# On the Efficiency of All-Pay Mechanisms

George Christodoulou*, Alkmini Sgouritsa, and Bo Tang

University of Liverpool, UK

**Abstract.** We study the inefficiency of mixed equilibria, expressed as the price of anarchy, of all-pay auctions in three different environments: combinatorial, multi-unit and single-item auctions. First, we consider item-bidding combinatorial auctions where $m$ all-pay auctions run in parallel, one for each good. For fractionally subadditive valuations, we strengthen the upper bound from 2 [23] to 1.82 by proving some structural properties that characterize the mixed Nash equilibria of the game. Next, we design an all-pay mechanism with a randomized allocation rule for the multi-unit auction. We show that, for bidders with submodular valuations, the mechanism admits a unique, 75% efficient, pure Nash equilibrium. The efficiency of this mechanism outperforms all the known bounds on the price of anarchy of mechanisms used for multi-unit auctions. Finally, we analyze single-item all-pay auctions motivated by their connection to contests and show tight bounds on the price of anarchy of social welfare, revenue and maximum bid.

## 1 Introduction

It is a common economic phenomenon in competitions that agents make irreversible investments without knowing the outcome. *All-pay* auctions are widely used in economics to capture such situations, where all players, even the losers, pay their bids. For example, a lobbyist can make a monetary contribution in order to influence decisions made by the government. Usually the group invested the most increases their winning chances, but all groups have to pay regardless of the outcome. In addition, all-pay auctions have been shown useful to model rent seeking, political campaigns and R&D races. There is a well-known connection between all-pay auctions and *contests* [21]. In particular, the all-pay auction can be viewed as a single-prize contest, where the payments correspond to the effort that players make in order to win the competition.

In this paper, we study the efficiency of mixed Nash equilibria in all-pay auctions with complete information, from a worst-case analysis perspective, using the *price of anarchy* [16] as a measure. As social objective, we consider the *social welfare*, i.e. the sum of the bidders' valuations. We study the equilibria induced from all-pay mechanisms in three fundamental resource allocation scenarios; combinatorial auctions, multi-unit auctions and single-item auctions.

In a combinatorial auction a set of items are allocated to a group of selfish individuals. Each player has different preferences for different subsets of items

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and this is expressed via a *valuation set* function. A multi-unit auction can be considered as an important special case, where there are multiple copies of a single good. Hence the valuations of the players are not set functions, but depend only on the number of copies received. Multi-unit auctions have been extensively studied since the seminal work by Vickrey [24]. As already mentioned, all-pay auctions have received a lot of attention for the case of a single item, as they model all-pay contests and procurements via contests.

### 1.1 Contribution

*Combinatorial Auctions.* Our first result is on the price of anarchy of simultaneous all-pay auctions with item-bidding that was previously studied by Syrgkanis and Tardos [23]. For fractionally subadditive valuations, it was previously shown that the price of anarchy was at most 2 [23] and at least $e/(e-1) \approx 1.58$ [8]. We narrow further this gap, by improving the upper bound to 1.82. In order to obtain the bound, we come up with several structural theorems that characterize mixed Nash equilibria in simultaneous all-pay auctions.

*Multi-unit Auctions.* Our next result shows a novel use of all-pay mechanisms to the multi-unit setting. We propose an all-pay mechanism with a randomized allocation rule inspired by Kelly's seminal proportional allocation mechanism [15]. We show that this mechanism admits a *unique*, 75% efficient *pure* Nash equilibrium and no other mixed Nash equilibria exist, when bidders' valuations are submodular. As a consequence, the price of anarchy of our mechanism outperforms all current price of anarchy bounds of prevalent multi-unit auctions including uniform price [18] and discriminatory [14] auctions, with bound $e/(e-1)$.

*Single-item Auctions.* Finally, we study the efficiency of a single-prize contest that can be modeled as a single-item all-pay auction. We show a tight bound on the price of anarchy for mixed equilibria which is approximately 1.185. By following previous study on the procurement via contest, we further study two other standard objectives, *revenue* and *maximum bid*. We evaluate the performance of all-pay auctions in the prior-free setting, i.e. no distribution over bidders' valuation is assumed. We show that both the revenue and the maximum bid of any mixed Nash equilibrium are at least as high as $v_2/2$, where $v_2$ is the second highest valuation. In contrast, the revenue and the maximum bid in some mixed Nash equilibrium may be less than $v_2/2$ when using reward structure other than allocating the entire reward to the highest bidder. This result coincides with the optimal crowdsourcing contest developed in [6] for the setting with prior distributions. We also show that in conventional procurements (modeled by first-price auctions), $v_2$ is exactly the revenue and maximum bid in the worst equilibrium. So procurement via all-pay contests is a 2-approximation to the conventional procurement in the context of worst-case equilibria.

