Skein theory and the Murphy operators

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Abstract

The Murphy operators in the Hecke algebra H_n of type A are explicit commuting elements whose sum generates the centre. They can be represented by simple tangles in the Homfly skein theory version of H_n. In this paper I present a single tangle which represents their sum, and which is obviously central. As a consequence it is possible to identify a natural basis for the Homfly skein of the annulus, $\mathcal{C}$.

Symmetric functions of the Murphy operators are also central in H_n. I define geometrically a homomorphism from $\mathcal{C}$ to the centre of each algebra H_n, and find an element in $\mathcal{C}$, independent of n, whose image is the mth power sum of the Murphy operators. Generating function techniques are used to describe images of other elements of $\mathcal{C}$ in terms of the Murphy operators, and to demonstrate relations among other natural skein elements.

Keywords: skein theory; Murphy operators; power sums; symmetric functions; annulus; Hecke algebras.

Introduction.

The Hecke algebra H_n of type A is a deformed version of the group algebra $\mathbf{C}[S_n]$ of the symmetric group. It has a simple skein theory model in terms of n-tangles and the Homfly skein relations.

Certain sums of transpositions,

$$m(j)=\sum_{i=1}^{j-1}(i\,j)\in\mathbf{C}[S_n],\ j=2,\ldots,n,$$

known as Jucys-Murphy elements, appeared in the work of Jucys [7] in 1974, and later in work of Murphy [14]. These elements all commute, and every symmetric polynomial in them can be shown to lie in the *centre* of the algebra $\mathbf{C}[S_n]$. For example $m(2) = (1\,2), m(3) = (1\,3) + (2\,3)$ and

$$m(2)m(3) = (1\,3\,2) + (1\,2\,3) = m(3)m(2).$$

Dipper and James [4] used a simple deformation of the transpositions to define analogous elements $M(j) \in H_n$, which they called the *Murphy operators*. These elements again all commute, and symmetric polynomials in them belong to the centre of H_n. Dipper and James showed that for generic values of the deformation parameter these account for the whole of the centre. Katriel, Abdessalam and Chakrabarti [8] noted that in fact any central element can be expressed as a polynomial in just the sum $M = \sum_{j=2}^{n} M(j)$ of the Murphy operators. This is a stronger result than for the non-deformed algebra $\mathbf{C}[S_n]$, although the centre has the same dimension in each case, given by $\pi(n)$, the number of partitions of n.

In this paper I present a skein theory version of the Murphy operators and their sum, finishing with an elegant representation of their power sums and other symmetric polynomials. Starting in section 2 with a choice of n-string braids corresponding to the transpositions I exhibit a braid $T(j)$ representing each of the individual Murphy operators $M(j)$, following Ram [15], and then a very natural simple n-tangle $T^{(n)}$ which represents their sum M, up to a linear combination with the identity element in H_n in each case. From the tangle viewpoint it becomes immediately clear that M is central.

The Homfly skein of the annulus, $\mathcal{C}$, has been exploited for many years, for example in the work of Jun Murakami, and myself and Short, in constructing and analysing Homfly-based invariants by the use of satellites. These invariants cover the same ground as the quantum $sl(N)$ invariants for all choices of module and have been developed variously in algebraic and skein theoretic ways, for example by Wenzl [17], Aiston and myself [13] and Blanchet [3].

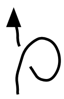
In section 3 I show how the tangle view of the Murphy operators gives a direct means for identifying the best linear basis for $\mathcal{C}$ when studying these quantum invariants and establishing its multiplicative properties. I then introduce a rather overlooked homomorphism from $\mathcal{C}$ to the centre of H_n. Using generating function methods I show how to realise other symmetric functions of the Murphy operators as images of explicit elements of $\mathcal{C}$. The principal result is the identification of an element $P_m \in \mathcal{C}$, independent of n, whose image determines the mth power sum of the Murphy operators in H_n.

Section 4 gives the final details of the results of section 3, based on a further underused skein, $\mathcal{A}$, whose unexpected algebraic properties allow for some satisfyingly clean proofs.

1 The skein models.

The skein theory model of H_n is based on the framed Homfly skein relations, in their simplest form

 $-$  $= (s - s^{-1})$ 

and  $= v^{-1}$  .

The Homfly skein $\mathcal{S}(F)$ of a planar surface F, with some designated input and output boundary points, is defined as linear combinations of oriented tangles in F, modulo these two local relations, and Reidemeister moves II and III. The coefficient ring can be taken as $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with powers of $s^k - s^{-k}$ in the denominators.

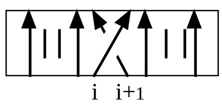
Every skein admits a *mirror map*, $\overline{} : \mathcal{S}(F) \to \mathcal{S}(F)$ induced by switching all crossings in a tangle, coupled with inverting v and s in Λ.

Write $R_n^n(v, s)$ for the skein $\mathcal{S}(F)$ of n-tangles, where F is a rectangle with n inputs at the bottom and n outputs at the top. Composing n-tangles induces a product which makes $R_n^n(v, s)$ into an algebra. It has a linear basis of $n!$ elements, and is isomorphic to the Hecke algebra $H_n(z)$. This algebra has a presentation with generators $\{\sigma_i\}, i = 1, \ldots, n-1$ satisfying the braid relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \end{aligned}$$

and the quadratic relations $\sigma_i^2 = z\sigma_i + 1$.

