occur which are defined by

$$s_n(z) = \frac{1}{\sqrt{3}} \Im \left[\text{Li}_n \left(\frac{e^{iz}}{\sqrt{3}} \right) \right]. \tag{2.17}$$

To assist the evaluation of the quark mass anomalous dimensions numerically we note

$$\begin{aligned}
\zeta(3) &= 1.20205690, \quad \psi'\left(\frac{1}{3}\right) = 10.09559713, \quad \psi'''\left(\frac{1}{3}\right) = 488.1838167, \quad s_2\left(\frac{\pi}{2}\right) = 0.32225882 \\
s_2\left(\frac{\pi}{6}\right) &= 0.22459602, \quad s_3\left(\frac{\pi}{2}\right) = 0.32948320, \quad s_3\left(\frac{\pi}{6}\right) = 0.19259341. \tag{2.18}
\end{aligned}$$

Throughout we use $a = g^2/(16\pi^2)$ as the coupling constant in keeping with conventions used in previous articles. Equipped with these expressions we are now in a position to analyse their values at the Banks-Zaks fixed point.

3 Fixed points.

Before carrying out that analysis we concentrate in this section on various aspects of fixed points in the QCD β -function. For the moment we review the situation in the $\overline{\text{MS}}$ scheme partly as this forms the discussion but partly as this was the scheme in which the Banks-Zaks fixed point was explored initially, [3, 5]. Although the four loop $\overline{\text{MS}}$ β -function is available, and will be used later, for the moment the three loop result of [19] is sufficient for the present discussion and is

$$\beta^{\overline{\text{MS}}}(a) = - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 - \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3$$

$$\begin{aligned}
& + [2830C_A^2T_F N_f - 2857C_A^3 + 1230C_A C_F T_F N_f - 316C_A T_F^2 N_f^2 \\
& - 108C_F^2 T_F N_f - 264C_F T_F^2 N_f^2] \frac{a^4}{54} + O(a^5)
\end{aligned} \tag{3.1}$$

in four dimensions. The observation of [5] was basically that at two loops for a range of values of N_f the β -function has a non-trivial zero which we formally denote by a_2 . This arises when the first term of (3.1) is negative and when the second term is positive. For a sufficiently large number of massless quarks asymptotic freedom is lost and the theory becomes like QED. When a real positive non-trivial solution exists then this is termed the Banks-Zaks fixed point. As it occurs for that part of the β -function which is scheme independent then it should be a universal property of the theory. However, with the inclusion of higher order terms in $\beta(a)$ not only will the location of the fixed point be refined but its specific value will be scheme dependent. So, for example, denoting the three and four loop Banks-Zaks fixed points by a_3 and a_4 respectively, then these would depend on the renormalization scheme which $\beta(a)$ was expressed in. In the case of a_4 and higher fixed points there could be more than one non-trivial root of $\beta(a) = 0$. The Banks-Zaks one is always regarded as the one closest to the origin. Some remarks are apt on the scheme dependence of the range of N_f for which the conformal window exists. From (3.1) the upper limit of the range is determined by the one loop coefficient while the two loop term gives the lower limit. For $SU(3)$ the lower limit is $N_f = 9$. If the Banks-Zaks fixed point is related to the breaking of chiral symmetry then it would appear to be ruled out in this scenario. However, the conformal window discussed so far is deduced from a perturbative analysis and, moreover, the value of the critical coupling for low values of N_f in the window are outside the region of perturbative credibility. Indeed it could be the case that when lower values of N_f are analysed non-perturbatively then the lower boundary of the window could be reduced. A second aspect of the lower end is that it derives from the two loop coefficient of $\beta(g)$. In mass independent renormalization schemes where there is a single coupling constant this term is scheme independent, [68]. However, in MOM schemes with a non-zero α this term is both α and scheme dependent. In the Landau gauge the two loop terms of each of the MOM β -functions reduce to the same value as (3.1). This may not be the case in other gauges such as a non-linear gauge. For instance, in [69] the two loop renormalization group functions have been deduced in the corresponding MOM schemes for the maximal abelian gauge, [70, 71, 72]. From those results using the two loop term in the corresponding β -functions the lower bound of the conformal window for two of the schemes drops to $N_f = 8$. Again this is a perturbative observation in a region where the location of the fixed point lies outside the range of validity of perturbation theory. So it does not imply that the lower limit of the conformal window is lower ahead of a full non-perturbative analysis. Though the lattice study of [8] has provided evidence that the low end of the window can accommodate this value. Not only has the location of the fixed point been studied in various schemes in [17, 31, 32] but the values of the quark mass anomalous dimension at the Banks-Zaks fixed point have been estimated in the same schemes. As these critical exponents are renormalization group invariant it should be the case that with sufficiently high accuracy the scheme dependence evident in lower loop estimates should wash out. That has been observed in [17] for certain values of N_f in the window where the Banks-Zaks fixed point exists. This is invariably for large values of N_f close to the upper boundary. For lower values of N_f the value of a_n ceases to be small and so estimates of critical exponents would be outside the perturbative region. For N_f in the intermediate part of the range it may be the case that the higher order corrections restore a_n to perturbative reliability. In addition certain schemes may remain within the perturbative region better than others. This is one question we aim to analyse.

In respect of these points it is worth noting the structure of the MOM scheme β -functions we will use to deduce the quark mass critical exponents at a_n . As these expressions have been given elsewhere, [44, 45], and are equally as cumbersome as (2.16), we record the expression for

the MOMh scheme in the Landau gauge as it is the more compact of the three. It is, [44, 45],

$$\begin{aligned}
\beta^{\text{MOMh}}(a, 0) = & - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 - \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3 \\
& + \left[[18817920 + 103680\pi^2 - 16422912\zeta(3) - 155520\psi'(\frac{1}{3})] N_f T_F C_A C_F \right. \\
& + [29167776 + 3729024\pi^2 + 29568\pi^4 + 11562912\zeta(3) - 5593536\psi'(\frac{1}{3}) \\
& + 7200\pi^2\psi'(\frac{1}{3}) - 5400(\psi'(\frac{1}{3}))^2 - 11988\psi'''(\frac{1}{3}) - 31726080s_2(\frac{\pi}{6}) \\
& + 63452160s_2(\frac{\pi}{2}) + 52876800s_3(\frac{\pi}{6}) - 42301440s_3(\frac{\pi}{2}) + 78880\pi^3\sqrt{3} \\
& + 881280\ln(3)\pi\sqrt{3} - 73440\ln^2(3)\pi\sqrt{3}] N_f T_F C_A^2 \\
& + [-4105728 - 705024\pi^2 - 3981312\zeta(3) + 1057536\psi'(\frac{1}{3}) + 5971968s_2(\frac{\pi}{6}) \\
& - 11943936s_2(\frac{\pi}{2}) - 9953280s_3(\frac{\pi}{6}) + 7962624s_3(\frac{\pi}{2}) - 14848\pi^3\sqrt{3} \\
& - 165888\ln(3)\pi\sqrt{3} + 13824\ln^2(3)\pi\sqrt{3}] N_f^2 T_F^2 C_A \\
& + [-5723136 + 5971968\zeta(3)] N_f^2 T_F^2 C_F - 559872 N_f T_F C_F^2 \\
& + [-35200008 - 4741632\pi^2 - 81312\pi^4 - 1689336\zeta(3) + 7112448\psi'(\frac{1}{3}) \\
& - 19800\pi^2\psi'(\frac{1}{3}) + 14850(\psi'(\frac{1}{3}))^2 + 32967\psi'''(\frac{1}{3}) + 42083712s_2(\frac{\pi}{6}) \\
& - 84167424s_2(\frac{\pi}{2}) - 70139520s_3(\frac{\pi}{6}) + 56111616s_3(\frac{\pi}{2}) - 104632\pi^3\sqrt{3} \\
& \left. - 1168992\ln(3)\pi\sqrt{3} + 97416\ln^2(3)\pi\sqrt{3}] C_A^3 \right] \frac{a^4}{279936} + O(a^5) \quad (3.2)
\end{aligned}$$

