Explosion of escaping endpoints of exponential maps

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Abstract

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In 1988, Mayer proved that for exponential maps $f_a = e^z + a$ where $a < -1$, the set $E(f_a)$ of endpoints is totally disconnected but $E(f_a)$ with the point at infinity is connected. This result raises the question: Do these topological properties hold for the set of escaping endpoints, in the sense of Schleicher and Zimmer [SZ03], for exponential maps?
As an answer to this question we proved that for exponential maps where the singular value escapes or is accessible from the escaping set of f_a, the set of escaping endpoints is totally disconnected but the set of escaping endpoints with the point at infinity is connected, as one of our main results. Then we proved that the latter topological property, the set of escaping endpoints with the point at infinity is connected, could be generalized for all exponential maps f_a, where $a \in \mathbb{C}$. A further question which then arose was: Are there some cases in which the set of endpoints and the set of escaping endpoints of exponential maps are connected? We answered this question by showing that there could be a case in which the sets are connected.

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Chapter 1

Introduction

1.1 Introduction

In topology, surprising properties might appear as we venture deeper into the study of some topological spaces. One of these surprising properties is the existence of explosion points, which is the main subject in our thesis. In a metric space X an *explosion point* is a point which keeps the space connected but without which the space is totally separated. A topological space X is called *totally separated* if for every two distinct points in X there exist nonempty, closed and open sets whose intersection is empty and whose union contains X, such that each point belongs to a different set. Note that every totally separated space is totally disconnected but every totally disconnected space is not necessarily totally separated.

According to [Wil27], in 1921 Knaster and Kuratowski, gave an example of a totally disconnected space which becomes connected when adding a certain point. Then in 1927 Wilder [Wil27] gave the first example of a space having an explosion point. Another famous example of a space having an explosion point, which is one of the important keys for our thesis, is the set of endpoints of a *Lelek fan*, together with its top [Lel61, §9] (more details and properties of Lelek fans are given in Section 2.3). Then in 1988 Mayer [May90] showed that explosion points can be obtained in the iteration of transcendental entire functions. This result together with the work of Schleicher and Zimmer [SZ03], is the inspiration of our research.

The study of the iteration (dynamics) of holomorphic functions, in general, was started in the early twentieth century by Fatou and Julia. They concluded that the complex plane divides into two invariant sets due to the behavior of a holomorphic function under iteration. The first set which consists of all points at which the function f has stable behavior under iteration, is called the Fatou set $F(f)$. The other set, called the Julia set $J(f)$, is the complement of $F(f)$ and consists of the points near which the function has chaotic behavior. Their study concentrated on rational functions, then later Fatou extended some of his results to entire functions [Ber93]. About 60 years later, the investigation and the study of the iteration of holomorphic functions arose again. Most of these studies concentrated on the Julia set where the mysterious and interesting dynamics appears, more details and definitions on holomorphic dynamics can be found in Chapter 3.

In our thesis we are working on the Julia sets of exponential maps of the form:

$$f_a\colon \mathbb{C} \mapsto \mathbb{C}; z \to e^z + a \quad \text{for } a \in \mathbb{C}.$$

The dynamics of the exponential maps for $a \in (-\infty, -1)$ are well understood [DG87]. One of the main fact about it is that $J(f_a)$ consists of uncountably many curves, each connecting a finite point in the Julia set, called an endpoint, to infinity. The set of all of these endpoints is denoted by $E(f_a)$.

For general $a \in \mathbb{C}$ the Julia set $J(f_a)$ does not have the same structure as for $a < -1$. On the other hand a natural concept of "hairs" or "rays" generalizing the curves mentioned above still exists. Schleicher and Zimmer [SZ03] have shown that the escaping sets of exponential maps are organized in curves tending to infinity. Some of these curves land at points in the Julia set, with such points being called escaping endpoints and being denoted by $\tilde{E}(f_a)$.

In 1988 Mayer [May90] proved that (for $a < -1$) the set $E(f_a)$ of endpoints is totally separated but $E(f_a) \cup \{\infty\}$ is connected. These two great results raise the question:

For an exponential map f_a, is ∞ an explosion point for the set $\tilde{E}(f_a) \cup \{\infty\}$?

In order to answer this question we had to study the connectedness of the set of escaping endpoints with and without the point at infinity. Then we obtained our three main theorems. As a first step to introduce our theorems, we will define the concepts we use in our thesis as follows; see Section 3.2 for a more detailed discussion.

Definition 1.1.1 (Types of escaping points).
The set of escaping points *of f_a is denoted by*

$$I(f_a) := \{z \in \mathbb{C} \colon f_a^n(z) \to \infty\}.$$

We say that a point $z_0 \in \mathbb{C}$ is on a hair *if there exists an arc $\gamma \colon [-1, 1] \to I(f_a)$ such that $\gamma(0) = z_0$.*

We say that a point $z_0 \in \mathbb{C}$ is an endpoint *if z_0 is not on a hair and there is an arc $\gamma \colon [0, 1] \to \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(t) \in I(f_a)$ for all $t > 0$. The set of all endpoints is denoted by $E(f_a)$, while $\tilde{E}(f_a) := E(f_a) \cap I(f_a)$ denotes the set of* escaping endpoints.

Definition 1.1.2 (Singular value).
. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an entire function. A critical value is a point $w = f(z)$ with $f'(z) = 0$; the point z is a critical point. An asymptotic value is a point $w \in \mathbb{C}$ such that there exists a curve $\gamma \colon [0, \infty) \to \mathbb{C}$ so that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to w$ as $t \to \infty$. The set of singular values of f is the closed set

$$S(f) := \overline{\{\text{ critical and asymptotic values}\}}.$$

In our study of the connectedness of the set of escaping endpoints of exponential maps f_a, we first prove the following theorem which shows that ∞ is an explosion point of the set $\tilde{E}(f_a) \cup \{\infty\}$ for some parameters a.

Theorem 1.1.3.
Let f_a be an exponential map. If the singular value a, which is an asymptotic value of f_a, satisfies one of the following conditions:

1. *the singular value a is on a hair, or*

2. *the singular value a is an endpoint of f_a,*

then ∞ is an explosion point for the set $\tilde{E}(f_a) \cup \{\infty\}$.

From the definition of explosion points, the proof of this Theorem requires two steps: First, we must show that $\tilde{E}(f_a) \cup \{\infty\}$ is connected, and then we must prove that $\tilde{E}(f_a)$ is totally separated.

It turns out that the former topological property can be generalized for all exponential maps f_a, where $a \in \mathbb{C}$, and this result is our second theorem in the thesis:

Theorem 1.1.4.
Let f_a be an exponential map where $a \in \mathbb{C}$. Then $\tilde{E}(f_a) \cup \{\infty\}$ is connected.

We proved this theorem by using a topological model for the dynamics which was developed by Rempe-Gillen in [Rem06], and then infer the general case using a conjugacy result also proved in [Rem06].

In Theorem 1.1.3, because of the conditions on the parameter a, we were able to use combinatorial techniques to prove that $\tilde{E}(f_a)$ is totally separated. On the other hand, without using combinatorial techniques, the question depends on extremely difficult questions concerning the possible accumulation behavior of rays in the dynamical plane. For example, it is widely believed that a dynamic ray cannot accumulate on the entire complex plane [Rem03, Question 7.1.33], but this question remains open, as far as we are aware. The following result indicates that proving that ∞ is an explosion point for $\tilde{E}(f_a)$, $a \in \mathbb{C}$, is at least as difficult.

Theorem 1.1.5.

Suppose that, for some parameter $a \in \mathbb{C}$, the set $I(f_a)$ has a path-connected component that is dense in the Julia set. Then $\tilde{E}(f_a)$ and $E(f_a)$ are connected.

Idea of the proofs

In Chapter 6 we will prove Theorem 1.1.4. We will do so using an explicit topological model for the Julia set of f_a, which was introduced in [Rem06]. We review its definition and our desired results in Chapter 4 by showing that this model contains an invariant subset W such that $W \cup \{\infty\}$ is homeomorphic to a Lelek fan. Then we transfer this result to the dynamical plane of any exponential map using a general conjugacy result from [Rem06]. After that, in Chapter 7 we will prove Theorem 1.1.3. We proved the results in this order because Theorem 1.1.4 is general for all $a \in \mathbb{C}$. In order to prove Theorem 1.1.3 we have to show that for the chosen parameters a, in the theorem, the set of escaping endpoints $\tilde{E}(f_a)$ is totally separated. To do so we review and apply some combinatorial methods from [Rem07], [RS09] in Chapter 7 before using them to prove that $\tilde{E}(f_a)$ is totally separated and prove Theorem 1.1.3.

Finally, in Chapter 8 we will prove Theorem1.1.5. In order to do so we will use the fact that in the Julia set of any exponential map there exists a Cantor bouquet with ∞ as its top. More details about Cantor bouquets can be found in Section 2.4.

Basic notation

As usual, the complex plane is denoted to $\mathbb{C}$ and the Riemann sphere is denoted by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Also the set of natural numbers is denoted to $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{\infty\}$. Note that, throughout our thesis, we consider the family $(f_a)_{a\in\mathbb{C}}$ of exponential maps defined by $f_a(z) = e^z + a$.

Chapter 2

Preliminaries

2.1 General Topology

In this section, in order to explain all the notation we will include in our thesis, we will introduce and explain some topological concepts we will use in our proofs in this thesis. We expect that the reader will have some basic topological background and in particular is familiar with the concepts of open, connected, simply connected and compact sets and with closures, complements and boundaries. A comprehensive topological background can be found in [Dug78], or in any topology textbook.

In general for a connected set M, it is well known that the image of the set M under a continuous map is connected. In the following lemma we will show that the preimage of certain connected set under an entire map is connected set.

Lemma 2.1.1.
Let f be an entire function. If $M \subset \hat{\mathbb{C}}$ is connected and $\infty \in M$, then $f^{-1}(M) \cup \{\infty\}$ is connected.

Proof. Assume for a contradiction that $f^{-1}(M) \cup \{\infty\}$ is disconnected. Hence from the definition of disconnected sets, there exist two nonempty, open sets U and V such that $U \cap V = \emptyset$, $U \cup V \supseteq f^{-1}(M) \cup \{\infty\}, V \cap (f^{-1}(M) \cup \{\infty\}) \neq \emptyset$

and $U \cap (f^{-1}(M) \cup \{\infty\}) \neq \emptyset$. If $\infty \in V$, then V contains a neighborhood of ∞, that is, for a large $R > 0$, $V \supseteq \{z \in \mathbb{C} : |z| > R\}$. Hence $U \subseteq \{z \in \mathbb{C} : |z| \leq R\}$. Thus U is bounded. Therefore the closure of U, $\overline{U}$, is bounded and since it is closed then, from a well known topological property of bounded sets that a bounded closed set is compact, we obtain that it is compact. Note that the continuous image of $\overline{U}$ under f intersects the set M, that is, $M \cap f(\overline{U}) \neq \emptyset$. Now we claim that $M \cap \partial f(U) = \emptyset$.

Assume for a contradiction that $M \cap \partial f(U) \neq \emptyset$ and so there exists at least one x such that $x \in M \cap \partial f(U)$. Because $x \in \partial f(U) \subset f(\partial U)$, $x \in f(\partial U)$. Hence $f^{-1}(M) \cap \partial U \neq \emptyset$. However that is a contradiction with the assumptions, hence $M \cap \partial f(U) = \emptyset$. Thus $f^{-1}(M) \cup \{\infty\}$ is connected. □

An important concept in our research is that of an explosion point of a set, of which we gave a brief explanation in the introduction. We will define it formally in this section. In order to do so, we must define a totally separated space. Before that we remind the reader that for any two points $x, y \in X$, where X is a topological space, a closed, connected set $A \subset X$, where $x, y \notin A$, separates these two points if they belong to different component of the complement of A.

Definition 2.1.2 (Totally separated)**.**
A topological space X is said to be totally separated *if, for every two points $x_1, x_2 \in X$, there exist open sets $U, V \subset X$ such that $X = U \cup V$, $U \cap V = \emptyset$, $x_1 \in U$ and $x_2 \in V$.*

Now we are able to define an explosion point.

Definition 2.1.3 (Explosion point)**.**
Let X be a topological space and $x \in X$. We say that x is an explosion point *of the space X if X is connected but $X \setminus \{x\}$ is totally separated.*

In topology one of the main theorems concerning separations is the Plane Separation Theorem (VI 3.1 in [Why42]). In our thesis we use one of the consequences of Plane Separation Theorem which is Janiszewski's theorem. The theorem states that if two compact sets, which have a connected intersection, do not separate two points, then the union of these sets does not separate them either.

Theorem 2.1.4 (S Janiszewski [Dug78, p.362]).
Let A, B be compact sets in $\mathbb{C}$ and let x, y be two points of $\mathbb{C}$. Assume:

- *A does not separate x and y;*
- *B does not separate x and y; and*
- *$A \cap B$ is connected.*

Then $A \cup B$ does not separate x and y.

Now we will introduce an important topological theorem which is the Boundary Bumping Theorem, which we will use in the proof of our last result Theorem 1.1.5.

Before we state the theorem we will introduce a useful topological concept:

Definition 2.1.5 (Continuum).
A continuum *is a nonempty, compact, connected metric space.*

A nonempty, closed and connected subset of a continuum is called a subcontinuum.

Example 2.1.6.
Some simple examples of continua are:

- *The unit disc $\overline{\mathbb{D}}$ is a continuum in $\mathbb{C}$.*
- *The closed interval $[0, 1]$ is a continuum in $\mathbb{R}$.*

In Section 2.3 we will give a brief explanation about continua and their properties.

Theorem 2.1.7 (Boundary Bumping Theorem [Nad92]).
Let X be a continuum, and let E be a proper subset of X. Let K be a component

of E. If E is open in X, then

$$\overline{K} \setminus E \neq \emptyset.$$

If E is closed in X, then

$$K \cap \overline{(X - E)} \neq \emptyset.$$

The proof of the Boundary Bumping theorem can be found in [Nad92, p.75].

2.2 Product Topology

The natural topology on the space $\mathbb{Z}^{\mathbb{N}_0}$ is the product topology, as defined e.g; in [Kel75, p.90]. The product topology will play a main role in our thesis, because the topological model $J(\mathcal{F})$, which we will use in some of our proofs, is a subset of the product space $\mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}$. The following observation gives one of the properties of product spaces which we will use in our thesis:

Observation 2.2.1.
Let T and X_i, for $i \in \mathbb{N}_0$, be topological spaces, and consider the infinite product

$$X = T \times \prod_{i=0}^{\infty} X_i$$

(with the product topology).

Let Y be another topological space, together with a function $\lambda\colon Y \to T$. If $\underline{x}^0 = x_0x_1x_2\ldots \in \prod_{i=0}^{\infty} X_i$, then the map $f\colon Y \to X; y \to (\lambda(y), \underline{x}^0)$ is continuous if and only if λ is continuous.

Proof. Let τ be the projection map of X onto T and τ_i be the projection map of X onto X_i for all $i \in \mathbb{N}_0$. Then by using [Mor89, Proposition 10.1.8] it follows that the map f is continuous if and only if each mapping $\tau \circ f : Y \to T$ and $\tau_i \circ f : Y \to X_i$ is continuous. The maps $\tau_i \circ f$ are continuous since each one of them is a constant map. Hence the map $f : Y \to X$ is continuous if and only if the map λ is continuous. □

Definition 2.2.2 (Neighborhoods product topology).
Let $X = T \times \prod_{i=0}^{\infty} X_i$ *be as above. Let* $x = (t^s, \underline{s}) \in X$*, where* $\underline{s} = s_0 s_1 s_2 \ldots$*, then for all* $\epsilon > 0$ *there exists* $M > 0$ *such that the set* $N_{\epsilon,M}(x) := \{(t^u, \underline{u}) \in X : d_T(t - t^u) < \epsilon$ *and* $u_i = s_i$ *for all* $i < M\}$ *is called a neighborhood of* x*.*
Note that every product topology is metrizable.

More about the product topology and its properties can be found in topological texts e.g.; [Dug78], [Kel75].

2.3 Topological Fans

In this section we give a brief topological background for topological fans and define and explain the fan which A. Lelek has shown in [Lel61] as an example of a smooth fan with a certain condition on the set of its endpoints. This fan is called *a Lelek fan.*

First we recall some definitions which will be used to define a Lelek fan.

Definition 2.3.1 (Hereditarily unicoherent).
A continuum S *is called a* unicoherent *continuum if for any two subcontinuua* A *and* B *of* S *which satisfy* $S = A \cup B$*, then* $A \cap B$ *must be connected.*

If each of it is subcontinuua is unicoherent then it is said to be hereditarily unicoherent*.*

Example 2.3.2.
Any closed interval $[a, b]$ *in* $\mathbb{R}$ *is a hereditarily unicoherent continuum.*

A simple example of a continuum that is not unicoherent is the unit circle $\mathbb{S}$*.*

The following two results are well known.

Corollary 2.3.3.
If $S \subset \mathbb{C}$ is locally connected hereditarily unicoherent continuum, then it does not contain any subset which disconnects the sphere $\widehat{\mathbb{C}}$.

Proof. Assume for a contradiction that S is locally connected hereditarily unicoherent continuum containing a subset which disconnects the sphere. Then we could obtain a simple closed curve $C \subset S$, and the curve C is homeomorphic to the unit circle $\mathbb{S}$ which is not a unicoherent continuum. Thus C is not a unicoherent subcontinuum of S and this is a contradiction since S is a hereditarily unicoherent continuum. □

Definition 2.3.4 (Dendroid)**.**
A dendroid *is an arcwise-connected hereditarily unicoherent continuum.*

Lemma 2.3.5.
Let X be a dendroid, then X is uniquely arcwise-connected.

The proof of this lemma follows from the fact that if there exists a dendroid which is not uniquely arcwise-connected, then we obtain a simple closed curve which is homeomorphic to the unit circle which is not a unicoherent continuum. This contradictions the fact that a dendroid is a hereditarily unicoherent continuum.

Observation 2.3.6.
Let X be a dendroid, then every subcontinuum of X is a dendroid [Nad92, p.192].

Definition 2.3.7 (Ramification point)**.**
Let S be a dendroid. A point $z_0 \in S$ is said to be a ramification point *of the dendroid S if there are three arcs az_0, bz_0 and cz_0 in S and the intersection of any two of these three arcs is the point $\{z_0\}$.*

Observation 2.3.8.
Let X be a continuum and $K \subset X$ be a subcontinuum. If x_0 is a ramification

point of K then it is a ramification point of X too.

