

# MOM renormalization group functions in the maximal abelian gauge

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**Abstract.** The one loop 3-point vertex functions of QCD in the maximal abelian gauge (MAG) are evaluated at the fully symmetric point at one loop. As a consequence the theory is renormalized in the various momentum (MOM) schemes which are defined by the trivalent vertices, as well as in the  $\overline{MS}$  scheme. From these the *two* loop renormalization group functions in the MOM schemes are derived using the one loop conversion functions. In parallel we repeat the analysis for the Curci-Ferrari gauge which corresponds to the MAG in a specific limit. The relation between the  $\Lambda$  parameters in different schemes is also provided.

# 1 Introduction.

One of the outstanding problems in quantum field theory is to understand the mechanism behind quark and gluon confinement. The former are the building blocks of hadronic states while the latter are the quanta which mediate the strong nuclear force. Unlike other fundamental particles in the standard model neither quarks nor gluons are seen in nature as isolated states. Though at high energy quarks behave as asymptotically free entities and to all intents and purposes are seen through their interaction within deep inelastic scattering, for example. However, this high energy property of asymptotic fundamentality does not persist at low energies. Instead infrared slavery dominates and single free quarks cannot be isolated. In other words the full quark and gluon propagators do not have simple poles at a zero or non-zero mass. There have been many attempts to explain the absence of free quark and gluon states. For background see, for instance, the review article [1]. One framework which has received attention is that where the infrared dynamics is based on an abelian theory involving magnetic monopoles, [1, 2, 3, 4, 5]. In a parallel of what occurs in superconductivity, colour charge is confined when the monopoles condense to produce an Abrikosov-Nielsen-Olesen string, [1, 2, 3, 4]. A main key in the whole picture is the underlying abelian structure within a theory which has a non-abelian colour group. Thus the actual mathematical structure of the colour group of the underlying quantum field theory describing the strong force plays an important role, [5]. This is either Yang-Mills theory which describes purely gluons or Quantum Chromodynamics (QCD) when quarks are included and involve the non-abelian Lie group  $SU(3)$ . The abelian monopoles are associated with the quanta derived from the centre of the colour group. These are believed to dominate the infrared dynamics, [2, 3, 4, 5]. In other words the contribution from the remaining off-diagonal sector quanta are negligible.

To understand this picture further from a quantum field theory viewpoint requires accessing each sector of the colour group. However, in the canonical formulation of Yang-Mills or QCD using a linear covariant gauge fixing, one has no direct access to examining separate centre or off-diagonal gluon dynamics. Moreover, one requires techniques to study the field theory non-perturbatively. One useful method is that of Schwinger-Dyson equations. In this approach the aim is to solve the tower of coupled  $n$ -point functions, usually in a particular approximation, that allows clean access to the problem at hand. Though for an abelian monopole analysis one has to have a way of making contact with the centre directly. One way of achieving this is to choose an appropriate gauge fixing. One such gauge is the maximal abelian gauge (MAG), [4, 6, 7]. It is a nonlinear covariant gauge fixing where the centre and off-diagonal gluons are gauge fixed differently, [4, 6, 7]. While this has been used in Schwinger-Dyson analyses, such as [8, 9, 10], and several lattice studies, such as [11, 12, 13, 14, 15, 16], there has not been as much activity in the MAG compared with the Landau gauge. Encouraging results have emerged such as differing infrared behaviours of the centre and off-diagonal gluon and ghost propagators. Although the focus has primarily been on 2-point functions, more recently Landau gauge studies have turned to vertex functions and specifically 3-point functions, [17]. These functions have been studied at several momentum configurations. The two main ones are the asymmetric and the symmetric points. The former is easier to simulate on the lattice whereas the latter has relatively noisier signals. However, at the symmetric point there are no infrared problems since the momentum configuration is non-exceptional in contrast to the asymmetric point.

The issue of the subtraction point of the vertices is related to the area of renormalization schemes. In [18] the momentum (MOM) subtraction schemes were introduced for the 3-point vertices of QCD where the focus was on linear covariant gauges. This family of schemes are mass dependent renormalization schemes which are physical. The actual subtraction is such that after renormalization the Lorentz channel of the 2 and 3-point functions containing the divergences is

unity at the renormalization point. This original analysis of [18] was extended to the next loop order recently in [19]. Given that the lattice measures vertex functions non-perturbatively and requires matching to the high energy behaviour, the more loop order information available at high energy allows one to have reduced error estimates on infrared measurements. In addition Schwinger-Dyson analyses of Green's functions requires matching. This was in part the motivation of [19]. Based on this and the interest in the infrared structure of QCD in the MAG, it is therefore the purpose of this article to provide an analysis of the 3-point vertex functions of QCD in the MAG at one loop. This will extend earlier work on the MAG for various colour groups, [20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Moreover, it will be a complete parallel of [18] for the linear covariant gauge fixings. We will also provide the symmetric point 3-point vertices both in the  $\overline{\text{MS}}$  scheme as well as the various MOM schemes associated with the trivalent vertices. One major consequence is that the two loop renormalization group functions will be deduced in the MOM schemes. This is because the mappings of the various parameters between the schemes are derived from the one loop analysis. Hence these conversion functions between schemes are used in conjunction with the renormalization group equation and the known two loop  $\overline{\text{MS}}$  renormalization group functions, [30]. As checks on the results we will compare with the nonlinear covariant gauge known as the Curci-Ferrari gauge, [31]. In a certain limit the MAG is equivalent to this gauge and we have performed the full analysis in the Curci-Ferrari gauge. By taking the limit from the MAG we will be able to verify agreement. Indeed the Curci-Ferrari gauge is of interest in its own right as it has a special feature. Originally it was noted in [31] that a BRST invariant gluon mass could be included in the Lagrangian. Clearly it is not gauge invariant but it was regarded as a useful tool for potentially modelling gluon mass. Indeed lattice and Schwinger-Dyson analyses in recent years have indicated that the Landau gauge gluon propagator freezes in the infrared to a finite non-zero value. The initial observations in this respect can be found in [32, 33, 34, 35, 36, 37, 38, 39, 40]. This freezing would correspond to some type of effective gluon mass. Moreover, studies in the MAG on the lattice suggest a similar phenomena but with differing masses for centre and off-diagonal gluons, [13, 14, 15, 16]. This splitting of masses in the infrared is believed to be symptomatic of the dominance of the abelian sector.

The paper is organized as follows. We provide all the relevant background to the MAG in section 2 including group theory identities we use when the centre is identified, the renormalization group scheme conversion functions and the computational setup for the symmetric point analysis. The subsequent sections are devoted to the explicit results. The  $\overline{\text{MS}}$  amplitudes are given in section 3. The two loop renormalization group functions, amplitudes and conversion functions for the three MOM schemes are given respectively in sections 4, 5 and 6. The results for the related Curci-Ferrari gauge are presented in section 7 with the derivation of the relation between the  $\Lambda$  parameters given in section 8. Concluding remarks are given in section 9. An appendix collects the tensor basis of the vertex functions and the explicit forms of the associated projection matrices.

## 2 Background.

We devote this section to reviewing the key properties of the MAG as well as the calculational techniques which we use. First, as noted the MAG is a particular gauge fixing where the gluons are allocated to two parts of the colour group, [4, 6, 7]. Those deriving from the centre are named diagonal or centre gluons while those which are not part of this abelian subgroup are termed off-diagonal. Given this we use the same notation as [29, 30] using letters  $a, b$  and  $c$  as off-diagonal indices but  $i, j, k$  and  $l$  as indices for gluons and other fields associated with the

centre. Capital letters are reserved for the adjoint indices of the full colour group. Thus we decompose the group valued gauge field,  $\mathcal{A}_\mu = A_\mu^A T^A$  into

$$\mathcal{A}_\mu = A_\mu^a T^a + A_\mu^i T^i \quad (2.1)$$

where  $T^A$  are the group generators. As we will be summing over colour indices we define the dimensions of the diagonal sector as  $N_A^d$  in the adjoint representation and  $N_A^o$  for the off-diagonal sector. Thus  $1 \leq i \leq N_A^d$ ,  $1 \leq a \leq N_A^o$  and  $1 \leq A \leq N_A$  where  $N_A$  is the dimension of the adjoint representation in the full group. The dimension of the fundamental is  $N_F$ . So

$$N_A^d + N_A^o = N_A . \quad (2.2)$$

As an example for  $SU(N_c)$  we have  $N_A^d = N_c - 1$  and  $N_A^o = N_c(N_c - 1)$ . Though we will work throughout with an arbitrary colour group and only specify  $SU(3)$  in certain cases. In this notation the gauge invariant QCD Lagrangian is, [29],

$$L = -\frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu} + i\bar{\psi} \not{D} \psi + L_{\text{gf}} \quad (2.3)$$

where  $G_{\mu\nu}^A$  is the usual field strength and there are  $N_f$  massless quarks  $\psi$ . Translating this to the MAG situation the field strength splits into two terms since

$$L = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4} G_{\mu\nu}^i G^{i\mu\nu} + i\bar{\psi} \not{D} \psi + L_{\text{gf}}^{\text{MAG}} . \quad (2.4)$$

The main difference between this Lagrangian and the canonical covariant gauge fixing term is that in  $L_{\text{gf}}^{\text{MAG}}$  the prescription to fix the gauge for the off-diagonal gluons is different from that of the diagonal gluons, [4, 6, 7]. As this construction has been discussed elsewhere we record the full gauge fixed MAG as

$$\begin{aligned} L_{\text{gf}}^{\text{MAG}} = & -\frac{1}{2\alpha} \left( \partial^\mu A_\mu^a \right)^2 - \frac{1}{2\bar{\alpha}} \left( \partial^\mu A_\mu^i \right)^2 + \bar{c}^a \partial^\mu \partial_\mu c^a + \bar{c}^i \partial^\mu \partial_\mu c^i \\ & + g \left[ f^{abk} A_\mu^a \bar{c}^k \partial^\mu c^b - f^{abc} A_\mu^a \bar{c}^b \partial^\mu c^c - \frac{1}{\alpha} f^{abk} \partial^\mu A_\mu^a A_\nu^b A^{k\nu} - f^{abk} \partial^\mu A_\mu^a c^b \bar{c}^k \right. \\ & \quad \left. - \frac{1}{2} f^{abc} \partial^\mu A_\mu^a \bar{c}^b c^c - 2 f^{abk} A_\mu^k \bar{c}^a \partial^\mu \bar{c}^b - f^{abk} \partial^\mu A_\mu^k \bar{c}^b c^c \right] \\ & + g^2 \left[ f_d^{acbd} A_\mu^a A^\mu \bar{c}^b c^d - \frac{1}{2\alpha} f_o^{akbl} A_\mu^a A^\mu A_\nu^k A^{l\nu} + f_o^{adcj} A_\mu^a A^\mu \bar{c}^c c^d \right. \\ & \quad - \frac{1}{2} f_o^{ajcd} A_\mu^a A^\mu \bar{c}^c c^d + f_o^{ajcl} A_\mu^a A^\mu \bar{c}^c c^l + f_o^{alcj} A_\mu^a A^\mu \bar{c}^c c^l \\ & \quad - f_o^{cjd} A_\mu^i A^\mu \bar{c}^c c^d - \frac{\alpha}{4} f_d^{abcd} \bar{c}^a \bar{c}^b c^c c^d - \frac{\alpha}{8} f_o^{abcd} \bar{c}^a \bar{c}^b c^c c^d + \frac{\alpha}{8} f_o^{acbd} \bar{c}^a \bar{c}^b c^c c^d \\ & \quad \left. - \frac{\alpha}{4} f_o^{abcl} \bar{c}^a \bar{c}^b c^c c^l + \frac{\alpha}{4} f_o^{acbl} \bar{c}^a \bar{c}^b c^c c^l - \frac{\alpha}{4} f_o^{albc} \bar{c}^a \bar{c}^b c^c c^l + \frac{\alpha}{2} f_o^{akbl} \bar{c}^a \bar{c}^b c^k c^l \right] . \quad (2.5) \end{aligned}$$

It is worth noting at this stage we are basing this on the more general modified MAG discussed in [21]. Though the interpolating parameter,  $\zeta$ , which is apparent in [21] and [22] is set to the specific value for the MAG itself. Its interpolating property is not relevant for this article. In addition to this, given the nature of this construction we need to make comments relevant to our analysis. First, there are two gauge parameters,  $\alpha$  and  $\bar{\alpha}$ . The latter is the parameter associated with the centre gluons and as such only appears in the quadratic term. It is necessary in order to construct the centre gluon propagator and is set to zero thereafter. In other words that sector is gauge fixed in the Landau gauge, [22, 29]. For the off-diagonal gluons the gauge parameter appears in the interactions as well as the quadratic term. Moreover, it cannot be set

to zero after the propagator has been constructed since several interactions would have singular couplings. Thus a non-zero  $\alpha$  is retained throughout. Though we note that aside from the gauge parameter renormalization all the other renormalization group functions are finite in the  $\alpha \rightarrow 0$  limit. For the quartic terms we use a compact notation for the structure functions, [30],

$$f_d^{ABCD} = f^{iAB} f^{iCD} \quad , \quad f_o^{ABCD} = f^{eAB} f^{eCD} . \quad (2.6)$$

In other words the subscript denotes whether the summed index is from the centre or off-diagonal sector. In this respect it is worth noting one consequence of the Lie algebra. If

$$[T^A, T^B] = i f^{ABC} T^C \quad (2.7)$$

then

$$f^{ijk} = 0 \quad , \quad f^{ijc} = 0 \quad (2.8)$$

and

$$[T^a, T^j] = i f^{ajc} T^c . \quad (2.9)$$

These are important when it comes to performing the group theory associated with Feynman diagrams. In addition to the  $\alpha$  dependence in (2.5) the gauge fixing requires Faddeev-Popov ghosts,  $c^A$ . These are associated with each colour sector. It is worth noting that while ordinarily an abelian gauge theory does not have coupled ghosts this statement only applies to the case where the gauge fixing is linear. For instance, in the 't Hooft-Veltman gauge, [41], there are interacting Faddeev-Popov ghosts. The situation is the same here in that the MAG, being a nonlinear gauge fixing, produces centre ghosts which couple in a non-trivial fashion. Moreover, there are quartic ghost terms. These together with the other interactions do not spoil renormalizability which has been established in [20, 22, 23, 24, 26, 29]. As part of the renormalization we note that the renormalization constants for the fields and the parameters are

$$\begin{aligned} A_o^{a\mu} &= \sqrt{Z_A} A^{a\mu} \quad , \quad A_o^{i\mu} = \sqrt{Z_{A^i}} A^{i\mu} \quad , \quad c_o^a = \sqrt{Z_c} c^a \quad , \quad \bar{c}_o^a = \sqrt{Z_c} \bar{c}^a \\ c_o^i &= \sqrt{Z_{c^i}} c^i \quad , \quad \bar{c}_o^i = \frac{\bar{c}^i}{\sqrt{Z_{c^i}}} \quad , \quad \psi_o = \sqrt{Z_\psi} \psi \\ g_o &= \mu^\epsilon Z_g g \quad , \quad \alpha_o = Z_\alpha^{-1} Z_A \alpha \quad , \quad \bar{\alpha}_o = Z_{\alpha^i}^{-1} Z_{A^i} \bar{\alpha} \end{aligned} \quad (2.10)$$

where the index  $i$  on objects in the subscript are to indicate the centre sector and there is no summation over this label when it is repeated. Bare quantities are denoted by the subscript  $_o$ . We use the same conventions as [30]. In particular we dimensionally regularize in  $d = 4 - 2\epsilon$  dimensions where  $\epsilon$  is the regularizing parameter and the mass scale  $\mu$  is introduced to ensure the coupling constant is dimensionless in  $d$ -dimensions. We have included the abelian gauge parameter renormalization for completeness but it will not be discussed here as the Landau gauge will be chosen for that sector. Also we have provided (2.10) to highlight that there is a nontrivial aspect to the renormalization of QCD in the MAG. When one fixes a gauge the remnant of the original gauge symmetry becomes manifest in the Slavnov-Taylor identities via the underlying BRST symmetry. These place certain constraints on the renormalization constants which must be respected in any computation and renormalization scheme. Those identities for the ordinary linear covariant gauge fixing are well known and can be established systematically by the algebraic renormalization technique [42]. However, when that method is applied to the MAG the diagonal ghost and anti-ghost renormalization constants are not defined in the canonical way, (2.10), [29]. This has been checked to three loops in  $\overline{\text{MS}}$  in [30]. Therefore, we have to allow for this in our definitions. Moreover, to determine the centre ghost renormalization at one loop requires a two loop renormalization of a *vertex* function. A second consequence of the Slavnov-Taylor identities is that the centre gluon wave function

renormalization constant is not an independent renormalization. It is related to the coupling constant renormalization, [29], and as such provides an independent check on any computation. For completeness the relevant renormalization group functions we will consider here in the various MOM schemes are

$$\begin{aligned}
\gamma_A(a, \alpha) &= \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_A + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_A \\
\gamma_\alpha(a, \alpha) &= \left[ \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_\alpha - \gamma_A(a, \alpha) \right] \left[ 1 - \alpha \frac{\partial}{\partial \alpha} \ln Z_\alpha \right]^{-1} \\
\gamma_{A^i}(a, \alpha) &= \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_{A^i} + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_{A^i} \\
\gamma_c(a, \alpha) &= \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_c + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_c \\
\gamma_\psi(a, \alpha) &= \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_\psi + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_\psi
\end{aligned} \tag{2.11}$$

where the form of  $\gamma_\alpha(a, \alpha)$  is due to the fact that unlike a linear covariant gauge fixing  $Z_\alpha$  is not unity. The quantity  $a$  is defined to be  $a = g^2/(16\pi^2)$ . Also we have included  $\alpha$  dependence on the  $\beta$ -function since in mass dependent renormalization schemes such as the MOM cases we consider here the  $\beta$ -function is gauge dependent.