at three loops. Like (2.16) at three loops the presence of the underlying symmetric point masters are evident. We note that we are effectively quoting the full expression given in [45] but with a modification. In the three loop term of equation (5.28) in [45] an additional numerical object, Σ , was present which was a combination of harmonic polylogarithms. When [45] appeared it was not apparent that this was not an independent quantity and has since been shown to correspond to, [73],

$$\Sigma = \frac{1}{36}\psi''' \left(\frac{1}{3} \right) - \frac{2\pi^4}{27} \quad (3.3)$$

in the notation of the previous renormalization group equations. We have substituted (3.3) in the original expression of [45] for consistency here. In comparing (3.1) and (3.2) one can see that there is a structural question to be addressed. If when one computes the critical exponent for, say, the quark mass anomalous dimension in $\overline{\text{MS}}$ and MOMh at the Banks-Zaks fixed point then both expressions ought to be the same. This is because ultimately the critical exponent is a physical quantity and hence a renormalization group invariant. It is independent of the renormalization scheme in which it is determined. However, given the form of both β -functions this cannot be the case. Indeed this is one of the motivations for examining the critical exponents at the Banks-Zaks fixed point in MOM schemes. These are clearly in a different class from the point of view of the numerology when compared with the schemes analysed in [17] which were $\overline{\text{MS}}$, RI' and mMOM. The coefficients appearing in the renormalization group functions of these three schemes are from the set

$$\{\mathbb{Q}, \pi^2, \zeta(3), \zeta(5)\} \quad (3.4)$$

to four loops. By contrast the basis for the MOM scheme coefficients to three loops is

$$\left\{ \mathbb{Q}, \pi^2, \zeta(3), \zeta(4), \psi'(\frac{1}{3}), \psi'''(\frac{1}{3}), s_2(\frac{\pi}{2}), s_2(\frac{\pi}{6}), s_3(\frac{\pi}{2}), s_3(\frac{\pi}{6}), \frac{\ln^2(3)\pi}{\sqrt{3}}, \frac{\ln(3)\pi}{\sqrt{3}}, \frac{\pi^3}{\sqrt{3}} \right\}. \quad (3.5)$$

The aim would be to see if the numerical values for the exponents in various schemes show the consistency which would indicate renormalization group invariance.

This problem can also be illustrated in the context of another fixed point which is present in the QCD β -function but is usually discussed in the context of scalar field theories. It is the Wilson-Fisher fixed point, [74, 75, 76, 77], which occurs in the d -dimensional β -function. For QCD the latter, which will be denoted by $\beta_d^{\overline{\text{MS}}}(a)$ in $\overline{\text{MS}}$, is related to (3.1) by

$$\beta_d^{\overline{\text{MS}}}(a) = \frac{1}{2}(d-4) + \beta^{\overline{\text{MS}}}(a). \quad (3.6)$$

Irrespective of whether there is a Banks-Zaks fixed point or not in QCD, there will be Wilson-Fisher fixed point in $d < 4$ dimensions when the one loop term of $\beta(a)$ is *positive*. In scalar theories this fixed point has proved useful in obtaining estimates for critical exponents in *three* dimensions through, for example, resummation techniques. However, such critical exponents are also renormalization group invariant and therefore the explicit expressions should be equivalent. It has been possible to check this for certain scalar theories, [78]. The same analysis can be studied here with (3.2) but the numeric structure of the renormalization group functions would appear to suggest otherwise given the number bases indicated above. This is not the case due to a subtle feature which is absent in (3.2). We have correctly introduced the concept of the Wilson-Fisher fixed point for the $\overline{\text{MS}}$ scheme. For the MOMh scheme the situation is completely parallel except that one cannot merely replace the scheme label in (3.6) by the scheme label for MOMh. This is because the MOM schemes are defined by the inclusion of finite parts in the renormalization constants. In the derivation of (3.2) the final step is to set $\epsilon = 0$ whence (3.2) emerges. However, the d -dimension MOMh β -function has ϵ -dependent coefficients unlike the $\overline{\text{MS}}$ scheme. In the latter scheme there are no finite contributions to the renormalization constants. So to check that the MOMh Wilson-Fisher fixed point actually delivers expressions for critical exponents which are equivalent for different schemes one has to perform the analysis fully in d -dimensions. As the MOM renormalization group functions have not been recorded in d -dimensions for each of the three MOM schemes of Celmaster and Gonsalves we do so here for an interested reader. Though to save space we record the expressions numerically and provide the full analytic d -dimensional expressions in the accompanying data file. In that file the three loop terms in the MOM renormalization group functions are not provided as the ϵ dependent terms at that loop order are derived from the *finite* parts of the three loop renormalization constants. These are not known at present. We do include the three loop coefficients in the data file for the quark mass operator in the MOM schemes as they appear here for the first time. So, for instance, in the Landau gauge for the colour group $SU(3)$ we have

$$\begin{aligned} \beta^{\text{MOMq}}(a, 0) &= -\epsilon a + [0.666667N_f - 11.000000 + 1.111111N_f\epsilon - 16.715775\epsilon] a^2 \\ &\quad + [12.666667N_f - 102.000000 + 91.930102\epsilon N_f - 385.483952\epsilon] a^3 + O(a^3) \\ \gamma_A^{\text{MOMq}}(a, 0) &= [0.666667N_f - 6.500000 + 1.111111N_f\epsilon - 8.083333\epsilon] a \\ &\quad + [9.411706N_f - 46.639132 + 62.308328N_f\epsilon - 311.747527\epsilon] a^2 + O(a^3) \\ \gamma_\psi^{\text{MOMq}}(a, 0) &= [-1.333333N_f + 22.333333 - 4.666667N_f\epsilon + 50.928412\epsilon] a^2 + O(a^3) \\ \gamma_c^{\text{MOMq}}(a, 0) &= [-2.250000 - 3.000000\epsilon] a \\ &\quad + [0.750000N_f - 13.202007 + 8.541667N_f\epsilon - 102.724216\epsilon] a^2 + O(a^3) \\ \gamma_{\bar{\psi}\psi}^{\text{MOMq}}(a, 0) &= [-4.000000 - 0.645519\epsilon] a \\ &\quad + [-1.791876N_f - 7.570942 + 7.309836N_f\epsilon - 34.841722\epsilon] a^2 + O(a^3) \end{aligned} \quad (3.7)$$

for the MOMq scheme and

$$\begin{aligned} \beta^{\text{MOMggg}}(a, 0) &= -\epsilon a + [0.666667N_f - 11.000000 + 3.416806N_f\epsilon - 26.492489\epsilon] a^2 \\ &\quad + [12.666667N_f - 102.000000 + 7.974346N_f^2\epsilon + 42.091196N_f\epsilon \end{aligned}$$