### 1.2 Related work

The inefficiency of Nash equilibria in auctions has been a well-known fact (see e.g. [17]). Existence of efficient equilibria of simultaneous sealed bid auctions in full information settings was first studied by Bikhchandani [3]. Christodoulou, Kovács and Schapira [7] initiated the study of the (Bayesian) price of anarchy of simultaneous auctions with item-bidding. Several variants have been studied since then [2, 12, 11], as well as multi-unit auctions [14, 18].

Syrgkanis and Tardos [23] proposed a general smoothness framework for several types of mechanisms and applied it to settings with fractionally subadditive bidders obtaining several upper bounds (e.g., first price auction, all-pay auction, and multi-unit auction). Christodoulou et al. [8] constructed tight lower bounds for first-price auctions and showed a tight price of anarchy bound of 2 for all-pay auctions with subadditive valuations. Roughgarden [20] presented an elegant methodology to provide price of anarchy lower bounds via a reduction from the hardness of the underlying optimization problems.

All-pay auctions and contests have been studied extensively in economic theory. Baye, Kovenock and de Vries [1], fully characterized the Nash equilibria in single-item all-pay auction with complete information. The connection between all-pay auctions and crowdsourcing contests was proposed in [9]. Chawla et al. [6] studied the design of optimal crowdsourcing contest when agents' value are drawn independently from a specific distribution.

## 2 Preliminaries

In a *combinatorial auction*, $n$ *players* compete on $m$ *items*. Every player (or *bidder*) $i \in [n]$ has a valuation function $v_i : \{0,1\}^m \to \mathbb{R}^+$ which is monotone and normalized, that is, $\forall S \subseteq T \subseteq [m]$, $v_i(S) \leq v_i(T)$, and $v_i(\emptyset) = 0$. The outcome of the auction is represented by a tuple of $(\mathbf{X}, \mathbf{p})$ where $\mathbf{X} = (X_1, \dots, X_n)$ specifies the allocation of items ($X_i$ is the set of items allocated to player $i$) and $\mathbf{p} = (p_1, \dots, p_n)$ specifies the buyers' payments ($p_i$ is the payment of player $i$ for the allocation $\mathbf{X}$). In the *simultaneous item-bidding* auction, every player $i \in [n]$ submits a non-negative bid $b_{ij}$ for each item $j \in [m]$. The items are then allocated by independent auctions, i.e. the allocation and payment rule for item $j$ only depend on the players' bids on item $j$. In a simultaneous *all-pay* auction the allocation and payment for each player is determined as follows: each item $j \in [m]$ is allocated to the bidder $i^*$ with the highest bid for that item, i.e. $i^* = \arg\max_i b_{ij}$, and each bidder $i$ is charged an amount equal to $p_i = \sum_{j \in [m]} b_{ij}$. It is worth mentioning that, for any bidder profile, there always exists a tie-breaking rule such that mixed equilibria exist [22]. Actually, our results hold for arbitrary tie-breaking rule. For completeness, we specify a tie-breaking rule where the mechanism will allocate the item to a winner picked uniformly from all highest bidders as in [1].

**Definition 1 (Valuations).** *Let $v : 2^{[m]} \to \mathbb{R}$ be a valuation function. Then $v$ is called a)* additive*, if $v(S) = \sum_{j \in S} v(\{j\})$; b)* submodular*, if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$; c)* fractionally subadditive *or* XOS*, if $v$ is determined by a finite set of additive valuations $\xi_k$ such that $v(S) = \max_k \xi_k(S)$.*

The classes of the above valuations are in increasing order of inclusion.

*Multi-unit Auction.* In a multi-unit auction, $m$ copies of an item are sold to $n$ bidders. Here, bidder $i$ 's valuation is a function that depends on the number of copies he gets. That is $v_i : \{0, 1, \ldots, m\} \to \mathbb{R}^+$ and it is non-decreasing and normalized, with $v_i(0) = 0$. We say a valuation $v_i$ is *submodular*, if it has non-increasing marginal values, i.e. $v_i(s+1) - v_i(s) \geq v_i(t+1) - v_i(t)$ for all $s \leq t$.