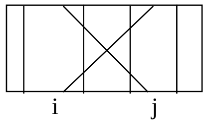
In R_n^n the generators are the elementary braids

$\sigma_i =$ 

and the parameter z is $s - s^{-1}$, giving the alternative form $(\sigma_i - s)(\sigma_i + s^{-1}) = 0$ for the quadratic relations.

In the special case $z = 0$ the Hecke algebra reduces to $\mathbf{C}[S_n]$, with σ_i becoming the transposition $(i\; i+1)$. For general i, j we can view the transposition $(i\,j)$ diagrammatically

$$(i\,j) \;=$$



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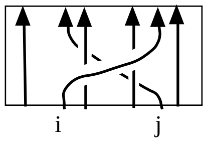
The corresponding positive permutation braid

$$\omega_{(i\,j)} \;=$$



with all crossings positive realises this transposition, as in [13]. Regarded via R_n^n as an element of $H_n(z)$, it becomes the transposition $(i\,j)$ when we set $z = 0$.

Definition. The *Murphy operator* $M(j) \in R_n^n,\ j = 2, \ldots, n$ is defined as

$$M(j) = \sum_{i=1}^{j-1} \omega_{(i\,j)}.$$

These elements certainly project to the Jucys-Murphy elements $m(j)$ in $\mathbf{C}[S_n]$. With a bit of algebraic work they can be shown to commute, and their sum

$$M = \sum_{j=2}^{n} M(j) \quad = \sum_{i<j} \omega_{(i\,j)}$$

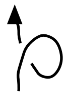
can be shown to lie in the centre of R_n^n. These facts are immediately obvious from the skein representatives $T(j)$ and $T^{(n)}$, discussed shortly.

1.1 Variants of the Hecke algebras.

A simple adjustment of the skein relations, as in [13], allows for a skein model of H_n whose parameters can be readily adapted to match any of the different appearances of the algebra.

Extend the coefficient ring to include an invertible parameter x, and define the skein $R_n^n(x, v, s)$ by linear combinations of oriented n-tangles modulo the relations

x^{-1}  $- \; x$  $= \; (s - s^{-1})$ 

and  $= \; xv^{-1}$  .

There is a natural algebra homomorphism $R_n^n(v, s) \to R_n^n(x, v, s)$ induced by replacing each tangle T in the skein $R_n^n(v, s)$ by $x^{wr(T)}T$ in $R_n^n(x, v, s)$, where $wr(T)$ is the writhe of T.

We then get corresponding versions of the Murphy elements in our new skein; for example the new version of $T(j)$ is represented by $x^{2j-2}T(j)$ in $R_n^n(x, v, s)$.

The parameters in $R_n^n(x, v, s)$ can be adjusted to match the exact version of H_n under study. Taking $x = s, q = s^2, v = 1$ we get the common algebraist's version $H_n(q)$ of [4], while the endomorphism rings arising from the fundamental representation of $sl(N)_q$ and its standard R-matrix correspond to the choice $x = s^{-1/N}, v = s^{-N}$ with $s = e^{h/2}$.

The choice $x = v$ gives the Homfly skein relation which is invariant under Reidemeister move I, and $x = q^r, s = q, v = a$ gives the version used by Kawagoe in [9].

The elements $M(j)$ above do indeed become the Murphy operators used by Dipper and James under the isomorphism of $R_n^n(v, s)$ with the algebraic version $H_n(q)$ of the Hecke algebra where $x = s, q = s^2, v = 1$.

2 Geometric views of the Murphy operators.

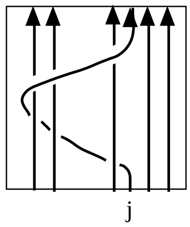
The Murphy operators $M(j)$ and their sum M have simple skein representatives as shown here, using $R_n^n(v, s)$ as the model for the Hecke algebra H_n.

Theorem 2.1 (Ram). *The Murphy operator $M(j)$ can be represented in H_n by a single braid $T(j)$, up to linear combination with the identity.*

Theorem 2.2 (Morton). *The sum M of the Murphy operators can be represented in H_n by a single tangle $T^{(n)}$, again up to linear combination with the identity.*

Proof of theorem 2.1: In [15] Ram notes that $M(j)$ can *almost* be represented by the single braid $T(j)$ pictured, where 'almost' means that $M(j)$ is a linear combination of $T(j)$ and the identity.

Set

$T(j) =$ 

in H_n, for each $j \leq n$ including the case $T(1) = 1$. Skein theory shows quickly that $T(j) - 1 = zM(j)$, giving

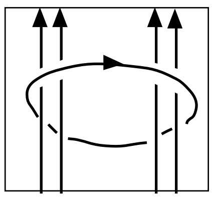
$$M(j) = \frac{T(j) - 1}{s - s^{-1}}.$$

□

The elements $T(j)$ will do equally well in place of $M(j)$, so long as $z \neq 0$, in other words away from $\mathbf{C}[S_n]$.