Having defined the renormalization group functions in relation to the renormalization constants for a particular scheme we now recall the formalism which relates the expressions between two different schemes. First we note that the parameters such as the coupling constant and the gauge parameter are associated with a scheme and therefore their values differ between schemes. They are related via their respective renormalization constants. In particular\*

$$g_{\text{MOMi}}(\mu) = \frac{Z_g^{\overline{\text{MS}}}}{Z_g^{\text{MOMi}}} g_{\overline{\text{MS}}}(\mu) \quad , \quad \alpha_{\text{MOMi}}(\mu) = \frac{Z_A^{\overline{\text{MS}}} Z_\alpha^{\text{MOMi}}}{Z_A^{\text{MOMi}} Z_\alpha^{\overline{\text{MS}}}} \alpha_{\overline{\text{MS}}}(\mu) \tag{2.12}$$

where we use MOMi to label a typical MOM scheme and use these as well as the  $\overline{\text{MS}}$  scheme to illustrate the formalism for converting between schemes. However, it is important to realise that the explicit relation between the parameters is found recursively. This is because on the right hand side of each of the equations of (2.12) the MOMi renormalization constant is a function of the parameters in that scheme. Therefore these have to be mapped order by order in the perturbative expansion to the reference scheme, which will be  $\overline{\text{MS}}$  throughout, prior to extracting the parameter relation at a particular loop order. Otherwise one would not have a relation between parameters which is finite with respect to the regularization. Once the mapping of the parameters from one scheme to another has been established it is possible to define conversion functions for all the renormalization group functions. These are similar to (2.12) and are given by

$$C_g^{\text{MOMi}}(a, \alpha) = \frac{Z_g^{\text{MOMi}}}{Z_g^{\overline{\text{MS}}}} \quad , \quad C_\phi^{\text{MOMi}}(a, \alpha) = \frac{Z_\phi^{\text{MOMi}}}{Z_\phi^{\overline{\text{MS}}}} \tag{2.13}$$

where  $\phi$  denotes the field associated with the anomalous dimension and the arguments of the conversion functions are the  $\overline{\text{MS}}$  parameters as this is the reference scheme. Though for the gauge parameter we define

$$C_\alpha^{\text{MOMi}}(a, \alpha) = \frac{Z_\alpha^{\text{MOMi}} Z_A^{\overline{\text{MS}}}}{Z_\alpha^{\overline{\text{MS}}} Z_A^{\text{MOMi}}} \tag{2.14}$$

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\*The second equation corrects an obvious error in the corresponding relation in [19].

as the conversion function. Again the perturbative expansion of each conversion function is finite with respect to  $\epsilon$  at each order once the parameter mapping has been applied. Equipped with these then the relations between the renormalization group functions in the various schemes are

$$\beta^{\text{MOMi}}(a_{\text{MOMi}}, \alpha_{\text{MOMi}}) = \left[ \beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMi}}}{\partial a_{\overline{\text{MS}}}} + \alpha_{\overline{\text{MS}}} \gamma_{\alpha}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMi}}}{\partial \alpha_{\overline{\text{MS}}}} \right]_{\overline{\text{MS}} \rightarrow \text{MOMi}} \quad (2.15)$$

and

$$\begin{aligned} \gamma_{\phi}^{\text{MOMi}}(a_{\text{MOMi}}, \alpha_{\text{MOMi}}) &= \left[ \gamma_{\phi}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) + \beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial}{\partial a_{\overline{\text{MS}}}} \ln C_{\phi}^{\text{MOMi}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right. \\ &\quad \left. + \alpha_{\overline{\text{MS}}} \gamma_{\alpha}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_{\phi}^{\text{MOMi}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right]_{\overline{\text{MS}} \rightarrow \text{MOMi}} \end{aligned} \quad (2.16)$$

where  $\phi$  also includes  $\alpha$  here now and the subscript  $\overline{\text{MS}} \rightarrow \text{MOMi}$  indicates there is a mapping of the parameters after the evaluation of the quantity. The  $\overline{\text{MS}}$  parameters in the square parentheses are mapped to those of the MOMi scheme. We have written (2.15) in this particular form in order to indicate its derivation originates from the renormalization group formalism but the two derivatives can be simply related to  $C_g^{\text{MOMi}}(a, \alpha)$ . From (2.15) and (2.16) it is clear from examining the  $a$  dependence that to deduce the two loop renormalization group functions in the MOMi scheme only the one loop conversion functions are needed as the two loop  $\overline{\text{MS}}$  renormalization group functions are known. In essence the conversion functions derive from the finite parts of the vertex functions after renormalization which we will deduce as part of our computations for each of the 3-point vertices.

Given the structure of the trivalent vertices in the Lagrangian and our aim of computing in the MOM setup it appears that there are six such possible schemes. This is in contrast to the linear covariant gauge fixing where there are three schemes deriving from the triple gluon, ghost-gluon and quark-gluon vertices. However, in the MAG there are only three rather than the potential six MOM schemes. This is because the Slavnov-Taylor identity renders the vertices involving the centre gluons effectively trivial. The coupling constant renormalization constant derived from these vertices is already determined by this identity. Thus the three schemes we will focus on are those which are completely parallel to those of [18] where the gluon is off-diagonal. Given this we recall the computational setup which will be completely parallel to [19]. First we decompose each vertex function at the symmetric subtraction point into the scalar amplitudes with their associated Lorentz tensor basis. Factoring off the overall colour tensors for each vertex function using

$$\begin{aligned} \left\langle A_{\mu}^a(p) A_{\nu}^b(q) A_{\sigma}^c(r) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= f^{abc} \Sigma_{\mu\nu\sigma}^{\text{ggg}}(p, q) \Big|_{p^2=q^2=-\mu^2} \\ \left\langle c^a(p) \bar{c}^b(q) A_{\sigma}^c(r) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= f^{abc} \Sigma_{\sigma}^{\text{cgg}}(p, q) \Big|_{p^2=q^2=-\mu^2} \\ \left\langle \psi^i(p) \bar{\psi}^j(q) A_{\sigma}^c(r) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= T_{ij}^c \Sigma_{\sigma}^{\text{qg}}(p, q) \Big|_{p^2=q^2=-\mu^2} \end{aligned} \quad (2.17)$$

then we write

$$\begin{aligned} \Sigma_{\mu\nu\sigma}^{\text{ggg}}(p, q) \Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^{14} \mathcal{P}_{(k)\mu\nu\sigma}^{\text{ggg}}(p, q) \Sigma_{(k)}^{\text{ggg}}(p, q) \\ \Sigma_{\sigma}^{\text{cgg}}(p, q) \Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^2 \mathcal{P}_{(k)\sigma}^{\text{cgg}}(p, q) \Sigma_{(k)}^{\text{cgg}}(p, q) \end{aligned}$$

$$\Sigma_{\sigma}^{\text{qqg}}(p, q) \Big|_{p^2=q^2=-\mu^2} = \sum_{k=1}^6 \mathcal{P}_{(k)\sigma}^{\text{qqg}}(p, q) \Sigma_{(k)}^{\text{qqg}}(p, q). \quad (2.18)$$

Throughout we use  $p$  and  $q$  as the two independent external momenta which will be the incoming momenta for the ghost and quark lines in their respective cases. The third external momentum is  $r$  where

$$r = -p - q. \quad (2.19)$$

The symmetric point is then defined as

$$p^2 = q^2 = r^2 = -\mu^2 \quad (2.20)$$

which implies

$$pq = \frac{1}{2}\mu^2. \quad (2.21)$$

To determine each scalar amplitude within a vertex function we use the same projection method and tensor basis as [19] where the explicit derivation is detailed. The explicit forms of the tensors and projection matrices  $\mathcal{M}_{kl}^i$ , where  $i$  denotes the vertex, are given for completeness in appendix A. Though we recall that

$$\begin{aligned} f^{abc}\Sigma_{(k)}^{\text{ggg}}(p, q) &= \mathcal{M}_{kl}^{\text{ggg}} \left( \mathcal{P}_{(l)}^{\text{ggg}\mu\nu\sigma}(p, q) \left\langle A_{\mu}^a(p) A_{\nu}^b(q) A_{\sigma}^c(r) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2} \\ f^{abc}\Sigma_{(k)}^{\text{ccg}}(p, q) &= \mathcal{M}_{kl}^{\text{ccg}} \left( \mathcal{P}_{(l)}^{\text{ccg}\sigma}(p, q) \left\langle c^a(p) \bar{c}^b(q) A_{\sigma}^c(r) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2} \\ T_{ij}^c \Sigma_{(k)}^{\text{qqg}}(p, q) &= \mathcal{M}_{kl}^{\text{qqg}} \left( \mathcal{P}_{(l)}^{\text{qqg}\sigma}(p, q) \left\langle \psi^i(p) \bar{\psi}^j(q) A_{\sigma}^c(r) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2} \end{aligned} \quad (2.22)$$

are the linear combinations for each Lorentz channel. For the quark-gluon vertex we use the generalized  $\gamma$ -matrices denoted by  $\Gamma_{(n)}$  and defined by

$$\Gamma_{(n)}^{\mu_1 \dots \mu_n} = \gamma^{[\mu_1} \dots \gamma^{\mu_n]} \quad (2.23)$$

where a factor  $1/n!$  is understood in the total antisymmetrization. Properties of these matrices have been detailed in [43, 44, 45, 46, 47]. In this basis which spans the space of  $\gamma$ -matrices there is a natural partition due to

$$\text{tr} \left( \Gamma_{(m)}^{\mu_1 \dots \mu_m} \Gamma_{(n)}^{\nu_1 \dots \nu_n} \right) \propto \delta_{mn} I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n} \quad (2.24)$$

which is evident in (A.6).

Once all the vertices have been decomposed into their Lorentz scalars we have to reduce the large number of Feynman integrals to a form in which they can be evaluated. We have chosen to use the Laporta approach, [48]. This method allows one to construct all the integration by parts identities for a minimal set of basic topologies. Suitably chosen these cover all possible topologies which arise in the vertex functions. Once the relations between all the integrals are known then they can be algebraically solved to a small set of master graphs. Ordinarily these have to be determined by non-integration by parts methods. In practical terms the Laporta algorithm has been coded in several packages. We have used REDUZE, [49], which uses GINAC, [50], and built the necessary database. At one loop there is one basis topology. For each vertex function we have used QGRAF, [51], to generate all the Feynman graphs and then mapped them on to the basic topologies. We have appended colour and Lorentz indices in the initial steps too. Throughout we have used FORM, [52, 53], as the computational tool to handle the algebra symbolically. For the triple off-diagonal vertex there are 23 one loop graphs. The ghost-gluon



vertex has 16 graphs and there are 5 for the quark-gluon vertex. Briefly for the 2-point functions we have used MINCER, [54, 55], in order to evaluate the small number of straightforward graphs.

Given that we are working in the MAG it is worthwhile recalling some of the group theory identities which we have had to use, [30]. As the colour group has been split into two sectors we have to be careful in implementing this symbolically. Useful in this instance is the set facility in FORM in order to treat centre and off-diagonal indices separately. The starting point for deriving any group identities for the split Lie algebra is the original identities. First, the usual Casimirs are defined by

$$f^{ACD}f^{BCD} = C_A\delta^{AB} \quad , \quad T^AT^A = C_F I \quad , \quad \text{Tr}(T^AT^B) = T_F\delta^{AB} \quad (2.25)$$

where  $I$  is the identity. The former gives the non-trivial results

$$C_A\delta^{ab} = f^{acd}f^{bcd} + 2f^{acj}f^{bcj} \quad , \quad C_A\delta^{ij} = f^{icd}f^{jcd} \quad (2.26)$$

if one recalls the structure functions can only have at most one centre index. These imply

$$\begin{aligned} f^{iab}f^{iab} &= N_A^d C_A \quad , \quad f^{abc}f^{abc} = [N_A^o - 2N_A^d] C_A \\ f^{acj}f^{bcj} &= \frac{N_A^d}{N_A^o} C_A \delta^{ab} \quad , \quad f^{acd}f^{bcd} = \frac{[N_A^o - 2N_A^d]}{N_A^o} C_A \delta^{ab} . \end{aligned} \quad (2.27)$$

The remaining equations of (2.25) give the simple expressions

$$\text{Tr}(T^AT^b) = T_F\delta^{ab} \quad , \quad \text{Tr}(T^aT^i) = 0 \quad , \quad \text{Tr}(T^iT^j) = T_F\delta^{ij} \quad (2.28)$$

as well as

$$T^iT^i = \frac{T_F}{N_F} N_A^d I \quad , \quad T^aT^a = \left[ C_F - \frac{T_F}{N_F} N_A^d \right] I . \quad (2.29)$$

The Jacobi identity

$$0 = f^{ABE}f^{CDE} + f^{BCE}f^{ADE} + f^{CAE}f^{BDE} \quad (2.30)$$

provides more results which we needed such as

$$\begin{aligned} f^{apq}f^{bpr}f^{cqr} &= \frac{[N_A^o - 3N_A^d]}{2N_A^o} C_A f^{abc} \quad , \quad f^{apq}f^{bpi}f^{cqi} = \frac{N_A^d}{2N_A^o} C_A f^{abc} \\ f^{ipq}f^{bpr}f^{cqr} &= \frac{[N_A^o - 2N_A^d]}{2N_A^o} C_A f^{ibc} \quad , \quad f^{ipq}f^{bpj}f^{cqj} = \frac{N_A^d}{N_A^o} C_A f^{ibc} . \end{aligned} \quad (2.31)$$

A useful relation between dimensions is

$$C_F N_F = (N_A^d + N_A^o) T_F \quad (2.32)$$

which is required usually for simplifying algebra from the quark sector. These basic results and others have been coded within a FORM module and applied prior to the integrals being mapped to the basic topologies. This is because as was noted in [30] the group theory for some graphs is zero. Hence in such cases there is no need for a calculation to be performed.

### 3 $\overline{\text{MS}}$ scheme.

As a preliminary to the MOM scheme computations we first record the results for the amplitudes in the  $\overline{\text{MS}}$  scheme. This is the basic reference scheme. Indeed to deduce the two loop MOM scheme renormalization group functions using the conversion functions, the two loop  $\overline{\text{MS}}$  results are necessary. Therefore, for completeness we note that these are<sup>†</sup>, [29, 30],

$$\begin{aligned}
\gamma_A(a) &= \frac{1}{6N_A^o} \left[ N_A^o \left( (3\alpha - 13)C_A + 8T_F N_f \right) + N_A^d (-3\alpha + 9)C_A \right] a \\
&+ \frac{1}{48N_A^{o2}} \left[ N_A^{o2} \left( (6\alpha^2 + 66\alpha - 354)C_A^2 + 240C_A T_F N_f + 192C_F T_F N_f \right) \right. \\
&\quad + N_A^o N_A^d \left( (3\alpha^2 + 210\alpha + 331)C_A^2 - 80C_A T_F N_f \right) \\
&\quad \left. + N_A^{d2} \left( (15\alpha^2 - 6\alpha - 33)C_A^2 \right) \right] a^2 + O(a^3) \\
\gamma_\alpha(a) &= \frac{1}{12\alpha N_A^o} \left[ N_A^o \left( (-3\alpha^2 + 26\alpha)C_A - 16\alpha T_F N_f \right) + N_A^d (-6\alpha^2 - 36\alpha - 36)C_A \right] a \\
&+ \frac{1}{48\alpha N_A^{o2}} \left[ N_A^{o2} \left( (-3\alpha^3 - 51\alpha^2 + 354\alpha)C_A^2 - 240\alpha C_A T_F N_f - 192\alpha C_F T_F N_f \right) \right. \\
&\quad + N_A^o N_A^d \left( (-27\alpha^3 - 339\alpha^2 - 647\alpha - 928)C_A^2 \right. \\
&\quad \quad \left. + (160\alpha + 512)C_A T_F N_f \right) \\
&\quad \left. + N_A^{d2} (-30\alpha^3 - 366\alpha^2 + 294\alpha + 2016)C_A^2 \right] a^2 + O(a^3) \\
\gamma_{A^i}(a) &= \frac{1}{3} [4T_F N_f - 11C_A] a \\
&+ \frac{1}{3} \left[ -34C_A^2 + 20C_A T_F N_f + 12C_F T_F N_f \right] a^2 + O(a^3) \\
\gamma_c(a) &= \frac{1}{4N_A^o} \left[ N_A^o (\alpha - 3)C_A + N_A^d (-2\alpha - 6)C_A \right] a \\
&+ \frac{1}{96N_A^{o2}} \left[ N_A^{o2} \left( (6\alpha^2 - 6\alpha - 190)C_A^2 + 80C_A T_F N_f \right) \right. \\
&\quad + N_A^o N_A^d \left( (-42\alpha^2 - 126\alpha - 347)C_A^2 + 160C_A T_F N_f \right) \\
&\quad \left. + N_A^{d2} (12\alpha^2 - 588\alpha + 510)C_A^2 \right] a^2 + O(a^3) \\
\gamma_{c^i}(a) &= \frac{1}{4N_A^o} \left[ N_A^o (-\alpha - 3)C_A + N_A^d (-2\alpha - 6)C_A \right] a \\
&+ \frac{1}{96N_A^{o2}} \left[ N_A^{o2} \left( (-6\alpha^2 - 66\alpha - 190)C_A^2 + 80C_A T_F N_f \right) \right. \\
&\quad + N_A^o N_A^d \left( (-54\alpha^2 - 354\alpha - 323)C_A^2 + 160C_A T_F N_f \right) \\
&\quad \left. + N_A^{d2} (-60\alpha^2 - 372\alpha + 510)C_A^2 \right] a^2 + O(a^3) \\
\gamma_\psi(a) &= \frac{\alpha N_A^o T_F}{N_F} a \\
&+ \frac{1}{4N_F} \left[ (-\alpha^2 + 22\alpha + 23)C_A C_F N_F + (\alpha^2 - 14\alpha + 2)N_A^o C_A T_F \right. \\
&\quad \left. - 6C_F^2 N_F - 8C_F N_f T_F N_F \right] a^2 + O(a^3) . \tag{3.1}
\end{aligned}$$

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<sup>†</sup>Electronic versions of all the MAG renormalization group functions, conversion functions and the  $\overline{\text{MS}}$  amplitudes for each of the three vertices and the vertex associated with its MOMi scheme are available in the attached data file.

