

Renormalization group functions of QCD in the minimal MOM scheme

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Abstract. We provide the full set of renormalization group functions for the renormalization of QCD in the minimal MOM scheme to four loops for the colour group $SU(N_c)$.

1 Introduction.

The main renormalization scheme used in quantum field theory is the modified minimal subtraction, $\overline{\text{MS}}$, scheme introduced in [1, 2]. It has many elegant features which can be exploited to determine the renormalization group functions to a very high loop order. One of these is that of calculability. Briefly, only the divergences with respect to the regulator are removed, together with a specific finite part, $\ln(4\pi e^{-\gamma})$ where γ is the Euler-Mascheroni constant [2]. Ordinarily for conventional perturbation theory calculations one uses dimensional regularization in $d = 4 - 2\epsilon$ dimensions and ϵ plays the role of the regulator. Given this one need only consider the underlying massless quantum field theory safe in the knowledge that in this renormalization scheme the divergences will be mass independent. Thus, since massless Feynman graphs are significantly easier to compute than massive ones, then one can extract the ultraviolet divergences to high loop order. Moreover, using the scheme in gauge theories with massless fields gauge symmetry is preserved, [1]. Thus $\overline{\text{MS}}$ has been established as the favoured scheme for many years. However, for certain problems it is not necessarily the best choice. For instance, in lattice gauge theory computations it is not practical to implement since, for example, it is expensive for calculating Green's functions involving derivatives. Instead physical schemes such as the modified regularization invariant (RI') scheme have been introduced, [3, 4]. These are asymmetric in that the definition is related to the choice of momentum configuration of 3-point functions. Related types of physical schemes, but not motivated by lattice considerations, are the momentum subtraction schemes of [5] denoted by MOM. These differ from RI' schemes in that the 3-point function momentum configuration is completely symmetric. Hence they do not suffer from infrared issues as the configuration does not have exceptional momenta. Both RI' and MOM schemes differ from $\overline{\text{MS}}$ in that finite pieces are absorbed into the renormalization constants which therefore depend on external momentum scales. As a corollary they are more difficult to calculate in analytically to high loop order. Irrespective of which scheme one chooses to use for an analysis, through the structure of the renormalization group equation it is possible to relate results. Thus within perturbation theory one can compute the conversion functions which allow one to map, for example, the coupling constant defined in one scheme to that in another. The other parameters and renormalization group functions can equally be related by the same formalism.

A more recent development has been the introduction of another variant within the RI' and MOM family of physical renormalization schemes, [6]. It is called the minimal MOM scheme and is motivated by a property of the ghost-gluon vertex of QCD in the Landau gauge. This property is the non-renormalization of the vertex, [7]. However, the scheme is an extension of the concept beyond this specific gauge in a way which preserves a definition of the coupling constant in terms of the ghost and gluon form factors, [6]. This effective running coupling constant has been the subject of intense interest in recent years due to interesting features at medium and low energies which were noted earlier in [8]. For instance, it is believed that there are dimension two deviations from the expected running when compared to pure perturbation theory. The most recent work, [9], appears to reaffirm this property. With the minimal MOM scheme the effective coupling constant is not only simple to define but from a practical point of view does not require full knowledge of the ghost-gluon vertex function as would be necessary in other schemes [6]. Indeed in spite of being a non-exceptional momentum configuration it is numerically harder to extract a clean signal from the lattice for a fully symmetric vertex such as in the MOM context. In developing the minimal MOM scheme, [6], the four loop QCD β -function was determined for $SU(N_c)$ not only in the Landau gauge but also for a particular formulation of a linear covariant gauge. The full set of renormalization group functions as far as they could be calculated were not given. Therefore, it is the purpose of this note to provide

the wave function renormalization group functions to as far a loop order as is possible. For an arbitrary colour group this is to three loops for the wave functions and four loops for the β -function and quark mass anomalous dimension. Though we will also provide the former to four loops for $SU(N_c)$. We will do this in two ways. The first is the direct evaluation of all the three loop Green's functions where the minimal MOM renormalization scheme definition is implemented directly. The second is by construction of the associated conversion functions and use of the renormalization group equations. This will serve as a check on our computations and allow us to deduce four loop information. One of the reasons for the direct renormalization is that it provides a non-trivial independent check on the results of [6]. There the β -function was adduced from known finite parts of Green's functions given in [10]. As a separate exercise we choose to work in a minor variation of the original minimal MOM scheme and that is to renormalize the gauge parameter, α , in the full ethos of the minimal MOM scheme. In [6] the renormalization of the gauge parameter is completely equivalent to that of the $\overline{\text{MS}}$ scheme. Though our results will equate for the Landau gauge. A similar issue for α arises in the RI' case [11]. Finally, given the interest in the behaviour of the effective coupling constant in the Landau gauge and its power law deviation, we will provide the minimal MOM anomalous dimension of the operator thought to be associated with the dimension two correction which is $\frac{1}{2}A_\mu^a A^{a\mu}$ where A_μ^a is the gluon. This is in order to allow one to perform a complete renormalization group running analysis in the minimal MOM scheme for such infrared problems.

The paper is organized as follows. We recall the definition and properties of the minimal MOM scheme in section 2 before recording our results in the subsequent section. These include the renormalization group functions and the conversion functions for an arbitrary colour group. For $\overline{\text{MS}}$ renormalization group functions which are only known at four loops for $SU(N_c)$, we provide the corresponding minimal MOM scheme results in section 4 together with those for the dimension two operator anomalous dimension. A conclusion is provided in section 5.

2 Formalism.

We begin by recalling the definition of the minimal MOM scheme, [6]. First, if we denote bare quantities in the QCD Lagrangian by the subscript o , then in our notation the renormalization constants, Z_i , are given by

$$\begin{aligned} A_o^{a\mu} &= \sqrt{Z_A} A^{a\mu} \quad , \quad c_o^a = \sqrt{Z_c} c^a \quad , \quad \psi_o^{iI} = \sqrt{Z_\psi} \psi^{iI} \\ \alpha_o &= Z_\alpha^{-1} Z_A \alpha \quad , \quad m_o = m Z_m \quad , \quad g_o = \mu^\epsilon Z_g g \end{aligned} \quad (2.1)$$

where A_μ^a is the gluon, c^a is the Faddeev-Popov ghost and ψ^i is the quark. The indices have the ranges $1 \leq a \leq N_A$, $1 \leq i \leq N_F$ and $1 \leq I \leq N_f$ where N_F and N_A are the respective dimensions of the fundamental and adjoint representations of the colour group and N_f is the number of massive quarks each of the same mass m . The coupling constant is g and α is the gauge parameter of the linear covariant gauge. The Landau gauge corresponds to $\alpha = 0$. We use the above definition of the renormalization of α to be consistent with [11]. Throughout we use dimensional regularization in $d = 4 - 2\epsilon$ spacetime dimensions and the mass scale μ is introduced to ensure the coupling constant is dimensionless in d -dimensions. With these formal definitions of the renormalization constants they are then determined explicitly by specifying a scheme to absorb the infinities in the various 2 and 3-point functions of the theory. For instance, $\overline{\text{MS}}$ corresponds to removing only the poles in ϵ together with a certain finite part at some subtraction point.

For momentum subtraction schemes, denoted generally by MOM, the scheme is defined such that at the subtraction point the poles in ϵ together with all the finite part are absorbed into the

renormalization constant, [5]. For the QCD Lagrangian this produces several different schemes since there are several vertices which one can use to define the coupling constant renormalization. Choosing one, say, means that the remaining vertex functions are finite and consistent with the Slavnov-Taylor identities. The variation on this approach introduced in [6] is that the 2-point functions are renormalized using the MOM criterion of [5] but the 3-point vertices are treated differently. Specifically, to ease comparison with lattice analyses the completely symmetric subtraction point of [5] is not used. Instead the asymmetric point is used where the external momentum of an external leg is nullified. Moreover, partly motivated by the non-renormalization of the ghost-gluon vertex in the Landau gauge, [7], the coupling constant renormalization is defined by ensuring that this vertex renormalization constant is the *same* as the $\overline{\text{MS}}$ one. One benefit of this, [6], is that to define the scheme one only needs to know the vertex structure in the $\overline{\text{MS}}$ scheme which reduces work for non-perturbative applications. In our notation, (2.1), this corresponds to, [6],

$$Z_g^{\overline{\text{MS}}} \sqrt{Z_A^{\overline{\text{MS}}} Z_c^{\overline{\text{MS}}}} = Z_g^{\text{mMOM}} \sqrt{Z_A^{\text{mMOM}}} Z_c^{\text{mMOM}} \quad (2.2)$$

where mMOM denotes the minimal MOM scheme. Though, in this formal definition it is important to appreciate that the variables g and α on either side of the equation are in different schemes. We note that throughout our convention is that when a scheme is specified as a label on a quantity then it is a function of the parameters g and α in that scheme. With (2.2) then all the renormalization constants of massless QCD are defined for the minimal MOM scheme, [6]. As noted earlier in [6] the gauge parameter renormalization was treated as an $\overline{\text{MS}}$ one rather than define it as the full MOM renormalization as used in [5]. Therefore, we will follow the approach of [5] here and have a minimal MOM α . From the practical point of view our results in the Landau gauge will be the same and differ only in the α dependent part.

The procedure we have used is to apply the MINCER algorithm, [12], to the massless QCD Lagrangian and compute all the 2-point functions as well as the ghost-gluon vertex at the asymmetric point. The quark mass anomalous dimension will be discussed later. This algorithm evaluates massless three loop 2-point functions to the finite part in dimensional regularization. It has been encoded, [13], in the symbolic manipulation language FORM, [14], which is our main computational tool. The Feynman diagrams are generated by QGRAF, [15], and the output converted into FORM input notation. As all graphs are evaluated to the finite parts then we can extract the explicit renormalization constants in the minimal MOM. We do this first by renormalizing the 2-point functions before defining the coupling constant renormalization via (2.2). Then one proceeds to the next loop order. In using the definition of the coupling constant renormalization we have to relate the parameters between the schemes. For the gauge parameter this is given by

$$\alpha_{\text{mMOM}}(\mu) = \frac{Z_A^{\overline{\text{MS}}}}{Z_A^{\text{mMOM}}} \alpha_{\overline{\text{MS}}}(\mu) \quad (2.3)$$

where we used the fact that we are in a linear covariant gauge which implies $Z_\alpha = 1$. To ensure a finite expression in ϵ emerges the parameters within Z_A^{mMOM} have to be converted to their $\overline{\text{MS}}$ partners. This is achieved order by order in perturbation theory. We have determined these to three loops and, with $a = g^2/(16\pi^2)$, found

$$\begin{aligned} a^{\text{mMOM}} &= a + \left[9\alpha^2 C_A + 18\alpha C_A + 169C_A - 80N_f T_F \right] \frac{a^2}{36} \\ &+ \left[405\alpha^3 C_A^2 - 486\zeta(3)\alpha^2 C_A^2 + 2835\alpha^2 C_A^2 + 3564\zeta(3)\alpha C_A^2 + 2421\alpha C_A^2 \right. \\ &\quad - 1440\alpha C_A N_f T_F - 6318\zeta(3)C_A^2 + 76063C_A^2 - 10368\zeta(3)C_A N_f T_F \\ &\quad \left. - 50656C_A N_f T_F + 20736\zeta(3)C_F N_f T_F - 23760C_F N_f T_F + 6400N_f^2 T_F^2 \right] \frac{a^3}{1296} \end{aligned}$$