$$\begin{aligned}
& - 517.221499\epsilon] a^3 + O(a^4) \\
\gamma_A^{\text{MOMggg}}(a, 0) &= [0.666667N_f - 6.500000 + 1.111111N_f\epsilon - 8.083333\epsilon] a \\
& + [1.537130N_f^2 - 12.093123N_f + 16.909511 + 2.561884N_f^2\epsilon \\
& + 32.807608N_f\epsilon - 232.719087\epsilon] a^2 + O(a^3) \\
\gamma_\psi^{\text{MOMggg}}(a, 0) &= [- 1.333333N_f + 22.333333 - 4.666667N_f\epsilon + 50.928412\epsilon] a^2 + O(a^3) \\
\gamma_c^{\text{MOMggg}}(a, 0) &= [- 2.250000 - 3.000000\epsilon] a \\
& + [- 4.437814N_f + 8.795600 + 1.624581N_f\epsilon - 73.394073\epsilon] a^2 + O(a^3) \\
\gamma_{\bar{\psi}\psi}^{\text{MOMggg}}(a, 0) &= [- 4.000000 - 0.645519\epsilon] a \\
& + [- 11.014658N_f + 31.535915 + 5.821466N_f\epsilon - 28.530668\epsilon] a^2 \\
& + O(a^3) \\
\beta^{\text{MOMh}}(a, 0) &= - \epsilon a + [0.666667N_f - 11.000000 + 1.111111N_f\epsilon - 18.548275\epsilon] a^2 \\
& + [12.666667N_f - 102.000000 + 88.675121N_f\epsilon - 595.803097\epsilon] a^3 + O(a^4) \\
\gamma_A^{\text{MOMh}}(a, 0) &= [0.666667N_f - 6.500000 + 1.111111N_f\epsilon - 8.083333\epsilon] a \\
& + [8.190039N_f - 34.727877 + 60.272216N_f\epsilon - 296.934812\epsilon] a^2 + O(a^3) \\
\gamma_\psi^{\text{MOMh}}(a, 0) &= [- 1.333333N_f + 22.333333 - 4.666667N_f\epsilon + 50.928412\epsilon] a^2 + O(a^3) \\
\gamma_c^{\text{MOMh}}(a, 0) &= [- 2.250000 - 3.000000\epsilon] a \\
& + [0.750000N_f - 9.078880 + 8.541667N_f\epsilon - 97.226714\epsilon] a^2 + O(a^3) \\
\gamma_{\bar{\psi}\psi}^{\text{MOMh}}(a, 0) &= [- 4.000000 - 0.645519\epsilon] a \\
& + [-1.791876N_f - 0.240939 + 7.309836N_f\epsilon - 33.658808\epsilon] a^2 + O(a^3) \quad (3.8)
\end{aligned}$$

for the other two MOM schemes. Of course this procedure can be reverse engineered if one knew the β -function in one scheme to L loops and to $(L - 1)$ in another scheme. In this instance some information on the latter β -function can be adduced about the L -loop term from the renormalization group invariance of the underlying critical exponent. Though this is essentially reflective of the use of the conversion functions to establish the anomalous dimensions at the next order in a scheme in the context we used earlier. Finally, we note that we have checked that the *same* critical exponents emerge at the Wilson-Fisher fixed point in the MOM schemes as in $\overline{\text{MS}}$ to $O(\epsilon^3)$ as expected. The situation for the Banks-Zaks fixed point is not the same primarily because it is a purely *four* dimensional fixed point. Clearly the MOM critical exponents at the Banks-Zaks fixed point will involve the numbers in the basis (3.5) in contrast to the basis (3.4) for the schemes studied in [17]. However, it is also partly due to the fact that when we compute the estimates of the quark mass critical exponent, for instance, we are endeavouring to use the perturbation theory of QCD which is valid in a region near the origin. Provided one is within the perturbative region of that theory then information about the exponents of the theory underpinning the Banks-Zaks fixed point can be obtained and should be comparable across schemes. However, to truly understand the renormalization group invariance via a scheme analysis of the Banks-Zaks fixed point one would first have to construct the quantum field theory which is in the same universality class and then renormalize it within the various schemes. That theory is not yet available as far as we are aware.

4 Results.

Having discussed the nature of the two main critical points in the QCD renormalization group functions we turn now to the problem we will analyse with them which is the evaluation of the

quark mass anomalous dimension at the Banks-Zaks fixed point. This will be carried out for a variety of colour groups with the quarks in various representations. The usual case where the quarks are in the fundamental representation will form the main part of the analysis. However, for theories beyond the Standard Model, the analysis of [17] also included quarks in the adjoint representation as well as in the two-index symmetric and antisymmetric representations for the RI' and minimal MOM schemes. We will therefore provide results for these representations too in order to have as large a picture as possible on where the convergence is best. For the explicit values of the various colour group Casimirs for these representations we refer the reader to Appendix B of [17]. It is worth noting that the window for a Banks-Zaks fixed point depends on the particular representation and for some of these there is a much smaller range of N_f values for an infrared fixed point than that for quarks in the fundamental representation. Although the main interest is the quark mass anomalous dimension due to its relation to the conformal window, for a convergence analysis an equally useful critical exponent to analyse is that relating to the critical slope of the β -function which is usually denoted by ω . Its main role is as a measure of corrections to scaling. Therefore, we will be providing evaluations of this exponent for the same quark representations as the quark mass anomalous dimension. In order to present our results we need to introduce our notation.

First, we formally define the Landau gauge β -function in the scheme \mathcal{S} by

$$\beta^{\mathcal{S}}(a, 0) = \sum_{r=1}^{\infty} \beta_r^{\mathcal{S}} a^{r+1} \quad (4.1)$$

and the β -function partial sums by

$$\beta_n^{\mathcal{S}}(a, 0) = \sum_{r=1}^n \beta_r^{\mathcal{S}} a^{r+1} . \quad (4.2)$$

Then for each scheme the Banks-Zaks fixed point a_L at the L th loop order is defined as the first non-trivial zero of

$$\beta_L^{\mathcal{S}}(a_L, 0) = 0 . \quad (4.3)$$

From this we define the critical exponent ω at the L th loop as

$$\omega_L = 2\beta'_L(a_L, 0) . \quad (4.4)$$

For the Landau gauge quark mass anomalous dimension $\gamma_{\bar{\psi}\psi}^{\mathcal{S}}(a, 0)$ we define the critical exponent by a similar process. We let the perturbative expression be

$$\gamma_{\bar{\psi}\psi}^{\mathcal{S}}(a, 0) = \sum_{r=1}^{\infty} \gamma_r^{\mathcal{S}} a^r \quad (4.5)$$

and then the corresponding partial sums are

$$\gamma_{\bar{\psi}\psi n}^{\mathcal{S}}(a, 0) = \sum_{r=1}^n \gamma_r^{\mathcal{S}} a^r . \quad (4.6)$$

Denoting the quark mass anomalous dimension exponent by ρ then its evaluation at the L th loop fixed point is ρ_L where

$$\rho_L = - 2\gamma_{\bar{\psi}\psi L}^{\mathcal{S}}(a_L, 0) \quad (4.7)$$

for each scheme. The definition of ρ coincides with that of [17] so that there is a direct comparison. However, given that we are defining the β -function consistent with the conventions used in [45],

the values of the location of the fixed points differ by a factor of 4π from those of [17]. Further, in presenting our results we use a similar form of tables but perform the evaluation to six decimal places. This is partly to compare the convergence for certain cases. The format of the results tables parallels [17] in that we present the $\overline{\text{MS}}$ and mMOM results in a combined table since there are results to four loops for these two schemes. Subsequently the results for the same quantity in the three MOM schemes are given. The order within each choice of quark representation is fixed point location, ω and ρ . Though in one instance we include results for the 't Hooft scheme of [79]. Briefly the renormalization group functions of this scheme are defined as that part which is renormalization scheme independent. For the β -function this is the two loop part and for the quark mass anomalous dimension it is the one loop term, [70]. Our final general comment on the tables of results concerns the situation with the mMOM scheme. It transpires that in [41] there was an error in the derivation of the four loop quark mass anomalous dimension. Specifically the four loop term of the $\overline{\text{MS}}$ anomalous dimension was inadvertently subtracted in the corresponding derivation using (2.13). Therefore, the results of [17] have been corrected in an erratum using the erratum for [41]. For completeness we also include the results for the correct version of the mMOM four loop quark mass anomalous dimension to the accuracy we are working to.