Though the three loop results are also available, [30].

Next the full one loop amplitudes for each of the three vertex functions which we have calculated here in  $\overline{\text{MS}}$  are

$$\begin{aligned}
\Sigma_{(1)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} &= \Sigma_{(2)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} = -\frac{1}{2} \Sigma_{(3)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} = -\Sigma_{(4)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \frac{1}{2} \Sigma_{(5)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} = -\Sigma_{(6)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= 1 + \left[ -72\psi'(\frac{1}{3})\alpha^2 C_A N_A^d + 36\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 90\psi'(\frac{1}{3})\alpha C_A N_A^d \right. \\
&\quad - 162\psi'(\frac{1}{3})\alpha C_A N_A^o - 702\psi'(\frac{1}{3})C_A N_A^d + 138\psi'(\frac{1}{3})C_A N_A^o \\
&\quad - 384\psi'(\frac{1}{3})N_f N_A^o T_F - 81\alpha^3 C_A N_A^d + 27\alpha^3 C_A N_A^o + 48\pi^2 \alpha^2 C_A N_A^d \\
&\quad + 810\alpha^2 C_A N_A^d - 24\pi^2 \alpha^2 C_A N_A^o - 405\alpha^2 C_A N_A^o - 60\pi^2 \alpha C_A N_A^d \\
&\quad + 243\alpha C_A N_A^d + 108\pi^2 \alpha C_A N_A^o - 243\alpha C_A N_A^o + 468\pi^2 C_A N_A^d \\
&\quad + 2916 C_A N_A^d - 92\pi^2 C_A N_A^o - 243 C_A N_A^o + 256\pi^2 N_f N_A^o T_F \\
&\quad \left. + 1296 N_f N_A^o T_F \right] \frac{a}{648 N_A^o} + O(a^2) \\
\Sigma_{(7)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} &= 2 \Sigma_{(9)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} = -2 \Sigma_{(11)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} = -\Sigma_{(14)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[ 108\psi'(\frac{1}{3})\alpha^5 C_A N_A^d - 36\psi'(\frac{1}{3})\alpha^5 C_A N_A^o - 324\psi'(\frac{1}{3})\alpha^4 C_A N_A^d \right. \\
&\quad + 162\psi'(\frac{1}{3})\alpha^4 C_A N_A^o + 324\psi'(\frac{1}{3})\alpha^3 C_A N_A^d - 108\psi'(\frac{1}{3})\alpha^3 C_A N_A^o \\
&\quad + 1296\psi'(\frac{1}{3})\alpha^2 C_A N_A^d - 456\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 768\psi'(\frac{1}{3})\alpha^2 N_f N_A^o T_F \\
&\quad + 216\psi'(\frac{1}{3})\alpha C_A N_A^d + 270\psi'(\frac{1}{3})C_A N_A^d - 72\pi^2 \alpha^5 C_A N_A^d - 324\alpha^5 C_A N_A^o \\
&\quad + 24\pi^2 \alpha^5 C_A N_A^o + 108\alpha^5 C_A N_A^o + 216\pi^2 \alpha^4 C_A N_A^d + 810\alpha^4 C_A N_A^d \\
&\quad - 108\pi^2 \alpha^4 C_A N_A^o - 405\alpha^4 C_A N_A^o - 216\pi^2 \alpha^3 C_A N_A^d - 1377\alpha^3 C_A N_A^d \\
&\quad + 72\pi^2 \alpha^3 C_A N_A^o + 1458\alpha^3 C_A N_A^o - 864\pi^2 \alpha^2 C_A N_A^d + 891\alpha^2 C_A N_A^d \\
&\quad + 304\pi^2 \alpha^2 C_A N_A^o - 873\alpha^2 C_A N_A^o - 512\pi^2 \alpha^2 N_f N_A^o T_F - 576\alpha^2 N_f N_A^o T_F \\
&\quad \left. - 144\pi^2 \alpha C_A N_A^d - 243\alpha C_A N_A^d - 180\pi^2 C_A N_A^d + 243 C_A N_A^d \right] \frac{a}{972\alpha^2 N_A^o} \\
&\quad + O(a^2) \\
\Sigma_{(8)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} &= -\Sigma_{(13)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[ -108\psi'(\frac{1}{3})\alpha^5 C_A N_A^d + 36\psi'(\frac{1}{3})\alpha^5 C_A N_A^o + 540\psi'(\frac{1}{3})\alpha^4 C_A N_A^d \right. \\
&\quad - 270\psi'(\frac{1}{3})\alpha^4 C_A N_A^o - 270\psi'(\frac{1}{3})\alpha^3 C_A N_A^d + 378\psi'(\frac{1}{3})\alpha^3 C_A N_A^o \\
&\quad + 1242\psi'(\frac{1}{3})\alpha^2 C_A N_A^d - 390\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 384\psi'(\frac{1}{3})\alpha^2 N_f N_A^o T_F \\
&\quad + 216\psi'(\frac{1}{3})\alpha C_A N_A^d + 270\psi'(\frac{1}{3})C_A N_A^d + 72\pi^2 \alpha^5 C_A N_A^d \\
&\quad + 567\alpha^5 C_A N_A^d - 24\pi^2 \alpha^5 C_A N_A^o - 189\alpha^5 C_A N_A^o - 360\pi^2 \alpha^4 C_A N_A^d \\
&\quad - 2268\alpha^4 C_A N_A^d + 180\pi^2 \alpha^4 C_A N_A^o + 1134\alpha^4 C_A N_A^o + 180\pi^2 \alpha^3 C_A N_A^d \\
&\quad + 648\alpha^3 C_A N_A^d - 252\pi^2 \alpha^3 C_A N_A^o - 243\alpha^3 C_A N_A^o - 828\pi^2 \alpha^2 C_A N_A^d \\
&\quad + 1053\alpha^2 C_A N_A^d + 260\pi^2 \alpha^2 C_A N_A^o - 1206\alpha^2 C_A N_A^o - 256\pi^2 \alpha^2 N_f N_A^o T_F \\
&\quad + 1008\alpha^2 N_f N_A^o T_F - 144\pi^2 \alpha C_A N_A^d - 243\alpha C_A N_A^d - 180\pi^2 C_A N_A^d \\
&\quad \left. + 243 C_A N_A^d \right] \frac{a}{972\alpha^2 N_A^o} + O(a^2) \\
\Sigma_{(10)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} &= -\Sigma_{(12)}^{\text{ggg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[ 216\psi'(\frac{1}{3})\alpha^3 C_A N_A^d - 72\psi'(\frac{1}{3})\alpha^3 C_A N_A^o - 864\psi'(\frac{1}{3})\alpha^2 C_A N_A^d \right.
\end{aligned}$$

$$\begin{aligned}
& + 432\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 594\psi'(\frac{1}{3})\alpha C_A N_A^d - 486\psi'(\frac{1}{3})\alpha C_A N_A^o \\
& + 54\psi'(\frac{1}{3})C_A N_A^d - 66\psi'(\frac{1}{3})C_A N_A^o + 384\psi'(\frac{1}{3})N_f N_A^o T_F \\
& - 144\pi^2 \alpha^3 C_A N_A^d - 891\alpha^3 C_A N_A^d + 48\pi^2 \alpha^3 C_A N_A^o + 297\alpha^3 C_A N_A^o \\
& + 576\pi^2 \alpha^2 C_A N_A^d + 3078\alpha^2 C_A N_A^d - 288\pi^2 \alpha^2 C_A N_A^o - 1539\alpha^2 C_A N_A^o \\
& - 396\pi^2 \alpha C_A N_A^d - 2025\alpha C_A N_A^d + 324\pi^2 \alpha C_A N_A^o + 1701\alpha C_A N_A^o \\
& - 36\pi^2 C_A N_A^d - 162C_A N_A^d + 44\pi^2 C_A N_A^o + 333C_A N_A^o \\
& - 256\pi^2 N_f N_A^o T_F - 1584N_f N_A^o T_F \Big] \frac{a}{972N_A^o} + O(a^2) \tag{3.2}
\end{aligned}$$

for the triple gluon vertex. Those for the other two vertices are

$$\begin{aligned}
\Sigma_{(1)}^{\text{cgg}}(p, q) \Big|_{\overline{\text{MS}}} &= - \Sigma_{(2)}^{\text{cgg}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \frac{1}{2} + \left[ 18\psi'(\frac{1}{3})\alpha N_A^d - 6\psi'(\frac{1}{3})\alpha N_A^o - 33\psi'(\frac{1}{3})N_A^d + 15\psi'(\frac{1}{3})N_A^o - 12\alpha N_A^d \pi^2 \right. \\
&\quad - 27\alpha N_A^d + 4\alpha N_A^o \pi^2 + 27\alpha N_A^o + 22N_A^d \pi^2 + 27N_A^d - 10N_A^o \pi^2 \\
&\quad \left. + 81N_A^o \right] \frac{C_A a}{216N_A^o} + O(a^2) \tag{3.3}
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_{(1)}^{\text{qgq}}(p, q) \Big|_{\overline{\text{MS}}} &= -1 + \left[ 6\psi'(\frac{1}{3})\alpha^2 C_A N_F N_A^d - 3\psi'(\frac{1}{3})\alpha^2 C_A N_F N_A^o - 12\psi'(\frac{1}{3})\alpha C_A N_F N_A^d \right. \\
&\quad + 12\psi'(\frac{1}{3})\alpha C_A N_F N_A^o + 48\psi'(\frac{1}{3})\alpha N_A^{o2} T_F + 30\psi'(\frac{1}{3})C_A N_F N_A^d \\
&\quad + 39\psi'(\frac{1}{3})C_A N_F N_A^o - 24\psi'(\frac{1}{3})C_F N_F N_A^o - 4\pi^2 \alpha^2 C_A N_F N_A^d \\
&\quad - 54\alpha^2 C_A N_F N_A^d + 2\pi^2 \alpha^2 C_A N_F N_A^o + 27\alpha^2 C_A N_F N_A^o \\
&\quad + 8\pi^2 \alpha C_A N_F N_A^d - 8\pi^2 \alpha C_A N_F N_A^o - 32\pi^2 \alpha N_A^{o2} T_F \\
&\quad - 216\alpha N_A^{o2} T_F - 20\pi^2 C_A N_F N_A^d - 162C_A N_F N_A^d - 26\pi^2 C_A N_F N_A^o \\
&\quad \left. - 351C_A N_F N_A^o + 16\pi^2 C_F N_F N_A^o + 216C_F N_F N_A^o \right] \frac{a}{108N_F N_A^o} \\
&\quad + O(a^2)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(2)}^{\text{qgq}}(p, q) \Big|_{\overline{\text{MS}}} &= \Sigma_{(5)}^{\text{qgq}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[ 6\psi'(\frac{1}{3})\alpha^2 C_A N_F N_A^d - 3\psi'(\frac{1}{3})\alpha^2 C_A N_F N_A^o + 24\psi'(\frac{1}{3})\alpha N_A^{o2} T_F \right. \\
&\quad + 6\psi'(\frac{1}{3})C_A N_F N_A^d + 15\psi'(\frac{1}{3})C_A N_F N_A^o - 24\psi'(\frac{1}{3})C_F N_F N_A^o \\
&\quad - 4\pi^2 \alpha^2 C_A N_F N_A^d - 36\alpha^2 C_A N_F N_A^d + 2\pi^2 \alpha^2 C_A N_F N_A^o + 18\alpha^2 C_A N_F N_A^o \\
&\quad - 36\alpha C_A N_F N_A^d + 36\alpha C_A N_F N_A^o - 16\pi^2 \alpha N_A^{o2} T_F - 72\alpha N_A^{o2} T_F \\
&\quad - 4\pi^2 C_A N_F N_A^d + 36C_A N_F N_A^d - 10\pi^2 C_A N_F N_A^o - 126C_A N_F N_A^o \\
&\quad \left. + 16\pi^2 C_F N_F N_A^o + 144C_F N_F N_A^o \right] \frac{a}{54N_F N_A^o} + O(a^2)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(3)}^{\text{qgq}}(p, q) \Big|_{\overline{\text{MS}}} &= \Sigma_{(4)}^{\text{qgq}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[ 6\psi'(\frac{1}{3})\alpha C_A N_F N_A^d - 6\psi'(\frac{1}{3})\alpha C_A N_F N_A^o + 24\psi'(\frac{1}{3})\alpha N_A^{o2} T_F \right. \\
&\quad + 6\psi'(\frac{1}{3})C_A N_F N_A^d + 6\psi'(\frac{1}{3})C_A N_F N_A^o - 18\alpha^2 C_A N_F N_A^d + 9\alpha^2 C_A N_F N_A^o \\
&\quad - 4\pi^2 \alpha C_A N_F N_A^d - 45\alpha C_A N_F N_A^d + 4\pi^2 \alpha C_A N_F N_A^o + 45\alpha C_A N_F N_A^o \\
&\quad - 16\pi^2 \alpha N_A^{o2} T_F - 36\alpha N_A^{o2} T_F - 4\pi^2 C_A N_F N_A^d + 45C_A N_F N_A^d \\
&\quad \left. - 4\pi^2 C_A N_F N_A^o - 90C_A N_F N_A^o + 72C_F N_F N_A^o \right] \frac{a}{54N_F N_A^o} + O(a^2)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(6)}^{\text{qg}}(p, q) \Big|_{\overline{\text{MS}}} &= \left[ -6\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^d + 3\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^o - 12\psi'(\tfrac{1}{3})\alpha C_A N_A^d + 12\psi'(\tfrac{1}{3})\alpha C_A N_A^o \right. \\
&\quad - 6\psi'(\tfrac{1}{3})C_A N_A^d + 33\psi'(\tfrac{1}{3})C_A N_A^o - 24\psi'(\tfrac{1}{3})C_F N_A^o + 4\pi^2\alpha^2 C_A N_A^d \\
&\quad - 2\pi^2\alpha^2 C_A N_A^o + 8\pi^2\alpha C_A N_A^d - 8\pi^2\alpha C_A N_A^o + 4\pi^2 C_A N_A^d - 22\pi^2 C_A N_A^o \\
&\quad \left. + 16\pi^2 C_F N_A^o \right] \frac{a}{54N_A^o} + O(a^2) \tag{3.4}
\end{aligned}$$

where  $\psi(z)$  is the derivative of the logarithm of the Euler  $\Gamma$ -function and  $\zeta(z)$  is the Riemann zeta function. In the course of deriving these we have verified that the one loop  $\overline{\text{MS}}$  anomalous dimensions are in agreement with those found in [29, 30]. There the vertex functions were renormalized at the asymmetric point where the momentum of one of the gluon lines for the triple gluon vertex, and the gluon line in the remaining vertices was nullified. While this is an exceptional momentum configuration it is still possible to extract the  $\overline{\text{MS}}$  renormalization constants. In changing the subtraction point to the symmetric one the same  $\overline{\text{MS}}$  renormalization correctly emerged. It is worth noting that essentially the contributions to the Lorentz channels containing the poles in  $\epsilon$  will form the basis for the MOM renormalization. One minor check on the expressions is that the correct symmetry structure for each vertex emerged. In other words the relations between the various amplitudes for the triple off-diagonal gluon vertex, for instance, are consistent with expectations based on [19]. In defining the basis of Lorentz tensors we made no assumptions about the structure which should be present.