$$\begin{aligned}
& + \left[-10692\zeta(3)\alpha^4 C_A^3 + 8505\zeta(5)\alpha^4 C_A^3 + 41067\alpha^4 C_A^3 - 59292\zeta(3)\alpha^3 C_A^3 \right. \\
& \quad - 4860\zeta(5)\alpha^3 C_A^3 + 293301\alpha^3 C_A^3 - 138024\zeta(3)\alpha^2 C_A^3 - 85050\zeta(5)\alpha^2 C_A^3 \\
& \quad + 1315035\alpha^2 C_A^3 + 46656\zeta(3)\alpha^2 C_A^2 N_f T_F - 322056\alpha^2 C_A^2 N_f T_F \\
& \quad + 3355020\zeta(3)\alpha C_A^3 - 860220\zeta(5)\alpha C_A^3 + 1277496\alpha C_A^3 \\
& \quad - 2115072\zeta(3)\alpha C_A^2 N_f T_F - 581760\alpha C_A^2 N_f T_F + 373248\zeta(3)\alpha C_A C_F N_f T_F \\
& \quad - 427680\alpha C_A C_F N_f T_F + 331776\zeta(3)\alpha C_A N_f^2 T_F^2 + 32256\alpha C_A N_f^2 T_F^2 \\
& \quad - 6552900\zeta(3)C_A^3 - 1896615\zeta(5)C_A^3 + 42074947C_A^3 - 4499712\zeta(3)C_A^2 N_f T_F \\
& \quad + 2488320\zeta(5)C_A^2 N_f T_F - 38975424C_A^2 N_f T_F + 15303168\zeta(3)C_A C_F N_f T_F \\
& \quad + 3732480\zeta(5)C_A C_F N_f T_F - 23755968C_A C_F N_f T_F + 3068928\zeta(3)C_A N_f^2 T_F^2 \\
& \quad + 9209280C_A N_f^2 T_F^2 + 4603392C_F^2 N_f T_F - 7464960\zeta(5)C_F^2 N_f T_F T_F \\
& \quad + 1482624C_F^2 N_f T_F - 6469632\zeta(3)C_F N_f^2 T_F^2 \\
& \quad \left. + 8065152C_F N_f^2 T_F^2 - 512000N_f^3 T_F^3 \right] \frac{\alpha^4}{46656} + O(\alpha^5) \tag{2.4}
\end{aligned}$$

and

$$\begin{aligned}
\alpha^{\text{mMOM}} & = \alpha + \left[-9\alpha^2 C_A - 18\alpha C_A - 97C_A + 80N_f T_F \right] \frac{\alpha a}{36} \\
& + \left[18\alpha^4 C_A^2 - 18\alpha^3 C_A^2 + 190\alpha^2 C_A^2 - 320\alpha^2 C_A N_f T_F - 576\zeta(3)\alpha C_A^2 + 463\alpha C_A^2 \right. \\
& \quad - 320\alpha C_A N_f T_F + 864\zeta(3)C_A^2 - 7143C_A^2 + 2304\zeta(3)C_A N_f T_F + 4248C_A N_f T_F \\
& \quad \left. - 4608\zeta(3)C_F N_f T_F + 5280C_F N_f T_F \right] \frac{\alpha a^2}{288} \\
& + \left[-486\alpha^6 C_A^3 + 1944\alpha^5 C_A^3 + 4212\zeta(3)\alpha^4 C_A^3 - 5670\zeta(5)\alpha^4 C_A^3 - 18792\alpha^4 C_A^3 \right. \\
& \quad + 12960\alpha^4 C_A^2 N_f T_F + 48276\zeta(3)\alpha^3 C_A^3 - 6480\zeta(5)\alpha^3 C_A^3 - 75951\alpha^3 C_A^3 \\
& \quad - 8640\alpha^3 C_A^2 N_f T_F - 52164\zeta(3)\alpha^2 C_A^3 + 2916\zeta(4)\alpha^2 C_A^3 + 124740\zeta(5)\alpha^2 C_A^3 \\
& \quad + 92505\alpha^2 C_A^3 - 129600\zeta(3)\alpha^2 C_A^2 N_f T_F - 147288\alpha^2 C_A^2 N_f T_F \\
& \quad + 248832\zeta(3)\alpha^2 C_A C_F N_f T_F - 285120\alpha^2 C_A C_F N_f T_F - 38400\alpha^2 C_A N_f^2 T_F^2 \\
& \quad - 1303668\zeta(3)\alpha C_A^3 + 11664\zeta(4)\alpha C_A^3 + 447120\zeta(5)\alpha C_A^3 + 354807\alpha C_A^3 \\
& \quad + 698112\zeta(3)\alpha C_A^2 N_f T_F - 312336\alpha C_A^2 N_f T_F + 248832\zeta(3)\alpha C_A C_F N_f T_F \\
& \quad - 285120\alpha C_A C_F N_f T_F - 221184\zeta(3)\alpha C_A N_f^2 T_F^2 + 55296\alpha C_A N_f^2 T_F^2 \\
& \quad + 2007504\zeta(3)C_A^3 + 8748\zeta(4)C_A^3 + 1138050\zeta(5)C_A^3 - 10221367C_A^3 \\
& \quad + 1505088\zeta(3)C_A^2 N_f T_F - 279936\zeta(4)C_A^2 N_f T_F - 1658880\zeta(5)C_A^2 N_f T_F \\
& \quad + 9236488C_A^2 N_f T_F - 5156352\zeta(3)C_A C_F N_f T_F + 373248\zeta(4)C_A C_F N_f T_F \\
& \quad - 2488320\zeta(5)C_A C_F N_f T_F + 9293664C_A C_F N_f T_F - 884736\zeta(3)C_A N_f^2 T_F^2 \\
& \quad - 1343872C_A N_f^2 T_F^2 - 3068928\zeta(3)C_F^2 N_f T_F + 4976640\zeta(5)C_F^2 N_f T_F \\
& \quad \left. - 988416C_F^2 N_f T_F + 2101248\zeta(3)C_F N_f^2 T_F^2 - 2842368C_F N_f^2 T_F^2 \right] \frac{\alpha a^3}{31104} \\
& + O(a^4) \tag{2.5}
\end{aligned}$$

where $\zeta(z)$ is the Riemann zeta function. The group Casimirs are defined by

$$\text{Tr} \left(T^a T^b \right) = T_F \delta^{ab} \quad , \quad T^a T^a = C_F I \quad , \quad f^{acd} f^{bcd} = C_A \delta^{ab} \tag{2.6}$$

where T^a are the generators of the colour group whose structure functions are f^{abc} . In (2.4) and (2.5) the variables on the right hand side are in the $\overline{\text{MS}}$ scheme. For the Landau gauge it

is easy to see that then the parameters coincide. We have checked that (2.4) agrees with the alternative definition of the mapping given in [6] based on the actual finite parts of the gluon and ghost 2-point functions after their $\overline{\text{MS}}$ renormalization.

While we will perform a direct evaluation of the renormalization constants in the minimal MOM, there are several checks which will be carried out. One is to exploit properties of the renormalization group equation which allows one to map the anomalous dimensions deduced in each scheme via conversion functions which are denoted by $C_i(a, \alpha)$ where i will be a label corresponding to a field or a parameter. First, we will perform the explicit renormalization in the minimal MOM and deduce the anomalous dimensions directly. Then we will compute the conversion functions and from these construct the anomalous dimensions indirectly. Thus if we define the conversion functions by

$$C_g^{\text{mMOM}}(a, \alpha) = \frac{Z_g^{\text{mMOM}}}{Z_g^{\overline{\text{MS}}}} \quad , \quad C_\phi^{\text{mMOM}}(a, \alpha) = \frac{Z_\phi^{\text{mMOM}}}{Z_\phi^{\overline{\text{MS}}}} \quad (2.7)$$

where $\phi \in \{A, c, \psi\}$, then the minimal MOM renormalization group functions are given by

$$\beta^{\text{mMOM}}(a_{\text{mMOM}}, \alpha_{\text{mMOM}}) = \left[\beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial a_{\text{mMOM}}}{\partial a_{\overline{\text{MS}}}} + \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial a_{\text{mMOM}}}{\partial \alpha_{\overline{\text{MS}}}} \right]_{\overline{\text{MS}} \rightarrow \text{mMOM}} \quad (2.8)$$

and

$$\gamma_\phi^{\text{mMOM}}(a_{\text{mMOM}}, \alpha_{\text{mMOM}}) = \left[\gamma_\phi^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) + \beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial}{\partial a_{\overline{\text{MS}}}} \ln C_\phi^{\text{mMOM}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) + \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_\phi^{\text{mMOM}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right]_{\overline{\text{MS}} \rightarrow \text{mMOM}} \quad (2.9)$$

Here $\overline{\text{MS}} \rightarrow \text{mMOM}$ means that after computing the right hand side the expression will be a function of $\overline{\text{MS}}$ variables and these must therefore be converted to minimal MOM ones. The relations are given by inverting (2.4) and (2.5). One benefit of this formalism is that it can be exploited to produce the *four* loop anomalous dimensions and β -function. The reason for this is that the three loop conversion functions give a four loop contribution to the minimal MOM anomalous dimensions and β -function and as the $\overline{\text{MS}}$ versions of these are known, [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], then the left hand side can be deduced at four loops.

3 Results.

We now formally record our results. The one loop expressions will be the same as the $\overline{\text{MS}}$ ones since that term is scheme independent. This includes the β -function as we are using a mass dependent renormalization scheme and only in mass independent schemes is the two loop term scheme independent. Moreover, the β -function will be gauge dependent for the same reason. Therefore, we have*

$$\beta^{\text{mMOM}}(a, \alpha) = - [11C_A - 4N_f T_F] \frac{a^2}{3}$$

* A data file is attached which gives an electronic version of all our expressions.