Proof. The proof of this observation follows directly from the definition of ramification points Definition 2.3.7. □

Now we can define a fan in terms of a dendroid.

Definition 2.3.9 (Fan)**.**
A fan *is a dendroid with exactly one ramification point.*

Such a ramification point is called the *top* of the fan.

Example 2.3.10.
The Cantor fan (the collection of the straight lines connecting the points $(0, c)$, where c is a point in the Cantor set, to the point $(\frac{1}{2}, \frac{1}{2})$ in $\mathbb{R}^2$) is a topological fan. Figure 2.1 is courtesy of Pavel Pyrih [CKP01].

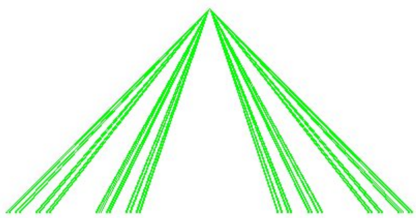

Figure 2.1: Cantor fan.

Definition 2.3.11 (Smooth fan)**.**
A fan with a top t is said to be a smooth fan *if, for each sequence $\{q_n\}_{n=1}^{\infty}$*

of its points converging to a point q, the arcs tq_n converge with respect to the Hausdorff metric to the arc tq.

Definition 2.3.12.
An endpoint *of a fan is a point that is the endpoint of each arc containing it.*

Now we have defined the concepts of a smooth fan and the endpoints of a fan we can introduce a Lelek fan:

Definition 2.3.13 (Lelek fan)**.**
A Lelek fan *is a smooth fan such that the set of its endpoints is a dense subset of the fan.*

Figure 2.2 is courtesy of Pavel Pyrih [CKP01].

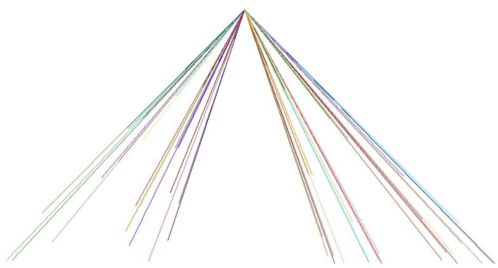

Figure 2.2: Lelek fan.

Note that the Cantor fan is a smooth fan, but its set of endpoints is not dense. However, a Lelek fan can be obtained as a subset of the Cantor fan by shortening some of the hairs inductively (see Figure 2.2). We will not give the details here.
The following Theorem is obtained in [Cha89], [AO93] as a Corollary, it gives some properties of an arbitrary smooth fan Y in terms of a Lelek fan.

Theorem 2.3.14.
Let Y be a smooth fan, then the following are equivalent:

1. *The fan Y is homeomorphic to the fan defined by Lelek in [Lel61, §9];*

2. *the set of endpoints of Y is dense;*

3. *the set of the endpoints of Y together with the top is connected.*

This theorem give us an abbreviated way to prove that a topological fan is homeomorphic to a Lelek fan by satisfying one of these equivalent conditions.

2.4 Straight brush and Cantor bouquet

In this section, we will review some topological concepts and results from [AO93] and [BJR12]. Straight brushes are defined as subsets of the product $[0,\infty)\times\mathbb{R}\setminus\mathbb{Q}$ in [AO93]. First we will present the formal definition of a straight brush from [AO93] but as it is presented in [BJR12].

Definition 2.4.1 (Straight brush).
A subset X of $[0,\infty)\times\mathbb{R}\setminus\mathbb{Q}$ is called a straight brush if the following properties hold:

1. *The set X is a closed subset of $\mathbb{R}^2$.*

2. *For every point $(x, y) \in X$ there exists $t_y \geq 0$ such that $\{x\colon (x, y) \in X\} = [t_y, \infty)$. The set $[t_y, \infty) \times \{y\}$ is called the hair attached at y and the point (t_y, y) is called its endpoint.*

3. *The set $\{y\colon (x, y) \in X$ for some $x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in X$ there exist two sequences of hairs attached respectively at $\alpha_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\alpha_n < y < \gamma_n$, $\alpha_n, \gamma_n \to y$ and $t_{\alpha_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.*

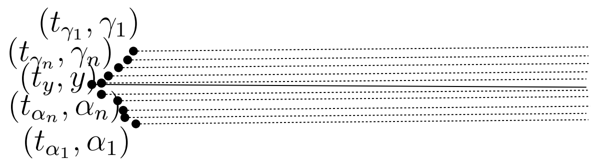

Figure 2.3: Convergence of hairs from above and below (condition 3) in a straight brush.

Some topological properties of straight brushes can be found in [BO90]. In particular, any two straight brushes are homeomorphic. In fact, in [AO93] and [BJR12] it has been shown that any two straight brushes are even *ambiently* homeomorphic, which implies that the vertical order of the hairs is preserved under the homeomorphism.

Observation 2.4.2.
From the discussion in [AO93, 1.12] we obtain that a straight brush X union the point at infinity is a Lelek fan.

Definition 2.4.3 (Comb)**.**
A comb *is a continuum X containing an arc B (called the base of the comb X) such that*

- *The closure of every component of $X \setminus B$ is an arc, with exactly one endpoint at the set B. In particular, for every $x \in X \setminus B$, there exists a unique arc $\gamma_x \colon [0,1] \to X$ with $\gamma_x(0) = x, \gamma_x(t) \notin B$ for $t < 1$ and*

$\gamma_x(1) \in B$.

- *The intersection of any two arcs is empty.*
- *The set $X \setminus B$ is dense in X.*

This definition will lead us to define another important concept from [AO93]; we will use the definitions of [BJR12]

Definition 2.4.4 (Hairy arc)**.**
A hairy arc *is a comb with base B and a total order $\prec$ on B (generating the topology of B) such that the following holds. If $b \in B$ and x belongs to the hair attached at b i.e.; the component of the comb with an endpoint b, then there exist sequences x_n^+, x_n^-, attached at points $b_n^+, b_n^- \in B$, such that $b_n^- \prec b \prec b_n^+$ and $x_n^+, x_n^- \to x$ as $n \to \infty$.*

A set $A \subset \mathbb{R}^2$ is called a one sided hairy arc *if A is topologically a hairy arc and there exists a Jordan arc $C \subset \mathbb{R}^2 \setminus A$ connecting the two endpoints of the base B of A such that the connected components of $A \setminus B$ belong to the same connected component of $\mathbb{R}^2 \setminus (B \cup C)$.*

Theorem 2.4.5 ([BJR12])**.**
Any two hairy arcs are homeomorphic.

Furthermore, any two one sided hairy arcs $X_1, X_2 \subset \mathbb{R}^2$ are ambiently homeomorphic. In other words , any homeomorphism $\theta\colon X_1 \to X_2$ extends to a homeomorphism $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$. In particular, any two straight brushes are ambiently homeomorphic.

This theorem implies that, for any straight brush Y, one can add a base $B = \{(\infty, y)\colon y \in [-\infty, +\infty]\}$ to obtain a hairy arc. Conversely, for any hairy arc X, the set $X \setminus B$, where B is the base of X, is homeomorphic to a straight brush [BJR12, p5].

Now we can define a Cantor bouquet as in [Dev99], [BJR12]:

Definition 2.4.6 (Cantor bouquet)**.**
A Cantor bouquet is any subset of the plane that is ambiently homeomorphic

to a straight brush. A well known example of a Cantor bouquet is the Julia set of an exponential map $\mathbb{C} \to \mathbb{C} : z \mapsto \exp z + a$ *where* $a \in (-\infty, -1)$ *(see Section 3.2). In Figure 2.4 The Julia set of the function* $z \mapsto \exp z - 2$, *this set is known to be a Cantor bouquet [AO93]. Figure 2.4 is courtesy of Lasse Rempe-Gillen.*

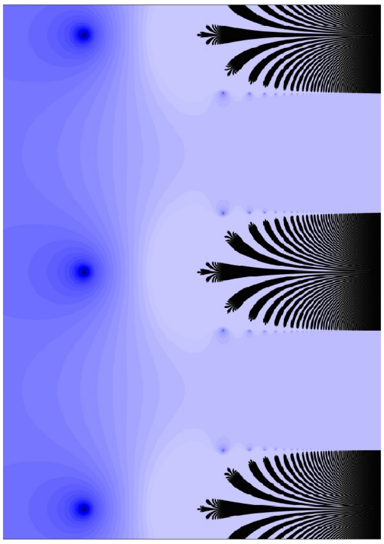

Figure 2.4: The Julia set of the function $z \mapsto \exp z - 2$.

For a Cantor bouquet C which is ambiently homeomorphic to a straight brush B, let E_C be the set of the homeomorphic image of endpoints of B in C, the endpoints of C, and let $\hat{E}_C = E_C \cup \{\infty\}$. Then we obtain the following corollary.

Corollary 2.4.7.
Let C *be a Cantor bouquet. Then the set* E_C *is totally separated, but* $\hat{E}_C$ *is connected .*

Proof. By definition, C is homeomorphic to a straight brush and any two hairs in a straight brush can be separated by a straight horizontal line in $\mathbb{R}^2$ at a suitable rational height, and any hair contains exactly one endpoint. So E_C is totally separated. Also by Observation 2.4.2 we obtain that $\hat{E}_C$ is a Lelek fan. So from Theorem 2.3.14 we obtain that $\hat{E}_C$ is connected. □

Chapter 3

Dynamics of exponential maps

3.1 Dynamics of holomorphic maps

The study of the dynamics of holomorphic maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ was started in the early twentieth century by Pierre Fatou and Gaston Julia [Ber93].

They divided the plane into two disjoint invariant sets. One is now called the Fatou set $F(f)$ which consists of all points near which the function f has stable behavior under iteration. The other set is called the Julia set $J(f)$, which is the complement of $F(f)$. It consists of the points near which the map has chaotic behavior.

More precisely, the Fatou set $F(f)$ consists of all points near which the family of iterates of the map f is a *normal family* of functions. The complement of the Fatou set is the Julia set $J(f)$ and because of its points' action under the iteration, most of the interesting and mysterious dynamics appear on the Julia set $J(f)$.

About 60 years after the work of Fatou and Julia, the study of holomorphic dynamics was renewed due to the introduction of computer graphics and new techniques from other areas. Most of the work at this point focused on the dynamics of rational functions but there were some important papers on the dynamics of transcendental entire functions, most notably by Noel Baker who had been working on this area for many years. Recently, work on the dynamics of transcendental entire functions has increased [Ber93].

In this chapter we will review some results and define some dynamical concepts which will give us a brief introduction to the Julia set of holomorphic functions, specifically the exponential maps. In order to do so we must first review some concepts from complex analysis which we use to define the Julia and the Fatou set of holomorphic functions. More and well explained complex analysis can be found e.g. in [Nee97]. In the following text an entire function, analytic on the whole complex plane, will be denoted as $f : \mathbb{C} \to \mathbb{C}$.

Definition 3.1.1 (Normality).
Let $U \subset \mathbb{C}$ be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$. A family $\mathcal{F}$ is said to be normal *on U if every sequence (f_n) of functions in $\mathcal{F}$ has a subsequence which converges locally uniformly to a holomorphic function f on U.*

Definition 3.1.2 (Equicontinuity).
A family $\mathcal{F}$ is said to be equicontinuous *at $z \in \mathbb{C}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_S(f(z), f(z_1)) < \epsilon, \text{ for all } |z - z_1| < \delta$$

and all $f \in \mathcal{F}$, where d_S is the spherical metric.

A family $\mathcal{F}$ is said to be equicontinuous *at U if it is equicontinuous in every point of U.*

These two definitions are connected to each other by an important theorem, Arzela-Ascoli's Theorem :

Theorem 3.1.3 (Arzela-Ascoli Theorem).
Let U be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of entire functions $f : \mathbb{C} \to \mathbb{C}$. A family $\mathcal{F}$ is normal on U if and only if it is equicontinuous in U.

Next we will state a theorem which plays an important role in complex dynamics.

Theorem 3.1.4 (Montel's Theorem).
Let $a, b \in \hat{\mathbb{C}}$, U be a domain in $\hat{\mathbb{C}}$, and $\mathcal{F}$ be a family of holomorphic functions on U which omit the two values $\{a, b\}$. Then $\mathcal{F}$ is normal or equivalently, by Arzela-Ascoli's theorem, is equicontinuous.

Now we have stated the main definitions we need, we can define the Julia and Fatou sets.

Definition 3.1.5 (Fatou set).
Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. A point $z \in \mathbb{C}$ is in the Fatou set *$F(f)$ if there exists a neighborhood U_z of z such that the family $\mathcal{F}$ of the iterates of f is equicontinuous at every point of U (or normal in U).*

Definition 3.1.6 (Julia set).
Let f be a holomorphic function. The Julia set *of f, denoted by $J(f)$, is the complement of the Fatou set:*

$$J(f) \colon = \mathbb{C} \setminus F(f).$$

Now we will remind the reader of some properties of the Julia set of a holomorphic function.

Lemma 3.1.7 ([Ber93]).
Let f be a holomorphic function. Then

- *$F(f)$ is open and $J(f)$ is closed.*
- *$F(f)$ and $J(f)$ are completely invariant; i.e. $z \in F(f) \Leftrightarrow f(z) \in F(f)$ and similarly $z \in J(f) \Leftrightarrow f(z) \in J(f)$.*
- *$F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$.*

Definition 3.1.8 (Backward orbit)**.**
Let f be a holomorphic function and $z_0 \in \mathbb{C}$. The forward orbit *of z_0, denoted by $O^+(z_0)$, is defined as*

$$O^+(z_0) := \{f^n(z_0)\colon n \geq 0\}.$$

The backward orbit *of z_0, denoted $O^-(z_0)$, is defined as*

$$O^-(z_0) := \bigcup_{n\geq 1} f^{-n}(z_0) = \bigcup_{n\geq 1}\{z\colon f^n(z) = z_0\}.$$

A point $z \in \mathbb{C}$ is called an *exceptional point* if $O^-(z)$ is finite.

Lemma 3.1.9 ([Ber93, Lemma 4 p.156])**.**
Let f be a holomorphic function. If $z \in J(f)$ and z is not an exceptional point, then $J(f) = \overline{O^-(z)}$.

This lemma shows that the Julia set of a function f is the smallest closed, invariant subset of $\mathbb{C}$ containing at least two points.

Lemma 3.1.10 ([Ber93, p.155])**.**
Let f be a holomorphic function. Then either $J(f) = \mathbb{C}$ or $J(f)$ has no interior.

The properties which have been stated in this section can be found (with proofs) in standard books on complex dynamics e.g. see [CG93].

3.2 Dynamics of exponential maps

In this section we will focus on the properties of Julia set of exponential maps $f_a\colon \mathbb{C} \to \mathbb{C}; z \to e^z + a$ where $a \in \mathbb{C}$.

For $a = 0$ the Julia set of such a map was studied in 1981 by Misiurewicz [Mis81]. He proved that $J(e^z) = \mathbb{C}$. This result was conjectured by Fatou in 1926.

In the case where $a \in (-\infty, -1)$, it is well known [DG87] that $J(f_a)$ consists of uncountably many curves, each connecting a finite endpoint to infinity. For $-\infty < a < -1$ it was proved in [AO93] that the Julia set of the exponential map f_a is a Cantor bouquet.

For $a \in \mathbb{C}$ it is known [SZ03] that in the Julia set there still exists a collection of rays to ∞ called *dynamic rays* on which the iterates of f_a tend to infinity. In many cases, not all rays have an endpoint [Rem07]. So in order to understand the dynamics of exponential maps we must recall the definition of dynamic rays. Dynamic rays of an exponential map f_a are defined in the set of escaping points of f_a, which is the set of points with real components escape to infinity under iteration of f_a.

Definition 3.2.1 ([MB12, Dynamic rays]).
Let f_a be an exponential map. A dynamic ray *of f_a is a maximal injective continuous curve $g\colon (0,\infty) \to I(f_a)$ such that*

1. $\lim_{t\to\infty} \operatorname{Re} f_a^n(g(t)) = \infty$ *uniformly in n, and*
2. *for all $t_0 > 0$, $\lim_{n\to\infty} \operatorname{Re} f_a^n(g(t)) = \infty$ uniformly for $t \geq t_0$.*

If additionally $z_0 := \lim_{t\to 0} g(t)$ is defined, then we say that g lands *at z_0.*

In the introduction we showed some types of escaping points see Definition 1.1.1. To remind the reader: the set of endpoints of f_a is denoted $E(f_a)$, while the set of escaping endpoints is denoted

$$\tilde{E}(f_a) = I(f_a) \cap E(f_a).$$

A critical property of the set of escaping points of exponential map is presented in the next theorem.

Theorem 3.2.2 ([Rem06, Proposition 4.2]).
Let f_a be an exponential map, where $a \in \mathbb{C}$. Suppose that $\gamma : [0,1] \to I(f_a)$ is a continuous injective map (a curve) in $I(f_a)$. Then for all $R > 0$ there exists $n > 0$ such that $Re(f_a^n(\gamma(t))) > R$ for all $t \in [0,1]$.

Chapter 4

The topological model $J(\mathcal{F})$

In [AO93] it was shown that the Julia set of an exponential map $J(f_a)$ where $a \in (-\infty, -1)$ is a Cantor bouquet. We know from Definition 2.4.6 that a Cantor bouquet is ambiently homeomorphic to a straight brush. Here we will use the same straight brush which was constructed by Rempe-Gillen in [Rem06] which does not depend on the parameter a. This straight brush is our topological model $J(\mathcal{F})$. Rempe-Gillen defined the model $J(\mathcal{F})$ as a subset of the product space $\mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}$. The dynamics of this model is similar to the dynamics of $J(f_a)$ for $a \in (-\infty, -1)$. Then in [Rem06, Theorem 1.1] Rempe-Gillen showed that for exponential maps f_a where $a \in \mathbb{C}$ the Julia sets near the point at infinity are similar, and there exists a conjugucy between the exponential maps in certain subsets of their Julia sets. Hence an exponential map in a certain subset of its Julia set is conjugate to the model function $\mathcal{F}$.