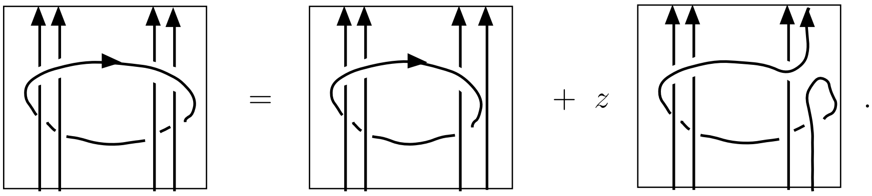
It is clear geometrically that the elements $T(j)$ all commute. It is not immediately clear that the *sum* of these elements is in the centre of H_n, although their product is the full twist, a well-known central element.

Proof of theorem 2.2: Set

$T^{(n)} =$ 

in $R_n^n \cong H_n(z)$, with the coefficient ring extended to include $v^{\pm 1}$.

Apply the skein relation at one crossing to get



Hence $T^{(n)} = T^{(n-1)} + zv^{-1}T(n)$, using the standard inclusion of H_{n-1} in H_n to interpret $T^{(n-1)}$ as an element of H_n. Then

$$T^{(n)} - T^{(0)} = zv^{-1}\sum_{j=1}^{n} T(j),$$

by induction on n. Here $T^{(0)}$ is just a scalar multiple of the identity, represented by a single disjoint simple loop alongside the identity braid. Since a trivial loop in any Homfly skein contributes the scalar $\delta = (v^{-1} - v)/z$ we can rewrite the equation as

$$T^{(n)} - \frac{v^{-1} - v}{z} = zv^{-1}\sum_{j=1}^{n} T(j) = z^2v^{-1}M + nzv^{-1}.$$

Then

$$T^{(n)} = z^2v^{-1}M + \left(nzv^{-1} + \frac{v^{-1} - v}{z}\right) \times 1,$$

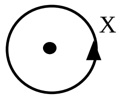
and hence, again away from $z = 0$, we can write M as a linear combination of $T^{(n)}$ and 1. □

From this representation it is quite obvious that $T^{(n)}$, and thus M, is in the centre of H_n.

There is a known set of idempotent elements, E_λ, one for each partition λ of n, which were originally described algebraically by Gyoja [5]. Skein pictures of these based on the Young diagram for λ were given by Aiston and myself [13]. We showed there, in a skein based counterpart of the results of [8], that $ME_\lambda = m_\lambda E_\lambda$ for an explicit scalar m_λ and that $m_\lambda \neq m_\mu$ when $\lambda \neq \mu$. It follows at once that $T^{(n)}E_\lambda = t_\lambda E_\lambda$, where the scalars t_λ are different for each partition λ.

3 The skein of the annulus.

The Homfly skein of the annulus, $\mathcal{C}$, as discussed in [12] and originally in [16], is defined as linear combinations of diagrams in the annulus, modulo the Homfly skein relations. The element $X \in \mathcal{C}$ will be indicated on a diagram as



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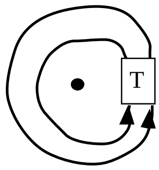
We shall make use of the results of section 2 to study $\mathcal{C}$, and also to find simple skein representatives for the power sums of the Murphy operators in H_n in a way which is independent of n.

The best known relation of $\mathcal{C}$ with the Hecke algebra H_n is the closure map $R_n^n \to \mathcal{C}$, induced by taking a tangle T to its closure $\hat{T}$ in the annulus, defined by

$\hat{T} =$ 

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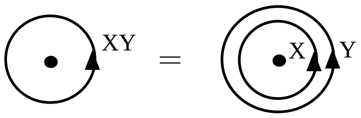
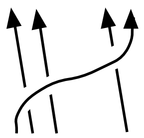
This is a linear map, whose image we will call $\mathcal{C}_n$.

The skein $\mathcal{C}$ has a product induced by placing one annulus outside another, under which $\mathcal{C}$ becomes a commutative algebra;



.

Write $A_m \in \mathcal{C}$ for the closure of the m-braid

$\sigma_{m-1} \cdots \sigma_2 \sigma_1 =$ 

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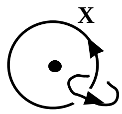
An explicit spanning set of $\mathcal{C}_n$ consists of the monomials in $\{A_j\}$ of total weight n, where A_j has weight j. Such monomials correspond bijectively with partitions of n.

A very simple skein theory construction determines a natural linear map $\varphi : \mathcal{C} \to \mathcal{C}$, induced by taking a diagram X in the annulus and linking it once with a simple loop to get

$\varphi(X) =$  .

It is instructive to look at the eigenvectors of φ. Clearly φ carries $\mathcal{C}_n$ into itself. What is more, if we take an element $S \in H_n$ with closure $\hat{S} \in \mathcal{C}_n$ and compose it with the central element $T^{(n)}$ then $ST^{(n)}$ has closure $\varphi(\hat{S})$.

Theorem 3.1 *The eigenvalues of $\varphi|\mathcal{C}_n$ are all distinct.*

Proof: Take $S = E_\lambda$ to see that the closure of $T^{(n)}E_\lambda$ is $t_\lambda \hat{E}_\lambda$ and also $\varphi(\hat{E}_\lambda)$. The element $Q_\lambda = \hat{E}_\lambda \in \mathcal{C}_n$ is then an eigenvector of φ with eigenvalue t_λ. Now there are $\pi(n)$ of these eigenvectors, (the number of partitions of n), and the eigenvalues are all distinct, by [13]. Since $\mathcal{C}_n$ is spanned by $\pi(n)$ elements we can deduce that the elements Q_λ form a basis for $\mathcal{C}_n$, and that the eigenspaces of φ are all 1-dimensional. □

It follows that *any* element of $\mathcal{C}_n$ which is an eigenvector of φ must be a multiple of some Q_λ.