$$\begin{aligned}
& + \left[-3\alpha^3 C_A^2 + 10\alpha^2 C_A^2 - 8\alpha^2 C_A N_f T_F + 13\alpha C_A^2 - 8\alpha C_A N_f T_F - 136C_A^2 \right. \\
& \quad \left. + 80C_A N_f T_F + 48C_F N_f T_F \right] \frac{a^3}{12} \\
& + \left[-165\alpha^4 C_A^3 + 24\alpha^4 C_A^2 N_f T_F + 108\zeta(3)\alpha^3 C_A^3 - 189\alpha^3 C_A^3 \right. \\
& \quad - 144\alpha^3 C_A^2 N_f T_F - 468\zeta(3)\alpha^2 C_A^3 + 2175\alpha^2 C_A^3 + 144\zeta(3)\alpha^2 C_A^2 N_f T_F \\
& \quad - 1656\alpha^2 C_A^2 N_f T_F - 864\alpha^2 C_A C_F N_f T_F - 1188\zeta(3)\alpha C_A^3 + 3291\alpha C_A^3 \\
& \quad - 1776\alpha C_A^2 N_f T_F - 1152\alpha C_A C_F N_f T_F + 5148\zeta(3)C_A^3 - 38620C_A^3 \\
& \quad + 6576\zeta(3)C_A^2 N_f T_F + 32144C_A^2 N_f T_F - 16896\zeta(3)C_A C_F N_f T_F \\
& \quad + 20512C_A C_F N_f T_F - 3072\zeta(3)C_A N_f^2 T_F^2 - 4416C_A N_f^2 T_F^2 \\
& \quad \left. - 576C_F^2 N_f T_F + 6144\zeta(3)C_F N_f^2 T_F^2 - 5888C_F N_f^2 T_F^2 \right] \frac{a^4}{288} \\
& + \left[864\zeta(3)\alpha^5 C_A^4 - 3780\zeta(5)\alpha^5 C_A^4 - 11745\alpha^5 C_A^4 + 1728\alpha^5 C_A^3 N_f T_F \right. \\
& \quad + 32472\zeta(3)\alpha^4 C_A^4 + 4140\zeta(5)\alpha^4 C_A^4 - 81549\alpha^4 C_A^4 \\
& \quad - 1440\zeta(3)\alpha^4 C_A^3 N_f T_F - 5040\zeta(5)\alpha^4 C_A^3 N_f T_F + 7200\alpha^4 C_A^3 N_f T_F \\
& \quad + 7776\alpha^4 C_A^2 C_F N_f T_F + 47052\zeta(3)\alpha^3 C_A^4 + 19800\zeta(5)\alpha^3 C_A^4 \\
& \quad - 81873\alpha^3 C_A^4 + 18432\zeta(3)\alpha^3 C_A^3 N_f T_F + 1440\zeta(5)\alpha^3 C_A^3 N_f T_F \\
& \quad - 67752\alpha^3 C_A^3 N_f T_F - 7776\alpha^3 C_A^2 C_F N_f T_F - 397368\zeta(3)\alpha^2 C_A^4 \\
& \quad + 152280\zeta(5)\alpha^2 C_A^4 + 1028898\alpha^2 C_A^4 - 36576\zeta(3)\alpha^2 C_A^3 N_f T_F \\
& \quad - 1098936\alpha^2 C_A^3 N_f T_F + 639360\zeta(3)\alpha^2 C_A^2 C_F N_f T_F \\
& \quad - 790272\alpha^2 C_A^2 C_F N_f T_F + 73728\zeta(3)\alpha^2 C_A^2 N_f^2 T_F^2 + 133632\alpha^2 C_A^2 N_f^2 T_F^2 \\
& \quad + 20736\alpha^2 C_A C_F^2 N_f T_F - 221184\zeta(3)\alpha^2 C_A C_F N_f^2 T_F^2 \\
& \quad + 211968\alpha^2 C_A C_F N_f^2 T_F^2 - 2400708\zeta(3)\alpha C_A^4 + 987660\zeta(5)\alpha C_A^4 \\
& \quad + 1719423\alpha C_A^4 + 1655712\zeta(3)\alpha C_A^3 N_f T_F - 254880\zeta(5)\alpha C_A^3 N_f T_F \\
& \quad - 1817880\alpha C_A^3 N_f T_F + 798336\zeta(3)\alpha C_A^2 C_F N_f T_F \\
& \quad - 1030752\alpha C_A^2 C_F N_f T_F - 617472\zeta(3)\alpha C_A^2 N_f^2 T_F^2 + 391488\alpha C_A^2 N_f^2 T_F^2 \\
& \quad + 31104\alpha C_A C_F^2 N_f T_F - 331776\zeta(3)\alpha C_A C_F N_f^2 T_F^2 \\
& \quad + 317952\alpha C_A C_F N_f^2 T_F^2 + 98304\zeta(3)\alpha C_A N_f^3 T_F^3 - 24576\alpha C_A N_f^3 T_F^3 \\
& \quad + 5509416\zeta(3)C_A^4 + 3090780\zeta(5)C_A^4 - 22106704C_A^4 \\
& \quad - 1217376\zeta(3)C_A^3 N_f T_F - 5178960\zeta(5)C_A^3 N_f T_F + 23501280C_A^3 N_f T_F \\
& \quad - 7050240\zeta(3)C_A^2 C_F N_f T_F - 6082560\zeta(5)C_A^2 C_F N_f T_F \\
& \quad + 17477280C_A^2 C_F N_f T_F - 1654272\zeta(3)C_A^2 N_f^2 T_F^2 \\
& \quad + 1474560\zeta(5)C_A^2 N_f^2 T_F^2 - 5719680C_A^2 N_f^2 T_F^2 \\
& \quad - 7907328\zeta(3)C_A C_F^2 N_f T_F + 12165120\zeta(5)C_A C_F^2 N_f T_F \\
& \quad - 607104C_A C_F^2 N_f T_F + 4755456\zeta(3)C_A C_F N_f^2 T_F^2 \\
& \quad + 2211840\zeta(5)C_A C_F N_f^2 T_F^2 - 10861056C_A C_F N_f^2 T_F^2 \\
& \quad + 344064\zeta(3)C_A N_f^3 T_F^3 + 229376C_A N_f^3 T_F^3 - 476928C_F^3 N_f T_F \\
& \quad + 3538944\zeta(3)C_F^2 N_f^2 T_F^2 - 4423680\zeta(5)C_F^2 N_f^2 T_F^2 + 267264C_F^2 N_f^2 T_F^2 \\
& \quad \left. - 884736\zeta(3)C_F N_f^3 T_F^3 + 1327104C_F N_f^3 T_F^3 - 2433024\zeta(3)\frac{d_{AA}^{(4)}}{N_A} \right]
\end{aligned}$$

$$\begin{aligned}
& + 92160 \frac{d_{AA}^{(4)}}{N_A} + 5750784 \zeta(3) \frac{d_{FA}^{(4)}}{N_A} N_f - 589824 \frac{d_{FA}^{(4)}}{N_A} N_f \\
& - 1769472 \zeta(3) \frac{d_{FF}^{(4)}}{N_A} N_f^2 + 811008 \frac{d_{FF}^{(4)}}{N_A} N_f^2 \left] \frac{a^5}{10368} + O(a^6) . \quad (3.1)
\end{aligned}$$

As the four loop $\overline{\text{MS}}$ β -function was computed for an arbitrary colour group, [23], the general colour group Casimirs appear. In our notation they are defined by

$$d_{FF}^{(4)} = d_F^{abcd} d_F^{abcd} , \quad d_{FA}^{(4)} = d_F^{abcd} d_A^{abcd} , \quad d_{AA}^{(4)} = d_A^{abcd} d_A^{abcd} \quad (3.2)$$

and the totally symmetric rank 4 colour tensor is defined by, [27],

$$d_R^{abcd} = \frac{1}{6} \text{Tr} \left(T^a T^{(b} T^c T^{d)} \right) \quad (3.3)$$

where the group generators are in the R representation.

For the anomalous dimensions only the three loop $\overline{\text{MS}}$ expressions are known for an arbitrary colour group, [21]. Thus to the same order the minimal MOM expressions are

$$\begin{aligned}
\gamma_A^{\text{mMOM}}(a, \alpha) &= [3\alpha C_A - 13C_A + 8N_f T_F] \frac{a}{6} \\
&+ \left[-6\alpha^3 C_A^2 + 17\alpha^2 C_A^2 - 16\alpha^2 C_A N_f T_F + 17\alpha C_A^2 - 16\alpha C_A N_f T_F - 170C_A^2 \right. \\
&\quad \left. + 136C_A N_f T_F + 96C_F N_f T_F \right] \frac{a^2}{24} \\
&+ \left[-165\alpha^4 C_A^3 + 24\alpha^4 C_A^2 N_f T_F + 54\zeta(3)\alpha^3 C_A^3 - 126\alpha^3 C_A^3 - 144\alpha^3 C_A^2 N_f T_F \right. \\
&\quad - 576\zeta(3)\alpha^2 C_A^3 + 1761\alpha^2 C_A^3 + 144\zeta(3)\alpha^2 C_A^2 N_f T_F - 1512\alpha^2 C_A^2 N_f T_F \\
&\quad - 864\alpha^2 C_A C_F N_f T_F - 774\zeta(3)\alpha C_A^3 + 102\alpha C_A^3 - 288\zeta(3)\alpha C_A^2 N_f T_F \\
&\quad - 600\alpha C_A^2 N_f T_F - 1152\alpha C_A C_F N_f T_F + 3456\zeta(3)C_A^3 - 23032C_A^3 \\
&\quad + 6288\zeta(3)C_A^2 N_f T_F + 21320C_A^2 N_f T_F - 16896\zeta(3)C_A C_F N_f T_F \\
&\quad + 19648C_A C_F N_f T_F - 3072\zeta(3)C_A N_f^2 T_F^2 - 2496C_A N_f^2 T_F^2 \\
&\quad \left. - 576C_F^2 N_f T_F + 6144\zeta(3)C_F N_f^2 T_F^2 - 5888C_F N_f^2 T_F^2 \right] \frac{a^3}{288} + O(a^4) \\
\gamma_c^{\text{mMOM}}(a, \alpha) &= [\alpha - 3] \frac{C_A a}{4} + [3\alpha^2 C_A - 3\alpha C_A - 34C_A + 8N_f T_F] \frac{C_A a^2}{16} \\
&+ \left[54\zeta(3)\alpha^3 C_A^2 - 45\alpha^3 C_A^2 - 36\zeta(3)\alpha^2 C_A^2 + 216\alpha^2 C_A^2 - 48\alpha^2 C_A N_f T_F \right. \\
&\quad + 42\zeta(3)\alpha C_A^2 + 109\alpha C_A^2 + 96\zeta(3)\alpha C_A N_f T_F - 152\alpha C_A N_f T_F \\
&\quad + 564\zeta(3)C_A^2 - 5196C_A^2 + 96\zeta(3)C_A N_f T_F + 3608C_A N_f T_F + 288C_F N_f T_F \\
&\quad \left. - 640N_f^2 T_F^2 \right] \frac{C_A a^3}{192} + O(a^4) \\
\gamma_\psi^{\text{mMOM}}(a, \alpha) &= \alpha C_F a + C_F \left[3\alpha^2 C_A + 6\alpha C_A + 25C_A - 6C_F - 8N_f T_F \right] \frac{C_F a^2}{4} \\
&+ \left[18\zeta(3)\alpha^3 C_A^2 + 27\alpha^3 C_A^2 - 24\alpha^3 C_A C_F - 90\zeta(3)\alpha^2 C_A^2 + 123\alpha^2 C_A^2 \right. \\
&\quad - 36\alpha^2 C_A C_F + 48\alpha^2 C_A N_f T_F - 618\zeta(3)\alpha C_A^2 + 395\alpha C_A^2 + 72\alpha C_A C_F \\
&\quad + 192\zeta(3)\alpha C_A N_f T_F - 64\alpha C_A N_f T_F - 1470\zeta(3)C_A^2 + 3843C_A^2 \\
&\quad + 576\zeta(3)C_A C_F - 1260C_A C_F + 384\zeta(3)C_A N_f T_F - 1840C_A N_f T_F \\
&\quad \left. + 72C_F^2 - 96C_F N_f T_F + 128N_f^2 T_F^2 \right] \frac{C_F a^3}{48} + O(a^4) . \quad (3.4)
\end{aligned}$$

We have checked explicitly that the gauge parameter satisfies

$$\gamma_\alpha^{\text{mMOM}}(a, \alpha) = -\gamma_A^{\text{mMOM}}(a, \alpha) \quad (3.5)$$

which is a check on our calculation.