In this chapter we review the definition of the topological model $J(\mathcal{F})$ and present some properties of $J(\mathcal{F})$ in Section 4.3. The main reason for introducing the model $J(\mathcal{F})$ is that in our proof of Theorem 1.1.4 we use certain subsets of the topological model $J(\mathcal{F})$.

4.1 Introduction

In order to introduce $J(\mathcal{F})$, we need to show first how integer sequences are related to iteration under an exponential map f_a where $a \in \mathbb{C}$. We will do so by giving first a brief outline of the structure of the Julia set of the exponential map f_{-2} that is because Rempe-Gillen showed in [Rem06] that there is a conjugacy between f_{-2} and the model function $\mathcal{F}$. The interval $(-\infty, -1)$ is contained in the Fatou set, hence the Julia set is disjoint from the preimages of this interval, which are the horizontal straight lines in $\mathbb{C}$ of the form $\{\operatorname{Im} z = (2j-1)\pi\}$. Thus the Julia set of f_{-2}, $J(f_{-2})$, is completely contained in the strips:

$$S_j := \{z : \operatorname{Im} z \in ((2j-1)\pi, (2j+1)\pi), \text{for } j \in \mathbb{Z}\}.$$

Hence, by using this fact and the fact that the Julia set is invariant, for any point $z \in J(f_{-2})$ there is an associated sequence of integers $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ such that $\underline{s} = s_0 s_1 s_2 \ldots$, such that $f_{-2}^k(z) \in S_{s_k}$ for all k. This sequence of integers is called the *(infinite) external address* of z. It turns out that the connected components of the Julia set $J(f_{-2})$ are horizontal curves consisting of escaping points and an endpoint, and the points in each curve have the same external address also different curves correspond to different addresses.

The model space $J(\mathcal{F})$ is situated in the product space $[0, \infty) \times \mathbb{Z}^{\mathbb{N}_0}$. So the elements of this space are pairs $(t, \underline{s})$ where $\underline{s}$ is a sequence of integers and t is a nonnegative real number. In our thesis we will use T for the projection to the first component i.e., $T(t, \underline{s}) = t$.

To define our model $J(\mathcal{F})$ we will first define the function $F(t)$ and state some properties of it and then define a model function $\mathcal{F}$.

$F\colon [0, \infty) \to [0, \infty); F(t) = \exp(t) - 1$. The following lemmas will give us a brief explanation about the iteration of the map F:

Lemma 4.1.1.
Let $t > 0$, then $F^n(t) \to \infty$ for $n \to \infty$.

Proof. Because $t > 0$ and $F(t) > t$ for all $t > 0$ then we have

$$t < F(t) < F^2(t) < F^3(t) < \ldots < F^n(t) < \ldots$$

Hence as $n \to \infty$ and since the sequence $F^n(t)$ is increasing then it has a limit. Then by the continuity of the map F, that implies that the limit point must be ∞. □

The following property of the map F plays a significant role in proving some properties of the space $J(\mathcal{F})$:

Lemma 4.1.2.
If $t > 0$ *then*

$$F(t + \delta) \geq F(t) + F(\delta),$$

for $\delta \geq 0$.

Proof. If $t > 0$ and $\delta \geq 0$ then

$$\begin{aligned}
F(t + \delta) - F(\delta) &= e^{t+\delta} - 1 - (e^{\delta} - 1) \\
&= e^t e^{\delta} - e^{\delta} \\
&= e^{\delta}(e^t - 1) \\
&= e^{\delta} F(t) \\
&\geq F(t).
\end{aligned}$$

Hence,

$$F(t + \delta) \geq F(t) + F(\delta).$$

□

Now let us define a model function:

$$\mathcal{F} \colon \mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}; \mathcal{F}(t, \underline{s}) := (F(t) - 2\pi|s_1|, \sigma(\underline{s})),$$

where σ is the shift map and s_1 is the second entry of $\underline{s}$.

Lemma 4.1.3.
The model function $\mathcal{F}\colon \mathbb{Z}^{\mathbb{N}_0} \times [0,\infty) \to \mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}$ *is a continuous function.*

The proof of this lemma follows directly from the fact that the shift map is continuous in the product space of integer sequences, and the exponential map is continuous.
Note that the size of the image of a point under $\mathcal{F}$ is approximately the exponential of its real part. If $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and $t \geq 0$ with $T((t,\underline{s})) \geq 0$, then

$$\frac{1}{\sqrt{2}} F(t) \leq |\mathcal{F}(t,\underline{s})| \leq F(t).$$

4.2 The topological model $J(\mathcal{F})$

Definition 4.2.1 (The model space $J(\mathcal{F})$)**.**
The model space $J(\mathcal{F})$ *is a subspace of the product space we considered above and is defined as*

$$J(\mathcal{F}) := \{(t,\underline{s}) \in [0,\infty) \times \mathbb{Z}^{\mathbb{N}_0} \colon \mathcal{F}^n(t,\underline{s}) \in [0,\infty) \times \mathbb{Z}^{\mathbb{N}_0} \text{ for all } n \geq 0\}.$$

Definition 4.2.2 (The escaping set of the model space $J(\mathcal{F})$)**.**
The set of escaping points *of* $J(\mathcal{F})$ *is defined as*

$$I(\mathcal{F}) := \{(t,\underline{s}) \in J(\mathcal{F}) \colon T(\mathcal{F}^n(t,\underline{s})) \to \infty\}.$$

Note that $J(\mathcal{F})$ consists of the points of $\mathbb{Z}^{\mathbb{N}_0} \times [0,\infty)$ which always get a positive second component when iterated under $\mathcal{F}$ infinitely often i.e., for all $n > 0$, $\mathcal{F}^n(t,\underline{s})$ is defined and $T(\mathcal{F}^n(t,\underline{s})) \geq 0$.
Our model lives in the space $\mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}$ (with the product topology).

Definition 4.2.3 (Exponentially bounded address)**.**

An address $\underline{s}$ *is said to be* exponentially bounded *if there exists* $x > 0$ *such that* $|s_i| \leq F^i(x)$ *for all* $i \geq 0$*. Note that for any* $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ *if there is* $t > 0$ *with* $(t, \underline{s}) \in J(\mathcal{F})$*, then* $\underline{s}$ *is exponentially bounded.*

Definition 4.2.4 (Slow and fast addresses)**.**
A sequence $\underline{s}$ *is said to be* slow *if there exists* $x > 0$ *such that for all* $n_0 > 0$ *there exists* $n \geq n_0$ *such that* $|s_{n+i}| \leq F^i(x)$ *for all* $i \geq 0$*. However, a sequence* $\underline{s}$ *is called* fast *if it is not slow.*

Definition 4.2.5.
Let $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$*. We define*

$$t_{\underline{s}} := \begin{cases} \min\{t \geq 0\colon (t, \underline{s}) \in J(\mathcal{F})\} & \textit{if } \underline{s} \textit{ is exponentially bounded} \\ \infty & \textit{otherwise.} \end{cases}$$

If $\underline{s}$ *is exponentially bounded, then* $(t_{\underline{s}}, \underline{s})$ *is called an* endpoint *of* $J(\mathcal{F})$*, and if* $(t_{\underline{s}}, \underline{s}) \in I(\mathcal{F})$*, then it is called an* escaping endpoint*. We write* E *and* $\tilde{E}$ *for the sets of endpoints and escaping endpoints, respectively.*

Lemma 4.2.6 ([Rem06])**.**
Let $\underline{s}$ *be an external address and define*

$$t^*_{\underline{s}} := \sup_{k \geq 1} F^{-k}(2\pi|s_k|)$$

Then $\underline{s}$ *is exponentially bounded if and only if* $t^*_{\underline{s}} < \infty$ *in which case* $t^*_{\underline{s}} \leq t_{\underline{s}} \leq t^*_{\underline{s}} + 1$.

Observation 4.2.7.
Every exponentially bounded address is either slow or fast. We observe also that an exponentially bounded address $\underline{s}$ *is fast if and only if the endpoint,* $(t_{\underline{s}}, \underline{s})$*, of the curve at this address, if it exists, is an escaping point. Because otherwise we will obtain a contradiction to the definition of the fast address Definition 4.2.4 or to Lemma 4.2.6.*

Lemma 4.2.8.
The space $J(\mathcal{F})$ *is closed in* $\mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}$.

Proof. The proof follows by using the facts that under a continuous map the preimage of a closed set is closed, and that the intersection of countably many closed sets is closed. □

Definition 4.2.9.
The space $\hat{J}(\mathcal{F}) = J(\mathcal{F}) \cup \{\infty\}$ *is the one point compactification of* $J(\mathcal{F})$. *Then* $\hat{J}(\mathcal{F})$ *is compact by the definition.*

Observation 4.2.10.
From the discussion in [AO93, Section 1.1] and [Dev99, §3.2] we conclude that there is a homomorphism between the space $\mathbb{Z}^{\mathbb{N}_0}$ *and the space of irrationals* $\mathbb{R} \setminus \mathbb{Q}$, $\varphi : \mathbb{Z}^{\mathbb{N}_0} \to \mathbb{R} \setminus \mathbb{Q}$. *Moreover, this homeomorphism is also order-isomorphic between* $\mathbb{Z}^{\mathbb{N}_0}$ *with the lexicographic order and the space* $\mathbb{R} \setminus \mathbb{Q}$ *with the natural order. Thus the space* $J(\mathcal{F})$ *can be thought of as a subset of* $\mathbb{R}^2$, *via the embedding* $(t, \underline{s}) \to (t, \varphi(\underline{s}))$.

Corollary 4.2.11.
The space $\hat{J}(\mathcal{F})$ *is metrizable.*

Proof. That means we need to prove that $\hat{J}(\mathcal{F})$ is homeomorpic to a metric space. So it follows directly from Observation 4.2.10. □

Lemma 4.2.12.
Let $\hat{J}(\mathcal{F})$ *be as above. Then a set* U *is* neighborhood of ∞ *in* $\hat{J}(\mathcal{F})$ *if and only if there exist* $R > 0$ *and* $M > 0$ *such that*

$$\{(t, \underline{s}) \in J(\mathcal{F})\colon |s_0| > M \ \text{or}\ t > R\} \subset U. \tag{4.1}$$

Proof. We will prove the first direction by contrapositon: Suppose that U does not satisfy (4.1). Then for all $R > 0$, $M > 0$ and $N = \max(M, R)$ there is $(t^N, \underline{s}^N) \in J(\mathcal{F}) \setminus U$ such that $|s_0^N| > N$ or $|t^N| > N$. We claim that the

sequence $(t^N, \underline{s}^N)$ has no convergent subsequence in $J(\mathcal{F})$. This follows from the assumption that $|s_0^N| > N$ or $|t^N| > N$.
Hence $J(\mathcal{F}) \setminus U$ is not contained in a compact subset of $J(\mathcal{F})$. Thus U is not a neighborhood of ∞ in $\hat{J}(\mathcal{F})$.
In the other direction, it is enough to show that there exist $R > 0$ and $M > 0$ such that the set $J(\mathcal{F})_{R,M} = \{(t, \underline{s}) \in J(\mathcal{F})\colon |s_0| \leq M \text{ and } t \leq R\}$ is compact. As the set $J(\mathcal{F})_{R,M}$ is a metric space, it is enough to prove sequential compactness to prove that $J(\mathcal{F})_{R,M}$ is compact. □

Observation 4.2.13.
Compact, connected subsets of $\hat{J}(\mathcal{F})$ are homeomorphic to either closed intervals or a union of curves each one of which contains the point at infinity.

From [Rem06] and the definition of $t_{\underline{s}}$ in (Definition 4.2.5) we obtain that $J(\mathcal{F})$ consists of horizontally parallel curves.
Each curve starts at a unique sequence. Hence the points with the same first entries lie in one curve and escape to ∞ as their second entries grow. For a sequence $\underline{s}$ the line with the points with first entries $\underline{s}$ is called the hair at $\underline{s}$. The intersection of the closures of any two different curves is the point at ∞.

Lemma 4.2.14.
The space $J(\mathcal{F})$ is a straight brush under the embedding in Observation 4.2.10.

Proof. Let $\vartheta\colon \mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}^2$ be the embedding from Observation 4.2.10, such that for any $(t, \underline{s}) \in J(\mathcal{F})$, $\vartheta((t, \underline{s})) = (t, \varphi(\underline{s}))$, where φ is the homeomorphic map from Observation 4.2.10. Hence to prove this lemma it is enough to show that the image of $J(\mathcal{F})$ under this embedding, the set $\vartheta(J(\mathcal{F}))$, is a straight brush.

Thus we claim that $\vartheta(J(\mathcal{F}))$ is a straight brush. To prove that we need to show that all the properties in Definition 2.4.1 are satisfied for $\vartheta(J(\mathcal{F}))$.

1. $\vartheta(J(\mathcal{F}))$ is closed in $\mathbb{R}^2$. Let $\vartheta((x^n)_{n=0}^{\infty})$ be a convergent sequence in $\vartheta(J(\mathcal{F}))$ such that $\vartheta((x^n)_{n=0}^{\infty}) \to z$ where $z \in \mathbb{R}^2$. We claim that $z \in$

$\vartheta(J(\mathcal{F}))$. We prove our claim by assuming for a contradiction that $z \notin \vartheta(J(\mathcal{F}))$ hence $z \in (\mathbb{R} \times \mathbb{Q})$ which contradicts the compactification of $\hat{J}(\mathcal{F})$.

2. The second and the third conditions of Definition 2.4.1 are: For every point $z = (x, y) \in \vartheta(J(\mathcal{F}))$ there exists $t_q \geq 0$ such that $\{y\colon (x, y) \in \vartheta(J(\mathcal{F}))\} = x_y, \infty)$, and the set $\{y\colon (x, y) \in \vartheta(J(\mathcal{F}))$ for some $x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. These two conditions are follow directly from the definitions of the model $J(\mathcal{F})$ and the homeomorphic map φ.

Hence the set $\vartheta(J(\mathcal{F}))$ is a straight brush. Thus $J(\mathcal{F})$ is a straight brush. □

Lemma 4.2.15.
The space $\hat{J}(\mathcal{F})$ is a Lelek fan.

Proof. In Lemma 4.2.14 we showed that $J(\mathcal{F})$ is a straight brush. Hence from Observation 2.4.2 we obtain that the space $\hat{J}(\mathcal{F})$ is a Lelek fan. □

4.3 Properties of the model $J(\mathcal{F})$

In this section we will present some properties of some subsets of the topological model $J(\mathcal{F})$. We will focus on some of the dense subsets of $J(\mathcal{F})$. Note that most of these properties can be obtained just by using Lemma 4.2.14. However we do not use this because we want to understand the space $J(\mathcal{F})$ in more detail.

Theorem 4.3.1.
Let $x \in J(\mathcal{F}) \setminus X$. Then x is an endpoint.

Proof. Assume for a contradiction that $\tilde{x} = (t, \underline{s}) \in J(\mathcal{F}) \setminus X$, where $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and $t > 0$, but it is not an endpoint. Let $x = (t_{\underline{s}}, \underline{s})$ be the endpoint of the curve at the sequence $\underline{s}$, then $t > t_s$ and hence there is $\delta > 0$ such that $t = t_{\underline{s}} + \delta$.

Set $T_n := T(\mathcal{F}^n(x))$ and $\tilde{T}_n := T(\mathcal{F}^n(\tilde{x}))$. We claim that $\tilde{T}_n - T_n \geq F^n(\delta)$. By the induction:

- for $n = 0$, we have $t = t_{\underline{s}} + \delta$. Hence we can write $t - t_{\underline{s}} \geq \delta$.
- Assume it holds for $n - 1$ so that

$$\tilde{T}_{n-1} - T_{n-1} \geq F^{n-1}(\delta).$$

- For n,

$$\begin{aligned}\tilde{T}_n &= T(\mathcal{F}^n(\tilde{x})) \\ &= F(\tilde{T}_{n-1}) - 2\pi|s_n| \\ &\geq F(T_{n-1} + F^{n-1}(\delta)) - 2\pi|s_n| \\ &\geq F(T_{n-1}) + F^n(\delta) - 2\pi|s_n| \\ &= T_n + F^n(\delta).\end{aligned}$$

  Hence

$$\tilde{T}_n - T_n \geq F^n(\delta).$$

From Lemma 4.1.1 we have $F^n(\delta) \to \infty$ as $n \to \infty$. Hence, the distance between $\tilde{T}_n$ and T_n is expanding. Thus we have $\tilde{T}_n \to \infty$ or $T_n \to \infty$ both of which imply that $\tilde{x}$ is an escaping point. This contradicts the assumption. Thus $\tilde{x}$ must be the endpoint x itself. □

Definition 4.3.2 (Inverse orbit).
Let $x \in J(\mathcal{F})$. We define the inverse orbit of x in $J(\mathcal{F})$, denoted $O^-(x)$, as:

$$O^-(x) :\ = \{y \in J(\mathcal{F}) \colon \textit{there exists } n > 0 \colon \mathcal{F}^n(y) = x\}.$$

We also write

$$O^-(x) = \bigcup_{j \geq 0} \mathcal{F}^{-j}(\{x\}).$$

Now we study the properties of an inverse orbit of a point x in $J(\mathcal{F})$. In order to study the behavior of the inverse orbit of $x \in J(\mathcal{F})$ we will state and prove some propositions which will show how the image of a closed subset of $\mathbb{Z}^{\mathbb{N}_0} \times [0, \infty)$, which we choose to be a box, will expand under the model function $\mathcal{F}$.

Before we present our next propositions we will define a subset of the space $\mathbb{Z}^{\mathbb{N}_0} \times [0, \infty)$.

Definition 4.3.3.

Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$, $n > 0$, and $t_1, t_2 \in (0, \infty)$ such that $0 < t_1 < t_2$ we define a box $B_{t_1,t_2,\underline{s}^0,n} \subset \mathbb{Z}^{\mathbb{N}_0} \times [0, \infty)$ as

$$B_{t_1,t_2,\underline{s}^0,n} := \{(t, \underline{s})\colon s_j = s_j^0 \text{ for } j \leq n \text{ and } t \in (t_1, t_2)\}.$$

Proposition 4.3.4.

Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$, $n > 0$, and $t_1, t_2 \in (0, \infty)$ where $0 < t_1 < t_2$ such that $B_{t_1,t_2,\underline{s}^0,n}$ as defined above. If

$$\tilde{t_1} := F(t_1) - 2\pi s_1^0 = T(\mathcal{F}(t_1, \underline{s}^0)),$$

$$\tilde{t_2} := F(t_2) - 2\pi s_1^0 = T(\mathcal{F}(t_2, \underline{s}^0)),$$

then

$$\mathcal{F}\big(B_{t_1,t_2,\underline{s}^0,n}\big) = B_{\tilde{t_1},\tilde{t_2},\sigma(\underline{s}^0),n-1}.$$

Proof.

$$\mathcal{F}(B_{t_1,t_2,\underline{s}^0,n}) = \mathcal{F}(\{(t, \underline{s}) : s_j = s_j^0 \text{ for } j \leq n \text{ and } t \in (t_1, t_2)\}).$$

By the definition of $\mathcal{F}$,

$$\begin{aligned}\mathcal{F}(B_{t_1,t_2,\underline{s}^0,n}) &= \{(F(t)-2\pi|s_1|,\sigma(\underline{s})) : s_j = s_j^0 \text{ for } j \leq n-1 \\ &\quad \text{and } F(t)-2\pi|s_1| \in (F(t_1)-2\pi|s_1|, F(t_2)-2\pi|s_1|)\} \\ &= B_{\tilde{t_1},\tilde{t_2},\sigma(\underline{s}^0),n-1}\end{aligned}$$

where

$$\tilde{t_1} := F(t_1) - 2\pi s_1^0 = T(\mathcal{F}(t_1,\underline{s}^0)),$$

$$\tilde{t_2} := F(t_2) - 2\pi s_1^0 = T(\mathcal{F}(t_2,\underline{s}^0)).$$

□

Proposition 4.3.5.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$, *let* $t_2 > t_1 \geq t_{\underline{s}}$, *let* $n > 0$ *and* $m \leq n$. *Then*

$$\mathcal{F}^m\big(B_{t_1,t_2,\underline{s}^0,n}\big) = B_{t_1^m,t_2^m,\sigma^m(\underline{s}^0),n-m},$$

where

$$t_1^m := T(\mathcal{F}^m(t_1,\underline{s}^0)), \quad t_2^m := T(\mathcal{F}^m(t_2,\underline{s}^0)).$$

Proof. We will prove it by induction on m.

- It holds for $m = 0$ by the definition of $B_{t_1,t_2,\underline{s}^0,n}$.
- Assume it holds for $m-1$:

$$\mathcal{F}^{m-1}\big(B_{t_1,t_2,\underline{s}^0,n}\big) = B_{t_1^{m-1},t_2^{m-1},\sigma^m(\underline{s}^0),n-(m-1)}.$$

- Now we will show that it holds for m :

$$\begin{aligned}\mathcal{F}^m\big(B_{t_1,t_2,\underline{s}^0,n}\big) &= \mathcal{F}\big(\mathcal{F}^{m-1}\big(B_{t_1,t_2,\underline{s}^0,n}\big)\big) \\ &= \mathcal{F}\big(B_{t_1^{m-1},t_2^{m-1},\sigma^m(\underline{s}^0),n-(m-1)}\big) \text{ from the induction.}\end{aligned}$$

Hence from Proposition 4.3.4

$$\mathcal{F}^m\left(B_{t_1,t_2,\underline{s}^0,n}\right) = B_{t_1^m,t_2^m,\sigma^m(\underline{s}^0),n-m},$$

where

$$t_1^m := T(\mathcal{F}^m(t_1,\underline{s}^0)), \quad t_2^m := T(\mathcal{F}^m(t_2,\underline{s}^0)).$$

Thus we proved our proposition. □

Proposition 4.3.6.
Suppose $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and $T \geq 0$. There exists $R > 0$ for all t_1 and t_2 where $0 < t_1 < t_2$ such that $t_2 - t_1 > R$, and there is $m \in \mathbb{Z}$ such that

$$(T, m\underline{s}) \in \mathcal{F}(B_{t_1,t_2,\underline{s}^0,0}) \quad \textit{for all } \underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}.$$

Proof. If $(t,\underline{u}) \in B_{t_1,t_2,\underline{s}^0,0}$ then $t \in (t_1,t_2)$, $u_0 = s_0^0$ and
$\mathcal{F}(t,\underline{u}) = (F(t) - 2\pi|u_1|, \sigma(\underline{u}))$.

Note that $\sigma(\underline{u})$ can be any integer sequence which satisfies the condition that $F(t) - 2\pi|u_1| = T$. Choose $R > 0$ such that $F(R) \geq T$ and $F(R) \geq 2\pi$, hence $F(t_2) - F(t_1) \geq F(R) \geq 2\pi$. Then we can choose $t > 0$ such that $t_1 \leq t < t_2$ and $F(t) = T + 2\pi m$ for some $m \in \mathbb{Z}$, $m \geq 0$. Thus choose $\underline{u} = s_0^0 m s_0 s_1 s_2 s_3 = s_0^0 m\underline{s}$ where $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and t as above, then

$$\mathcal{F}(t,\underline{u}) = (T, m\underline{s}) \in \mathcal{F}(B_{t_1,t_2,\underline{s}^0,0}).$$

Hence we proved the proposition. □

Theorem 4.3.7.
Let $x \in J(\mathcal{F})$. Then the inverse orbit $O^-(x)$ as defined in 4.3.2 is dense in $J(\mathcal{F})$.

Proof. Let $y = (t_0, \underline{s}^0) \in J(\mathcal{F})$. We have to show that for every $N \in \mathbb{N}_0$ and $\epsilon > 0$ there is $\tilde{y} = (\underline{\tilde{s}}, \tilde{t}) \in O^-(x)$ such that $\underline{s}^0$ and $\underline{\tilde{s}}$ agree in the first N entries and $|\tilde{t} - t_0| \leq \epsilon$.

Let $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and $\tau \geq 0$ such that $x = (\tau, \underline{s})$. Then $\mathcal{F}^{-1}(x) = \{(F^{-1}(\tau + 2\pi|s_0|, m\underline{s}) : m \in \mathbb{Z}\}$. Set $T \coloneqq F^{-1}(\tau + 2\pi|s_0|)$ then by Proposition 4.3.6 there is $R > 0$ such that

$\mathcal{F}^{-1}(x) \cap \mathcal{F}(B_{t_1,t_2,\underline{s},0}) \neq \emptyset$ for all $\underline{s} \in \mathbb{Z}^{\mathbb{N}_0}$ and all $0 < t_1 < t_2$ with $t_2 > R + t_1$.

Set $t_1 = t_0, t_2 = t_0 + \epsilon$ and $t_j^m = T(\mathcal{F}^m(t_j, \underline{s}^0))$ for $m \geq 0$ and $j \in \{1, 2\}$.

Now we claim that for a large enough m we have $t_2^m - t_1^m > R$. Because $t_2 - t_1 = \epsilon > 0$ then from lemma 4.1.1 for a large m we obtain

$$F^m(\epsilon) > R.$$

Hence from Lemma 4.1.2

$$t_2^m - t_1^m > R,$$

as claimed.

Now for a sufficiently large N we choose $n > N$ such that $t_2^n - t_1^n \geq F^m(\epsilon)$.

Then

$$\mathcal{F}^n(B_{t_1,t_2,\underline{s}^0,n}) = B_{t_1^n,t_2^n,\sigma^n(\underline{s}^0),0},$$

by Proposition 4.3.5.

Hence,

$$\mathcal{F}^{n+1}(B_{t_1,t_2,\underline{s}^0,n}) \cap \mathcal{F}^{-1}(x) \neq \emptyset,$$

by Proposition 4.3.6.

Therefore, there is $\tilde{y} = (\tilde{t}, \tilde{\underline{s}}) \in B_{t_1,t_2,\underline{s}^0,n}$ such that

$$\mathcal{F}^{n+1}(\tilde{y}) \in \mathcal{F}^{-1}(x)$$

and hence

$$\mathcal{F}^{n+2}(\tilde{y}) = x.$$

We have $\tilde{t} \in [t_1, t_2] = [t_0, t_0 + \epsilon]$ so

$$|\tilde{t} - t_0| \leq \epsilon$$

and $\tilde{s}$ agrees with $\underline{s}^0$ in the first n-th entries so $\tilde{y}$ has the desired properties, and hence the proof is done. □

Theorem 4.3.8.
The set of endpoints is dense in $J(\mathcal{F})$.

Proof. The preimage of any endpoint is a set of endpoints; let $(t_{\underline{s}}, \underline{s})$ be an endpoint, then

$$\mathcal{F}^{-1}(t_{\underline{s}}, \underline{s}) = \{(F^{-1}(t_{\underline{s}} + 2\pi|s_0|, m\underline{s})) : m \in \mathbb{Z}\},$$

is a set of endpoints.

Hence the inverse orbit of an endpoint contains only endpoints. By Theorem 4.3.7, we obtain that such an inverse orbit of endpoints is dense in $J(\mathcal{F})$. Thus the set of all endpoints is dense in $J(\mathcal{F})$. □

Theorem 4.3.9.
The set of periodic points is dense in $J(\mathcal{F})$.

To prove this Theorem we first need to present some lemmas. These lemmas build on the topological concept which was introduced in Lemma 4.2.6.

The following lemmas will be the main steps in the proof of Theorem 4.3.9.

Lemma 4.3.10.
Let $c > 0$ and let $x \in J(\mathcal{F})$ where $x = (t, \underline{s}) \in \mathbb{Z}^{\mathbb{N}_0} \times [0, \infty)$. Suppose that for all $n > 0$, we have a sequence of endpoints $x^n = (t_{\underline{s}^n}, \underline{s}^n) \in J(\mathcal{F})$, and a sequence $k_n \to \infty$, such that:

1. *$s_j^n = s_j$ for all $j \leq k_n$.*

2. For the value t^* which introduced in Lemme 4.2.6 we have:
$|t^*_{\sigma^{k_n}(\underline{s}^n)} - T(\mathcal{F}^{k_n}(x))| \leq c.$

then $x^n \to x$.

Proof. From the first condition and the fact that $k_n \to \infty$ it follows that $\underline{s}^n$ converges to $\underline{s}$ as $n \to \infty$. Also from the second condition we obtain that $t^*_{\sigma^{k_n}(\underline{s}^n)} < \infty$. Thus from Lemma 4.2.6 we obtain that $\sigma^{k_n}(\underline{s}^n)$ is an exponentially bounded address, and

$$t^*_{\sigma^{k_n}(\underline{s}^n)} \leq t_{\sigma^{k_n}(\underline{s}^n)} \leq t^*_{\sigma^{k_n}(\underline{s}^n)} + 1.$$

Thus
$$|t_{\sigma^{k_n}(\underline{s}^n)} - T(\mathcal{F}^{k_n}(x))| \leq c + 1.$$

Hence
$$|t_{\underline{s}^n} - t| \leq F^{-k_n}(|t_{\sigma^{k_n}(\underline{s}^n)} - T(\mathcal{F}^{k_n}(x))|) \leq F^{-k_n}(c+1).$$

That is by using the fact that $F(x-y) \leq F(x) - F(y)$ for $y, x \in [0, \infty)$ and 1.

By Lemma 4.1.1, we obtain that

$$F^{-k_n}(c) \to 0 \text{ as } k_n \to \infty \text{ for all } c > 0.$$

Therefore $t_{\underline{s}^n} \to t$ as $k_n \to \infty$. Hence $(t_{\underline{s}^n}, \underline{s}^n) \to (t, \underline{s})$. □

Lemma 4.3.11.
Let $\underline{s}$ be an exponentially bounded address, and $x := (t_{\underline{s}}, \underline{s}) \in J(\mathcal{F})$. For each n let $\underline{s}^n$ be the periodic address of period n defined by $s^n_i = s_i$ for $i < n$. Then $(t_{\underline{s}^n}, \underline{s}^n) \to x$.

Proof. As a first step of the proof we claim that $t^*_{\underline{s}^n} \to t^*_{\underline{s}}$ as $n \to \infty$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, $|t^*_{\underline{s}^n} - t^*_{\underline{s}}| < \epsilon$.

Then we claim that generally for a fixed k:
$t^*_{\sigma^k(\underline{s}^n)} \to t^*_{\sigma^k(\underline{s})}$ as $n \to \infty$.

We start to prove the first claim that $t^*_{\underline{s}^n} \to t^*_{\underline{s}}$ as $n \to \infty$, where

$$t^*_{\underline{s}} = \sup_{k \geq 1} F^{-k}(2\pi|s_k|),$$

and

$$t^*_{\underline{s}^n} = \sup_{l \geq 1} F^{-l}(2\pi|s^n_l|).$$

Because $\underline{s}^n$ is a periodic sequence, let $s^n_l = s_i$ where $l = i(\mathrm{mod}\, n)$ and $i < n$ so $l = i + mn$ for some $m \geq 0$.

Hence,

$$\begin{aligned} t^*_{\underline{s}^n} &= F^{-l}(2\pi|s^n_l|) \\ &= F^{-(i+mn)}(2\pi|s_i|) \\ &\leq F^{-i}(2\pi|s_i|) \leq t^*_{\underline{s}}. \end{aligned}$$

Now assume for a contradiction that $t^*_{\underline{s}^n} \nrightarrow t^*_{\underline{s}}$. Then there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is $n > N$ such that

$$|t^*_{\underline{s}} - t^*_{\underline{s}^n}| > \epsilon.$$

This implies that

$$t^*_{\underline{s}^n} \leq t^*_{\underline{s}} - \epsilon$$

and hence (from the definition of t^*) we obtain that

$$F^{-k}(2\pi|s_k|) \leq t^*_{\underline{s}} - \epsilon \text{ for all } k \leq N.$$

This is a contradiction with the definition of the supremum $t^*_{\underline{s}}$. Hence

$$t^*_{\underline{s}^n} \to t^*_{\underline{s}}.$$

Now our second claim is that, for a fixed k, $t^*_{\sigma^k(\underline{s}^n)} \to t^*_{\sigma^k(\underline{s})}$ as $n \to \infty$. Since $\underline{s}^n$ is a periodic sequence, let $s^n_{k+m} = s_{k+i}$ where $m = i(\mathrm{mod}(n-k))$ and

$i < n-k$ so $m = i + j(n-k)$ for some $j \geq 0$. Then with similar steps to the first case we first get $t^*_{\sigma^k(\underline{s}^n)} \leq t^*_{\sigma^k(\underline{s})}$ and analogously we obtain that

$$t^*_{\sigma^k(\underline{s}^n)} \to t^*_{\sigma^k(\underline{s})}.$$

That is, for all $\epsilon > 0$ there exists $N > 0$ such that for all $k > N$

$$|t^*_{\sigma^k(\underline{s}^n)} - t^*_{\sigma^k(\underline{s})}| < \epsilon$$

and since $\underline{s}$ is exponentially bounded then so are $\underline{s}^n$ and $\sigma^k(\underline{s}^n)$. Hence, by Lemma 4.2.6 $t_{\sigma^k(\underline{s}^n)}$ is near $t^*_{\sigma^k(\underline{s}^n)}$:

$$|t_{\sigma^k(\underline{s}^n)} - t^*_{\sigma^k(\underline{s})}| \leq \epsilon + 1.$$

Then from Lemma 4.3.10 we obtain

$$(t_{\underline{s}^n}, \underline{s}^n) \to (t_{\underline{s}}, \underline{s}) \text{ as } n \to \infty.$$

Thus we proved the lemma. □

Now we will state the proof of Theorem 4.3.9 and first we will restate the Theorem.

Theorem 4.3.12.
The set of periodic points is dense in $J(\mathcal{F})$.

Proof. The proof follows directly from previous results, that is : in Lemma 4.3.8 we showed that the set of endpoints is dense in $J(\mathcal{F})$. Then in Theorem 4.3.11 we proved that each endpoint of $J(\mathcal{F})$ is a limit of periodic points in $J(\mathcal{F})$. Hence from the properties of density, we obtain that the periodic points are also dense in $J(\mathcal{F})$. □

4.4 Conjugacy between $\mathcal{F}$ and f_a

In this section we will present the relation between the topological model $J(\mathcal{F})$ and the Julia set of the exponential map f_a, where $a \in \mathbb{C}$.

In [Rem06] Rempe-Gillen shows that there is a conjugacy between $\mathcal{F}$ and f_a, which is defined on a suitable subset of the topological model $J(\mathcal{F})$.

Now to recall the conjugacy we first define a subset of $J(\mathcal{F})$ where the conjugacy is defined:

Definition 4.4.1.
For Q large enough we define

$$Y_Q := \{(t, \underline{s}) \in J(\mathcal{F})\colon T(\mathcal{F}^n(t, \underline{s})) \geq Q \text{ for all } n\}.$$

Now we will present some properties of the set Y_Q, as lemmas, which play a significant roles in our thesis more precisely in the proof of Theorem 1.1.5.

Lemma 4.4.2.
The set Y_Q, defined above, is a closed subset of $J(\mathcal{F})$.

Proof. The proof of the lemma follows from the definition of Y_Q. That is because Y_Q is the intersection of closed sets. □

Lemma 4.4.3.
Let Y_Q be as above. Then the set $\tilde{E}_{Y_Q} := \tilde{E} \cap Y_Q$ is dense in Y_Q.

Proof. Let $x = (t, \underline{s}) \in Y_Q$ be a non-endpoint. We aim to show that for all $\epsilon > 0$, $N \in \mathbb{N}$ there exist $n > N$ and $x^n = (t_{\underline{s}^n}, \underline{s}^n) \in \tilde{E}_{Y_Q}$ such that: $s_i = s_i^n$ for all $i < n$, and $|t - t_{\underline{s}^n}| < \epsilon$. We will prove this by using Lemma 4.3.10, we will show that in Y_Q there exists a sequence of points which satisfies the conditions in Lemma 4.3.10. Now, to start, let us define a sequence $\underline{s}^n$:

$$s_j^n := \begin{cases} s_j, & \text{if } j < n, \\ \left\lceil \frac{F(T(\mathcal{F}^{j-1}(x)))}{2\pi} \right\rceil, & \text{if } j \geq n. \end{cases}$$

Let $\delta = t - t_{\underline{s}}$: note that $\delta > 0$ because x is not an endpoint. Then by induction we obtain that $T(\mathcal{F}^m(x)) \geq F^m(\delta)$. Then by Lemma 4.1.2 we obtain that $\underline{s}$ is fast in the sense of Definition 4.2.4. That is for all $y > 0$ there exists $m_0 > 0$ such that for all $m \geq m_0$ there exists $i \geq 0$ such that $|s_{m+i}| \geq F^i(y)$. Hence from the definition of $\underline{s}^n$ we obtain that it is a fast sequence in the since of Definition 4.2.4.