We turn now to a discussion of the results in the individual Tables. For the fundamental representation the fixed point locations are given in Tables 1 and 2 and we make no comment on them as comparison between schemes of the location is not fully meaningful. One role they play is to give an indication as to where the fixed point is becoming reasonably stable for certain values of N_f . Then one would hope that the corresponding critical exponents could be converging. For instance, from Table 1 it would appear that for $N_f \geq 13$ the fixed point has reached a plateau for each scheme from the stability at three and four loops. It was noted in [17] that the convergence is best at the upper end of the window for the infrared fixed point. This is because one is still in the region where the coupling constant has a small value. For smaller values of N_f the perturbative results do not appear to be reliable. Throughout our analysis we are broadly in agreement with this point of view. As $N_f = 12$ is the value which is of intense interest in the lattice community the perturbative results may not be competitive with that analysis. Given this we are not in a position to indicate whether the same range of N_f values for perturbative reliability are valid in the MOM case as we only have three loop results as is evident in Table 2. In both instances, another way of examining convergence and the relevant N_f window is to examine the renormalization group invariant critical exponents. For ω these are given in Tables 3 and 4 for the five schemes we are interested in. In this and further remarks our discussion will always concentrate on the $N_c = 3$ case, unless otherwise indicated, due to the relation to QCD but for $N_c = 2$ and 3 parallel remarks will apply but for different N_f values. From Tables 3 and 4 the three loop values of ω are all in accord for $N_f = 16$ as expected although the MOMggg scheme is slightly lower. Indeed this is the case for lower values of N_f . For instance, when $N_f = 13$ the four loop $\overline{\text{MS}}$ and mMOM values of ω are similar to the three loop ones of MOMq and MOMh. However, when $N_f = 12$ this relative convergence is absent as expected.

In general there is a parallel picture for the ρ in Tables 5 and 6. Before concentrating on the five schemes we are interested in, in the former Table we have included an additional column for the 't Hooft scheme. This was not needed for ω since the two loop $\overline{\text{MS}}$ column in Table 3 corresponds to that scheme. However, for the quark mass anomalous dimension case it appears evident that for a large range of N_f in the fixed point window the estimates lie well away from those of our five schemes. This is not surprising given the way the series is defined. Focusing now on our five schemes the three loop $N_f = 16$ values are comparable although again the MOMggg value appears to be the outlier here being on the higher side which is also reflected at lower values of N_f . At $N_f = 13$ the four loop $\overline{\text{MS}}$ and MOMq values are similar. The mMOM values are higher but appear to be slowly decreasing. We note that with the previous wrong result the mMOM

four loop estimates were all higher than the three loop value. This is not the case now and is in fact reversed when the correct four loop expression is used. At $N_f = 12$ the situation is similar to ω . However, in this instance there are various lattice estimates for what we have termed ρ . For instance, one particular analysis gives the value of 0.235(15) from [14] and another more recent study gives 0.235(46) from [15]. In either case these values are lower than any of the three or four loop perturbative estimates and so we reinforce the observation of [17] that non-perturbative properties may be beginning to dominate the window at this point. One interesting feature of this N_f value is that if the lattice estimate is roughly correct the four loop $\overline{\text{MS}}$ value of ρ is the closest. However, in terms of convergence the *three* loop MOMq and MOMh values are smaller than the corresponding three loop $\overline{\text{MS}}$ one. Hence one hope would be that a four loop analysis in these two schemes may produce a better estimate in comparison to [14, 15] than the four loop $\overline{\text{MS}}$ one. In some sense since we are evaluating the quark mass anomalous dimension exponent it might be expected that the MOMq scheme would produce the more reliable value. That the MOMh value is competitive may seem surprising but given the similar structure of the Feynman graphs within the vertex functions defining each of the MOMq and MOMh schemes this would appear to be the main explanation. In each case one renormalizes the same number of graphs in the respective vertex renormalizations, [45], and the graphs are effectively the same structure topologically when examined in detail.

Our final remark on the estimates of ρ for $N_f = 12$ specifically concerns the use of the *five* loop quark mass anomalous dimension which was recently determined in [50] in $\overline{\text{MS}}$ but specifically for $N_c = 3$. Although the five loop β -function is not available we have carried out a tentative analysis using the expression given in [50]. The corresponding results are presented in Table 7 where we have used an additional notation, ρ_{5l} . This indicates the use of the five loop $\overline{\text{MS}}$ quark mass anomalous dimension of [50] but evaluated with the corresponding values of the l -loop fixed points given in the $\overline{\text{MS}}$ columns of Table 1. The reason for using the three and four loop fixed point values is that if there is perturbative convergence it would be hoped that these would bound the actual five loop value which is as yet unknown. As $a_4 > a_3$ we have assumed without any justification that there is such an alternating convergence. So if these are the bounding values the same reasoning would be that ρ_{53} and ρ_{54} would bound the actual five loop value. This would appear to be the case for $N_f = 16$ as well as down to $N_f = 13$ when comparing between schemes. If this reasoning applied to $N_f = 12$ then the value of Table 7 would appear to be significantly different from the lattice estimates. By contrast another way of expressing this is to determine the value of a_5 which would be required to give the central value of [14, 15] of $\rho_5 = 0.235$. From the five loop expression of [50] we would have to have $a_5 = 0.028376$ in our conventions which is significantly lower than the three and four loop values we used to obtain the $N_f = 12$ estimates in Table 7. In other words it is well inside the region where perturbation theory is valid and suggests that non-perturbative properties are the drive behind the two consistent lattice estimates. Such a large drop in the critical coupling value from successive loop orders is not seen in $\overline{\text{MS}}$ for $N_c = 3$ even for smaller values of N_f . Finally, for fundamental quarks we note that the $SU(2)$ colour group has been studied on the lattice for values of N_f in the range $6 \leq N_f \leq 10$, [10]. All our estimates for ρ are in good agreement with the $N_f = 10$ value of 0.08 given in [10]. This is in keeping with the $SU(3)$ case as this is at the upper end of the conformal window. The lower end of the $SU(2)$ window is a current topic of study which has not reached consensus yet, [12, 13]. For example, in [12] the $N_f = 6$ value of ρ is in the range [0.26, 0.74]. Of the schemes we have analysed only the three loop MOMq and MOMh estimates lie comfortably within this band. This apparent agreement should be taken with caution due to the limit of perturbative credibility and lack of convergence as well as the effect four loop corrections could have if the situation in $\overline{\text{MS}}$ is a guide.

Although the main interest in the Banks-Zaks fixed point stems from its possible connection

with a phase transition associated with chiral symmetry breaking in QCD when the quarks are in the fundamental representation, for the purposes of analysing possible theories beyond the Standard Model like [17] we will consider the quarks in other representations. In this instance we will make brief remarks as there is mostly a parallel situation in these cases. When the quarks are in the adjoint representation the corresponding fixed point locations and critical exponents are provided in Tables 8 to 13. We stress that this not a supersymmetric version of QCD as there are not equal numbers of Bose and Fermi degrees of freedom. For the three colour values we considered for the fundamental representation there is only one non-trivial infrared fixed point and then it is only present for $N_f = 2$. In the MOM schemes for both ω and ρ the same feature emerges in that the three loop estimates are N_c independent for the $SU(N_c)$ colour group. The N_c dependence becomes apparent at four loops from Tables 10 and 12. For ω the three loop value of ω is around the same estimates of the four loop $\overline{\text{MS}}$ and mMOM values. By contrast for ρ the three loop MOMq estimates are competitive with the two four loop results except possibly for $N_c = 2$. This may be due to the origin of the operator being a scalar quark bilinear.