## 4 MOMggg scheme.

Having discussed the structure of the 3-point vertices in the  $\overline{\text{MS}}$  scheme at one loop in detail we can now renormalize in each of the MOM schemes defined by the same vertices. Given that the method and results for each of the MOMggg, MOMh and MOMq schemes are all effectively the same we focus on the former and present the full analytic results of the amplitudes for the vertex defining each scheme. For the other two cases we give condensed versions in the subsequent sections as the full results are in the data file. With the finite parts of the Green's functions being available we define the MOMggg scheme in the MAG in the same way as in QCD, [18], by ensuring that after renormalization there are no  $O(a)$  corrections to the Lorentz channels containing the divergences in  $\epsilon$ . In other words for the first six amplitudes there are no  $O(a)$  parts at the symmetric point but the remaining eight amplitudes can have  $O(a)$  contributions. Given this and the  $\overline{\text{MS}}$  results we find that the mappings of the parameters between the schemes are

$$\begin{aligned}
\alpha_{\text{MOMggg}} &= a + \left[ -72\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^d + 36\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^o + 90\psi'(\tfrac{1}{3})\alpha C_A N_A^d \right. \\
&\quad - 162\psi'(\tfrac{1}{3})\alpha C_A N_A^o - 702\psi'(\tfrac{1}{3})C_A N_A^d + 138\psi'(\tfrac{1}{3})C_A N_A^o - 384\psi'(\tfrac{1}{3})N_f N_A^o T_F \\
&\quad - 81\alpha^3 C_A N_A^d + 27\alpha^3 C_A N_A^o + 48\pi^2\alpha^2 C_A N_A^d + 324\alpha^2 C_A N_A^o \\
&\quad - 24\pi^2\alpha^2 C_A N_A^o - 162\alpha^2 C_A N_A^o - 60\pi^2\alpha C_A N_A^d - 243\alpha C_A N_A^o \\
&\quad + 108\pi^2\alpha C_A N_A^o + 243\alpha C_A N_A^o + 468\pi^2 C_A N_A^d - 92\pi^2 C_A N_A^o \\
&\quad \left. + 2376 C_A N_A^o + 256\pi^2 N_f N_A^o T_F - 864 N_f N_A^o T_F \right] \frac{a^2}{324N_A^o} + O(a^3) \\
\alpha_{\text{MOMggg}} &= \alpha + \left[ 18\alpha^3 C_A N_A^d - 9\alpha^3 C_A N_A^o + 54\alpha^2 C_A N_A^d - 36\alpha^2 C_A N_A^o + 234\alpha C_A N_A^d \right. \\
&\quad \left. - 97\alpha C_A N_A^o + 80\alpha N_f N_A^o T_F + 90 C_A N_A^d \right] \frac{a}{36N_A^o} + O(a^2). \tag{4.1}
\end{aligned}$$

Given the nature of the one loop 2-point functions it transpires that the gauge parameter mapping is the same for all schemes. This is because the effect the scheme choice makes on the

renormalization of the gauge parameter does not occur until two loops. The same comment applies to the conversion functions for the field renormalizations. Therefore, in order to construct the two loop renormalization group functions we need only record the conversion function for the coupling constants. For MOMggg we have

$$\begin{aligned}
C_g^{\text{MOMggg}}(a, \alpha) = & 1 + \left[ 72\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^d - 36\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^o - 90\psi'(\tfrac{1}{3})\alpha C_A N_A^d \right. \\
& + 162\psi'(\tfrac{1}{3})\alpha C_A N_A^o + 702\psi'(\tfrac{1}{3})C_A N_A^d - 138\psi'(\tfrac{1}{3})C_A N_A^o \\
& + 384\psi'(\tfrac{1}{3})N_f N_A^o T_F + 81\alpha^3 C_A N_A^d - 27\alpha^3 C_A N_A^o - 48\alpha^2 C_A N_A^d \pi^2 \\
& - 324\alpha^2 C_A N_A^o + 24\alpha^2 C_A N_A^o \pi^2 + 162\alpha^2 C_A N_A^d + 60\alpha C_A N_A^d \pi^2 \\
& + 243\alpha C_A N_A^o - 108\alpha C_A N_A^o \pi^2 - 243\alpha C_A N_A^d - 468 C_A N_A^d \pi^2 \\
& \left. + 92 C_A N_A^o \pi^2 - 2376 C_A N_A^o - 256 N_f N_A^o \pi^2 T_F + 864 N_f N_A^o T_F \right] \frac{a}{648 N_A^o} \\
& + O(a^2). \tag{4.2}
\end{aligned}$$

For the other conversion functions we do not label them with the scheme but note that like  $C_g^{\text{MOMggg}}(a, \alpha)$  the variables on the left hand side are the  $\overline{\text{MS}}$  ones as is our convention. Thus we have

$$\begin{aligned}
C_A(a, \alpha) = & 1 + \left[ -18\alpha^2 C_A N_A^d + 9\alpha^2 C_A N_A^o - 18\alpha C_A N_A^d + 18\alpha C_A N_A^o - 108 C_A N_A^d + 97 C_A N_A^o \right. \\
& \left. - 80 N_f N_A^o T_F \right] \frac{a}{36 N_A^o} + O(a^2) \\
C_c(a, \alpha) = & 1 + C_A \left[ 2N_A^d + N_A^o \right] \frac{a}{N_A^o} + O(a^2) \\
C_\psi(a, \alpha) = & 1 - \frac{\alpha N_A^o T_F a}{N_F} + O(a^2). \tag{4.3}
\end{aligned}$$

Having determined the conversion functions it is straightforward to apply the renormalization group formalism to construct the two loop MOMggg renormalization group functions. For the  $\beta$ -function we find

$$\begin{aligned}
\beta^{\text{MOMggg}}(a, \alpha) = & [-11C_A + 4N_f T_F] \frac{a^2}{3} \\
& + \left[ 288\psi'(\tfrac{1}{3})\alpha^3 C_A^2 N_A^{d^2} - 72\psi'(\tfrac{1}{3})\alpha^3 C_A^2 N_A^{o^2} + 1548\psi'(\tfrac{1}{3})\alpha^2 C_A^2 N_A^{d^2} \right. \\
& - 1878\psi'(\tfrac{1}{3})\alpha^2 C_A^2 N_A^d N_A^o + 786\psi'(\tfrac{1}{3})\alpha^2 C_A^2 N_A^{o^2} \\
& + 768\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^d N_f N_A^o T_F - 384\psi'(\tfrac{1}{3})\alpha^2 C_A N_f N_A^{o^2} T_F \\
& + 648\psi'(\tfrac{1}{3})\alpha C_A^2 N_A^{d^2} + 1860\psi'(\tfrac{1}{3})\alpha C_A^2 N_A^d N_A^o \\
& - 1404\psi'(\tfrac{1}{3})\alpha C_A^2 N_A^{o^2} - 480\psi'(\tfrac{1}{3})\alpha C_A N_A^d N_f N_A^o T_F \\
& + 864\psi'(\tfrac{1}{3})\alpha C_A N_f N_A^{o^2} T_F - 1080\psi'(\tfrac{1}{3})C_A^2 N_A^{d^2} \\
& + 1944\psi'(\tfrac{1}{3})C_A^2 N_A^d N_A^o + 486\alpha^4 C_A^2 N_A^{d^2} + 81\alpha^4 C_A^2 N_A^d N_A^o \\
& - 81\alpha^4 C_A^2 N_A^{o^2} - 192\pi^2 \alpha^3 C_A^2 N_A^{d^2} + 1620\alpha^3 C_A^2 N_A^{d^2} \\
& - 3078\alpha^3 C_A^2 N_A^d N_A^o + 48\pi^2 \alpha^3 C_A^2 N_A^{o^2} + 1026\alpha^3 C_A^2 N_A^{o^2} \\
& + 1296\alpha^3 C_A N_A^d N_f N_A^o T_F - 432\alpha^3 C_A N_f N_A^{o^2} T_F - 1032\pi^2 \alpha^2 C_A^2 N_A^{d^2} \\
& - 4374\alpha^2 C_A^2 N_A^{d^2} + 1252\pi^2 \alpha^2 C_A^2 N_A^d N_A^o + 8289\alpha^2 C_A^2 N_A^d N_A^o \\
& - 524\pi^2 \alpha^2 C_A^2 N_A^{o^2} - 3051\alpha^2 C_A^2 N_A^{o^2} - 512\pi^2 \alpha^2 C_A N_A^d N_f N_A^o T_F \\
& - 3456\alpha^2 C_A N_A^d N_f N_A^o T_F + 256\pi^2 \alpha^2 C_A N_f N_A^{o^2} T_F + 1728\alpha^2 C_A N_f N_A^{o^2} T_F \\
& \left. - 432\pi^2 \alpha C_A^2 N_A^{d^2} - 4860\alpha C_A^2 N_A^{d^2} - 1240\pi^2 \alpha C_A^2 N_A^d N_A^o \right]
\end{aligned}$$

$$\begin{aligned}
& - 1134\alpha C_A^2 N_A^d N_A^o + 936\pi^2 \alpha C_A^2 N_A^{o2} + 2106\alpha C_A^2 N_A^{o2} \\
& + 320\pi^2 \alpha C_A N_A^d N_f N_A^o T_F + 1296\alpha C_A N_A^d N_f N_A^o T_F - 576\pi^2 \alpha C_A N_f N_A^{o2} T_F \\
& - 1296\alpha C_A N_f N_A^{o2} T_F + 720\pi^2 C_A^2 N_A^{d2} + 2916 C_A^2 N_A^{d2} \\
& - 1296\pi^2 C_A^2 N_A^d N_A^o - 2916 C_A^2 N_A^d N_A^o - 14688 C_A^2 N_A^{o2} \\
& + 8640 C_A N_f N_A^{o2} T_F + 5184 C_F N_f N_A^{o2} T_F \Big] \frac{a^3}{1296 N_A^{o2}} + O(a^4). \quad (4.4)
\end{aligned}$$

The anomalous dimensions are

$$\begin{aligned}
\gamma_A^{\text{MOMggg}}(a, \alpha) = & \left[ - 3\alpha C_A N_A^d + 3\alpha C_A N_A^o + 9 C_A N_A^d - 13 C_A N_A^o + 8 N_f N_A^o T_F \right] \frac{a}{6 N_A^o} \\
& + \left[ - 432\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^{d2} + 648\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^d N_A^o \right. \\
& - 216\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^{o2} + 1836\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^{d2} \\
& - 4032\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^d N_A^o + 1908\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^{o2} \\
& + 1152\psi'(\frac{1}{3})\alpha^2 C_A N_A^d N_f N_A^o T_F - 576\psi'(\frac{1}{3})\alpha^2 C_A N_f N_A^{o2} T_F \\
& - 5832\psi'(\frac{1}{3})\alpha C_A^2 N_A^{d2} + 10296\psi'(\frac{1}{3})\alpha C_A^2 N_A^d N_A^o \\
& - 5040\psi'(\frac{1}{3})\alpha C_A^2 N_A^{o2} - 3744\psi'(\frac{1}{3})\alpha C_A N_A^d N_f N_A^o T_F \\
& + 4896\psi'(\frac{1}{3})\alpha C_A N_f N_A^{o2} T_F + 12636\psi'(\frac{1}{3})C_A^2 N_A^{d2} \\
& - 20736\psi'(\frac{1}{3})C_A^2 N_A^d N_A^o + 3588\psi'(\frac{1}{3})C_A^2 N_A^{o2} \\
& + 18144\psi'(\frac{1}{3})C_A N_A^d N_f N_A^o T_F - 12192\psi'(\frac{1}{3})C_A N_f N_A^{o2} T_F \\
& + 6144\psi'(\frac{1}{3})N_f^2 N_A^{o2} T_F^2 - 486\alpha^4 C_A^2 N_A^{d2} + 648\alpha^4 C_A^2 N_A^d N_A^o \\
& - 162\alpha^4 C_A^2 N_A^{o2} + 288\pi^2 \alpha^3 C_A^2 N_A^{d2} + 6318\alpha^3 C_A^2 N_A^{d2} \\
& - 432\pi^2 \alpha^3 C_A^2 N_A^d N_A^o - 6966\alpha^3 C_A^2 N_A^d N_A^o + 144\pi^2 \alpha^3 C_A^2 N_A^{o2} \\
& + 1674\alpha^3 C_A^2 N_A^{o2} + 1296\alpha^3 C_A N_A^d N_f N_A^o T_F - 432\alpha^3 C_A N_f N_A^{o2} T_F \\
& - 1224\pi^2 \alpha^2 C_A^2 N_A^{d2} + 9477\alpha^2 C_A^2 N_A^{d2} + 2688\pi^2 \alpha^2 C_A^2 N_A^d N_A^o \\
& + 2025\alpha^2 C_A^2 N_A^d N_A^o - 1272\pi^2 \alpha^2 C_A^2 N_A^{o2} - 3078\alpha^2 C_A^2 N_A^{o2} \\
& - 768\pi^2 \alpha^2 C_A N_A^d N_f N_A^o T_F - 2592\alpha^2 C_A N_A^d N_f N_A^o T_F \\
& + 384\pi^2 \alpha^2 C_A N_f N_A^{o2} T_F + 1296\alpha^2 C_A N_f N_A^{o2} T_F + 3888\pi^2 \alpha C_A^2 N_A^{d2} \\
& + 34020\alpha C_A^2 N_A^{d2} - 6864\pi^2 \alpha C_A^2 N_A^d N_A^o - 6048\alpha C_A^2 N_A^d N_A^o \\
& + 3360\pi^2 \alpha C_A^2 N_A^{o2} - 270\alpha C_A^2 N_A^{o2} + 2496\pi^2 \alpha C_A N_A^d N_f N_A^o T_F \\
& + 3024\alpha C_A N_A^d N_f N_A^o T_F - 3264\pi^2 \alpha C_A N_f N_A^{o2} T_F - 3024\alpha C_A N_f N_A^{o2} T_F \\
& - 8424\pi^2 C_A^2 N_A^{d2} + 8019 C_A^2 N_A^{d2} + 13824\pi^2 C_A^2 N_A^d N_A^o \\
& + 16119 C_A^2 N_A^d N_A^o - 2392\pi^2 C_A^2 N_A^{o2} - 5310 C_A^2 N_A^{o2} \\
& - 12096\pi^2 C_A N_A^d N_f N_A^o T_F - 6480 C_A N_A^d N_f N_A^o T_F + 8128\pi^2 C_A N_f N_A^{o2} T_F \\
& + 4608 C_A N_f N_A^{o2} T_F + 15552 C_F N_f N_A^{o2} T_F - 4096\pi^2 N_f^2 N_A^{o2} T_F^2 \\
& \left. + 2304 N_f^2 N_A^{o2} T_F^2 \right] \frac{a^2}{3888 N_A^{o2}} + O(a^3) \quad (4.5)
\end{aligned}$$

$$\begin{aligned}
\gamma_\alpha^{\text{MOMggg}}(a, \alpha) = & \left[ - 6\alpha^2 C_A N_A^d - 3\alpha^2 C_A N_A^o - 36\alpha C_A N_A^d + 26\alpha C_A N_A^o - 16\alpha N_f N_A^o T_F \right. \\
& \left. - 36 C_A N_A^d \right] \frac{a}{12\alpha N_A^o}
\end{aligned}$$