Because we have $s_j = s_j^n$ for all $j < n$, and for all $j \geq 0$ we have

$$t_{\sigma^j(\underline{s}^n)} \geq t^*_{\sigma^j(\underline{s})} \geq F^{-1}(2\pi|s_{j+1}|) \geq T(\mathcal{F}^j(x)) \geq Q.$$

Hence $\underline{s}^n \in \tilde{E}_{Y_Q}$.

Now we will apply the conditions of Lemma 4.3.10 to our point x^n.

The first condition follows directly from the definition of the sequence $\underline{s}^n$. In order to apply the second condition we first note that for $j \geq n$ we have:

$$F(T(\mathcal{F}^j(x) - 2\pi)) \leq 2\pi|s^n_{j+1}| \leq F(T(\mathcal{F}^j(x) + 2\pi)).$$

Hence

$$T(\mathcal{F}^j(x)) - 2\pi \leq F^{-1}(2\pi|s^n_{j+1}| \leq T(\mathcal{F}^j(x)) + 2\pi.$$

This implies

$$T(\mathcal{F}^j(x)) - 2\pi \leq t^*_{\sigma^j(\underline{s}^n)} \leq T(\mathcal{F}^j(x)) + 2\pi.$$

Hence

$$|t^*_{\sigma^j(\underline{s}^n)} - T(\mathcal{F}^j(x))| \leq 2\pi.$$

This is just the second condition and hence x^n satisfies the conditions in Lemma 4.3.10. We deduce that for any non-endpoint $x \in Y_Q$ there exists $n > 0$ such that $x^n \in \tilde{E}_{Y_Q}$ is in a small neighborhood of x, that is

$$x^n \to x.$$

Hence $\tilde{E}_{Y_Q}$ is dense in Y_Q. □

Corollary 4.4.4.
Let $x^0 = (t^0, \underline{s}^0) \in Y_Q$. Then there exist sequences $(x^{n+} = (t_{\underline{s}^{n+}}, \underline{s}^{n+}))_{n=1}^{\infty}$ and $(x^{n-} = (t_{\underline{s}^{n-}}, \underline{s}^{n-}))_{n=1}^{\infty}$ in $\tilde{E}_{Y_Q}$ such that $\underline{s}^{n-} < \underline{s}^0 < \underline{s}^{n+}$ for all n, with respect to lexicographical order and such that $x^{n-}, x^{n+} \to x^0$.

The proof of the corollary follows analogously from the proof of Lemma 4.4.3.

Now we will present a Theorem from [Rem06, Theorem 4.2 and 4.3],. This Theorem introduces a conjugucy between the space Y_Q and the Julia set of exponential maps f_a, where $a \in \mathbb{C}$.

Theorem 4.4.5 ([Rem06])**.**
Let $a \in \mathbb{C}$, and consider the exponential map $f_a\colon \mathbb{C} \to \mathbb{C}; f_a(z) = e^z + a$. If Q is sufficiently large, then there exists a closed forward-invariant set $K \subset J(f_a)$ and a homeomorphism $\mathfrak{g}\colon Y_Q \to K$ with the following properties.

(a) $\mathfrak{g}$ *is a conjugacy between* $\mathcal{F}$ *and* f*; i.e.* $\mathfrak{g} \circ \mathcal{F} = f_a \circ \mathfrak{g}$.

(b) $\operatorname{Re} \mathfrak{g}(x) \to +\infty$ *as* $T(x) \to \infty$.

(c) The map $\mathfrak{g}$ preserves vertical ordering. That is, if $\underline{s}^1, \underline{s}^2 \in \mathbb{Z}^{\mathbb{N}_0}$ are exponentially bounded addresses such that $\underline{s}^1 < \underline{s}^2$ with respect to lexicographical ordering, then the curve $t \mapsto \mathfrak{g}(\underline{s}^1, t)$ tends to infinity below the curve $\mathfrak{g}(\underline{s}^2, t)$.

(d) There is a number $R > 0$ with the following property: if $z \in \mathbb{C}$ such that $Ref_a^n(z) \geq R$ for all $n \geq 0$, then $z \in K$.

Lemma 4.4.6.
Let $a \in \mathbb{C}$ be arbitrary, and let $\mathfrak{g}$ and K be as in Theorem 4.4.5. Let $z \in I(f_a)$, and $n \geq 0$ be sufficiently large such that $f_a^n(z) \in K$. Then either

1. *$\mathfrak{g}^{-1}(f_a^n(z)) \in \tilde{E}$, and z is an escaping endpoint of f_a in the sense of Definition 1.1.1.*

2. *$\mathfrak{g}^{-1}(f_a^n(z)) \in X \setminus \tilde{E}$, and z is on a hair in the sense of Definition 1.1.1.*

Proof. We first prove the second case. Assume that $x_n := \mathfrak{g}^{-1}(f_a^n(z))$ is a non-endpoint for $J(\mathcal{F})$, and is a non-endpoint for Y_Q, where Q is as in Definition 4.4.1. Hence, x_n is on a hair in $J(\mathcal{F})$, because $J(\mathcal{F})$ is a straight brush and $x_n \in J(\mathcal{F})$ is not an endpoint. Thus it is immediate that $f_a^n(z)$ is on a hair, by the conjugacy $\mathfrak{g}$. Then we apply an inverse of f_a^{-n} restricted to certain branches, and hence we obtain an arc in $I(f_a)$ containing z as an interior point, as required.

Now we prove the first case. If $x_n \in \tilde{E}$, where x_n as above, then we see analogously that z is accessible from the escaping set $I(f_a)$. Hence it only remains to prove that, if $\gamma \subset I(f_a)$ is an arc, then γ cannot contain z in its interior.

Assume for a contradiction that $\gamma \subset I(f_a)$ is an arc, and γ contains z in its interior. Hence $f_a^n(z)$ is in the interior of the curve $\gamma_n = f_a^n(\gamma)$. Then from what we proved we obtain that $\mathfrak{g}^{-1}(f_a^n(z)) \in X \setminus \tilde{E}$ but that contradicts the assumption that $x_n \in \tilde{E}$. Hence we have proved the Lemma. □

Chapter 5

Subsets of the model $J(\mathcal{F})$

In this chapter, we will provide the main steps of our proof of Theorem 1.1.4. That is, we prove that, for a certain subset of the topological model $J(\mathcal{F})$ the set of escaping endpoints, together with infinity, is indeed a connected set. We do so by showing that it can be written as a union of endpoints of Lelek fans, which have the same top point, together with the top.

As mentioned, this result together with some theorems and results from [Rem06], will imply the desired result for $a \in (-\infty, -1)$.

5.1 Fans in the topological model $J(\mathcal{F})$

In this section we study the connectivity of the set of escaping endpoints of the explicit topological model $J(\mathcal{F})$, from the previous chapter, to study the dynamics for the Julia set of f_a, where $a \in (-\infty, -1)$. We will define a subset of the model $J(\mathcal{F})$ aiming to show that the closure of this set in the Riemann sphere is homeomorphic to a Lelek fan. Hence this subset is a Lelek fan.

Definition 5.1.1.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$ be a sequence of integers. Then we define $\mathcal{A}(\underline{s}^0) \subset \mathbb{Z}^{\mathbb{N}_0}$ and $J(\mathcal{F})_{\underline{s}^0} \subset J(\mathcal{F})$ by

$$\mathcal{A}(\underline{s}^0) := \{\underline{s} \in \mathbb{Z}^{\mathbb{N}_0} : |s_j| \geq |s_j^0| \text{ for all } j\} \quad \text{and}$$

$$J(\mathcal{F})_{\underline{s}^0} := \{(t, \underline{s}) \in J(\mathcal{F}) : \underline{s} \in \mathcal{A}(\underline{s}^0)\}.$$

The following result is the main step in the proof of Theorem 1.1.4.

Theorem 5.1.2.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$. Then the space $\hat{J}(\mathcal{F}_{\underline{s}^0}) := J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is a Lelek fan.

In this section, we begin the proof of Theorem 5.1.2 by showing that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a smooth fan (Proposition 5.1.9 below). Let us begin with some preliminary lemmas.

Lemma 5.1.3.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$, then the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is compact.

Proof. We claim the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a closed subset of $\hat{J}(\mathcal{F})$. We will prove it by showing that the complement of $J(\mathcal{F})_{\underline{s}^0}$ is open in $J(\mathcal{F})$. This means we must show that for $(t, \underline{s}) \in J(\mathcal{F}) \setminus J(\mathcal{F})_{\underline{s}^0}$ there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that the set $B_{N,\epsilon} = \{(t_1, \underline{s}^1) \in J(\mathcal{F}) : |s_i^1| = |s_i| \text{ for } i \leq N, |t - t_1| < \epsilon\}$ is a subset of $J(\mathcal{F}) \setminus J(\mathcal{F})_{\underline{s}^0}$.

The complement of $J(\mathcal{F})_{\underline{s}^0}$ is

$$C = \{(t, \underline{s}) \in J(\mathcal{F}) \colon \text{ there exists } j \geq 0 \text{such that } |s_j| < |s_j^0|\}.$$

Let $(t, \underline{s}) \in C$. Then there exists $j \geq 0$ such that $|s_j| < |s_j^0|$. Choose $N \colon = j+1$ and $\epsilon \colon = 1$. Then the set $\{(t_1, \underline{s}^1) \in J(\mathcal{F}) : |s_i| = |s_i^1| \text{ for } i \leq N, |t - t_1| < \epsilon\}$ is not an empty set, and since its points have the jth entries of their sequences less than $|s_j^0|$, it follows that the set is a subset of C. Hence C is an open subset of $J(\mathcal{F})$, and so the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a closed subset of $\hat{J}(\mathcal{F})$.

Because $\hat{J}(\mathcal{F})$ is compact and by using the fact that a closed subset of a compact space is compact [Dug78, Theorem 1.4, p224], we obtain that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is compact. □

Lemma 5.1.4.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$. *Then the space* $\hat{J}(\mathcal{F}_{\underline{s}^0})$ *is path-connected. In particular it is connected.*

Proof. The space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is the union of arcs

$\{(t, \underline{s}) \in J(\mathcal{F})\colon \underline{s} \in \mathcal{A}(\underline{s}^0), t \geq t_{\underline{s}}\} \cup \{\infty\}$. Each arc is path connected and all of these arcs have the point ∞ in common. Hence the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a path-connected space [Dug78, p.115]. Thus $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is connected (by using the fact that any path-connected space is connected). □

Lemma 5.1.5.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$. *Then the space* $\hat{J}(\mathcal{F}_{\underline{s}^0})$ *is a continuum.*

Proof. We aim to prove that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a continuum i.e; a connected, compact metric space. Recall that in Section 4.3 we discussed the topology of $\hat{J}(\mathcal{F}) = J(\mathcal{F}) \cup \{\infty\}$; in particular, the space $\hat{J}(\mathcal{F}_{\underline{s}^0}) \subset \hat{J}(\mathcal{F})$ is a (non-empty) metric space. Furthermore, $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is compact by Lemma 5.1.3. Later in Lemma 5.1.4 we proved that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is connected. Hence it follows that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a continuum. □

Lemma 5.1.6.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$. *Then the space* $\hat{J}(\mathcal{F}_{\underline{s}^0})$ *is hereditarily unicoherent.*

Proof. In Lemma 4.2.15 we showed that the space $\hat{J}(\mathcal{F})$ is a Lelek fan and hence it is a hereditarily unicoherent continuum i.e. every subcontinuum of it is a unicoherent continuum. Since $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a subcontinuum of $\hat{J}(\mathcal{F})$, it follows that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a hereditarily unicoherent continuum too. □

In order to show that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a fan we must prove that it is a dendroid i.e., an arcwise connected hereditarily unicoherent continuum.

Lemma 5.1.7.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$. *Then the space* $\hat{J}(\mathcal{F}_{\underline{s}^0})$ *is uniquely arcwise connected.*

Proof. We claim $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is arcwise connected. Previously we showed in Lemma 5.1.4 that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is path connected and in Lemma 5.1.5 that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a metric space, hence the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a Hausdorff space. Thus it follows that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is arcwise connected. Therefore it is a dendroid i.e. an arcwise connected hereditarily unicoherent continuum. Thus from Lemma 2.3.5 it follows that $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is uniquely arcwise connected. □

From this lemma we have $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is uniquely arcwise connected, and so from the definition of the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ for any point $x_1 = (t_1, \underline{s}^1) \in J(\mathcal{F})(\underline{s}^0)$, the unique arc which connects the point x_1 to ∞ is the set

$$\gamma_{x_1\infty} = \{(t, \underline{s}^1) : t \in [t_1, \infty)\} \cup \{\infty\}, \qquad (*)$$

which is the image of the unit interval $[0, 1]$ under a homeomorphic map we denote as $\lambda_{x_1\infty}$

$$\lambda_{x_1\infty} : [0, 1] \to \gamma_{x_1\infty}.$$

For any two distinct points $x_1 =, x_2 = (t_2, \underline{s}^2) \in J(\mathcal{F})_{\underline{s}^0}$, let $\gamma_{x_1\infty}$ and $\gamma_{x_2\infty}$ be the arcs that connect x_1 to ∞ and x_2 to ∞ respectively. Then the unique arc which connects x_1 to x_2 has two possible cases:

1. If $\underline{s}^1 \neq \underline{s}^2$ then the arc $\gamma_{x_1x_2}$ is the image of $[0, 1]$ under a homeomorphic map, which maps the interval $[0, 1]$ to $\gamma_{x_1\infty} \cup \gamma_{x_2\infty}$. Hence $\lambda_{x_1x_2} : [0, 1] \to \gamma_{x_1\infty} \cup \gamma_{x_2\infty}$, where $\lambda_{x_1x_2}(0) = x_1, \lambda_{x_1x_2}(1) = x_2$ and $\lambda_{x_1x_2}(1/2) = \infty$.
2. If $\underline{s}^1 = \underline{s}^2$ then the arc $\gamma_{x_1x_2}$ is the image of $[0, 1]$ under a homeomorphic map, which maps the interval $[0, 1]$ to $\gamma_{x_1x_2} = \{(t, \underline{s}^1) : t \in [t_i, t_j]\}$ where $t_i < t_j$ and $\{i, j\} = \{1, 2\}$.

Lemma 5.1.8.
Let $\underline{s}^0 \in \mathbb{Z}^{\mathbb{N}_0}$ be an exponentially bounded address. Then the space $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ has exactly one ramification point ∞.

Proof. In the proof of Lemma 4.2.15 we showed that the space $\hat{J}(\mathcal{F})$ is a Lelek fan with ∞ as its top. Hence by Observation 2.3.8 the space $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ only has one ramification point which is ∞. The fact that ∞ is a ramification point of $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ follows directly from the definition of $J(\mathcal{F})_{\underline{s}^0}$ and from Definition 2.3.7. □

Now we are ready to prove the main result of this section which is one of the main steps to prove Lemma 5.1.2.

Proposition 5.1.9.
The space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a smooth fan.

Proof. First we show that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a fan. To prove that, we must show that it is a uniquely arcwise connected, hereditarily unicoherent continuum with exactly one ramification point. In Lemma 5.1.6 we proved the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a hereditarily unicoherent continuum. Then in Lemma 5.1.7 we proved that it is uniquely arcwise connected and in Lemma 5.1.8 we showed it has exactly one ramification point. Hence it follows that the space $\hat{J}(\mathcal{F}_{\underline{s}^0})$ is a fan.

In Lemma 4.2.15 we showed that the space $\hat{J}(\mathcal{F})$ is a Lelek fan and hence it is a smooth fan. Since the fan $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is a subset of the smooth fan $\hat{J}(\mathcal{F})$, it is also smooth. □

5.2 Density of endpoints in the fans defined for fast addresses

To study the density of the endpoints of $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ we will use lemmas, results and definitions from a previous chapter (Chapter 4 in which we recalled the definition and properties of the topological model $J(\mathcal{F})$ from [Rem06]).

Lemma 5.2.1.
Let $J(\mathcal{F})_{\underline{s}^0}$ be as defined in 5.1.1 . Then the set of endpoints $EJ(\mathcal{F})_{\underline{s}^0}$ of

$J(\mathcal{F})_{\underline{s}^0}$ is dense in $J(\mathcal{F})_{\underline{s}^0}$.

Proof. In this proof we will show that for any $x \in J(\mathcal{F})_{\underline{s}^0}$ there exists a sequence $x^n \in J(\mathcal{F})_{\underline{s}^0}$ of endpoints which satisfies the conditions in Lemma 4.3.10 (with $k_n = n$) and hence converges to x. This shows that the set of endpoints of $J(\mathcal{F})_{\underline{s}^0}$ is dense.

Let $x = (t, \underline{s}) \in J(\mathcal{F})_{\underline{s}^0}$, and let $\underline{s}^n \in \mathbb{Z}^{\mathbb{N}_0}$ be a sequence of addresses such that

1. $|s_i^n| = |s_i|$ for $i \le n$ and
2. $|s_i^n| \ge |s_i|$ for all $i > n$.

We shall show that, under certain additional conditions on $\underline{s}^n$ (see (5.2)), the endpoints $x^n \coloneqq (t_{\underline{s}^n}, \underline{s}^n)$ satisfy the conditions of Lemma 4.3.10.