The elements $h_i \in \mathcal{C}$, discussed below, arise as Q_λ for the Young diagram λ consisting of a single row with i cells. Kawagoe [9] constructed an element s_λ of $\mathcal{C}_n$ for each partition λ of n, as a Schur polynomial of the sequence $\{h_i\}$, in Macdonald's context of symmetric functions, [11]. He then used skein theory to show that each s_λ is also an eigenvector of the map φ. His elements s_λ can now be identified almost at once with the elements Q_λ, by theorem 3.1.

This gives an immediate proof of the multiplicative properties of the elements Q_λ in the skein of the annulus, since the elements s_λ automatically multiply according to the Littlewood-Richardson rules for Young diagrams [11]. Conversely it guarantees that Kawagoe's elements s_λ are eigenvectors for a wide range of skein maps on $\mathcal{C}$ as well as φ, using the skein properties of E_λ from [13].

Recently Lukac, [10], has been able to simplify the proof that s_λ is an eigenvector of φ, by using algebraic properties of the skein $\mathcal{A}$ of the annulus with an input on one boundary and an output on the other. Discussion and use of the skein $\mathcal{A}$ in section 4 leads to simple descriptions for other symmetric functions of the Murphy operators, derived from a direct interplay between the skein of the annulus and the centre of the Hecke algebras.

3.1 Symmetric functions and the skein of the annulus.

In this section I recall some explicit results about elements in the Hecke algebras and their closure in $\mathcal{C}$, and their interpretation in the context of symmetric functions, following the methods of Macdonald. This leads to a simple description in terms of generating functions of some interrelations in $\mathcal{C}$, and also of the elements to be used later to represent the power sums of the Murphy operators.

The starting point is the description of the two simplest idempotents in H_n, corresponding to the single row and column Young diagrams. These are given algebraically in [6]; here I use the skein version as described in [12] in terms of the positive permutation braids ω_π, $\pi \in S_n$. Define two quasi-idempotents by

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} \omega_\pi, \quad b_n = \sum_{\pi \in S_n} (-s)^{-l(\pi)} \omega_\pi,$$

where $l(\pi) = wr(\omega_\pi)$ is the writhe of the braid ω_π.

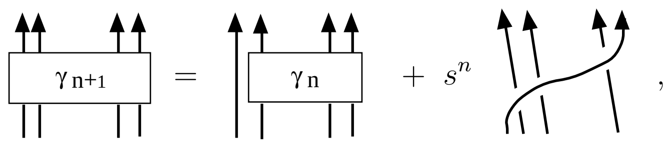
The following result is straightforward.

Lemma 3.2 *We can write*

$$a_n = a_{n-1}\gamma_n,$$

where $\gamma_n = 1 + s\sigma_{n-1} + s^2\sigma_{n-1}\sigma_{n-2} + \cdots + s^{n-1}\sigma_{n-1}\cdots\sigma_1$.

We have $\gamma_{n+1} = 1 + s\sigma_n\gamma_n$, and also an immediate skein relation



in R^{n+1}_{n+1}.

The next lemma is proved in [13].

Lemma 3.3 *For any braid $\beta \in B_n$ we have $a_n\beta = \varphi_s(\beta)a_n = \beta a_n$, where $\varphi_s(\beta) = s^{wr(\beta)}$.*

Similar results, with s replaced by $-s^{-1}$, hold for b_n.

The element a_n then satisfies

$$a_n^2 = \varphi_s(a_n)a_n = \varphi_s(a_{n-1})\varphi_s(\gamma_n)a_n.$$

Since $\varphi_s(\gamma_n) = 1 + s^2 + \cdots + s^{2n-2} = s^{n-1}[n]$, we have an immediate corollary.

Corollary 3.4 *We can write*

$$s^{n-1}[n]h_n = h_{n-1}\gamma_n,$$

where $h_n = a_n/\varphi_s(a_n)$ is the true idempotent.

The element $h_n \in H_n$ is the idempotent which corresponds to the single row with n cells. The single column idempotent is given from h_n by using $-s^{-1}$ in place of s.

With a slight abuse of notation write $h_n \in \mathcal{C}$ in place of $\hat{h}_n$ for the closure of this element in $\mathcal{C}$, and $e_n \in \mathcal{C}$ for the closure of the single column idempotent.

Remark. Aiston uses the notation Q_{c_n} and Q_{d_n} in [1], in place of e_n and h_n which are used here to suggest the terminology and techniques of symmetric functions from [11].

Write

$$H(t) = 1 + \sum_{n=1}^{\infty} h_n t^n$$

for the generating function of the elements $\{h_n\}$, regarded as a formal power series with coefficients in $\mathcal{C}$, and similarly set $E(t) = 1 + \sum_{n=1}^{\infty} e_n t^n$. In [1] Aiston showed that

Theorem 3.5

$$E(-t)H(t) = 1,$$

as power series in $\mathcal{C}$.