*Nash equilibrium and price of anarchy.* We use $b_i$ to denote a pure strategy of player $i$ which might be a single value or a vector, depending on the auction. So, for the case of $m$ simultaneous auctions, $b_i = (b_{i1}, \ldots, b_{im})$. We denote by $\mathbf{b}_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ the strategies of all players except for $i$. Any *mixed strategy* $B_i$ of player $i$ is a probability distribution over pure strategies.

For any profile of strategies, $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{X}(\mathbf{b})$ denotes the allocation under the strategy profile $\mathbf{b}$. The valuation of player $i$ for the allocation $\mathbf{X}(\mathbf{b})$ is denoted by $v_i(\mathbf{X}(\mathbf{b})) = v_i(\mathbf{b})$. The *utility* $u_i$ of player $i$ is defined as the difference between her valuation and payment: $u_i(\mathbf{X}(\mathbf{b})) = u_i(\mathbf{b}) = v_i(\mathbf{b}) - p_i(\mathbf{b})$.

**Definition 2 (Nash equilibria).** *A bidding profile $\mathbf{b} = (b_1, \ldots, b_n)$ forms a pure Nash equilibrium if for every player i and all bids $b'_i$, $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$. Similarly, a mixed bidding profile $\mathbf{B} = \times_i B_i$ is a mixed Nash equilibrium if for all bids $b'_i$ and every player $i$, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$. Clearly, any pure Nash equilibrium is also a mixed Nash equilibrium.*

Our global objective is to maximize the sum of the valuations of the players for their received allocations, i.e., to maximize the *social welfare* $SW(\mathbf{X}) = \sum_{i\in[n]} v_i(X_i)$. So $\mathbf{O}(\mathbf{v}) = \mathbf{O} = (O_1, \ldots, O_n)$ is an *optimal allocation* if $SW(\mathbf{O}) = \max_{\mathbf{X}} SW(\mathbf{X})$. In Sect. 5, we also study two other objectives: the *revenue*, which equals the sum of the payments, $\sum_i p_i$, and the *maximum payment*, $\max_i b_i$. We also refer to the maximum payment as the *maximum bid.*

**Definition 3 (Price of anarchy).** *Let $\mathcal{I}([n], [m], \mathbf{v})$ be the set of all instances, i.e. $\mathcal{I}([n], [m], \mathbf{v})$ includes the instances for every set of bidders and items and any possible valuation functions. The mixed price of anarchy, PoA, of a mechanism is defined as*

$$PoA = \max_{I\in\mathcal{I}} \max_{\mathbf{B}\in\mathcal{E}(I)} \frac{SW(\mathbf{O})}{\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{X}(\mathbf{b}))]} \enspace ,$$

*where $\mathcal{E}(I)$ is the class of mixed Nash equilibria for the instance $I \in \mathcal{I}$. The pure PoA is defined as above but restricted in the class of pure Nash equilibria.*

Let $\mathbf{B} = (B_1, \ldots, B_n)$ be a profile of mixed strategies. Given the profile $\mathbf{B}$, we fix the notation for the following *cumulative distribution functions (CDF):* $G_{ij}$ is the CDF of the bid of player $i$ for item $j$; $F_j$ is the CDF of the highest bid for item $j$ and $F_{ij}$ is the CDF of the highest bid for item $j$ if we exclude the bid of player $i$. Observe that $F_j = \prod_k G_{kj}$ and $F_{ij} = \prod_{k\neq i} G_{kj}$. We also use $\varphi_{ij}(x)$ to denote the probability that player $i$ gets item $j$ by bidding $x$. Then, $\varphi_{ij}(x) \leq F_{ij}(x)$. When we refer to a single item, we may drop the index $j$. Whenever it is clear from the context, we will use shorter notation for expectations, e.g. we use $\mathbb{E}[u_i(\mathbf{b})]$ instead of $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$, or even $SW(\mathbf{B})$ to denote $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{X}(\mathbf{b}))]$.

## 3 Combinatorial Auctions

In this section we prove an upper bound of 1.82 for the mixed price of anarchy of simultaneous all-pay auctions when bidders' valuations are fractionally subadditive (XOS). This result improves over the previously known bound of 2 due to [23]. We first state our main theorem and present the key ingredients. Then we prove these ingredients in the following subsections. Due to space limitation, we give proofs of the lemmas and theorems in the full version.