Fix n and $i > n$, and set $m \coloneqq i - n > 0$. By using Lemma 4.2.6 we obtain:

$$F^{-m}(2\pi|s_i|) \le t^*_{\sigma^n(\underline{s})} \le t_{\sigma^n(\underline{s})} \le T(\mathcal{F}^n(x)).$$

Because F is a strictly increasing map it follows that

$$2\pi|s_i| \le F^m(T(\mathcal{F}^n(x))).$$

Hence

$$|s_i| \le F^m(T(\mathcal{F}^n(x)))/2\pi. \tag{5.1}$$

In order to apply Lemma 4.3.10, we have to make sure that $t^*_{\sigma^n(\underline{s}^n)}$ is close to $T(\mathcal{F}^n(x))$. To do so we use equation (5.1) and take $|s_i^n|$ close to $F^m(T(\mathcal{F}^n(x)))/2\pi$ for all $i > n$. By taking $|s_i^n|$ to be the smaller integer that is greater or equal to $F^m(T(\mathcal{F}^n(x)))/2\pi$ this will ensure that $t^*_{\sigma^n(\underline{s}^n)}$ and $T(\mathcal{F}^n(x))$ are close, and also satisfy the condition 2:

$$F^m(T(\mathcal{F}^{k_n}(x)))/2\pi \le |s_i^n| \le F^m(T(\mathcal{F}^{k_n}(x)))/2\pi + 1. \tag{5.2}$$

Therefore by (5.1) $|s_i^n| \ge |s_i|$ for all $i > n$. It follows from (5.2) that

$$F^m(T(\mathcal{F}^n(x))) \leq 2\pi|s_i^n| \leq F^m(T(\mathcal{F}^n(x))) + 2\pi.$$

Now we will apply F^{-m} to the previous equation to obtain

$$T(\mathcal{F}^n(x) \leq F^{-m}(2\pi|s_i^n|) \leq (T(\mathcal{F}^n(x)) + 2\pi.$$

Since $t^*_{\sigma^n(\underline{s}^n)}$ is just the supremum of $F^{-m}(2\pi|s_i^n|)$, we have

$$T(\mathcal{F}^n(x)) \leq t^*_{\sigma^n(\underline{s}^n)} \leq T(\mathcal{F}^n(x)) + 2\pi$$

which implies

$$|t^*_{\sigma^n(\underline{s}^n)} - T(\mathcal{F}^n(x))| \leq 2\pi.$$

Hence the choice of $(t_{\underline{s}^n}, \underline{s}^n)$ satisfies the conditions in Lemma 4.3.10, and hence the set of endpoints of $J(\mathcal{F})_{\underline{s}^0}$ is dense. □

Now we have proved the main steps in separate lemmas we will prove Theorem 5.1.2.

Proof. [Theorem 5.1.2] A Lelek fan is a smooth fan with dense set of endpoints. From Proposition 5.1.9 we know that the space $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is a smooth fan, and in Lemma 5.2.1 we proved that the set of endpoints $E(J(\mathcal{F})_{\underline{s}^0})$ of $J(\mathcal{F})_{\underline{s}^0}$ is dense in $J(\mathcal{F})_{\underline{s}^0}$. Hence, from the definition of a Lelek fan, we obtain that the space $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is a Lelek fan. □

Note that from the definitions and the properties of a Lelek fan and straight brushes we obtain the following corollary.

Corollary 5.2.2.
The space $J(\mathcal{F})_{\underline{s}^0}$ is homemorphic to a straight brush.

Lemma 5.2.3.
Let Y_Q be as defined in Definition 4.4.1 and Q as in Theorem 4.4.5, and let

$\underline{s}^0$ be a fast address, $(\underline{s}^0, t_{\underline{s}^0}) \in \tilde{E}J(\mathcal{F})_{\underline{s}^0}$. If $t_{\underline{s}^0} \geq Q$ then the space $J(\mathcal{F})_{\underline{s}^0}$, as defined in Definition 5.1.1, is a subset of Y_Q.

Proof. The proof of this lemma follows from the definitions of the spaces $J(\mathcal{F})_{\underline{s}^0}$ and Y_Q. □

5.3 Endpoints in the fans defined for fast addresses

In this section we aim to study the connectedness of the set of endpoints of the space $J(\mathcal{F})_{\underline{s}^0}$ together with the point at infinity which is the top of the smooth fan $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$. Then using our result we will study the connectedness of the set of escaping endpoints of the whole topological model $J(\mathcal{F})$ together with ∞.

Corollary 5.3.1.
Let $J(\mathcal{F})_{\underline{s}^0}$ be the space defined in 5.1.1. Then the set $E(J(\mathcal{F})_{\underline{s}^0}) \cup \{\infty\}$ is connected.

Proof. This follows directly from Theorem 5.1.2 and Theorem 2.3.14. □

Theorem 5.3.2.
Let $\tilde{E}$ be the set of escaping endpoints in $J(\mathcal{F})$:

$$\tilde{E} := \{(t_{\underline{s}}, \underline{s}) \in J(\mathcal{F}) : \underline{s} \text{ is fast}\}.$$

(Recall Observation 4.2.7.) Then

1. *$\tilde{E}$ is totally disconnected.*
2. *$\tilde{E} \cup \{\infty\}$ is connected.*

Proof. 1. Define a map $\zeta : \tilde{E} \to \mathbb{Z}^{\mathbb{N}_0}; (t_s, \underline{s}) \mapsto \underline{s}$.

The map ζ is the projection map and hence it is continuous and from the definition of ζ we see that it is injective. Since the space $\mathbb{Z}^{\mathbb{N}_0}$ is totally disconnected then using the fact that the preimage of a totally disconnected set under a continuous injective map is totally disconnected, we deduce that the set $\tilde{E}$ is totally disconnected.

2. Let us write the set $\tilde{E}$ as:

$$\tilde{E} = \bigcup_{\underline{s}^0 \text{ fast}} \Big(E(J(\mathcal{F})_{\underline{s}^0}) \Big).$$

Note that the equality holds because $(t_{\underline{s}^0}, \underline{s}^0) \in J(\mathcal{F})_{\underline{s}^0}$ for all $\underline{s}^0$. From Corollary 5.3.1 we have $E(J(\mathcal{F})_{\underline{s}^0}) \cup \{\infty\}$ is connected for each $\underline{s}^0$. Thus the set

$$\tilde{E} \cup \{\infty\} = \bigcup_{\underline{s}^0 \text{fast}} \Big(E(J(\mathcal{F})_{\underline{s}^0}) \Big) \cup \{\infty\}$$

is connected. Here we use the fact the the union of connected sets which have at least one common point is connected.

□

Chapter 6

Connectivity of Escaping Endpoints of f_a together with ∞

In this chapter we study the set of escaping endpoints of an exponential map $f_a : z \to e^z + a$, where $a \in \mathbb{C}$, and show that this set, together with the point at infinity, is connected. We will use the topological model for the dynamics of f_a on the Julia set, which was introduced in Chapter 4. More precisely, we will use properties of the spaces $J(\mathcal{F}_{\underline{s}^0}) \subset Y_Q \subset J(\mathcal{F})$ which were defined in Definition 4.4.1 and Definition 5.1.1. Then we use the conjugacy $\mathfrak{g}$ from Theorem 4.4.5 to transfer these properties to subsets of the Julia sets $J(f_a) \cup \{\infty\}$ where $a \in \mathbb{C}$, and conclude the desired result.

6.1 Escaping endpoints of the exponential maps f_a

Recall that for the family of exponential maps defined by $f_a(z) = \exp(z) + a$, where $a \in \mathbb{C}$, the set of escaping points of f_a is denoted by

$$I(f_a) = \{z \in \mathbb{C} : f_a^n(z) \to \infty\}.$$

As we mentioned in Chapter 3, according to [SZ03] the escaping points are organized on curves called dynamic rays, which tend to infinity under iteration of f_a. In this section we will use some concepts we introduced in Section 3.2 and we expect the reader to be aware of these concepts. The main goal in this section is to prove Theorem 1.1.4. In order to do so we will prove the following Theorem as a main step to proving Theorem 1.1.4.

Theorem 6.1.1.
Let $a \in \mathbb{C}$. Then there is an invariant set $A \subset \tilde{E}(f_a)$ such that ∞ is an explosion point of $A \cup \{\infty\}$.

Proof. The first idea of the proof is to choose a fast address $\underline{s}^0$ such that $X(\underline{s}^0)$ is subset of Y_Q, where Y_Q is as defined in Definition 4.4.1.

Let $\underline{s}^0$ be a fast address such that $t_{\underline{s}^0} \geq Q$ and $|s_i^0| \geq F(Q+1)$ for all i. Then the set $J(\mathcal{F})_{\underline{s}^0}$ is a closed subset of Y_Q (see Lemma 5.2.3).

Recall that the space $J(\mathcal{F})_{\underline{s}^0} \cup \{\infty\}$ is a smooth fan (Proposition 5.1.9) and the set of the endpoints $E(J(\mathcal{F})_{\underline{s}^0})$ is dense in $J(\mathcal{F})_{\underline{s}^0}$ (Thoerem 5.2.1).

Now we will use the conjugacy $\mathfrak{g} : J(\mathcal{F}) \to J(f_a)$, which is defined in Theorem 4.4.5, to generate $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$, a homeomorphic image of $J(\mathcal{F})_{\underline{s}^0}$. We have $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0}) \in I(f_a)$, since $X(\underline{s}^0) \subset Y_Q$ and $\mathfrak{g}(Y_Q)$ must be in the escaping set of f_a [Rem06, Theorem 4.2].

From the definition of the map $\mathfrak{g}$, we observe that, for $(t, \underline{s}) \in X$ $\mathfrak{g}(t, \underline{s}) \to \infty$ as $T(t, \underline{s}) \to \infty$. Since $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$ is a homeomorphic image of $J(\mathcal{F})_{\underline{s}^0}$ then $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$ has dense endpoints and the set of these endpoints, denoted A, is totally disconnected while $A \cup \{\infty\}$ is connected. Hence ∞ is an explosion point of the set $A \cup \{\infty\}$, where $A \subset \tilde{E}(f_a)$ for all $a \in \mathbb{C}$, and this proves the Theorem. □

Now we can prove Theorem 1.1.4 but to remind the reader we will restate it first.

Theorem 6.1.2.

Let f_a be an exponential map where $a \in \mathbb{C}$. Then $\tilde{E}(f_a) \cup \{\infty\}$ is connected.

Proof. The set $A \cup \{\infty\}$ in Theorem 6.1.1 is a subset of the set of the escaping endpoints of f_a, where $a \in \mathbb{C}$, together with the point at infinity. Thus by Lemma 2.1.1, taking the preimages of $A \cup \{\infty\}$ will give us a connected subset of the escaping endpoints together with the point at infinity. Then by the topological property that a topological space with a connected dense subset is connected itself, and because the set of the union of the preimages of $A \cup \{\infty\}$ is dense in $\tilde{E}(f_a) \cup \{\infty\}$, since the backward orbit of any point in $J(f_a)$ other than a is dense in $J(f_a)$ (by Montel's theorem), it follows that the set $\tilde{E}(f_a) \cup \{\infty\}$ is connected for all $a \in \mathbb{C}$. $\square$

Chapter 7

Connectivity of escaping endpoints of exponential map with accessible parameter

7.1 Introduction

In this chapter we focus our study on the set of escaping endpoints of an exponential map with a singular value which either escapes to infinity under iteration of the map f_a or which is an endpoint. Such a singular value is accessible from the set of escaping points.

Definition 7.1.1 (Accessible).
Let f_a be an exponential map, with the property that $a \in J(f_a)$. We say that a is an accessible *parameter if a is either on a dynamic ray or the landing point of a dynamic ray.*

If $\underline{s}$ is the external address of such a ray, we write $\operatorname{addr}(a) = \underline{s}$.

Our main goal in this chapter is to prove the rest of Theorem 1.1.3. In Theorem 1.1.4 we proved the first half of Theorem 1.1.3. So to complete the proof of Theorem 1.1.3 we need to prove that the sets of escaping endpoints of the exponential maps in the theorem are totally separated.

In order to study the behaviour of the set of escaping endpoints for an exponential map f_a with an accessible parameter we will assign combinatorics, following the steps and the structures in [RS08] and [Rem07], to curves of escaping points of f_a in the dynamical plane.

7.2 Combinatorics of exponential maps

As we mentioned above, to study the topological behavior for the escaping endpoints of an exponential map f_a, where a is accessible, we will assign combinatorics and in order to do so, in this section we will define some concepts we will use later in our proof.

To start our section we will give further definitions, in addition to those given in Chapter 4.

Definition 7.2.1 (Intermediate external address)**.**
A sequence is called an intermediate *external address if it is a finite sequence of the form*

$$\underline{s} = s_0 s_1 s_2 \dots s_{n-2} \infty,$$

where $n \geq 2$*,* $s_i \in \mathbb{Z}$ *for all* $i < n-2$ *and* $s_{n-2} \in \mathbb{Z} + \frac{1}{2}$*.*

The sequence $\underline{s}$ is of length n. The set of all infinite external addresses and intermediate external addresses will be denoted by Γ.

Remark 7.2.2 ([RS08])**.**
For every $\underline{s} = s_0 s_1 s_2 \dots$ *and* $\underline{s}' = (s_0+1) s_1 s_2 \dots$*, the shift map* $\sigma : [\underline{s}, \underline{s}') \to \Gamma$ *preserves the cyclic order of* Γ*.*

A cyclic order is a ternary relation that generalises the idea of cyclic order in the circle, in similar manner as the notion of an order relation generalizes the properties of the usual order on a real line,, See [Nov82] for the formal definition.

The space Γ is order-complete with respect to the lexicographic order. The one point compactification space $\overline{\Gamma} = \Gamma \cup \{\infty\}$ carries a complete circular ordering. We can think of ∞ as an intermediate external address of length 1. Note that by Remark 7.2.2 the shift map $\sigma \colon \Gamma \to \overline{\Gamma}$ is locally order-preserving map [Rem07].

Definition 7.2.3 (Surrounding addresses)**.**
In Γ *two addresses* $\underline{r}, \tilde{\underline{r}} \in \Gamma$ surround *an address* $\underline{s}$*, if* $\underline{s}$ *belongs to the bounded component of* $\Gamma \setminus \{\underline{r}, \tilde{\underline{r}}\}$*.*

Next we will review the methods of combinatorics for curves in the dynamical plane of exponential maps, which are introduced in [SZ03] and [RS08].

Definition 7.2.4 (Itinerary)**.**
Let f_a *be an exponential map, where* a *is an accessible parameter, and* $\operatorname{addr}(a) = \underline{s}$*. If* $z \in \mathbb{C}$*, we can assign to* z *an* itinerary

$$\operatorname{itin}(z) := \operatorname{itin}_{\underline{s}}(z) := \underline{u} := u_0 u_1 u_2 u_3 \dots$$

where $u_j = m$ *if* $f_a^j(z) \in U_m$*, where* U_m *is the strip bounded by* $g_{m\underline{s}}$ *and* $g_{m+1\underline{s}}$*, and* $u_j = \binom{m}{m-1}$ *if* $f_a^j(z) \in g_{m\underline{r}}$*.*

If $z \in g_{\underline{r}}$ *for some integer sequence* $\underline{r}$*, then the itinerary entries of* z *satisfy*

$$u_j = \begin{cases} m & \text{if } m\underline{s} < \sigma^j(\underline{r}) < (m+1)\underline{s} \\ \binom{m}{m-1} & \text{if } \sigma^j(\underline{r}) = m\underline{s}. \end{cases} \tag{7.1}$$

For any infinite external address $\underline{r}$ and every infinite external address $\underline{s}$ we can define an address $\operatorname{itin}_{\underline{s}}(\underline{r})$ by the formula 7.1.
In other words, because a is accessible then the preimage of the ray $g_{\underline{s}}$, which contains a or having a as its landing point, consists of countably many horizontal curves (rays at the addresses $i\underline{s}$ for $i \in \mathbb{Z}$). These curves divide the dynamical plain into strips U_i and any address whose associated ray lies in this strip has i as a first entry of its itinerary.

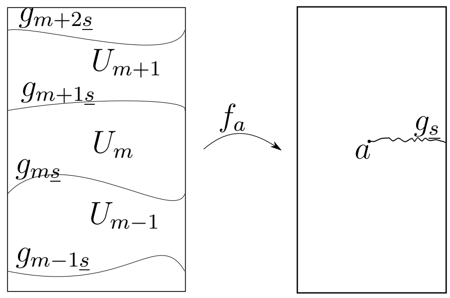

Figure 7.1: Itineraries.

The next figure illustrates that the j-th entry in the itinerary of a point $z \in g_{\underline{r}}$ is the number m such that $f_a^j(z) \in U_m$, or equivalently $m\underline{s} < \sigma^j(\underline{r}) < (m+1)\underline{s}$.

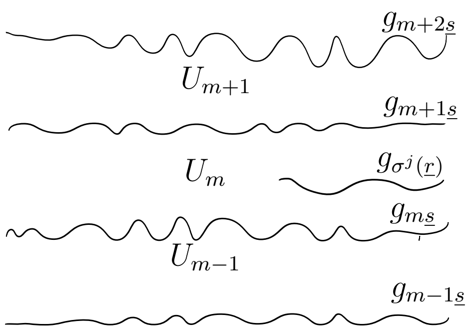

Figure 7.2: Relationship between external addresses and itineraries.

Definition 7.2.5 (Kneading sequence)**.**
The kneading sequence *of an external address* $\underline{s}$*, is the itinerary* $\mathbb{K}(\underline{s}) := \text{itin}_{\underline{s}}(\underline{s})$.

In [RS08, Remark 1, p11] Rempe-Gillen and Schleicher give us the main idea about how to think of $\underline{s}$ as being in the combinatorial parameter plane while the sequence $\underline{r}$ is in the combinatorial dynamic plane associated with $\underline{s}$.

It can be observed from the previous two definitions that for f_a, where a is an accessible parameter, and $\text{addr}(a) = \underline{s}$, every point in an exponentially bounded address has an itinerary with respect to $\underline{s}$ and points with the same address have the same itinerary. The question is now whether can two different addresses have the same itinerary with respect to $\underline{s}$? The answer that they can was already well-known and later it was proved by Rempe-Gillen in [Rem07] as a Lemma. In this Chapter, as an introduction to our result, we shall present that lemma which we used as a main step in our next result in the next section.

Lemma 7.2.6 ([Rem07, Lemma 2.3])**.**
Let $\underline{s}$ be an external address and $\underline{u} = \mathbb{K}(\underline{s})$. Suppose that $\underline{r} \neq \underline{\tilde{r}}$ are two addresses sharing the same itinerary $\underline{\tilde{u}} = \mathrm{itin}_{\underline{s}}(\underline{r}) = \mathrm{itin}_{\underline{s}}(\underline{\tilde{r}})$, and let $m \geq 0$ with $r_m \neq \tilde{r}_m$.

Then for every $k \geq 1$, there exists $0 \leq j \leq k$ such that $\sigma^{m+k}(\underline{r})$ and $\sigma^{m+k}(\underline{\tilde{r}})$ surround $\sigma^j(\underline{s})$. In particular,

$$\tilde{u}_{m+k} \in \{u_0, \ldots, u_{k-1}\} \text{ for all } k \geq 1.$$

In the next Section we use this Lemma to conclude that if two addresses sharing an itinerary then these addresses must be slow in the sense of Definition 4.2.4.