If we regard the elements h_n formally as the nth complete symmetric functions in a large number N of variables $x_1, \ldots, x_N$, setting

$$H(t) = \prod_{i=1}^{N} \frac{1}{1 - x_i t} \ ,$$

then $E(t) = \prod(1 + x_i t)$ and e_n is the nth elementary symmetric function in the variables.

Following Macdonald, the Schur function $s_\lambda(x_1, \ldots, x_n)$ for each partition λ can be expressed as a polynomial in the elements h_n which does not depend on N for large enough N, and is therefore determined formally by the series $H(t)$ and λ as an element s_λ of $\mathcal{C}$. This is the element s_λ used by Kawagoe, which is identified with the idempotent closures Q_λ by Lukac in [10].

The $\pi(n)$ elements $\{s_\lambda\}$ with $|\lambda| = n$ form a linear basis for $\mathcal{C}_n$, and can be expressed in terms of the monomials of weight n in $\{h_r\}$ by the classical Jacobi-Trudy formulae [11].

Indeed the skein $\mathcal{C}^+$, defined as $\cup \mathcal{C}_n$, is spanned by all monomials in $\{h_r\}$, and can be interpreted as the ring of symmetric polynomials in an unlimited number of variables $\{x_i\}$.

Monomials in the geometrically simpler closed braid elements $\{A_m\}$ also span $\mathcal{C}_n$, and it is thus interesting to relate these directly to $\{h_r\}$. An attractive formula connecting these two generating sets can be derived from corollary 3.4.

Theorem 3.6 *Write* $A(t) = 1 + z\sum_{m=1}^{\infty} A_m t^m$, *with* $z = s - s^{-1}$.
Then

$$A(t) = \frac{H(st)}{H(s^{-1}t)} = H(st)E(-s^{-1}t).$$

The proof will be given in section 4.

Lemma 3.7 *The elements* $\{h_n\}$ *are invariant under the mirror map switching crossings and inverting* s *and* v.

Proof : In H_n we have $\sigma_i a_n = \varphi_s(\sigma_i) a_n = s a_n$. Then $\overline{\sigma_i a_n} = \sigma_i^{-1} \overline{a}_n = s^{-1} \overline{a}_n$ giving $\sigma_i \overline{a}_n = s \overline{a}_n$. It follows that $\beta \overline{a}_n = \varphi_s(\beta) \overline{a}_n$ for any β and so $a_n \overline{a}_n = \varphi_s(a_n) \overline{a}_n$.

On the other hand $a_n \overline{a}_n = a_n \varphi_s(\overline{a}_n)$ by lemma 3.3, so that

$$h_n = \frac{a_n}{\varphi_s(a_n)} = \frac{\overline{a}_n}{\varphi_s(\overline{a}_n)} = \overline{h}_n.$$

□

Corollary 3.8 *The inverse series to* $A(t) = 1 + z\sum A_m t^m$ *is* $\overline{A}(t) = 1 - z\sum \overline{A}_m t^m$*, where* $\overline{A}_m$ *is the negative closed braid corresponding to* A_m*.*

Proof : Apply the mirror map in theorem 3.6 to get

$$\overline{A}(t) = \overline{H(st)} \ / \ \overline{H(s^{-1}t)} = H(s^{-1}t) \ / \ H(st) = A(t)^{-1}.$$

□

In our later description of the power sums of Murphy operators we use the element $P_m \in \mathcal{C}$ which can be interpreted formally as the mth power sum, $P_m = \sum x_i^m$, of the variables. This is determined unambiguously as a polynomial in $\{h_r\}$, and hence an element of $\mathcal{C}$, by Newton's power series equation

$$\sum_{m=1}^{\infty} \frac{P_m}{m} t^m = \ln H(t).$$

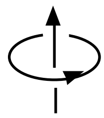
Aiston showed in [2] that $[m]P_m$ is the sum of m closed m-string braids given by switching the first i crossings of the closed braid A_m from positive to negative, for $i = 0, \ldots, m-1$. Her proof depends on some quantum group translations to identify s_λ and Q_λ, and to write A_m in terms of s_λ. In another article I give a direct skein theory proof of her result, using theorem 3.6.

3.2 Symmetric functions of the Murphy operators.

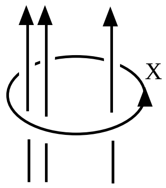
The main result in this paper, besides theorem 2.2 and its consequences, is an expression for the mth power sum of the Murphy operators in H_n in terms of the element $P_m \in \mathcal{C}$.

For this I use a previously unremarked relation between the Hecke algebras and the skein of the annulus. This relation takes the form of a very natural homomorphism ψ_n from $\mathcal{C}$ to the centre of each algebra H_n.

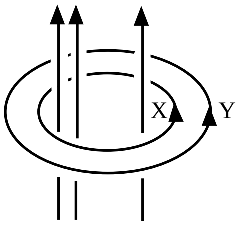
The diagram

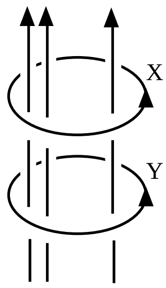
$D =$ 

determines a map $\psi_n : \mathcal{C} \to H_n$, induced by placing $X \in \mathcal{C}$ around the circle in D and the identity of H_n on the arc, to get

$\psi_n(X) =$  $\in H_n.$

It is clear that

$\psi_n(XY) =$ 

$=$ 

$= \psi_n(X)\psi_n(Y),$

so that ψ_n is an algebra homomorphism. Moreover the elements $\psi_n(X)$ obviously all lie in the centre of H_n, and in fact it follows from theorem 3.9 below that the image of ψ_n consists af all symmetric polynomials in the Murphy operators and so, by [4], makes up the whole of the centre in the generic case.