**Theorem 4.** *The mixed price of anarchy for simultaneous all-pay auctions with fractionally subadditive (XOS) bidders is at most* 1.82*.*

*Proof.* Given a valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, let $\mathbf{O} = (O_1, \ldots, O_n)$ be a fixed optimal solution, that maximizes the social welfare. We can safely assume that $\mathbf{O}$ is a partition of the items. Since $v_i$ is an XOS valuation, let $\xi_i^{O_i}$ be a maximizing additive function with respect to $O_i$. For every item $j$ we denote by $o_j$ item $j$'s contribution to the optimal social welfare, that is, $o_j = \xi_i^{O_i}(j)$, where $i$ is such that $j \in O_i$. The optimal social welfare is thus $SW(\mathbf{O}) = \sum_j o_j$. In order to bound the price of anarchy, we consider only items with $o_j > 0$, as it is without loss of generality to omit items with $o_j = 0$.

For a fixed mixed Nash equilibrium $\mathbf{B}$, recall that by $F_j$ and $F_{ij}$ we denote the CDFs of the maximum bid on item $j$ among all bidders, with and without the bid of bidder $i$, respectively. For any item $j \in O_i$, let $A_j = \max_{x \geq 0} \{F_{ij}(x) o_j - x\}$.

As a key part of the proof we use the following two inequalities that bound from below the social welfare in any mixed Nash equilibrium $\mathbf{B}$.

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \left( A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx \right) \;, \tag{1}$$

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx \;. \tag{2}$$

Inequality (1) suffices to provide a weaker upper bound of 2 (see [8]). The proof of (2) is much more involved, and requires a deeper understanding of the equilibria properties of the induced game. We postpone their proofs in Sect. 3.1 (Lemma 5) and Sect. 3.2 (Lemma 6), respectively. By combining (1) and (2),

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \cdot \sum_j \left( A_j + \int_0^{o_j - A_j} \left( 1 - F_j(x) + \lambda \cdot \sqrt{F_j(x)} \right) dx \right) \;, \tag{3}$$

for every $\lambda \geq 0$. It suffices to bound from below the right-hand side of (3) with respect to the optimal social welfare. For any cumulative distribution function $F$, and any positive real number $v$, let

$$R(F, v) \stackrel{\text{def}}{=} A + \int_0^{v-A} (1 - F(x)) dx + \lambda \cdot \int_0^{v-A} \sqrt{F(x)} dx \;,$$

where $A = \max_{x \geq 0}\{F(x) \cdot v - x\}$. Inequality (3) can then be rewritten as $SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j)$. Finally, we show a lower bound of $R(F, v)$ that holds for any CDF $F$ and any positive real $v$.

$$R(F, v) \geq \frac{3 + 4\lambda - \lambda^4}{6} \cdot v \ . \tag{4}$$

The proof of (4) is given in Sect. 3.3 (Lemma 9). Finally, we obtain that for any $\lambda > 0$,

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j) \geq \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot \sum_j o_j = \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot SW(\mathbf{O}) \ .$$

By taking $\lambda = 0.56$, we conclude that the price of anarchy is at most 1.82. $\square$

### 3.1 Proof of Inequality (1)

This section is devoted to the proof of the following lower bound. Recall that the definition $o_j$ is from the definition of XOS functions.

**Lemma 5.** $SW(\mathbf{B}) \geq \sum_{j \in [m]} (A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx)$.

*Proof.* Recall that $A_j = \max_{x_j \geq 0} \{F_{ij}(x) o_j - x_j\}$. We can bound bidder $i$'s utility in the Nash equilibrium $\mathbf{B}$ by $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$. To see this, consider the deviation for bidder $i$, where he bids only for items in $O_i$, namely, for each item $j$, he bids the value $x_j$ that maximizes the expression $F_{ij}(x_j) o_j - x_j$. Since for any obtained subset $T \subseteq O_i$, he has value $v_i(T) \geq \sum_{j \in T} o_j$, and the bids $x_j$ must be paid in any case, the expected utility with these bids is at least $\sum_{j \in O_i} \max_{x_j \geq 0} (F_{ij}(x) o_j - x_j) = \sum_{j \in O_i} A_j$. With $\mathbf{B}$ being an equilibrium, we infer that $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$. By summing up over all bidders,

$$\begin{aligned} SW(\mathbf{B}) &= \sum_{i \in [n]} u_i(\mathbf{B}) + \sum_{i \in [n]} \sum_{j \in [m]} \mathbb{E}[b_{ij}] \geq \sum_{j \in [m]} A_j + \sum_{j \in [m]} \sum_{i \in [n]} \mathbb{E}[b_{ij}] \\ &\geq \sum_{j \in [m]} (A_j + \mathbb{E}[\max_{i \in [n]} \{b_{ij}\}]) \geq \sum_{j \in [m]} \left( A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx \right) \ . \end{aligned}$$