7.3 Escaping endpoints of exponential map with accessible parameter.

In this section we will prove Theorem 1.1.3. This is one of the main results in our thesis. We will use Lemma 7.2.6 to establish two supplementary results which together will imply the rest of the Theorem which we want to prove. The first result is general and it shows that if there exist two fast non equal addresses, then their itineraries are not equal. The second states that for an exponential map with an accessible singular value, if there exist two addresses where itineraries are not equal then the dynamic rays (curves) at these addresses can be separated by a curve.

Lemma 7.3.1.

Let $\underline{s}$ be an exponentially bounded address, and $\underline{u} = \mathbb{K}(\underline{s})$. Let $\underline{r}, \underline{\tilde{r}} \in \mathbb{Z}^{\mathbb{N}_0}$ be exponentially bounded addresses. If $\underline{r} \neq \underline{\tilde{r}}$ and both are fast, then

$$\operatorname{itin}_{\underline{s}}(\underline{r}) \neq \operatorname{itin}_{\underline{s}}(\underline{\tilde{r}}).$$

Proof. $\underline{s}$ is an exponentially bounded address, so there exists $y > 0$ such that $|s_i| \leq F^i(y)$ for all $i \geq 0$.

We will prove the contrapositive. Assuming that $\operatorname{itin}_{\underline{s}}(\underline{r}) = \operatorname{itin}_{\underline{s}}(\underline{\tilde{r}}) = \underline{v}$ and $\underline{r} \neq \underline{\tilde{r}}$, we claim that the addresses $\underline{r}$ and $\underline{\tilde{r}}$ are slow. Precisely we will show that there exists $x > 0$ such that, for all $n_0 > 0$, there exists $n \geq n_0$ such that $|r_{n+i}|, |\tilde{r}_{n+i}| \leq F^i(x)$ for all $i \geq 0$.

Set $x = y + 2$. Let $n_0 > 0$. Now we claim $\sigma^{n_0}(\underline{r}) \neq \sigma^{n_0}(\underline{\tilde{r}})$.

Assume for a contradiction that $\sigma^{n_0}(\underline{r}) = \sigma^{n_0}(\underline{\tilde{r}})$. Because $\underline{r} \neq \underline{\tilde{r}}$ and $\sigma^{n_0}(\underline{r}) = \sigma^{n_0}(\underline{\tilde{r}})$, there exists $j \geq 0$ with $j < n_0$ such that $r_j \neq \tilde{r}_j$ but $r_i = \tilde{r}_i$ for all $i > j$. By definition $\sigma^j(\underline{r})$ and $\sigma^j(\underline{\tilde{r}})$ have different 0-th itinerary entry, which contradicts the assumption that $\operatorname{itin}_{\underline{s}}(\underline{r}) = \operatorname{itin}_{\underline{s}}(\underline{\tilde{r}})$. Hence $\sigma^{n_0}(\underline{r}) \neq \sigma^{n_0}(\underline{\tilde{r}})$.

Let $m \geq n_0$ be such that $r_m \neq \tilde{r}_m$. By Lemma 7.2.6 we obtain that, for every $k \geq 0$, there exists $l_k \geq 0$ such that $v_{m+k} = u_{l_k}$ where $0 \leq l_k \leq k - 1$.

Set $n = m + 1$, and $k \geq 0$. Then

$$\begin{aligned}
\max(|r_{n+k}|, |\tilde{r}_{n+k}|) \leq |v_{n+k}| + 1 &= |v_{m+1+k}| + 1 \\
&= |u_{l_k+1}| + 1 \\
&\leq |s_{l_k+1}| + 1 + 1 \\
&\leq F^{l_k+1}(y) + 2 \\
&\leq F^{l_k+1}(y + 2) \\
&= F^{l_k+1}(x) \\
&\leq F^k(x).
\end{aligned}$$

Thus the address $\underline{r}$ is slow.

Hence the lemma is proved. □

In the previous lemma we proved that the itineraries of any two different addresses are different and hence have two different escaping endpoints. Now in the next lemma our main question which we want to discuss and answer is: If we have two external addresses with different itineraries then what is the relation between the rays at these chosen external addresses?

Lemma 7.3.2.
Let f_a be an exponential map, where a is accessible and $\underline{s} = \operatorname{addr}(a)$. Let $\underline{r}, \tilde{\underline{r}} \in \mathbb{Z}^{\mathbb{N}_0}$. If $\operatorname{itin}_{\underline{s}}(\underline{r}) \neq \operatorname{itin}_{\underline{s}}(\tilde{\underline{r}})$ and $\sigma^n(\underline{r}), \sigma^n(\tilde{\underline{r}}) \neq \underline{s}$ for all $n \geq 1$, then $g_{\underline{r}}$ and $g_{\tilde{\underline{r}}}$ can be separated by a closed curve in the escaping set with no endpoints.

Proof. Set $\underline{v} = \operatorname{itin}_{\underline{s}}(\underline{r})$ and $\tilde{\underline{v}} = \operatorname{itin}_{\underline{s}}(\tilde{\underline{r}})$. Note from Definition 7.2.4 that these are the joint itinraries of points on $g_{\underline{r}}$ and respectively $g_{\tilde{\underline{r}}}$.

Because $\operatorname{itin}_{\underline{s}}(\underline{r}) \neq \operatorname{itin}_{\underline{s}}(\tilde{\underline{r}})$, there exists an integer $j \geq 0$ such that $v_j \neq \tilde{v}_j$ and $v_i = \tilde{v}_i$ for all $i < j$. Hence $\sigma^j(\underline{r}) \neq \sigma^j(\tilde{\underline{r}})$. Assume without loss of generality that $\tilde{v}_j < v_j$. Then by the definition of itineraries, we obtain
$v_j\underline{s} < \sigma^j(\underline{r}) < (v_j + 1)\underline{s}$,

Let $g_{v_j\underline{s}}$ be the dynamic ray at $v_j\underline{s}$. This ray lands at ∞ and hence has no endpoints. Define $\mathcal{H}_Q := \{z \in \mathbb{C} \colon \operatorname{Re} z > Q\}$ where $Q > 0$. Then for Q large enough we have $\mathcal{H}_Q \setminus g_{v_j\underline{s}}$ has two connected components say U and V (from a discussion in [RS08, p.7]) and ([RS09, p.114]). Here U contains all tails of rays with addresses less than $v_j\underline{s}$ and V is the complement of the closure of U in $\mathcal{H}_Q$. Thus $g_{\sigma^j(\underline{r})} \subset V$ and $g_{\sigma^j(\tilde{\underline{r}})} \subset U$, and hence $g_{\sigma^j(\underline{r})}$ and $g_{\sigma^j(\tilde{\underline{r}})}$ are separated by $g_{v_j\underline{s}}$. This is a curve in the escaping set $I(f_a)$ containing no endpoints.

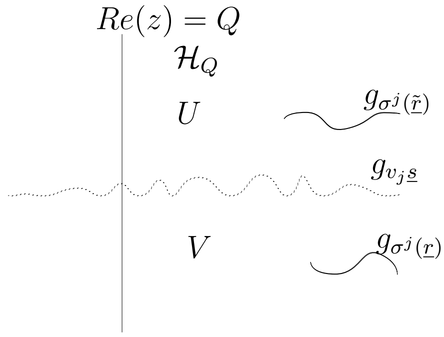

Figure 7.3: The difference in the itineraries of two distinct external sequences means that, for some j, their j-th iterates are separated.

Now the main step of the proof: we claim that for $i \leq j$ there exists a simple closed curve which separates $g_{\sigma^i(\underline{r})}$ and $g_{\sigma^i(\underline{\tilde{r}})}$ in the escaping set $I(f_a)$.

We will prove by induction for $n = 0, \ldots j$ that $g_{\sigma^i(\underline{r})}$ and $g_{\sigma^i(\underline{\tilde{r}})}$, where $i = j - n$, are separated by a Jordan curve with no endpoints.

- Basis of the induction: if $n = 0$ then we have $i = j$, hence our claim holds by what was proved and we have $g_{\sigma^i(\underline{r})}$ and $g_{\sigma^i(\underline{\tilde{r}})}$ are separated by the Jordan curve $\gamma_j = g_{v_j\underline{s}} \cup \{\infty\}$.

- Induction hypothesis: assume that the claim holds for n, hence there exists a simple closed curve $\gamma_i = g_{v_i\underline{s}}$, where $i = j - n$, separating $g_{\sigma^{j-n}(\underline{r})}$ and $g_{\sigma^{j-n}(\underline{\tilde{r}})}$, and lying in the escaping set $I(f_a)$. In order to prove our last step we define $U \subset \mathcal{H}_Q \backslash \gamma_i$ to be the component which contains $g_{\sigma^i(\underline{r})}$ and bounded by the simple closed curve γ_i, and V to be the complement of the closure of U in $\mathcal{H}_Q$. Assume without loss of generality that $a \notin U$.

- Next step: we aim to prove the result for $n + 1$. From the previous discussion one can observe that $g_{\sigma^i(\underline{r})} \cap \gamma_i = \emptyset$. Now let us define a map Φ to be the branch of f_a^{-1} on U which maps $g_{\sigma^i(\underline{r})}$ to $g_{\sigma^{i-1}(\underline{r})}$. Note that such a branch exists because we assumed that $a \notin U$.

  Then from the assumption and because Φ is continuous, $\Phi(U)$ is open, connected and is contained in the strip $U_{v_{i-1}}$ which is bounded by the curves $g_{v_{i-1}\underline{s}}$ and $g_{(v_{i-1}+1)\underline{s}}$. Because γ_i is a simple closed curve through ∞ then its preimages under f_a are also simple closed curves through ∞. In addition, because U is open and bounded by γ_i, $\Phi(U)$ is bounded by one or two preimages of γ_i but one of them separates the preimages of the curves $g_{v_{i-1}\underline{s}}$ and $g_{(v_{i-1}+1)\underline{s}}$, say γ_{i-1}; i.e. $\Phi(\gamma_i) = f_a^{-1}|_U(\gamma_i) = \gamma_{i-1}$.

Now we claim that $g_{\sigma^{i-1}(\underline{\tilde{r}})} \cap \Phi(U) = \emptyset$. Assume for a contradiction that $g_{\sigma^{i-1}(\underline{\tilde{r}})} \cap \Phi(U) \neq \emptyset$ and so there exists $x \in g_{\sigma^{i-1}(\underline{\tilde{r}})} \cap \Phi(U)$. Hence $f_a(x) \in g_{\sigma^{i}(\underline{\tilde{r}})} \cap U$ but that contradicts our assumption that $g_{\sigma^{i}(\underline{\tilde{r}})}$ is contained in V. Thus we obtain that $g_{\sigma^{i-1}(\underline{\tilde{r}})} \cap \Phi(U) = \emptyset$. Hence γ_{i-1}, a preimage of γ_i in $U_{v_{i-1}}$, separates $g_{\sigma^{i-1}(\underline{r})}$ and $g_{\sigma^{i-1}(\underline{\tilde{r}})}$.

If $a \in \gamma_i$ then the preimage of γ_i, γ_{i-1}, is a ray landing at ∞ and we writ it as $\gamma_{i-1} = g_{\underline{s}} \cup g_{\text{add}^-(\underline{s})}$, where $\text{add}^-(\underline{s})$ is the negative address of $\underline{s}$ landing at ∞. Then because γ_i separates $g_{\sigma^{i}(\underline{r})}$ and $g_{\sigma^{i}(\underline{\tilde{r}})}$ then from Janiszewski's Theorem [Dug78] either $g_{\underline{s}}$ or $g_{\text{add}^-(\underline{s})}$ separates $g_{\sigma^{i}(\underline{r})}$ and $g_{\sigma^{i}(\underline{\tilde{r}})}$. Then a similar discussion holds and we conclude that γ_{i-1}, a preimage of γ_i in $U_{v_{i-1}}$, separates $g_{\sigma^{i-1}(\underline{r})}$ and $g_{\sigma^{i-1}(\underline{\tilde{r}})}$.

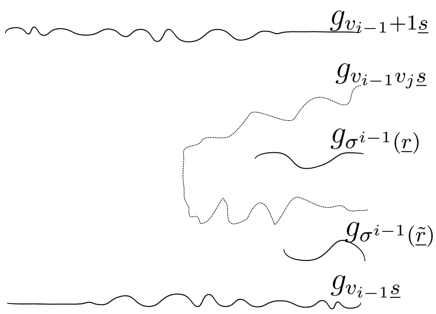

Figure 7.4: The preimage of Figure 7.3, which shows that the curves $g_{\sigma^{i-1}(\underline{r})}$ and $g_{\sigma^{i-1}(\underline{\tilde{r}})}$ are separated in $I(f_a)$.

Thus we prove our claim that for $i \leq j$ there exist a Jordan curve separating $g_{\sigma^{i}(\underline{r})}$ and $g_{\sigma^{i}(\underline{\tilde{r}})}$ in the escaping set $I(f_a)$. Hence the lemma is proved.

□

Theorem 7.3.3.

Let f_a be an exponential map, where a is an accessible parameter. Then the set of escaping endpoints $\tilde{E}(f_a)$ is totally separated.

Proof. First let us set $\underline{s} = \text{addr}(a)$, $\underline{s}$ is the external address of a dynamical ray which contains a or has a as its landing point.

Let $x, \tilde{x} \in \tilde{E}(f_a)$ be two distinct escaping endpoints such that $\text{addr}(x) = \underline{r}$ and $\text{addr}(\tilde{x}) = \underline{\tilde{r}}$, where $\underline{r}, \underline{\tilde{r}}$ are sequences of integers.
Because the points x and $\tilde{x}$ are distinct then $\underline{r} \neq \underline{\tilde{r}}$, otherwise we would obtain a dynamic ray at the address $\underline{r}$ with two endpoints, which contradicts the fact that each ray has at most one endpoint.

Since x and $\tilde{x}$ are escaping points and they are the endpoints of the rays which contain them, then from the Observation 4.2.7, we obtain that $\underline{r}$ and $\underline{\tilde{r}}$ are fast.

Because $\underline{r} \neq \underline{\tilde{r}}$ and both are fast, then, from Lemma 7.3.1, we obtain that $\text{itin}_{\underline{s}}(\underline{r}) \neq \text{itin}_{\underline{s}}(\underline{\tilde{r}})$. Hence, as a result of Lemma 7.3.2, we obtain that the curves g_r and $g_{\tilde{r}}$ are separated by a curve in the escaping set which contains no endpoint.

Thus, for any two distinct, escaping endpoints $x, \tilde{x}$ there exists a closed connected curve in $\mathbb{C} \setminus \tilde{E}(f_a)$ which separates the points x and $\tilde{x}$. Therefore, the set of escaping endpoints $\tilde{E}(f_a)$ is totally separated. □

Now to remind the reader of our main theorem of this chapter which is Theorem 1.1.3 we will restate it in order to prove it.

Theorem 7.3.4.

Let f_a be an exponential map, where a is an accessible parameter . Then ∞ is an explosion point for the set $\tilde{E}(f_a) \cup \{\infty\}$.

Proof. The proof is just a combination of two of the main results of our research. First note that in Theorem 6.1.2 we obtained that for an exponential map f_a, the union of the set of escaping endpoints $\tilde{E}(f_a)$ with the point at infinity is connected for all parameters a. Then in our previous Theorem 7.3.3

we obtained that for exponential maps f_a, where a is an accessible parameter, the set $\tilde{E}(f_a)$ is totally separated.

Thus for exponential maps f_a, where a is accessible, the point at ∞ is an explosion point of the set $\tilde{E}(f_a) \cup \{\infty\}$. $\square$

Chapter 8

Connectedness of $E(f_a)$ and $\tilde{E}(f_a)$

8.1 Introduction to the proof of Theorem 1.1.5

In this chapter we aim to determine whether there is any special hypothesis under which the set of endpoints $E(f_a)$ and the set of escaping endpoints $\tilde{E}(f_a)$ of the exponential map f_a are connected. Before we study the connectivity of the set of endpoints and the set of escaping endpoints of f_a, note that from theorem 7.3.3 we obtain that if a is an escaping point or a landing point of an escaping curve then the set of escaping endpoints of f_a is totally separated. For that reason in this chapter we will omit exponential maps with accessible parameters.

We shall start our study with some brief background for the proof of our theorem.

Lemma 8.1.1.
Let Y be a straight brush. Let $K \subset \mathbb{R}$ be a closed connected set not containing ∞, and $K \not\subset Y$. Let $h : [0, \infty) \to Y$ be a curve in Y and $0 < t_1 < t < t_2$ such that $h(t) \in K$, $h(t_1), h(t_2) \notin K$ and $h(0)$ is the endpoint of the curve h. Then K must intersect $E(Y)$, the set of endpoints of Y.

Proof. Assume without loss of generality that Y is our model $J(\mathcal{F})_{\underline{s}^0}$, which is defined in Definition 5.1.1. Assume for a contradiction that there exists K a subset of $\mathbb{R}$, as above, but $K \cap E(J(\mathcal{F})_{\underline{s}^0}) = \emptyset$.

Let $R \subset \mathbb{R}^2$ be an open rectangle such that $h(t), h(0) \in R$, with the horizontal boundaries of R being in the complement of $J(\mathcal{F})_{\underline{s}^0}$, very close and parallel to the curve h, and such that one vertical boundary contains $h(t_2)$ and the other is in the complement of $J(\mathcal{F})_{\underline{s}^0}$. Hence $K \cap R \neq \emptyset$.

Let K_R be the connected component of $K \cap R$ such that $h(t) \in K_R$. Then as a result of the Boundary bumping Theorem 2.1.7 we obtain that $\overline{K}_R$ intersects the horizontal boundaries of R. Note that K does not intersect the vertical boundaries of R because K is closed and $h(t_1), h(t_2) \notin K$.