The next representation of interest is the $2S$ representation which corresponds to a double index symmetric representation. The results for this case are given in Tables 14 to 19 where there are only two fixed points for $N_c = 3$ and 4 and again for low values of N_f . The two critical exponents have similar properties to the fundamental representation case. For the larger of the two values of N_f there appears to be a convergent result when comparing the four loop results of [17] and the three loop MOM scheme results except possibly for the MOMggg scheme. For two flavours there is no clear pattern for either of the exponents or values of N_c . Indeed for ρ all bar one estimate is larger than unity. Finally, for the $2A$ representation, which is the antisymmetric double index partner to $2S$, the results are included in the Tables 20 to 25. While there are more fixed points for $N_c = 4$ we do not present results for $N_c = 3$. This is because in this representation the colour group Casimirs are precisely equal to their corresponding values in the fundamental representation and we have commented on those results already. Though we do note that in the context of model building or considering extensions to current theories quarks could be considered as being in the $2A$ representation rather than the fundamental one. In terms of the critical exponents for $2A$ the situation for ρ appears to parallel our discussion for the differing behaviours of $N_f = 12$ and 13. However, here the boundary appears to be at $N_f = 7$ and 8. We would have to exclude the MOMggg results from this analysis as it again seems to be an outlier for ρ . In terms of lattice analysis there has been an investigation for $N_f = 6$, [11], which is the lower boundary of the conformal window from the perturbative analysis. An estimate for ρ lies in the range $[0.3, 0.35]$ for which only the four loop $\overline{\text{MS}}$ estimate is close to.

We close this section by making some general comments on the analysis and try to give a perspective on the reliability of the perturbative estimates. In focusing the discussion so far on the comparison within a representation it may miss some key features. For instance, for ρ as a general rule it appears that when the value of ρ_2 is in the region of 1 or larger then the higher loop estimates appear to be unreliable. By this we mean that the value appears to be at odds with estimates in other schemes. However, we need to be clear in saying this in that we are not suggesting that for that scheme the exponent does not converge. For values of N_f close to the upper boundary of the window in all the schemes the corresponding scheme estimates for ρ clearly are in line with other schemes. What is probably the case is that more terms in the loop expansion for that particular scheme are needed in order to see the convergence. In the main the MOMggg scheme appeared mostly to be in this outlier class. This is not unreasonable due to the nature of the MOMggg scheme. It is based on ensuring that the triple gluon vertex has no $O(a)$ corrections at the completely symmetric point. Therefore, with the associated renormalization group functions their content is necessarily weighted by gluonic rather than quark contributions. For the quark mass anomalous dimension, therefore, the quark content is not dominant.

5 Discussion.

It is worth making several general comments on our analysis. In [17] the evaluation of the quark mass anomalous dimensions at the Banks-Zaks fixed point was examined in the conformal window for a set of renormalization schemes. We have extended that analysis here to a different set of schemes which are the momentum subtraction schemes of Celmaster and Gonsalves, [43, 44]. This is an important exercise since the analytic structure of the respective set of schemes is different from the point of view of the specific numbers which appear. Ultimately critical exponents which have been determined from the renormalization group functions at criticality are renormalization group invariants and the values have to be independent of the renormalization scheme used to determine the anomalous dimensions. In this respect we have demonstrated this for the MOM QCD renormalization group functions at the Wilson-Fisher fixed point in $d = 4 - 2\epsilon$ dimensions. This is not a trivial exercise as the d -dimensional renormalization group functions are required in the MOM case in order to observe the renormalization group invariance in d -dimensions. For the Banks-Zaks fixed point the situation is different with regard to the invariance. Until the quantum field theory which drives the Banks-Zaks fixed point is found then at present a numerical evaluation of the critical exponents order by order in the loop expansion is the only tool available. In other words there will be a theory in the same universality class as QCD at the Banks-Zaks fixed point where direct computation of its anomalous dimensions in various schemes ought to be the way to see the renormalization invariance of the critical exponents. Having said this on the whole, despite the differing numeric natures of the renormalization group functions in MOM schemes versus those of the $\overline{\text{MS}}$, RI' and mMOM schemes analysed in [17], the scheme dependence appears to disappear for values of N_f near the upper end of the conformal window for the various quark representations we have considered. This is where perturbation theory is at its most reliable. One interesting point is when there are $N_f = 12$ fundamental flavours for $SU(3)$. On the whole the quark mass anomalous dimension appears to be converging slowly towards recent values measured on the lattice, [14, 15]. For the MOMq scheme the *three* loop estimate of ρ is closer than the corresponding $\overline{\text{MS}}$ value. Whether there is faster convergence for this particular scheme remains to be seen in the absence of a full four loop computation. Given the nature of this scheme, which is founded on the quark-gluon vertex, it may be the case that the quark mass anomalous dimension in this scheme does indeed have the best convergence. However, these remarks need to be tempered by the observations in [17] where it was noted that $N_f = 12$ may be the point where non-perturbative features become dominant. A measure of that can be seen in the evaluation of the stability critical exponent ω . In Tables 3 and 4 for $N_f = 13$ the value of ω appears to be consistent across all the schemes considered except for MOMggg. The values for ρ for the same N_f accord with this. For $N_f = 12$ the estimates of ω have a broader range across the schemes.

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	F	\overline{MS}			mMOM		
N_c	N_f	a_2	a_3	a_4	a_2	a_3	a_4
2	6	0.909091	0.130937	0.190588	0.909091	0.100122	0.088677
2	7	0.225352	0.083898	0.096318	0.225352	0.067933	0.062904
2	8	0.100000	0.054773	0.060487	0.100000	0.046821	0.045404
2	9	0.047337	0.033280	0.035339	0.047337	0.030031	0.029984
2	10	0.018349	0.015622	0.015944	0.018349	0.014878	0.014954
3	9	0.416667	0.081803	0.085291	0.416667	0.064438	0.054935
3	10	0.175676	0.060824	0.064860	0.175676	0.049421	0.044230
3	11	0.098214	0.046039	0.049832	0.098214	0.038603	0.036070
3	12	0.060000	0.034607	0.037434	0.060000	0.029962	0.028981
3	13	0.037234	0.025191	0.026853	0.037234	0.022535	0.022329
3	14	0.022124	0.017070	0.017793	0.022124	0.015786	0.015838
3	15	0.011364	0.009818	0.010001	0.011364	0.009383	0.009431
3	16	0.003311	0.003162	0.003170	0.003311	0.003118	0.003121
4	12	0.281690	0.060040	0.060411	0.281690	0.047748	0.040336
4	13	0.147239	0.048027	0.049944	0.147239	0.039016	0.034347
4	14	0.092219	0.038926	0.041445	0.092219	0.032328	0.029529
4	15	0.062291	0.031616	0.034072	0.062291	0.026858	0.025323
4	16	0.043478	0.025488	0.027490	0.043478	0.022159	0.021442
4	17	0.030558	0.020179	0.021580	0.030558	0.017964	0.017724
4	18	0.021136	0.015460	0.016291	0.021136	0.014097	0.014086
4	19	0.013962	0.011175	0.011573	0.013962	0.010440	0.010493
4	20	0.008316	0.007218	0.007350	0.008316	0.006907	0.006943
4	21	0.003758	0.003511	0.003530	0.003758	0.003438	0.003446

Table 1. Location of Banks-Zaks critical points for \overline{MS} and mMOM at two, three and four loops.