$$\begin{aligned}
& + \left[ -432\psi'(\frac{1}{3})\alpha^4 C_A^2 N_A^{d^2} + 108\psi'(\frac{1}{3})\alpha^4 C_A^2 N_A^{o^2} - 2052\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^{d^2} \right. \\
& \quad + 2466\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^d N_A^o - 1422\psi'(\frac{1}{3})\alpha^3 C_A^2 N_A^{o^2} \\
& \quad - 1152\psi'(\frac{1}{3})\alpha^3 C_A N_A^d N_f N_A^o T_F + 576\psi'(\frac{1}{3})\alpha^3 C_A N_f N_A^{o^2} T_F \\
& \quad - 3564\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^{d^2} - 8154\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^d N_A^o \\
& \quad + 4626\psi'(\frac{1}{3})\alpha^2 C_A^2 N_A^{o^2} - 864\psi'(\frac{1}{3})\alpha^2 C_A N_A^d N_f N_A^o T_F \\
& \quad - 3744\psi'(\frac{1}{3})\alpha^2 C_A N_f N_A^{o^2} T_F - 22032\psi'(\frac{1}{3})\alpha C_A^2 N_A^{d^2} \\
& \quad + 17388\psi'(\frac{1}{3})\alpha C_A^2 N_A^d N_A^o - 3588\psi'(\frac{1}{3})\alpha C_A^2 N_A^{o^2} \\
& \quad - 25056\psi'(\frac{1}{3})\alpha C_A N_A^d N_f N_A^o T_F + 12192\psi'(\frac{1}{3})\alpha C_A N_f N_A^{o^2} T_F \\
& \quad - 6144\psi'(\frac{1}{3})\alpha N_f^2 N_A^{o^2} T_F^2 - 25272\psi'(\frac{1}{3})C_A^2 N_A^{d^2} \\
& \quad + 4968\psi'(\frac{1}{3})C_A^2 N_A^d N_A^o - 13824\psi'(\frac{1}{3})C_A N_A^d N_f N_A^o T_F - 486\alpha^5 C_A^2 N_A^{d^2} \\
& \quad - 81\alpha^5 C_A^2 N_A^d N_A^o + 81\alpha^5 C_A^2 N_A^{o^2} + 288\pi^2 \alpha^4 C_A^2 N_A^{d^2} - 1944\alpha^4 C_A^2 N_A^{d^2} \\
& \quad + 3078\alpha^4 C_A^2 N_A^d N_A^o - 72\pi^2 \alpha^4 C_A^2 N_A^{o^2} - 945\alpha^4 C_A^2 N_A^{o^2} \\
& \quad - 1296\alpha^4 C_A N_A^d N_f N_A^o T_F + 432\alpha^4 C_A N_f N_A^{o^2} T_F + 1368\pi^2 \alpha^3 C_A^2 N_A^{d^2} \\
& \quad - 6804\alpha^3 C_A^2 N_A^{d^2} - 1644\pi^2 \alpha^3 C_A^2 N_A^d N_A^o - 7614\alpha^3 C_A^2 N_A^d N_A^o \\
& \quad + 948\pi^2 \alpha^3 C_A^2 N_A^{o^2} + 4050\alpha^3 C_A^2 N_A^{o^2} + 768\pi^2 \alpha^3 C_A N_A^d N_f N_A^o T_F \\
& \quad + 2592\alpha^3 C_A N_A^d N_f N_A^o T_F - 384\pi^2 \alpha^3 C_A N_f N_A^{o^2} T_F \\
& \quad - 1296\alpha^3 C_A N_f N_A^{o^2} T_F + 2376\pi^2 \alpha^2 C_A^2 N_A^{d^2} - 49086\alpha^2 C_A^2 N_A^{d^2} \\
& \quad + 5436\pi^2 \alpha^2 C_A^2 N_A^d N_A^o + 8775\alpha^2 C_A^2 N_A^d N_A^o - 3084\pi^2 \alpha^2 C_A^2 N_A^{o^2} \\
& \quad - 108\alpha^2 C_A^2 N_A^{o^2} + 576\pi^2 \alpha^2 C_A N_A^d N_f N_A^o T_F - 4752\alpha^2 C_A N_A^d N_f N_A^o T_F \\
& \quad + 2496\pi^2 \alpha^2 C_A N_f N_A^{o^2} T_F + 3456\alpha^2 C_A N_f N_A^{o^2} T_F + 14688\pi^2 \alpha C_A^2 N_A^{d^2} \\
& \quad - 10206\alpha C_A^2 N_A^{d^2} - 11592\pi^2 \alpha C_A^2 N_A^d N_A^o - 22599\alpha C_A^2 N_A^d N_A^o \\
& \quad + 2392\pi^2 \alpha C_A^2 N_A^{o^2} + 5310\alpha C_A^2 N_A^{o^2} + 16704\pi^2 \alpha C_A N_A^d N_f N_A^o T_F \\
& \quad + 15552\alpha C_A N_A^d N_f N_A^o T_F - 8128\pi^2 \alpha C_A N_f N_A^{o^2} T_F \\
& \quad - 4608\alpha C_A N_f N_A^{o^2} T_F - 15552\alpha C_F N_f N_A^{o^2} T_F + 4096\pi^2 \alpha N_f^2 N_A^{o^2} T_F^2 \\
& \quad - 2304\alpha N_f^2 N_A^{o^2} T_F^2 + 16848\pi^2 C_A^2 N_A^{d^2} + 116640 C_A^2 N_A^{d^2} \\
& \quad - 3312\pi^2 C_A^2 N_A^d N_A^o - 14904 C_A^2 N_A^d N_A^o + 9216\pi^2 C_A N_A^d N_f N_A^o T_F \\
& \quad \left. + 10368 C_A N_A^d N_f N_A^o T_F \right] \frac{a^2}{3888\alpha N_A^{o^2}} + O(a^3) \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{MOMggg}}(a, \alpha) & = \left[ -2\alpha N_A^d + \alpha N_A^o - 6N_A^d - 3N_A^o \right] \frac{C_A a}{4N_A^o} \\
& + \left[ -288\psi'(\frac{1}{3})\alpha^3 C_A N_A^{d^2} + 288\psi'(\frac{1}{3})\alpha^3 C_A N_A^d N_A^o - 72\psi'(\frac{1}{3})\alpha^3 C_A N_A^{o^2} \right. \\
& \quad - 504\psi'(\frac{1}{3})\alpha^2 C_A N_A^{d^2} - 828\psi'(\frac{1}{3})\alpha^2 C_A N_A^d N_A^o + 540\psi'(\frac{1}{3})\alpha^2 C_A N_A^{o^2} \\
& \quad - 1728\psi'(\frac{1}{3})\alpha C_A N_A^{d^2} + 552\psi'(\frac{1}{3})\alpha C_A N_A^d N_A^o - 1248\psi'(\frac{1}{3})\alpha C_A N_A^{o^2} \\
& \quad - 1536\psi'(\frac{1}{3})\alpha N_A^d N_f N_A^o T_F + 768\psi'(\frac{1}{3})\alpha N_f N_A^{o^2} T_F - 8424\psi'(\frac{1}{3})C_A N_A^{d^2} \\
& \quad - 2556\psi'(\frac{1}{3})C_A N_A^d N_A^o + 828\psi'(\frac{1}{3})C_A N_A^{o^2} - 4608\psi'(\frac{1}{3})N_A^d N_f N_A^o T_F \\
& \quad - 2304\psi'(\frac{1}{3})N_f N_A^{o^2} T_F - 324\alpha^4 C_A N_A^{d^2} + 270\alpha^4 C_A N_A^d N_A^o \\
& \quad \left. - 54\alpha^4 C_A N_A^{o^2} + 192\pi^2 \alpha^3 C_A N_A^{d^2} + 972\alpha^3 C_A N_A^{d^2} \right]
\end{aligned}$$



$$\begin{aligned}
& -192\pi^2\alpha^3 C_A N_A^d N_A^o - 2106\alpha^3 C_A N_A^d N_A^o + 48\pi^2\alpha^3 C_A N_A^{o2} \\
& + 648\alpha^3 C_A N_A^{o2} + 336\pi^2\alpha^2 C_A N_A^{d2} + 5184\alpha^2 C_A N_A^{d2} \\
& + 552\pi^2\alpha^2 C_A N_A^d N_A^o - 1944\alpha^2 C_A N_A^d N_A^o - 360\pi^2\alpha^2 C_A N_A^{o2} \\
& - 648\alpha^2 C_A N_A^{o2} + 1152\pi^2\alpha C_A N_A^{d2} - 10368\alpha C_A N_A^{d2} \\
& - 368\pi^2\alpha C_A N_A^d N_A^o - 144\alpha C_A N_A^d N_A^o + 832\pi^2\alpha C_A N_A^{o2} \\
& - 1710\alpha C_A N_A^{o2} + 1024\pi^2\alpha N_A^d N_f N_A^o T_F - 576\alpha N_A^d N_f N_A^o T_F \\
& - 512\pi^2\alpha N_f N_A^{o2} T_F + 288\alpha N_f N_A^{o2} T_F + 5616\pi^2 C_A N_A^{d2} + 17010 C_A N_A^{d2} \\
& + 1704\pi^2 C_A N_A^d N_A^o - 1485 C_A N_A^d N_A^o - 552\pi^2 C_A N_A^{o2} - 378 C_A N_A^{o2} \\
& + 3072\pi^2 N_A^d N_f N_A^o T_F + 864 N_A^d N_f N_A^o T_F + 1536\pi^2 N_f N_A^{o2} T_F \\
& + 432 N_f N_A^{o2} T_F \Big] \frac{C_A a^2}{2592 N_A^2} + O(a^3) \tag{4.7}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\psi^{\text{MOMggg}}(a, \alpha) &= \frac{\alpha N_A^o T_F a}{N_F} \\
&+ \left[ 72\psi'(\frac{1}{3})\alpha^3 C_A C_F N_F - 108\psi'(\frac{1}{3})\alpha^3 C_A N_A^o T_F - 90\psi'(\frac{1}{3})\alpha^2 C_A C_F N_F \right. \\
&\quad + 252\psi'(\frac{1}{3})\alpha^2 C_A N_A^o T_F + 702\psi'(\frac{1}{3})\alpha C_A C_F N_F - 840\psi'(\frac{1}{3})\alpha C_A N_A^o T_F \\
&\quad + 384\psi'(\frac{1}{3})\alpha N_f N_A^o T_F^2 + 81\alpha^4 C_A C_F N_F - 108\alpha^4 C_A N_A^o T_F \\
&\quad - 48\pi^2\alpha^3 C_A C_F N_F - 486\alpha^3 C_A C_F N_F + 72\pi^2\alpha^3 C_A N_A^o T_F \\
&\quad + 729\alpha^3 C_A N_A^o T_F + 60\pi^2\alpha^2 C_A C_F N_F - 162\alpha^2 C_A C_F N_F \\
&\quad - 168\pi^2\alpha^2 C_A N_A^o T_F + 324\alpha^2 C_A N_A^o T_F - 468\pi^2\alpha C_A C_F N_F \\
&\quad + 648\alpha C_A C_F N_F + 560\pi^2\alpha C_A N_A^o T_F - 1017\alpha C_A N_A^o T_F \\
&\quad - 256\pi^2\alpha N_f N_A^o T_F^2 + 144\alpha N_f N_A^o T_F^2 + 2025 C_A C_F N_F \\
&\quad \left. - 486 C_F^2 N_F - 648 C_F N_F N_f T_F \right] \frac{a^2}{324 N_F} + O(a^3) . \tag{4.8}
\end{aligned}$$

The explicit forms of the associated amplitudes are

$$\begin{aligned}
\Sigma_{(1)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} &= \Sigma_{(2)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} = -\frac{1}{2} \Sigma_{(3)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= -\Sigma_{(4)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} = \frac{1}{2} \Sigma_{(5)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= -\Sigma_{(6)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} = 1 + O(a^2) \\
\Sigma_{(7)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} &= 2 \Sigma_{(9)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} = -2 \Sigma_{(11)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= -\Sigma_{(14)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= \left[ 108\psi'(\frac{1}{3})\alpha^5 C_A N_A^d - 36\psi'(\frac{1}{3})\alpha^5 C_A N_A^o - 324\psi'(\frac{1}{3})\alpha^4 C_A N_A^d \right. \\
&\quad + 162\psi'(\frac{1}{3})\alpha^4 C_A N_A^o + 324\psi'(\frac{1}{3})\alpha^3 C_A N_A^d - 108\psi'(\frac{1}{3})\alpha^3 C_A N_A^o \\
&\quad + 1296\psi'(\frac{1}{3})\alpha^2 C_A N_A^d - 456\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 768\psi'(\frac{1}{3})\alpha^2 N_f N_A^o T_F \\
&\quad + 216\psi'(\frac{1}{3})\alpha C_A N_A^d + 270\psi'(\frac{1}{3})\alpha C_A N_A^o - 72\pi^2\alpha^5 C_A N_A^d - 324\alpha^5 C_A N_A^d \\
&\quad + 24\pi^2\alpha^5 C_A N_A^o + 108\alpha^5 C_A N_A^o + 216\pi^2\alpha^4 C_A N_A^d + 810\alpha^4 C_A N_A^d \\
&\quad - 108\pi^2\alpha^4 C_A N_A^o - 405\alpha^4 C_A N_A^o - 216\pi^2\alpha^3 C_A N_A^d - 1377\alpha^3 C_A N_A^d \\
&\quad \left. + 72\pi^2\alpha^3 C_A N_A^o + 1458\alpha^3 C_A N_A^o - 864\pi^2\alpha^2 C_A N_A^d + 891\alpha^2 C_A N_A^d \right]
\end{aligned}$$

$$\begin{aligned}
& + 304\pi^2\alpha^2 C_A N_A^o - 873\alpha^2 C_A N_A^o - 512\pi^2\alpha^2 N_f N_A^o T_F - 576\alpha^2 N_f N_A^o T_F \\
& - 144\pi^2\alpha C_A N_A^d - 243\alpha C_A N_A^d - 180\pi^2 C_A N_A^d + 243 C_A N_A^d \Big] \frac{a}{972\alpha^2 N_A^o} \\
& + O(a^2) \\
\Sigma_{(8)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} &= - \Sigma_{(13)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= \left[ -108\psi'(\frac{1}{3})\alpha^5 C_A N_A^d + 36\psi'(\frac{1}{3})\alpha^5 C_A N_A^o + 540\psi'(\frac{1}{3})\alpha^4 C_A N_A^d \right. \\
&\quad - 270\psi'(\frac{1}{3})\alpha^4 C_A N_A^o - 270\psi'(\frac{1}{3})\alpha^3 C_A N_A^d + 378\psi'(\frac{1}{3})\alpha^3 C_A N_A^o \\
&\quad + 1242\psi'(\frac{1}{3})\alpha^2 C_A N_A^d - 390\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 384\psi'(\frac{1}{3})\alpha^2 N_f N_A^o T_F \\
&\quad + 216\psi'(\frac{1}{3})\alpha C_A N_A^d + 270\psi'(\frac{1}{3}) C_A N_A^d + 72\pi^2\alpha^5 C_A N_A^d \\
&\quad + 567\alpha^5 C_A N_A^d - 24\pi^2\alpha^5 C_A N_A^o - 189\alpha^5 C_A N_A^o - 360\pi^2\alpha^4 C_A N_A^d \\
&\quad - 2268\alpha^4 C_A N_A^d + 180\pi^2\alpha^4 C_A N_A^o + 1134\alpha^4 C_A N_A^o + 180\pi^2\alpha^3 C_A N_A^d \\
&\quad + 648\alpha^3 C_A N_A^d - 252\pi^2\alpha^3 C_A N_A^o - 243\alpha^3 C_A N_A^o \\
&\quad - 828\pi^2\alpha^2 C_A N_A^d + 1053\alpha^2 C_A N_A^d + 260\pi^2\alpha^2 C_A N_A^o - 1206\alpha^2 C_A N_A^o \\
&\quad - 256\pi^2\alpha^2 N_f N_A^o T_F + 1008\alpha^2 N_f N_A^o T_F - 144\pi^2\alpha C_A N_A^d - 243\alpha C_A N_A^d \\
&\quad \left. - 180\pi^2 C_A N_A^d + 243 C_A N_A^d \right] \frac{a}{972\alpha^2 N_A^o} + O(a^2) \\
\Sigma_{(10)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} &= - \Sigma_{(12)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= \left[ 216\psi'(\frac{1}{3})\alpha^3 C_A N_A^d - 72\psi'(\frac{1}{3})\alpha^3 C_A N_A^o - 864\psi'(\frac{1}{3})\alpha^2 C_A N_A^d \right. \\
&\quad + 432\psi'(\frac{1}{3})\alpha^2 C_A N_A^o + 594\psi'(\frac{1}{3})\alpha C_A N_A^d - 486\psi'(\frac{1}{3})\alpha C_A N_A^o \\
&\quad + 54\psi'(\frac{1}{3}) C_A N_A^d - 66\psi'(\frac{1}{3}) C_A N_A^o + 384\psi'(\frac{1}{3}) N_f N_A^o T_F \\
&\quad - 144\pi^2\alpha^3 C_A N_A^d - 891\alpha^3 C_A N_A^d + 48\pi^2\alpha^3 C_A N_A^o + 297\alpha^3 C_A N_A^o \\
&\quad + 576\pi^2\alpha^2 C_A N_A^d + 3078\alpha^2 C_A N_A^d - 288\pi^2\alpha^2 C_A N_A^o - 1539\alpha^2 C_A N_A^o \\
&\quad - 396\pi^2\alpha C_A N_A^d - 2025\alpha C_A N_A^d + 324\pi^2\alpha C_A N_A^o + 1701\alpha C_A N_A^o \\
&\quad - 36\pi^2 C_A N_A^d - 162 C_A N_A^d + 44\pi^2 C_A N_A^o + 333 C_A N_A^o \\
&\quad \left. - 256\pi^2 N_f N_A^o T_F - 1584 N_f N_A^o T_F \right] \frac{a}{972 N_A^o} + O(a^2). \tag{4.9}
\end{aligned}$$

Again we observe that the same symmetries emerge as in the  $\overline{\text{MS}}$  case which is a minor check on the computation.