Having provided the anomalous dimensions we have checked that they are completely reproduced using the conversion function approach. The explicit forms of these functions are

$$\begin{aligned} C_g(a, \alpha) = & 1 + \left[-9\alpha^2 C_A - 18\alpha C_A - 169C_A + 80N_f T_F \right] \frac{a}{72} \\ & + \left[243\alpha^4 C_A^2 - 648\alpha^3 C_A^2 + 1944\zeta(3)\alpha^2 C_A^2 - 1242\alpha^2 C_A^2 - 4320\alpha^2 C_A N_f T_F \right. \\ & \quad - 14256\zeta(3)\alpha C_A^2 + 8568\alpha C_A^2 - 2880\alpha C_A N_f T_F + 25272\zeta(3)C_A^2 - 218569C_A^2 \\ & \quad + 41472\zeta(3)C_A N_f T_F + 121504C_A N_f T_F - 82944\zeta(3)C_F N_f T_F + 95040C_F N_f T_F \\ & \quad \left. - 6400N_f^2 T_F^2 \right] \frac{a^2}{10368} \\ & + \left[-3645\alpha^6 C_A^3 + 21870\alpha^5 C_A^3 + 33048\zeta(3)\alpha^4 C_A^3 - 68040\zeta(5)\alpha^4 C_A^3 \right. \\ & \quad - 183951\alpha^4 C_A^3 + 97200\alpha^4 C_A^2 N_f T_F + 754272\zeta(3)\alpha^3 C_A^3 + 38880\zeta(5)\alpha^3 C_A^3 \\ & \quad - 1501740\alpha^3 C_A^3 - 155520\alpha^3 C_A^2 N_f T_F + 206064\zeta(3)\alpha^2 C_A^3 + 680400\zeta(5)\alpha^2 C_A^3 \\ & \quad - 710235\alpha^2 C_A^3 - 1026432\zeta(3)\alpha^2 C_A^2 N_f T_F - 1887840\alpha^2 C_A^2 N_f T_F \\ & \quad + 2239488\zeta(3)\alpha^2 C_A C_F N_f T_F - 2566080\alpha^2 C_A C_F N_f T_F - 172800\alpha^2 C_A N_f^2 T_F^2 \\ & \quad - 20977056\zeta(3)\alpha C_A^3 + 6881760\zeta(5)\alpha C_A^3 + 3407958\alpha C_A^3 \\ & \quad + 11259648\zeta(3)\alpha C_A^2 N_f T_F - 4231296\alpha C_A^2 N_f T_F + 1492992\zeta(3)\alpha C_A C_F N_f T_F \\ & \quad - 1710720\alpha C_A C_F N_f T_F - 2654208\zeta(3)\alpha C_A N_f^2 T_F^2 + 778752\alpha C_A N_f^2 T_F^2 \\ & \quad + 39610296\zeta(3)C_A^3 + 15172920\zeta(5)C_A^3 - 206477857C_A^3 \\ & \quad + 21036672\zeta(3)C_A^2 N_f T_F - 19906560\zeta(5)C_A^2 N_f T_F + 170325744C_A^2 N_f T_F \\ & \quad - 80372736\zeta(3)C_A C_F N_f T_F - 29859840\zeta(5)C_A C_F N_f T_F \\ & \quad + 141862464C_A C_F N_f T_F - 14598144\zeta(3)C_A N_f^2 T_F^2 - 28289280C_A N_f^2 T_F^2 \\ & \quad - 36827136\zeta(3)C_F^2 N_f T_F + 59719680\zeta(5)C_F^2 N_f T_F - 11860992C_F^2 N_f T_F \\ & \quad \left. + 31850496\zeta(3)C_F N_f^2 T_F^2 - 41711616C_F N_f^2 T_F^2 + 512000N_f^3 T_F^3 \right] \frac{a^3}{746496} \\ & + O(a^4) \end{aligned} \quad (3.6)$$

$$\begin{aligned} C_A(a, \alpha) = & 1 + \left[9\alpha^2 C_A + 18\alpha C_A + 97C_A - 80N_f T_F \right] \frac{a}{36} \\ & + \left[810\alpha^3 C_A^2 + 2430\alpha^2 C_A^2 + 5184\zeta(3)\alpha C_A^2 + 2817\alpha C_A^2 - 2880\alpha C_A N_f T_F \right. \\ & \quad - 7776\zeta(3)C_A^2 + 83105C_A^2 - 20736\zeta(3)C_A N_f T_F - 69272C_A N_f T_F \\ & \quad \left. + 41472\zeta(3)C_F N_f T_F - 47520C_F N_f T_F + 12800N_f^2 T_F^2 \right] \frac{a^2}{2592} \\ & + \left[-12636\zeta(3)\alpha^4 C_A^3 + 17010\zeta(5)\alpha^4 C_A^3 + 64638\alpha^4 C_A^3 - 51516\zeta(3)\alpha^3 C_A^3 \right. \\ & \quad + 19440\zeta(5)\alpha^3 C_A^3 + 322947\alpha^3 C_A^3 + 203148\zeta(3)\alpha^2 C_A^3 - 8748\zeta(4)\alpha^2 C_A^3 \\ & \quad - 374220\zeta(5)\alpha^2 C_A^3 + 1094553\alpha^2 C_A^3 + 15552\zeta(3)\alpha^2 C_A^2 N_f T_F \\ & \quad - 303912\alpha^2 C_A^2 N_f T_F + 4636764\zeta(3)\alpha C_A^3 - 34992\zeta(4)\alpha C_A^3 \\ & \quad - 1341360\zeta(5)\alpha C_A^3 + 1457685\alpha C_A^3 - 3670272\zeta(3)\alpha C_A^2 N_f T_F \\ & \quad \left. - 890064\alpha C_A^2 N_f T_F + 746496\zeta(3)\alpha C_A C_F N_f T_F - 855360\alpha C_A C_F N_f T_F \right] \end{aligned}$$

$$\begin{aligned}
& + 663552\zeta(3)\alpha C_A N_f^2 T_F^2 + 64512\alpha C_A N_f^2 T_F^2 - 7531056\zeta(3)C_A^3 \\
& - 26244\zeta(4)C_A^3 - 3414150\zeta(5)C_A^3 + 44961125C_A^3 - 7293888\zeta(3)C_A^2 N_f T_F \\
& + 839808\zeta(4)C_A^2 N_f T_F + 4976640\zeta(5)C_A^2 N_f T_F - 49928712C_A^2 N_f T_F \\
& + 23514624\zeta(3)C_A C_F N_f T_F - 1119744\zeta(4)C_A C_F N_f T_F \\
& + 7464960\zeta(5)C_A C_F N_f T_F - 37099872C_A C_F N_f T_F + 5971968\zeta(3)C_A N_f^2 T_F^2 \\
& + 13873536C_A N_f^2 T_F^2 + 9206784\zeta(3)C_F^2 N_f T_F - 14929920\zeta(5)C_F^2 N_f T_F \\
& + 2965248C_F^2 N_f T_F - 12939264\zeta(3)C_F N_f^2 T_F^2 + 16130304C_F N_f^2 T_F^2 \\
& - 1024000N_f^3 T_F^3 \Big] \frac{a^3}{93312} + O(a^4) \tag{3.7}
\end{aligned}$$

$$\begin{aligned}
C_c(a, \alpha) & = 1 + C_A a \\
& + \left[-36\zeta(3)\alpha^2 C_A + 72\alpha^2 C_A + 72\zeta(3)\alpha C_A - 21\alpha C_A \right. \\
& \quad \left. - 180\zeta(3)C_A + 1943C_A - 760N_f T_F \right] \frac{C_A a^2}{192} \\
& + \left[-11178\zeta(3)\alpha^3 C_A^2 - 4860\zeta(5)\alpha^3 C_A^2 + 29241\alpha^3 C_A^2 - 56862\zeta(3)\alpha^2 C_A^2 \right. \\
& \quad + 1458\zeta(4)\alpha^2 C_A^2 + 34020\zeta(5)\alpha^2 C_A^2 + 102789\alpha^2 C_A^2 + 254826\zeta(3)\alpha C_A^2 \\
& \quad + 5832\zeta(4)\alpha C_A^2 - 63180\zeta(5)\alpha C_A^2 - 3510\alpha C_A^2 - 67392\zeta(3)\alpha C_A N_f T_F \\
& \quad + 42984\alpha C_A N_f T_F - 728082\zeta(3)C_A^2 + 4374\zeta(4)C_A^2 - 63180\zeta(5)C_A^2 \\
& \quad + 4329412C_A^2 - 100224\zeta(3)C_A N_f T_F - 139968\zeta(4)C_A N_f T_F - 2650192C_A N_f T_F \\
& \quad + 684288\zeta(3)C_F N_f T_F + 186624\zeta(4)C_F N_f T_F - 1165104C_F N_f T_F \\
& \quad \left. + 27648\zeta(3)N_f^2 T_F^2 + 330304N_f^2 T_F^2 \right] \frac{C_A a^3}{31104} + O(a^4) \tag{3.8}
\end{aligned}$$

and

$$\begin{aligned}
C_\psi(a, \alpha) & = 1 - \alpha C_F a \\
& + \left[-9\alpha^2 C_A + 8\alpha^2 C_F + 24\zeta(3)\alpha C_A - 52\alpha C_A + 24\zeta(3)C_A \right. \\
& \quad \left. - 82C_A + 5C_F + 28N_f T_F \right] \frac{C_F a^2}{8} \\
& + \left[1728\zeta(3)\alpha^3 C_A^2 - 11880\alpha^3 C_A^2 - 5184\zeta(3)\alpha^3 C_A C_F + 12312\alpha^3 C_A C_F \right. \\
& \quad + 3456\zeta(3)\alpha^3 C_F^2 - 5184\alpha^3 C_F^2 + 25272\zeta(3)\alpha^2 C_A^2 - 972\zeta(4)\alpha^2 C_A^2 \\
& \quad - 6480\zeta(5)\alpha^2 C_A^2 - 63747\alpha^2 C_A^2 - 31104\zeta(3)\alpha^2 C_A C_F + 59616\alpha^2 C_A C_F \\
& \quad + 181440\zeta(3)\alpha C_A^2 - 1944\alpha\zeta(4)C_A^2 - 12960\zeta(5)\alpha C_A^2 - 358191\alpha C_A^2 \\
& \quad + 57024\zeta(3)\alpha C_A C_F - 103680\zeta(5)\alpha C_A C_F + 85536\alpha C_A C_F \\
& \quad - 41472\zeta(3)\alpha C_A N_f T_F + 124056\alpha C_A N_f T_F - 11016\alpha C_F^2 - 28512\alpha C_F N_f T_F \\
& \quad + 678024\zeta(3)C_A^2 + 22356\zeta(4)C_A^2 - 213840\zeta(5)C_A^2 - 1274056C_A^2 \\
& \quad - 228096\zeta(3)C_A C_F - 31104\zeta(4)C_A C_F + 103680\zeta(5)C_A C_F + 215352C_A C_F \\
& \quad - 89856\zeta(3)C_A N_f T_F + 760768C_A N_f T_F + 31536C_F^2 - 82944\zeta(3)C_F N_f T_F \\
& \quad \left. + 68256C_F N_f T_F - 100480N_f^2 T_F^2 \right] \frac{C_F a^3}{5184} + O(a^4) . \tag{3.9}
\end{aligned}$$

We use the convention that the variables on the right hand side are in the $\overline{\text{MS}}$ scheme.