We already know from theorem 2.2 that the sum $\sum T(j)$ is essentially $T^{(n)}$, and we can write $T^{(n)} = \psi(X)$ with $X = h_1$, represented by a single string around the core of the annulus. We may then look for elements X_2 with $\psi(X_2) = \sum T(j)^2$ and more generally $\psi(X_m) = \sum T(j)^m$.

These can certainly be found for each fixed n, using for example a suitable polynomial in h_1 which depends on n. Much more surprising is the result of theorem 3.9, that there exists an element X_m *independent of* n with the property that

$$\psi_n(X_m) = \sum_{j=1}^{n} T(j)^m,$$

up to a multiple of the identity. Consequently any other symmetric function of the Murphy operators can be achieved as $\psi(X)$ where the choice of X is essentially independent of n.

In fact, we can take $X_m = P_m$, to get the following explicit result, proved in section 4.

Theorem 3.9 *For any n we have*

$$\psi_n(P_m) - \psi_0(P_m) = (s^m - s^{-m})v^{-m} \sum_{j=1}^{n} T(j)^m.$$

This is particularly satisfactory in that P_m itself arises as a power sum, although not of any identifiable objects in the ring $\mathcal{C}$ itself.

Since any element $X \in \mathcal{C}^+$ can be written as a polynomial in $\{P_m\}$ we can then express $\psi(X)$ as a symmetric function of the Murphy operators $T(j)$. It is useful to be able to write the elements $\psi(h_i)$ in terms of the Murphy operators; theorem 3.9 leads to a compact expression using formal power series with coefficients in the centre of the Hecke algebra.

I shall do this in terms of the *Murphy series* $HM(t)$ of the Hecke algebra, whose coefficients, lying in the centre of H_n, are defined to be the complete symmetric functions of the elements $T(j)$. Thus the Murphy series in H_n can be written explicitly as

$$HM(t) = \prod_{j=1}^{n} (1 - T(j)t)^{-1}.$$

The series has a formal inverse, $EM(-t)$, where the coefficients of $EM(t) = \prod(1 + T(j)t)$ are the elementary symmetric functions of the elements $T(j)$, also in the centre of H_n.

Theorem 3.10

$$\psi_n(H(t)) = \psi_0(H(t)) \frac{HM(sv^{-1}t)}{HM(s^{-1}v^{-1}t)}.$$

Proof : It is enough to establish that the logarithms of the two sides are equal. Now

$$\ln(\psi_n(H(t))) = \psi_n(\ln(H(t))) = \sum_{m=1}^{\infty} \frac{\psi_n(P_m)}{m} t^m,$$

while

$$\ln(HM(t)) = \sum_{m=1}^{\infty} \frac{\sum T(j)^m}{m} t^m.$$

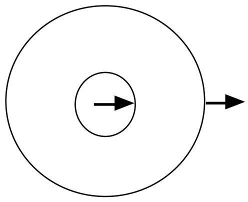
The coefficients of t^m in $\ln(\psi_n(H(t)))$ and in

$$\begin{aligned} & \ln\left(\psi_0(H(t))\frac{HM(sv^{-1}t)}{HM(s^{-1}v^{-1}t)}\right) \\ = & \ln(\psi_0(H(t))) + \ln\left(HM(sv^{-1}t)\right) - \ln\left(HM(s^{-1}v^{-1}t)\right) \end{aligned}$$

are equal, by theorem 3.9. □

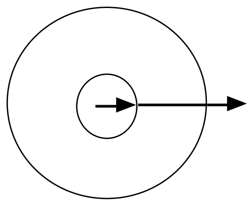
4 The annulus with two boundary points.

To prove theorems 3.6 and 3.9 I introduce a third skein $\mathcal{S}(F)$, whose underlying surface is the annulus with a single input on one boundary component and corresponding output on the other, as shown,



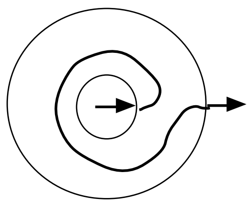
.

Write $\mathcal{A}$ for this skein, whose elements are represented as linear combinations of oriented tangles in the annulus consisting of a single arc and a number of closed curves. As for the skein $\mathcal{C}$ of the annulus there is a product on $\mathcal{A}$ induced by placing one annulus outside the other. The identity element $e \in \mathcal{A}$ is represented by the tangle



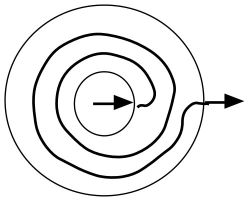
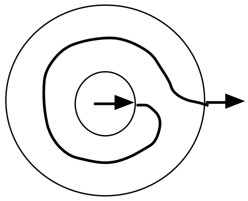
.