The first equality holds because $\sum_i \mathbb{E}_{\mathbf{b}}[v_i(\mathbf{b})] = \sum_i \mathbb{E}_{\mathbf{b}}[u_i(\mathbf{b}) + \sum_{j \in [m]} b_{ij}]$. The second inequality follows because $\sum_i b_{ij} \geq \max_i b_{ij}$ and the last one is implied by the definition of the expected value of any positive random variable. $\square$

### 3.2 Proof of Inequality (2)

Here, we prove the following lemma for any mixed Nash equilibrium $\mathbf{B}$.

**Lemma 6.** $SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx$.

First we show a useful lemma that holds for XOS valuations. We will further use the technical Proposition 8.

**Lemma 7.** *For any fractionally subadditive (XOS) valuation function $v$,*

$$v(S) \geq \sum_{j \in [m]} \left(v(S) - v(S \setminus \{j\})\right) \ .$$

*Proof.* Let $\xi$ be a maximizing additive function of $S$ for the XOS valuation $v$. By definition, $v(S) = \xi(S)$ and for every $j$, $v(S \setminus \{j\}) \geq \xi(S \setminus \{j\})$. Then, $\sum_{j \in [m]} (v(S) - v(S \setminus \{j\})) \leq \sum_{j \in S} (\xi(S) - \xi(S \setminus \{j\})) = \sum_{j \in S} \xi(j) = v(S)$. □

**Proposition 8.** *For any integer $n \geq 2$, any positive reals $G_i \leq 1$ and positive reals $g_i$, for $1 \leq i \leq n$,*

$$\sum_{i=1}^{n} \frac{g_i}{\sum_{k \neq i} \frac{g_k}{G_k}} \geq \sqrt{\prod_{i=1}^{n} G_i} \ .$$

We are now ready to prove Lemma 6. We only state a proof sketch here to illustrate the main ideas.

*Proof (Sketch of Lemma 6).* Recall that $G_{ij}$ is the CDF of the bid of player $i$ for item $j$. For simplicity, we assume $G_{ij}(x)$ is continuous and differentiable, with $g_{ij}(x)$ being the PDF of player $i$'s bid for item $j$. First, we define the *expected marginal valuation* of item $j$ w.r.t player $i$,

$$v_{ij}(x) \stackrel{\text{def}}{=} \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}}[v_i(X_i(\mathbf{b}) \cup \{j\}) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \ .$$

Given the above definition and a careful characterization of mixed Nash equilibria, we are able to show $F_{ij}(x) \cdot v_{ij}(x) = \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x]$ and $\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx}$ for any $x$ in the support of $G_{ij}$. Let $g_{ij}(x)$ be the derivative of $G_{ij}(x)$. Using Lemma 7, we have

$$\begin{aligned} SW(\mathbf{B}) &= \sum_i \mathbb{E}[v_i(X_i(\mathbf{b}))] \geq \sum_i \sum_j \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})] \\ &\geq \sum_i \sum_j \int_0^{o_j - A_j} \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \cdot g_{ij}(x) dx \\ &\geq \sum_i \sum_j \int_0^{o_j - A_j} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) dx \ , \end{aligned}$$

where the second inequality follows by the law of total probability. By using the facts that $F_{ij}(x) = \prod_{k \neq i} G_{kj}(x)$ and $\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx}$, for any $x > 0$ such that $g_{ij}(x) > 0$ ($x$ is in the support of player $i$) and $F_j(x) > 0$, we obtain

$$F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) = \frac{F_{ij}(x) \cdot g_{ij}(x)}{\frac{dF_{ij}}{dx}(x)} = \frac{\prod_{k \neq i} G_{kj}(x) \cdot g_{ij}(x)}{\sum_{k \neq i} \left( g_{kj} \cdot \prod_{s \neq k \wedge s \neq i} G_{sj} \right)} = \frac{g_{ij}(x)}{\sum_{k \neq i} \frac{g_{kj}(x)}{G_{kj}(x)}} \ .$$