While [6] provided the renormalization group functions for massless QCD it is possible to deduce the quark mass anomalous dimension to four loops for an arbitrary colour group. This

requires the conversion function for the quark mass renormalization and the four loop $\overline{\text{MS}}$ anomalous dimension. The latter has been provided in [24] and [25]. To deduce the former we work in the massless theory but renormalize the associated mass operator by inserting it in a quark 2-point function at zero momentum insertion. This was the procedure used in the original three loop $\overline{\text{MS}}$ renormalization of [28, 29]. We then use the renormalization condition that there is no finite part at the subtraction point. In this computational setup we can still use the MINCER algorithm, [12, 13]. Thus we can deduce the renormalization constant and hence the three loop quark mass conversion function which is

$$\begin{aligned}
C_m(a, \alpha) = & 1 + C_F [-\alpha - 4] a \\
& + \left[-18\alpha^2 C_A + 24\alpha^2 C_F - 84\alpha C_A + 96\alpha C_F + 432\zeta(3)C_A - 1285C_A \right. \\
& \quad \left. - 288\zeta(3)C_F + 57C_F + 332N_f T_F \right] \frac{C_F a^2}{24} \\
& + \left[-13122\alpha^3 C_A^2 + 15552\alpha^3 C_A C_F - 7776\alpha^3 C_F^2 + 8748\zeta(3)\alpha^2 C_A^2 - 71685\alpha^2 C_A^2 \right. \\
& \quad - 23328\zeta(3)\alpha^2 C_A C_F + 89424\alpha^2 C_A C_F + 46656\zeta(3)\alpha^2 C_F^2 - 31104\alpha^2 C_F^2 \\
& \quad + 103032\zeta(3)\alpha C_A^2 - 357777\alpha C_A^2 - 334368\zeta(3)\alpha C_A C_F + 573804\alpha C_A C_F \\
& \quad - 31104\zeta(3)\alpha C_A N_f T_F + 113400\alpha C_A N_f T_F - 46656\zeta(3)\alpha C_F^2 - 30132\alpha C_F^2 \\
& \quad + 62208\zeta(3)\alpha C_F N_f T_F - 123120\alpha C_F N_f T_F + 3368844\zeta(3)C_A^2 - 466560\zeta(5)C_A^2 \\
& \quad - 6720046C_A^2 - 2493504\zeta(3)C_A C_F + 155520\zeta(5)C_A C_F + 2028348C_A C_F \\
& \quad - 532224\zeta(3)C_A N_f T_F + 186624\zeta(4)C_A N_f T_F + 3052384C_A N_f T_F \\
& \quad + 451008\zeta(3)C_F^2 + 933120\zeta(5)C_F^2 - 2091096C_F^2 - 331776\zeta(3)C_F N_f T_F \\
& \quad - 186624\zeta(4)C_F N_f T_F + 958176C_F N_f T_F - 27648\zeta(3)N_f^2 T_F^2 \\
& \quad \left. - 240448N_f^2 T_F^2 \right] \frac{C_F a^3}{7776} + O(a^4). \tag{3.10}
\end{aligned}$$

Equipped with this and the result of [24, 25] we find the minimal MOM quark mass anomalous dimension is

$$\begin{aligned}
\gamma_m^{\text{mMOM}}(a, \alpha) = & -3C_F a + \left[\alpha^2 C_A - 67C_A - 6C_F + 8N_f T_F \right] \frac{C_F a^2}{4} \\
& + \left[-3\alpha^3 C_A^2 + 24\alpha^3 C_A C_F - 54\zeta(3)\alpha^2 C_A^2 + 411\alpha^2 C_A^2 + 108\alpha^2 C_A C_F \right. \\
& \quad - 48\alpha^2 C_A N_f T_F + 396\zeta(3)\alpha C_A^2 + 15\alpha C_A^2 + 72\alpha C_A C_F + 48\alpha C_A N_f T_F \\
& \quad + 5634\zeta(3)C_A^2 - 10095C_A^2 - 4224\zeta(3)C_A C_F + 244C_A C_F \\
& \quad - 1152\zeta(3)C_A N_f T_F + 3888C_A N_f T_F - 3096C_F^2 + 1536\zeta(3)C_F N_f T_F \\
& \quad \left. + 736C_F N_f T_F - 384N_f^2 T_F^2 \right] \frac{C_F a^3}{48} \\
& + \left[-1134\alpha^4 C_A^3 C_F \zeta(3) + 2835\alpha^4 C_A^3 C_F \zeta(5) - 10125\alpha^4 C_A^3 C_F \right. \\
& \quad + 8586\alpha^4 C_A^2 C_F^2 + 648\alpha^4 C_A^2 C_F N_f T_F - 2592\alpha^4 C_A C_F^3 \\
& \quad - 35316\zeta(3)\alpha^3 C_A^3 C_F - 1620\zeta(5)\alpha^3 C_A^3 C_F + 45522\alpha^3 C_A^3 C_F \\
& \quad - 15552\zeta(3)\alpha^3 C_A^2 C_F^2 + 43740\alpha^3 C_A^2 C_F^2 - 9072\alpha^3 C_A^2 C_F N_f T_F \\
& \quad + 31104\zeta(3)\alpha^3 C_A C_F^3 - 7776\alpha^3 C_A C_F^3 - 485352\zeta(3)\alpha^2 C_A^3 C_F \\
& \quad - 28350\zeta(5)\alpha^2 C_A^3 C_F + 893025\alpha^2 C_A^3 C_F + 167832\zeta(3)\alpha^2 C_A^2 C_F^2 \\
& \quad + 134784\alpha^2 C_A^2 C_F^2 + 85536\zeta(3)\alpha^2 C_A^2 C_F N_f T_F - 307152\alpha^2 C_A^2 C_F N_f T_F \\
& \quad + 160704\zeta(3)\alpha^2 C_A C_F^3 + 246888\alpha^2 C_A C_F^3 - 82944\zeta(3)\alpha^2 C_A C_F^2 N_f T_F \\
& \quad \left. - 90720\alpha^2 C_A C_F^2 N_f T_F + 31104\alpha^2 C_A C_F N_f^2 T_F^2 - 41472\zeta(3)\alpha^2 C_F^3 N_f T_F \right]
\end{aligned}$$

$$\begin{aligned}
& + 799956\zeta(3)\alpha C_A^3 C_F - 286740\zeta(5)\alpha C_A^3 C_F + 295551\alpha C_A^3 C_F \\
& - 417744\zeta(3)\alpha C_A^2 C_F^2 + 223668\alpha C_A^2 C_F^2 - 658368\zeta(3)\alpha C_A^2 C_F N_f T_F \\
& + 115416\alpha C_A^2 C_F N_f T_F - 824256\zeta(3)\alpha C_A C_F^3 + 432864\alpha C_A C_F^3 \\
& + 463104\zeta(3)\alpha C_A C_F^2 N_f T_F - 250560\alpha C_A C_F^2 N_f T_F \\
& + 165888\zeta(3)\alpha C_A C_F N_f^2 T_F^2 - 46080\alpha C_A C_F N_f^2 T_F^2 \\
& + 248832\zeta(3)\alpha C_F^3 N_f T_F + 20736\alpha C_F^3 N_f T_F - 110592\zeta(3)\alpha C_F^2 N_f^2 T_F^2 \\
& + 27648\alpha C_F^2 N_f^2 T_F^2 + 16036470\zeta(3)C_A^3 C_F - 6334605\zeta(5)C_A^3 C_F \\
& - 10139319C_A^3 C_F - 10029096\zeta(3)C_A^2 C_F^2 + 3421440\zeta(5)C_A^2 C_F^2 \\
& - 2188530C_A^2 C_F^2 - 15748128\zeta(3)C_A^2 C_F N_f T_F + 2737152\zeta(4)C_A^2 C_F N_f T_F \\
& + 4147200\zeta(5)C_A^2 C_F N_f T_F + 8403640C_A^2 C_F N_f T_F + 2208384\zeta(3)C_A C_F^3 \\
& + 6842880\zeta(5)C_A C_F^3 - 4669704C_A C_F^3 + 7091712\zeta(3)C_A C_F^2 N_f T_F \\
& - 2737152\zeta(4)C_A C_F^2 N_f T_F + 1244160\zeta(5)C_A C_F^2 N_f T_F \\
& - 2214048C_A C_F^2 N_f T_F + 2405376\zeta(3)C_A C_F N_f^2 T_F^2 \\
& - 995328\zeta(4)C_A C_F N_f^2 T_F^2 - 2128192C_A C_F N_f^2 T_F^2 - 1741824\zeta(3)C_F^4 \\
& - 817128C_F^4 + 4935168\zeta(3)C_F^3 N_f T_F - 7464960\zeta(5)C_F^3 N_f T_F \\
& + 3509568C_F^3 N_f T_F - 1327104\zeta(3)C_F^2 N_f^2 T_F^2 + 995328\zeta(4)C_F^2 N_f^2 T_F^2 \\
& - 605568C_F^2 N_f^2 T_F^2 + 147456\zeta(3)C_F N_f^3 T_F^3 - 2048C_F N_f^3 T_F^3 \\
& + 1244160\zeta(3)\frac{d_{FA}^{(4)}}{N_F} - 165888\frac{d_{FA}^{(4)}}{N_F} - 2488320\zeta(3)\frac{d_{FF}^{(4)}}{N_F} N_f \\
& + 331776\frac{d_{FF}^{(4)}}{N_F} N_f \Big] \frac{a^4}{5184} + O(a^5) \tag{3.11}
\end{aligned}$$

which involves the same rank 4 Casimirs as the β -function. We have checked the three loop part by the direct evaluation of the anomalous dimension using the minimal MOM renormalization constant. Given that we are considering a relatively new scheme we have also renormalized the flavour non-singlet vector current. This is important since it is a conserved physical current and its anomalous dimension is zero to all orders in perturbation theory. This is true in all schemes but we have checked this explicitly to three loops by repeating the above quark mass operator renormalization but using the vector current, $\bar{\psi}\gamma^\mu\psi$, instead. With the minimal MOM quark wave function renormalization constants and isolating the Lorentz channel of the Green's function with the inserted current corresponding to the transverse part, we have checked that the vector current renormalization constant is unity to three loops in the minimal MOM scheme. Thus the Slavnov-Taylor identity has been checked to this loop order with the above renormalization.

For more practical purposes it is useful to provide the explicit numerical expressions for $SU(3)$. Thus we have

$$\begin{aligned}
C_g(a, \alpha) &= 1 + \left[-0.375000\alpha^2 - 0.750000\alpha + 0.555556N_f - 7.041667 \right] a \\
&+ \left[0.210937\alpha^4 - 0.562500\alpha^3 - 0.625000\alpha^2 N_f + 0.950346\alpha^2 - 0.416667\alpha N_f \right. \\
&\quad \left. - 7.437954\alpha - 0.1543210N_f^2 + 24.491186N_f - 163.359911 \right] a^2 \\
&+ \left[-0.131836\alpha^6 + 0.791016\alpha^5 + 0.585937\alpha^4 N_f - 7.768301\alpha^4 - 0.937500\alpha^3 N_f \right. \\
&\quad \left. - 20.064612\alpha^3 - 0.173611\alpha^2 N_f^2 - 18.480594\alpha^2 N_f + 8.788748\alpha^2 \right. \\
&\quad \left. - 2.423078\alpha N_f^2 + 56.307562\alpha N_f - 530.662942\alpha + 0.085734N_f^3 \right] a^3
\end{aligned}$$

$$\begin{aligned}
& -47.581830N_f^2 + 1099.935641N_f - 5176.895449] a^3 + O(a^4) \\
C_A(a, \alpha) = & 1 + [0.750000\alpha^2 + 1.500000\alpha - 1.111111N_f + 8.083333] a \\
& + [2.812500\alpha^3 + 8.437500\alpha^2 - 1.666667\alpha N_f + 31.418274\alpha + 1.234568N_f^2 \\
& - 53.912928N_f + 256.103491] a^2 \\
& + [19.411733\alpha^4 + 81.359870\alpha^3 - 13.754707\alpha^2 N_f + 272.349881\alpha^2 \\
& + 6.929489\alpha N_f^2 - 254.788106\alpha N_f + 1621.114903\alpha - 1.371742N_f^3 \\
& + 171.267648N_f^2 - 2601.166373N_f + 9357.562431] a^3 + O(a^4) \\
C_c(a, \alpha) = & 1 + 3.000000a + [1.346529\alpha^2 + 3.072567\alpha - 5.937500N_f + 80.935770] a^2 \\
& + [9.344565\alpha^3 + 61.885373\alpha^2 - 5.501305\alpha N_f + 211.462123\alpha + 8.765877N_f^2 \\
& - 431.804136N_f + 2945.691833] a^3 + O(a^4) \\
C_\psi(a, \alpha) = & 1 - 1.333333\alpha a + [-2.722222\alpha^2 - 11.575317\alpha + 2.333333N_f - 25.464206] a^2 \\
& + [-16.906900\alpha^3 - 72.363802\alpha^2 + 23.739312\alpha N_f - 317.382214\alpha \\
& - 6.460905N_f^2 + 246.442650N_f - 1489.980500] a^3 + O(a^4) \\
C_m(a, \alpha) = & 1 + [-1.333333\alpha - 5.333333] a \\
& + [-1.222222\alpha^2 - 6.888889\alpha + 9.222222N_f - 149.040228] a^2 \\
& + [-11.953704\alpha^3 - 44.682370\alpha^2 + 14.024098\alpha N_f - 269.395219\alpha \\
& - 11.731930N_f^2 + 713.333651N_f - 5598.952656] a^3 + O(a^4). \tag{3.12}
\end{aligned}$$

Clearly it would appear that the series have large corrections at three loops. Though that for the quark wave function is best.