Now let $\tilde{R} \subset \overline{R}$ be a closed rectangle such that $h(t) \in \tilde{R}$, the horizontal boundaries of $\tilde{R}$ are subsets of the horizontal boundaries of R and there exists $\epsilon > 0$ such that the vertical boundaries of $\tilde{R}$ intersect the points $h(t_1 + \epsilon)$ and $h(t_2 - \epsilon)$. Note that we can choose ϵ such that $h(t_1 + \epsilon), h(t_2 - \epsilon) \notin K$, this is from the fact that $\mathbb{R} \setminus K$ is open. Because $\overline{R}$ and $\tilde{R}$ are rectangles, we obtain that the complement of $\tilde{R}$ in $\overline{R}$ has two different connected components. Hence the set $(\partial R \cup \overline{K}_R) \cup \tilde{R}$ separates these two components. Set R^- to be the half open rectangle $\overline{R} \setminus v_{t_2}$ where v_{t_2} is the vertical boundary of R which contains $h(t_2)$.

Hence we obtain that the set $(\partial(R^-) \cup \overline{K}_R) \cup \tilde{R}$ separates the points $h(0)$ and $h(t_2)$. In particular, it separates $h(0)$ from all the points $h(l)$ where $l > t_2 - \epsilon$ including the point ∞.

Now because $(\partial(R^-) \cup \overline{K}_R) \cap \tilde{R}$ is connected, we can use Janiszewski's Theorem 2.1.4. Thus by using Janiszewski's Theorem 2.1.4 we obtain that either $\tilde{R}$ separates the points $h(0)$ and ∞ or $(\partial(R^-) \cup \overline{K}_R)$ separates these points. But $\tilde{R}$ does not because $\tilde{R}$ might contain some endpoint of $J(\mathcal{F})_{\underline{s}^0}$. Thus $\partial(R^-) \cup \overline{K}_R$ separates the points $h(0)$ and ∞. Because $\overline{K}_R$ does not contain endpoints of $J(\mathcal{F})_{\underline{s}^0}$, and because $\partial(R^-)$ is in the complement of $J(\mathcal{F})_{\underline{s}^0}$, we obtain that $\partial(R^-) \cup \overline{K}_R$ separates an endpoint of $J(\mathcal{F})_{\underline{s}^0}$ from the point ∞.

However this is a contradiction. Hence K must contain endpoints of $J(\mathcal{F})_{\underline{s}^0}$; that is $K \cap E(J(\mathcal{F})_{\underline{s}^0}) \neq \emptyset$. □

8.2 Cantor bouquets in the Julia sets of exponential functions

As a main step in our proof of Theorem 1.1.5 we will use Cantor bouquets which are in the Julia sets of exponential functions. In order to prove the existence of a Cantor bouquet in the Julia set of an exponential map f_a, we will use a subset of our topological model $J(\mathcal{F})$, more precisely, a subset of the space Y_Q, which was defined in Definition 4.4.1. We will define such a set and then prove that its image under the map $\mathfrak{g}$, which was defined in Theorem 4.4.5, is a Cantor bouquet.

Note that the map $\mathfrak{g}$ in Theorem 4.4.5 extends to a homeomorphism $\mathbb{R}^2 \to \mathbb{C}$, where $J(\mathcal{F})_{\underline{s}^0} \subset J(\mathcal{F})$ is considered embedded in $\mathbb{R}^2$ as in 4.2.10.

Lemma 8.2.1.
Let $J(\mathcal{F})_{\underline{s}^0}$ be the space in Lemma 5.2.3. Then the set $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$ is a Cantor bouquet.

Proof. The proof of this lemma follows from previous results. Using Theorem 5.1.2 and Corollary 4.4.4 and the result of Theorem 4.4.5 that the map $\mathfrak{g}$ preserves the vertical order we conclude that the space $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$ satisfies the conditions in [ARG15, Theorem 2.8]. Hence we obtain that $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$ is a Cantor bouquet. □

8.3 The proof of Theorem 1.1.5

In order to prove Theorem 1.1.5, our last theorem in this thesis, we will first show that far an exponential map f_a, any closed connected set which intersects a curve at an exponentially bounded address must intersect a distinct curve at a fast address. Then from the later intersection we obtain a closed set with certain conditions which intersect the cantor bouquet in $J(f_a)$ which is a subset of $I(f_a)$ close to infinity. The existence of such a set contradicts some properties of the cantor bouquet, and that is how we will prove our theorem.

Lemma 8.3.1.
Let f_a be an exponential map, where $a \in \mathbb{C}$. Let $g_{\underline{s}}$ be a curve at a slow address $\underline{s}$ and let K be a closed connected subset of $\mathbb{C}$. If $K \not\subset g_{\underline{s}}$ and there exist $0 < t_1 < t < t_2$ such that $g_{\underline{s}}(t) \in K$ and $g_{\underline{s}}(t_1), g_{\underline{s}}(t_2) \notin K$, then K must intersect a curve at a fast address.

Proof. Let $n_0 > 0$, then there exists an intermediate address $\underline{s}^{n_0} = s_0 s_1 \dots s_{n_0} + 1/2\infty$ close to $\underline{s}$. Note that the curve $g_{\underline{s}^{n_0}}$ is in the complement of the set of all curves at infinite external addresses, and it is close to the curve $g_{\underline{s}}$ if n_0 is large. Because K is a closed set, there exists ϵ_0 such that N_{ϵ_0,t_1}, N_{ϵ_0,t_1} are two neighborhoods of $g_{\underline{s}}(t_1)$ and $g_{\underline{s}}(t_2)$, respectively, and $K \cap N_{\epsilon_0,t_i} = \emptyset$, where $i = 1, 2$. Now we can choose n_0 large enough such that $g_{\underline{s}^{n_0}}$ intersects N_{ϵ_0,t_1} and N_{ϵ_0,t_2} at the points $g_{\underline{s}^0}(t_1')$ and $g_{\underline{s}^0}(t_2')$, respectively. Note that $g_{\underline{s}^0}(t_1')$ and $g_{\underline{s}^0}(t_2')$ are not in $\overline{K}$. Then we can obtain two vertical lines from $g_{\underline{s}}(t_1), g_{\underline{s}}(t_2)$ to the curve $g_{\underline{s}^{n_0}}$ such that these two lines are in the complement of K. Thus we obtain a closed rectangle R between the curves $g_{\underline{s}^{n_0}}$ and $g_{\underline{s}}$. Now let K_t be a connected component of $K \cap R$ which contains the point $g_{\underline{s}}(t)$. Hence by the Boundary bumping Theorem 2.1.7 we obtain that the set $\overline{K}_t$ intersects the boundaries of R but K does not intersect the vertical boundaries, and hence $\overline{K}_t$ does not intersect the verticals. Therefore, $\overline{K}_t$ intersects the two horizontal boundaries of R.
Now let $\epsilon > 0$ and let us define a sequence of integers $\underline{\tilde{s}}$ as $\tilde{s}_i := \lfloor s_i + F^i(\epsilon) \rfloor$. Note that the sequence $\underline{\tilde{s}}$ is a fast address that is obtained from Definition 4.2.4 and Lemma 4.1.1. Let us choose ϵ small enough such that $F^{n_0}(\epsilon) < 1$. Then, $\underline{\tilde{s}}$ will be very close to $\underline{s}$, even closer than $\underline{s}^{n_0}$. This is derived from the lexicographic order of integer sequences. Hence the curve $g_{\underline{\tilde{s}}}$ is closer to the

curve $g_{\underline{s}}$ than the curve $g_{\underline{s}^{n_0}}$.

Now we claim that $\overline{K_t}$ intersects the curve $g_{\underline{\tilde{s}}}$.

We will prove it directly. Because $g_{\underline{\tilde{s}}}$ is between the curves $g_{\underline{s}^{n_0}}$ and $g_{\underline{s}}$ then either $g_{\underline{\tilde{s}}}$ intersects one of these two curves, or it intersects the set $\overline{K_t}$. If the first case holds then we will conclude that either $g_{\underline{\tilde{s}}}$ intersects a curve in its complement or that the sequence $\underline{\tilde{s}}$ is not fast hence $\epsilon = 0$. However these all contradict the assumptions. Thus $g_{\underline{\tilde{s}}}$ intersects the set $\overline{K_t}$. Hence we proved our lemma that the set $\overline{K_t}$ intersects $g_{\underline{\tilde{s}}}$, a curve at a fast sequence $\underline{\tilde{s}}$.

Note that if we assumed that $\underline{s}^{n_0} = s_0 s_1 \dots s_{n_0} - 1/2\infty$ and use it to obtain a rectangle then we define $\underline{\tilde{s}}$ as $\tilde{s}_i := \lceil s_i - F^i(\epsilon) \rceil$ and the rest follows analogously. □

Lemma 8.3.2.
Let f_a be an exponential map, where $a \in \mathbb{C}$ is not accessible. Let $z_1, z_2 \in \mathbb{C}$ are escaping endpoints that are in the accumulation set of the same ray $g_{\underline{u}}$, where $\underline{u}$ is an exponentially bounded address then z_1, z_2 can not be separated in $\tilde{E}$.

Proof. Let z_1, z_2 be as in the assumption then we say that can not be separated in $\tilde{E}$ i.e. there are no open, nonempty sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$, $\tilde{E}(f_a) \subset U_1 \cup U_2$ and $z_1 \in U_1, z_2 \in U_2$. Or by using [NEW51, Theorem V.14.3] this means that there is no any closed connected set separating them.

To prove our lemma we assume for a contradiction that there exist z_1, z_2 which satisfy the previous conditions but are separated in $\tilde{E}$. Thus there exists a simple closed connected set A such that $A \cap \tilde{E}(f_a) = \emptyset$ and $\tilde{E}(f_a)$ intersects at least two different components say U_1 and U_2, of the complement of A.

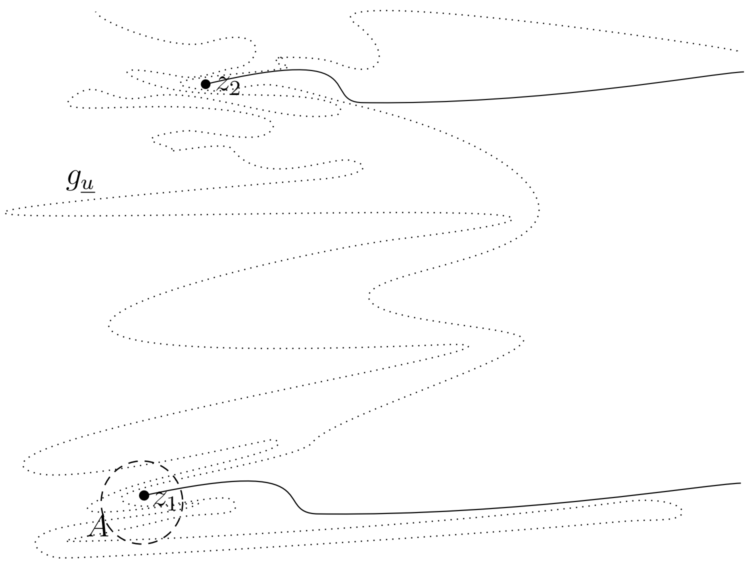

Figure 8.1: Two escaping endpoints in the accumulation set of $g_{\underline{u}}$.

Therefore, because z_1, z_2 are in the accumulation set of $g_{\underline{u}}$, the ray $g_{\underline{u}}$ intersects the set A. Let V be a connected component of $A \cap g_{\underline{u}}$ which is also a compact sub-piece of $g_{\underline{u}}$, that is because $g_{\underline{u}}$ must leave the set A in order to get close to the point z_2.

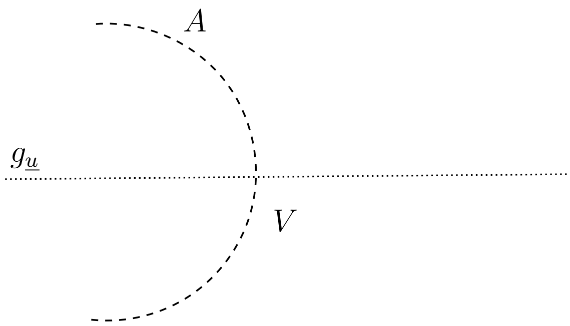

Figure 8.2: The intersection of the set A and $g_{\underline{u}}$.

Let also m be large enough, such that $V_m = f_a^m(V)$ is in a Cantor bouquet which exists in $J(f_a)$ close to infinity [BJR12, Theorem 1.6]. In our case we choose the Cantor bouquet to be $\mathfrak{g}(J(\mathcal{F})_{\underline{s}^0})$. Note that V_m exists because of the fact that $V \subset g_{\underline{u}}$ and $g_{\underline{u}} \subset I(f_a)$.

Then $V_m \subset I(f_a)$ is the m-th image of V under the map f_a. Define $\epsilon := \frac{\min d(f_a^i(a), V_m)}{2}$ for $i \leq m$, hence there exists $N_{\epsilon,m}$, an ϵ neighborhood of V_m and every point in $N_{\epsilon,m}$ is at a distance less than ϵ from some point of V_m, such that $f_a^i(a) \notin N_{\epsilon,m}$ for all $i \leq m$.

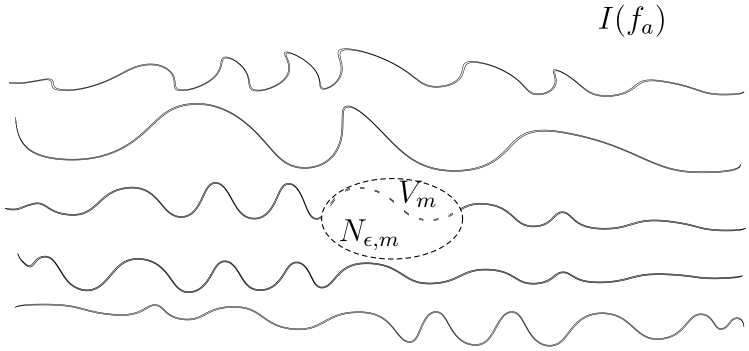

Figure 8.3: The m-th image of the set V under f_a.

From the previous assumption now we can take the preimage of $N_{\epsilon,m}$, for m times and restrict these preimages on the branches of f_a^{-1} which satisfy $f_a^{-m}(V_m) = V$. Hence we obtain an open set N_ϵ that contains the set V and $A \cap N_\epsilon \neq \emptyset$. This is due to the assumption that $V \subset A$. Thus there exists a connected component of $A \cap N_\epsilon$, different to V, intersecting V. Assume that K is a connected component of $A \cap N_\epsilon$ such that $K \cap V \neq \emptyset$. Then from the Boundary bumping Theorem 2.1.7, we obtain that $\overline{K}$, the closure of K, intersects the boundary $\partial(N_\epsilon)$.

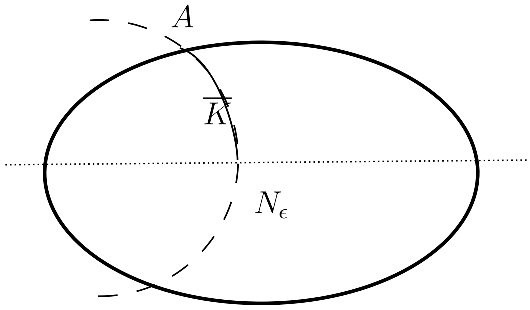

Figure 8.4: A connected component of the intersection of A and N_ϵ.

Then by Lemma 8.3.1 $\overline{K}$ must intersect a curve at a fast address say $g_{\underline{v}}$ and there exist $0 < t_1' < t' < t_2'$ such that $g_{\underline{v}}(t') \in \overline{K}$ but $g_{\underline{v}}(t_i') \notin \overline{K}$ where $i = 1, 2$. Note that $\overline{K} \subset \overline{V} \subset g_{\underline{u}}$ hence $\overline{K} \cap \tilde{E}(f_a) = \emptyset$.

Now we will apply the map f_a to the set $\overline{N}_\epsilon$ m times. Hence we obtain a closed set $\overline{K}_m$ such that $\overline{K}_m \cap g_{\sigma^m(\underline{v})} \neq \emptyset$ and $g_{\sigma^m(\underline{v})}(t') \in \overline{K}$ but $g_{\sigma^m(\underline{v})}(t_i') \notin \overline{K}$ where $i = 1, 2$. Note that $\sigma^m(\underline{v})$ is a fast address.

Finally, because any Cantor bouquet is ambiently homeomorphic to a straight brush, in our proof we have $J(\mathcal{F})_{\underline{s}^0}$ as a straight brush such that the Cantor bouquet, which is a subset of $J(f_a)$, is ambiently homeomorphic to $J(\mathcal{F})_{\underline{s}^0}$. Thus there exists an ambiently homeomorphic map α which maps the Cantor bouquet to $J(\mathcal{F})_{\underline{s}^0}$. Hence by applying the map α we obtain a similar statement in $J(\mathcal{F})_{\underline{s}^0}$ as in the Cantor bouquet. Set $\alpha(\overline{K}_m) = \overline{K}_x$, $\alpha(g_{\sigma^m(\underline{v})}) = h^{\sigma^m(\underline{v})}$ and $\alpha(N_\epsilon) = N_x$.
Thus now in $J(\mathcal{F})_{\underline{s}^0}$ there exists a closed connected set $\overline{K}_x$ which does not contain any endpoint of $J(\mathcal{F})(\underline{s}^0)$, intersects $h^{\sigma^m(\underline{v})}$ and contains the point $h^{\sigma^m(\underline{v})}(t')$, but $h^{\sigma^m(\underline{v})}(t_i') \notin \overline{K}_x$ where $i = 1, 2$. Hence we obtain a contradiction to Lemma 8.1.1.

Thus there exists no such A as in our assumption exists and hence z_1 and z_2 can not be separated in $\tilde{E}$. □

Now we will restate our last result in our thesis which studies the connected behavior of the set of endpoints and the set of escaping endpoints of f_a under certain conditions.

Theorem 8.3.3.
Let f_a be an exponential map, where $a \in \mathbb{C}$ is not accessible. If the set of escaping points $I(f_a)$ has a path-connected component that is dense in the Julia set, then $E(f_a)$ and $\tilde{E}(f_a)$ are connected.

Proof. Assume for a contradiction that $E(f_a)$ and $\tilde{E}(f_a)$ are not connected. Hence there exist at least two distinct escaping endpoints which are in the accumulation set of the path-connected component which is dense in $J(f_a)$ but these two points can be separated in $\tilde{E}$, which contradicts Lemma 8.3.2.

Thus if $I(f_a)$ has a path-connected component that is dense in the Julia set, then $E(f_a)$ and $\tilde{E}(f_a)$ are connected. □

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