For every $x > 0$, we use Proposition 8 only over the set $S$ of players with $g_{ij}(x) > 0$. After summing over all bidders we get,

$$\sum_{i\in[n]} F_{ij}(x)\cdot v_{ij}(x)\cdot g_{ij}(x) \geq \sum_{i\in S} \frac{g_{ij}(x)}{\sum_{k\neq i,k\in S}\frac{g_{kj}}{G_{kj}}} \geq \sqrt{\prod_{i\in S} G_{ij}(x)} \geq \sqrt{F_j(x)} \ .$$

The above inequality also holds for $F_j(x) = 0$. Finally, by merging the above inequalities, we conclude that $SW(\mathbf{B}) \geq \sum_{j\in[m]} \int_0^{o_j - A_j} \sqrt{F_j(x)}dx$. □

### 3.3 Proof of Inequality (4)

In this section we prove the following technical lemma.

**Lemma 9.** *For any CDF $F$ and any real $v > 0$, $R(F, v) \geq \frac{3+4\lambda-\lambda^4}{6}v$.*

In order to obtain a lower bound for $R(F, v)$ as stated in the lemma, we show first that we can restrict attention to cumulative distribution functions of a simple special form, since these constitute worst cases for $R(F, v)$. In the next lemma, for an arbitrary CDF $F$ we will define a simple piecewise linear function $\hat{F}$ that satisfies the following two properties:

$$\int_0^{v-A}(1-\hat{F}(x))dx = \int_0^{v-A}(1-F(x))dx \ ; \quad \int_0^{v-A}\sqrt{\hat{F}(x)}dx \leq \int_0^{v-A}\sqrt{F(x)}dx \ .$$

Once we establish this, it is convenient to lower bound $R(\hat{F}, v)$ for the given type of piecewise linear functions $\hat{F}$.

**Lemma 10.** *For any CDF $F$ and real $v > 0$, there always exists another CDF $\hat{F}$ such that $R(F, v) \geq R(\hat{F}, v)$ that, for $A = \max_{x\geq 0}\{F(x)\cdot v - x\}$, is defined by*

$$\hat{F}(x) = \begin{cases} 0 & , \textit{ if } x \in [0, x_0] \\ \frac{x+A}{v} & , \textit{ if } x \in (x_0, v-A] \end{cases} \ .$$

Now we are ready to proceed with the proof of Lemma 9.

*Proof (of Lemma 9).* By Lemma 10, for any fixed $v > 0$, we only need to consider the CDF's in the following form: for any positive $A$ and $x_0$ such that $x_0 + A \leq v$,

$$F(x) = \begin{cases} 0 & , \text{ if } x \in [0, x_0] \\ \frac{x+A}{v} & , \text{ if } x \in (x_0, v-A] \end{cases} \ .$$

Clearly, $\max_{x\geq 0}\{F(x)\cdot v - x\} = A$. Let $t = \frac{A+x_0}{v}$. Then

$$\begin{aligned} R(F, v) &= A + \int_0^{v-A}(1 - F(x))dx + \lambda\cdot\int_0^{v-A}\sqrt{F(x)}dx \\ &= A + v - A - \frac{v}{2}\cdot\left(\frac{x+A}{v}\right)^2\Bigg|_{x_0}^{v-A} + \lambda\cdot\frac{2v}{3}\cdot\left(\frac{x+A}{v}\right)^{\frac{3}{2}}\Bigg|_{x_0}^{v-A} \\ &= v - \frac{v}{2}\cdot(1-t^2) + \lambda\cdot\frac{2v}{3}\cdot(1-t^{\frac{3}{2}}) = v\cdot\left(\frac{1}{2}(1+t^2) + \frac{2\lambda}{3}(1-t^{\frac{3}{2}})\right) \ . \end{aligned}$$

By optimizing over $t$, the above formula is minimized when $t = \lambda^2 \leq 1$. That is,

$$R(F, v) \geq v \cdot \left( \frac{1}{2}(1 + \lambda^4) + \frac{2\lambda}{3}(1 - \lambda^3) \right) = \frac{3 + 4\lambda - \lambda^4}{6} \cdot v \ . \qquad \square$$

## 4 Multi-unit Auctions

In this section, we propose a randomized all-pay mechanism for the multi-unit setting, where $m$ identical items are to be allocated to $n$ bidders. Markakis and Telelis [18] and de Keijzer et al. [14] have studied the price of anarchy for several multi-unit auction formats. The current best upper bound obtained was 1.58 for both pure and mixed Nash equilibria.