4 $SU(N_c)$.

Although we have given the minimal MOM scheme results to as high a loop order as is possible for an *arbitrary* colour group, it is possible to provide the complete set at *four* loops for the case of $SU(N_c)$. This is because the four loop $\overline{\text{MS}}$ anomalous dimensions of the gluon, ghost and quark are known for this colour group for an arbitrary linear covariant gauge fixing, [24, 26]. Using the electronically available data files associated with the latter article we have extended the various three loop minimal MOM results using the same method. This is also possible since we have the mapping for the gauge parameter between the two schemes at three loops. Thus for the gluon we have

$$\begin{aligned}
\gamma_A^{\text{mMOM}}(a, \alpha) = & [3\alpha N_c - 13N_c + 4N_f] \frac{a}{6} \\
& + [-6\alpha^3 N_c^3 + 17\alpha^2 N_c^3 - 8\alpha^2 N_c^2 N_f + 17\alpha N_c^3 - 8\alpha N_c^2 N_f - 170N_c^3 \\
& + 92N_c^2 N_f - 24N_f] \frac{a^2}{24N_c} \\
& + [-165\alpha^4 N_c^5 + 12\alpha^4 N_c^4 N_f + 54\zeta(3)\alpha^3 N_c^5 - 126\alpha^3 N_c^5 - 72\alpha^3 N_c^4 N_f \\
& - 576\zeta(3)\alpha^2 N_c^5 + 1761\alpha^2 N_c^5 + 72\zeta(3)\alpha^2 N_c^4 N_f - 972\alpha^2 N_c^4 N_f \\
& + 216\alpha^2 N_c^2 N_f - 774\zeta(3)\alpha N_c^5 + 102\alpha N_c^5 - 144\zeta(3)\alpha N_c^4 N_f - 588\alpha N_c^4 N_f]
\end{aligned}$$

$$\begin{aligned}
& + 288\alpha N_c^2 N_f + 3456\zeta(3)N_c^5 - 23032N_c^5 - 1080\zeta(3)N_c^4 N_f + 15500N_c^4 N_f \\
& - 1360N_c^3 N_f^2 + 4224\zeta(3)N_c^2 N_f - 4768N_c^2 N_f - 768\zeta(3)N_c N_f^2 + 736N_c N_f^2 \\
& - 72N_f] \frac{a^3}{288N_c^2} \\
& + \left[1728\zeta(3)\alpha^5 N_c^7 - 7560\zeta(5)\alpha^5 N_c^7 - 23490\alpha^5 N_c^7 + 1728\alpha^5 N_c^6 N_f \right. \\
& + 52389\zeta(3)\alpha^4 N_c^7 - 2250\zeta(5)\alpha^4 N_c^7 - 130590\alpha^4 N_c^7 - 1440\zeta(3)\alpha^4 N_c^6 N_f \\
& - 5040\zeta(5)\alpha^4 N_c^6 N_f + 10440\alpha^4 N_c^6 N_f + 3888\zeta(3)\alpha^4 N_c^5 + 2430\zeta(5)\alpha^4 N_c^5 \\
& - 3888\alpha^4 N_c^4 N_f + 38214\zeta(3)\alpha^3 N_c^7 + 40410\zeta(5)\alpha^3 N_c^7 - 20727\alpha^3 N_c^7 \\
& + 23616\zeta(3)\alpha^3 N_c^6 N_f + 1440\zeta(5)\alpha^3 N_c^6 N_f - 86328\alpha^3 N_c^6 N_f \\
& + 1944\zeta(3)\alpha^3 N_c^5 + 53460\zeta(5)\alpha^3 N_c^5 + 3888\alpha^3 N_c^4 N_f - 937188\zeta(3)\alpha^2 N_c^7 \\
& + 656370\zeta(5)\alpha^2 N_c^7 + 1387395\alpha^2 N_c^7 + 324144\zeta(3)\alpha^2 N_c^6 N_f \\
& - 30240\zeta(5)\alpha^2 N_c^6 N_f - 1222596\alpha^2 N_c^6 N_f - 18432\zeta(3)\alpha^2 N_c^5 N_f^2 \\
& + 93888\alpha^2 N_c^5 N_f^2 + 93312\zeta(3)\alpha^2 N_c^5 + 145800\zeta(5)\alpha^2 N_c^5 - 5832\alpha^2 N_c^5 \\
& - 319680\zeta(3)\alpha^2 N_c^4 N_f + 373104\alpha^2 N_c^4 N_f + 55296\zeta(3)\alpha^2 N_c^3 N_f^2 \\
& - 52992\alpha^2 N_c^3 N_f^2 + 5184\alpha^2 N_c^2 N_f - 3727014\zeta(3)\alpha N_c^7 \\
& + 1557630\zeta(5)\alpha N_c^7 + 1030995\alpha N_c^7 + 1860768\zeta(3)\alpha N_c^6 N_f \\
& - 142560\zeta(5)\alpha N_c^6 N_f - 1254696\alpha N_c^6 N_f - 400896\zeta(3)\alpha N_c^5 N_f^2 \\
& + 192288\alpha N_c^5 N_f^2 + 872856\zeta(3)\alpha N_c^5 - 461700\zeta(5)\alpha N_c^5 - 23328\alpha N_c^5 \\
& + 24576\zeta(3)\alpha N_c^4 N_f^3 - 6144\alpha N_c^4 N_f^3 - 383616\zeta(3)\alpha N_c^4 N_f \\
& + 428544\alpha N_c^4 N_f + 82944\zeta(3)\alpha N_c^3 N_f^2 - 79488\alpha N_c^3 N_f^2 + 7776\alpha N_c^2 N_f \\
& + 8025711\zeta(3)N_c^7 + 4451400\zeta(5)N_c^7 - 27205691N_c^7 \\
& - 5654160\zeta(3)N_c^6 N_f - 4647600\zeta(5)N_c^6 N_f + 23030340N_c^6 N_f \\
& + 642816\zeta(3)N_c^5 N_f^2 + 737280\zeta(5)N_c^5 N_f^2 - 4246944N_c^5 N_f^2 \\
& - 2563488\zeta(3)N_c^5 - 4949910\zeta(5)N_c^5 + 142344N_c^5 - 24576\zeta(3)N_c^4 N_f^3 \\
& + 188672N_c^4 N_f^3 + 8424000\zeta(3)N_c^4 N_f - 2730240\zeta(5)N_c^4 N_f \\
& - 7459344N_c^4 N_f - 1810944\zeta(3)N_c^3 N_f^2 + 552960\zeta(5)N_c^3 N_f^2 \\
& + 2377728N_c^3 N_f^2 + 110592\zeta(3)N_c^2 N_f^3 - 165888N_c^2 N_f^3 \\
& - 1976832\zeta(3)N_c^2 N_f + 3041280\zeta(5)N_c^2 N_f - 326736N_c^2 N_f \\
& - 221184\zeta(3)N_c N_f^2 - 552960\zeta(5)N_c N_f^2 + 337536N_c N_f^2 \\
& \left. + 59616N_f] \frac{a^4}{20736N_c^3} + O(a^5) \right. \tag{4.1}
\end{aligned}$$

where we have substituted the $SU(N_c)$ values for C_F and C_A . Similarly, the ghost anomalous dimension is

$$\begin{aligned}
\gamma_c^{\text{mMOM}}(a, \alpha) &= N_c[\alpha - 3] \frac{a}{4} \\
& + N_c \left[3\alpha^2 N_c - 3\alpha N_c - 34N_c + 4N_f \right] \frac{a^2}{16} \\
& + \left[54\zeta(3)\alpha^3 N_c^3 - 45\alpha^3 N_c^3 - 36\zeta(3)\alpha^2 N_c^3 + 216\alpha^2 N_c^3 - 24\alpha^2 N_c^2 N_f \right. \\
& + 42\zeta(3)\alpha N_c^3 + 109\alpha N_c^3 + 48\zeta(3)\alpha N_c^2 N_f - 76\alpha N_c^2 N_f + 564\zeta(3)N_c^3 \\
& \left. - 5196N_c^3 + 48\zeta(3)N_c^2 N_f + 1876N_c^2 N_f - 160N_c N_f^2 - 72N_f \right] \frac{a^3}{192}
\end{aligned}$$

$$\begin{aligned}
& + \left[3843\zeta(3)\alpha^4 N_c^5 + 2070\zeta(5)\alpha^4 N_c^5 - 8052\alpha^4 N_c^5 + 72\alpha^4 N_c^4 N_f \right. \\
& \quad - 3456\zeta(3)\alpha^4 N_c^3 + 1890\zeta(5)\alpha^4 N_c^3 + 8298\zeta(3)\alpha^3 N_c^5 - 6390\zeta(5)\alpha^3 N_c^5 \\
& \quad - 9411\alpha^3 N_c^5 - 576\zeta(3)\alpha^3 N_c^4 N_f + 192\alpha^3 N_c^4 N_f + 10152\zeta(3)\alpha^3 N_c^3 \\
& \quad - 18900\zeta(5)\alpha^3 N_c^3 - 108\zeta(3)\alpha^2 N_c^5 - 32790\zeta(5)\alpha^2 N_c^5 + 73071\alpha^2 N_c^5 \\
& \quad - 2448\zeta(3)\alpha^2 N_c^4 N_f + 3360\zeta(5)\alpha^2 N_c^4 N_f - 32388\alpha^2 N_c^4 N_f + 2880\alpha^2 N_c^3 N_f^2 \\
& \quad + 3888\zeta(3)\alpha^2 N_c^3 - 22680\zeta(5)\alpha^2 N_c^3 + 648\alpha^2 N_c^3 + 1296\alpha^2 N_c^2 N_f \\
& \quad - 40458\zeta(3)\alpha N_c^5 + 26790\zeta(5)\alpha N_c^5 + 88801\alpha N_c^5 + 1152\zeta(3)\alpha N_c^4 N_f \\
& \quad - 12480\zeta(5)\alpha N_c^4 N_f - 42248\alpha N_c^4 N_f + 3072\zeta(3)\alpha N_c^3 N_f^2 + 3136\alpha N_c^3 N_f^2 \\
& \quad - 1080\zeta(3)\alpha N_c^3 - 52380\zeta(5)\alpha N_c^3 + 2592\alpha N_c^3 - 1728\zeta(3)\alpha N_c^2 N_f \\
& \quad + 3600\alpha N_c^2 N_f + 310041\zeta(3)N_c^5 + 192240\zeta(5)N_c^5 - 1888893N_c^5 \\
& \quad - 91728\zeta(3)N_c^4 N_f - 59040\zeta(5)N_c^4 N_f + 997068N_c^4 N_f + 13824\zeta(3)N_c^3 N_f^2 \\
& \quad - 141984N_c^3 N_f^2 - 526176\zeta(3)N_c^3 + 549990\zeta(5)N_c^3 + 14904N_c^3 \\
& \quad + 3840N_c^2 N_f^3 + 54720\zeta(3)N_c^2 N_f - 34560\zeta(5)N_c^2 N_f - 104928N_c^2 N_f \\
& \quad \left. - 4608\zeta(3)N_c N_f^2 + 18816N_c N_f^2 - 432N_f \right] \frac{a^4}{4608N_c} + O(a^5) . \quad (4.2)
\end{aligned}$$