## 5 MOMh scheme.

Having recorded the results for the triple gluon vertex at length we briefly present the results for the ghost-gluon vertex in numerical form in order to save space. The full analytic expressions are given in the attached data file. Given the nature of the MOMh scheme the amplitudes are effectively trivial since

$$\Sigma_{(1)}^{\text{ccg}}(p, q) \Big|_{\text{MOMh}} = - \Sigma_{(2)}^{\text{ccg}}(p, q) \Big|_{\text{MOMh}} = \frac{1}{2} + O(a^2). \tag{5.1}$$

This is because of the symmetry of the original ghost-gluon vertex and the definition of the MOMh scheme. With the coupling constant conversion function for  $SU(3)$

$$C_g^{\text{MOMh}}(a, \alpha) = 1 + \left[ -0.125000\alpha^2 - \alpha + 0.555556N_f - 10.432318 \right] a + O(a^2) \tag{5.2}$$

we can deduce the two loop MOMh anomalous dimensions. They are

$$\begin{aligned}
\beta^{\text{MOMh}}(a, \alpha) &= [0.666667N_f - 11.000000] a^2 \\
&\quad + \left[ -0.625000\alpha^3 - 0.333333\alpha^2 N_f - 0.750000\alpha^2 - 1.333333\alpha N_f \right. \\
&\quad \left. + 5.500000\alpha + 12.666667N_f - 108.000000 \right] a^3 + O(a^4) \\
\gamma_A^{\text{MOMh}}(a, \alpha) &= [\alpha + 0.666667N_f - 5.000000] a \\
&\quad + \left[ -0.625000\alpha^3 - 0.333333\alpha^2 N_f + 0.125000\alpha^2 - 1.333333\alpha N_f \right. \\
&\quad \left. + 7.093698\alpha + 5.479132N_f - 3.468488 \right] a^2 + O(a^3) \\
\gamma_\alpha^{\text{MOMh}}(a, \alpha) &= \left[ -1.250000\alpha^2 - 0.666667\alpha N_f + 3.500000\alpha - 3.000000 \right] \frac{a}{\alpha} \\
&\quad + \left[ 0.625000\alpha^4 + 0.333333\alpha^3 N_f - 0.250000\alpha^3 + 1.333333\alpha^2 N_f - 7.023372\alpha^2 \right. \\
&\quad \left. - 3.979132\alpha N_f - 2.297058\alpha + 2.000000N_f + 15.093907 \right] \frac{a^2}{\alpha} + O(a^3) \\
\gamma_c^{\text{MOMh}}(a, \alpha) &= [0.250000\alpha - 3.750000] a \\
&\quad + \left[ 0.187500\alpha^2 - 7.945326\alpha + 1.250000N_f - 0.726366 \right] a^2 + O(a^3) \\
\gamma_\psi^{\text{MOMh}}(a, \alpha) &= \alpha a + \left[ 0.500000\alpha^2 - 0.281302\alpha - 1.333333N_f + 22.333333 \right] a^2 \\
&\quad + O(a^3)
\end{aligned} \tag{5.3}$$

which agree with the explicit direct two loop computation.

## 6 MOMq scheme.

For the MOMq scheme we also give the results in numerical form for  $SU(3)$ . Though the non-channel 1 amplitudes are non-trivial here since

$$\begin{aligned}
\Sigma_{(1)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} &= -1.000000 + O(a^2) \\
\Sigma_{(2)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} &= \Sigma_{(5)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} = \left[ 0.138008\alpha^2 + 1.562605\alpha - 1.540716 \right] a + O(a^2) \\
\Sigma_{(3)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} &= \Sigma_{(4)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} = \left[ 0.166667\alpha^2 + 1.781302\alpha - 0.826284 \right] a + O(a^2) \\
\Sigma_{(6)}^{\text{qqg}}(p, q) \Big|_{\text{MOMq}} &= \left[ 0.195326\alpha^2 + 1.562605\alpha + 3.971621 \right] a + O(a^2).
\end{aligned} \tag{6.1}$$

The associated coupling constant conversion function is

$$C_g^{\text{MOMq}}(a, \alpha) = 1 + \left[ 0.027337\alpha^2 + 0.843907\alpha + 0.555556N_f - 7.381259 \right] a + O(a^2) \tag{6.2}$$

from which we deduce that the two loop  $SU(3)$  renormalization group functions are

$$\begin{aligned}
\beta^{\text{MOMq}}(a, \alpha) &= [0.666667N_f - 11.000000] a^2 \\
&\quad + \left[ 0.136686\alpha^3 + 0.072899\alpha^2 N_f + 1.727047\alpha^2 + 1.125210\alpha N_f \right. \\
&\quad \left. - 5.579304\alpha + 12.666667N_f - 96.936557 \right] a^3 + O(a^4) \\
\gamma_A^{\text{MOMq}}(a, \alpha) &= [\alpha + 0.666667N_f - 5.000000] a \\
&\quad + \left[ -0.320326\alpha^3 - 0.130217\alpha^2 N_f + 2.289442\alpha^2 + 1.125210\alpha N_f \right. \\
&\quad \left. - 5.243258\alpha + 9.547210N_f - 33.979074 \right] a^2 + O(a^3)
\end{aligned}$$

$$\begin{aligned}
\gamma_\alpha^{\text{MOMq}}(a, \alpha) &= \left[ -1.250000\alpha^2 - 0.666667\alpha N_f + 3.500000\alpha - 3.000000 \right] \frac{a}{\alpha} \\
&\quad + \left[ 0.244157\alpha^4 + 0.130217\alpha^3 N_f - 3.793408\alpha^3 - 1.125210\alpha^2 N_f - 2.657691\alpha^2 \right. \\
&\quad \left. - 8.047210\alpha N_f + 7.996908\alpha + 2.000000 N_f - 3.212444 \right] \frac{a^2}{\alpha} + O(a^3) \\
\gamma_c^{\text{MOMq}}(a, \alpha) &= [0.250000\alpha - 3.750000] a \\
&\quad + \left[ 0.076169\alpha^3 - 0.033075\alpha^2 - 20.249101\alpha + 1.250000 N_f \right. \\
&\quad \left. - 23.609305 \right] a^2 + O(a^3) \\
\gamma_\psi^{\text{MOMq}}(a, \alpha) &= \alpha a + \left[ 0.304674\alpha^3 + 4.187814\alpha^2 + 5.820815\alpha - 1.333333 N_f + 22.333333 \right] a^2 \\
&\quad + O(a^3). \tag{6.3}
\end{aligned}$$

Unlike MOMh the quark anomalous dimension is cubic in the gauge parameter.

## 7 Curci-Ferrari gauge.

One interesting property of the maximal abelian gauge is that in the formal limit  $N_A^d/N_A^o \rightarrow 0$  the Lagrangian becomes equivalent to gauge fixing QCD in the nonlinear Curci-Ferrari gauge, [31]. More specifically the Lagrangian for the choice of the Curci-Ferrari gauge is, [31],

$$\begin{aligned}
L^{\text{CF}} &= -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 - \bar{c}^a \partial^\mu D_\mu c^a + i\bar{\psi}^i \not{D} \psi^i \\
&\quad + \frac{g}{2} f^{abc} \partial^\mu A_\mu^a \bar{c}^b c^c + \frac{\alpha g^2}{8} f^{eab} f^{ecd} \bar{c}^a c^b \bar{c}^c c^d. \tag{7.1}
\end{aligned}$$

Here the colour indices have the formal range  $1 \leq a \leq N_A$  where  $N_A$  is the dimension of the adjoint representation of the colour group and  $\alpha$  is the associated gauge parameter. This gauge choice differs from the usual linear covariant gauge fixed Lagrangian in that there is a quartic ghost vertex and the ghost-gluon vertex is different. The former vertex does not exclude renormalizability which can be seen using simple power counting with the proof given in [20, 22, 23, 24, 26, 29] which provides the relations between the various renormalization constants. The gauge parameter differs also from that of the linear gauge fixing but when  $\alpha = 0$  then (7.1) reduces to the Landau gauge Lagrangian, [31]. It is straightforward to deduce that the Curci-Ferrari gauge is a particular limit of (2.5) by examining that Lagrangian and omitting any interaction with a centre field. In some respects given the non-renormalization of certain aspects of those fields the centre could be regarded as analogous to a background field in the context of the background field gauge, [56, 57, 58, 59, 60, 61, 62, 63].

Given the close relation between this gauge and the MAG, we have renormalized (7.1) in the three MOM schemes defined by the 3-point vertices. We did this directly and independently of the MAG and its limit to the Curci-Ferrari gauge in order to have an independent check on our computations. In other words we verify that the limit of the MAG renormalization group functions and amplitudes agree when  $N_A^d/N_A^o \rightarrow 0$ . Therefore, for completeness we record the direct results for the Curci-Ferrari gauge in the three MOM schemes. First, we recall that the relevant two loop  $\overline{\text{MS}}$  renormalization group functions are, [64, 65],

$$\begin{aligned}
\gamma_A(a) &= [(3\alpha - 13)C_A + 8T_F N_f] \frac{a}{6} \\
&\quad + \left[ (\alpha^2 + 11\alpha - 59) C_A^2 + 40C_A T_F N_f + 32C_F T_F N_f \right] \frac{a^2}{8} + O(a^3)
\end{aligned}$$

$$\begin{aligned}
\gamma_\alpha(a) &= - [(3\alpha - 26)C_A + 16T_F N_f] \frac{a}{12} \\
&\quad - \left[ (\alpha^2 + 17\alpha - 118) C_A^2 + 80C_A T_F N_f + 64C_F T_F N_f \right] \frac{a^2}{16} + O(a^3) \\
\gamma_c(a) &= (\alpha - 3)C_A \frac{a}{4} + \left[ (3\alpha^2 - 3\alpha - 95) C_A^2 + 40C_A T_F N_f \right] \frac{a^2}{48} + O(a^3) \\
\gamma_\psi(a) &= \frac{\alpha C_F}{4} a + \frac{1}{4} \left[ (8\alpha + 25)C_A C_F - 6C_F^2 - 8C_F T_F N_f \right] a^2 + O(a^3). \tag{7.2}
\end{aligned}$$

For the MOM results we first record the mappings for the parameters at one loop for each scheme. The three coupling constant mappings are

$$\begin{aligned}
a_{\text{MOMggg}} &= a + \left[ 36\psi'(\frac{1}{3})\alpha^2 C_A - 162\psi'(\frac{1}{3})\alpha C_A + 138\psi'(\frac{1}{3})C_A - 384\psi'(\frac{1}{3})N_f T_F \right. \\
&\quad \left. + 27\alpha^3 C_A - 24\pi^2 \alpha^2 C_A - 162\alpha^2 C_A + 108\pi^2 \alpha C_A + 243\alpha C_A \right. \\
&\quad \left. - 92\pi^2 C_A + 2376C_A + 256\pi^2 N_f T_F - 864N_f T_F \right] \frac{a^2}{324} + O(a^3) \\
a_{\text{MOMh}} &= a + \left[ -12\psi'(\frac{1}{3})\alpha C_A + 30\psi'(\frac{1}{3})C_A + 27\alpha^2 C_A + 8\pi^2 \alpha C_A \right. \\
&\quad \left. + 108\alpha C_A - 20\pi^2 C_A + 669C_A - 240N_f T_F \right] \frac{a^2}{108} + O(a^3) \\
a_{\text{MOMq}} &= a + \left[ 6\psi'(\frac{1}{3})\alpha^2 C_A - 24\psi'(\frac{1}{3})\alpha C_A - 96\psi'(\frac{1}{3})\alpha C_F - 78\psi'(\frac{1}{3})C_A \right. \\
&\quad \left. + 48\psi'(\frac{1}{3})C_F - 4\pi^2 \alpha^2 C_A - 27\alpha^2 C_A + 16\pi^2 \alpha C_A + 54\alpha C_A \right. \\
&\quad \left. + 64\pi^2 \alpha C_F + 216\alpha C_F + 52\pi^2 C_A + 993C_A - 32\pi^2 C_F - 432C_F \right. \\
&\quad \left. - 240N_f T_F \right] \frac{a^2}{108} + O(a^3). \tag{7.3}
\end{aligned}$$

Subsequently we can deduce that the coupling constant conversion functions in each scheme are

$$\begin{aligned}
C_g^{\text{MOMggg}}(a, \alpha) &= 1 + \left[ -36\psi'(\frac{1}{3})\alpha^2 C_A + 162\psi'(\frac{1}{3})\alpha C_A - 138\psi'(\frac{1}{3})C_A + 384\psi'(\frac{1}{3})N_f T_F \right. \\
&\quad \left. - 27\alpha^3 C_A + 24\pi^2 \alpha^2 C_A + 162\alpha^2 C_A - 108\pi^2 \alpha C_A - 243\alpha C_A \right. \\
&\quad \left. + 92\pi^2 C_A - 2376C_A - 256\pi^2 N_f T_F + 864N_f T_F \right] \frac{a}{648} + O(a^2) \\
C_g^{\text{MOMh}}(a, \alpha) &= 1 + \left[ 12\psi'(\frac{1}{3})\alpha C_A - 30\psi'(\frac{1}{3})C_A - 27\alpha^2 C_A - 8\pi^2 \alpha C_A \right. \\
&\quad \left. - 108\alpha C_A + 20\pi^2 C_A - 669C_A + 240N_f T_F \right] \frac{a}{216} + O(a^2) \\
C_g^{\text{MOMq}}(a, \alpha) &= 1 + \left[ -6\psi'(\frac{1}{3})\alpha^2 C_A + 24\psi'(\frac{1}{3})\alpha C_A + 96\psi'(\frac{1}{3})\alpha C_F + 78\psi'(\frac{1}{3})C_A \right. \\
&\quad \left. - 48\psi'(\frac{1}{3})C_F + 4\pi^2 \alpha^2 C_A + 27\alpha^2 C_A - 16\pi^2 \alpha C_A - 54\alpha C_A \right. \\
&\quad \left. - 64\pi^2 \alpha C_F - 216\alpha C_F - 52\pi^2 C_A - 993C_A + 32\pi^2 C_F + 432C_F \right. \\
&\quad \left. + 240N_f T_F \right] \frac{a}{216} + O(a^2). \tag{7.4}
\end{aligned}$$

At one loop the gauge parameter mapping is the same in each scheme, similar to the MAG, and thus we have

$$\alpha_{\text{MOMi}} = \alpha + \left[ -9\alpha^2 C_A - 36\alpha C_A - 97C_A + 80N_f T_F \right] \frac{\alpha a}{36} + O(a^2). \tag{7.5}$$

Equally the conversion functions for the wave function renormalization constants are the same in each scheme and are

$$\begin{aligned}
C_A(a, \alpha) &= 1 + \left[ 9\alpha^2 C_A + 18\alpha C_A + 97C_A - 80N_f T_F \right] \frac{a}{36} + O(a^2) \\
C_c(a, \alpha) &= 1 + C_A a + O(a^2) \\
C_\psi(a, \alpha) &= 1 - \alpha C_F a + O(a^2). \tag{7.6}
\end{aligned}$$

As before we have checked that the scheme independent one loop parts of the renormalization group functions correctly emerge in our direct evaluation. Equipped with these and the one loop conversion functions which derive from the finite parts of the Green's functions we find the following results for the renormalization group functions. First, for the MOMggg scheme we have

$$\begin{aligned}
\beta^{\text{MOMggg}}(a, \alpha) &= [-11C_A + 4N_f T_F] \frac{a^2}{3} \\
&+ \left[ -72\psi'(\frac{1}{3})\alpha^3 C_A^2 + 786\psi'(\frac{1}{3})\alpha^2 C_A^2 - 384\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F \right. \\
&\quad - 1404\psi'(\frac{1}{3})\alpha C_A^2 + 864\psi'(\frac{1}{3})\alpha C_A N_f T_F - 81\alpha^4 C_A^2 + 48\pi^2 \alpha^3 C_A^2 \\
&\quad + 1026\alpha^3 C_A^2 - 432\alpha^3 C_A N_f T_F - 524\pi^2 \alpha^2 C_A^2 - 3051\alpha^2 C_A^2 \\
&\quad + 256\pi^2 \alpha^2 C_A N_f T_F + 1728\alpha^2 C_A N_f T_F + 936\pi^2 \alpha C_A^2 + 2106\alpha C_A^2 \\
&\quad - 576\pi^2 \alpha C_A N_f T_F - 1296\alpha C_A N_f T_F - 14688C_A^2 + 8640C_A N_f T_F \\
&\quad \left. + 5184C_F N_f T_F \right] \frac{a^3}{1296} + O(a^4) \\
\gamma_A^{\text{MOMggg}}(a, \alpha) &= [3\alpha C_A - 13C_A + 8N_f T_F] \frac{a}{6} \\
&+ \left[ -108\psi'(\frac{1}{3})\alpha^3 C_A^2 + 954\psi'(\frac{1}{3})\alpha^2 C_A^2 - 288\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F \right. \\
&\quad - 2520\psi'(\frac{1}{3})\alpha C_A^2 + 2448\psi'(\frac{1}{3})\alpha C_A N_f T_F + 1794\psi'(\frac{1}{3})C_A^2 \\
&\quad - 6096\psi'(\frac{1}{3})C_A N_f T_F + 3072\psi'(\frac{1}{3})N_f^2 T_F^2 - 81\alpha^4 C_A^2 + 72\pi^2 \alpha^3 C_A^2 \\
&\quad + 837\alpha^3 C_A^2 - 216\alpha^3 C_A N_f T_F - 636\pi^2 \alpha^2 C_A^2 - 1539\alpha^2 C_A^2 \\
&\quad + 192\pi^2 \alpha^2 C_A N_f T_F + 648\alpha^2 C_A N_f T_F + 1680\pi^2 \alpha C_A^2 - 135\alpha C_A^2 \\
&\quad - 1632\pi^2 \alpha C_A N_f T_F - 1512\alpha C_A N_f T_F - 1196\pi^2 C_A^2 - 2655C_A^2 \\
&\quad + 4064\pi^2 C_A N_f T_F + 2304C_A N_f T_F + 7776C_F N_f T_F - 2048\pi^2 N_f^2 T_F^2 \\
&\quad \left. + 1152N_f^2 T_F^2 \right] \frac{a^2}{1944} + O(a^3) \\
\gamma_\alpha^{\text{MOMggg}}(a, \alpha) &= [-3\alpha C_A + 26C_A - 16N_f T_F] \frac{a}{12} \\
&+ \left[ 108\psi'(\frac{1}{3})\alpha^3 C_A^2 - 1422\psi'(\frac{1}{3})\alpha^2 C_A^2 + 576\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F \right. \\
&\quad + 4626\psi'(\frac{1}{3})\alpha C_A^2 - 3744\psi'(\frac{1}{3})\alpha C_A N_f T_F - 3588\psi'(\frac{1}{3})C_A^2 \\
&\quad + 12192\psi'(\frac{1}{3})C_A N_f T_F - 6144\psi'(\frac{1}{3})N_f^2 T_F^2 + 81\alpha^4 C_A^2 - 72\pi^2 \alpha^3 C_A^2 \\
&\quad - 945\alpha^3 C_A^2 + 432\alpha^3 C_A N_f T_F + 948\pi^2 \alpha^2 C_A^2 + 4050\alpha^2 C_A^2 \\
&\quad - 384\pi^2 \alpha^2 C_A N_f T_F - 1296\alpha^2 C_A N_f T_F - 3084\pi^2 \alpha C_A^2 - 108\alpha C_A^2 \\
&\quad + 2496\pi^2 \alpha C_A N_f T_F + 3456\alpha C_A N_f T_F + 2392\pi^2 C_A^2 + 5310C_A^2 \\
&\quad - 8128\pi^2 C_A N_f T_F - 4608C_A N_f T_F - 15552C_F N_f T_F + 4096\pi^2 N_f^2 T_F^2 \\
&\quad \left. - 2304N_f^2 T_F^2 \right] \frac{a^2}{3888} + O(a^3) \\
\gamma_c^{\text{MOMggg}}(a, \alpha) &= [\alpha - 3] \frac{C_A a}{4} \\
&+ \left[ -36\psi'(\frac{1}{3})\alpha^3 C_A + 270\psi'(\frac{1}{3})\alpha^2 C_A - 624\psi'(\frac{1}{3})\alpha C_A + 384\psi'(\frac{1}{3})\alpha N_f T_F \right. \\
&\quad + 414\psi'(\frac{1}{3})C_A - 1152\psi'(\frac{1}{3})N_f T_F - 27\alpha^4 C_A + 24\pi^2 \alpha^3 C_A + 324\alpha^3 C_A \\
&\quad - 180\pi^2 \alpha^2 C_A - 324\alpha^2 C_A + 416\pi^2 \alpha C_A - 855\alpha C_A - 256\pi^2 \alpha N_f T_F \\
&\quad \left. + 144\alpha N_f T_F - 276\pi^2 C_A - 189C_A + 768\pi^2 N_f T_F + 216N_f T_F \right] \frac{C_A a^2}{1296}
\end{aligned}$$