We propose a *randomized* all-pay mechanism that induces a *unique pure* Nash equilibrium, with an improved price of anarchy bound of 4/3. We call the mechanism Random proportional-share allocation mechanism (PSAM), as it is a randomized version of Kelly's celebrated proportional-share allocation mechanism for divisible resources [15]. The mechanism works as follows (illustrated as Mechanism 1).

Each bidder submits a non-negative real $b_i$ to the auctioneer. After soliciting all the bids from the bidders, the auctioneer associates a real number $x_i$ with bidder $i$ that is equal to $x_i = \frac{m \cdot b_i}{\sum_{i \in [n]} b_i}$. Each player pays their bid, $p_i = b_i$. In the degenerate case, where $\sum_i b_i = 0$, then $x_i = 0$ and $p_i = 0$ for all $i$.

We turn the $x_i$'s to a random allocation as follows. Each bidder $i$ secures $\lfloor x_i \rfloor$ items and gets one more item with probability $x_i - \lfloor x_i \rfloor$. An application of the Birkhoff-von Neumann decomposition theorem guarantees that given an allocation vector $(x_1, x_2, \ldots, x_n)$ with $\sum_i x_i = m$, one can always find a randomized allocation[1] with random variables $X_1, X_2, \ldots, X_n$ such that $\mathbb{E}[X_i] = x_i$ and $\Pr[\lfloor x_i \rfloor \leq X_i \leq \lceil x_i \rceil] = 1$ (see for example [10, 4]).

We next show that the game induced by the Random PSAM when the bidders have submodular valuations is *isomorphic* to the game induced by Kelly's mechanism for a single divisible resource when bidders have piece-wise linear concave valuations.

**Theorem 11.** *Any game induced by the Random PSAM applied to the multi-unit setting with submodular bidders is* isomorphic *to a game induced from Kelly's mechanism applied to a single divisible resource with piece-wise linear concave functions.*

*Proof.* For each bidder $i$'s submodular valuation function $f_i : \{0, 1, \ldots, m\} \to R^+$, we associate a concave function $g_i : [0, 1] \to R^+$ such that,

$$\forall x \in [0, m], \ \ g_i(x/m) = f_i(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (f_i(\lfloor x \rfloor + 1) - f_i(\lfloor x \rfloor)) \ . \qquad (5)$$

[1] As an example, assume $x_1 = 2.5, x_2 = 1.6, x_3 = 1.9$. One can define a random allocation such that assignments $(3, 2, 1)$, $(3, 1, 2)$ and $(2, 2, 2)$ occur with probabilities 0.1, 0.4, and 0.5 respectively.

**Mechanism 1:** Random PSAM

**Input**: Total number of items $m$ and all bidders' bid $b_1, b_2, \ldots, b_n$
**Output**: Ex-post allocations $X_1, X_2, \ldots, X_n$ and payments $p_1, p_2, \ldots, p_n$
**if** $\sum_{i\in[n]} b_i > 0$ **then**
  **foreach** *bidder* $i = 1, 2, \ldots, n$ **do**
    $x_i \leftarrow \frac{m\cdot b_i}{\sum_{i\in[n]} b_i}$;
    $p_i \leftarrow b_i$;
  Sample $\{X_i\}_{i\in[n]}$ from $\{x_i\}_{i\in[n]}$ by using Birkhoff-von Neumann decomposition theorem such that $\lfloor x_i \rfloor \le X \le \lceil x_i \rceil$ and the expectation of sampling $X_i$ is $x_i$;
**else** Set $\mathbf{X} = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$;
Return $X_i$ and $p_i$ for all $i \in [n]$;

Essentially, $g_i$ is the piecewise linear function that comprises the line segments that connect $f_i(k)$ with $f_i(k+1)$, for all nonnegative integers $k$. It is easy to see that $g_i$ is concave if $f_i$ is submodular. We use identity functions as the bijections $\phi^i$ in the definition of game isomorphism. Therefore, it suffices to show that, for any pure strategy profile $\mathbf{b}$, $u_i(\mathbf{b}) = u'_i(\mathbf{b})$, where $u_i$ and $u'_i$ are the bidder $i$'s utility functions in the first and second game, respectively. Let $x_i = \frac{m\cdot b_i}{\sum_i b_i}$, then

$$\begin{aligned} u_i(\mathbf{b}) &= (x_i - \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor + 1) + (1 - x_i + \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor) - b_i \\ &= f_i(\lfloor x_i \rfloor) + (x_i - \lfloor x_i \rfloor)(f_i(\lfloor x_i \rfloor + 1) - f_i(\lfloor x_i \rfloor)) - b_i \\ &= g_i\left(\frac{x_i}{m}\right) - b_i = g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i = u'_i(\mathbf{b}) \ . \end{aligned}$$