	F	MOMq		MOMggg		MOMh	
N_c	N_f	a_2	a_3	a_2	a_3	a_2	a_3
2	6	0.909091	0.079453	0.909091	0.075345	0.909091	0.100010
2	7	0.225352	0.060047	0.225352	0.051522	0.225352	0.069384
2	8	0.100000	0.044163	0.100000	0.035988	0.100000	0.048379
2	9	0.047337	0.029574	0.047337	0.023848	0.047337	0.031152
2	10	0.018349	0.014999	0.018349	0.012674	0.018349	0.015317
3	9	0.416667	0.051906	0.416667	0.047997	0.416667	0.064858
3	10	0.175676	0.042853	0.175676	0.037161	0.175676	0.050466
3	11	0.098214	0.035202	0.098214	0.029277	0.098214	0.039778
3	12	0.060000	0.028357	0.060000	0.023018	0.060000	0.031047
3	13	0.037234	0.021938	0.037234	0.017681	0.037234	0.023405
3	14	0.022124	0.015687	0.022124	0.012809	0.022124	0.016367
3	15	0.011364	0.009437	0.011364	0.008032	0.011364	0.009655
3	16	0.003311	0.003136	0.003311	0.002914	0.003311	0.003156
4	12	0.281690	0.038650	0.281690	0.035451	0.281690	0.048181
4	13	0.147239	0.033425	0.147239	0.029214	0.147239	0.039802
4	14	0.092219	0.028879	0.092219	0.024372	0.092219	0.033229
4	15	0.062291	0.024786	0.062291	0.020409	0.062291	0.027756
4	16	0.043478	0.020992	0.043478	0.017023	0.043478	0.022982
4	17	0.030558	0.017383	0.030558	0.014013	0.030558	0.018663
4	18	0.021136	0.013876	0.021136	0.011238	0.021136	0.014641
4	19	0.013962	0.010408	0.013962	0.008578	0.013962	0.010810
4	20	0.008316	0.006940	0.008316	0.005922	0.008316	0.007105
4	21	0.003758	0.003461	0.003758	0.003134	0.003758	0.003498

Table 2. Location of Banks-Zaks critical points for MOMq, MOMggg and MOMh at two and three loops.

	F	\overline{MS}			mMOM		
N_c	N_f	ω_2	ω_3	ω_4	ω_2	ω_3	ω_4
2	6	6.060606	1.620106	0.974775	6.060606	1.261453	1.245537
2	7	1.201878	0.728326	0.676986	1.201878	0.615403	0.618233
2	8	0.400000	0.318182	0.299703	0.400000	0.286878	0.289100
2	9	0.126233	0.115100	0.110454	0.126233	0.109360	0.109439
2	10	0.024465	0.023925	0.023541	0.024465	0.023590	0.023507
3	9	4.166667	1.475455	1.464386	4.166667	1.189101	1.165667
3	10	1.522523	0.871775	0.853407	1.522533	0.736141	0.736306
3	11	0.720238	0.516977	0.498035	0.720238	0.454913	0.459085
3	12	0.360000	0.295517	0.282328	0.360000	0.269774	0.272234
3	13	0.173759	0.155581	0.149130	0.173759	0.146681	0.147243
3	14	0.073746	0.069899	0.067812	0.073746	0.067695	0.067572
3	15	0.022727	0.022307	0.021975	0.022727	0.022037	0.021957
3	16	0.002208	0.002203	0.002198	0.002208	0.002200	0.002198
4	12	3.755869	1.430447	1.429308	3.755897	1.165365	1.140669
4	13	1.766871	0.964661	0.954675	1.766861	0.812318	0.809419
4	14	0.983670	0.655163	0.639277	0.983670	0.568776	0.572539
4	15	0.581387	0.440398	0.424261	0.581387	0.393264	0.397364
4	16	0.347826	0.288274	0.275809	0.347826	0.264197	0.266663
4	17	0.203718	0.180219	0.172523	0.203718	0.169115	0.170002
4	18	0.112726	0.104596	0.100807	0.112726	0.100224	0.100263
4	19	0.055846	0.053622	0.052223	0.055846	0.052293	0.052131
4	20	0.022176	0.021789	0.021468	0.022176	0.021539	0.021457
4	21	0.005010	0.004989	0.004965	0.005010	0.004974	0.004964

Table 3. Critical exponent ω for the Banks-Zaks critical point for \overline{MS} and mMOM at two, three and four loops.

	F	MOMq		MOMggg		MOMh	
N_c	N_f	ω_2	ω_3	ω_2	ω_3	ω_2	ω_3
2	6	6.060606	1.013077	6.060606	0.962970	6.060606	1.260113
2	7	1.201878	0.555171	1.201878	0.486742	1.201878	0.626165
2	8	0.400000	0.275290	0.400000	0.236097	0.400000	0.293412
2	9	0.126233	0.108457	0.126233	0.095150	0.126233	0.111475
2	10	0.024465	0.023649	0.024465	0.022125	0.024465	0.023797
3	9	4.166667	0.973459	4.166667	0.904648	4.166667	1.196201
3	10	1.522523	0.652189	1.522533	0.575996	1.522533	0.749100
3	11	0.720238	0.423769	0.720238	0.365393	0.720238	0.465266
3	12	0.360000	0.259872	0.360000	0.223235	0.360000	0.276171
3	13	0.173759	0.144437	0.173759	0.125839	0.173759	0.149791
3	14	0.073746	0.067504	0.073746	0.060674	0.073746	0.068753
3	15	0.022727	0.022074	0.022727	0.020774	0.022727	0.022213
3	16	0.002208	0.002201	0.002208	0.002176	0.002208	0.002203
4	12	3.755869	0.959967	3.755869	0.885870	3.755869	1.174951
4	13	1.766871	0.711138	1.766871	0.631571	1.766871	0.826128
4	14	0.983670	0.519614	0.983670	0.451225	0.983670	0.581177
4	15	0.581387	0.370624	0.581387	0.318564	0.581387	0.402677
4	16	0.347826	0.254787	0.347826	0.219046	0.347826	0.270526
4	17	0.203718	0.165850	0.203718	0.144003	0.203718	0.172852
4	18	0.112726	0.099425	0.112726	0.088003	0.112726	0.102081
4	19	0.055846	0.052229	0.055846	0.047543	0.055846	0.053001
4	20	0.022176	0.021569	0.022176	0.020338	0.022176	0.021706
4	21	0.005010	0.004979	0.005010	0.004872	0.005010	0.004986

Table 4. Critical exponent ω for Banks-Zaks critical point for MOMq, MOMggg and MOMh at two and three loops.

	F	$\overline{\text{MS}}$			mMOM			't Hooft
N_c	N_f	ρ_2	ρ_3	ρ_4	ρ_2	ρ_3	ρ_4	ρ
2	6	33.171488	0.924853	- 4.019013	39.576446	1.034933	0.893430	4.090909
2	7	2.674073	0.456824	0.032536	3.118429	0.523238	0.455155	1.014085
2	8	0.751875	0.272074	0.203618	0.849375	0.300337	0.279549	0.450000
2	9	0.275060	0.160546	0.157402	0.299149	0.168800	0.165956	0.213018
2	10	0.091049	0.073829	0.074794	0.095005	0.074836	0.075064	0.082569
3	9	19.768519	1.061659	- 0.143490	23.356481	1.191042	0.979184	3.333333
3	10	4.189838	0.646806	0.155885	4.882518	0.734781	0.620806	1.405405
3	11	1.613131	0.439241	0.249686	1.846779	0.492300	0.436592	0.785714
3	12	0.772800	0.311751	0.253328	0.866400	0.340313	0.317156	0.480000
3	13	0.404469	0.220154	0.209757	0.442979	0.233293	0.226367	0.297872
3	14	0.212450	0.146369	0.147421	0.226917	0.151029	0.150241	0.176991
3	15	0.099690	0.082573	0.083600	0.103736	0.083547	0.083816	0.090909
3	16	0.027187	0.025833	0.025895	0.027550	0.025868	0.025896	0.026490
4	12	17.296915	1.107600	0.058357	20.371702	1.243981	1.009616	3.169014
4	13	5.380895	0.755292	0.192015	6.275170	0.855872	0.712621	1.656442
4	14	2.445332	0.552297	0.258813	2.817397	0.622351	0.537602	1.037464
4	15	1.318886	0.420081	0.280672	1.498346	0.466289	0.419073	0.700779
4	16	0.778444	0.324942	0.268806	0.870599	0.353508	0.329838	0.489130
4	17	0.480849	0.250606	0.234022	0.528704	0.266804	0.256937	0.343774
4	18	0.300568	0.188596	0.186947	0.324580	0.196704	0.193870	0.237781
4	19	0.183246	0.134334	0.136002	0.194211	0.137668	0.137526	0.157068
4	20	0.102410	0.085397	0.086461	0.106473	0.086356	0.086657	0.093555
4	21	0.043993	0.040685	0.040877	0.044858	0.040801	0.040884	0.042273

Table 5. Quark mass critical exponent at the Banks-Zaks critical point for the $\overline{\text{MS}}$ and mMOM schemes at two, three and four loops and the 't Hooft scheme.