Write $a \in \mathcal{A}$ for the element represented by

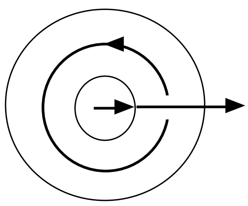
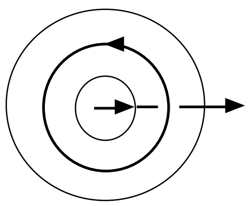


.

From this we can construct a^m for each $m \in \mathbf{Z}$, giving for example

$a^2 =$  , $a^{-1} =$  .

There are *two* bilinear products, $l : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ and $r : \mathcal{A} \times \mathcal{C} \to \mathcal{A}$ induced by placing an element of $\mathcal{C}$ respectively under or over an element of $\mathcal{A}$. For example, $h_1 \in \mathcal{C}$, represented by a single counterclockwise loop, gives

$l(h_1, e) =$  , $r(e, h_1) =$  .

From the skein relation in $\mathcal{A}$ we have $l(h_1, e) - r(e, h_1) = (s - s^{-1})a$.

4.1 Algebraic properties of $\mathcal{A}$.

Kawagoe [9] and other authors have used a skein which is linearly isomorphic to $\mathcal{A}$, based on the annulus with input and output on the *same* component. The significant advantage of $\mathcal{A}$ lies in its algebraic properties.

Both l and r are algebra homomorphisms, since it is clear diagrammatically that $l(c_1, a_1)l(c_2, a_2) = l(c_1c_2, a_1a_2)$, while r behaves similarly.

Theorem 4.1 *The algebra $\mathcal{A}$ is commutative.*

Proof : Unlike the case of $\mathcal{C}$ this is not immediately clear. Using standard skein theory techniques we can represent any element of $\mathcal{A}$ as a linear combination of tangles consisting of a totally descending arc lying over a number of closed curves. Each such tangle represents $l(c_m, a^m) = l(c_m, e)a^m$ for some m and some $c_m \in \mathcal{C}$. The general element of $\mathcal{A}$ can then be written as a Laurent polynomial

$$\sum_{m \in \mathbf{Z}} l(c_m, e)a^m$$

in a, with coefficients in the commutative subalgebra $l(\mathcal{C}, e) \subset \mathcal{A}$. Since a commutes with $l(\mathcal{C}, e)$ it follows that any two elements of $\mathcal{A}$ commute. $\square$

The subalgebras $l(\mathcal{C}, e)$ and $r(e, \mathcal{C})$ are both isomorphic to $\mathcal{C}$, but they are not equal. Their difference determines a sort of commutator map $[\ , e] : \mathcal{C} \to \mathcal{A}$, defined by $[c, e] = l(c, e) - r(e, c)$.

We have already noted that $[h_1, e] = (s - s^{-1})a$. We can deduce theorem 3.9 about the power sums of Murphy operators from the next theorem.

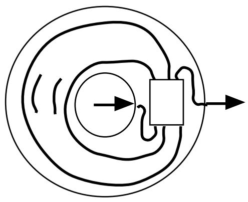
Theorem 4.2 *For $m \geq 1$ we have $[P_m, e] = (s^m - s^{-m})a^m$.*

4.2 Skein interaction between H_n, $\mathcal{A}$ and $\mathcal{C}$.

We can make use of wiring diagrams, as described in [13], to induce linear maps between skeins. A *wiring diagram* W is an inclusion of one surface F into another F', along with a fixed diagram of curves in F' which avoid F, and connect any distinguished input and output points on the boundaries of both, respecting orientation. A wiring W induces a linear map, denoted $\mathcal{S}(W) : \mathcal{S}(F) \to \mathcal{S}(F')$, using W to extend any diagram in F to a diagram in F'.

To avoid over-elaborate notation I shall write W in place of $\mathcal{S}(W)$ where there is no risk of confusion.

The diagram

$W_n =$ 

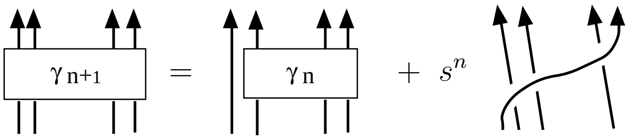
with n strings running around the annulus then induces a linear map $W_n : R_{n+1}^{n+1} \to \mathcal{A}$, by inserting an $(n+1)$-tangle in the empty box.

When an element $S \in H_n$ is included in H_{n+1} in the standard way, using their skein versions, we can see from a diagram that $W_n(S) = W_{n-1}(S)a$. Then $W_n(1) = a^n$, where $a \in \mathcal{A}$ is the element shown above.

Theorem 4.3 *The elements $W_n(h_{n+1}), W_n(h_n)$ and $l(h_n, e)$ in $\mathcal{A}$ satisfy the linear relation*

$$[n+1]W_n(h_{n+1}) = s^{-1}[n]W_n(h_n) + l(h_n, e).$$

Proof : The relation



gives $W_n(h_n\gamma_{n+1}) = W_n(\gamma_n h_n) + s^n W_n(h_n\sigma_n\cdots\sigma_1)$, after taking γ_n from the second diagram around the strings of the wiring.