$$\begin{aligned}
& + O(a^3) \\
\gamma_\psi^{\text{MOMggg}}(a, \alpha) &= \alpha C_F a \\
& + \left[ -36\psi'(\tfrac{1}{3})\alpha^3 C_A + 162\psi'(\tfrac{1}{3})\alpha^2 C_A - 138\psi'(\tfrac{1}{3})\alpha C_A + 384\psi'(\tfrac{1}{3})\alpha N_f T_F \right. \\
& \quad - 27\alpha^4 C_A + 24\pi^2 \alpha^3 C_A + 243\alpha^3 C_A - 108\pi^2 \alpha^2 C_A + 162\alpha^2 C_A \\
& \quad + 92\pi^2 \alpha C_A - 369\alpha C_A - 256\pi^2 \alpha N_f T_F + 144\alpha N_f T_F + 2025 C_A \\
& \quad \left. - 486 C_F - 648 N_f T_F \right] \frac{C_F a^2}{324} + O(a^3). \tag{7.7}
\end{aligned}$$

The results for the scheme based on the ghost vertex are similar since

$$\begin{aligned}
\beta^{\text{MOMh}}(a, \alpha) &= [-11C_A + 4N_f T_F] \frac{a^2}{3} \\
& + \left[ 18\psi'(\tfrac{1}{3})\alpha^2 C_A^2 - 156\psi'(\tfrac{1}{3})\alpha C_A^2 + 96\psi'(\tfrac{1}{3})\alpha C_A N_f T_F - 81\alpha^3 C_A^2 \right. \\
& \quad - 12\pi^2 \alpha^2 C_A^2 + 540\alpha^2 C_A^2 - 432\alpha^2 C_A N_f T_F + 104\pi^2 \alpha C_A^2 + 1404\alpha C_A^2 \\
& \quad - 64\pi^2 \alpha C_A N_f T_F - 864\alpha C_A N_f T_F - 7344 C_A^2 + 4320 C_A N_f T_F \\
& \quad \left. + 2592 C_F N_f T_F \right] \frac{a^3}{648} + O(a^4) \\
\gamma_A^{\text{MOMh}}(a, \alpha) &= [3\alpha C_A - 13C_A + 8N_f T_F] \frac{a}{6} \\
& + \left[ 36\psi'(\tfrac{1}{3})\alpha^2 C_A^2 - 246\psi'(\tfrac{1}{3})\alpha C_A^2 + 96\psi'(\tfrac{1}{3})\alpha C_A N_f T_F + 390\psi'(\tfrac{1}{3})C_A^2 \right. \\
& \quad - 240\psi'(\tfrac{1}{3})C_A N_f T_F - 81\alpha^3 C_A^2 - 24\pi^2 \alpha^2 C_A^2 + 459\alpha^2 C_A^2 - 432\alpha^2 C_A N_f T_F \\
& \quad + 164\pi^2 \alpha C_A^2 + 675\alpha C_A^2 - 64\pi^2 \alpha C_A N_f T_F - 864\alpha C_A N_f T_F - 260\pi^2 C_A^2 \\
& \quad - 2484 C_A^2 + 160\pi^2 C_A N_f T_F + 2376 C_A N_f T_F \\
& \quad \left. + 2592 C_F N_f T_F \right] \frac{a^2}{648} + O(a^3) \\
\gamma_\alpha^{\text{MOMh}}(a, \alpha) &= [-3\alpha C_A + 26C_A - 16N_f T_F] \frac{a}{12} \\
& + \left[ -36\psi'(\tfrac{1}{3})\alpha^2 C_A^2 + 402\psi'(\tfrac{1}{3})\alpha C_A^2 - 192\psi'(\tfrac{1}{3})\alpha C_A N_f T_F - 780\psi'(\tfrac{1}{3})C_A^2 \right. \\
& \quad + 480\psi'(\tfrac{1}{3})C_A N_f T_F + 162\alpha^3 C_A^2 + 24\pi^2 \alpha^2 C_A^2 - 675\alpha^2 C_A^2 \\
& \quad + 864\alpha^2 C_A N_f T_F - 268\pi^2 \alpha C_A^2 - 1107\alpha C_A^2 + 128\pi^2 \alpha C_A N_f T_F \\
& \quad + 1728\alpha C_A N_f T_F + 520\pi^2 C_A^2 + 4968 C_A^2 - 320\pi^2 C_A N_f T_F \\
& \quad \left. - 4752 C_A N_f T_F - 5184 C_F N_f T_F \right] \frac{a^2}{1296} + O(a^3) \\
\gamma_c^{\text{MOMh}}(a, \alpha) &= [\alpha - 3] \frac{C_A a}{4} \\
& + \left[ 12\psi'(\tfrac{1}{3})\alpha^2 C_A - 66\psi'(\tfrac{1}{3})\alpha C_A + 90\psi'(\tfrac{1}{3})C_A - 8\pi^2 \alpha^2 C_A + 108\alpha^2 C_A \right. \\
& \quad \left. + 44\pi^2 \alpha C_A - 81\alpha C_A - 60\pi^2 C_A - 432 C_A + 216 N_f T_F \right] \frac{C_A a^2}{432} + O(a^3) \\
\gamma_\psi^{\text{MOMh}}(a, \alpha) &= \alpha C_F a \\
& + \left[ 12\psi'(\tfrac{1}{3})\alpha^2 C_A - 30\psi'(\tfrac{1}{3})\alpha C_A - 8\pi^2 \alpha^2 C_A + 27\alpha^2 C_A + 20\pi^2 \alpha C_A \right. \\
& \quad \left. + 675 C_A - 162 C_F - 216 N_f T_F \right] \frac{C_F a^2}{108} + O(a^3). \tag{7.8}
\end{aligned}$$

Finally for the MOMq scheme the results are more involved since

$$\beta^{\text{MOMq}}(a, \alpha) = [-11C_A + 4N_f T_F] \frac{a^2}{3}$$

$$\begin{aligned}
& + \left[ -18\psi'(\frac{1}{3})\alpha^3 C_A^2 + 192\psi'(\frac{1}{3})\alpha^2 C_A^2 + 144\psi'(\frac{1}{3})\alpha^2 C_A C_F \right. \\
& \quad - 96\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F - 312\psi'(\frac{1}{3})\alpha C_A^2 - 1248\psi'(\frac{1}{3})\alpha C_A C_F \\
& \quad + 192\psi'(\frac{1}{3})\alpha C_A N_f T_F + 768\psi'(\frac{1}{3})\alpha C_F N_f T_F + 12\pi^2 \alpha^3 C_A^2 + 81\alpha^3 C_A^2 \\
& \quad - 128\pi^2 \alpha^2 C_A^2 - 783\alpha^2 C_A^2 - 96\pi^2 \alpha^2 C_A C_F - 324\alpha^2 C_A C_F \\
& \quad + 64\pi^2 \alpha^2 C_A N_f T_F + 432\alpha^2 C_A N_f T_F + 208\pi^2 \alpha C_A^2 + 702\alpha C_A^2 \\
& \quad + 832\pi^2 \alpha C_A C_F + 2808\alpha C_A C_F - 128\pi^2 \alpha C_A N_f T_F - 432\alpha C_A N_f T_F \\
& \quad - 512\pi^2 \alpha C_F N_f T_F - 1728\alpha C_F N_f T_F - 7344C_A^2 + 4320C_A N_f T_F \\
& \quad \left. + 2592C_F N_f T_F \right] \frac{a^3}{648} + O(a^4) \\
\gamma_A^{\text{MOMq}}(a, \alpha) & = [3\alpha C_A - 13C_A + 8N_f T_F] \frac{a}{6} \\
& + \left[ -18\psi'(\frac{1}{3})\alpha^3 C_A^2 + 150\psi'(\frac{1}{3})\alpha^2 C_A^2 + 288\psi'(\frac{1}{3})\alpha^2 C_A C_F \right. \\
& \quad - 48\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F - 78\psi'(\frac{1}{3})\alpha C_A^2 - 1392\psi'(\frac{1}{3})\alpha C_A C_F \\
& \quad + 192\psi'(\frac{1}{3})\alpha C_A N_f T_F + 768\psi'(\frac{1}{3})\alpha C_F N_f T_F - 1014\psi'(\frac{1}{3})C_A^2 \\
& \quad + 624\psi'(\frac{1}{3})C_A C_F + 624\psi'(\frac{1}{3})C_A N_f T_F - 384\psi'(\frac{1}{3})C_F N_f T_F \\
& \quad + 12\pi^2 \alpha^3 C_A^2 + 81\alpha^3 C_A^2 - 100\pi^2 \alpha^2 C_A^2 - 81\alpha^2 C_A^2 - 192\pi^2 \alpha^2 C_A C_F \\
& \quad - 648\alpha^2 C_A C_F + 32\pi^2 \alpha^2 C_A N_f T_F + 52\pi^2 \alpha C_A^2 - 999\alpha C_A^2 + 928\pi^2 \alpha C_A C_F \\
& \quad + 4104\alpha C_A C_F - 128\pi^2 \alpha C_A N_f T_F - 432\alpha C_A N_f T_F - 512\pi^2 \alpha C_F N_f T_F \\
& \quad - 1728\alpha C_F N_f T_F + 676\pi^2 C_A^2 + 1728C_A^2 - 416\pi^2 C_A C_F - 5616C_A C_F \\
& \quad \left. - 416\pi^2 C_A N_f T_F - 216C_A N_f T_F + 256\pi^2 C_F N_f T_F + 6048C_F N_f T_F \right] \frac{a^2}{648} \\
& + O(a^3) \\
\gamma_\alpha^{\text{MOMq}}(a, \alpha) & = [-3\alpha C_A + 26C_A - 16N_f T_F] \frac{a}{12} \\
& + \left[ 18\psi'(\frac{1}{3})\alpha^3 C_A^2 - 228\psi'(\frac{1}{3})\alpha^2 C_A^2 - 288\psi'(\frac{1}{3})\alpha^2 C_A C_F + 96\psi'(\frac{1}{3})\alpha^2 C_A N_f T_F \right. \\
& \quad + 390\psi'(\frac{1}{3})\alpha C_A^2 + 2640\psi'(\frac{1}{3})\alpha C_A C_F - 384\psi'(\frac{1}{3})\alpha C_A N_f T_F \\
& \quad - 1536\psi'(\frac{1}{3})\alpha C_F N_f T_F + 2028\psi'(\frac{1}{3})C_A^2 - 1248\psi'(\frac{1}{3})C_A C_F \\
& \quad - 1248\psi'(\frac{1}{3})C_A N_f T_F + 768\psi'(\frac{1}{3})C_F N_f T_F - 12\pi^2 \alpha^3 C_A^2 + 152\pi^2 \alpha^2 C_A^2 \\
& \quad + 567\alpha^2 C_A^2 + 192\pi^2 \alpha^2 C_A C_F + 648\alpha^2 C_A C_F - 64\pi^2 \alpha^2 C_A N_f T_F \\
& \quad - 260\pi^2 \alpha C_A^2 + 1269\alpha C_A^2 - 1760\pi^2 \alpha C_A C_F - 6912\alpha C_A C_F \\
& \quad + 256\pi^2 \alpha C_A N_f T_F + 864\alpha C_A N_f T_F + 1024\pi^2 \alpha C_F N_f T_F + 3456\alpha C_F N_f T_F \\
& \quad - 1352\pi^2 C_A^2 - 3456C_A^2 + 832\pi^2 C_A C_F + 11232C_A C_F \\
& \quad \left. + 832\pi^2 C_A N_f T_F + 432C_A N_f T_F - 512\pi^2 C_F N_f T_F - 12096C_F N_f T_F \right] \frac{a^2}{1296} \\
& + O(a^3) \\
\gamma_c^{\text{MOMq}}(a, \alpha) & = [\alpha - 3] \frac{C_A a}{4} \\
& + \left[ -6\psi'(\frac{1}{3})\alpha^3 C_A + 42\psi'(\frac{1}{3})\alpha^2 C_A + 96\psi'(\frac{1}{3})\alpha^2 C_F + 6\psi'(\frac{1}{3})\alpha C_A \right. \\
& \quad - 336\psi'(\frac{1}{3})\alpha C_F - 234\psi'(\frac{1}{3})C_A + 144\psi'(\frac{1}{3})C_F + 4\pi^2 \alpha^3 C_A + 54\alpha^3 C_A \\
& \quad - 28\pi^2 \alpha^2 C_A - 64\pi^2 \alpha^2 C_F - 216\alpha^2 C_F - 4\pi^2 \alpha C_A - 567\alpha C_A + 224\pi^2 \alpha C_F \\
& \quad \left. + 1080\alpha C_F + 156\pi^2 C_A + 540C_A - 96\pi^2 C_F - 1296C_F + 216N_f T_F \right] \frac{a^2}{432} \\
& + O(a^3)
\end{aligned}$$



$$\begin{aligned}
\gamma_\psi^{\text{MOMq}}(a, \alpha) &= \alpha C_F a \\
&+ \left[ -6\psi'(\tfrac{1}{3})\alpha^3 C_A + 24\psi'(\tfrac{1}{3})\alpha^2 C_A + 96\psi'(\tfrac{1}{3})\alpha^2 C_F + 78\psi'(\tfrac{1}{3})\alpha C_A \right. \\
&\quad - 48\psi'(\tfrac{1}{3})\alpha C_F + 4\pi^2\alpha^3 C_A + 54\alpha^3 C_A - 16\pi^2\alpha^2 C_A + 81\alpha^2 C_A \\
&\quad - 64\pi^2\alpha^2 C_F - 216\alpha^2 C_F - 52\pi^2\alpha C_A - 324\alpha C_A + 32\pi^2\alpha C_F \\
&\quad \left. + 432\alpha C_F + 675C_A - 162C_F - 216N_f T_F \right] \frac{C_F a^2}{108} + O(a^3). \tag{7.9}
\end{aligned}$$

We have concentrated on the renormalization group functions for the Curci-Ferrari gauge. The explicit form of the various amplitudes can be deduced from the MAG expressions given in the data file in the  $N_A^d/N_A^o \rightarrow 0$  limit. We have checked that these agree with the direct evaluation performed in the Curci-Ferrari gauge itself.