		MOMq		MOMggg		MOMh	
N_c	N_f	ρ_2	ρ_3	ρ_2	ρ_3	ρ_2	ρ_3
2	6	17.262397	0.461381	45.730994	0.861480	13.977991	0.305679
2	7	1.925820	0.346755	4.202074	0.542352	1.724000	0.304515
2	8	0.649692	0.247039	1.201675	0.336642	0.609951	0.234852
2	9	0.262282	0.156674	0.409221	0.189326	0.253377	0.153796
2	10	0.090649	0.073686	0.116219	0.079602	0.089311	0.073373
3	9	11.561746	0.553462	26.804208	0.954324	9.016606	0.375534
3	10	2.978729	0.452897	6.257561	0.701362	2.526293	0.377682
3	11	1.312033	0.364656	2.514774	0.516777	1.170622	0.330763
3	12	0.689329	0.285218	1.204608	0.374024	0.636553	0.270097
3	13	0.383454	0.212345	0.607462	0.259356	0.363130	0.206167
3	14	0.208960	0.144860	0.297076	0.165375	0.201785	0.142818
3	15	0.099806	0.082504	0.125435	0.088262	0.097913	0.082094
3	16	0.027285	0.025840	0.029663	0.026154	0.027124	0.025826
4	12	10.475472	0.586353	23.326276	0.984386	8.082819	0.401265
4	13	3.761930	0.503058	7.835301	0.780943	3.108222	0.406260
4	14	1.906259	0.428513	3.724747	0.622031	1.649824	0.375454
4	15	1.116733	0.360679	2.047092	0.493265	0.999731	0.331200
4	16	0.701301	0.298087	1.203586	0.386035	0.644300	0.281973
4	17	0.453285	0.239729	0.725617	0.294985	0.425128	0.231347
4	18	0.292424	0.184975	0.434301	0.216727	0.278953	0.181016
4	19	0.181893	0.133522	0.248856	0.149155	0.176016	0.131953
4	20	0.102711	0.085353	0.128262	0.091032	0.100626	0.084913
4	21	0.044214	0.040704	0.049797	0.041635	0.043788	0.040652

Table 6. Quark mass critical exponent at the Banks-Zaks critical point for MOMq, MOMggg and MOMh at two and three loops.

		\overline{MS}	
N_c	N_f	ρ_{53}	ρ_{54}
3	9	- 0.370415	- 0.596381
3	10	0.198718	0.105449
3	11	0.289590	0.266959
3	12	0.262582	0.268132
3	13	0.205572	0.215243
3	14	0.143001	0.148548
3	15	0.082153	0.083692
3	16	0.025828	0.025895

Table 7. Estimates of quark mass critical exponent at the Banks-Zaks critical point for the \overline{MS} at five loops using the three and four loop critical coupling.

G		$\overline{\text{MS}}$			mMOM		
N_c	N_f	a_2	a_3	a_4	a_2	a_3	a_4
2	2	0.050000	0.036525	0.035814	0.050000	0.033778	0.031703
3	2	0.033333	0.024350	0.024537	0.033333	0.022519	0.021491
4	2	0.025000	0.018263	0.018596	0.025000	0.016889	0.016217

Table 8. Location of Banks-Zaks critical points for $\overline{\text{MS}}$ and mMOM at two, three and four loops for the quarks in the adjoint representation.

G		MOMq		MOMggg		MOMh	
N_c	N_f	a_2	a_3	a_2	a_3	a_2	a_3
2	2	0.050000	0.032037	0.050000	0.026198	0.050000	0.035416
3	2	0.033333	0.021358	0.033333	0.017465	0.033333	0.023611
4	2	0.025000	0.016019	0.025000	0.013099	0.025000	0.017708

Table 9. Location of Banks-Zaks critical points for MOMq, MOMggg and MOMh at two and three loops for the quarks in the adjoint representation.

G		$\overline{\text{MS}}$			mMOM		
N_c	N_f	ω_2	ω_3	ω_4	ω_2	ω_3	ω_4
2	2	0.200000	0.185475	0.187427	0.200000	0.178949	0.183383
3	2	0.200000	0.185475	0.184637	0.200000	0.178949	0.182466
4	2	0.200000	0.185475	0.183419	0.200000	0.178949	0.182086

Table 10. Critical exponent ω for the Banks-Zaks critical point for $\overline{\text{MS}}$ and mMOM at two, three and four loops for the quarks in the adjoint representation.

G		MOMq		MOMggg		MOMh	
N_c	N_f	ω_2	ω_3	ω_2	ω_3	ω_2	ω_3
2	2	0.200000	0.174187	0.200000	0.154678	0.200000	0.182985
3	2	0.200000	0.174187	0.200000	0.154678	0.200000	0.182985
4	2	0.200000	0.174187	0.200000	0.154678	0.200000	0.182985

Table 11. Critical exponent ω for Banks-Zaks critical point for MOMq, MOMggg and MOMh at two and three loops for the quarks in the adjoint representation.

G		$\overline{\text{MS}}$			mMOM		
N_c	N_f	ρ_2	ρ_3	ρ_4	ρ_2	ρ_3	ρ_4
2	2	0.820000	0.543233	0.499621	0.885000	0.569034	0.520679
3	2	0.820000	0.543233	0.522652	0.885000	0.569034	0.537795
4	2	0.820000	0.543233	0.531736	0.885000	0.569034	0.544255

Table 12. Quark mass critical exponent at the Banks-Zaks critical point for the $\overline{\text{MS}}$ and mMOM schemes at two, three and four loops for the quarks in the adjoint representation.

G		MOMq		MOMggg		MOMh	
N_c	N_f	ρ_2	ρ_3	ρ_2	ρ_3	ρ_2	ρ_3
2	2	0.843280	0.523076	1.119867	0.563241	0.725384	0.493780
3	2	0.843279	0.523076	1.119867	0.563241	0.725384	0.493780
4	2	0.843280	0.523076	1.119867	0.563241	0.725384	0.493780

Table 13. Quark mass critical exponent at the Banks-Zaks critical point for MOMq, MOMggg and MOMh at two and three loops for the quarks in the adjoint representation.

$2S$		$\overline{\text{MS}}$			mMOM		
N_c	N_f	a_2	a_3	a_4	a_2	a_3	a_4
3	2	0.067010	0.039795	0.037400	0.067010	0.036641	0.031345
3	3	0.006757	0.006290	0.006324	0.006757	0.006133	0.006137
4	2	0.076923	0.038610	0.034993	0.076923	0.035879	0.028481
4	3	0.012085	0.010266	0.010429	0.012085	0.009773	0.009706

Table 14. Location of Banks-Zaks critical points for $\overline{\text{MS}}$ and mMOM at two, three and four loops for quarks in the $2S$ representation.