Finally, the quark anomalous dimension is

$$\begin{aligned}
\gamma_\psi^{\text{mMOM}}(a, \alpha) & = \alpha [N_c^2 - 1] \frac{a}{2N_c} \\
& + \left[3\alpha^2 N_c^4 - 3\alpha^2 N_c^2 + 6\alpha N_c^4 - 6\alpha N_c^2 + 22N_c^4 - 4N_c^3 N_f - 19N_c^2 \right. \\
& \quad \left. + 4N_c N_f - 3 \right] \frac{a^2}{8N_c^2} \\
& + \left[18\zeta(3)\alpha^3 N_c^6 + 15\alpha^3 N_c^6 - 18\zeta(3)\alpha^3 N_c^4 - 3\alpha^3 N_c^4 - 12\alpha^3 N_c^2 \right. \\
& \quad - 90\zeta(3)\alpha^2 N_c^6 + 105\alpha^2 N_c^6 + 24\alpha^2 N_c^5 N_f + 90\zeta(3)\alpha^2 N_c^4 \\
& \quad - 87\alpha^2 N_c^4 - 24\alpha^2 N_c^3 N_f - 18\alpha^2 N_c^2 - 618\zeta(3)\alpha N_c^6 + 431\alpha N_c^6 \\
& \quad + 96\zeta(3)\alpha N_c^5 N_f - 32\alpha N_c^5 N_f + 618\zeta(3)\alpha N_c^4 - 467\alpha N_c^4 - 96\zeta(3)\alpha N_c^3 N_f \\
& \quad + 32\alpha N_c^3 N_f + 36\alpha N_c^2 - 1182\zeta(3)N_c^6 + 3231N_c^6 + 192\zeta(3)N_c^5 N_f \\
& \quad - 944N_c^5 N_f + 32N_c^4 N_f^2 + 894\zeta(3)N_c^4 - 2637N_c^4 - 192\zeta(3)N_c^3 N_f \\
& \quad \left. + 968N_c^3 N_f - 32N_c^2 N_f^2 + 288\zeta(3)N_c^2 - 576N_c^2 - 24N_c N_f - 18 \right] \frac{a^3}{96N_c^3} \\
& + \left[330\zeta(3)\alpha^4 N_c^8 + 75\zeta(5)\alpha^4 N_c^8 - 151\alpha^4 N_c^8 - 12\alpha^4 N_c^7 N_f - 420\zeta(3)\alpha^4 N_c^6 \right. \\
& \quad - 45\zeta(5)\alpha^4 N_c^6 + 280\alpha^4 N_c^6 + 12\alpha^4 N_c^5 N_f + 42\zeta(3)\alpha^4 N_c^4 - 30\zeta(5)\alpha^4 N_c^4 \\
& \quad - 105\alpha^4 N_c^4 + 48\zeta(3)\alpha^4 N_c^2 - 24\alpha^4 N_c^2 + 1116\zeta(3)\alpha^3 N_c^8 - 180\zeta(5)\alpha^3 N_c^8 \\
& \quad - 33\alpha^3 N_c^8 - 192\zeta(3)\alpha^3 N_c^7 N_f + 184\alpha^3 N_c^7 N_f - 1620\zeta(3)\alpha^3 N_c^6 \\
& \quad + 180\zeta(5)\alpha^3 N_c^6 + 663\alpha^3 N_c^6 + 192\zeta(3)\alpha^3 N_c^5 N_f - 184\alpha^3 N_c^5 N_f \\
& \quad + 360\zeta(3)\alpha^3 N_c^4 - 558\alpha^3 N_c^4 + 144\zeta(3)\alpha^3 N_c^2 - 72\alpha^3 N_c^2 - 1662\zeta(3)\alpha^2 N_c^8 \\
& \quad - 192\zeta(3)N_c^3 N_f + 3130\zeta(5)\alpha^2 N_c^8 - 5767\alpha^2 N_c^8 - 384\zeta(3)\alpha^2 N_c^7 N_f \\
& \quad - 160\zeta(5)\alpha^2 N_c^7 N_f + 2580\alpha^2 N_c^7 N_f - 96\alpha^2 N_c^6 N_f^2 + 1518\zeta(3)\alpha^2 N_c^6 \\
& \quad - 4510\zeta(5)\alpha^2 N_c^6 + 6307\alpha^2 N_c^6 + 576\zeta(3)\alpha^2 N_c^5 N_f + 160\zeta(5)\alpha^2 N_c^5 N_f \\
& \quad - 2940\alpha^2 N_c^5 N_f + 96\alpha^2 N_c^4 N_f^2 + 144\zeta(3)\alpha^2 N_c^4 + 1380\zeta(5)\alpha^2 N_c^4 \\
& \quad \left. - 558\alpha^2 N_c^4 - 192\zeta(3)\alpha^2 N_c^3 N_f + 360\alpha^2 N_c^3 N_f + 18\alpha^2 N_c^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - 33680\zeta(3)\alpha N_c^8 + 17420\zeta(5)\alpha N_c^8 + 14292\alpha N_c^8 + 7040\zeta(3)\alpha N_c^7 N_f \\
& - 3200\zeta(5)\alpha N_c^7 N_f - 3752\alpha N_c^7 N_f + 320\alpha N_c^6 N_f^2 + 49248\zeta(3)\alpha N_c^6 \\
& - 37500\zeta(5)\alpha N_c^6 - 13642\alpha N_c^6 - 9408\zeta(3)\alpha N_c^5 N_f + 5760\zeta(5)\alpha N_c^5 N_f \\
& + 2632\alpha N_c^5 N_f - 192\alpha N_c^4 N_f^2 - 15568\zeta(3)\alpha N_c^4 + 20080\zeta(5)\alpha N_c^4 \\
& - 122\alpha N_c^4 + 2368\zeta(3)\alpha N_c^3 N_f - 2560\zeta(5)\alpha N_c^3 N_f + 1024\alpha N_c^3 N_f \\
& - 128\alpha N_c^2 N_f^2 - 528\alpha N_c^2 + 96\alpha N_c N_f - 172560\zeta(3)N_c^8 + 61875\zeta(5)N_c^8 \\
& + 252104N_c^8 + 43392\zeta(3)N_c^7 N_f - 12000\zeta(5)N_c^7 N_f - 118572N_c^7 N_f \\
& - 1536\zeta(3)N_c^6 N_f^2 + 15392N_c^6 N_f^2 + 118002\zeta(3)N_c^6 - 29085\zeta(5)N_c^6 \\
& - 225318N_c^6 - 640N_c^5 N_f^3 - 36864\zeta(3)N_c^5 N_f + 480\zeta(5)N_c^5 N_f \\
& + 120868N_c^5 N_f + 2304\zeta(3)N_c^4 N_f^2 - 16592N_c^4 N_f^2 + 66462\zeta(3)N_c^4 \\
& - 55830\zeta(5)N_c^4 - 19731N_c^4 + 640N_c^3 N_f^3 - 6912\zeta(3)N_c^3 N_f \\
& + 11520\zeta(5)N_c^3 N_f + 7152N_c^3 N_f - 768\zeta(3)N_c^2 N_f^2 + 1200N_c^2 N_f^2 \\
& - 2304\zeta(3)N_c^2 + 7680\zeta(5)N_c^2 - 3974N_c^2 + 384\zeta(3)N_c N_f - 9448N_c N_f \\
& - 9600\zeta(3) + 15360\zeta(5) - 3081] \frac{a^4}{384N_c^4} + O(a^5) . \tag{4.3}
\end{aligned}$$

For practical purposes it is perhaps more appropriate to provide the explicit numerical values for all renormalization group functions at four loops for the $SU(3)$ colour group. Thus

$$\begin{aligned}
\beta^{\text{mMOM}}(a, \alpha) &= [0.666667N_f - 11.000000] a^2 \\
&+ \left[- 2.250000\alpha^3 - \alpha^2 N_f + 7.500000\alpha^2 - \alpha N_f + 9.750000\alpha \right. \\
&\quad \left. + 12.666667N_f - 102.000000 \right] a^3 \\
&+ \left[0.375000\alpha^4 N_f - 15.468750\alpha^4 - 2.250000\alpha^3 N_f - 5.547924\alpha^3 \right. \\
&\quad \left. - 29.170372\alpha^2 N_f + 151.166003\alpha^2 - 35.750000\alpha N_f + 174.652162\alpha \right. \\
&\quad \left. - 19.383310N_f^2 + 625.386670N_f - 3040.482287 \right] a^4 \\
&+ \left[2.250000\alpha^5 N_f - 114.265701\alpha^5 + 4.816305\alpha^4 N_f - 298.616620\alpha^4 \right. \\
&\quad \left. - 61.925145\alpha^3 N_f - 37.364446\alpha^3 + 43.033474\alpha^2 N_f^2 - 1495.393156\alpha^2 N_f \right. \\
&\quad \left. + 5540.175086\alpha^2 + 3.385091\alpha N_f^3 - 83.916446\alpha N_f^2 - 152.826933\alpha N_f \right. \\
&\quad \left. - 1111.191853\alpha + 27.492640N_f^3 - 1625.402243N_f^2 + 24423.330550N_f \right. \\
&\quad \left. - 100541.058601 \right] a^5 + O(a^6) \\
\gamma_A^{\text{mMOM}}(a, \alpha) &= [1.500000\alpha + 0.666667N_f - 6.500000] a \\
&+ \left[- 2.250000\alpha^3 - \alpha^2 N_f + 6.375000\alpha^2 - \alpha N_f + 6.375000\alpha \right. \\
&\quad \left. + 11.166667N_f - 63.750000 \right] a^2 \\
&+ \left[0.375000\alpha^4 N_f - 15.468750\alpha^4 - 2.250000\alpha^3 N_f - 5.727087\alpha^3 \right. \\
&\quad \left. - 26.920372\alpha^2 N_f + 100.182677\alpha^2 - 22.784256\alpha N_f - 77.661754\alpha \right. \\
&\quad \left. - 14.383310N_f^2 + 444.852414N_f - 1769.783563 \right] a^3 \\
&+ \left[2.250000\alpha^5 N_f - 114.265701\alpha^5 + 3.972555\alpha^4 N_f - 270.114333\alpha^4 \right. \\
&\quad \left. - 72.936261\alpha^3 N_f + 287.225188\alpha^3 + 31.783474\alpha^2 N_f^2 \right. \\
&\quad \left. - 1126.940395\alpha^2 N_f + 3789.309068\alpha^2 + 3.385091\alpha N_f^3 \right.
\end{aligned}$$