Note that $g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i$ is player $i$'s utility, under $\mathbf{b}$, in Kelly's mechanism. □

We next show an equivalence between the optimal welfare.

**Lemma 12.** *The optimum social welfare in the multi-unit setting, with submodular valuations $\mathbf{f} = (f_1, \ldots, f_n)$, is equal to the optimal social welfare in the divisible resource allocation setting with concave valuations $\mathbf{g} = (g_1, \ldots g_n)$, where $g_i(x/m) = f_i(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (f_i(\lfloor x \rfloor + 1) - f_i(\lfloor x \rfloor))$.*

Theorem 11 and Lemma 12, allow us to obtain the existence and uniqueness of the pure Nash equilibrium, as well as the price of anarchy bounds of Random PSAM by the corresponing results on Kelly's mechanism for a single divisible resource [13]. Moreover, it can be shown that there are no other mixed equilibria by adopting the arguments of [5] for Kelly's mechanism. The main conclusion of this section is summarized in the following Corollary.

**Corollary 13.** *Random PSAM induces a* unique pure *Nash equilibrium when applied to the multi-unit setting with submodular bidders. Moreover, the price of anarchy of the mechanism is* exactly $4/3$.

## 5 Single item auctions

In this section, we study mixed Nash equilibria in the single item all-pay auction. First, we measure the inefficiency of mixed Nash equilibria, showing tight results for the price of anarchy. En route, we also show that the price of anarchy is $8/7$ for two players. Then we analyze the quality of two other important criteria, the *expected revenue (the sum of bids)* and the quality of the expected *highest submission (the maximum bid)*, which is a standard objective in crowdsourcing contests [6]. For these objectives, we show a tight lower bound of $v_2/2$, where $v_2$ is the second highest value among all bidders' valuations. In the following, we drop the word expected while referring to the revenue or to the maximum bid.

We quantify the loss of revenue and the highest submission in the worst-case equilibria. We show that the all-pay auction achieves a 2-approximation comparing to the conventional procurement (modeled as the first price auction), when considering worst-case mixed Nash equilibria; we show that the revenue and the maximum bid of the conventional procurement equals $v_2$ in the worst case. We also consider other structures of rewards allocation and conclude that allocating the entire reward to the highest bidder is the only way to guarantee the approximation factor of 2. Roughly speaking, allocating all the reward to the top prize is the optimal way to maximize the maximum bid and revenue among all the prior-free all-pay mechanisms where the designer has no prior information about the participants.

Due to the lack of space we give the proofs of theorems and lemmas in the full version.

**Theorem 14.** *The mixed price of anarchy of the single item all-pay auction is* 1.185*.*

**Theorem 15.** *In any mixed Nash equilibrium of the single-item all-pay auction, the revenue and the maximum bid are at least half of the second highest valuation.*

**Lemma 16.** *For any $\epsilon > 0$, there exists a valuation vector $\mathbf{v} = (v_1, \ldots, v_n)$, such that in a mixed Nash equilibrium of the induced single-item all-pay auction, the revenue and the maximum bid is at most $v_2/2 + \epsilon$.*

Finally, the next theorem indicates that allocating the entire reward to the highest bidder is the best choice. In particular a prior-free all-pay mechanism is presented by a probability vector $\mathbf{q} = (q_i)_{i \in [n]}$, with $\sum_{i \in [n]} q_i = 1$, where $q_i$ is the probability that the $i^{th}$ highest bidder is allocated the item, for every $i \leq n$. Note that the reward structure considered here does not depend on the index of the bidder, i.e. the mechanisms are anonymous.

**Theorem 17.** *For any prior-free all-pay mechanism that assigns the item to the highest bidder with probability strictly less than $1$, i.e. $q_1 < 1$, there exists a valuation profile and mixed Nash equilibrium such that the revenue and the maximum bid are strictly less than $v_2/2$.*

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