$2S$		MOMq		MOMggg		MOMh	
N_c	N_f	a_2	a_3	a_2	a_3	a_2	a_3
3	2	0.067010	0.033185	0.067010	0.026936	0.067010	0.038706
3	3	0.006757	0.006043	0.006757	0.005449	0.006757	0.006272
4	2	0.076923	0.031380	0.076923	0.025513	0.076923	0.037977
4	3	0.012085	0.009452	0.012085	0.008038	0.012085	0.010154

Table 15. Location of Banks-Zaks critical points for MOMq, MOMggg and MOMh at two and three loops for quarks in the $2S$ representation.

$2S$		\overline{MS}			mMOM		
N_c	N_f	ω_2	ω_3	ω_4	ω_2	ω_3	ω_4
3	2	0.580756	0.484962	0.494313	0.580756	0.461475	0.470733
3	3	0.013514	0.013449	0.013385	0.013514	0.013398	0.013391
4	2	1.025641	0.771209	0.784341	1.025641	0.733643	0.730358
4	3	0.064451	0.062991	0.062225	0.064451	0.062094	0.062379

Table 16. Critical exponent ω for the Banks-Zaks critical point for \overline{MS} and mMOM at two, three and four loops for quarks in the $2S$ representation.

$2S$		MOMq		MOMggg		MOMh	
N_c	N_f	ω_2	ω_3	ω_2	ω_3	ω_2	ω_3
3	2	0.580756	0.432782	0.580756	0.373054	0.580756	0.477139
3	3	0.013514	0.013363	0.013514	0.013007	0.013514	0.013444
4	2	1.025641	0.666122	1.025641	0.567521	1.025641	0.762733
4	3	0.064451	0.061393	0.064451	0.057224	0.064451	0.062807

Table 17. Critical exponent ω for the Banks-Zaks critical point for the MOMq, MOMggg and MOMh schemes at two and three loops for quarks in the $2S$ representation.

$2S$		\overline{MS}			mMOM		
N_c	N_f	ρ_2	ρ_3	ρ_4	ρ_2	ρ_3	ρ_4
3	2	2.442844	1.284021	1.122151	2.694805	1.422422	1.210883
3	3	0.143809	0.132625	0.133158	0.147386	0.133175	0.133159
4	2	4.815089	2.077658	1.787181	5.365385	2.436574	1.949337
4	3	0.380719	0.313071	0.314964	0.399558	0.318680	0.315594

Table 18. Quark mass critical exponent at the Banks-Zaks critical point for the \overline{MS} and mMOM schemes at two, three and four loops for quarks in the $2S$ representation.

$2S$		MOMq		MOMggg		MOMh	
N_c	N_f	ρ_2	ρ_3	ρ_2	ρ_3	ρ_2	ρ_3
3	2	2.440100	1.088873	3.194973	1.123601	1.837734	0.959833
3	3	0.148363	0.133049	0.166564	0.135940	0.142239	0.132247
4	2	4.616444	1.554419	5.894166	1.548531	3.166038	1.294776
4	3	0.399558	0.313149	0.485641	0.326803	0.363762	0.305782

Table 19. Quark mass critical exponent at the Banks-Zaks critical point for the MOMq, MOMggg and MOMh schemes at two and three loops for quarks in the $2S$ representation.

		2A		$\overline{\text{MS}}$			mMOM	
N_c	N_f	a_2	a_3	a_4	a_2	a_3	a_4	
4	6	0.172414	0.052865	0.061243	0.172414	0.044308	0.038398	
4	7	0.070796	0.034771	0.039931	0.070796	0.029895	0.028047	
4	8	0.035714	0.022840	0.025409	0.035714	0.020324	0.020083	
4	9	0.017937	0.013814	0.014662	0.017937	0.012777	0.012908	
4	10	0.007194	0.006401	0.006518	0.007194	0.006164	0.006212	

Table 20. Location of Banks-Zaks critical points for $\overline{\text{MS}}$ and mMOM at two, three and four loops for quarks in the 2A representation.

		2A		MOMq		MOMggg		MOMh	
N_c	N_f	a_2	a_3	a_2	a_3	a_2	a_3	a_2	a_3
4	6	0.172414	0.036860	0.172414	0.032342	0.172414	0.045374	0.172414	0.045374
4	7	0.070796	0.027053	0.070796	0.022389	0.070796	0.031004	0.070796	0.031004
4	8	0.035714	0.019325	0.035714	0.015630	0.035714	0.021179	0.035714	0.021179
4	9	0.017937	0.012543	0.017937	0.010258	0.017937	0.013285	0.017937	0.013285
4	10	0.007194	0.006161	0.007194	0.005338	0.007194	0.006332	0.007194	0.006332

Table 21. Location of Banks-Zaks critical points for MOMq, MOMggg and MOMh at two and three loops for quarks in the 2A representation.

		2A		$\overline{\text{MS}}$			mMOM	
N_c	N_f	ω_2	ω_3	ω_4	ω_2	ω_3	ω_4	
4	6	2.298851	1.193609	1.109724	2.298851	1.029719	1.022181	
4	7	0.755162	0.559626	0.511494	0.755162	0.503114	0.508341	
4	8	0.285714	0.248588	0.229893	0.285714	0.232661	0.233704	
4	9	0.095665	0.090611	0.086504	0.095665	0.087749	0.087236	
4	10	0.019185	0.018951	0.018660	0.019185	0.018791	0.018680	

Table 22. Critical exponent ω for the Banks-Zaks critical point for $\overline{\text{MS}}$ and mMOM at two, three and four loops for quarks in the 2A representation.

		2A		MOMq		MOMggg		MOMh	
N_c	N_f	ω_2	ω_3	ω_2	ω_3	ω_2	ω_3	ω_2	ω_3
4	6	2.298851	0.877863	2.298851	0.781571	2.298851	1.050754	2.298851	1.050754
4	7	0.755162	0.466867	0.755162	0.402103	0.755162	0.516585	0.755162	0.516585
4	8	0.285714	0.225543	0.285714	0.195358	0.285714	0.238387	0.285714	0.238387
4	9	0.095665	0.087014	0.095665	0.078133	0.095665	0.089231	0.095665	0.089231
4	10	0.019185	0.018789	0.019185	0.017908	0.019185	0.018909	0.019185	0.018909

Table 23. Critical exponent ω for the Banks-Zaks critical point for the MOMq, MOMggg and MOMh schemes at two and three loops for quarks in the 2A representation.

2A		$\overline{\text{MS}}$			mMOM		
N_c	N_f	ρ_2	ρ_3	ρ_4	ρ_2	ρ_3	ρ_4
4	6	9.782501	1.381815	0.292995	11.318371	1.566192	1.377240
4	7	2.191767	0.695302	0.435137	2.484143	0.769888	0.703235
4	8	0.801977	0.401949	0.368304	0.884885	0.429906	0.414671
4	9	0.330860	0.228000	0.231646	0.353918	0.235533	0.235585
4	10	0.116993	0.101120	0.102557	0.121047	0.101969	0.102620

Table 24. Quark mass critical exponent at the Banks-Zaks critical point for the $\overline{\text{MS}}$ and mMOM schemes at two, three and four loops for quarks in the 2A representation.

2A		MOMq		MOMggg		MOMh	
N_c	N_f	ρ_2	ρ_3	ρ_2	ρ_3	ρ_2	ρ_3
4	6	7.054427	0.805527	12.794194	1.154631	5.180018	0.582067
4	7	1.882686	0.560447	3.197151	0.730752	1.566644	0.491650
4	8	0.761721	0.375009	1.184460	0.452628	0.681294	0.353174
4	9	0.330392	0.225207	0.459282	0.253314	0.310104	0.219754
4	10	0.118476	0.101235	0.142790	0.106250	0.115212	0.100632

Table 25. Quark mass critical exponent at the Banks-Zaks critical point for the MOMq, MOMggg and MOMh schemes at two and three loops for quarks in the 2A representation.