$$\begin{aligned}
& -124.724674\alpha N_f^2 + 1081.645997\alpha N_f - 6926.344667\alpha + 22.492640N_f^3 \\
& - 1141.450868N_f^2 + 14846.203053N_f - 54060.225189] a^4 + O(a^5) \\
\gamma_c^{\text{mMOM}}(a, \alpha) &= [0.750000\alpha - 2.250000] a \\
& + [1.687500\alpha^2 - 1.687500\alpha + 0.750000N_f - 19.125000] a^2 \\
& + [2.799995\alpha^3 - 1.125000\alpha^2 N_f + 24.289587\alpha^2 - 0.857872\alpha N_f \\
& + 22.427774\alpha - 2.500000N_f^2 + 90.267128N_f - 635.349362] a^3 \\
& + [0.421875\alpha^4 N_f - 26.892596\alpha^4 - 2.931942\alpha^3 N_f - 121.006552\alpha^3 \\
& + 5.625000\alpha^2 N_f^2 - 185.757176\alpha^2 N_f + 648.958841\alpha^2 + 13.337341\alpha N_f^2 \\
& - 314.266897\alpha N_f + 1090.833574\alpha + 2.500000N_f^3 - 241.975687N_f^2 \\
& + 4788.563749N_f - 23240.416706] a^4 + O(a^5) \\
\gamma_\psi^{\text{mMOM}}(a, \alpha) &= 1.333333\alpha a + [3.000000\alpha^2 + 6.000000\alpha - 1.333333N_f + 22.333333] a^2 \\
& + [9.492589\alpha^3 + 2.000000\alpha^2 N_f - 0.296280\alpha^2 + 6.949789\alpha N_f \\
& - 78.967792\alpha + 0.888889N_f^2 - 59.211534N_f + 459.481285] a^3 \\
& + [-0.750000\alpha^4 N_f + 61.650284\alpha^4 - 2.924683\alpha^3 N_f + 210.615979\alpha^3 \\
& - 2.000000\alpha^2 N_f^2 + 121.134099\alpha^2 N_f - 869.566016\alpha^2 + 6.962963\alpha N_f^2 \\
& + 77.834748\alpha N_f - 1553.524496\alpha - 4.444444N_f^3 + 281.560058N_f^2 \\
& - 4934.050066N_f + 20300.851595] a^4 + O(a^5) \\
\gamma_m^{\text{mMOM}}(a, \alpha) &= -4.0a + [\alpha^2 + 1.333333N_f - 69.666667] a^2 \\
& + [1.916667\alpha^3 - 2.000000\alpha^2 N_f + 98.522232\alpha^2 + 2.000000\alpha N_f \\
& + 130.753633\alpha - 2.666667N_f^2 + 152.122739N_f - 1520.596003] a^3 \\
& + [0.750000\alpha^4 N_f - 36.419738\alpha^4 - 10.500000\alpha^3 N_f + 127.577470\alpha^3 \\
& + 6.000000\alpha^2 N_f^2 - 345.848075\alpha^2 N_f + 3588.203465\alpha^2 + 20.550022\alpha N_f^2 \\
& - 551.791827\alpha N_f + 5040.515124\alpha + 5.632797N_f^3 - 156.909331N_f^2 \\
& - 1073.781658N_f - 9337.969739] a^4 + O(a^5) \tag{4.4}
\end{aligned}$$

We have checked that the Landau gauge expression for the β -function agrees with that of [6].

One interesting consequence of these expressions is that we can provide the anomalous dimension of a particular dimension two operator which is

$$\mathcal{O} = \frac{1}{2} A_\mu^a A^{a\mu} - \alpha \bar{c}^a c^a . \tag{4.5}$$

It is known, [30, 31, 32], that \mathcal{O} has a novel renormalization property. In the Landau gauge the anomalous dimension of \mathcal{O} is the sum of the gluon and ghost anomalous dimensions. Moreover, in an arbitrary linear and nonlinear covariant gauge there is a simple generalization of this Slavnov-Taylor identity which was established in [33]. This was based on the observation given in [34] which were explicit three loop $\overline{\text{MS}}$ computations. The operator is of interest as it was an attempt to have a gluon mass term in the Lagrangian which while not gauge invariant is in fact BRST invariant, [35]. It has seen renewed interest more recently, since it is believed to be the origin of dimension two power corrections in the running of an effective coupling constant in the low energy limit, [8, 9]. In [26] the four loop $\overline{\text{MS}}$ Landau gauge result was given. However, the

arbitrary $\overline{\text{MS}}$ expression for a linear covariant gauge fixing was not recorded. As the gluon and ghost propagators are examined in the minimal MOM scheme, [36], it is worth providing the renormalization for the operators explicitly. Though for reasons of space we provide the $SU(3)$ expression[†]

$$\begin{aligned}
\gamma_{\mathcal{O}}^{\overline{\text{MS}}}(a, \alpha) \Big|_{SU(3)} &= [27\alpha + 8N_f - 105] \frac{a}{12} \\
&+ \left[108\alpha^2 + 567\alpha + 548N_f - 4041 \right] \frac{a^2}{48} \\
&+ \left[12393\alpha^3 + 4374\zeta(3)\alpha^2 + 52488\alpha^2 - 22356\alpha N_f + 17496\zeta(3)\alpha \right. \\
&\quad \left. + 268272\alpha - 12080N_f^2 - 28512\zeta(3)N_f + 437304N_f + 13122\zeta(3) \right. \\
&\quad \left. - 2041389 \right] \frac{a^3}{1728} \\
&+ \left[-3011499\zeta(3)\alpha^4 + 4133430\zeta(5)\alpha^4 + 8030664\alpha^4 + 20824614\zeta(3)\alpha^3 \right. \\
&\quad \left. + 708588\zeta(4)\alpha^3 - 10431990\zeta(5)\alpha^3 + 39936807\alpha^3 - 2939328\zeta(3)\alpha^2 N_f \right. \\
&\quad \left. + 314928\zeta(4)\alpha^2 N_f - 8669268\alpha^2 N_f + 93612348\zeta(3)\alpha^2 \right. \\
&\quad \left. - 3779136\zeta(4)\alpha^2 - 18305190\zeta(5)\alpha^2 + 159478227\alpha^2 \right. \\
&\quad \left. + 3359232\zeta(3)\alpha N_f^2 - 2796768\alpha N_f^2 - 60046272\zeta(3)\alpha N_f \right. \\
&\quad \left. - 11337408\zeta(4)\alpha N_f - 127188144\alpha N_f + 612180666\zeta(3)\alpha \right. \\
&\quad \left. - 43696260\zeta(4)\alpha - 513923130\zeta(5)\alpha + 1146415923\alpha + 497664\zeta(3)N_f^3 \right. \\
&\quad \left. - 846464N_f^3 - 19558656\zeta(3)N_f^2 - 6158592\zeta(4)N_f^2 - 107248896N_f^2 \right. \\
&\quad \left. - 289945440\zeta(3)N_f + 104451120\zeta(4)N_f + 313061760\zeta(5)N_f \right. \\
&\quad \left. + 2082893580N_f + 72636831\zeta(3) - 46766808\zeta(4) - 1908463680\zeta(5) \right. \\
&\quad \left. - 7232776173 \right] \frac{a^4}{373248} + O(a^5) \tag{4.6}
\end{aligned}$$

for non-zero α . Equipped with this then the equivalent minimal MOM scheme expression is

$$\begin{aligned}
\gamma_{\mathcal{O}}^{\text{mMOM}}(a, \alpha) \Big|_{SU(3)} &= [27\alpha + 8N_f - 105] \frac{a}{12} \\
&+ \left[-108\alpha^3 - 48\alpha^2 N_f + 387\alpha^2 - 48\alpha N_f + 225\alpha + 572N_f - 3978 \right] \frac{a^2}{48} \\
&+ \left[648\alpha^4 N_f - 26730\alpha^4 - 3888\alpha^3 N_f + 21870\zeta(3)\alpha^3 - 31347\alpha^3 \right. \\
&\quad \left. + 3888\zeta(3)\alpha^2 N_f - 53136\alpha^2 N_f - 102060\zeta(3)\alpha^2 + 337770\alpha^2 \right. \\
&\quad \left. - 3888\zeta(3)\alpha N_f - 36180\alpha N_f - 115182\zeta(3)\alpha + 43011\alpha \right. \\
&\quad \left. - 1536\zeta(3)N_f^2 - 27328N_f^2 - 29088\zeta(3)N_f + 959652N_f \right. \\
&\quad \left. + 696924\zeta(3) - 4993812 \right] \frac{a^3}{1728} \\
&+ \left[93312\alpha^5 N_f + 279936\zeta(3)\alpha^5 - 1224720\zeta(5)\alpha^5 - 3805380\alpha^5 \right. \\
&\quad \left. - 77760\zeta(3)\alpha^4 N_f - 272160\zeta(5)\alpha^4 N_f + 557928\alpha^4 N_f \right. \\
&\quad \left. + 11078613\zeta(3)\alpha^4 + 1341360\zeta(5)\alpha^4 - 27025488\alpha^4 \right. \\
&\quad \left. + 1135296\zeta(3)\alpha^3 N_f + 77760\zeta(5)\alpha^3 N_f - 4591728\alpha^3 N_f \right. \\
&\quad \left. + 13097214\zeta(3)\alpha^3 + 1319490\zeta(5)\alpha^3 - 10218393\alpha^3 \right. \\
&\quad \left. - 221184\zeta(3)\alpha^2 N_f^2 + 1817280\alpha^2 N_f^2 + 14990832\zeta(3)\alpha^2 N_f \right.
\end{aligned}$$

[†]The full expression for $SU(N_c)$ is given in the attached data file.

$$\begin{aligned}
& - 816480\zeta(5)\alpha^2 N_f - 71613396\alpha^2 N_f - 149908644\zeta(3)\alpha^2 \\
& + 83215350\zeta(5)\alpha^2 + 277974261\alpha^2 + 147456\zeta(3)\alpha N_f^3 \\
& - 36864\alpha N_f^3 - 6801408\zeta(3)\alpha N_f^2 + 3556224\alpha N_f^2 \\
& + 98413056\zeta(3)\alpha N_f - 10730880\zeta(5)\alpha N_f - 75346200\alpha N_f \\
& - 617646222\zeta(3)\alpha + 259312590\zeta(5)\alpha + 231547167\alpha \\
& - 73728\zeta(3)N_f^3 + 1125120N_f^3 + 8977920\zeta(3)N_f^2 \\
& + 14254080\zeta(5)N_f^2 - 82945888N_f^2 - 276910992\zeta(3)N_f \\
& - 280604160\zeta(5)N_f + 1438122060N_f + 1437422031\zeta(3) \\
& + 816720570\zeta(5) - 5780555523] \frac{a^4}{41472} + O(a^5) \tag{4.7}
\end{aligned}$$

for $SU(3)$.

5 Discussion.

We have provided all the renormalization group functions in QCD in the minimal momentum subtraction scheme introduced in [6]. To do this we have explicitly renormalized the theory and applied the renormalization prescription given in [6] to define the scheme. While [6] concentrated on the β -function the other renormalization group functions are required for other problems such as the infrared structure of propagators and therefore we have provided that information. Currently the results are known at four loops for the $SU(N_c)$ colour group and at three loops for a general group. One feature which differs from [6] rests in the renormalization of the gauge parameter. In [6] α was renormalized in the $\overline{\text{MS}}$ way whereas here we have chosen to follow a fuller approach and renormalize the gauge parameter according to the same criterion as all the 2-point functions. While this differs from [6] both sets of results agree in the Landau gauge which is the main gauge of interest for practical studies of the infrared dynamics of the gluon and ghost.

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