## 8 $\Lambda$ parameters.

Having provided all the one loop structure for the MAG and the Curci-Ferrari gauge for the MOM schemes we now briefly discuss the relation between the  $\Lambda$  parameters in the MOM schemes to those in the  $\overline{\text{MS}}$  scheme. This parameter sets the fundamental scale in QCD and corresponds to the boundary between infrared and ultraviolet physics. However, its actual value depends on the renormalization scheme one is considering. Though one remarkable feature of this non-perturbative quantity is that the ratio between  $\Lambda$  parameters in different schemes can be determined exactly from a *one* loop computation. In [18] those relations for the various MOM schemes were determined and we repeat that analysis here for the MAG and Curci-Ferrari gauges. In essence the ratio of parameters reflects the first term of the coupling constant conversion function. First, we define

$$\frac{\Lambda^{\text{MOMi}}}{\Lambda^{\overline{\text{MS}}}} = \exp \left[ \frac{\lambda^{\text{MOMi}}(\alpha, N_f)}{b_0} \right] \tag{8.1}$$

where

$$b_0 = \frac{22}{3}C_A - \frac{8}{3}T_F N_f \tag{8.2}$$

originates from the one loop  $\beta$ -function. Then for each of the three MOM schemes in the MAG we have

$$\begin{aligned}
\lambda^{\text{MOMggg}}(\alpha, N_f) &= \frac{1}{324N_A^o} \left[ -72\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^d + 36\psi'(\tfrac{1}{3})\alpha^2 C_A N_A^o + 90\psi'(\tfrac{1}{3})\alpha C_A N_A^d \right. \\
&\quad - 162\psi'(\tfrac{1}{3})\alpha C_A N_A^o - 702\psi'(\tfrac{1}{3})C_A N_A^d + 138\psi'(\tfrac{1}{3})C_A N_A^o \\
&\quad - 384\psi'(\tfrac{1}{3})N_f N_A^o T_F - 81\alpha^3 C_A N_A^d + 27\alpha^3 C_A N_A^o + 48\pi^2\alpha^2 C_A N_A^d \\
&\quad + 324\alpha^2 C_A N_A^d - 24\pi^2\alpha^2 C_A N_A^o - 162\alpha^2 C_A N_A^o - 60\pi^2\alpha C_A N_A^d \\
&\quad - 243\alpha C_A N_A^d + 108\pi^2\alpha C_A N_A^o + 243\alpha C_A N_A^o + 468\pi^2 C_A N_A^d \\
&\quad \left. - 92\pi^2 C_A N_A^o + 2376C_A N_A^o + 256\pi^2 N_f N_A^o T_F - 864N_f N_A^o T_F \right] \\
\lambda^{\text{MOMh}}(\alpha, N_f) &= \frac{1}{108N_A^o} \left[ 36\psi'(\tfrac{1}{3})\alpha C_A N_A^d - 12\psi'(\tfrac{1}{3})\alpha C_A N_A^o - 66\psi'(\tfrac{1}{3})C_A N_A^d \right. \\
&\quad + 30\psi'(\tfrac{1}{3})C_A N_A^o - 54\alpha^2 C_A N_A^d + 27\alpha^2 C_A N_A^o - 24\pi^2\alpha C_A N_A^d \\
&\quad - 108\alpha C_A N_A^d + 8\pi^2\alpha C_A N_A^o + 108\alpha C_A N_A^o + 44\pi^2 C_A N_A^d \\
&\quad \left. + 162C_A N_A^d - 20\pi^2 C_A N_A^o + 669C_A N_A^o - 240N_f N_A^o T_F \right]
\end{aligned}$$

$$\begin{aligned}
\lambda^{\text{MOMq}}(\alpha, N_f) = & \frac{1}{108N_F N_A^\circ} \left[ -12\psi'(\tfrac{1}{3})\alpha^2 C_A N_F N_A^d + 6\psi'(\tfrac{1}{3})\alpha^2 C_A N_F N_A^\circ \right. \\
& + 24\psi'(\tfrac{1}{3})\alpha C_A N_F N_A^d - 24\psi'(\tfrac{1}{3})\alpha C_A N_F N_A^\circ - 96\psi'(\tfrac{1}{3})\alpha N_A^{\circ 2} T_F \\
& - 60\psi'(\tfrac{1}{3})C_A N_F N_A^d - 78\psi'(\tfrac{1}{3})C_A N_F N_A^\circ + 48\psi'(\tfrac{1}{3})C_F N_F N_A^\circ \\
& + 8\pi^2\alpha^2 C_A N_F N_A^d + 54\alpha^2 C_A N_F N_A^d - 4\pi^2\alpha^2 C_A N_F N_A^\circ \\
& - 27\alpha^2 C_A N_F N_A^\circ - 16\pi^2\alpha C_A N_F N_A^d - 54\alpha C_A N_F N_A^d \\
& + 16\pi^2\alpha C_A N_F N_A^\circ + 54\alpha C_A N_F N_A^\circ + 64\pi^2\alpha N_A^{\circ 2} T_F \\
& + 216\alpha N_A^{\circ 2} T_F + 40\pi^2 C_A N_F N_A^d + 52\pi^2 C_A N_F N_A^\circ \\
& + 993C_A N_F N_A^\circ - 32\pi^2 C_F N_F N_A^\circ - 432C_F N_F N_A^\circ \\
& \left. - 240N_F N_f N_A^\circ T_F \right] . \tag{8.3}
\end{aligned}$$

While these are the explicit results it is perhaps more instructive to compare with the Landau gauge results of [18]. Therefore we have provided the values for the same choice of  $N_f$  and  $\alpha$  given in [18] for each scheme for  $SU(3)$  in Table 1. Though it is important to note that our  $\alpha$  is not the same parameter as in [18] and also the distinction between the  $\overline{\text{MS}}$  and  $\overline{\text{MS}}$  results of [18]. Interestingly for certain choices of  $\alpha$  and  $N_f$  the ratio is less than unity.

$\alpha$	$N_f$	MOMggg	MOMh	MOMq
0	0	2.3583	2.5816	1.9562
0	1	2.1127	2.6008	1.9359
0	2	1.8642	2.6228	1.9129
0	3	1.6167	2.6484	1.8869
0	4	1.3668	2.6784	1.8572
0	5	1.1239	2.7140	1.8229
1	0	2.0664	2.8596	1.8073
1	3	1.3739	3.0010	1.7128
1	4	1.1480	3.0655	1.6729
1	5	0.9298	3.1429	1.6271
3	3	0.9591	4.1883	1.3858
3	4	0.7787	4.3939	1.3308
-2	4	1.8624	2.2372	2.2445

Table 1. Values of  $\lambda^{\text{MOMi}}(\alpha, N_f)$  for the MAG in  $SU(3)$ .

We have repeated the analysis for the Curci-Ferrari gauge and the parallel results are presented in Table 2. Those for the MOMggg and MOMq schemes are equivalent to those of the linear covariant gauge fixing of [18]. This is because the coupling constant mapping is the same for both cases despite the fact that the ghost-gluon vertex is different. This does not affect the one loop vertices since the differences cancel out. However, this is not the case for the MOMh scheme since the quartic ghost vertex contributes to the mapping for all  $\alpha$  and in the Landau gauge case the differences in the ghost-gluon vertex are significant. However, the same increase and decrease of the ratio with  $\alpha$  and  $N_f$  is parallel to that for the standard linear covariant gauge fixing results of [18].

$\alpha$	$N_f$	MOMggg	MOMh	MOMq
0	0	3.3341	2.6588	2.1379
0	1	3.0543	2.6837	2.1277
0	2	2.7644	2.7123	2.1163
0	3	2.4654	2.7456	2.1032
0	4	2.1587	2.7846	2.0881
0	5	1.8471	2.8312	2.0706
1	0	2.8957	2.9893	1.9075
1	3	2.0751	3.1684	1.8296
1	4	1.7921	3.2505	1.7964
1	5	1.5088	3.3496	1.7581
3	3	1.8392	5.4177	1.3110
3	4	1.5732	5.8018	1.2533
-2	4	2.5437	2.6772	2.6597

Table 2. Values of  $\lambda^{\text{MOMi}}(\alpha, N_f)$  for the Curci-Ferrari gauge in  $SU(3)$ .

## 9 Discussion.

We make some comments on our analysis. First, we have provided all the information on the 3-point vertex functions relevant for the definition of the MOM schemes for the maximal abelian gauge. This is an analysis parallel to that of [18] for QCD fixed in the canonical linear covariant gauge. One motivation was to provide this data in relation to future lattice analyses of the vertex functions in the infrared in order to have precision matching at high energy. Moreover, the explicit values of the amplitudes will be useful for assisting overlap with Schwinger-Dyson studies. Several features which were observed in [30] are again present. One is the relation to the Curci-Ferrari gauge in that results from the latter can be derived from the MAG in the replica-like limit where the centre of the group is formally excluded. However, we have verified that the results in this limit are consistent with the direct calculation in the Curci-Ferrari gauge itself. Given properties of the renormalization group equation the one loop conversion functions for relating parameters in the MOM schemes to those of the  $\overline{\text{MS}}$  scheme have allowed us to compute the *two* loop renormalization group functions in each of the three MOM schemes. These have direct parallels with those of [30] since they are based on the triple gluon, ghost-gluon and quark-gluon vertices. Though an essential difference here is that with the split nature of the colour group in the MAG, it is the vertices with the off-diagonal gluons which are relevant. This is due in part to the fact that there are Slavnov-Taylor identities which ensure that the structure with vertices with centre gluons are predetermined. Indeed this is not unrelated to the fact these gluons are similar to the background fields of the background field gauge of [56, 57, 58, 59, 60, 61] with the off-diagonal gluons corresponding to the quantum fluctuations. Whether this scenario is significant in the picture of abelian monopoles underlying a picture of colour confinement would be interesting to investigate.

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## A Tensor basis.

In this appendix we record for completeness the tensor basis for each of the three 3-point vertices, using the same notation as [19]. For the triple gluon vertex we use the original tensor basis with the basis tensors

$$\begin{aligned}
\mathcal{P}_{(1)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\mu\nu}p_\sigma, & \mathcal{P}_{(2)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\nu\sigma}p_\mu, & \mathcal{P}_{(3)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\sigma\mu}p_\nu \\
\mathcal{P}_{(4)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\mu\nu}q_\sigma, & \mathcal{P}_{(5)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\nu\sigma}q_\mu, & \mathcal{P}_{(6)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\sigma\mu}q_\nu \\
\mathcal{P}_{(7)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}p_\mu p_\nu p_\sigma, & \mathcal{P}_{(8)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}p_\mu p_\nu q_\sigma, & \mathcal{P}_{(9)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}p_\mu q_\nu p_\sigma \\
\mathcal{P}_{(10)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu p_\nu p_\sigma, & \mathcal{P}_{(11)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}p_\mu q_\nu q_\sigma, & \mathcal{P}_{(12)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu p_\nu q_\sigma \\
\mathcal{P}_{(13)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu q_\nu p_\sigma, & \mathcal{P}_{(14)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu q_\nu q_\sigma. & & 
\end{aligned} \tag{A.1}$$

For the associated projection matrix we partition it into submatrices for ease of presentation. With the general form

$$\mathcal{M}^{\text{ggg}} = -\frac{1}{27(d-2)} \begin{pmatrix} \mathcal{M}_{11}^{\text{ggg}} & \mathcal{M}_{12}^{\text{ggg}} & \mathcal{M}_{13}^{\text{ggg}} \\ \mathcal{M}_{21}^{\text{ggg}} & \mathcal{M}_{22}^{\text{ggg}} & \mathcal{M}_{23}^{\text{ggg}} \\ \mathcal{M}_{31}^{\text{ggg}} & \mathcal{M}_{32}^{\text{ggg}} & \mathcal{M}_{33}^{\text{ggg}} \end{pmatrix}$$

then each of the submatrices are

$$\begin{aligned}
\mathcal{M}_{11}^{\text{ggg}} &= \begin{pmatrix} 36 & 0 & 0 & 18 & 0 & 0 \\ 0 & 36 & 0 & 0 & 18 & 0 \\ 0 & 0 & 36 & 0 & 0 & 18 \\ 18 & 0 & 0 & 36 & 0 & 0 \\ 0 & 18 & 0 & 0 & 36 & 0 \\ 0 & 0 & 18 & 0 & 0 & 36 \end{pmatrix}, & \mathcal{M}_{12}^{\text{ggg}} &= \begin{pmatrix} 48 & 24 & 24 & 24 \\ 48 & 24 & 24 & 24 \\ 48 & 24 & 24 & 24 \\ 24 & 48 & 12 & 12 \\ 24 & 12 & 12 & 48 \\ 24 & 12 & 48 & 12 \end{pmatrix} \\
\mathcal{M}_{13}^{\text{ggg}} &= \begin{pmatrix} 12 & 12 & 48 & 24 \\ 48 & 12 & 12 & 24 \\ 12 & 48 & 12 & 24 \\ 24 & 24 & 24 & 48 \\ 24 & 24 & 24 & 48 \\ 24 & 24 & 24 & 48 \end{pmatrix}, & \mathcal{M}_{21}^{\text{ggg}} &= \begin{pmatrix} 48 & 48 & 48 & 24 & 24 & 24 \\ 24 & 24 & 24 & 48 & 12 & 12 \\ 24 & 24 & 24 & 12 & 12 & 48 \\ 24 & 24 & 24 & 12 & 48 & 12 \end{pmatrix} \\
\mathcal{M}_{22}^{\text{ggg}} &= \begin{pmatrix} 64(d+1) & 32(d+1) & 32(d+1) & 32(d+1) \\ 32(d+1) & 32(2d-1) & 16(d+1) & 16(d+1) \\ 32(d+1) & 16(d+1) & 32(2d-1) & 16(d+1) \\ 32(d+1) & 16(d+1) & 16(d+1) & 32(2d-1) \end{pmatrix} \\
\mathcal{M}_{23}^{\text{ggg}} &= \begin{pmatrix} 16(d+4) & 16(d+4) & 16(d+4) & 8(d+10) \\ 8(4d+1) & 8(4d+1) & 8(d+4) & 16(d+4) \\ 8(4d+1) & 8(d+4) & 8(4d+1) & 16(d+4) \\ 8(d+4) & 8(4d+1) & 8(4d+1) & 16(d+4) \end{pmatrix} \\
\mathcal{M}_{31}^{\text{ggg}} &= \begin{pmatrix} 12 & 48 & 12 & 24 & 24 & 24 \\ 12 & 12 & 48 & 24 & 24 & 24 \\ 48 & 12 & 12 & 24 & 24 & 24 \\ 24 & 24 & 24 & 48 & 48 & 48 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{32}^{\text{ggg}} &= \begin{pmatrix} 16(d+4) & 8(4d+1) & 8(4d+1) & 8(d+4) \\ 16(d+4) & 8(4d+1) & 8(d+4) & 8(4d+1) \\ 16(d+4) & 8(d+4) & 8(4d+1) & 8(4d+1) \\ 8(d+10) & 16(d+4) & 16(d+4) & 16(d+4) \end{pmatrix} \\
\mathcal{M}_{33}^{\text{ggg}} &= \begin{pmatrix} 32(2d-1) & 16(d+1) & 16(d+1) & 32(d+1) \\ 16(d+1) & 32(2d-1) & 16(d+1) & 32(d+1) \\ 16(d+1) & 16(d+1) & 32(2d-1) & 32(d+1) \\ 32(d+1) & 32(d+1) & 32(d+1) & 64(d+1) \end{pmatrix}. \tag{A.2}
\end{aligned}$$

The situation for the remaining vertices is simple as the basis of each involve fewer tensors. For the ghost-gluon vertex we have

$$\mathcal{P}_{(1)\sigma}^{\text{ccg}}(p, q) = p_\sigma \quad , \quad \mathcal{P}_{(2)\sigma}^{\text{ccg}}(p, q) = q_\sigma \tag{A.3}$$

where

$$\mathcal{M}^{\text{ccg}} = -\frac{1}{3} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \tag{A.4}$$

is the projection matrix. Finally, the quark-gluon vertex basis is

$$\begin{aligned}
\mathcal{P}_{(1)\sigma}^{\text{qqg}}(p, q) &= \gamma_\sigma \quad , \quad \mathcal{P}_{(2)\sigma}^{\text{qqg}}(p, q) = \frac{p_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}_{(3)\sigma}^{\text{qqg}}(p, q) = \frac{p_\sigma \not{q}}{\mu^2} \quad , \\
\mathcal{P}_{(4)\sigma}^{\text{qqg}}(p, q) &= \frac{q_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}_{(5)\sigma}^{\text{qqg}}(p, q) = \frac{q_\sigma \not{q}}{\mu^2} \quad , \quad \mathcal{P}_{(6)\sigma}^{\text{qqg}}(p, q) = \frac{1}{\mu^2} \Gamma_{(3)\sigma pq} \quad . \tag{A.5}
\end{aligned}$$

which leads to the projection matrix

$$\mathcal{M}^{\text{qqg}} = \frac{1}{36(d-2)} \begin{pmatrix} 9 & 12 & 6 & 6 & 12 & 0 \\ 12 & 16(d-1) & 8(d-1) & 8(d-1) & 4(d+2) & 0 \\ 6 & 8(d-1) & 4(4d-7) & 4(d-1) & 8(d-1) & 0 \\ 6 & 8(d-1) & 4(d-1) & 4(4d-7) & 8(d-1) & 0 \\ 12 & 4(d+2) & 8(d-1) & 8(d-1) & 16(d-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix}. \tag{A.6}$$

We have used the convention that when a momentum is contracted with a Lorentz index then that momentum appears instead of the index in the tensor.

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