Now $\gamma_n h_n = s^{n-1}[n]h_n$ and $W_n(h_n\sigma_n\cdots\sigma_1) = l(h_n, e)$. Combined with the equation $s^n[n+1]h_{n+1} = h_n\gamma_{n+1}$ from corollary 3.4 the result follows at once. □

Now write $Y_n = [n+1]W_n(h_{n+1})$ and set $Y(t) = \sum_{n=0}^{\infty} Y_n t^n$.

Corollary 4.4 *As power series with coefficients in $\mathcal{A}$ we have*

$$l(H(t), e) = (e - s^{-1}at)Y(t).$$

Proof : Since $W_n(h_n) = W_{n-1}(h_n)a = aW_{n-1}(h_n)$ the relation can be written $Y_n = s^{-1}aY_{n-1} + l(h_n, e)$. This gives $Y(t) = s^{-1}atY(t) + l(H(t), e)$, and hence the result. □

The mirror map, switching crossings and inverting s and v, when applied to $l(X, e)$ for any $X \in \mathcal{C}$ gives $r(e, \overline{X})$. Now $\overline{H(t)} = H(t)$ and so $\overline{Y(t)} = Y(t)$, giving

$$r(e, H(t)) = (e - sat)Y(t).$$

Subtracting this from the equation in corollary 4.4 then gives $[H(t), e] = (s - s^{-1})atY(t)$, a result used by Lukac in [10].

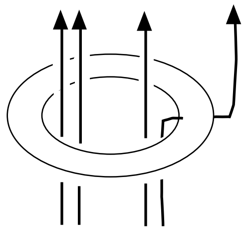
Proof of theorem 4.2: Take logarithms of the equations and then subtract. Then

$$\begin{aligned} \ln(e - s^{-1}at) - \ln(e - sat) &= \ln(l(H(t), e)) - \ln(r(e, H(t))) \\ &= l(\ln(H(t)), e) - r(e, \ln(H(t))) \\ &= \sum_{m=1}^{\infty} [\frac{P_m}{m} t^m, e]. \end{aligned}$$

Now $\ln(e - s^{-1}at) = -\sum_{m=1}^{\infty} s^{-m}a^m t^m/m$, and it only remains to compare the coefficients of t^m to see that $[P_m, e] = (s^m - s^{-m})a^m$. □

Proof of theorem 3.9: Use the wiring

V_n =



with $n-1$ strings passing through the annulus to carry $\mathcal{A}$ into R_n^n. Then for any X in $\mathcal{C}$ we have $V_n(l(X,e)) = \psi_n(X)$ and $V_n(r(e,X)) = \psi_{n-1}(X)$ with the standard inclusion. We can also see that $V_n(a) = v^{-1}T(n)$, and extend this to $V_n(a^m) = v^{-m}T(n)^m$. Then $\psi_n(P_m) - \psi_{n-1}(P_m) = (s^m - s^{-m})v^{-m}T(n)^m$.

Induction on n completes the proof that

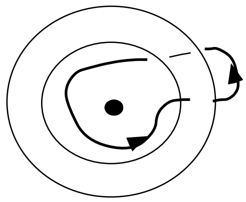
$$\psi_n(P_m) - \psi_0(P_m) = (s^m - s^{-m})v^{-m}\sum_{j=1}^{n} T(j)^m.$$

□

To conclude this paper I give the proof of theorem 3.6, using one final wiring, this time to induce a map from $\mathcal{A}$ to $\mathcal{C}$.

Write

C =



for the wiring diagram shown, which provides a form of closure for an element of $\mathcal{A}$, and determines a linear map $C : \mathcal{A} \to \mathcal{C}$. It is clear that $C(a^m) = A_{m+1}$ for every $m \geq 0$. We can also see readily that

$$C(F\ r(e,X)) = C(r(F,X)) = C(F)X$$

for every $F \in \mathcal{A}$ and $X \in \mathcal{C}$.

Proof of theorem 3.6: The relation $(e - sat)Y(t) = r\,(e, H(t))$ gives

$$Y(t) = \left(\sum_{m=0}^{\infty} s^m a^m t^m\right) r\,(e, H(t))\,.$$

Then

$$\begin{aligned} C\left(Y(t)\right) &= C\left(\sum_{m=0}^{\infty} s^m a^m t^m\right) H(t) \\ &= \left(\sum_{m=0}^{\infty} s^m A_{m+1} t^m\right) H(t), \end{aligned}$$

giving

$$zstC\left(Y(t)\right) = \left(A(st) - 1\right) H(t).$$

Now $C\left(W_n(h_{n+1})\right)$ is the closure of the $(n+1)$-tangle $\sigma_1\sigma_2\cdots\sigma_n h_{n+1} = s^n h_{n+1}$. Then $C(Y_n) = [n+1]s^n h_{n+1} \in \mathcal{C}$. It follows that

$$\begin{aligned} zstC\left(Y(t)\right) &= \sum_{n=0}^{\infty}(s^{n+1} - s^{-(n+1)})s^{n+1}h_{n+1}t^{n+1} \\ &= \sum_{n=0}^{\infty}(s^{2n+2} - 1)h_{n+1}t^{n+1} \\ &= H(s^2t) - H(t). \end{aligned}$$

This gives $H(s^2t) - H(t) = A(st)H(t) - H(t)$, and thus $H(st) = A(t)H(s^{-1}t)$, to complete the proof. □

Version 1.2, February 2001.

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